Contents

Chapter 1

Countable sets

1.1 Notation

$\mathbb{N} = \mathbb{N} \setminus \{0\}$

1.2 Def

S is **infinitely countable** if $\exists S \to \mathbb{N}$ bijection, **countable** if S is finite or inf-countable

Remark

- for sequence $< S_n >_{n\in\mathbb{N}}$
$$\begin{aligned} \mathbb{N} &\to S \\ n &\mapsto S_n \end{aligned}$$
- if $S \neq \varnothing$ then TFAE:
  - S is countable
  - $\exists$ surjection $\mathbb{N} \to S$
  - $\exists$ injection $S \to \mathbb{N}$
- $\mathbb{Q}$ is inf-countable
- if $m \in \mathbb{N}_0 S_1, \cdots, S_m$ are countable. Then $\prod_{j=1}^{m} S_j$ is countable.

1.3 Cantor Theorem

$\mathbb{N}$ is not equinumberous with $\wp(\mathbb{N})$

Proof

$\wp(\mathbb{N}) \cong \{0,1\}^{\mathbb{N}}$ if $A \in \wp(\mathbb{N})$ then

$$\begin{aligned} \mathbb{1}_A : \mathbb{N} &\to \{0,1\} \\ n &\mapsto \begin{cases} 1 \text{ if } n \in A \\ 0 \text{ if } n \notin A \end{cases} \end{aligned}$$

the identify of A:

$$\begin{aligned} \wp(\mathbb{N}) &\to \{0,1\}^{\mathbb{N}} \\ A &\mapsto \mathbb{1}_A \end{aligned}$$

is a bijection

$$\{0,1\}^{\mathbb{N}} = \mathscr{F}(\mathbb{N}; \{0,1\})$$

Remark

A,B be sets. $\mathscr{F}(A; B)$ is the set of all functions from A to B.

Proof

Assume that $\exists$ bijection

$$\begin{aligned} \mathbb{N} &\to \wp(\mathbb{N}) \\ n &\mapsto f_n \end{aligned}$$

Define

$$\begin{aligned} f : \mathbb{N} &\to \{0,1\} \\ n &\mapsto \begin{cases} 0 \text{ if } f_n(n) = 1 \\ 1 \text{ if } f_n(n) = 0 \end{cases} \end{aligned}$$

$f \in \mathscr{F}(\mathbb{N}; \{0,1\})$ thus $\exists m \in \mathbb{N}$ s.t. $f = f_m$. Then $f_m(m)$ broken.

Chapter 2

Number Series

2.1 Def

$\sum\limits_{n=0}^{+\infty} a_n$ is **commutatively convergent** (CC) if for each permutation ϕ of $\mathbb{N}$ the series $\sum\limits_{n=0}^{+\infty} a_{\phi(n)}$ converges.

Remark

A.C. is **absolutely convergent**.
C. is **convergent**. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection.

- if $\sum\limits_{n=0}^{+\infty} a_n$ is A.C. then $\sum\limits_{n=0}^{+\infty} a_n$ C.
- $\sum\limits_{n=0}^{+\infty} \frac{(-1)^n}{n}$ C. but not A.C. or C.C.

2.2 Riemann Theorem

Let $\sum\limits_{n=0}^{+\infty} a_n$ be a convergent series in $\mathbb{R}$ TFAE:

- $\sum\limits_{n=0}^{+\infty} a_n$ is not A.C.
- $\forall s \in \mathbb{R}$ $\exists$ permutation of $\mathbb{N}$ s.t.

$$\sum_{n=0}^{+\infty} a_{\phi(n)} = s$$

- $\forall s \in \mathbb{R} \cup \{-\infty, +\infty\}$ $\exists$permutation of $\mathbb{N}$ s.t.

$$\sum_{n=0}^{+\infty} a_{\phi(n)} = s$$

Chapter 3

Kurzneil-Henstock integral

3.1 Def

Cell is a non-degenerated interval

3.2 Nested cell theorem

If $< I_n >_{n\in\mathbb{N}}$ is a decreasing sequence ($I_{n+1} \subseteq I_n$) of compact cells s.t.

$$\lim_{N\to+\infty} diam I_n = 0$$

then $\exists x \in \mathbb{R}$

$$\bigcap_{n\in\mathbb{N}} I_n = \{x\}$$

3.3 Exercises

Every cell is uncountable.

3.4 Def

Two cells are **non-overlapping** if their intersection either empty or a singleton.

3.5 Exercises

If I_1, I_2, I_3 are pairwise non-overlapping, then

$$I_1 \cap I_2 \cup I_3 = \varnothing$$

3.6 Lemma

If I is a compact cell and $N \in \mathbb{N}_0$ are pairwise non-overlapping cells s.t. $\bigcup_{n=1}^{N} I_n = I$ then renumbering them if necessary, we may get:

$$\min I = \min I_1$$
$$\max I_n = \min I_{n+1}$$
$$\max I_N = \max I$$

3.7 Def

A **partial division** Δ of I is a finite set consisting of non-overlapping compact sub-cells of I. If

$$\bigcup \Delta = I$$

it's called a **division** of I

3.8 Lemma

If Δ is a partial division of I, then there exists a partial Δ' of I s.t. $\Delta \cap \Delta'$ is a division of I

3.9 Def

A **gauge** on I is a function

$$\delta : I \to \mathbb{R}$$

such that $\forall x \in I \ \delta(x) > 0$

Remark

If $\delta_1, \cdots, \delta_N$ are gauges on I then

$$\delta(x) = \min\{\delta_1(x), \cdots, \delta_N(x)\}$$

is also a gauge.

3.10 Def

A **partial P-division** of a compact cell I, is a finite Π of pairs (J, x) s.t.

- $J \subseteq I$
- J is a compact cell

- $x \in J$
- $\forall (J_1, x), (J_2, x_2) \in \Pi$ if $J_1 \neq J_2$ then J_1, J_2 are non-overlapping

x is cal tag of the pair.

3.11 Def

Given a partial P-division Π of I define

$$body(\Pi) = \bigcup \{J : (J, x) \in \Pi\}$$

A **P-division** Π of I is a partial P-division s.t. $body(\Pi) = I$

3.12 Lemmas

- If $\Pi_1, \cdots, \Pi_N$ are partial P-divisions of I s.t. for each $n, m \in \{1, \cdots, N\}, n \neq m$ $body\Pi_n$ and $body\Pi_m$ are either disjoint or their intersection is a singleton, then $\bigcup\limits_{n=1}^{N} \Pi_n$ is a partial P-division of I.
- If Π is a partial P-division of I and $\xi \in I$ then there're at most 2 $(J, x) \in \Pi$ s.t. $x = \xi$

3.13 Def

Let δ be a gauge on I and Π a (partial) P-division of I, we say that Π is δ-finite if

$$\forall (J, x) \in \Pi \quad J \subseteq [x - \delta(x), x + \delta(x)]$$

3.14 Def

If $f : I \to \mathbb{R}$ and Π is a (partial) P-division then the **Riemann sum** is defined as

$$S(\Pi, f) := \sum_{(J,x) \in \Pi} f(x) \, |J|$$

3.15 Def

Let $f : I \to \mathbb{R}$ f is **KH-integrable** on I if $\exists r \in \mathbb{R}, \forall \epsilon > 0 \exists$ gauge δ on I $\forall \delta$-finite P-division Π of I

$$|S(\Pi, f) - r| < \epsilon$$

3.16 Prop

r is unique

Proof

Assume that r_1 and r_2. Fix $\epsilon > 0$. For $i = 1, 2$, there's a gauge δ_i on I s.t. if Π is a δ_i-finite P-division of I then

$$|S(\Pi, f) - r_i| < \epsilon$$

$$\begin{aligned} |r_1 - r_2| &= |r_1 - S(\Pi, f) + S(\Pi, f) - r_2| \\ &\leq |r_1 - S(\Pi, f)| + |S(\Pi, f) - r_2| \\ &< 2\epsilon \end{aligned}$$

Let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ then δ is a gauge on I. If Π is δ-finite then it's δ_1-finite and δ_2-finite.

3.17 Cousin Theorem

I be a compact cell and δ a gauge on I. Then there exists a δ-finite then P-division of I.

Proof

assume there's no. Then divide I into I_l, I_r by middle. Then either I_l, I_r has no δ-finite division. Then we get a decreasing sequence $(I_n)_{n\in\mathbb{N}}$ by keeping dividing. According to nested theorem, get their intersection a singleton x. Notice that x is a point of I, for $N \in \mathbb{N}$ big enough

$$diam I_N = 2^{-N} \cdot diam I < \delta(x)$$

then $\Pi = \{(I_N, x)\}$ is a δ-finite P-division of I_N.

3.18 Notation

$$r = \int_I f = \int_I f(x)\mathrm{d}x$$

if $I = [a, b]$

$$r = \int_a^b f = \int_a^b f(x)\mathrm{d}x$$

3.19 Prop of Riemann Sum

linearity $\forall \Pi$(partial)P-division, $\forall f_1, f_2 : I \to \mathbb{R},\ \forall \alpha \in \mathbb{R}$

$$S(\Pi, \alpha f_1 + f_2) = \alpha S(\Pi, f_1) + S(\Pi, f_2)$$

monotonicity

$$f_1 \leq f_2 \ \Rightarrow\ S(\Pi, f_1) \leq S(\Pi, f_2)$$

additivity if Π_1, Π_2 are partial P-division of I and $(body\Pi_1) \cap (body\Pi_2)$ is either empty or a finite set, then $\forall f$

$$S(\Pi_1 \cup \Pi_2) = S(\Pi_1, f) + S(\Pi_2, f)$$

3.20 Prop of KH-integral

I a compact cell

3.21 Prop: Constant functions

If $f : I \to \mathbb{R}$ is constant then $f \in KH(I)$ and $\int_I f = y \cdot |I|$. (y is the constant value of f)

Proof

$\forall \Pi$ P-division of I

$$S(\Pi, f) = \sum_{(J,x)\in\Pi} f(x)\,|J| = y \sum_{(J,x)\in\Pi} |J| = y\,|I|$$

3.22 Theorem

$KH(I)$ is a vector space and $KH(I) \to \mathbb{R}, f \mapsto \int_I f$ is linear and monotone.

Proof

$0, \mathbb{1}_I \in KH(I)$
If $f_1, f_2 \in KH(I)$ and $\alpha \in \mathbb{R}$, we want to show that $\alpha f_1 + f_2 \in KH(I)$ and

$$\int_I (\alpha f_1 + f_2) = \alpha \int_I f_1 + \int_I f_2$$

Let $\epsilon > 0$, δ_1 be a gauge on I, $\frac{\epsilon}{2(|\alpha|+1)}$-adapted to f_1 and δ_2 $\frac{\epsilon}{2}$-adapted. Def

$$\delta = \min\{\delta_1, \delta_2\}$$

Let Π be a δ-finite P-division of I

$$\begin{aligned}\left|S(\Pi,\alpha S(\Pi,f_1)+S(\Pi,f_2))-(\alpha\int_I f_1+\int_I f_2)\right| &= \left|\alpha S(\Pi,f_1)+S(\Pi,f_2)-(\alpha\int_I f_1+\int_I f_2)\right|\\ &\leq |\alpha|\left|S(\Pi,f_1)-\int_I f_1\right|+\left|S(\Pi,f_2)-\int_I f_2\right|\\ &\leq |\alpha|\frac{\epsilon}{2(|\alpha|+1)}+\frac{\epsilon}{2}\leq\epsilon\end{aligned}$$

3.23 Cauchy criterion

Let $f : I \to \mathbb{R}$, TFAE:

- $f \in KH(I)$
- $\forall\epsilon > 0$ $\exists$gauge δ on I s.t. $\forall\Pi, \Pi$ is δ-finite P-division of I

$$\left|S(\Pi,f)-\int_I f\right|<\epsilon$$

Proof

$1 \Rightarrow 2$

trivial

$2 \Rightarrow 1$

For each $n \in \mathbb{N}_0$, we apply hypothesis (2) with $\epsilon = \frac{1}{n}$ and we obtain a gauge δ_n, define

$$\hat{\delta_n} = \min_{i=1}^{n} \delta_i$$

choose Π_n a $\hat{\delta_n}$-finite

Let $r_n := S(\Pi_n, f)$. We show that $\langle r_n\rangle_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Let $0 < p < q \in \mathbb{N}_0$

$$|r_p - r_q| = |S(\Pi_p, f) - S(\Pi_q, f)| < \frac{1}{p}$$

Name $r := \lim_{n\to+\infty} r_n$, now we show that f is KH-integrable with $\int_I f = r$. Let $\epsilon > 0$, choose $n_0 \in \mathbb{N}_0$ large enough for $\frac{1}{n_0} < \epsilon$. We claim that $\hat{\delta_n}$ is a gauge with integrability of f. $\forall\Pi\hat{\delta_n}$-finite, for each $n \geq n_0$, we have:

$$\begin{aligned}|S(\Pi,f)-r| &\leq |S(\Pi,f)-S(\Pi_n,f)|+|S(\Pi_n,f)-r|\\ &\leq \frac{1}{n_0}+|r_n-r|\\ &\leq \epsilon+\epsilon\end{aligned}$$

3.24 Example: Dirichlet function

$$f : \mathbb{R} \to \mathbb{R} : \mathbb{1}_{\mathbb{Q}}$$

Let I be a compact cell, we want to show

$$f\mid_I \in KH(I) \qquad \int_I f\mid_i = 0$$

We deal with $S(\Pi, \mathbb{1}_{\mathbb{Q}}) = \sum_{(J,x)\in\Pi, x\in\mathbb{Q}} |J|$. For $\mathbb{Q}$ countable:

$$\exists q : \mathbb{N} \overset{q}{\cong} I \cap \mathbb{Q}$$

Let $\epsilon > 0$, we define δ on $I \cap \mathbb{Q}$ as follows:

- If $x \in I \cap \mathbb{Q}$, then $x = q(n)$ for some n and let $\delta(x) = \frac{\epsilon}{2^n}$
- If $x \in I \setminus \mathbb{Q}$, then define $\delta(x) = 1$

Let Π be δ-finite,

$$\begin{aligned} S(\Pi, \mathbb{1}_{\mathbb{Q}}) &= \sum_{(J,x)\in\Pi, x\in\mathbb{Q}} |J| \\ &\leq \sum_{n=0}^{\infty} \sum_{(J,x)\in\Pi, x\in\mathbb{Q}} |J| \\ &\leq \sum_{n=0}^{\infty} 2 \cdot 2 \cdot \frac{\epsilon}{2^n} = 8\epsilon \end{aligned}$$

Exercises

If $\mathbb{1}_{\mathbb{Q}\cap I}$ Riemann integrable?

3.25 Theorem

Let $f \in KH(I), g : I \to \mathbb{R}$ s.t. $\{f \neq g\}$ is countable. Then $g \in KH(I)$ and $\int_I g = \int_I f$

3.26 Theorem: subordinate P-division

Let I be a compact cell and Δ be a division of I. There exists a gauge δ on I satisfying the following properties:
$\forall\delta$-finite P-division Π of I,

- $\forall K \in \Delta,\ \exists$P-division Π_K of K

- There exists a P-division $\tilde{\Pi}$ of I s.t.

  A $\tilde{\Pi} = \bigcup_{K \in \Delta} \Pi_K$

  B $\forall f : I \to \mathbb{R}$
  $$S(\Pi, f) = S(\tilde{\Pi}, f) = \sum_{K \in \Delta} S(\Pi_K, f_K)$$

  C For every gauge η on I, if Π is η-finite, then each Π_K is $\eta \mid_K$-finite, $K \in \Delta$

Proof

$$\delta(x) = \begin{cases} dist(x, F) & \text{if } x \notin F \\ dist(x, F \setminus \{x\}) \text{if } x \in F \end{cases}$$

3.27 Finite-additivity

Let $\{I_1, \cdots, I_N\}$ be a division of a compact cell I and $f : I \to \mathbb{R}$ TFAE

- $f \in KH(I)$
- $f \mid_{I_n} \in KH(I_n), \forall n \in \{1, \cdots, N\}$, In this case,
$$\int_I f = \sum_{I_n} f \mid_{I_n}$$

Proof

$1 \Rightarrow 2$

Let $J \subseteq I$ be a compact cell and assume

$$I = J_1 \sqcup J \sqcup J_2 \quad (J_1 \leq J \leq J_2)$$

We want to show that $f \mid_J \in KH(J)$. Apply Cauchy criterion for this. Let $\epsilon > 0$ We need to find a gauge δ_0 on J s.t. Π_0 is $delta_0$-finite P-division, then

$$|S(\Pi_0, f \mid_J) - S(\Pi_0, f \mid_J)| < \epsilon$$

For $\epsilon > 0$, $\exists \delta$ gauge on I ϵ-adapted to f. We define:

- $\delta_1 = \delta \mid_{J_1}$ then Π_1 is δ_1-finite
- $\delta_0 = \delta \mid_J$ then Π_0 is δ_0-finite
- $\delta_2 = \delta \mid_{J_2}$ then Π_2 is δ_2-finite

so
$$S(\Pi, f) = S(\Pi_1, f \mid_{J_1}) + S(\Pi_0, f \mid_J) + S(\Pi_2, f \mid_{J_2})$$

$2 \Rightarrow 1$

trivial

3.28 Theorem

If $f \in KH(I)$ and $J \subseteq I$ is a compact cell, then $f|_J \in KH(I)$ and

$$\int_J f|_J = \int_I \mathbb{1}_J \cdot f$$

3.29 Def: step function

$f : I \to \mathbb{R}$ is a **step function** if there exists a division Δ of I s.t. $\forall J \in \Delta$, $f|_{\mathring{J}}$ is constant.

3.30 Theorem

Every step function on I is JH-integrable.

3.31 Theorem

If (f_n) a sequence in $KH(I)$ that converges uniformly on I to $f : I \to \mathbb{R}$, then $f \in KH(I)$

3.32 Def:regulated function

A **regulated function** $f : I \to \mathbb{R}$ is a function which is a limit of a sequence of step functions.

3.33 Corollary

Every regulated function on I is KH-integrable.

3.34 Prop

- Every continuous function $f : I \to \mathbb{R}$ is regulated
- Every monotone function $f : I \to \mathbb{R}$ is regulated.

Chapter 4

Fundamental theorem of calculus

4.1 Theorem

If $F : I \to \mathbb{R}$ is diff. (differentiable) everywhere, then $F' \in KH(I)$ and

$$\int_I F' = F(\max I) - F(\min I)$$

4.1.1 Lemma

If f is diff. at $x \in I$ then$\forall \epsilon\, \exists \delta > 0$ s.t. $\forall y \leq x \leq z, y, z \in I$, $\max\{|y - x|, |x - z|\} < \delta$, then

$$|F(z) - F(y) - F'(x)(z - y)| < \epsilon |z - y|$$

Proof of lemma

$$\begin{aligned}
&|F(z) - F(x) + F(x) - F(y) - F'(x)(z - x + x - y)| \\
\leq &\epsilon |z - x| + \epsilon |y - x| \\
= &\epsilon |y - z|
\end{aligned}$$

Proof

Let $\epsilon > 0$, $\forall x \in I$, there exists $\delta(x) > 0$ s.t. $\forall$ compact cell $J \subseteq I$, with $x \in J \subseteq [x - \delta(x), x + \delta(x)]$

$$|F(\max J) - F(\min J) - F'(x) |J|| < \epsilon |J|$$

If Π is a δ-finite P-division of I. We want to show

$$|S(\Pi, F') - F(\max I) + F(\min I)| < \epsilon |I|$$

Basically

$$S(\Pi, F') = \sum_{(J,x)\in\Pi} F'(x)\,|J|$$

$$F(\max I) - F(\min I) = \sum_{(J,x)\in\Pi} \left(F(\max J) - F(\min J)\right)$$

$$\begin{aligned} &|S(\Pi, F') - F(\max I) + F(\min I)| \\ \leq &\sum_{(J,x)\in\Pi} |F'(x)\,|J| - F(\max J) + F(\min J)| \\ \leq &\epsilon\,|I| \end{aligned}$$

Chapter 5

Change of variables

5.1 Theorem: change of variable

$$I \xrightarrow{\phi} \tilde{I} \xrightarrow{f} \mathbb{R}$$

I and $\tilde{I}$ be compact cells, $\phi : I \leftrightarrow \tilde{I}$ be a (monotone) bijection which is diff. everywhere on I. If $f \in KH(\tilde{I})$ then $(f \circ \phi)\,|\phi'| \in KH(I)$ and

$$\int_I (f \circ \phi)\,|\phi'| = \int_{\tilde{I}} f\,|f'|$$

Proof

Let $\epsilon > 0$, exists a gauge $\tilde{\delta}$ on $\tilde{I}$ s.t. if $\tilde{\Pi}$ is a $\tilde{\delta}$-finite P-division, then

$$\left| S(\tilde{\Pi}, f) - \int_{\tilde{I}} f\,|f| \right| < \epsilon$$

If Π is any P-division, then we can associate with if $\tilde{\Pi} = \{(\phi(J), \phi(x)) \mid (J, x) \in \Pi\}$ which is a P-division of $\tilde{I}$

Since ϕ is uniformly continuous on I, there exists $\eta :]0, +\infty[\to]0, +\infty[$ s.t. $\forall \delta > 0, \forall x, y \in I$ have

$$|x - y| \leq \eta(\delta) \;\Rightarrow\; |\phi(x) - \phi(y)| \leq \delta$$

Only a different interpretation of uniformly continuous. We define a gauge δ_1 on I:

$$\delta_1(x) = \eta \circ \tilde{\delta} \circ \phi(x)$$

Remark

If Π is a δ_1-finite P-division of I then $\tilde{\Pi}$ is a $\tilde{\delta}$-finite P-division of $\tilde{I}$

$$J = [y, z] \subseteq [x - \delta_1(x), x + \delta_1(x)]$$

$$\max\{|y-x|\,,|x-z|\} \leq \delta_1(x) = \eta \circ \tilde{\delta} \circ \phi(x)$$
$$\max\{|\phi(y)-\phi(x)|\,,|\phi(x)-\phi(z)|\} \leq \tilde{\delta} \circ \phi(x)$$

Given $x \in I$, we define $\epsilon(x) = \frac{\epsilon}{1+|f\circ\phi(x)|}$. By lemma4.1.1, there exists a $\delta_2(x) > 0$ s.t. if $J = [y,z] \subseteq [x-\delta_2(x), x+\delta_2(x)] \subseteq I$ contains x and then

$$\begin{aligned} ||\phi(J)| - |\phi'(x)|\cdot|J|| &= ||\phi(y)-\phi(z)| - |\phi'(x)|\cdot|z-y|| \\ &= |\phi(z)-\phi(y)-\phi'(x)(z-y)| \\ &< \epsilon(x)\,|z-y| = \epsilon(x)\,|J| \end{aligned}$$

Define a gauge δ on I by $\delta = \min\{\delta_1, \delta_2\}$. If Π is a δ-finite P-division of I then

$$\begin{aligned} \left|\int_{\tilde{I}} f - S(\Pi, (f\circ\phi)\cdot|\phi'|)\right| &\leq \left|\int_{\tilde{I}} f - S(\tilde{\Pi}, f)\right| + \left|S(\tilde{\Pi}, f) - S(\Pi, (f\circ\phi)\cdot|\phi'|)\right| \\ &\leq \sum_{(J,x)\in\Pi} |f\circ\phi(x)|\cdot||\phi(J)| - |\phi'(x)|\cdot|J|| \\ &\leq \sum_{(J,x)\in\Pi} |f\circ\phi(x)|\cdot\epsilon(x)\cdot|J| \\ &\leq \epsilon\,|I| \end{aligned}$$

Chapter 6

Integral on the real line

6.1 Saks-Henstock's theorem

Let I be a compact cell and $f \in KH(I)$ and $\epsilon > 0$ and δ a gauge on I which is ϵ-adapted to f. If Π is a partial δ-finite P-division of I then:

- $$\left| \sum_{(J,x)\in\Pi} \left(\int_J f \mid_J - f(x)\,|J| \right) \right| \leq \epsilon$$

- $$\sum_{(J,x)\in\Pi} \left| \int_J f \mid_J - f(x)\,|J| \right|$$

Proof

$1 \Rightarrow 2$

Given Π define

$$\Pi^+ = \Pi \cap \left\{ (J,x) \mid \int_J f \mid_J - f(x)\,|J| \geq 0 \right\}$$

$$\Pi^- = \Pi \cap \left\{ (J,x) \mid \int_J f \mid_J - f(x)\,|J| < 0 \right\}$$

let $\pi = \Pi^+ \sqcup \Pi^-$, then

$$\sum_{(J,x)\in\Pi^+} \left| \int_J f \mid_J - f(x)\,|J| \right| + \left| \sum_{(J,x)\in\Pi^+} \int_J f \mid_J - f(x)\,|J| \right| \leq \epsilon$$

the same for Π^-

prove (1)

$\Delta_\Pi = \{J \mid (J,x) \in \Pi\}$ is a partial division of I. There exists another partial division Δ' of I s.t. $\Delta \cup \Delta_\Pi$ is a division of I.

Let $\eta > 0$, $\forall K \in \Delta'$, there exists a gauge δ_K on K, η-adapted to $f\mid_K \in KH(K)$. Define $\tilde{\delta}_K(x) = \min\{\delta_K(x), \delta(x)\}, x \in K$, a gauge on K. Let Π_K be a $\delta\delta_K$-finite P-division of K. Then

$$\left| \int_K - S(\Pi_K, f) \right| < \eta$$

Define $\tilde{\Pi} = \Pi \cup \left(\bigcup_{K \in \Delta'} \Pi_K \right)$ is a P-division of I and is δ-finite. Since δ is a ϵ-adpated to f and $\tilde{\Pi}$ is δ-finite on I, we have:

$$\left| \int_I f - S(\tilde{\Pi}, f) \right| < \epsilon$$

$$S(\tilde{\Pi}, f) = \sum_{(J,x)\in\Pi} f(x)\,|J| + \sum_{K\in\Delta'} S(\Pi_K, f)$$

$$\int_I f = \sum_{(J,x)\in\Pi} \int_I f\mid_J + \sum_K f\mid_K$$

then

$$\begin{aligned}
\left| \sum_{(J,x)\in\Pi} \int_J f\mid_J - f(x)\,|J| \right| &\le \left| \int_I f - S(\tilde{\Pi}, f) \right| + \left| \sum_{K\in\Delta'} \left(\int_K f - S(\Pi_K, f) \right) \right| \\
&< \epsilon + \sum_{K\in\Delta'} \left| \int_K f - S(\Pi_K, f) \right| \\
&\le \epsilon + \eta \cdot (card \Delta')
\end{aligned}$$

6.2 Hake Theorem

Let I be a compact cell, $f : I \to \mathbb{R}$ and for $0 < \eta < |I|$, put

$$I_\eta = [\eta + \min I, \max I]$$

TFAE

- $f \in KH(I)$
- $\forall \eta$,

$$f\mid_{I_\eta} \in KH(I_\eta) \text{ and } \lim_{\eta\to 0} \int_{I_\eta} f\mid_{I_\eta} \text{ exists}$$

In this case,

$$\int_I f = \lim_{eta\to 0} \int_{I_\eta} f\mid_{I_\eta}$$

6.3 Corollary

If $f \in KH(I)$, then the **indefinite integral** of f

$$\begin{aligned} F : I &\to \mathbb{R} \\ x &\mapsto \begin{cases} \int_{[\min I, x]} f & \text{if} x > \min I \\ 0 & \text{if} x = \min I \end{cases} \end{aligned}$$

is continuous by Hake Theorem6.2

$$\int f := F$$

6.4 Prop

TFAE

- $f \in KH(I)$
- $\exists$ continuous function $F : I \to \mathbb{R}$ s.t. $\forall \epsilon > 0$ $\exists$ a gauge δ on I, $\forall$ partial δfinite P-division Π of I:

$$\sum_{(J,x) \in \Pi} |f(x)\,|J| - F(\max J) + F(\min J)| < \epsilon$$

6.5 Def: KH-integral

A function $f : \mathbb{R} \to \mathbb{R}$ is KH-integrable if:
$\exists F : \mathbb{R} \to \mathbb{R}$ and $\lim_{x \to -\infty} F(x)$ and $\lim_{x \to +\infty} F(x)$ exists. $\forall \epsilon > 0$ $\exists$ a gauge δ on $\mathbb{R}$ s.t. $\forall \Pi$ partial δ-finite P-division :

$$\sum_{(J,x) \in \Pi} |f(x)\,|J| - F(\max J) + F(\min J)| < \epsilon$$

and define

$$\int_{\mathbb{R}} f := \lim_{x \to +\infty} F(x) - \lim_{x \to -\infty} F(x)$$

Chapter 7

Monotone Converges & Lebesgue's Measure

I be a **closed** cell.

7.1 Def: AKH-integrable

$$AKH(I) = KH(I) \cap \{f \mid |f| \in KH(I)\}$$

7.2 Prop

TFAE

- $f \in AKH(I)$
- $$f^+ := \max\{f, 0\} \in KH(I) \quad f^- \min\{f, 0\} \in KH(I)$$

Proof

$1 \Rightarrow 2$

$$f^+ = \frac{|f| + f}{2} \quad f^- = \frac{|f| - f}{2}$$

$2 \Rightarrow 1$

$$f = f^+ - f^- \quad |f| = f^+ + f^-$$

7.3 Prop

Let $f \in AKH(I)$, then

1 $\left|\int_I f\right| \leq \int_I |f|$

2 $\forall J \subseteq I$, closed cell,

$$f\mid_J \in AKH(J)$$

7.4 Theorem

Let $f \in KH(I)$, then

$$f \in AKH(I)$$

iff

$$\sup\left\{\sum_{K\in\Delta}\left|\int_K f\mid_K\right| \middle| \Delta \text{ is a partial division of } I\right\} < +\infty$$

Proof

$\Rightarrow$

Trivial

$\Leftarrow$

Let $\epsilon > 0$. There exists a partial division Δ of I s.t.

$$v(f) < \frac{\epsilon}{2} + \sum_{K\in\Delta}\left|\int_K f\mid_K\right|$$

WLOG(without loss of generality), we assume that Δ is a division of I.

Let δ_1 be the gauge associated with Δ in the subordinate P-division theorem 3.26, δ_2 be $\frac{\epsilon}{4}$-adapted to f. Define

$$\delta = \min\{\delta_1, \delta_2\}$$

we claim that

$$|S(\Pi, |f|) - v(f)| < \epsilon$$

whenever Π is a δ-finite P-division of I.

Let $\Pi_K, K \in \Delta$, a P-divsion coming from the subordinate P-division theorem 3.26. Since Π is δ_1-finite

$$\begin{aligned}\sum_{(J,x)\in\Pi}\left|\int_J f\right| \le v(f) & \\ & \le \frac{\epsilon}{2} + \sum_{K\in\Delta}\left|\int_K f\right| \\ & = \frac{\epsilon}{2} + \sum_{K\in\Delta}\left|\sum_{(J,x)\in\Pi_K}\int_J f\right| \\ & \le \frac{\epsilon}{2} + \sum_{K\in\Delta}\sum_{(J,x)\in\Pi_K}\left|\int_J f\right| \\ & = \frac{\epsilon}{2} + \sum_{(J,x)\in\Pi}\left|\int_J f\right|\end{aligned}$$

$\Rightarrow$

$$\left|v(f) - \sum_{(J,x)\in\Pi}\left|\int_J f\right|\right| < \frac{\epsilon}{2}$$

Since Π is δ_2-finite

$$\begin{aligned}\left|S(\Pi,|f|) - \sum_{(J,x)\in\Pi}\left|\int_J f\right|\right| &= \left|\sum_{(J,x)\in\Pi}|f(x)|\,|J| - \left|\int_J f\right|\right| \\ &\le \sum_{(J,x)\in\Pi}\left||f(x)|\,|J| - \left|\int_J f\right|\right| \\ &\le \sum_{(J,x)\in\Pi}\left|f(x)\,|J| - \int_J f\right| \\ &\le \frac{\epsilon}{2} \quad \text{by Saks-Henstock's theorem6.1}\end{aligned}$$

7.5 Prop(comparison test)

$f, g \in KH(I)$. If $|f| \le g$, then $|f| \in KH(I)$

Proof

If $K \subseteq I$ is a sub-cell, then $\left|\int_K f\right| \le \int_K g$

$$\sum_{K\in\Delta}\left|\int_K f\right| \le \sum_{K\in\Delta}\int_K g \le \int_I g < +\infty$$

Then finish by theorem 7.4

7.6 Prop

$AKH(I)$ is a vector space

Proof

- $0 \cdot \mathbb{1}_I \in AKH(I)$
- Let $f, g \in AKH(I), \alpha \in \mathbb{R}$

$$|\alpha f + g| \leq |\alpha| \cdot |f| + |g|$$

7.7 Def

- $$\|\cdot\| : AKH(I) \to \mathbb{R}$$

  is a semi norm

- A sequence $(F_N : i \to \mathbb{R})$ of functions is **increasing** if $(\forall x \in I)(\forall n \in \mathbb{N})$:

$$f_n(x) \leq f_{n+1}(x)$$

- (f_n) **converges pointwisely** to $f : I \to \mathbb{R}$ if $\forall x \in I$

$$f(x) = \lim_{n \to +\infty} f_n(x)$$

7.8 Monotone convergence theorem

Let (f_n) be a sequence in $KH(I)$ s.t.

A (f_n) is increasing

B (f_n) converges pointwisely to $f : I \to \mathbb{R}$

C $\sup\limits_{n \in \mathbb{N}} \int_I f_n \leq +\infty$

then $f \in KH(I)$ and

$$\int_I f = \lim_{n \to +\infty} \int_I f_n$$

Proof(R. Henstock)

Since $f_n \leq f_{n+1}$,

$$\int_I f_n \leq \int_I f_{n+1} \quad \forall n \in \mathbb{N}$$

Then $(\int_I f_n)$ is a increasing sequence in $\mathbb{R}$ and is bounded by (C). Thus it's convergent

$$r := \lim_{n \to +\infty} \int_I f_n$$

We'll check that f is KH-integrable on I by showing that f satisfies the def of KH-integral with r.

Let $\epsilon > 0$

- $(\exists n_0 \in \mathbb{N})(\forall n \geq n_0 \in \mathbb{N})$

$$r - \frac{\epsilon}{3} < \int_I f_n \leq r$$

- with (B): $(\forall x \in I)(\exists n(x) \geq n_0)(\forall n > n_0 \in \mathbb{N})$:

$$f(x) - \frac{\epsilon}{3\,|I|} < f_n(x) \leq f(x)$$

- Let $\epsilon_n = \frac{\epsilon}{3 \cdot 2^{n+2}}$. $\forall n \in \mathbb{N}$ there's a gauge δ_n on I which ϵ_n-adapted to f_n. Let$\delta(x) = \delta_{n(x)}(x)$

Let Π be a P-division of I:

$$\begin{aligned}
S(\Pi, f) = &\sum_{(J,x)\in\Pi} f(x)\,|J| \\
&= \sum_{(J,x)\in\Pi} \big(f(x) - f_{n(x)}(x)\big)\,|J| + \sum_{(J,x)\in\Pi} \left(f_{n(x)}(x)\,|J| - \int_J f_{n(x)}\right) + \sum_{(J,x)\in\Pi} \int_J f_{n(x)} \\
|S(\Pi, f) - r| &\leq \sum_{(J,x)\in\Pi} \frac{\epsilon}{3\,|I|}\,|J| + \underbrace{\sum_{(J,x)\in\Pi} \frac{\epsilon}{3 \cdot 2^{n(x)+2}}}_{\text{Saks-Henstock theorem}} + \underbrace{\left| r - \sum_{(J,x)\in\Pi} \int_J f_n(x) \right|}_{\text{monotonicity}} \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
\end{aligned}$$

Chapter 8

Lebesgue's measure

8.1 Def: integrability

$A \subseteq \mathbb{R}$ is **integrable** if $\nVdash_A \in KH(\mathbb{R})$ and define

$$L^1(A) = \int_{\mathbb{R}} \mathbb{1}_A$$

the **measure** of A.

8.2 Prop

- If I is a bounded cell then I is integrable and

$$L^1(I) = L^1(\mathring{I}) = L^1(\bar{I}) = \left|\bar{I}\right|$$

- If A is integrable, then

$$L^1 \geq 0$$

- If A and $B \supseteq A$ are integrable, then $B \setminus A$ is integrable and

$$L^1(B \setminus A) = L^1(B) - L^1(A)$$

- If A and B are integrable, then $A \cup B$ can $A \cap B$ are integrable.

- $N \in \mathbb{N}$ and $A_1, \cdots, A_N$ are disjoint integrable sets, then $\bigsqcup\limits_{i=1}^{N} A_i$ is integrable and

$$L^1(\bigsqcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} L^1(A_i)$$

- $(A_n)_{n\in\mathbb{N}}$ an increasing sequence of integrable sets and
$$\sup_{n\in\mathbb{N}} A_n < +\infty$$
then $\bigcup_{n\in\mathbb{N}} A_n$ is integrable and
$$L^1(\bigcup_{n\in\mathbb{N}} A_n) = \lim_{n\to+\infty} L^1(A_n)$$
- if $(A_n)_{n\in\mathbb{N}}$ disjoint sequence of integrable sets and $\sum_{n\in\mathbb{N}} L^1(A_n) < +\infty$, then $\bigsqcup_{n\in\mathbb{N}} A_n$ is integrable and
$$L^1(\bigsqcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} L^1(A_n)$$
- If $(A_n)_{n\in\mathbb{N}}$ a sequence of integrable sets s.t. $\sum_{n\in\mathbb{N}} L^1(A_n) < +\infty$, then $\bigcup_{n\in\mathbb{N}} A_n$ is integrable and
$$L^1(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} L^1(A_n)$$
- If $(A_n)_{n\in\mathbb{N}}$ a decreasing sequence of integrable sets, then $\bigcap_{n\in\mathbb{N}} A_n$ is integrable and
$$L^1(\bigcap_{n\in\mathbb{N}} A_n) = \lim_{n\to+\infty} L^1(A_n) = \inf_{n\in\mathbb{N}} A_n$$

8.3 Prop

- Each bounded open set if integrable
- Each bounded closed set is integrable

8.4 Def:measurable

Set $A \subseteq \mathbb{R}$ is **measurable** if $\forall I$ as compact cell $A \cap I$ is integrable.
$$\mathscr{M}(\mathbb{R}) = \wp(\mathbb{R}) \cap \{A \mid A \text{ measurable}\}$$
L^1 now is a mapping $L^1 : \mathscr{M}(\mathbb{R}) \to [0, +\infty]$ sending $A \in \mathscr{M}(\mathbb{R})$ to $\int_{\mathbb{R}} \mathbb{1}_A$ if A is integrable, otherwise $+\infty$

Remark

$$\begin{aligned} L^1 : \mathscr{M}(\mathbb{R}) &\to [0, +\infty] \\ A &\mapsto \begin{cases} \int_{\mathbb{R}} \mathbb{1}_A & \text{if A integrable} \\ +\infty & \text{otherwise} \end{cases} \end{aligned}$$

8.5 Prop

- $\varnothing \in \mathscr{M}(\mathbb{R})$
- If $A \in \mathscr{M}(\mathbb{R})$, then $\mathbb{R} \setminus \backslash A \mathscr{M}(\mathbb{R})$
- If (A_n) is a sequence in $\mathscr{M}(\mathbb{R})$ then

$$\bigcup_{n \in \mathbb{N}} A_n \in \mathscr{M}(\mathbb{R})$$

$$\bigcap_{n \in \mathbb{N}} A_n \in \mathscr{M}(\mathbb{R})$$

8.6 Lemma

If $A \subseteq B$ A and B are measurable and B is integrable, then A is integrable.

8.7 Theorem

- If A and B are both measurable and $A \subseteq B$ then $L^1(A) \leq L^1(B)$
- If (A_n) is a disjoint sequence in $\mathscr{M}(\mathbb{R})$ then

$$L^1(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} L^1(A_n)$$

- (A_n) increasing sequence in $\mathscr{M}(\mathbb{R})$

$$L^1(\bigcup_{n \in \mathbb{N}} A_n) = \sup_{n \in \mathbb{N}} L^1(A_n)$$

- (A_n) sequence in $\mathscr{M}(\mathbb{R})$

$$L^1(\bigcup_{n \in \mathbb{N}} A_n) \leq \sum_{n \in \mathbb{N}} L^1(A_n)$$

- All open and closed sets are measurable.

Chapter 9

Vitali sets

9.1 Def

Define a equivalence relation for any pair $x, y \in \mathbb{R}$

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$

9.2 Prop

Each equivalence class $[x]$ is dense in $\mathbb{R}$.

9.3 Def: Vitali set

$V \subseteq \mathbb{R}$ is a **Vitali set** if $\forall x \in \mathbb{R}$:

$$V \cap [x] \text{ is a singleton}$$

9.4 Prop

If V is a Vitali set and $q \in \mathbb{Q}$ then $q + V$ is a Vitali set.

9.5 Theorem

There exists a Vitali set.

Proof

Let $\wp := \wp(\mathbb{R}) \cap \{[x] \mid x \in \mathbb{R}\}$ be a partition of $\mathbb{R}$. Consider a selector (AC):

$$\Gamma : \wp \to \mathbb{R}$$

s.t $\forall C \in \wp$

$$\Gamma(C) \in C$$

Define

$$V := \mathbb{R} \cap \{\Gamma(C) \mid C \in \wp\} = \Im\Gamma$$

To finish the proof, check that $\forall C \in \wp$

- $V \cap C \neq \varnothing$
- $y_1, y_2 \in C \cap V \;\Rightarrow\; y_1 = y = 2$

9.6 Lemma

Let V be Vitali set, $q \in \mathbb{Q}$ ($V_q := q + V$)

A $\mathbb{R} = \bigcup\limits_{q \in \mathbb{Q}} V_q$

B $\forall q, r \in \mathbb{Q}$:

$$q \neq r \;\Rightarrow\; V_q \cap V_r = \varnothing$$

9.7 Theorem

If V is a Vitali set, $A \subseteq V$ is measurable, then $L^1(A) = 0$

Proof

For $n \in N_0$ define $A_0 := A \cap [-n, n]$ which is integrable. Define

$$Q := \mathbb{Q} \cap \{q \mid |q| \leq 1\}$$

and

$$B_{n,q} = q + A_n \subseteq q + A \subseteq q + V$$

By (B) in Lemma, the family $(B_{n,q})_{n,q \in Q}$ is disjoint. Moreover, $B_{n,q} \subseteq [-n - 1, n + 1]$ and $\forall q \in Q$

$$L^1(B_{n,q}) = L^1(A_n)$$

If $F \subseteq \mathbb{Q}$

$$(cardF)L^1(A_n) = \sum_{q \in F} L^1(B_{n,q}) = L^1\left(\bigcup_{q \in F} B_{n,q}\right)$$

Since F is infinite, $cardF$ can be chosen as large as Filip wishes, thus $L^1(A) = 0$

$$A = \bigcup_{n \in \mathbb{N}_0} A_n$$

and $(A_n)_{n \in \mathbb{N}_0}$ is increasing

$$L^1(A) = \lim_n L^1(A_n) = 0$$

9.8 Theorem

If V is Vitali set, then V is not measurable

Proof

If V were measurable, then $L^1(V) = 0$. By the previous theorem adn each V_q would be measurable with

$$L^1(V_q) = L^1(q+V) = L^1(V) = 0$$

By (A) of lemma9.6

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} V_q$$

$$L^1(\mathbb{R}) = \sum_{q \in \mathbb{Q}} L^1(V_q) = 0$$

Chapter 10

Measurable sharp

10.1 Def:refinement

Let X be a set and P, Q be two partitions of X. We say that Q is a **refinement** of P if each member of Q is contained in a member of P.

That's $\forall q \in Q$, $\exists p \in P$ s.t. $q \subseteq p$

10.2 Prop

A If Q is a refinement of P and $p \in P$, then

$$Q_p := Q \cap \{q \mid q \subseteq p\}$$

is a partition of P

B If Q is a refinement of P, R is a refinement of Q, then R is a refinement of P.

10.3 Def: dyodic cell

$I \subseteq \mathbb{R}$ is a **dyodic cell** there exists $k, j \in \mathbb{Z}$ s.t.

$$I(k,j) :=]\frac{j}{2^k}, \frac{j+1}{2^k}]$$

Remark

Notice j, k are uniquely determined by I. $L^1(I) = 2^{-k}$

$$genI := k$$

is called the **generation** of I

$$\mathscr{D}^1_k := \{\text{dyodic cells of generation } k\}$$

$$\mathscr{D}^1 := \bigcup_{k\in\mathbb{Z}} \mathscr{D}^1_k$$

10.4 Prop

A $\forall x \in \mathbb{R}, k, j \in \mathbb{Z}$

$$x \in I(k,j) \;\Leftrightarrow\; j = \left\lceil 2^k x \right\rceil - 1$$

B $\forall k \in \mathbb{Z}$, $\mathscr{D}^1_k$ is a partition of $\mathbb{R}$

C $\forall k, n \in \mathbb{Z}$

$$I(k,n) = I(k+1, 2n) \cup I(k+1, 2n+1)$$

Thus, $\mathscr{D}^1_{k+1}$ is a refinement of $\mathscr{D}^1_k$ and $\forall I \in \mathscr{D}^1_k$

$$card(\mathscr{D}^1_{k+1})_I = 2$$

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