# Contents

# Part I

# Set

# Chapter 1

# product

## 1.1 direct sum

$\bigoplus$ is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i\in I} V_i\{(x_i)_{i\in I} \in \prod_{i\in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0\}$$

# Chapter 2

# Ring

## 2.1 morphism

### Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \to B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

### Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
- 

### algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \to R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \to R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

### Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n\in\mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n\in\mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n\in\mathbb{N}} a_n T^n$ Such element is called a formal power series over k and $a_n$ is called the Coefficient of $T^n$ of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit $T^0$

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_n T^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]
$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$
let $F + G = (a_0 + b_0) + ...$
$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n$
Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n\right) (\sum\limits_{n\in\mathbb{N}} c_n T^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_p b_q c_l\right) T^n$
  is a trick applied on integral

Derivative:
let $F(T) \in k[T]$
We denote by $F'(T)$ *or* $\mathcal{D}(F(T))$ the formal power series
$\mathcal{D}(F) = \sum\limits_{n\in\mathbb{N}} (n+1) a_{n+1} T^n$
Properties:

- $\mathcal{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp
We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!} T^n$,which fulfil the differential equation $\mathcal{D}(exp(T)) = exp(T)$(interesting)
Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall (i,j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

# Part II

# Sequences

# Chapter 3

# Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i \in I} is\ a\ family\ of\ element\ in\ X, then\ \sup\limits_{i \in I} x_i\ denotes\ \sup\{x_i \mid i \in I\}(inX)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup\limits_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N},k\geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

# Chapter 4

# Interval

We fix a totally ordered set $(X, \leq)$
Notation:
If $(a, b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$
Def:
Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$,one has $[x, y] \subseteq I$ then we say that I is a interval in X
Example:
Let $(a, b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.
Let $\Lambda$ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X
- If $\bigcap\limits_{\lambda \in \Lambda} I_\lambda \neq \varnothing$, $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_] \mu$

- If $b \leq x \quad [a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a \quad [a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If a < x < b then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:
Let $(X, \leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.
Prop.
Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x, b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let $(X, \leq)$ be a totally ordered set .If $\forall (x, z) \in X \times X$,such that $x < z \quad \exists y \in X$ such that $x < y < z$,than we say that $(X, \leq)$ is thick

Prop.

Let $(X, \leq)$ be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[; ]a, b]; ]a, b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in ]a, b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b[\subseteq ]a, b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

# Chapter 5

# Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq$ *on* $\mathbb{R}$ *to* $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ become a totally ordered set, and $\mathbb{R} = ]-\infty, +\infty[$ Obviously,this is a thick totally ordered set.
We define:

- $\forall x \in ]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in ]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete
Example

Let $\Omega$ be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega)$, $\sup \mathscr{F} = \bigcup_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$
$\mathcal{AXION}$ :

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$
Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ *and* $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R}\backslash\{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$ ,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall(a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$ ,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup\limits_{x \in I} f(x) \quad b = \sup\limits_{y \in J} g(y) \quad c = \sup\limits_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup\limits_{x \in I, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup\limits_{x \in I} f(x))(\sup\limits_{x \in I} g(x))$

inf takes the similar($\leq \to \geq$) (provided when the sum are defined)

# Chapter 6

# Vector space

In this section:
K denotes a unitary ring.
Let 0 be zero element of K
1 be the unity of K

## 6.1 K-module

### 6.1.1 Def

Let $(V, +)$ be a commutative group.We call left/right K-module structure:
any mapping $\Phi$:$K \times V \to V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x))/\Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

### 6.1.2 Remark

Let $K^{op}$ be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a, b) \mapsto a + b$
- $K \times K \to K$
- $(a, b) \mapsto ba$

Then $K^{op}$ forms a unitary ring
Any left $K^{op}-module$ is a right K-module
Any right $K^{op}-module$ is a left K-module
$(K^{op})^{op}=K$

### 6.1.3 Notation

When we talk about a left/right K-module $(V,+)$,we often write its left K-module structure as $K\times V\to V \quad (a,x)\mapsto ax$

The axioms become:

$$\begin{aligned}
&\forall(a,b)\in K\times K,\forall x\in V \quad (ab)x=a(bx)/b(ax)\\
&\forall(a,b)\in K\times K,\forall x\in V \quad (a+b)x=ax+bx\\
&\forall a\in K,\forall(x,y)\in V\times V \quad a(x+y)=ax+ay\\
&\forall x\in V \quad 1x=x
\end{aligned}$$

### 6.1.4 K-vector space

If K is commutative,then $K^{op}=K$,so left K-module and right K-module structure are the same .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space

Let $\Phi:K\times V\to V$ be a left or right K-module structure

$$\forall x\in V,\Phi(\cdot,x):K\to V \quad (a\in K)\mapsto\Phi(a,x)$$

is a morphism of addition groups.Hence $\Phi(0,x)=0,\Phi(-a,x)=-\Phi(a,x)$
$\forall a\in K,\Phi(a,\cdot):V\to V$ is a morphism of groups.Hence $\Phi(a,0)=0,\Phi(a,-x)=-\Phi(a,x)(\cdot\ is\ a\ var)$

### 6.1.5 Association:

$\forall x\in K$

$$\begin{aligned}
(f(f+g)+h)(x)&=(f+g)(x)+h(x)=f(x)+g(x)+h(x)\\
(f+(g+h))(x)&=f(x)+((g+h)(x))=f(x)+g(x)+h(x)
\end{aligned}$$

Let $0:I\to K:x\mapsto 0 \quad \forall f\in K^I \quad f+0=f$
Let $-f:f+(-f)=0$
The mapping $K\times K^I\to K^I:(a,f)\mapsto af \quad (af)(x)=af(x)$ is a left K-module structure
The mapping $K\times K^I\to K^I:(a\in I)\mapsto((x\in I)\mapsto f(x)a) \quad (af)(x)=af(x)$ is a right K-module structure

### 6.1.6 Remark:

We can also write an element $\mu\ of\ K^I$ is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I\ by\ \mu$)
Then

$$(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I}$$
$$a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I}$$
$$(\mu_i)_{i\in I}a = (\mu_i a)_{i\in I}$$

## 6.2 sub K-module

### 6.2.1 Def

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub-K-module of V.

### 6.2.2 Example

Let I be a set .Let $K^{\bigoplus I}$ be the subset of $K^I$ composed of mappings $f : I \to K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub-K-module of $K^I$

In fact,$\forall (f, g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$
Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of $K^I$
$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

## 6.3 morphism of K-modules

### 6.3.1 Def

Let V and W be left K-module, A morphism of groups $\phi : V \to W$ is called a morphism of left K-modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

### 6.3.2 K-linear mapping

If K is commutative, a morphism of K-modules is also called a K-linear mapping. We denote by $\hom_{K-Mod}(V, W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of $W^V$

### 6.3.3 Theorem

Let V be a left K-module. Let I be a set.
The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I :\ \phi \to (\phi(e_i))_{i\in I}$ is a bijection where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

### 6.3.4 Remark:column

In the case where $I = 1, 2, 3, ..., n$ $V^I$ is denoted as $V^n$, $K^I$ is denoted as $K^n$ For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n\phi(e_i) = x_i$

We write this $\phi$ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$

## 6.4 kernel

### 6.4.1 Prop

Let G and H be groups and $f : G \to H$ be a morphism of groups

- $I_m(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- $f$ is injection iff $\ker(f) = \{e_G\}$

### 6.4.2 Def

$\ker(f)$ is called the kernel of $f$

### 6.4.3 Theorem

$f$ is injection iff $\ker(f) = \{e_G\}$

## Proof

Let $e_G$ and $e_H$ be neutral element of G and H respectively

(1) Let x and y be element of G
$f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H

(2) Let x and y be element of $\ker(f)$ One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H \quad e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$ So $\ker(f)$ is a subgroup of G

(3) Suppose that f is injection.
Since $f(E_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ Suppose that $\ker(f) = \{e_G\}$ If $f(x) = f(y)$then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$
Hence $xy^{-1} = e_G \Rightarrow x = y$

### 6.4.4 Def

Let (V,+) be a commutative group, I be a set. We define a composition law + on $V^I$ as follows

$$(x_i)_{i\in I} + (y_i)_{i\in I} \ := (x_i + y_i)_{i\in I}$$

Then $V^I$ forms a commutative group

### 6.4.5 Remark

Let E and F be left K-modules
$\hom_{K=Mod}(E,F) := \{$morphisms of left K-modules from E to F$\} \subseteq F^E$ is a subgroup of $F^E$
In fact f and g are elements of $\hom_{K-Mod}(E,F)$, then $f-g$ is also a morphism of left K-module
$(f-g)(x+y) = f(x+y) - g(x+y) = (f(x)+f(y)) - (g(x)+g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f-g)(x) + (f-y)(x)$

### 6.4.6 Theorem

Let V be a left K-module,I be a set The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I \ : \ \phi \mapsto (\phi(e_i))_i \in I$ is an isomorphism of groups, where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

### 6.4.7 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_I) + \psi(e_i)$
$\forall (\phi, psi) \in \hom_{K-Mod}(K^{\bigoplus I}, V)^2$
Hence $\Psi(\phi,\psi) = (\phi(e_i) + \psi(e_i))_{i\in I} = \Psi(\phi) + \Psi(\psi)$
So $\Psi$ is a morphism of groups

injectivity Let $\phi \in \hom_{K-Mod}(K^{\bigoplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$
Let $a = (a_i)_{i\in I} \in K^{\bigoplus I}$ One has $a = \sum_{i\in I} a_i e_i$
If fact,$\forall j \in I, a_j = \sum_{i\in I, a_i\neq 0} a_i e_i(j)$
Thus $\phi(a) = \sum_{i\in I, a_i \neq 0} a - I\phi(e_i) = 0$
Hence $\phi$ is the neutral element.

surjectivity Let $x = (x_i)_{i\in I} \in V^I$ We define $\phi_x : K^{\bigoplus I} \to V$ such that $\forall a = (a_i)_{i\in I} \in K^{\bigoplus I}, \phi_x(a) = \sum_{i\in I, a_i\neq 0} a_i x_i$
This is a morphism of left K-modules
$foralli \in I, \phi_x(e_i) = 1x_i = x_i$ So $\Psi(\phi_x) = x$

Suppose that K' is a unitary ring,and V is also equipped with a right K'-module structure, Then $\hom_{K-Mod}(K^{\bigoplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub-k'-module ,and $\Psi$ in the theorem is a right K'-module isomorphism

# Chapter 7

# Monotone mappings

## 7.1 Def

Let I and X be partially ordered sets,$f : I \to X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)/f(a) < f(b)$,then we say that $f$ is increasing/strictly increasing. decreasing takes similar way.
- If $f$ is (strictly) increasing or decreasing, we say that $f$ is (strictly) monotone.

## 7.2 Prop.

Let X,Y,Z be partially ordered sets.$f : X \to Y, g : Y \to Z$ be mappings

- If $f$ and $g$ have the same monotonicity, then $g \circ f$ is increasing
- If $f$ and $g$ have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

## 7.3 Def

Let $f$ be a function from a partially ordered set I to another partially ordered set .If $f \mid_{Dom(f)} \to X$ is (strictly) increasing/decreasing then we say that $f$ is (strictly) increasing/decreasing

## 7.4 Prop.

Let I and X be partially ordered sets. $f$ be function from I to X.

- If $f$ is increasing/decreasing and $f$ is injection, then $f$ is strictly increasing/decreasing
- Assume that I is totally ordered and $f$ is strictly monotone, then $f$ is injection

## 7.5 Prop

Let A be totally ordered set, B be a partially ordered set, $f$ be an injective function from A to B
If $f$ is increasing/decreasing ,then so is $f^{-1}$

## 7.6 Def

Let X and Y be partially ordered sets. $f : X \to Y$ be a bijection. If both $f$ and $f^{-1}$ are increasing ,then we say that $f$ is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \to Y$ is an isomorphism of partially ordered sets iff $f$ is a bijection and $f$ is increasing)

## 7.7 Prop.

Let I be a subset of $\mathbb{N}$ which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \to I$

## 7.8 Proof

### 7.8.1 bijection

We construct $f : \mathbb{N} \to I$ by induction as follows.
Let $f(0) = \min I$ Suppose that $f(0), ..., f(n)$ are constructed
then we take $f(n+1) := \min(I \backslash \{f(0), ..., f(n)\})$
Since $I \backslash \{f(0), ..., f(n-1)\} \supseteq I \backslash \{f(0), ..., f(n)\}$.Therefore $f(n) \le f(n+1)$
$Since f(n+1) \notin \{f(0), ..., f(n)\}$,we have $f(n) < f(n+1)$
Hence $f$ is strictly increasing and ths is injective
If $f$ is not surjective,then $I \backslash Im(f)$ has a element N.
Let $m = \min\{n \in \mathbb{N} \mid N \le f(n)\}$.
Since $N \notin Im(f), N < f(m)$.
So $m \neq 0$.Hence $f(m-1) < N < f(m) = \min(I \backslash \{f(0), ..., f(m-1)\})$
By definition,$N \in I \backslash Im(f) \subseteq I \backslash \{f(0), ..., f(m-1)\}$,
Hence $f(m) \le N$,causing contradiction.

### 7.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from $\mathbb{N}$ to $\mathbb{N}$

# Chapter 8

# sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

## 8.1 Def

Let X be a set.We call sequence in X parametrized by I a mapping from I to X.

## 8.2 Remark

If K is a unitary ring and E is a left K-module then the set of sequence $E^I$ admits a left-K-module structure. If $x = (x_n)_{n\in I}$ is a sequence in E, we define a sequence $\sum(x) := (\sum\limits_{i\in I, i\leq n} x_i)_{n\in\mathbb{N}}$,called the series associated with the sequence x.

## 8.3 Prop

$\sum : E^I \to E^{\mathbb{N}}$ is a morphism of left-K-module

## 8.4 proof

Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of $E^I$
$\sum\limits_{i\in I, i\leq n} (x_i + y_i) = (\sum\limits_{i\in I, i\leq n} x_i) + (\sum\limits_{i\in I, i\leq n} y_i), \lambda \sum\limits_{i\in I, i\leq n} x_i = \sum\limits_{i\in I, i\leq n} \lambda x_i$

## 8.5 Prop

Let I be a totally ordered set . X be a partially ordered set,$f : I \to X$ be a mapping ,$J \in I$ Assume that J does not have any upper bound in I

- If $f$ is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If $f$ is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

## 8.6 limit

### 8.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset.$\forall (x_i)_{n\in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$,we define:

$$\limsup_{n\in I, n\to+\infty} x_n := \inf_{n\in I}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n := \sup_{n\in I}(\inf_{i\in I, i\geq n} x_i)$$

If $\limsup_{n\in I, n\to+\infty} x_n = \liminf_{n\in I, n\to+\infty} x_n = l$, we then say that $(x_n)_{n\in I}$ tends to $l$ and that $l$ is the limit of $(x_n)_{n\in I}$. If in addition $(x_n)_{n\in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$,we say that $(x_n)_{n\in I}$ converges to $l$

### 8.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n\in I, n\to+\infty} = \inf_{n\in J}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n = \sup_{n\in J}(\inf_{i\in I, i\geq n} x_i)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i\in I}$ the limit superior and the limit inferior do not change.
In fact, if we take $J = \mathbb{N}\backslash\{0, ..., m\}$, then $\inf_{n\in J}(...)$ and $\sup_{n\in J}(...)$ only depends on the values of $x_i, i \in I, i \geq m$

### 8.6.3 Prop

$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$, $\liminf_{n\in I, n\to+\infty} x_n \leq \limsup_{n\in I, n\to+\infty} x_n$

### 8.6.4 Prop

Let $(x_n)_{n\in I} \in [-\infty, +\infty]^I$

$$\forall c \in \mathbb{R} \qquad \begin{aligned} \limsup_{n\in I, n\to+\infty}(x_n + c) &= (\limsup_{n\in I, n\to+\infty} x_n) + c \\ \liminf_{n\in I, n\to+\infty}(x_n + c) &= (\liminf_{n\in I, n\to+\infty} x_n) + c \end{aligned}$$

$$\forall c \in \mathbb{R}_{>0} \qquad \begin{aligned} \limsup_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \end{aligned}$$

$$\forall c \in \mathbb{R}_{<0} \qquad \begin{aligned} \limsup_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \end{aligned}$$

### 8.6.5 Prop

Let $(x_n)_{n\in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$,one has $x_n \leq y_n$ Then

$$\limsup_{n\in I, n\to+\infty}(x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

,

$$\liminf_{n\in I, n\to+\infty}(x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

### 8.6.6 Theorem

Let $(x_n)_{n\in I}, (y_n)_{n\in I}, (z_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$
Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n\in I}$ and $(z_n)_{n\in I}$ tend to the same limit $l$

Then $(y_n)_{n\in I}$ tends to $l$

### 8.6.7 Def

Let I be an infinite subset of $\mathbb{N}$, and $(x_n)_{n\in I}$ be a sequence in some set X. We call subsequence of $(x_n)_{n\in I}$ a sequence of the form $(x_n)_{n\in J}$,where J is an infinite subset of I

### 8.6.8 Prop

Let I and J be infinite subset of $\mathbb{N}$ such that $J \subseteq I$
$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$,one has

$$\liminf_{n\in I, n\to+\infty}(x_n) \leq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in I, n\to+\infty} (x_n) \geq \limsup_{n\in I, n\to+\infty} y_n$$

In particular, if $(x_n)_{n\in I}$ tends to $l \in [-\infty, +\infty]$,then $(x_n)_{n\in J}$ tends to $l$

### 8.6.9 Prop

$\forall n \in \mathbb{N}$,one has

$$\liminf_{n\in J, n\to+\infty} (x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in J, n\to+\infty} (x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

### 8.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_N)_{n\in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing,then $(x_N)_{i\in I}$ tends to $\sup\limits_{n\in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing,then $(x_N)_{i\in I}$ tends to $\inf\limits_{n\in I} x_n$

### 8.6.11 Notation

If a sequence $(x_N)_{n\in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim\limits_{n\in I, n\to} x_n$ denotes this limit $l$

### 8.6.12 Corollary

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{N}_{\geq 0}$ Then the series $\sum\limits_{n\in I} x_n$(the sequence $(\sum\limits_{i\in I, i\leq n})_{n\in\mathbb{N}})$ tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ It converges in $\mathbb{R}$ iff it is bounded from above (namely has an upper bound in $\mathbb{R}$)

### 8.6.13 Notation

If a series $\sum\limits_{n\in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum\limits_{n\in I} x_n$ to denote the limit

### 8.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n\in I}$ be a sequence in $[-\infty, +\infty]$ There exists a subsequence of $(x_n)_{n\in I}$ that tends to $\limsup\limits_{n\in I, n\to+\infty} x_n$ There exists a subsequence of $(x_n)_{n\in I}$ that rends to $\liminf\limits_{n\in I, n\to+\infty} x_n$

## Proof

Let $J = \{n \in I \mid \forall m \in I, \text{if } m \leq n \text{ then } x_m \leq x_n\}$

If $J$ is infinite, the sequence $(x_N)_{n\in J}$ is decreasing so it tends to $\inf\limits_{n\in J} x_n$

$\forall n \in J$ by definition $x_n = \sup\limits_{i\in I, i\geq n} x_i$ so $\limsup\limits_{n\in I, n\to+\infty} x_n = \inf\limits_{n\in J} \sup\limits_{i\in I, i\geq n} x_i = \inf\limits_{n\in J} x_n = \lim\limits_{n\in J, n\to+\infty} x_n$

Assume that $J$ is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$.Denote by $l = \sup\limits_{n\in I, n\geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup\limits_{i\in I, i\geq n_0} x_i \leq l$. If the strict inequality $\sup\limits_{i\in I, i\geq N} x_i < l$ holds, then $\sup\limits_{i\in I, i\geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup\limits_{i\in I, i\geq N} x_i$ We may also assume that $n$ is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality.

Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$ Thus $n \in J$ that contradicts the maximality of $n_0$

Therefore

$$l = \sup_{i\in I, i\geq N} x_i$$

, which leads to

$$\limsup_{n\in I, n\to+\infty} x_n = l$$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$,so $x_m < l$(since otherwise $x_m = \sup\limits_{i\in I, i\geq m} x_i$ and hence $m \in J$)Hence,$\forall$ *finite subset* $I'$ *of*$\{m \in I \mid m \geq n_0\}$

$\max\limits_{i\in I} x_i < l$ and hence $\exists n \in I$,such that $n > \max I'$,and $\max\limits_{i\in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in $I$

Let $n_0$ be as above. Let $f : \mathbb{N} \to I_{\geq n_0}$ be a surjective mapping.

If $n_j$ is chosen, we choose $n_{j+1} \in I$ such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is increasing

And

$$\sup_{j\in\mathbb{N}} x_{n_j} \leq \sup_{j\in\mathbb{N}} x_{f(j)} = \sup_{n\in I, n\geq n_0} x_n = l$$

$$l = \sup_{n\in I, n\geq n_0}$$

So $(x_{n_j})_{j\in\mathbb{N}}$ tends to $l$

# Chapter 9

# Cauchy sequence

## 9.1 Def

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{R}$
If $\inf_{N\in\mathbb{N}} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = \lim_{N\to+\infty} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = 0$ then we say that $(x_n)_{n\in I}$ is a Cauchy sequence

## 9.2 Prop

- If $(x_n)_{i\in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_N)_{i\in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite,$(x_n)_{n\in I}$ is a Cauchy sequence.
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n\in I}$ converges to $l$, then $(x_n)_{n\in J}$ converges to $l$ too.

## 9.3 Theorem: Completeness of real number

If $(x_n)_{n\in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in $\mathbb{R}$

### Proof

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$ So $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n\in I}$ converges to the same limit.

## 9.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

### 9.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in $\mathbb{R}$

# Chapter 10

# Comparison and Technics of Computation

## 10.1 Def

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be sequence in $\mathbb{R}$

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_N| \leq M|y_m|$ then we write $x_n = O(y_n), n \in I, n \to +\infty$
- If there exists $(\epsilon_n)_{n\in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim\limits_{n\in I, n\to+\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_N| \leq |\epsilon y_m|$, then we write $x_n = \circ(y_n), n \in I, n \to +\infty$
  Example:
  $$\lim_{n\to+\infty} \frac{1}{n} = 0$$

## 10.2 Prop.

Let I and X be partially ordered sets and $f : I \to X$ be an increasing/decreasing mapping. Let J ba a subset of I. Assume that any elements of I has an upper bound in J. Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

## 10.3 Theorem

Let I be a totally ordered set, $f : I \to [-\infty, +\infty]$ and $g : I \to [-\infty, +\infty]$ be two mappings that are both increasing/decreasing.Then the following equalities holds,provided that the sum on the right hand side of the equality is wel defined.

$$\sup_{x\in I, \{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\sup_{x\in I} f(x)) + (\sup_{y\in I} g(y))$$

$$\inf_{x\in I,\{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\inf_{x\in I} f(x)) + (\inf_{y\in I} g(y))$$

## Proof

We can assume $f$ and $g$ increasing. Let $a = \sup f(I)$,$b = \sup g(I)$
Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$
We equip A with the following order relation.

$$(x, y) \leq (x', y') \ iff \ x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.
Consider

$$h : A \to [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

$h$ is increasing.
Let $(x, y) \in A$. Assume that $x \leq y$
If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$
If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$
So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined,$(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if$x \geq y$, $(x, y)$ has also an upper bound in B. Therefore: $\sup h(A) = \sup h(B)$

## 10.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I,n\to+\infty} (x_n + y_n) \leq (\limsup_{n\in I,n\to+\infty} x_n) + (\limsup_{n\in I,n\to+\infty} y_n)$$

$$\liminf_{n\in I,n\to+\infty} (x_n + y_n) \geq (\liminf_{n\in I,n\to+\infty} x_n) + (\liminf_{n\in I,n\to+\infty} y_n)$$

## Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n\in I,n\geq N} x_n \quad B_N = \sup_{n\in I,n\geq N} y_n$. $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are decreasing, and $\limsup_{n\in I,n\to+\infty} x_n = \inf_{N\in\mathbb{N}} A_N \quad \limsup_{n\in I,n\to+\infty} y_n = \inf_{N\in\mathbb{N}} B_N$
By theorem:

$$\inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N = \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n\in I,n\geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.
Therefore

$$\inf_{N\in\mathbb{N}} C_N \leq \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N) = \inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N$$

## 10.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty,+\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) \geq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

### Proof

a tricky proof ?:

$$\limsup_{n\in I, n\to} x_n = \limsup_{n\in I, n\to}(x_n+y_n-y_n) \leq \limsup_{n\in I, n\to}(x_n+y_n) - \liminf_{n\in I, n\to} y_n$$

to have a true proof,only need to discuss conditions with $\infty$

## 10.6 Theorem

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ . Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n\in I}$ converges to some $i \in \mathbb{R}$.
Then:

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) = (\limsup_{n\in I, n\to+\infty} x_n) + l$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) = (\liminf_{n\in I, n\to+\infty} x_n) + l$$

## 10.7 Prop.

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Then:

$$\liminf_{n\in I, n\to+\infty} \max\{x_n, y_n\} = \max\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

$$\liminf_{n\in I, n\to+\infty} \min\{x_n, y_n\} = \min\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

### Proof

About the first inequality.Since $\max\{x_n, y_n\} \geq x_n \quad \max\{x_n, y_N\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset $J$ of $I$ such that

$$\lim_{n\in J, n\to+\infty} = \limsup_{n\in J, n\to+\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\} \quad J_1 = \{n \in J \mid x_n \leq y_n\}$
$J_1 \cup J_2 = J$ So either $J_1$ or $J_2$ is infinite
Suppose that $J_1$ is infinite, then

$$\lim_{n\in J, n\to} \max\{x_n, y_n\} = \lim_{n\in J_1, n\to} \max\{x_n, y_n\} = \lim_{n\in J, n\to} x_n \leq \limsup_{n\in I, n\to+\infty} x_n$$

If $J_2$ is infinite

$$\limsup_{n\in I, n\to+\infty} = \lim_{n\in J_2, n\to+\infty} \max\{x_n, y_n\} \leq \limsup_{n\in I, n\to+\infty} y_n$$

## 10.8 Theorem

Let $(a_N)_{n\in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_N)_{n\in I}$ converges to $l$
- $\limsup_{n\in I, n\to+\infty} |a_n - l| = 0$

### Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$
$$\limsup_{n\in I, n\to+\infty} |a_n - l| = \max\{(\limsup_{n\in I, n\to+\infty} a_n) - l, l - (\liminf_{n\in I, n\to+\infty} a_n)\}$$

(1) $\Rightarrow$ (2):
If $(a_n)_{n\in I}$ converges to $l$, then $\limsup_{n\in I, n\to+\infty} a_n = \liminf_{n\in I, n\to+\infty} a_n = l$

(2) $\Rightarrow$ (1):
If $\limsup_{n\in I, n\to+\infty} |a_n - l| = 0$ ,then $\limsup_{n\in I, n\to+\infty} a_n \leq l \leq \liminf_{n\in I, n\to+\infty} a_n$
Therefore: $\limsup_{n\in I, n\to+\infty} a_n = \liminf_{n\in I, n\to+\infty} a_n = l$

## 10.9 Remark

Let $(a_n)_{n\in I}$ be a sequence in $\mathbb{R}$, $l \in \mathbb{R}$
The sequence $(a_n)_{n\in I}$ converges to l iff $a_n - l = o(1), n \in I, n \to +\infty$

## 10.10 Calculates on O(),o()

### 10.10.1 Plus

Let $(a_n)_{n\in I}$ $(a'_n)_{n\in I}$ and $(b_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \to +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \to +\infty$

### 10.10.2 Transform

Let $(a_n)_{n\in I}$ and $(b_n)_{n\in I}$ be two sequence in $\mathbb{R}$ If $a_n = o(b_n), n \in I, n \to +\infty$,then $a_n = O(b_n), n \in I, n \to +\infty$

### 10.10.3 Transition

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ and $(c_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = O(c_n), n \in I, n \to +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$

### 10.10.4 Times

Let $(a_n)_{n\in I}, (b_n)_{n\in I}, (c_n)_{n\in I}, (d_n)_{n\in I}$ be sequences in $\mathbb{R}$

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = O(b_n d_n), n \in I, n \to +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = o(b_n d_n), n \in I, n \to +\infty$

## 10.11 On the limit

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ be elements of $\mathbb{R}^I$ that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n\in I}$ converges to $l + l'$
- $(a_n b_n)_{n\in I}$ converges to $ll'$

## 10.12 Prop

Let $a \in \mathbb{R}$ then $a^n = o(n!) \quad n \to +\infty$

### Proof

Let $N \in \mathbb{N}$ such that $|a| < N$
For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!}(\frac{|a|}{N})^n - N$$

And $0 < \frac{|a|}{<} 1 \Rightarrow \lim\limits_{n \to +\infty} (\frac{|a|}{N})^n = 0$. Therefore:

$$\lim_{n \to +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

## 10.13 Prop

$n! = o(n^n) \quad n \to +\infty$

### Proof

Let $N \in \mathbb{N}_{\geq 1}$
$0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim\limits_{n \to +\infty} \frac{n!}{n^n} = 0$

## 10.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of $\mathbb{R}^I$ If the series $\sum\limits_{n \in I} b_n$ converges absolutely and if $on = O(b_n) \quad n \to +\infty$Then $\sum\limits_{n \in I} a_n$ converges absolutely

### Proof

By definition $\sum\limits_{n \in I} |b_N| < +\infty$ If $|a_N| \leq M|b_N|$ fro $n \in I, n \geq N$ where $N \in \mathbb{N}$
Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

## 10.15 Theorem: d'Alembert ratio test

Let $(a_N)_{n \in \mathbb{N}} \in (\mathbb{R} \backslash \{0\})^{\mathbb{N}}$

- If $\limsup\limits_{n \to +\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum\limits_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf\limits_{n \to +\infty} |\frac{a_{n+1}}{a_n}| > 1$, then $\sum\limits_{n \in \mathbb{N}} a_n$ does not converge (diverges)

## Proof

### (1)

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n\to+\infty}|\frac{a_{n+1}}{a_n}| < \alpha < 1$, *alpha* isn't a lower bound of $(\sup_{n\geq N}|\frac{a_{n+1}}{a_n}|)_{N\in\mathbb{N}}$
So $\exists N \in \mathbb{N}$ such that $\sup_{n\geq N}|\frac{a_{n+1}}{a_n}| < \alpha$Hence for $n \geq N \quad |a_n| \leq \alpha^{n-N}|a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N}\frac{a_{N+2}}{a_{N+1}}...\frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n\in\mathbb{N}} = \frac{1}{1-\alpha} < +\infty$, $\sum_{n\in\mathbb{N}} a_n$ converge absolutely.

### 10.15.1 Lemma

If a series $\sum_{n\in\mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n\to+\infty} a_n = 0$

#### Proof

If $(\sum_{i=0}^{n} a_i)_{n\in\mathbb{N}}$ converges to some $l \in \mathbb{R}$ ,then $(\sum_{i=0}^{n-1} a_i)_{n\in\mathbb{N},n\geq 1}$ converges to $l$, too. Hence $\left(a_n = \left(\sum_{i=0}^{n} a_i\right) - \left(\sum_{i=0}^{n-1} a_i\right)\right)_{n\in\mathbb{N}}$ converges to $l - l = 0$

### 10.15.2 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n\to+\infty}|\frac{a_{n+1}}{a_n}| = \sup_{N\in\mathbb{N}}\inf_{n\geq N}|\frac{a_{n+1}}{a_n}|$
So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n\geq N}|\frac{a_{n+1}}{a_n}|$
$\forall n \in \mathbb{N}, n \geq N \quad |\frac{a_{n+1}}{a_n}| \geq \beta$
Hence $(|a_n|)_{n\in\mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N}|a_n|$
By the lemma: $\sum_{n\in\mathbb{N}} a_n$ diverges.

## 10.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \to +\infty$

## Proof

Let $\epsilon > 0$ such that $a = (1+\epsilon)^2$

$$a^n = (1+\epsilon)^{2n} = (1+\epsilon)^n(1+\epsilon)^n \geq (1+n\epsilon)(1+n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

### 10.16.1 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \to +\infty$

#### Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

### 10.16.2 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

#### Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

## 10.17 Theorem: Cauchy root test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $\alpha = \limsup\limits_{n\to+\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum\limits_{n\in\mathbb{N}} a_n$ converges absolutely.
- If $a > 1$ then $\sum\limits_{n\in\mathbb{N}} a_n$ diverges

### Proof

#### (1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_N|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum\limits_{n\in\mathbb{N}} a_n$ converges absolutely.

#### (2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \quad \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N}\ \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n\in\mathbb{N}}$ cannot converge to 0.

# Part III

# Axiom of choice

# Chapter 11

# Preparation

## 11.1 Statement of axiom of choice

For any set I and any family $(A_i)_{i\in I}$ of non-empty sets , there exists a mapping $f : I \to \bigcup\limits_{i\in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

## 11.2 Def

Let $(X, \leq)$ be a partially ordered set If $\forall A \subseteq X$ A is non-empty , there exists a least element of A then we say that $(X, \leq)$ is a well ordered set.

## 11.3 Theorem

For any set X, there exists an order relation $\leq$ on such that $(X, \leq)$ forms a well ordered set.

## 11.4 Zorn's lemma

Let $(X, \leq)$ be a partially ordered set . If $\forall A \subseteq X$ that is totally ordered with respect to $\leq$, there exists an upper bound of A inside X. Then , there exists a maximal element $x_0$ of $X(\forall y \in X, y > x_0$ does not hold)

## 11.5 Prop.

Let $(X, \leq)$ be a well ordered set , $y \notin X$. We extends $\leq$ to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

## 11.6 Proof

Let $A \subseteq X \cup \{y\}, A \neq \varnothing$. If $A = \{y\}$ then $Y$ is the least element of A. If $A \neq \{y\}$ then $B = A \backslash \{y\}$ is non-empty. Let $b$ be the least element of B. Since $b < y$ it's also the least element of A

## 11.7 Def: Initial Segment

Let $(X, \leq)$ be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S(X_{<s} \subseteq S)$, then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

## 11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X.

## 11.9 Prop.

Let $(X, \leq)$ be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X, then $\bigcup\limits_{i \in I} S_i$ is an initial segment of X

## 11.10 Proof

$\forall s \in \bigcup\limits_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup\limits_{i \in I} S_i$

## 11.11 Prop.

Let $(X, \leq)$ be a well ordered set.

(1) Let S be a proper initial segment of X, $x = \min(X \backslash S)$ Then $S = X_{<x}$

(2) $\begin{aligned} X &\to \wp(X) \\ x &\mapsto X_{<x} \end{aligned}$

(3) The set of all initial segments of X forms a well ordered subset of $(\wp(x), \subseteq)$

## 11.12 Proof

(1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction.Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \backslash S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

(2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \backslash X_{<x}$ So $X_{<x} \subsetneqq X_{<y}$

(3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \varnothing$. Then there exists $A \subseteq X$ such that $\mathcal{F} \backslash \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \varnothing$ then $\mathcal{F} = \{X\}$, and $\{X\}$is the least element of $\mathcal{F}$ . Otherwise $A \neq \varnothing$ and A has a least element a. Then by(2) $X_{<}a$ is the least element of $\mathcal{F}$

## 11.13 Lemma

Let $(X, \leq)$ be a well ordered set, $f : X \to X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

### Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \varnothing$, let a be the least element of A. By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since $f$ is strictly increasing . This shows $f(a) \in A$. But a is the least element of A, $f(a) < a$ cannot hold: contradiction.

## 11.14 Prop

Let $(X, \leq)$ be a well ordered set, S and T be two initial segment of X . If $f : S \to T$ is a bijection that's strictly increasing , then $S = T, f = Id_S$

### Proof

We may assume $T \subseteq S$.Let $l : T \to S$ be the induction mapping and $g = l \circ f : S \to S$. Since $g$ is strictly increasing , by the lemma ,$\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$
Apply the lemma to $f^{-1}$ we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

## 11.15 Def

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets. If $\exists f : X \to Y$ that's increasing and bijective, we say that $(X, \leq)$ and $(Y, \leq)$ are isomorphic

## 11.16 Def

Let $(X, \leq)$ and $(Y, \leq)$ be well ordered sets. If $(X, \leq)$ is isomorphic to an initial segment of Y. We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y, we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \prec X$

## 11.17 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

### Proof

We construct a correspondence $f$ from X to Y, such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$
By the last proposition of Oct. 11, $f$ is a function.

- If $a, b \in Dom(f)^2, a < b$,then $X_{<a} \subsetneqq X_{<b}$
  By definition, $Y_{<f(b)} \sim X_{<b} \quad Y_{<f(a)} \sim X_{<a}$
  Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y < f(b)$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_{<}f(B)$. We then get $f(a) < f(b)$. Thus $f$ is strictly increasing.

- Let $a \in Dom(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y \quad X_{<x} \sim Y_{<y}$ This shows that $x \in Dom(f)$. Hence $Dom(f)$ is an initial segment of X. Applying this to $f^{-1}$, we get : $Im(f) = Dom(f)$ is an initial segment of Y

- Either $Dom(f) = X$ or $Im(f) = Y$.
  Assume that $x \in X \backslash Dom(f), y \in Y \backslash Im(f)$ are respectively the least elements of $X \backslash Dom(f)$ and $Y \backslash Im(f)$.
  Then we get $Dom(f) = X_{<x}, Im(f) = Y_{<y}$.
  We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction

- 

Case 1 $Dom(f) = X, Im(f) \subsetneqq Y \quad X \prec Y$

Case 2 $Dom(f) \subsetneqq X, Im(f) = Y \quad X \succ Y$

Case 3 $Dom(f) = X, Im(f) = Y \quad X \sim Y$

## 11.18 Lemma

Let $(X, \leq)$ be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .

- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B, or B is a initial segment of A.

Let $Y = \bigcup_{A \in \mathfrak{S}} A$. Then $(Y, \leq)$ is a well ordered set, and ,$\forall A \in \mathfrak{S}, A$ is an initial segment of Y.

## Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$.Since $Y = \bigcup_{B \in \mathfrak{S}} B$,$\exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B. Hence $y \in A$. Contradiction

- Let $Z \subseteq Y, Z \neq \varnothing$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \mathfrak{S}$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$ , then A is an initial segment of B.

Since B is well ordered , if $m \not\leq z$ then $z < m$. Since $m \in A$, we het $z \in A$. Contradiction.

Therefore, m is the least element of Z.

# Chapter 12

# Zorn's lemma

Let $(X, \leq)$ be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X, the X has a maximal element(a maximal element m of $\{x \mid x > m\} = \varnothing$)

## 12.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega . \exists f(A)$ such that $\forall a \in A, a < f(A)$
Let

$$\omega = \{\text{well ordered subset of X}\}$$

. (guaranteed by axiom of choice)
Let $f : \omega \to X$ such that $f(A)$ is an upper bound of $A \in \omega$.
If $A \in \omega$ satisfies

$$\forall a \in Aa = f(A_{<a})$$

, we say that $A$ is a $f$-set
Let

$$\mathfrak{S} = \{f - sets\}$$

Note that

$$\varnothing \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cap \{f(A)\} \in \mathfrak{S}$$

In fact, if $a \in A$, then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If $a = f(A) \notin A$ then

$$(A \cup \{f(A)\})_{<a} = A$$

Let A and B be elements of $\mathfrak{S}$. Let I be the union of all common initial segments of A and B. This is also a common initial segment of A and B.
If $I \neq A$ and $I \neq B$, then

$$\exists (a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then $I \cup \{a\}$ is also a common initial segment of A and B, contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and $\forall A \in \mathfrak{S}$ is an initial segment of Y.

Since A is an initial segment of Y

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in AA_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If $f(y)$ is not a maximal element of X

$$\exists x \in X, f(y) < x$$

# Part IV

# Topology

# Chapter 13

# Absolute value and norms

## 13.1 Def

Let K be a field . By absolute value on K, we mean a mapping $|\cdot| : K \to \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\forall a \in K \quad |a| = 0$ iff $a = 0$

(2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$

(3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$(triangle inequality)

## 13.2 Notation

$\mathbb{Q}$ Take a prime num $p$ $\forall \alpha \in \mathbb{Q}\backslash\{0\}$ there exists a integer $ord_p(\alpha)\frac{a}{b}$, where $\begin{matrix} a \in \mathbb{Z}\backslash\{0\} \\ b \in \mathbb{N}\backslash\{0\} \end{matrix}, p \nmid a, p \nmid b$

## 13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \to \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & if\ \alpha \neq 0 \\ 0 & if\ \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on $\mathbb{Q}$

### Proof

(1) Obviously

(2) If $\alpha = p^{ord_p(\alpha)}\frac{a}{b}, \beta = p^{ord_p(\beta)}\frac{c}{d} \quad p \nmid abcd$
$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)}\frac{ac}{bd} \quad p \nmid ac, p \nmid bd$

(3) $\alpha + \beta = p^{ord_p(\alpha)}\frac{a}{b} + p^{ord_p(\beta)}\frac{c}{d}$
Assume $ord_p(\alpha) \geq ord_p(\beta)$
$\alpha + \beta$
$= p^{ord_p(\beta)}\left(p^{ord_p(\alpha)-ord_p(\beta)}\frac{a}{b} + \frac{c}{d}\right)$
$= p^{ord_p(\beta)}\frac{p^{ord_p(\alpha)-ord_p(\beta)}ad+bc}{bd} \quad p \nmid bd$
So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

Hence $ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$
So $|\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$
$\max\{|\alpha|_p, |\alpha|_p\} \leq |\alpha|_p, |\alpha|_p$

## 13.4 Def

Let $K$ be a filed and $|\cdot|$ be an absolute value. We call $(K, |\cdot|)$ a valued field.

# Chapter 14

# Quotient Structure

## 14.1 Def

Let X be a set and $\sim$ be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that $\sim$ is an equivalence relation

## 14.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of $x$ on $X$.Let $X/\sim$ be the set $\{[x] \mid x \in X\}$

## 14.3 Prop.

Let X be a set and $\sim$ be an equivalence relation on X

(1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$

(2) If $\alpha$ and $\beta$ are elements of $X/\sim$ such that $\alpha \neq \beta$ then $\alpha \cap \beta = \varnothing$

(3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

**Proof**

(1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ on has $x \sim y$
Therefore ,$x \sim z$ namely $z \in [x]$. This proves $y[] \subseteq [x]$. Moreover ,since $x \sim y$ , one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$

(2) Suppose that $\alpha \cap \beta \neq \varnothing, y \in \alpha \cap \beta$
By (1),$\alpha = [y], \beta = [y]$, Thus leads to a contradiction.

(3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup\limits_{\alpha \in X/\sim} \alpha$Hence $X \subseteq \bigcup\limits_{\alpha \in X/\sim} \alpha$.Conversely, $\forall \alpha \in X/\sim, \alpha$ is a subset of $X$. Hence $\bigcup\limits_{\alpha \in X/\sim} \alpha \subseteq X$.Then $X = \bigcup\limits_{\alpha \in X/\sim} \alpha$

## 14.4 Def

Let G be a group and X be a set
We call left/right action of G on X ant mapping $G \times X \to X : (g,x) \mapsto gx/(g,x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x \ / \ x1 = x$
- $\forall (g,h) \in G^2, x \in X \quad g(hx) = (gh)x \ / \ (xg)h = x(gh)$

## 14.5 Remark

If we denote by $G^{op}$ the set G equipped with the composition law :

$$G \times G \to G$$

$$(g,h) \mapsto hg$$

The a right action of G on X is just a left action of $G^{op}$ on X.

## 14.6 Prop

Let G be a group and X be a set . Assume given a left action of G on X. Then the binary relation $\sim$ on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

## 14.7 Notation on Equivalence Class

We denote by $G/X$ the set $X/\sim$ $\forall x \in X$ the equivalence class of $x$ is denoted as $Gx/xG$ or $orb_G(x)$ call the orbit of x under the action of G

## 14.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x.(y \sim x)$
- $\forall (x, y, z) \in X^3$,if $\exists (g, h) \in G^2$ , such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

## 14.9 Quotient set

Let X be a set and $\sim$ be an equivalence relation, the mapping $X \to X/\sim$: $(x \in X) \mapsto [x]$ is called the projection mapping.
$X/\sim$ is called the quotient set of X by equivalence relation $\sim$

### 14.9.1 Example

Let G be a group and H be a subgroup of G.Then the mapping

$$H \times G \to G$$

$$(h, g) \mapsto hg/(h, g) \mapsto gh$$

is a left/right action of H on G. Thus we obtain two quotient sets $H/G$ and $G/H$

## 14.10 Def

Let G be a group and H be a subgroup of G. Ig $\forall g \in G, h \in H \quad ghg^{-1} \in H$, Then we say that H is a normal subgroup of G

## 14.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G. In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

## 14.12 Prop

If G is commutative, any subgroup of G is normal

## 14.13 Theorem

Let G be a group and H be a normal subgroup of G. Then the mapping

$$G/H \times H/G \to G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set $G/H$ Moreover the projection mapping

$$\pi : G \to G/H$$

$$x \mapsto xH$$

is a morphism of groups.

### Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$ Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH)$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

## 14.14 Def

Let K be a unitary ring and E be a left K-module. We say that a subgroup F og $(E, +)$ is a left sub-K-module of E if $\forall (a, x) \in K \times F, ax \in F$

## 14.15 Prop

Let K be a unitary ring , E be a left K-module and F be a sub-K-module. Then the mapping

$$K \times (E/F) \to E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined , and defines a left-K-module structure on $E/F$. Moreover, the projection mapping $pi : E \to E/F$ is a morphism of left-K-modules

### Proof

Let x and x' be elements of E such that $[x] = [x']$, that meas: $x' - x \in F$ Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that $E/F$ forms a left K-module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, $\pi$ is a morphism of groups. Moreover $\forall x \in E, a \in K \quad \pi(ax) = [ax] = a[x] = a\pi(x)$

## 14.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$ (I is a left and right sub-K-module of A)

## 14.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \to A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , $A/I$ becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

### Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$ Therefore $a'b' \sim ab$

### 14.17.1 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of $\mathbb{Z}$
If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z} \quad adm = dma \in d\mathbb{Z}$
Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the reside class of n modulo d
If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

## 14.18 Theorem

Let $f : G \to H$ be a morphism of groups

(1) $Im(f)$ is a subgroup of H

(2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G

(3) The mapping

$$\begin{aligned} \widetilde{f} :& G/Ker(f) \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

(4) $f$ is injective iff $\ker(f) = \{1_G\}$

### Proof

(1) Let $\alpha$ and $\beta$ be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup

(2) Let x and y be elements of $\ker(f)$.
One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
Let $x \in \ker f, y \in G$.
One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup

(3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So $f$ is well defined.
Moreover $\widetilde{f}([x][y]) = \widetilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence $\widetilde{f}$ is a morphism of groups.
By definition $Im(\widetilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y, x \sin y$ that means $[x] = [y]$
Therefore $\widetilde{f}$ is injective.

(4) If $f$ is injective ,$\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f\{1_G\}$
Conversely , suppose that $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$

## 14.19 Theorem

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Then

(1) $Im(f)$ is a left-sub-K-module of F

(2) $\ker(f)$ is a left-sub-K-module of E

(3) $\begin{array}{l} \widetilde{f} : E/\ker f \to Im(f) \\ \quad [x] \mapsto f(x) \end{array}$ is a isomorphism of left K-modules

### Proof

(1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in Im(f)$

(2)

(3)

# Chapter 15

# Topology

## 15.1 Def

Let X be a set. We call topology on X any subset $\mathcal{J}$ of $\wp(x)$ that satisfies:

- $\varnothing \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitrary family of elements in $\mathcal{J}$, then $\bigcup\limits_{i \in I} u_i \in \mathcal{J}$
- If $u$ and $v$ are elements of $\mathcal{J}$, then $u \cap v \in \mathcal{J}$

## 15.2 Remark

If $(u_i)_i^n = 1$ is a finite family of elements of $\mathcal{J}$, then $\bigcap\limits_{i=1}^{n} u_i \in \mathcal{J}$(by induction, this follows from (3))

### 15.2.1 Example

$\{\phi, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

## 15.3 Def

Let X be a set. We call metric on $X$ any mapping $d : \begin{matrix} X \times X \to \mathbb{R}_{\geq 0} \\ (x, y) \mapsto d(x, y) \end{matrix}$,that satisfies

- $d(x, y) = 0$ iff x=y
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$(triangle inequality)

$(X, d)$ is called a metric space

### 15.3.1 Example

Let X be a set

$$d : X^2 \to \mathbb{R}_{\geq 0}$$
$$d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a metric

## 15.4 Def

Let $(X, d)$ be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ We call the open ball of radius $\epsilon$ centered at $x$

### 15.4.1 Example

Consider $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$,then $B(x, \epsilon) = ]x - \epsilon, x + \epsilon[$

## 15.5 Prop.

Let (X,d) be a metric space . let $\mathcal{J}_d$ be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x, \epsilon) \subseteq U$ THen $\mathcal{J}_d$ is a topology on X

### Proof

- $\varnothing \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of $\mathcal{J}_d$ Let $U = \bigcup_{i \in I} u_i$, $\forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, y) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of $\mathcal{J}_d$ Let $x \in U \cap V$ $\exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$ Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

## 15.6 Def

$\mathcal{J}_d$ is called the topology induced by the metric d

## 15.7 Def

We call topology space any pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a topology on X

Given a topological space $(X, \mathcal{J})$ If $U \in \mathcal{J}$ then we say that $U$ is an open subset of X. If $F \in \wp(X)$ such that $X \backslash F \in \mathcal{J}$, then we say that $F$ is closed subset of $X$
If there exists $d$ a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that $\mathcal{J}$ is metrizable

### 15.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact,m if d denote the metric defined as $d(x,y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$
$\forall x \in X \quad B(x,1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup\limits_{x \in A} \{x\} \in \mathcal{J}_d$

# Chapter 16

# Filter

## 16.1 Def

Let Xbe a set. We call filter if X any $\mathcal{F} \subseteq \wp(x)$ that satisfies:

(1) $\mathcal{F} \neq \varnothing, \varnothing \notin \mathcal{F}$

(2) $\forall A \in \mathcal{F}, \forall B \in \wp(X)$, if $A \subseteq B$, then $B \in \mathcal{F}$

(3) $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

### 16.1.1 Example

(1) Let $Y \subseteq X, Y \neq \varnothing$. $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y.

(2) Let X be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \backslash A \text{ is infinite}\}$$

is a filter called the Fréchet filter of X.

(3) Let $(X, \mathcal{J})$ be a topological space, $x \in X$ We call neighborhood of $x$ any $V \in \wp(X)$ such that $\exists u \in \mathcal{J}$, satisfying $x \in U \subseteq V$. Then $\mathcal{V} = \{\text{neighborhoods of x}\}$ is a filter.

## 16.2 Def: Filter Basis

Let X ba a set. $\mathscr{B} \subseteq \wp(X)$. If $\varnothing \notin \mathscr{B}$ and $\forall (B_1, b_2) \in \mathscr{B}^2, \exists B \in \mathscr{B}$, such that $B \subseteq B_1 \cap B_2$. We say that $\mathscr{B}$ is a filter basis.

### 16.2.1 Remark

If $\mathscr{B}$ is a filter basis, then $\mathcal{F}(\mathscr{B}) = \{A \subseteq X \mid \exists B \in \mathscr{B} \quad B \subseteq A\}$ is a filter

### Proof

$\varnothing \notin \mathcal{F}(\mathscr{B}), \mathcal{F}(\mathscr{B}) \neq \varnothing$ since $0 \neq B \subseteq \mathcal{F}(\mathscr{B})$. If $A \in \mathcal{F}(\mathscr{B}), A' \in \wp(X)$such that $A \subseteq A'$, then $\exists B \in \mathscr{B}$ such that $B \subseteq A \subseteq A'$. Hence $A' \in \mathcal{F}(\mathscr{B})$ If $A_1, A_2 \in \mathcal{F}(\mathscr{B})$, then $\exists (B_1, B_2) \in \mathscr{B}^2$ such that $B_1 \subseteq A_1, B_2 \subseteq A_2$. Since $\mathscr{B}$ is a filter basis, $\exists B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ Hence $A_1 \cap A_2 \in A_1 \cap A_2 \in \mathcal{F}(\mathscr{B})$

### 16.2.2 Example

- Let $Y \subseteq X, Y \neq \varnothing$
  $\mathscr{B} = \{Y\}$ is a filter basis. $\mathcal{F}(\mathscr{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let $(X, \mathcal{J})$ be a topological space $x \in X$. If $\mathscr{B}_x$ is a filter basis such that $\mathcal{F}(\mathscr{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$, then we say that $\mathscr{B}_x$ is a neighborhood basis of x

## 16.3 Remark

Let $\mathscr{B}_x$ is a neighborhood basis of x iff

- $\mathscr{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathscr{B}_x$ such that $U \subseteq V$
- Let $(X, d)$ be a metric space , $x \in X \forall \epsilon > 0$, Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

  Then

  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x
  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x

### 16.3.1 Example

$\mathcal{V}_x \cap \mathcal{J}$ is a neighborhood basis of x

## 16.4 Def

$V \in \wp(X)$ is called a neighborhood of $x$ if $\exists U \in \mathcal{J}$ such that $x \in U \subseteq V$

## 16.5 Remark

Let $(X, \mathcal{J})$ be a topological space, $x \in X$ and $\mathscr{B}_x$ a neighborhood basis os x. Suppose that $\mathscr{B}$ is countable . We choose a surjective mapping $(B_n)_{n\in\mathbb{N}}$ from $\mathbb{N}$ to$\mathscr{B}_x$. For any $n \in \mathbb{N}$, let $A_n = B_0 \cap B_1 \cap ... \cap B_n \in \mathcal{V}_x$ The sequence $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\{A_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of x.

## 16.6 Extra Episode

$\wp(\mathbb{N})$is NOT countable

Suppose that $f : \wp(\mathbb{N}) \to \mathbb{N}$ injective. Then $\exists g : \mathbb{N} \to \wp(\mathbb{N})$ surjective. Taking $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$. Since $g$ is surjective, $\exists a \in \mathbb{N}$ such that $A = g(a)$.

If $a \in A$, then $a \in g(a)$, hence $a \notin A$

If $a \notin A$, then $a \in g(a) = A$

Contradiction

## 16.7 Prop.

Let Y and R be sets, $g : Y \to E$ be a mapping ,

- If $\mathcal{F}$ is a filter of Y, then

  $$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

  is a filter on E

- If $\mathscr{B}$ is a filter basis of Y, then

  $$g(\mathscr{B}) := \{g(B) \mid B \in \mathscr{B}\}$$

  is a filter of E, and $\mathcal{F}(g(\mathscr{B})) = g_*(\mathcal{F}(\mathscr{B}))$

### Proof

(1) $E \in g_x(\mathcal{F})$ since $g^{-1}(E) = Y$
$\varnothing \notin g_x(\mathcal{F})$ since $g^{-1}(\varnothing) = \varnothing$

  If $A \in g_x(\mathcal{F})$ and $A' \supseteq A$, then $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{J}$, so $g^{-1}(A') \in \mathcal{J}$, Hence $A' \in g_x(\mathcal{F})$

  If $A_1, A_2 \in g_x(\mathcal{F})$. Then $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ Hence $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$. So $A_1 \cap A_2 \in g_x(\mathcal{F})$.

(2) Since $g$ is a mapping , and $\varnothing \notin \mathscr{B}$, we get $\varnothing \notin g(\mathscr{B})$,since $\mathscr{B} \neq \varnothing, g(\mathscr{B}) \neq \varnothing$.

Let $B_1, B_2 \in \mathscr{B}$, there exists $C \in \mathscr{B}$ such that $C \subseteq B_1 \cap B_2$. Hence $g(C) \subseteq g(B_1) \cap g(B_2)$, namely $g(\mathscr{B})$ is a filter basis.

# Chapter 17

# Limit point and accumulation point

We fix a topological space $(X, \mathcal{J})$

## 17.1 Def

Let $\mathcal{F}$ be a filter of X and $x \in X$

- If $\mathcal{V}_x \subseteq \mathcal{F}$ then we say that $x$ is an limit point of $\mathcal{F}$
- If $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \varnothing$, we say that $x$ is an accumulation point of $\mathcal{F}$

So any limit point of $\mathcal{F}$ is necessarily a accumulation point of $mathcalF$

## 17.2 Prop

Let $\mathscr{B}$ be a filter basis of X, $x \in X, \mathscr{B}_x$ a neighborhood basis of x. Then $x$ is an accumulation point of $\mathcal{F}(\mathscr{B})$ iff $\forall (B, U) \in \mathscr{B} \times \mathscr{B}_x, B \cap U \neq \varnothing$

### Proof

#### Necessity

Since $\mathscr{B} \subseteq \mathcal{F}(\mathscr{B}), \mathscr{B} \subseteq \mathcal{V}_x$, the necessity is true.

#### Sufficiency

Let $(A, V) \in \mathcal{F}(\mathscr{B}) \times \mathcal{V}_x$. There exist $B \in \mathscr{B}, U \in \mathscr{B}_x$, such that $B \subseteq A, U \subseteq V$. Hence $\varnothing \neq B \cap U \subseteq A \cap V$

## 17.3 Def

Let $Y \subseteq X, Y \neq \varnothing$. W call accumulation point of Y any accumulation point of the principal filter $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$.

## 17.4 Def

We denote by $\overline{Y} = \{\text{accumulation points of } Y\}$.,called the closure of $Y$ Note that $x \in \overline{Y}$ iff $\forall U \in \mathscr{B}_x, Y \cap U \neq \varnothing$

By convention $\overline{\varnothing} = \varnothing$

## 17.5 Prop

Let $Y \subseteq X$. Then $\overline{Y}$ is the smallest closed subset of X containing Y.

### Proof

$\forall x \in X \backslash \overline{Y}$, then there exists $U_x = \mathcal{V} \cap \mathcal{J}$, such that $Y \cap U_x = \varnothing$. Moreover , $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{J}$. This shows that $\forall y \in U_x, y \notin \overline{Y}$. Therefore $X \backslash \overline{Y} = \bigcap\limits_{x \in X \backslash \overline{Y}} U_x \in \mathcal{J}$

Let $Z \subseteq X$ be a closed subset that contain Y. Suppose that $\exists y \in \overline{Y} \backslash Z$. Then $U = X \backslash Z \in \mathcal{V}_y \cap \mathcal{J}$ and $U \cap Y \subseteq U \cap Z = \varnothing$. So $y \notin \overline{Y}$ contradiction. Hence $\overline{Y} \subseteq Z$.

## 17.6 Def: dense

Let $(X, \mathcal{J})$ be a topological space,$Y$ a subset of $X$. We call $Y$ is dense in $X$ if

$$\overline{Y} = Y$$

# Chapter 18

# Limit of mappings

## 18.1 Def

Let $(E, \mathcal{J}_E)$ be a topological space . $f : Y \to E$ a mapping , and $\mathcal{F}$ eb a filter of Y. If $a \in E$ is a limit point of $F_*(\mathcal{F})$ namely , $\forall \text{neighborhood} V \text{of} a, f^{-1}(V) \in \mathcal{F}$, then we say that $a$ is a limit of the filter $\mathcal{F}$ by $f$

## 18.2 Remark

Let $\mathscr{B}_a$ be a neighborhood basis of $a$. Then $\mathcal{V}_a \subseteq f_x(\mathcal{F})$, iff $\mathscr{B} \subseteq f_*(\mathcal{F})$ Therefore , $a$ is a limit of $\mathcal{F}$ by $f$ iff $\forall V \in \mathscr{B}_a, f^{-1}(V) \in \mathcal{F}$

### 18.2.1 Example

Let $(E, \mathcal{J}_E)$ be a topological space. $I \subseteq \mathbb{N}$ be an infinite subset, $x = (x_n)_{n \in I} \in E^I$. If the Fréchet filter $\mathcal{F}_{Fr}(I)$ has a limit $a \in E$ by the mapping $x : I \to E$, we say that $(x_n)_{n \in I}$ converges to a ,denote as

$$a = \lim_{n \in I, n \to +\infty} x_n$$

## 18.3 Remark

$a = \lim\limits_{n \in I, n \to +\infty} x_n$ iff, $\forall U \in \mathscr{B}_a$(where $\mathscr{B}_a$ is a neighborhood basis of $a$), $\exists N \in \mathbb{N}$ such that $x_n \in U$ for any $n \in I_{\geq N}$

Suppose that $\mathcal{J}_E$ is induced by a metric $d. \{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\} \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\} \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ are all neighborhood basis of $a$ .There fore , the following are equivalent

- $a = \lim\limits_{n \in i, n \to +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$

$(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\}$? unknown position )

## 18.4 Remark

We consider the metric d on $\mathbb{R}$ defined as

$$\forall (x, x) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of $\mathbb{R}$ defined by this metric is called the usual topology on $\mathbb{R}$

## 18.5 Prop

Let $(x_n)_{n \in I} \in \mathbb{R}^I$, where $I \subseteq \mathbb{N}$ is an infinite subset. Let $l \in \mathbb{R}$. The following statements are equivalent:

- The sequence $(x_n)_{n \in I}$ converges to $l$ in the topological space $\mathbb{R}$
- $\liminf\limits_{n \in I, n \to +\infty} x_n = \limsup\limits_{n \in I, n \to +\infty} x_n = l$
- $\limsup\limits_{n \in I, n \to} |x_n - l| = 0$

## 18.6 Theorem

Let $(X, d)$ be a metric space .Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be an element of $X^I$. Let $l \in X$. The following statements are equivalent:

- $(x_n)_{n \in I}$ converges to $l$
- $\limsup\limits_{n \in I, n \to +\infty} d(x_n, l) = 0$(equivalent to $\lim\limits_{n \in I, n \to +\infty} d(x, l) = 0$)

### Proof

(1) $\Rightarrow$ (2) The condition (1) is equivalent to $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$. We then get $\sup\limits_{n \in I_{geqN}} d(x, l) \leq \epsilon$. Therefore $\limsup\limits_{n \in I, n \to +\infty} d(x_n, l) \leq \epsilon$ We obtain that $\limsup\limits_{n \in I, n \to +\infty} = 0$

(2) $\Rightarrow$ (1) Let $\epsilon \in \mathbb{R}_{>0}$ If $\inf\limits_{N \in \mathbb{N}} \sup\limits_{n \in I_{\geq N}} d(x_n, l) = 0$. Then $\exists N \in \mathbb{N} \quad \sup\limits_{n \in I_{\leq N}} d(x_n, l) < \epsilon$. Hence ,$\forall n \in I_{\geq N} d(x_n, l) < \epsilon$. Since $\epsilon$ is arbitrary, $(*)$ is true, Hence (1) is also true .

## 18.7 Prop

Let $(X, \mathcal{J})$ be a topological space . $Y \subseteq X, p \in \overline{Y} \backslash Y$.Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of Y.

### Proof

Y is not empty otherwise $\overline{Y} = \varnothing$.

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$
  $\varnothing \notin \mathcal{V}_{p,Y}$ since $p \in \overline{Y}$
- Let $V \in \mathcal{V}_p$ and $A \subseteq Y$ such that $V \cap Y \subseteq A$. Let $U = V \cup (A \backslash (V \cap Y)) \in \mathcal{V}_p$ and $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let U and V be elements of $\mathcal{V}_p$ Let $W = U \cap V \in \mathcal{V}_p$ Then $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

## 18.8 Def

Let $(X, \mathcal{J}_x)$ and $(E, \mathcal{J}_E)$ be topological spaces, $Y \subseteq X, p \in \overline{Y} \backslash Y$, and $f : Y \to E$ be a mapping . If a is a limit point of $(F_*(\mathcal{V}_{p,Y}))$, then we say that a is a limit of $f$ when the variable $y \in Y$ tends to $p$, denoted as $a = \lim\limits_{y \in Y, y \to p} f(y)$

## 18.9 Remark

If $\mathscr{B}_a$ is a neighborhood basis of $a$. Then $a = \lim\limits_{y \in Y, y \to p} f(y)$ is equivalent to $\forall U \in \mathscr{B}_a \quad \exists V \in \mathcal{V}_p$ such that $Y \cap V \subseteq f_{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

## 18.10 Prop

Let X be a set, $\mathscr{B}$ be a filter basis, $\mathscr{G}$ be a filter. If $\mathscr{B} \subseteq \mathscr{G}$, then $\mathcal{F} \subseteq \mathscr{G}$.

### Proof

Let $V \in \mathcal{F}(\mathscr{B})$ By definition $\exists U \in \mathscr{B}$ such that $U \subseteq V$, since $U \in \mathscr{G}($ for $\mathscr{B} \subseteq \mathscr{G})$ and since $\mathscr{G}$ is a filter, $V \in G$

## 18.11 Theorem

Let $(X, \mathcal{J}_x)$ and $(E < \mathcal{J}_E)$ be topological spaces. $Y \subseteq X,\ p \in \overline{T} \backslash Y, a \in E$. We consider the following conditions.

(i) $a = \lim_{y \in Y, y \to p} f(y)$

(ii) $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ if $\lim_{n \to +\infty} y_n = p$ then $\lim_{n \to \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that $p$ has a countable neighborhood basis, then (i) and (ii) are equivalent.

### Proof

(1) Let $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ such that $p = \lim_{n \to +\infty} y_n$. For any $U \in \mathcal{V}_p, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N} \quad y \in U \cap Y . y_n \in U \cap Y$ Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \to +\infty} f(y_n) = a$$

(2) Assume that $p$ has a countable neighborhood basis . There exists a decreasing sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_P^{\mathbb{N}}$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of $p$.
Assume that (i) does not hold. Then there exists $U \in \mathcal{V}_a$ such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \backslash f^{-1}(U)$$

Therefore ,

$$\lim_{n \to +\infty} y_n = \varnothing$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However $f(y_n) \notin U$ for any $n \in \mathbb{N}$,so $(f(y_n))_{n \in \mathbb{N}}$ cannot converges to $a$.

## 18.12 Prop.

Let X be a set. If $(\mathcal{J}_i)_{i\in I}$ is a family of topologies on X, then $\mathcal{J} = \bigcap_{i\in I} \mathcal{J}_i$ is a topology. In particular. for any $\mathcal{A} \subseteq \wp(X)$, there is a smallest topology on X that contain $\mathcal{A}$

### 18.12.1 Proof

- $\forall i \in I \quad \{\varnothing, X\} \subseteq \mathcal{J}_i$ So $\{\varnothing, X\} \subseteq \mathcal{J}$
- Let $(u_j)_{j\in J}$ be a family of elements of $\mathcal{J}$ $\forall j \in J, i \in I \quad u_i \in \mathcal{J}_i$ So $\bigcup_{j\in J} u_j \in \mathcal{J}_i$ We then get $\bigcup_{j\in J} u_j \in \mathcal{J}$
- Let U and V be elements of $\mathcal{J}$ $\forall i \in I, \{u, v\} \subseteq \mathcal{J}_i$ So $U \cap V \in \mathcal{J}_i$. THerefore we get $U \cap V \in \mathcal{J}$ Let $\mathcal{A} \subseteq \wp(X)$ Let $\mathcal{J}(\mathcal{A}) = \bigcap_{\mathcal{J}\subseteq\wp(X)\text{a topology}\mathcal{A}\subseteq\mathcal{J}} \mathcal{J}$ Then $\mathcal{J}(\mathcal{A})$ is a topology. By definition, if $\mathcal{J}$ is a topology containing $\mathcal{A}$, then $\mathcal{J}(\mathcal{A}) \subseteq \mathcal{J}$ Hence $\mathcal{J}(\mathcal{A})$ is the smallest topology containing $\mathcal{A}$

# Chapter 19

# Continuity

## 19.1 Def

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces $f$ be a function from X to Y, $x \in Dom(f)$. If for any neighborhood U of $f(x)$, there exists a neighborhood V of $x$ such that $f(V) \subseteq U$. Then we say that $f$ is continuous at $x$. If $f$ is continuous at any $x \in Dom(f)$ then we say $f$ is continuous.

## 19.2 Remark

Let $\mathscr{B}_{f(x)}$ be a neighborhood basis of $f(x)$ If $\forall U \in \mathscr{B}_{f(x)}$ there exist $V \in \mathscr{B}_{f(x)} V_x$ such that $f(V) \subseteq U$, then $f$ is continuous at $x$ Suppose that X and Y are metric space. Then $f$ is continuous at $x$ iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in Dom(f) \quad d(y,x) < \delta \text{ implies } \ d(f(y), f(x)) < \epsilon$$

## 19.3 Theorem

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces, $f$ be a function from X to Y $x \in Dom(f)$ Consider the following condition

- $f$ is continuous at $x$
- $\forall (x_n)_{n \in \mathbb{N}} \in Dom(f)^{\mathbb{N}}$, if $\lim\limits_{n \to +\infty} x_n = x$, then $\lim\limits_{n \to +\infty} f(x_n) = f(x)$ THen (i) implies (ii) Moreover, if $x$ has a countable neighborhood basis, then (i) and (ii) are equivalent.

## 19.4 Proof

$(i) \Rightarrow (ii)$ Let $(x_n)_{n \in \mathbb{N}} \in Dom(f)^{\mathbb{N}}$ that converges to $x$ $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$ Since $\lim\limits_{n \to +\infty} x_n = x$,there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}$, $x_n \in V$.

Hence $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$. Thus $\lim_{n \to +\infty} f(x_n) = f(x)$

$(ii) \Rightarrow (i)$ under the hypothesis that $x$ has countable neighborhood basis. actually we will prove $NOT(i) \Rightarrow NOT(ii)$
Let $(V_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{V}_x$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of $x$
If (i) does not hold, then $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$ Pick $x_n \in V_n$ such that $f(x_n) \notin U \quad \forall N \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$. Hence $(x_n)_{n\in\mathbb{N}}$ converges to $x$. However, $f(x_n) \notin U$ for any $n$ So $(f(x_n))_{n\in\mathbb{N}}$ does not converges to $f(x)$. Therefore (ii) does not hold.

## 19.5 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$ be topological spaces. $f$ be a function from X to Y, $g$ be a function from Y to Z. Let $x \in Dom(g \circ f)$ If $f$ and $g$ are continuous at $x$. then $g \circ f$ is continuous at $x$ sectionProof Let $U \in \mathcal{V}_{g(f(x))}$ Since $g$ is continuous at $f(x)$:

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since $f$ is continuous at $x$:

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore, $g(f(V)) \subseteq g(W) \subseteq U$ Hence $g \circ f$ is continuous at $x$

## 19.6 Def

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$, If any element of $\mathcal{J}$ can be written as the union of a family of sets in $\mathscr{B}$ we say that $\mathscr{B}$ is a topological basis of $\mathcal{J}$

## 19.7 Prop

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$ $\mathscr{B}$ is a topological basis iff

$$\forall x \in X, \mathscr{B}_x := \{V \in \mathscr{B} \mid x \in V\}$$

is a neighborhood basis of $x$

## 19.8 Proof

$\Rightarrow$:

$$\forall x \in X \mathscr{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since $\mathscr{B}$ is a topological basis of $\mathcal{J}$,

$$\exists W \in \mathscr{B}, x \in W \subseteq V \subseteq U$$

Hence $\mathcal{V}_x$ is generated by $\mathscr{B}_x$

$\Leftarrow$ Let $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathscr{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

## 19.9 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces. $\mathscr{B}_Y$ be a topological basis of $\mathcal{J}_Y$ $f : X \to Y$ be a mapping. The following conditions are equivalent:

(1) $f$ is continuous

(2) $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$

(3) $\forall U \in \mathscr{B}, f^{-1}(U) \in \mathcal{J}_X$

### Proof

(1) $\Rightarrow$ (2)

Lemma Let $(X, \mathcal{J})$ be a topological space, $V \in \wp(X)$, THen $V \in \mathcal{J}$ iff $\forall x \in V, V$ is a neighborhood of $x$

Proof of lemma $\Rightarrow$ is by definition
$Leftarrow$:

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W - x \in \mathcal{J}$$

Let $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood W of $x$ such that $f(W) \subseteq U$ Since $f$ is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since $x$ is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2) $\Rightarrow$ (3) For (3) is a special situation of (2), it's natural.

(3) $\Rightarrow$ (1) Let $x \in X$

$$\forall U \in \mathscr{B}_Y \ s.t. \ f(x) \in U, f^{-1}(U)$$

is an open neighborhood of $x$, and

$$f(f^{-1}(U)) \subseteq U$$

Hence $f$ is continuous at $x$

## 19.10 Def

LEt X be a set ,$((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces. $\forall i \in I$ let $f_i : X \to Y_i$ be a mapping. We call initial topology of $(f_i)_{i \in I}$ on X the smallest topology on X making all $f_i$ continue

## 19.11 Remark

If $\mathcal{J}$ is the initial topology of $(f_i)_{i \in I}$, $\forall i \in I, U_i \in \mathcal{J}_i \quad f_i^{-1}(U_i) \in \mathcal{J}$ If $J \subseteq I$ is a finite subset, $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j$ then $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{J}$

## 19.12 Prop

$$\mathscr{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{is finite} (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology $\mathcal{J}$

## Proof

First

$$\mathscr{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{$subset V of X that can be written as the union of a family of sets in$\mathscr{B}\}$

- $\varnothing \in \mathcal{J}' \quad X \in \mathscr{B} \subseteq \mathcal{J}'$
- $\mathcal{J}'$ is stable by taking the union of any family of elements in $\mathcal{J}'$
- If $V_1, V_2$ are elements of $\mathcal{J}'$, then

  $$V_1 \cap V_2 \in \mathcal{J}'$$

  In fact, $V_1, V_2$ are of the form of the union of some sets of $\mathscr{B}$

The intersection of two elements of $\mathscr{B}$ is still a element of $\mathscr{B}$

$$\left(\bigcap_{j\in J} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j\in J'} f_j^{-1}(U_j')\right)$$
$$= \bigcap_{j\in J\cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J\backslash J' \\ U_j' & j \in J'\backslash J \\ U_j \cap U_j' & j \in J \cap J' \end{cases}$$
$$\left(\bigcap_{j\in J\backslash J'} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j\in J\cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U_j')\right) \cap \left(\bigcap_{j\in J'\backslash J} f_j^{-1}(U_j')\right)$$

So $\mathcal{J}'$ is a topology making all $f_i$ continuous.Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

## Example

Let $((Y_i, \mathcal{J}_i))_{i\in I}$ be topological spaces. $Y = \prod_{i\in I} Y_i$ and

$$\pi_i : \begin{matrix} Y \to Y_i \\ (y_j)_{j\in I} \mapsto y_i \end{matrix}$$

The product topology on Y is by definition the initial topology of $(\pi_i)_{i\in I}$

## 19.13 Theorem

Let X be a set , $((Y_i, \mathcal{J}_i))_{i\in I}$ be a family of topological spaces,

$$((f_i : X \to Y_i))_{i\in I}$$

be a family of mappings and we equip X with the initial topology $\mathcal{J}_X$ of $(f_i)_{i\in I}$
Let $(Z, \mathcal{J}_Z)$ be a topological space and

$$h : Z \to X$$

be a mapping. Then $h$ is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{is continuous}$$

### 19.13.1 Proof

$\Rightarrow$ If $h$ is continuous, since each $f_i$ is continuous, $f_i \circ h$ is also continuous.

$\Leftarrow$ Suppose that $\forall i \in I, f_i \circ h$ is continuous .Hence

$$\forall U_i \in \mathcal{J}_i, (f_i \circ h)_{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Let

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_j^{-1}(U_j) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathcal{J}_j \right\}$$

$\forall U \in \mathscr{B}$

$$h^{-1}(U) = \bigcap_{j\in J} h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Therefore, $h$ is continuous.

## 19.14 Remark

We keep the notation of the definition of initial topology If $\forall i \in I, \mathscr{B}_i$ is a topological basis of $\mathcal{J}_i$, then

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_i^{-1}(U_i) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathscr{B}_j \right\}$$

is also a topological basis of the initial topology,

### 19.14.1 Example

Let $((X_i, d_i))_{i\in\{1,...,n\}}$ be a family of metric spaces.

$$X = \prod_{i\in\{1,...,n\}} X_i$$

We define a mapping

$$d : \begin{array}{l} X \times X \to \mathbb{R}_{\geq 0} \\ ((x_i)_i \in \{1, ..., n\}(y_i)_{i\in\{1,...,n\}}) \mapsto \max\limits_{i\in\{1,...,n\}} d_i(x_i, y_i) \end{array}$$

$d$ is a metric on X. If $x = (x_i)_{i\in\{1,...,n\}}$ $y = (y_i)_{i\in\{1,...,n\}}$ $z = (z_i)_{i\in\{1,...,n\}}$ are elements of X, then

$$d(x, z) = \max_{i\in\{1,...,n\}} d_i(x_i, z_i) \leq \max_{i\in\{1,...,n\}} (d_i(x_i, y_i) + d(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i : \begin{array}{l} X \to X_i \\ (x_i)_{i\in\{1,...,n\}} \mapsto x_i \end{array}$$

is continuous. Hence the product topology $\mathcal{J}$ is contained in $\mathcal{J}_d$
Let $x = (x_i)_{i\in\{1,...,n\}} \in X, \epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i\in\{1,...,n\}} \mid \max_{i\in\{1,...,n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i\in\{1,...,n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i\in\{1,...,n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$

# Chapter 20

# Uniform continuity and convergency

## 20.1 Def

Let $(X, d)$ be a metric space. $\forall A \subseteq X$, we define

$$diam(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A.By convention

$$diam(\varnothing) := 0$$

If $diam(A) < +\infty$, we say that A is bounded

## 20.2 Remark

- If A is finite, then it's bounded
- If $A \subseteq B$ then $diam(A) \leq diam(B)$

## 20.3 Prop

Let $(X, d)$ be a metric space. $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$. Then

$$diam(A \cup B) \leq diam(A) + d(x_0, y_0) + diam(B)$$

In particular, if A,B are bounded, then $A \cup B$ is bounded.

## Proof

Let $(x, y) \in (A \cup B)^2$. If $\{x, y\} \subseteq A$, then $d(x, y) \leq diam(A)$
If $\{x, y\} \subseteq B$ then $diam(B) \geq d(x, y)$
If $x \in A, y \in B$,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq diam(A) + d(x_0, y_0) + diam(B)$$

Similarly if $x \in B, y \in A$

$$d(x, y) \leq diam(A) + d(x_0, y_0) + diam(B)$$

## 20.4 Def

Let $(X, d)$ be a metric space. $I \subseteq \mathbb{N}$ be an infinite subset, $(x_n)_{n \in I} \in X^I$. If

$$\forall \epsilon > 0 \; \exists N \in \mathbb{N} \quad diam(\{x_n \mid n \in I_{\geq \mathbb{N}}\}) \leq \epsilon$$

then we say that $(x_n)_{n \in I}$ is a Cauchy sequence.

## 20.5 Prop

(1) If $(x_n)_{n \in I}$ converges, then it's a Cauchy sequence.

(2) If $(x_n)_{n \in I}$ is a Cauchy sequence, $\{x_n \mid n \in I\}$ is bounded

(3) Suppose that $(x_n)_{n \in I}$ is a Cauchy sequenceIf there exists an infinite subset J of I such that $(x_n)_{n \in J}$ converges to some $x \in X$ ,then $(x_n)_{n \in I}$ converges to $x$

### 20.5.1 Proof

(1) trivial

(2) trivial

(3) Let $\epsilon > 0, \exists N \in \mathbb{N}$

$$diam(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence $(x_n)_{n \in I}$ converges to $x$

## 20.6 Def

Let $(X, d_X), (Y, d_Y)$ be metric space. $f$ be a function from X to Y. If $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in Dom(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x,y) \in Dom(f)^2, d(x,y) \leq \delta} d(f(x), f(y)) = 0$$

we say that $f$ is uniformly continuous.

## 20.7 Prop

Let $(X, d_X), (Y, d_Y)$ be metric spaces $f$ be a function from X to Y which is uniformly continuous.

(1) If $I \subseteq \mathbb{N}$ is finite, and $(x_n)_{n \in I}$ is a Cauchy sequence in $Dom(f)^I$ then $(f(x_n))_{n \in I}$ is Cauchy sequence

(2) $f$ is continuous

### 20.7.1 Proof

(1) $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in Dom(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that

$$\forall (n, m) \in I^2_{\geq N}, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

THerefore $(f(x_n))_{n \in I}$ is a Cauchy sequence.

(2) Let $(x_n)_{n \in I}$ be a sequence in $Dom(f)^{\mathbb{N}}$ that converges to $x \in Dom(f)$ We define$(y_n)_{n \in \mathbb{N}}$ as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then $(y_n)_{n \in \mathbb{N}}$ converges to $x$. Hence $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since $f$ is uniformly continuous, $(f(y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y.

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to $f(x)$. Hence $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x)$

## 20.8 Def

Let X be a set , $Z \subseteq X, (Y, d)$ be a metric space, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}$ and $f$ be functions from X to Y, having Z as their common domain of definition.

- If $\forall x \in Z, (f_n(x))_{n\in I}$ converges to $f(x)$ , we say that $(f_n)_{n\in I}$ converges pointwisely to $f$
- If
$$\lim_{n\in I, n\to+\infty} \sup_{x\in Z} d(f_n(x), f(x)) = 0$$
we say that $(f_n)_{n\in I}$ converges uniformly to $f$

## 20.9 Theorem

Let X and Y be metric space, $Z \subseteq X, I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}, f$ be functions from X to Y, having Z as domain of definition. Suppose that

- $(f_n)_{n\in I}$ converges uniformly to $f$
- each $f_n$ is uniformly continuous

Then $f$ is uniformly continuous.

### 20.9.1 Proof

$\forall n \in I$ let
$$A_n = \sup_{x\in Z} d(f_n(x), f(x))$$
$$\lim_{n\in I, n\to+\infty} A_n = 0$$
$\forall (x, y) \in Z^2, n \in I$
$$\begin{aligned} & d(f(x), f(y)) \\ \le & d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ \le & 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$
$$\inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) \le 2A_n + \inf_{(x,y)\in Z^2, d(x,y)\le\delta} d(f_n(x), f_n(y)) = 0$$
Hence
$$0 \le \inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) \le 2A_n$$
Take $\lim_{n\to+\infty}$, by squeeze theorem, we get
$$\inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) = 0$$

# 20.10 Theorem

Let X be a topological space, Y be a metric space, $Z \subseteq X, p \in Z$, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}$ and $f$ function from X to Y, having Z as domain of definition. Suppose that:

- $(f_n)_{n\in I}$ converges uniformly to $f$
- each $f_n$ is continuous at $p$

Then $f$ is continuous at $p$

## 20.10.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x\in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \ \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since $f_n$ is continuous $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{\mathcal{B}}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad & d(f(x)f(p)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ & \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$f(U) \subseteq \overline{\mathcal{B}}(f(p), \epsilon)$

## 20.10.2 Def

Let X Y be metric spaces , $f$ be a function from X to Y, $\epsilon > 0$. If

$$\forall (x, y) \in Dom(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that $f$ is $\epsilon$-Lipschitzian
If $\exists \epsilon > 0$ such that $f$ is $\epsilon$-Lipschitzian, then it's uniformly continuous.

# 20.11 Remark

If $f$ is Lipschitzian, then it's uniformly continuous.

# 20.12 Example

- Let $((X_i, d_i))_{i\in I}$ be metric space. $X = \prod_{i\in I} X_i$ where I is finite

$$d : \begin{array}{l} X \times X \to \mathbb{R}_{\geq 0} \\ d((x_i), (y_i)_{i\in I}) = \max_{i\in I} d_i(x_i, y_i) \end{array}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \to X_i$$

is Lipschitzian.($\forall x = (x_i)_{i \in I}, \forall x = (x_i)_{i \in I}$)

- Let $(X, d)$ be a metric space

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

# Part V

# Normed Vector Space

# Chapter 21

# Linear Algebra

We fix a unitary ring K

## 21.1 Def

Let M be a left K-module ,and let $x = (x_i)_{i \in I}$ be a family of elements of M. We define a morphism of left K-module as following:

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} && \to M \\ & (a_i)_{i\in I} && \mapsto \sum_{i\in I} a_i x_i (:= \sum_{i\in I, i\neq 0} a_i x_i) \end{aligned}$$

### 21.1.1 Notation

$$K^{\bigoplus I} := \{(a_i)_{i\in I} \in K^I \mid \exists J \subseteq I \text{finite,such that} a_i = 0 \text{ for } i \in I \setminus J\}$$
$$\varphi_x((a_i)_{i\in I} + (b_i)_{i\in I}) = \varphi_x((a_i)_{i\in I}) \varphi_x((b_i)_{i\in I})$$

## 21.2 Def

Ler M be a left K-module, I be a set, $x = (x_i)_{i\in I} \in M^I$ If

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} \to M \\ & (a_i)_{i\in I} \mapsto \sum_{i\in I} a_i x_i \end{aligned}$$

is

| | |
|---|---|
| injective | then we say $(x_i)_{i\in I}$ is K-linearly independent |
| surjective | then we say $(x_i)_{i\in I}$ is system of generator |
| a bijection | then we say $(x_i)_{i\in I}$ is a basis of M |

### Example

Let $e_i$ be the element $(\delta_{ij})_{j\in I}$ with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i\in I} \in (K^{\bigoplus I})^I$$

is a basis of $K^{\bigoplus I}$

## 21.3 Def

Let M be a left K-module

- If M bas a basis, we say that M is a free K-module
- If M has finite system of generated
  ($\exists a$ finite set I and a family $(x_i)_{i\in I} \in M^I$ that forms a system of generator),
  then we say that M is of finite type.

## 21.4 Remark

Let $x = (x_i)_{i\in\{1,\ldots,n\}} \in M^n$, where $n \in \mathbb{N}$

- $x$ is linearly independent iff

  $$\forall a \in K^n \quad \sum a_i x_i = 0$$

  implies

  $$a = 0$$

- $x$ is a system of generator iff for any element of M can be written in the form

  $$\sum b_i x_i \quad b \in K^n$$

  Such expression is called a K-linear combination of $x_1, ...x_n$

## 21.5 Theorem

Let K be a division ring ($0 \neq 1$ and $\forall k \in K \setminus \{0\}$ $k$ is invertible)
Let V be a left K-module of finite type and $(x_i)_{i\in I}$ be a system of generators of V. Then ,there exists a subset I of $\{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V. (In particular, V is a free K-module)

## Proof

(By induction on $n$)
If $n = 0$, then $V = \{0\}$
In this case $\varnothing$ is a basis of V

### Induction hypothesis

True for a system of generators of $n-1$ elements. Let $(x_i)_{i\in\{1,...,n\}}$ be a system of generators of V. If $(x_i)_{i\in\{1,...,n\}}$ is linearly independent, it's a basis. Otherwise, $\exists (a_i)_{i\in I} \in K^n$ such that

$$(a_i, ...a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose $a_n \neq 0$. Then

$$x_n = -a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i)$$

Since $(x_i)_{i\in\{1,...,n\}}$ is a system of generators, any elements of V can be written as

$$\begin{aligned}\sum b_i x_i &= (\sum_{i=1}^{n-1} b_i x_i) - b_n a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i)\\ &= \sum_{i=1}^{n-1}(b_i - b_n a_n^{-1} a_i)x_i\end{aligned}$$

Thus $(x_i)_{i\in\{1,...n\}}$ forms a system of generators . By the induction hypothesis, there exists $I \subseteq \{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V.

# 21.6 Theorem

Let K be a unitary ring and B be a left K-module. W be a left K-submodule of V. Let $(x_i)_{i=1}^n$ be an element of $W^n$

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where $(n, l) \in \mathbb{N}^2$ $\forall j \in \{1, ...l\}$ ,let $x_{n+j}$ be an element in the equivalence class $\alpha_j$

- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are linearly independent, then $(x_i)_{i=1}^{n+l}$ is also linearly independent
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are system of generators of W and $V/W$ respectively, then $(x_i)_{i=1}^{n+l}$ is also a system of generators
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are basis, then $(x_i)_{i=1}^{n+l}$ is also a basis

## Proof

(1) Suppose that $(b_i)_{i=1}^{n+l}$ such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \to V/W$$

be the projection morphism($\pi(x) = [x]$)

$$0 = \pi(\sum_{i=1}^{n+l} b_i x_i) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^{l} b_{n+j}\pi(x_{n+j}) = \sum_{j=1}^{l} b_{n+j}\alpha_j$$

$\{x_1, ...x_n\} \subseteq W$ So$\forall i \in \{1, ..., n\}$

$$\pi(x_i) = 0$$

Since $(\alpha_j)_{j=1}^{l}$ is linearly independent,

$$b_{n+1} = ... = b_{n+j} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since $(x_i)_{i=1}^{n}$ is linearly independent,

$$b_1 = ...b_n = 0$$

(2) Let $y \in V$. Then $\pi(y) \in V/W$. So there exists

$$(c_{n+1}, ..., c_{n+l}) \in K^l$$

such that

$$\begin{aligned}\pi(y) &= \sum_{j=1}^{l} c_{n+j}\alpha_j \\ &= \sum_{j=1}^{l} c_{n+j}\pi(x_{n+j}) \quad = \pi(\sum_{j=1}^{l} c_{n+j}x_{n+j})\end{aligned}$$

So

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) \in W$$

$\exists c \in K^n$ such that

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) = (\sum_{i=1}^{n} c_i x_i)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

## 21.7 Corollary

Let K be a division ring and V be a left K-module of finite type. If $(x_i)_{i=1}^n$ is a linearly independent family of elements of V($n \in \mathbb{N}$), then

$$\exists\ l \in \mathbb{N} \quad \exists\ (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of V

### Proof

Let W be the image of

$$\begin{aligned} \varphi_(x_i)_{i=1}^n :& K^n \to V \\ &(a_i)_{i=1}^n \mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left K-submodule of V.

Note that $(x_i)_{i=1}^n$ forms a basis of W.

$$\begin{aligned} \varphi_(x_i)_{i=1}^n :& K^n \to W \\ &\varphi_(x_i)_{i=1}^n(e_j) = x_j \in W \end{aligned}$$

Moreover ,since V is of finite type there exists $d \in \mathbb{N}$ and a surjective morphism of left K-modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \to V/W$$

is surjective.

Hence the composite morphism

$$\begin{tikzcd} K^d \arrow[r, "\psi"] \arrow[rr, bend right, "\pi\circ\psi"'] & V \arrow[r, "\pi"] & V/W \end{tikzcd}$$

is surjective. Thus $V/W$ is of finite type. There exist then a basis

$$(a_j)_{j=1}^{l}$$

of $V/W$.

Taking $x_{n+j} \in \alpha_j$ for $j \in \{1, ..., l\}$, we get a basis of V:

$$(x_i)_{i=1}^{n+l}$$

## 21.8 Def

Let K be a division ring and V be a left K-module of finite type. We call rank of V the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If K is a field $rk(V)$ is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of V.

## 21.9 Theorem

Let K be a division ring and V be a left K-module of finite type. Let W be a left K-submodule of V.

(1) W and $V/W$ are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of V has exactly $rk(V)$ elements

## 21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let $(x_i)_{i=1}^{n}$ be a basis of V. Let

$$\begin{aligned} \pi :& V \to V/W \\ & x \mapsto [x] \end{aligned}$$

In $(\pi(x_i))_{i=1}^n$ we extract a basis of $V/W$, say

$$(\pi(x_i))_{i=1}^l$$

For $j \in \{l+1, ..., n\}$,

$$\exists (b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^{l} b_{j,i}\pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^{l} b_{j,i}x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any $x \in W$,$\exists (a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^{n} a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^{l} a_i x_i + \sum_{j=l+1}^{n} a_j(y_j + \sum_{i=1}^{l} b_{j,i}x_i) \\ &= \sum_{j=l+1}^{n} a_j y_j + \sum_{i=1}^{l}(a_i + \sum_{j=l+1} a_j b_{j,i})x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^{l}(a_i + \sum_{j=l+1}^{n} a_j b_{j,i})\pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^{n} a_i y_i$$

Hence W is of finite type , and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let $(x_i)_{i=1}^n$ be a basis of V. Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of $V/W$.
We extract a subfamily , say $(x_i)_{i=1}^l$ such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of V/W.
For $j \in \{1, ..., l\}$, there exists:

$$(b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i}\pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i}x_i \in W$$

Let $x \in W$,$\exists (a_i)_{i=1}^n \in K^n$ let $x = \sum a_i x_i$,then

$$\begin{aligned} x &= (\sum_{i=1}^l a_i x_i) + (\sum_{j=l+1}^n a_j(y_j + \sum_{i=1}^l b_{j,i}x + i)) \\ &= (\sum_{i=1}^l a_i x_i) + (\sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i) + (\sum_{j=l+1}^n a_j y_j) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j.i})\pi(x_i)$$

Therefore $(y_j)_{j=l+1}^n$ is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + tk(V/W)$$

(2) All basis of V have $rk(V)$ elements.

We reason by induction on $rk(V)$

(1)

$$rk(V) = 0$$

In this case $V = \{0\}$ The only basis of V is $\varnothing$. So the statement holds.

(2) Assume that there exists $e \in V \setminus \{0\}$ such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of V is of the form

$$ae$$

where $a \in K\{0\}$

Let $(e_i)_{i=1}^m$ be a basis of V. We reason by induction on m to prove that

$$m = rk(V)$$

The cases where $m = 0$ or 1 are proved in (1)(2) respectively. Induction hypothesis: true for a basis of $< m$ elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : & V \to V/W \\ & x \mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of $V/W$.

If $(a_i)_{i=2}^m \in K^{m-1}$ such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for $(e_i)_{i=1}^m$ a basis of V,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of V/W. We then obtain that

$$rk(V/W) \leq m-1 \leq n-1$$

By the induction hypothesis,

$$m-1 = rk(V/W)$$

By (2), $rk(W) = 1$. Hence

$$m = (m-1)+1 = rk(V/W) + rk(W) = rk(V)$$

## 21.11 Prop

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Let I be a set and $(x_i)_{i\in I} \in E^I$

- If $(x_i)_{i\in I}$ is linearly independent and $f$ is injective, then $(f(x_i))_{i\in I}$ is linearly independent.
- If $(x_i)_{i\in I}$ is a system of generators and $f$ is surjective, then $(f(x_i))_{i\in I}$ is a system of generators.
- If $(x_i)_{i\in I}$ is a basis and $f$ is an isomorphism, then $(f(x_i))_{i\in I}$ is a basis.

### 21.11.1 Proof

$$\varphi_{(f(x_i))_{i\in I}} = f \circ \varphi_{(x)_{i\in I}}$$

# Chapter 22

# Matrices

We fix unitary ring K

## 22.1 Def

Let $n \in \mathbb{N}$ and V be a left K-module.

For any $(x_i)_{i=1}^n \in V^n$, we denote by $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ the morphism

$$\begin{aligned} \phi_{(x_i)_{i=1}^n} :& K^n \to V \\ (a_i)_{i=1}^n \mapsto & \sum_{i=1}^n a_i n_i \end{aligned}$$

### 22.1.1 Example

Suppose that $V = K^p (p \in \mathbb{N})$ Then each $x_i \in K^p$ is of the form $(x_{i,1}, ..., x_{i,p})$

Hence $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ can be written:

$$\begin{pmatrix} x_{1,1} & . & . & . & x_{1,p} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ x_{n,1} & . & . & . & x_{n,p} \end{pmatrix}$$

## 22.2 Def

Let $(n, p) \in \mathbb{N}^2$. We call $n$ by $p$ matrix of coefficient in K any morphism of left K-modules from $K^n$ to $K^p$

### 22.2.1 Example

- Denote by $I_n$ then identity mapping.Then $(e_i)_{i=1}^n$ is a basis of $K^n$ called the canonical basis of $K^n$

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, ..., a_n)) = \sum_{i=1}^n a_i e_i = (a_1, ..., a_n)$$

- Let $(x_1, ..., x_n) \in K^n$, Denote by

$$\begin{aligned} diag(x_1, ...x_n)(= \varphi_{(x_i e_i)_{i=1}^n}) : & K^n \to K^n \\ & (a_1, ..., a_n) \mapsto (a_1 x_1, ..., a_n x_n) \end{aligned}$$

## 22.3 Def

We denote by $M_{n,p}(K)$ the set of all $n$ by $p$ matrices of coefficients in K. For $(n, p, r) \in \mathbb{N}^3$,we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) \to & \quad M_{n,r}(K) \\ (A, B) \mapsto & \quad AB := B \circ A \end{aligned}$$

## 22.4 Calculate Matrices

Let K be a unitary ring , and V be a left K-module. Let $n \in \mathbb{N}$ and

$$x = (x_1, ..., x_n) \in V^n$$

### 22.4.1 Remind

$$\begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \varphi : (a_1, ..., a_n) \mapsto a_1x_1, ..., a_nx_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1,...,p\} \times \{1,...,n\}} \in M_{p,n}(K)$$

A is a morphism of left K-modules from $K^p$ to $K^n$ Recall that

$$A \begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let $(b_1, ..., b_n) \in K^p$

$$\begin{aligned} A((b_1, ..., b_n)) &= \sum_{i=1}^{p} b_i(a_{i,1}, ..., a_{i,n}) \\ \varphi(A((b_1, ..., b_n))) &= \sum_{i=1}^{p} b_i\varphi_x((a_{i,1}, ..., a_{i,n})) \\ &= \sum_{i=1}^{p} b_i(a_{i,1}x_1, ..., a_{i,n}x_n) \end{aligned}$$

Let $B = \{b_{ij}\}_{(i,j) \in \{1,...,n\} \times \{1,...,r\}} : K^n \to K^r$

$$AB = \left\{ \sum_{j=1}^{n} a_{lj}b_{jm} \right\}_{(l,m) \in \{1,...,p\} \times \{1,...,r\}}$$

# Chapter 23

# Transpose

We fix a unitary ring K

## 23.1 Def

Let E be a left-K-module. Denote by

$$E^\vee := \{\text{morphisms of left K-modules } E \to K\}$$

$\forall (f,g) \in E^\vee$ let

$$\begin{aligned} f+g : &E \to K \\ &x \mapsto f(x)+g(x) \end{aligned}$$

$(E^\vee, +)$ forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : &E \to K \\ &x \mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} &K \times E^\vee \to E^\vee \\ &(a,f) \mapsto fa : x \in E \to f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right K-module on $E^\vee$

## 23.2 Def

Let E and F be two left K-modules. $\varphi : E \to F$ be a morphism of left K-modules. We denote by

$$\varphi^\vee : F^\vee \to E^\vee$$

the morphism of right K-modules sending $g \in F^\vee$ to $g \circ \varphi \in E^\vee$
Actually $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

### 23.2.1 Example

Suppose that $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & . & . & . & b_{1,p} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ b_{n,1} & . & . & . & b_{n,p} \end{pmatrix}$$

$\varphi$ sends $(a_1, ..., a_n)$ to $\{\sum_{i=1}^{n} a_i b_i j\}_{j\in\{1,...,p\}}$ Let $g \in F^\vee \quad g : K^p \to K$, then $g$ is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$ sends $(a_1, ..., a_n)$ to $\sum_{i=1}^{p}(\sum_{i=1}^{n} a_i b_{ij} y_j)$

Assume that K is commutative. We denote by

$$\begin{aligned} \iota_p :&(K^p)^\vee \to K^p \\ &\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, ..., x_p) \end{aligned}$$

$$\begin{aligned} \iota_n :&(K^n)^\vee \to K^n \\ &\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, ..., x_n) \end{aligned}$$

are isomorphisms of K-modules

For any morphism of K-modules $\varphi : K^n \to K^p$, we denote by $\varphi^\tau$ the morphism of K-modules $K^p \to K^n$ given by $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{tikzcd}
(K^p)^\vee \arrow[r, "\varphi^\vee"] \arrow[d, "\iota_p", "\cong"'] \arrow[dr, phantom, "\circlearrowright"] & (K^n)^\vee \arrow[d, "\cong"'] \\
K^p \arrow[r, "\varphi^\tau"'] & K^n
\end{tikzcd}$$

$\varphi^\tau$ is called the transpose of $\varphi$

## 23.3 Prop

Let E,F,G be left K-modules. $\varphi : E \to F, \psi : F \to G$ be morphisms of left K-modules. Then $(\psi \circ \varphi)^\vee$ is equal to $\varphi^\vee \circ \psi^\vee$

### Proof

$\forall f \in G^\vee$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

## 23.4 Corollary

Assume that K is commutative. Let $n, p, q$ be neutral numbers. $A \in M_{n,p}(K), B \in M_{p,q}(K)$. Then

$$(AB)^\tau = B^\tau A^\tau$$

### Proof

$$A^tau = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^tau = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned}
B^\tau A^\tau &= A^\tau \circ B^\tau \\
&= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\
&= (AB)^tau
\end{aligned}$$

## 23.5 Remark

(1) For $A \in M_{n,p}(K)$, one has $(A^\tau)^\tau$

(2) We have a mapping

$$E \to (E^\vee)^\vee$$
$$x \mapsto ((f \in E^\vee) \mapsto f(x))$$

This is a K-linear mapping.

If K is a field and E is of finite dimension, this is a isomorphism of K-modules.

In fact, if $e = (e_i)_{i=1}^n$ is a basis of E over K.For $i \in \{1, ..., n\}$, let

$$e_i^\vee : E \to K$$
$$\lambda_1 e_1, ..., \lambda_n e_n \mapsto \lambda_i$$

is called the dual basis of $e$

$$\begin{tikzcd} K^n \arrow[d, "\varphi_e"', "\cong"] \arrow[dr, "\varphi_{e^\vee}"] & (K^n)^\vee \arrow[l, "\cong"', "\iota_n"] \arrow[d, "\cong"', "\varphi_e^\vee"] \\ E \arrow[r, "\cong"'] & E^\vee \end{tikzcd}$$

$(e^\vee)^\vee$ gives a basis of $(E^\vee)^\vee$ Hence $E \to (E^\vee)^\vee$ is an isomorphism.

# Chapter 24

# Linear Equation

We fix a unitary ring K.

## 24.1 Def

For $a = (a_1, ..., a_n) \in K^n \setminus \{(0, ..., 0)\}$. Denote by $j(a)$ the first index $j \in \{1, ..., n\}$ such that $a_j \neq 0$.Let $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$. We write A as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, ..., a_n^{(i)}) \in K^p$$

We say that A is of row echelon form if,$\forall i \in \{1, ..., n-1\}$ one of following conditions is satisfied.

- $a^{(i+1)} = (0, ..., 0)$
- $a^{(i)}, a^{(i+1)}$ are non-zero, and $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, ..., n\}$ such that $a^{(i)} \neq (0, ..., 0)$ ,one has

  $$a_{j(a^{(i)})}^{(i)} = 1$$

  and

  $$\forall k \in \{1, ..., n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that A is of reduced row echelon form.

## 24.2 Prop

Suppose that $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$ is of row echelon form. Then $\{i \in \{1, ..., n\} \mid a^{(i)} \neq (0, ..., 0)\}$ is of cardinal $\leq p$

### Proof

Let $k = card\{i \in \{1, ..., n\} \mid a^{(i)} \neq (0, ..., 0)\}$ $a^{(k+1)} = ... = a^{(n)} = (0, ..., 0)$ and $j(a^{(1)}) < j(a^{(2)}) < ... < j(a^{(k)})$ Hence

$$\{1, ..., k\} \to \{1, ..., p\}, i \mapsto j(a^{(i)})$$

is injection. So $k \leq p$

## 24.3 Linear Equation

Let $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$. Let V be a left K-module and $(b_1, ..., b_n) \in V^n$. We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

The set of $(x_1, .., x_p) \in V^p$ that satisfies (*) is called the solution set of (*)

## 24.4 Prop

Suppose that A is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, ..., n\} \mid (a_{i,1}, ..., a_{i,p}) \neq (0, ..., 0)\}$$

$$J_0(A) = \{1, ..., p\} \setminus \{j((a_{i,1}, ..., a_{i,p})) \mid i \in I(A)\}$$

- If $\exists i \in \{1, ..., n\} \setminus I(A)$ such that $b_i \neq 0$ then (*) does not have any solution in $K^n$
- Suppose that $\forall i \in \{1, ..., n\} \setminus I(A), b_i = 0$. Then (*) has at least one solution. Moreover

$$V^{J_0(A)} \to V^p$$
$$(z_k)_{k \in J_0(A)} \mapsto (x_1, ..., x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, ..., a_{i,p})) \end{cases}$$

is an injective mapping , whose image is equal to the set of solution of (*)

## 24.5 Prop

Let $m \in \mathbb{N}, S \in M_{m,n}(K)$. If $(x_1, ..., x_p) \in V^p$ is a solution of (*), then $(x_1, ..., x_p)$ is a solution of $(*)_S$ :

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

In the case where S is left invertible, namely there exist $R \in M_{n,m}(K)$ such that $RS = I_n \in M_{m,n}(K)$. Then (*) and $(*)_S$ have the same solution set.

## 24.6 Def

Let $G_n(K)$ be the set of $S \in M_{n,n}(K)$ that can be written as $U_1...U_N$(by convention $S = I_n$ where $N = 0$) where each $U_i$ is of one of the following forms.

- $P_\sigma$ where $\sigma \in \mathfrak{S}_n$
- $diag(r_1, ..., r_n)$ where each $r_i \in K$ is left invertible
- $S_{i,c}$ with $i \in \{1, ..., n\}$ $c = (c_1, ..., c_n) \in K^n, c_i = 0$

Let $p \in \mathbb{N}$ ,we say that $A \in M_{n,p}(K)$ is reducible by Gauss elimination if $\exists S \in G_n(K)$ such that $SA$ is of reduced row echelon form

## 24.7 Theorem

Assume that K is a division ring $\forall (n, p) \in \mathbb{N}$ any $A \in M_{n,p}(K)$ is reducible by Gauss elimination

### Proof

The case where $n = 0$ or $p = 0$ is trivial. We assume $n \geq 1, p \geq 1$ We write A as

$$\begin{pmatrix} \lambda_1 & \\ \vdots & B \\ \lambda_n & \end{pmatrix} \text{ where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If $\lambda_1 = ... = \lambda_n = 0$

  Applying the induction hypothesis to B, for $S \in G_n(K)$

$$SA = \begin{pmatrix} S\begin{pmatrix}\lambda_1 \\ \vdots \\ \lambda_n\end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 & \\ \vdots & SB \\ 0 & \end{pmatrix}$$

- Suppose that $(\lambda_1, ..., \lambda_n) \neq (0, ..., 0)$

  By permuting the rows we may assume $\lambda_1 \neq 0$. As K is division ring, by multiplying the first row by $\lambda_1^{-1}$ , we amy assume $\lambda_1 = 1$. We add $(-\lambda_i)$ times the first row to the $i^{th}$ row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 \cdots & \mu_p \\ 0 & & \\ \vdots & C & \\ 0 & & \end{pmatrix} \qquad \begin{array}{l} C \in M_{n-1,p-1}(K) \\ (\mu_2, ..., \mu_p) \in K^{p-1} \end{array}$$

  Applying the induction hypothesis to C, we say assume that C is of reduced row echelon form . For $i \in \{2, ..., k\}$ we add $-\mu_{j(c_i)}$ times the $i^{th}$ row of A to the first line to obtain a matrix of reduced row echelon form

# Chapter 25

# Normed Vector Space

## 25.1 Def

Let $(X, d)$ be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is an element of $X^{\mathbb{N}}$ such that

$$\lim_{N\to+\infty} \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} d(x_n, x_m) = 0$$

we say that $(x_n)_{n\in N}$ is a Cauchy sequence. If any Cauchy sequence in X converges, then we say that $(X, d)$ is complete.

Let $Cau(X, d)$ be the set of all Cauchy sequences in X. We define a binary relation $\sim$ on $Cau(X, d)$ as

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$$

iff

$$\lim_{n\to+\infty} d(x_n, y_n) = 0$$

## 25.2 Prop

$\sim$ is an equivalence relation.

### 25.2.1 Proof

$$\lim_{n\to+\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ be elements of $Cau(X, d)$. For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n\to+\infty} d(x_n, z_n) = 0$$

## 25.3 Def

$\hat{X} := Cau(X, d)\backslash \sim$

## 25.4 Def: The completion

The completion of $(X, d)$ is defined as

$$Cau(X)/\sim$$

and is denoted as

$$\hat{X}$$

## 25.5 Theorem

The mapping

$$\begin{aligned}\hat{d} :& \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0} \\ & (x, y) \mapsto \lim_{n\to+\infty} d(x_n, y_n)\end{aligned}$$

is well defined, and it's a metric on $\hat{X}$

### Proof

TO check that $\hat{d}$ is well defined , it suffices to prove that $\forall([x], [y]) \in \hat{X} \times \hat{X}$, $(d(x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy sequence and its limit doesn't depend on the choice of the representation $x$ and $y$
For $N \in \mathbb{N}$ and $(n, m) \in \mathbb{N}_{\geq N}$ for

$$\begin{aligned}d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m)\end{aligned}$$

one has,

$$|dd(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned}\sup_{(n,m\in\mathbb{N}_{\geq N})} |d(x_n, y_n) - d(x_m, y_m)| \leq & (\sup_{(n,m\in\mathbb{N}_{\geq N})} d(x_n, x_m)) \\ & + (\sup_{(n,m\in\mathbb{N}_{\geq N})} d(y_n, y_m))\end{aligned}$$

Taking $\lim_{N\to+\infty}$ we obtain that $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.
Hence it converges in $\mathbb{R}$. If $x' = (x'_n)_{n\in\mathbb{N}} \in [x], y' = (y'_n)_{n\in\mathbb{N}} \in [y]$ , thus

$$\lim_{n\to+\infty} d(x_n, x'_n) = \lim_{n\to+\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking $\lim_{n\to+\infty}$ we get

$$\lim_{n\to+\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(x'_n, y'_n)$$

In the following, we check that $\hat{d}$ is a metric

- $\hat{d}([x], [y]) = 0$ iff $[x] = [y]$: trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$: trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$:

$$\begin{aligned} d([x], [y]) &= \lim_{n\to+\infty} \\ &\leq \lim_{n\to+\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

## 25.6 Remark

Let

$$\begin{aligned} i_X : &X \to \hat{X} \\ &a \mapsto [(a, a, ...)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular, $i_x$ is injective(if $i_X(a) = i_X(b)$ then $d(a, b) = 0$ hence $a = b$)

## 25.7 Prop

$i_X(X)$ is dense in $\hat{X}$ (the closure of $i_X(X)$ in $\hat{X}$ is equal to $i_X(X)$ (or to say $\hat{X}$))

## Proof

Let $[x]$ be an equivalence class in $\hat{X}$. We claim that $\forall (x_n)_{n\in\mathbb{N}} \in [x]$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} i_X(x_n)$$

For any $N \in \mathbb{N}$

$$\begin{aligned} 0 \le \hat{d}(i_X(x_N), [x]) &= \lim_{n\to+\infty} d(x_N, x_n) \\ &\le \sup_{(n,m)\in\mathbb{N}^2_{\ge N}} d(x_n, x_m) \end{aligned}$$

Taking $\lim\limits_{N\to+\infty}$ we get

$$\lim_{N\to+\infty} \hat{d}(i_X(x_N), [x]) = 0$$

# 25.8 Theorem

$(\hat{X}, \hat{d})$ is a complete metric space

## Proof

Let $([x^{(N)}])_{N\in\mathbb{N}}$ be a Cauchy sequence in $\hat{X}$ , where $\forall N \in \mathbb{N}$, $x^{(N)} = (x_n^{(N)})_{n\in\mathbb{N}}$ is a Cauchy sequence
$\forall \epsilon > 0,\ \exists N_0 \in \mathbb{N}$ such that $\forall (k,l) \in \mathbb{N}_{\ge N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n\to+\infty} d(x_n^{(k)}, x_n^{(l)}) \le \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \le \frac{1}{N+1}$$

for any $(\mu, \nu) \in \mathbb{N}_{\ge\alpha(N)}$
Let $y_N = x_{\alpha(N)}^{(N)}$. Without loss of generality , we assume that

$$\alpha(0) \le \alpha(1) \le ...$$

Let $\epsilon > 0$ Take $N_0 \in \mathbb{N}$ such that

(1) $\forall (k,l) \in \mathbb{N},\ k,l \ge N_0$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \le \frac{\epsilon}{3}$$

(2)

$$\frac{1}{N_0+1} \le \frac{\epsilon}{3}$$

Let $(k,l) \in \mathbb{N}_{N_0}^2$,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since $\alpha(k) \geq N_0$, $\forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking $\lim\limits_{n\to+\infty}$ get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So $y = (y_N)_{N\in\mathbb{N}}$ is a Cauchy sequence. We check that

$$\lim_{N\to+\infty} \hat{d}([x^{(N)}], [y]) = 0$$

$$\begin{aligned} 0 &\leq \limsup_{N\to+\infty} \lim_{n\to+\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N\to+\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$d(x_n^{(N)}, y_n) \leq d(x_n^{(N)}, y_N) + d(y_n, y_N)$$

$$\limsup_{N\to+\infty} \lim_{n\to+\infty} d(x_n^{(N)}, y_n) \leq \limsup_{N\to+\infty} \left(\frac{1}{N+1} + \lim_{n\to+\infty} d(y_n, y_N)\right)$$

Since y is Cauchy sequence

$$\leq \limsup_{N\to+\infty} \lim_{n\to+\infty} d(y_n, y_N) = 0$$

## Example

Let $(K, |\cdot|)$ be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$ iff $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a,b) := |a-b|$$

$Cau(K)$ forms a commutative unitary ring.

$$(a_n)_{n\in\mathbb{N}} \sim (b_n)_{n\in\mathbb{N}}$$

iff

$$\lim_{n \to +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in Cau_0(K)$$

where

$$Cau_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of $Cau(K)$
Hence

$$\hat{K} = Cau(K) \setminus Cau_0(K)$$

is a quotient ring of $Cau(K)$
$|\cdot|$ extend to $\hat{K}$:

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \to +\infty} |a_n|$$

that forms an absolute value.

# Chapter 26

# Norms

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\to \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

## 26.1 Def

Let V be a vector space over K (K-module). We call seminorm on V any mapping

$$\begin{aligned} \|\cdot\| : &V \to \mathbb{R}_{\geq 0} \\ &s \mapsto \|\cdot\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V,\ \|s\| = 0$ iff $s = 0$

We say that $\|\cdot\|$ is a norm and $(V, \|\cdot\|)$ is normed space over K.

## 26.2 Remark

If $\|\cdot\|$ is a norm then

$$\begin{aligned} d : &V \times V \to \mathbb{R}_{\geq 0} \\ &(s, t) \mapsto \|s - t\| \end{aligned}$$

sectionDef Let $(V, \|\cdot\|)$ be a vector space over K equipped with a seminorm, and W be a vector space subspace of V (sub-K-module)

- The restriction of $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ to W forms a seminorm on W. It is a norm if $\|\cdot\|$ is a norm.
$$\begin{aligned}\|\cdot\|_W : &W \to \mathbb{R}_{\geq 0}\\ &x \mapsto \|x\|\end{aligned}$$
- The mapping
$$\begin{aligned}\|\cdot\|_{V/W} : &V/W \to \mathbb{R}_{\geq 0}\\ &\alpha \mapsto \inf_{s\in\alpha}\|s\|\\ \|[s]\|_{V/W} &= \inf_{w\in W}\|s+w\|\end{aligned}$$
is a seminorm on $V/W$
**Attention:** Even if $\|\cdot\|$ is a norm, $\|\cdot\|_{V/W}$ **might only be a seminorm**

## 26.3 Def

$\|\cdot\|_{V/W}$ is called the quotient seminorm of $\|\cdot\|$

## 26.4 Prop

Let $(V, \|\cdot\|)$ be a vector space over K, equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of V. Moreover, $\|\cdot\|_{V/N}$ is a norm

### Proof

If $(a, s) \in K \times N$ then $\|as\| = |a| \cdot \|s\| = 0$ so $as \in N$
If $(s_1, s_2) \in N \times N$ then $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$ so $s_1 + s_2 \in N$

### Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s\in\alpha}\|\lambda s\| = \inf_{s\in\alpha}|\lambda|\cdot\|s\| = |\lambda|\cdot\|\alpha\|_{V/W}\\ \|\alpha+\beta\| &= \inf_{s\in\alpha+\beta}\|s\| = \inf_{(x,y)\in\alpha\times\beta}\|x+y\|\\ &\leq \inf_{(x,y)\in\alpha\times\beta}(\|x\|+\|y\|)\\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let $\alpha \in V/N$ such that $\|\alpha\|_{W/N} = 0$ Let $s \in \alpha, \forall t \in N$

$$\|s+t\| \leq \|s\| + \|t\| = \|s\| = \|(s+t) + (-t)\| \leq \|s+t\| + \|-t\| = \|s+t\|$$

$$\|\alpha\|_{V/N} = \inf_{t\in N}\|s+t\| = \|s\|$$

Hence $\|\alpha\|_{V/N} = \|s\| = 0$ We obtain that $\alpha = N = [0]$

## 26.5 Def

Let $(V, \|\cdot\|)$ be a vector space over K,equipped with a seminorm. For any $x \in V$ and $r \geq 0$, we denote by

$$\mathcal{B}(x,r) = \{y \in V \mid \|y - x\| < r\}$$
$$\overline{\mathcal{B}}(x,r) = \{y \in V \mid \|y - x\| \leq r\}$$

## 26.6 Remark

If $N = \{s \in V, \|s\| = 0\}$ then when $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x,r)$$

$$x + N \subseteq \mathcal{B}(x,r)$$

## 26.7 Def

We equip the topology such that $\forall U \subseteq V, U$ is open iff $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

## 26.8 Prop

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorms. Let $f : V_1 \to V_2$ be a K-linear mapping

- If $f$ is continuous, $\forall s \in V_1$ if $\|s\|_1 = 0$ then $\|f(s)\|_2 = 0$
- If there exists $C > 0$ such that $\forall x \in V_1,\ \|f(x)\|_2 \leq C\|x\|_1$ then $f$ is continuous.

  The converse is true

when $|\cdot|$ is non-trivial

or $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$ is of finite type

### Proof

(1) Lemma If $(V, \|\cdot\|)$ is a vector space over K, equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let $s \in V \setminus N_{\|\cdot\|}$ Then $\|s\| > 0$.Let $\epsilon = \frac{\|s\|}{2},\ \forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq |\|s\| - \|s - x\|| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subsetneqq V \setminus N_{\|\cdot\|}$$

- Then $f^{-1}(N_{\|\cdot\|_2})$ is closed.
  Note that
  $$0 \in f^{-1}(N_{\|\cdot\|_2})$$
  hence
  $$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$
  $\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$
  $$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$
  and
  $$0 \in \mathcal{B}(x, \epsilon)$$
  Therefore $x \in \overline{\{0\}}$

(2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of $V_1$ that converges to some $x \in V_1$
Hence
$$\begin{aligned}\limsup_{n\to+\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n\to+\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n\to+\infty} C\|x_n - x\|_1 \\ &= C \limsup_{n\to+\infty} \|x_n - x\| \\ &= 0\end{aligned}$$

So $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. Hence $f$ is continuous at $x$
Assume that $|\cdot|$ is non-trivial and $f$ is continuous.Then
$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
is an open subset of $V_1$ containing $0 \in V_1$
So there exists $\epsilon > 0$ such that
$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
namely $\forall x \in V_1$ if $\|x\|_1 < \epsilon$ then $\|f(x)\|_2 < 1$
Since $|\cdot|$ si nontrivial, $\exists a \in K,\ 0 < |a| < 1$ We prove that $\forall x \in V_1$
$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|}\|x\|_1$$
If $\|x\|_1 = 0$ by (1) we obtain
$$\|f(x)\|_2 = 0$$
Suppose that $\|x\|_1 > 0$ then $\exists n \in \mathbb{Z}$ such that
$$\|a^n x\|_1 = |a|^n\|x\|_1 < \epsilon \leq \|a^{n-1}x\|_1 = |a|^{n-1}\|x\|_1$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned}\|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}}\frac{1}{|a|} \\ &\leq \frac{1}{\epsilon}\|x\|_1\frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|}\end{aligned}$$

## 26.9 Def: Operator Seminorm

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorm. We say that a K-linear mapping $f : V_1 \to V_2$ is bounded if there exists $C > 0$ that

$$\forall x \in V_1 \quad \|f(x)\| \leq C\|x\|_1$$

For a general K-linear mapping $f : V_1 \to V_2$ we denote

$$\|f\| := \begin{cases} \sup\limits_{x\in V_1, \|x\|_1 > 0} (\frac{\|f(x)\|_2}{\|x\|_1}) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

$f$ is bounded iff

$$\|f\| < +\infty$$

$\|f\|$ is called the operator seminorm of $f$

We denote by $\mathscr{L}(V_1, V_2)$ the set of all bounded K-linear mappings from $V_1$ to $V_2$

## 26.10 Prop

$\mathscr{L}(V_1, V_2)$ is a vector subspace of $Hom_K(V_1, V_2)$. Moreover $\|\cdot\|$ is a seminorm on $\mathscr{L}(V_1, V_2)$

### Proof

Let $f, g$ be elements of $\mathscr{L}(V_1, V_2)$

$$\begin{aligned}\|f+g\| &= \sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)+g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq (\sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1}) + (\sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1}) \\ &\leq +\infty\end{aligned}$$

Hence $f+g \in \mathscr{L}(V_1, V_2)$

Let $\lambda \in K,\ \lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x\in V_1, \|x\|_1>0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x\in V_1, \|x\|_1>0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

## 26.11 Remark

Let $f \in \mathscr{L}(V_1, V_2)$. Suppose that $\exists x \in V_1$ such that $f(x) \neq 0$. Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore $\|\cdot\|$ is a norm

## 26.12 Def

Let $(V, \|\cdot\|)$ be a normed vector space. If $V$ is complete with respect to the metric

$$\begin{aligned} d : & V \times V \to \mathbb{R}_{\geq 0} \\ & (x, y) \mapsto \|x-y\| \end{aligned}$$

then we say that $(V, \|\cdot\|)$ is a Banach space.

## 26.13 Theorem

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorm. If $(V_2, \|\cdot\|_2)$ is a Banach space, then

$$(\mathscr{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

## Proof

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(V_1, V_2)$.
$\forall x \in V_1$, the mapping

$$(f \in \mathscr{L}(V_1, V_2)) \mapsto f(x)$$

is $\|x\|_1$-Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\|\|x\|_1$$

So $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, for $V_2$ is complete, that converges to some $g(x) \in V_2$ Then we obtain a mapping $g : V_1 \to V_2$.We prove that $g$ is an element of $\mathscr{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

  $$g(x, y) = \lim_{n\to+\infty} f_n(x + y) = \lim_{n\to+\infty} f_n(x) + f_n(y)$$

  $$\begin{aligned}\|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \to +\infty)\end{aligned}$$

  So

  $$\lim_{n\to+\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

  $$g(\lambda x) = \lim_{n\to+\infty} f_n(\lambda x) = \lim_{n\to+\infty} \lambda f_n(x)$$

  $$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1)(n \to +\infty)$$

  So $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

  $$\|g(x)\| = \lim_{n\to+\infty} \|f_n(x)\| \leq (\lim_{n\to+\infty} \|f_n\|) \cdot \|x\|$$

  (because $\forall (a, b) \in V_2^2 \quad |\|a\| - \|b\|| \leq \|a - b\|$)Then

  $$|\|f_n(x)\| - \|g_n(x)\|| \leq \|f_n(x) - g_n(x)\| = o(1)\ (n \to +\infty)$$

  So $g \in \mathscr{L}(V_1, V_2)$

  $$\forall \epsilon > 0\ \exists N \in \mathbb{N}\ \forall (n, m) \in \mathbb{N}_{\geq N},\ \|f_n - f_m\| \leq \epsilon$$

  $\forall x \in V_1$

  $$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

  Taking $\lim_{n\to+\infty}$ we get

  $$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

  So $\forall n \in \mathbb{N}, n \geq N$

  $$\|f_n - g\| \leq \epsilon$$

# Chapter 27

# Differentiability

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\to \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

## 27.1 Def

Let X be a topological space and $p \in X$. Let $K$ be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K.

Let $f : X \to E$ be a mapping and $g : X \to \mathbb{R}_{\geq 0}$ be a non-negative mapping.

- We say that
  $$f(x) = O(g(x)) \ x \to p$$
  if there is a neighborhood V of $p$ in X and a constant $C > 0$ such that $\forall x \in V$
  $$\|f(x)\| \leq Cg(x)$$
- We say that
  $$f(x) = o(g(x)) \ x \to p$$
  if there exists a neighborhood V of $p$ in X and a mapping $\epsilon : V \to \mathbb{R}_{\geq 0}$ such that
  $$\lim_{x \in V, x \to p} \epsilon(x) = 0$$
  which is equivalent to
  $$\forall \delta > 0, \exists \text{ neighborhood U of } p \ U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$
  and $\forall x \in V$
  $$\|f(x)\| \leq \epsilon(x) g(x)$$

## 27.2 Def

Let $E$ and $F$ be normed vector space over $K$ $U \subseteq E$ be an open subset, $f : U \to F$ be a mapping and $p \in U$ If there exists $\varphi \in \mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \to p$$

we say that $f$ is differentiable at $p$, and $\varphi$ is the differential of $\varphi$ at $p$ Suppose that $|\cdot|$ is not trivial.$\varphi(x - p)$ also written as

$$d_p f$$

### Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \to p$$

means there exists an open neighborhood $V$ of $p$ with $V \subseteq U$ and a mapping $\epsilon V \to \mathbb{R}_{\geq 0}$ such that $\lim\limits_{x \to p} \epsilon(x) = 0$ and that $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

## 27.3 Prop

If $f$ is differentiable at $p$ ,then its differential at $p$ is unique

### Proof

Suppose that there exists $\varphi$ and $\psi$ in $\mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|\varphi - \psi\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\| (y - p)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \to p} \epsilon(y) = 0 \end{aligned}$$

## 27.4 Example

### 27.4.1

$$f : U \to F : \; f(x) = y_0 \; \forall x \in U$$

$\forall p \in U$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence $\forall x \in E$

$$d_p(f(x)) = 0$$

### 27.4.2

Let $f \in \mathscr{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence $d_p f = f$

### 27.4.3

$$\begin{aligned} A :& E \times E \to E \\ & (x, y) \mapsto x + y \end{aligned}$$

Let $E$ be a normed space. Then $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

### 27.4.4

$$\begin{aligned} m :& K \times E \to E \\ & (\lambda, x) \mapsto \lambda x \end{aligned}$$

Let $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when $(\lambda, x) \to (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

  is a K-linear mapping.

- $(\mu_1+\mu_2)p + a(y_1+y_2) = (\mu_1 p + a y_1) + (\mu_2 p + a y_2)$
- $b\mu p + a(by) = b(\mu p + ay)$
- $$\begin{aligned}\|\mu p + ay\| &\le |\mu|\,\|p\| + |a|\,\|y\| \\ &\le \max\{|\mu|\,,\|y\|\}(|a| + \|p\|)\end{aligned}$$

Hence $m$ is differentiable and $\forall(\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

## 27.5 Theorem:Chain rule

Let $E, F, G$ be normed vector spaces, $U \subseteq E, V \subseteq F$ be open subsets.

Let $f : U \to F,\ g : V \to G$ be mappings such that $f(U) \subseteq V$ Let $p \in U$ Assume that $f$ is differentiable at $p$ and $g$ differentiable at $f(p)$ Then $g \circ f$ is differentiable at $p$ and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

### Proof

Let $x \in U$ By definition

$$\begin{aligned} f(x) &= f(p) + d_p f(x-p) + o(\|x-p\|) \\ f(x) - f(p) &= O(\|x-p\|) \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x-p) + o(\|x-p\|)) + o(\|x-p\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x-p)) + o(\|x-p\|) \end{aligned}$$

So $g \circ f$ is differentiable at $p$ and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

## 27.6 Prop

Let $n$ be a positive integer $E, (F_i)_{i\in\{1,\dots,n\}}$ be normed vector spaces over K. $U \subseteq E$ an open subset, $p \in U$

$\forall i \in \{1, ..., n\}$ let $f_i : U \to F_i$ be a mapping. Let

$$f : U \to F = \prod F_i$$

be the mapping that sends $x \in U$ ot $(f_i(x))_{i\in\{1,\dots,n\}}$ We equip $F$ with the norm $\|\cdot\|$ defined as :

$$\left\|(y_i)_{i\in\{1,\dots,n\}}\right\| = \max_{i\in\{1,\dots,n\}} \|y_i\|$$

Then $f$ is differentiable at $p$ iff each $f_i$ is differentiable at $p$. Moreover, when this happen, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i\in\{1,\ldots,n\}}$$

## Proof

$\Leftarrow$ Suppose that $(f_i)_{i\in\{1,\ldots,n\}}$ are differentiable at $p$

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i\in\{1,\ldots,n\}} \\ &= (d_p f_i(x-p))_{i\in\{1,\ldots,n\}} + o(\|x-p\|) \end{aligned}$$

Therefore $f$ is differentiable at $p$ and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i\in\{1,\ldots,n\}}$$

$\Rightarrow$ Let

$$\begin{aligned} \pi_i : & F \to F_i \\ & (x_i)_{i\in\{1,\ldots,n\}} \mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has $\|\pi_i\| \leq 1$ because

$$\|x_i\| \leq \max_{i\in\{1,\ldots,n\}} \|x_i\| = \left\|(x_i)_{i\in\{1,\ldots,n\}}\right\|$$

$\pi_i$ is differentiable at $p$ then $\pi_i \circ f = f_i$ is differentiable at $p$

# 27.7 Def

Let $U$ be an open subset of $K$ and $(F, \|\cdot\|)$ be a normed vector space. If $f : U \to F$ is a mapping that is differentiable at some $p \in U$. We denote by $f'(p)$ the element

$$d_p f(1) \in F$$

called the derivative of $f$ at $p$

# 27.8 Corollary

Let $U$ and $V$ be open subsets of $K$, $(F, \|\cdot\|)$ be a normed vector space over K. $f : U \to K,\ g : V \to F$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If $f$ is differentiable at $p$ and $g$ is differentiable at $f(p)$ then

$$(g \circ f)'(p) = f'(p) g'(f(p))$$

## Proof

By definition

$$\begin{aligned} d_p(g\circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p)\cdot 1) \\ &= f'(p)\cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

# 27.9 Corollary

Let E and F be normed vector spaces, $U \subseteq E$ an open subset. $f : U \to L$ and $g : U \to F$ be mappings and $p \in U$ If both $f, g$ differentiable at $p$ then

$$\begin{aligned} fg :&U \to F \\ &x \mapsto f(x)g(x) \end{aligned}$$

is also differentiable at $p$ and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_pf(l) + g(p)d_pf(l)$$

## Proof

Consider

$$\begin{aligned} m :&K\times F \to F \\ &(a,y) \to ay \end{aligned}$$

We have shown $m$ is differentiable and

$$d_{a,y}m(b,z) = by = az$$

$fg$ is the following composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "fg"'] & K\times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m\circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p),g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

## 27.10 Corollary

Let $U$ be an open subset of $K$, $f, g$ be mappings from U to K and to a normed space F respectively.If $f, g$ are differentiable at $p \in U$ then

$$(fg)'(p) = d_p(fg)(1) = d_pf(1)g(p) + f(p)d_pg(1) = f'(p)g(p) + f(p)g'(p)$$

### Example

$$\begin{aligned} f_n :& K \to K \\ & x \mapsto x^n \end{aligned}$$

is differentiable at any $x \in K$

$$f'_n(x) = nx^{n-1}$$

#### Proof

$f_1 : K \to K$ is differentiable $\forall x \in K$

$$d_x f_1 = f_1$$

If $f'_n(x) = nx^{n-1}$ then

$$\begin{aligned} f'_{n+1}(x) &= (f_n f_1)'(x) \\ &= f_n(x)f'_1(x) + f'_n(x)f_1(x) \\ &= x^n + x'_n(x) = x^n + xnx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(l) &= l d_x f_n(1) \\ &= lnx^{n-1} \end{aligned}$$

## 27.11 Prop

Let $E, F, G$ be normed vector spaces. $U \subseteq E$ be an open subset, $\varphi \in \mathscr{L}(F, G), p \in U$ if $f : U \to E$ is differentiable at $p$ then so is $\varphi \circ f$. Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

### Proof

$\varphi$ is differentiable at $f(p)$ nad $d_{f(p)}\varphi = \varphi$

## 27.12 Corollary

Let E and F be normed vector spaces $U \in E$ be an open subset, $p \in U$. Let $f : U \to F$ and $g : U \to F$ be mappings that are differentiable at $p, (a, b) \in K \times K$. Then $af + bg$ is differentiable at $p$ and

$$d_p(af_bg) = ad_pf + bd_pg$$

### Proof

$af + bg$ is composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "ay+bz"'] & K \times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

## 27.13 Def: Equivalence of Norms

Let E be a vector space over K and $\|\cdot\|_1, \|\cdot\|_2$ be norms on E. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist constants $C_1, C_2 > 0$ such that $\forall s \in E$

$$C_1 \|S\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

## 27.14 Prop

If $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover $\|\cdot\|_1, \|\cdot\|_2$ defines the same topology on E.

### Proof

$$\|s\|_2 \leq C_2 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are continuous. So $\forall$ open subset U of $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in $(E, \|\cdot\|_1)$. Conversely if V is open in $(E, \|\cdot\|_1)$ then

$$V = Id_E^{-1}(V)$$

is open in $(E, \|\cdot\|_2)$

## 27.15 Remark

If $\|\cdot\|_1, \|\cdot\|_2$ are two norms on E that define the same topology on E, then they are equivalent (under the assumption that $|\cdot|$ is not trivial)

## 27.16 Prop

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces $\|\cdot\|'_E$ and $\|\cdot\|'_F$ be norms on E and F that are equivalent to $\|\cdot\|_E, \|\cdot\|_F$ respectively. Let $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping.

Let $p \in U$ Then $f$ is differentiable at $p$ with respect to $\|\cdot\|_E$ and $\|\cdot\|_F$ iff it's differentiable with respect to $\|\cdot\|'_E$ and $\|\cdot\|'_F$

Moreover the differentiable of $f$ at $p$ is not changed in the change of norms from $(\|\cdot\|_E, \|\cdot\|_F)$ to $(\|\cdot\|'_E, \|\cdot\|'_F)$

### Proof

$$\begin{tikzcd}
U \arrow[r, "Id_U"] \arrow[rrr, bend right=10, "f"'] & U \arrow[r, "f"] & F \arrow[r, "Id_F"] & F
\end{tikzcd}$$

$$(E, \|\cdot\|'_E) \qquad (E, \|\cdot\|_E) \qquad \|\cdot\|_F \qquad \|\cdot\|'_F$$

$$\begin{aligned}
d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\
&= Id_F \circ d_p f \circ Id_E \\
&= d_p f
\end{aligned}$$

$d'_p f : (E, \|\cdot\|'_E) \to (F, \|\cdot\|'_F)$

## 27.17 Theorem

Let V be a finite dimensional vector space over K. Then all norms on V are equivalent. Moreover V is complete with respect to any norm on V.

## Proof

Let $(e_i)_{i=1}^n$ be a basis of V(linear independent system of generators)The the mapping:

$$V \to \mathbb{R}_{\geq 0}$$
$$\sum_{i\in\{1,\dots,n\}} a_i e_i \mapsto \max_{i\in\{1,\dots,n\}}\{|a_i|\}$$

is a norm on V

Let $\|\cdot\|$ be another norm on V. One has

$$\begin{aligned}\left\|\sum_{i\in\{1,\dots,n\}} a_i e_i\right\| &\leq \sum_{1\in\{1,\dots,n\}} |a_i|\,\|e_i\| \\ &\leq (\sum_{i\in\{1,\dots,n\}} \|e_i\|) \max_{i\in\{1,\dots,n\}}\{|a_i|\}\end{aligned}$$

We reason by induction that there exists $C > 0$ such that

$$\max_{i\in\{1,\dots,n\}}\{|a_1|\} \leq C\left\|\sum_{i\in\{1,\dots,n\}} a_i e_i\right\|$$

The case where $n = 0$ is trivial.

n=1
$$\|a_1 e_1\| = |a_1|\,\|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension $< n$

Let
$$W = \{\sum_{i\in\{1,\dots,n-1\}} a_i e_i \mid (a_i)_{i\in\{1,\dots,n-1\}} \in K^{n-1}\}$$

equipped with $\|\cdot\|$ restricted to W

The induction hypothesis shows that W is complete. Hence it's closed in V. Let $Q = V/W$ and $\|\cdot\|_Q$ be the quotient norm on Q that's defined as

$$\forall \alpha \in Q \quad \|\alpha\|_Q = \inf_{s\in\alpha} \|s\|$$

– If $s \in V \setminus W, \exists \epsilon > 0$ such that

$$\overline{B}(s,\epsilon) \cap W = \phi$$

$\forall t \in W$,
$$s + t \notin \overline{B}(0,\epsilon)$$

since otherwise
$$-t \in W \cap \overline{B}(s,\epsilon)$$

Therefore
$$\|[s]\|_Q = \inf_{i\in W} \|s+t\| \geq \epsilon > 0$$

- $\forall \lambda \in K$

$$\|\lambda\alpha\|_Q = \inf_{s\in\alpha} \|\lambda s\| \quad = |\lambda|$$
$$\inf_{s\in\alpha} \|s\| = |\lambda| \cdot \|s\|_Q$$

-

$$\begin{aligned}\|\alpha+\beta\|_Q &= \inf_{s\in\alpha+\beta} \|s\| \\ &= \inf_{(x,y)\in\alpha\times\beta} \|x+y\| \\ &\leq \inf_{(x,y)\in\alpha\times\beta} (\|x\|+\|y\|) \\ &= \inf_{x\in\alpha} \|x\| + \inf_{y\in\beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some $A > 0$ such that $\forall (a_i)_{i\in\{1,\dots,n-1\}} \in K^{n-1}$

$$\max_{i\in\{1,\dots,n-1\}} \{|a_i|\} \leq A \left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\|$$

Take

$$s = \sum_{i\in\{1,\dots,n\}} a_i e_i \in V$$

Let $\alpha = [s] = a_n[e_n] \in Q$

$$\left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \leq \|s\| + |a_n| \cdot \|e_n\| \leq \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \, \|[e_n]\|_Q = |a_n| \inf_{t\in W} \|e_n + t\|$$

Take $e'_n \in V$ such that $[e'_n] = [e_n]$ and $\|e'_n\| \leq \|[e_n]\|_Q + \epsilon$

Note that $(e_1, \dots, e_{n-1}, e'_n)$ forms also basis of V over K. Hence by replacing $e_n$ by $e'_n$ we may assume that $\|e_n\| \leq \|[e_n]\|_Q + \epsilon$

$s = a_n e_n + t \in V$ with $t \in W$

$$\|s\| \geq \|[a_n e_n]\|_Q = |a_n| \, \|[e_n]\|_Q \geq B^{-1} |a_n| \cdot \|e_n\|$$

- If $\|a_n e_n\| < \frac{1}{2} \|t\|$

$$\|s\| \geq \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \geq \frac{1}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

- If $\|a_n e_n\| \geq \frac{1}{2} \|t\|$

$$\|s\| \geq B^{-1} |a_n| \cdot \|e_n\| \geq \frac{B^{-1}}{2} \|t\| \geq \frac{B^{-1} A}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

We take $C = \max\{B^{-1}\|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$ Then

$$\|s\| \geq C \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Another proof

completeness Under the norm $\max_{i\in\{1,...,n\}}$, a sequence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ is a Cauchy sequence iff $\forall i \in \{1,...,n\} \quad (a_i^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence. Since K is complete each $(a_i^{(k)})_{k\in\mathbb{N}}$ converges to some $a_i \in K$ Hence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ converges.

## 27.18 Prop

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed vector spaces over K. Assume that E is finite dimensional. Then any K-linear mapping $\varphi : E \to F$ is bounded.

### Proof

Let $(e_i)_{i=1}^n$ be a basis of E. For any two norms on E are equivalent.
$\forall(a_1, ..., a_n) \in K$

$$\left\|\sum_{i=1}^n a_ie_i\right\|_E = \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Then for any $s = \sum_{i=1}^n a_ie_i$

$$\|\varphi(s)\|_F = \left\|\sum_{i=1}^n a_ie_i\right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq (\sum_{i=1}^n \|\varphi(e_i)\|_F) \|s\|_E$$

## 27.19 Theorem

Let $E, F$ be normed vector spaces over a complete valued field, $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping . If $f$ is differentiable at $p$ then $f$ is continuous at $p$

### Proof

$$\begin{aligned}
f(x) &= f(p) + d_pf(x-p) + o(\|x-p\|) \\
&= f(p) + O(\|x-p\|) \\
&= f(p) + o(1) \quad x \to p \\
&\Rightarrow \lim_{x\to p} f(x) = f(p)
\end{aligned}$$

# Chapter 28

# Compactness

## 28.1 Def: cover

Let X be a topological space, $Y \subseteq X$ we call open cover of Y any family $(U_i)_{i \in I}$ open subset of X such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If $I$ is finite set, we say that $(U_i)_{i \in I}$ iss a finite open cover. If $J \subseteq I$ such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that $(U_j)_{j \in J}$ is a sub cover of $(U_i)_{i \in I}$

## 28.2 Def: compact

If any open cover of Y has a finite subcover ,we say that Y is quasi-compact. If in addition X is Hausdorff, namely $\forall (x, y) \in X \times X$ with $x \neq y$ $\exists$ open neighborhoods U and V of $x$ and $y$ such that $U \cap V = \varnothing$, we say that $Y$ is compact

## 28.3 Def

Let X be a set and $\mathcal{F}$ be a filter on X. If there does not exists any filter $\mathcal{F}'$ of X such that $\mathcal{F} \subsetneq \mathcal{F}'$, then we say that $\mathcal{F}$ is an ultrafilter.

**Zorn's lemma** implies that $\forall \mathcal{F}_{\prime}$ of X there exist an ultrafilter $\mathcal{F}$ if X containing $\mathcal{F}_0$

## 28.4 Prop

Let $\mathcal{F}$ be a filter on a set X. The following statements are equivalent.

(1) $\mathcal{F}$ is an ultrafilter

(2) $\forall A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$

(3) $\forall (A, B) \in \wp(X)^2$ if $A \cap B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$

### Proof

(1) $\Rightarrow$ (2) Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$ $\forall B \in \mathcal{F}$ one has

$$B \cap A \neq \varnothing$$

since otherwise $B \subseteq X \setminus A$ and hence $X \setminus A \in \mathcal{F}$ contradiction.

(2) $\Rightarrow$ (3) Suppose that $B \notin \mathcal{F}$ then $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So $A \in \mathcal{F}$

(3) $\Rightarrow$ (1) Suppose that $\mathcal{F}'$ is a filter such that $\mathcal{F} \subsetneqq \mathcal{F}'$ Take $A \in \mathcal{F}' \setminus \mathcal{F}$ Then by $X = A \cup (X \setminus A) \in \mathcal{F}$ Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \varnothing = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

## 28.5 Theorem

Let $(X, \mathcal{J})$ be a topological space . The following are equivalent

(1) X is quasi-compact

(2) Any filter of X has an accumulation point

(3) Any ultrafilter of X is converges.

### Proof

(1) $\Rightarrow$ (2) Assume that a filter $\mathcal{F}$ of X does not have any accumulation point. $\forall x \in X$ $\exists A_x \in \mathcal{F}$ $\exists$ open neighborhood $V_x$ of $x$ such that $A_x \cap V_x = \varnothing$ Since $X = \bigcup_{x \in X} V_x$ there is

$$\{x_1, ..., x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^{n} V_{x_i}$$

Take $B = \bigcap_{i=1}^{n} A_{x_i} \in \mathcal{F}$

$$B \cap X = B = \varnothing$$

Since $\forall i \; B \cap V_x = \varnothing$ contradiction.

(2) $\Rightarrow$ (3) Let $\mathcal{F}$ be an ultrafilter of X. By (2) there exist $x \in X$ such that $\mathcal{F} \cup \mathcal{V}_x$ generates a filter $\mathcal{F}'$ Since $\mathcal{F}$ is an ultrafilter $\mathcal{F} = \mathcal{F}'$ and hence $\mathcal{V}_x \subseteq \mathcal{F}$

(3) $\Rightarrow$ (1) Let $(U_i)_{i \in I}$ be an open cover of $X$ we suppose that this have no finite subcover. $\forall i \in I$ let

$$F_i = X \setminus U_i$$

For any $J \subseteq I$ finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \varnothing$$

Let $\mathcal{F}$ be the smallest filter on X that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let $\mathcal{F}'$ be ultrafilter containing $\mathcal{F}$. It has a limit point $x$ There exist $i \in I$ such that $x \in U_i$.Since $U_i$ is a neighborhood of $x$ and $\mathcal{V}_x \subseteq \mathcal{F}'$ we get $U_i \in \mathcal{F}'$ This is impossible since $F_i \in \mathcal{F}'$

## 28.6 Theorem

Let $(X, d)$ be a metric space. The following statements are equivalent:

(1) X is complete and $\forall \epsilon > 0 \; \exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

(2) X is compact

### Proof

(1) $\Rightarrow$ (2) Let $\mathcal{F}$ be an ultrafilter Let $\epsilon > 0$ and $\{x_1, ..., x_n\} \subseteq X$ such that

$$X = \bigcup_{i=1}^{n} \mathcal{B}(x, \epsilon)$$

There exists some $i \in \{1, ..., n\}$ such that $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$ That means $\mathcal{F}$ is a Cauchy filter (namely $\forall \delta > 0 \; \exists A \in \mathcal{F}$ of diameter $\leq \delta$) Since X is complete $\mathcal{F}$ has a limit point. So $\mathcal{F}$ is compact.

(2) ⇒ (1) Let $\epsilon > 0$ One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since X is compact $\exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$\mathcal{F}$ is an ultrafilter
$\Leftrightarrow \forall A \subseteq X \; A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$
$\Leftrightarrow \forall y \in \mathcal{F}$ if $y = A \cup B$ either $A \in \mathcal{F}$ or $B \in \mathcal{F}$
$\Leftrightarrow \forall Y \in \mathcal{F}$ if $Y = A_1 \cup A_2 \cup ... \cup A_n \; \exists i \in \{1, ..., n\}, A_i \in \mathcal{F}$

Let $\mathcal{F}$ be a Cauchy filter Let $x \in X$ be an accumulation point of $\mathcal{F}$ $\forall \epsilon > 0 \; \exists A \in \mathcal{F}$ with diameter $\leq \frac{\epsilon}{2}$ Note that $A \cup \mathcal{B}x, \frac{\epsilon}{2} \neq \varnothing$ Take $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2})$ $\forall z \in A$

$$\begin{aligned} d(x,z) &\leq d(x,y) + d(y,z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $A \subseteq \mathcal{B}(x, \epsilon)$ So $\mathcal{B}(x, \epsilon) \in \mathcal{F}$ This implies $\mathcal{V}_x \subseteq \mathcal{F}$

## 28.7 Lemma

Let $(X, d)$ be a metric space

(1) Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point $\mathcal{F}$ a limit point of $\mathcal{F}$

(2) X is complete iff any Cauchy filter of X has a limit point

### Proof

(1)

- Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point of $\mathcal{F}$ is a limit point of $\mathcal{F}$

(2) Suppose that X is complete.Let $\mathcal{F}$ be a Cauchy filter.$\forall n \in \mathbb{N}_{\geq 1}$ let $A_n \in \mathcal{F}$ such that $diam(A_n) \leq \frac{1}{n}$ Take $x_n \in \bigcap\limits_{k=1}^{n} A_k \in \mathcal{F}$ Then $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ is a Cauchy sequence since $\forall \epsilon > 0$ if we take $N \in \mathbb{N}$ with $\frac{1}{N} \leq \epsilon$ then $\forall (n,m) \in \mathbb{N}_{\geq N}$ $d(x_n, x_m \leq \frac{1}{N})$ Hence $(x_n)_{N \in \mathbb{N}_{\geq 1}}$ converges to some $x \in X$ Note that $x$ is an limit point of $\mathcal{F}$ since $\forall \epsilon > 0 \; \exists n \in \mathbb{N}$ with $A_n \subseteq \mathcal{B}(x, \epsilon)$ It suffices to take $n$ such that $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N},\ \{x_N, x_{N+1}, ...\} \subseteq A\}$$

This is a Cauchy filter on X since

$$\lim_{N\to+\infty} diam\{x_N, x_{N+1}, ...\} = 0$$

Hence $\mathcal{F}$ has a limit point $x \in X$ By definition $\forall U \in \mathcal{V}_x\ \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, ...\} \subseteq U$$

So $x = \lim_{n\to+\infty} x_n$

## 28.8 Prop

Let $f : X \to Y$ be a continuous mapping of topological spaces. If $A \subseteq X$ is quasi-compact then $f(A) \subseteq Y$ is also quasi-compact.

### Proof

Let $(V_i)_{i\in I}$ be an open cover of $f(A)$ Then

$$(f^{-1}(V_i))_{i\in I}$$

is an open cover of $A$ So $\exists J \subseteq I$ such that

$$A \subseteq \bigcup_{j\in J} f^{-1}(V_i)$$

This implies

$$f(A) \subseteq \bigcup_{j\in J} V_j$$

So $f(A)$ is quasi-compact.

## 28.9 Prop

Let X be a topological space and $A \subseteq X$ be a quasi-compact subset. For any closed subset $F$ of $X$ $A \cap F$ is quasi-compact.

### Proof

Let $(U_i)_{i\in I}$ be an open cover of $A \cap F$.Then

$$A \subseteq (\bigcup_{i\in I} U_i) \cup (X \setminus F)$$

Since A is quasi-compact there exist $J \subseteq I$ finite such that

$$A \subseteq (\bigcup_{j \in J} U_j) \cup (X \setminus F)$$

Hence $A \cap F \subseteq \bigcup_{j \in J} U_j$

## 28.10 Prop

Let X be a Hausdorff topological space. Any compact subset A of X is closed.

### Proof

Let $x \in X \setminus A$ $\forall y \in A, \exists$ open subsets $U_y$ nad $V_y$ such that $y \in U_y, x \in V_y$ and $U_y \cap V_y = \varnothing$ Since $A \subseteq \bigcup_{y \in A} U_y$ $\exists\{y_1, ..., y_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^{n} U_{y_i}$$

Let

$$U = \bigcup_{i=1}^{n} U_{y_i} \quad V = \bigcap_{i=1}^{n} V_{y_i}$$

These are open subset Moreover $A \subseteq U, x \in V$ and $U \cap V = \bigcup_{i=1}^{n} (U_{y_i} \cap V) = \varnothing$ In particular $x \in V \subseteq X \setminus A$ So $X \setminus A$ is open

## 28.11 Prop

Let X be a Hausdorff topological space and A and B be compact subsets of X such that $A \cap B = \varnothing$ Then there exists open subsets U and V such that

$$A \subseteq U, B \subseteq BandU \cap V = \varnothing$$

### proof

We have seen in the proof of the previous proposition that $\forall x \in B, \exists U_x, V_x$ open such that $A \subseteq U_x, x \in V_x$ and $U_x \cap V_x = \varnothing$ Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists\{x_1, ..., x_m\} \subseteq B$ such that

$$B \subseteq \bigcup_{i=1}^{n} V_{x_i}$$

We take

$$U = \bigcap_{i=1}^{m} U_{x_i} \quad V = \bigcup_{i=1}^{m} U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq U \quad U \cap V = \varnothing$$

## 28.12 Theorem

Let $(X, \mathcal{J})$ be a Hausdorff topological space If $(A_n)_{n\in\mathbb{N}}$ is a sequence of non-empty compact subsets of X such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq ...$$

Then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

### Proof

Suppose that

$$\bigcap_{n\in\mathbb{N}} A_n = \varnothing$$

then

$$A_0 \subseteq \bigcup_{n\in\mathbb{N}} (X \setminus A_n)$$

Since $A_0$ is compact, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^{N} (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^{N} A_n \\ &= X \setminus A_n \end{aligned}$$

So

$$A_n = \varnothing$$

## 28.13 Def

Let $(X, \tau)$ be a topological space if any sequence in X has a convergent subsequence, we say that X is sequentially compact.

### Example

By Bolzano-Weierstrass, any bounded sequence in $\mathbb{R}$ has a convergent subsequence. So any bounded and closed subset of $\mathbb{R}$ is sequentially compact.

### Note

bounded and closed together implies sequentially compact.

## 28.14 Theorem

Let $(X, d)$ be a metric space. Then the following statements are equivalent:

(1) $(X, d)$ is compact

(2) $(X, d)$ is sequentially compact

### Proof

$(1) \Rightarrow (2)$ Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Assume that no subsequence of $(x_n)_{n\in\mathbb{N}}$ converges in X. For any $p \in X$ there exists $\epsilon_p > 0$ such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For X is compact $\exists (p_i)_{i\in\{1,\dots,n\}}$

$$X \subseteq \bigcup_{i=1}^{n} \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^{n} \{n \in \mathbb{N}\ d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

$(2) \Rightarrow (1)$

prove $(X, d)$ is complete Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences $(x_{k_n})_{n\in\mathbb{N}}$ must converges to the same limit.

So $(X, d)$ is complete

If X is not covered by finitely many balls of radius $\epsilon$ we can construct a sequence $(x_{k_n})_{n\in\mathbb{N}}$ such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^{n} \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

## 28.15 Def

Let X be a Hausdorff topological space. If for any $x \in X$ there exist a compact neighborhood $\mathcal{C}_x$ we say that $X$ is locally compact.

### Example

$\mathbb{R}$ is locally compact.

## 28.16 Prop

Assume that $(K, |\cdot|)$ is a locally compact non-trivial valued field. field. Let $(E, \|\cdot\|)$ be a finite dimensional normed $K$-vector space. A subset $Y \subseteq E$ is compact iff it's closed and bounded.

### Proof

$\Rightarrow$ Let $Y \subseteq X$ be compact. Then for Y is Hausdorff, Y is closed. Moreover

$$Y \subseteq \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq ... \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^{n} \mathcal{B}(0, n_i)$$

$\Rightarrow$ Y is bounded.

$\Leftarrow$ We prove sequentially compact by a theorem proved before.
Let $(e_i)_{i=1}^{d}$ be a basis of E. Again we assume

$$\left\| \sum_{i=1}^{d} a_i e_i \right\| = \max_{i\in\{1,\dots,d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n\in\mathbb{N}} = (\sum_{i=1}^{d} a_i^{(n)} e_i)_{n\in\mathbb{N}}$$

Since $Y$ is bounded for any $i \in \{1, ..., d\}$ the sequence $(a_i^{(n)})$ is bounded. In particular we find $M > 0$ such that $\forall i \in \{1, ..., n\}$

$$\left|a_i^{(n)}\right| < M$$

Since $(K, |\cdot|)$ is locally compact, there exists a compact set $\mathcal{C} = \mathcal{C}_0 \subseteq K$ that's a neighborhood of 0. Let $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since $K$ is not trivially valued, then exists $a \in K$ such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$ is compact. We have the K-linear mapping

$$\begin{aligned} K \to &\quad K \\ y \mapsto &\quad ay \end{aligned}$$

is bounded, then continuous. Hence $a\mathcal{C}$ is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find $(I_i)_{i=1}^{d}$ are infinite subsets of $\mathbb{N}$ with

$$I_1 \supseteq ... \supseteq I_d$$

such that $(a_j)_{j\in I_i}^{(n)}$ converges to some $a_i \in K$. It follows that our original sequence has a convergent subsequence converges to $\sum_{i=1}^{d} a_i e_i$.

So $Y$ is sequentially compact.

## 28.17 Theorem

Let X be a topological space and $f : X \to \mathbb{R}$ be a continuous mapping. If $Y \subseteq X$ is a quasi-compact subset, then there exists $a \in Y$ and $b \in Y$ such that $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of $f$ to $y$ attains its maximum and minimum.

## Proof

$f(Y) \subseteq \mathbb{R}$ is a non-empty compact subset since $Y$ is quasi-compact and $\mathbb{R}$ is Hausdorff. Moreover, since $\mathbb{R}$ is locally compact. SO $f(Y)$ is bounded and closed.

Note that there exists sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ is $f(Y)$ that tends to $\sup f(Y)$ and $\inf f(Y)$ respectively. Since $f(Y)$ is closed, $\sup f(Y), \inf f(Y)$ belongs to $f(Y)$. So $f(Y)$ has a greatest and a least element.

# Chapter 29

# Mean Value Theorems

## 29.1 Rolle Theorem

Let $a, b$ be real numbers such that $a < b$ Let $f : [a, b] \to \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$ If $f(a) = f(b)$ then $\exists t \in ]a, b[$ such that

$$f'(t) = 0$$

### Proof

Since $[a, b]$ is closed and bounded then it's compact, $f$ attains its maximum and minimum. Let $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If $M \neq l$ $\exists t \in ]a, b[$ such that $f(t) = M$

$$f(t + x) = f(t) + f'(t)x + o(|X|)$$

$$f(t - x) = f(t) - f'(t)x + o(|X|)$$

$$0 \leq (f(t + x) - f(t))(f(t - x) - f(t))$$

$$= -f'(t)^2 x^2 + o(|x|^2)$$

$$0 \leq -f'(t)^2 + o(1) \quad x \to 0$$

Taking the limit when $x \to 0$ we get $f'(t)^2 = 0$

If $m \neq l$ then any $t \in ]a, b[$ such that $f(t) = m$ verifies $f'(t) = 0$

If $m = l = M$ $f$ is constant, so $\forall t \in ]a, b[, f'(t) = 0$

## 29.2 Mean value theorem(Lagrange)

Let $a, b$ be real numbers $a < b$, $f : [a, b] \to \mathbb{R}$ be a continuous mapping differentiable on $]a, b[$, then $\exists t \in ]a, b[$ such that

$$f(b) - f(a) = f'(t)(b - a)$$

### Proof

Let $g : [a, b] \to \mathbb{R}$ be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) = f(a) \quad g(b) = f(a)$ then apply Rolle Theorem to $g$ we get the proof.

## 29.3 Mean value inequality

Let $a, b$ be real numbers such that $a < b$ $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ $f : [a, b] \to E$ be a continuous mapping such that $f$ is differentiable on $]a, b[$ Then

$$\|f(b) - f(a)\| \leq (\sup_{x \in ]a,b[} \|f'(x)\|)(b - a)$$

### Proof

Suppose that

$$\sup_{x \in ]a,b[} \|f'(x)\| < +\infty$$

Let $M \in \mathbb{R}$ such that

$$M > \sup_{x \in ]a,b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition $J$ is an interval containing $a$, so $J$ is of form $[a, c[$ or $[a, c]$ Since $f$ is continuous by taking a sequence $(c_n)_{n \in \mathbb{N}}$ in $[a, b[$ that converges to $c$ we obtain

$$\begin{aligned} \|f(c) - f(a)\| &= \lim_{n \to +\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n \to +\infty} M(c_n - a) \\ &= M(c - a) \end{aligned}$$

Hence $c \in J$ namely $J = [a, c]$

$c > a$ We will prove that $c = b$ by contradiction

Suppose that $c < b$ $\forall h \in ]0, b - c[$

$$\begin{aligned} \|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h) \end{aligned}$$

Since $M > \|f'(c)\|, \exists h_0 > 0$ such that $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$

Hence

$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h)-f(c)\|+\|f(c)-f(a)\| \\ &\leq M(c_h-c+c-a) \\ &= M(c+h-a)\end{aligned}$$

So $c+h_0 \in J$ Contradiction. Thus

$$\|f(b)-f(a)\| \leq M(b-a)$$

for any $M > \sup\limits_{x\in]a,b[} \|f'(x)\|$ since $M$ is arbitrary the expected inequality holds.

$c=a$ In general, we apply the particular case (fis-extendable to a differentiable mapping at $a$) to $[\frac{a+b}{2}, b]$ and $[a, \frac{a+b}{2}]$ to get

$$\left\|f(b)-f(\frac{a+b}{2})\right\| \leq C\frac{b-a}{2}$$

$$\left\|f(\frac{a+b}{2})-f(a)\right\| \leq C\frac{b-a}{2}$$

with $C = \sup\limits_{x\in]a,b[} \|f'(x)\|$

Remark If $f$ is defined on an open neighborhood of $a$ and is differentiable at $a$ the the same arguments hold without the assumption

## 29.4 Theorem

Let I be an interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ be a continuous mapping, then $f(I)$ is an interval.

### Proof

Let $x \neq y$ be two elements of $f(I)$ Let $a, b$ elements of I such that $x = f(a)$ $y = f(b)$ without loss of generality, we assume $a < b$
Let $z \in \mathbb{R}$ such

$$(z-x)(z-y) \leq 0$$

We construct by induction three sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If $a_n, b_n, c_n$ are constructed, satisfying

$$c_n = \frac{1}{2}(a_n+b_n)$$

$$(z-f(a_n))(z-f(b_n)) \leq 0$$

we let

$$\begin{aligned}
(a_{n+1}, b_{n+1}) &= (a_n, c_n) && \text{if } (z - f(a_n))(z - f(c_n)) \leq 0 \\
(a_{n+1}, b_{n+1}) &= (c_n, b_n) && \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\
& && \quad ((z - f(c_n))(z - f(b_n)) \leq 0) \\
c_{n+1} &= \frac{a_{n+1} + b_{n+1}}{2}
\end{aligned}$$

The sequence $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are increasing and decreasing respectively and bounded, hence converges to some $l, m \in [a, b]$
Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \to 0(n \to +\infty)$$

So $l = m$,by $(z - f(a_n))(z - f(b_n)) \leq 0$ we obtain by letting $n \to +\infty$

$$(z - f(l))^2 \leq 0$$

So $z = f(l)$

## 29.5 Theorem(Heine)

Let I be an open interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a differentiable mapping. Then $f'(I)$ is an interval.

### Proof

Let $(a, b) \in I^2$ such that $a < b$. Consider the following mappings:

$$\begin{aligned}
g : [a, b] &\to \mathbb{R} \\
x &\mapsto \begin{cases} \dfrac{f(x) - f(a)}{x - a} & x \neq a \\ f'(a) & x = a \end{cases} \\
h : [a, b] &\to \mathbb{R} \\
x &\mapsto \begin{cases} \dfrac{f(b) - f(x)}{b - x} & x \neq b \\ f'(b) & x = b \end{cases}
\end{aligned}$$

$g, h$ are continuous $(\frac{f(x)-f(a)}{x-a} = f'(a) + o(1)\ x \to a)$
So $g([a, b])$ and $h([a, b])$ are intervals . Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$
$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that $g(b) = h(a)$ so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence $f'(I)$ is an interval.

# Chapter 30

# Fixed Point Theorem

## 30.1 Def

Let X be a set and $T : X \to X$ be a mapping. If $x \in X$ satisfies $T(x) = x$ we say that $x$ is a fixed point of $T$.

## 30.2 Def

Let $(X, d)$ be a metric space and $T : X \to X$ be a mapping. If $\exists \epsilon \in [0, 1[$ such that $T$ is $\epsilon$-Lipschitzian then we say that $T$ is a contraction.

## 30.3 Fixed Point Theorem

Let $(X, d)$ be a COMPLETE non-empty metric space, and $T : X \to X$ eb a contraction. Then $T$ has a unique fixed point. Moreover, $\forall x_n \in X$ if we let

$$x_{n+1} = T(x + n), x_0 \in X$$

then $(x_n)_{n\in\mathbb{N}}$ converges to the fixed point.

### Proof

If $p$ and $q$ are two fixed point of $T$, then

$$d(p, q) = d(T(p), T(q)) \le \epsilon d(p, q)$$

So $d(p, q) = 0$.
Let

$$x_{n+1} = T(x + n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \le \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any $N \in \mathbb{N}$, $\forall (n, m) \in \mathbb{N}^2_{\geq N} \quad n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1-\epsilon} d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1-\epsilon} d(x_0, x_1) \end{aligned}$$

So

$$\lim_{N \to +\infty} \sup_{(n,m) \in \mathbb{N}^2_{\geq N}} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges to some $p \in X$

$$d(T(p), p) = \lim_{n \to +\infty} d(T(x_n), x_n) = 0$$

since $d : X^2 \to \mathbb{R}_{\geq 0}$ is continuous.

# Part VI

# Higher differentials

# Chapter 31

# Multilinear mapping

Let K be a commutative cenitary ring.

## 31.1 Def

Let $n \in \mathbb{N}$, $V_1, ..., V_n, W$ be K-modules. We call n-linear mapping from $V_1 \times ... \times V_n$ to W any mapping $f : V_1 \times ... \times V_n \to W$ such that $\forall i \in \{1, ..., n\}$ $\forall (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in V_1 \times ... \times V_{i-1} \times V_{i+1} \times ... \times V_n$ the mapping

$$\begin{aligned} f(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) :& V_i \to W \\ & x_i \mapsto f(x_i) \end{aligned}$$

is a morphism of K-modules

We denote by $Hom^{(n)}(V_1 \times ... \times V_n, W)$ the set of all n-linear mappings from $V_1 \times ... \times V_n$ to W.

## 31.2 Example

$$\begin{aligned} K \times K &\to K \\ (a, b) &\mapsto ab \end{aligned}$$

is a 2-linear mapping (bilinear mapping)

## 31.3 Remark

$$Hom^{(0)}(\{0\}, W) := W(\text{by convention})$$

$$Hom^{(1)}(V_1, W) = Home(V_1, W) = \{\text{morphism of K-module from } V_1 \text{ to } W\}$$

## 31.4 Prop

Suppose that $n \geq 2$ For any $i \in \{1, ..., n-1\}$

$$Hom^{(n)}(V_1 \times ... \times V_n, W) \xrightarrow{\Phi} Hom^{(i)}(V_1 \times ... \times V_i, Hom^{(n-i)}(V_{i+1} \times ... \times V_n))$$
$$f \mapsto ((x_1, ..., x_i) \mapsto ((x_{i+1}, ..., x_n) \mapsto f(x_1, ..., x_n)))$$

is a bijection

### Proof

The inverse of $\Phi$ is given by

$$g \in Hom^{(i)}(V_1 \times ... \times V_i, Home^{(n-i)}(V_{i+1} \times ... \times V_n), W) \mapsto (((x_1, ..., x_n) \in V_1 \times ... \times V_n) \mapsto g(x_1, ..., x_i)(x_{i+1},$$

## 31.5 Remark

$Hom^{(n)}(V_1 \times ... \times V_n, W)$ is a sub-K-module of $W^{V_1 \times ... \times V_n}$ and $\Phi$ is an isomorphism of K-modules.

# Chapter 32

# Operator norm of Multilinear field

Let $(K, |\cdot|)$ be a complete valued field

## 32.1 Def

Let $V_1 \times ... \times V_n$ and $W$ be normed vector spaces over K. We define

$$\|\cdot\| : Hom^{(n)}(V_1 \times ... \times V_n, W) \to [0, +\infty]$$

as

$$\|f\| := \sup_{(x_1,...,x_n)\in V_1\times...\times V_n, x_1...x_n \neq 0} \frac{\|f(x_1, ..., x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

If $\|f\| < \infty$ we say that $f$ is bounded. We denote by $\mathscr{L}^{(n)}(V_1 \times ... \times V_n, W)$ the set of bounded n-linear mappings from $V_1 \times ... \times V_n$ to $W$.

## 32.2 Theorem

For any $i \in \{1, ..., n-1\}$, $\forall f \in \mathscr{L}^{(n)}(V_1 \times ... \times V_n, W)$ $\forall (x_1, ..., x_i) \in V_1 \times ... \times V_i$ the $(n-i)$-linear mapping

$$\begin{aligned} f(x_1, ..., x_i, \cdot) : &V_{i+1} \times ... \times V_n \to W \\ &(x_{i+1}, ..., x_n) \mapsto f(x_1, ..., x_n) \end{aligned}$$

belongs to $\mathscr{L}^{(n-i)}(V_{i+1} \times ... \times V_n, W)$. Moreover

$$\|f\| = \sup_{(x_1,...,x_n)\in V_1\times...\times V_n, x_1...x_n \neq 0} \frac{\|f(x_1, ..., x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

## Proof

$\forall (x_{i+1}, ..., x_n) \in V_{i+1} \times ... \times V_n$

$$\begin{aligned}\|f(x_1, ..., x_n)\| &\leq \|f\| \, \|x_1\| ... \|x_n\| \\ &= (\|f\| \, \|x_1\| ... \|x_i\|) \, \|x_{i+1}\| ... \|x_n\|\end{aligned}$$

So

$$\|f(x_1, ..., x_i, \cdot)\| \leq \|f\| \, \|x_1\| \, , ..., \|x_i\|$$

If we define

$$\|f\|' := \sup_{(x_1,...,x_i) \in V_1 \times ... \times V_i, x_1 ... x_i \neq 0} \frac{\|f(x_1, ..., x_i, \cdot)\|}{\|x_1\| \cdots \|x_i\|}$$

then

$$\|f\|' \leq \|f\|$$

## 32.3 Corollary

(1) $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$ is a vector subspace of $Hom^{(n)}(V_1 \times \cdots \times V_n, W)$

(2) $\|\cdot\|$ is a norm on $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$

(3) $\forall i \in \{1, ..., n\}$

$$\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W) \xrightarrow{\Phi} \mathscr{L}^{(n)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))$$

is a K-linear isomorphism that preserves operator norms.

$$\|f\| = \|\Phi(f)\|$$

### 32.3.1 Proof

Conversely $\forall (x_1, \cdot, x_n) \in V_1 \times \cdots \times V_n$ such that $x_1 \cdots x_n \neq 0$

$$\|f(x_1, ..., x_n)\| \leq \|f(x_1, ..., x_i, \cdot)\| \, \|x_{i+1}\| \cdots \|x_n\|$$

Hence

$$\frac{f(x_1, ..., x_n)}{\|x_1\| \cdots \|x_n\|} \leq \frac{\|f(x_1, ..., x_i, \cdot)\|}{\|x_1\| \cdots \|x_i\|} \leq \|f\|'$$

Taking sup, we get

$$\|f\| \leq \|f\|'$$

We reason by induction on $n$

$n = 1$

$$\mathscr{L}^{(1)}(V_1, W) = \mathscr{L}(V_1, W)$$

$i \in \{1, ..., n-1\}$ Suppose that the corollary is true for $m$-linear mappings with $m < n$ We consider the following diagram of mapping

To show that $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$ is a vector subspace, it suffices to check that $\forall g \in \mathscr{L}^{(i)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))$ one has $\left\|\Phi^{-1}(g)\right\| = \|g\| < +\infty$

$$\begin{aligned}\mathscr{L}^{(i)}(V_{i+1} \times \cdots \times V_n, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W)) &\subseteq Hom^{(i)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W)) \\ &\subseteq Hom^{(i)}(V_1 \times \cdots \times V_i, Hom^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))\end{aligned}$$

For any $(x_1, ..., x_n) \in V_1 \times \vdots \times V_n$

$$\begin{aligned}\left\|\Phi^{-1}(g)(x_1, ..., x_n)\right\| &= \|g(x_1, ..., x_i)(x_{i+1}, ...x_n)\| \\ &\leq \|g(x_1, ..., x_i)\| \, \|x_{i+1}\| \cdots \|x_n\| \\ &\leq \|g\| \, \|x_1\| \cdots \|x_i\| \, \|x_{i+1}\| \cdots \|x_n\|\end{aligned}$$

Therefore

$$\left\|\Phi^{-1}(g)\right\| \leq \|g\| = \left\|\Phi^{-1}(g)\right\|$$

# Chapter 33

# Higher differentials

We fix a complete non-trivial valued field $(K, |\cdot|)$ and normed K-vector space E and F.

## 33.1 Def

Let $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping

(1) If $f$ is continuous, we say that $f$ is of class $C^0$ and $f$ is 0-times differentiable

(2) If $f$ is differentiable on an open neighborhood $V \subseteq U$ of some point $p \in U$ and

$$\begin{aligned} df :& V \to \mathscr{L}(E, F) \\ & x \mapsto d_x f \end{aligned}$$

is n-times differentiable at $p$, then we say that $f$ is $(n+1)$-times differentiable at $p$. If $f$ is $(n+1)$-times differentiable at any point $p \in U$, we denote by

$$D^{n+1} f : U \to \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

the mapping that sends $x \in U$ to the image of $D^n(df)(x)$ by the K-linear bijection

$$\mathscr{L}^{(n)}(E^n, \mathscr{L}(E, F)) \to \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

$$df : U \to \mathscr{L}(E, F)$$

$$D^n(df) : U \to \mathscr{L}^{(n)}(E^n, \mathscr{L}(E, F)) \xrightarrow{\Phi} \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

If $D^{n+1} f$ is continuous, we say that $f$ is of class $C^{n+1}(n \geq 0)$ (Any mapping $f : U \to F$ is considered as o-times differential $D^0 f := f$)

## 33.2 Remark

If $f$ is n-times differentiable $\forall i \in \{1, ..., n-1\}$
$\forall p \in U, (h_1, ..., h_n) \in E^n$ one has

$$D^i(D^{n-i}f)(p)(h_1, ..., h_i)(h_{i+1}, ..., h_n) = D^n f(p)(h_1, ..., h_n)$$

$$D^{n-i}f : U \to \mathscr{L}^{(n-i)}(E^{n-i}, F)$$

$$D^i(D^{n-i}f) : \quad \begin{tikzcd} U \arrow[r] \arrow[dr, "D^n f"'] & \mathscr{L}^{(i)}(E^i, \mathscr{L}^(n-i)(E^{n-i}, F)) \arrow[d, leftrightarrow, "\cong"] \\ & \mathscr{L}^{(n)}(E^n, F) \end{tikzcd} \; U \to$$

## 33.3 Theorem

Assume that $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$
Let $f : U \to F$ be a mapping that is $(n+1)$-times differentiable on $U$. Let $p \in U$ and $h \in E$ such that $p + th \in U$ $\forall t \in [0, 1]$ Then

$$\left\| f(p+h) - f(p) - \sum_{k=1}^{n} \frac{1}{k!} D^k f(p)(h, ..., h) \right\| \leq$$
$$(\sup_{t \in ]0,1[} \frac{(1-y)^n}{n!} \left\| D^{n+1} f(p+th) \right\|) \cdot \|h\|^{n+1}$$

(Taylor-Lagrange formula)

## 33.4 Prop(Gronwall inequality)

Let F be a normed vector space over $\mathbb{R}$ $(a, b) \in \mathbb{R}^2, a < b$ Let $f : [a, b] \to F$ and $g : [a, b] \to \mathbb{R}$ be continuous mappings that are differentiable on $]a, b[$

Suppose that $\forall t \in ]a, b[$
$$\|f'(t)\| \leq g'(t)$$
then
$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

### Proof

Let $c \in ]a, b[$ Let $\epsilon > 0$ Let

$$J = \{t \in [c, b] \mid \forall s \in [c, t]\ , \|f(s) - f(c)\| \leq g(s) - g(c)\}$$

By definition $J$ is an interval.

Since $f, g$ are continuous, $J$ is a closed interval, hence $J$ is of the form $[c, t]$. If $t < b$ then for $h > 0$ Sufficiently small.

$$f(t+h) - f(t) = hf'(t) + o(h)$$

$$g(t+h) - g(t) = hg'(t) + o(h)$$

$\exists \delta > 0\ \forall h \in [0, \delta]$

$$\|f(t+h)\| \leq \|f'(t)\| \cdot h + \frac{\epsilon}{2} h$$

$$g(t+h) - g(t) \geq g'(t)h - \frac{\epsilon}{2} h$$

So

$$\|f(t+h) - f(t)\| \leq g(t+h) - g(t) + \epsilon h$$

Moreover

$$\|f(t) - f(c)\| \leq g(t) - g(c) + \epsilon(t-c)$$

$\Rightarrow$

$$\|f(t+h) - f(c)\| \leq g(t+h) - g(c) + \epsilon(t+h-c)$$

$\Rightarrow$

$$J \supseteq [c, t+\delta]$$

Contradiction, hence

$$\|f(b) - f(c)\| \leq g(b) - g(c) + \epsilon(b-c)$$

For the same reason

$$\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c-a)$$

Hence

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b-a)$$

Since $\epsilon > 0$ is arbitrary

$$\|f(b) - f(c)\| \leq g(b) - g(c)$$

Mean value theorem:

$$g(t) = (\sup(\|f'(\cdots)\|))$$

## 33.5 Theorem

Let $n \in \mathbb{N}$, $E, F$ be normed vector spaces over $\mathbb{R}$ $U \subseteq E$ open and $f : U \to F$ be a mapping that is $(n+1)$-times differentiable. Let $p \in U$ and $h \in E$ . Assume that $\forall \epsilon \in [0,1], p + th \in U$

Let

$$M = \sup_{t \in ]0,1[} \left\| D^{n+1} f(p+th) \right\|$$

Then

$$\left\| f(p+h) - \sum_{k=0}^{n} \frac{1}{k!} D^k f(p)(h, \cdots, h) \right\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

If $E = \mathbb{R}$ Then the formula become

$$\left\| f(p+h) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(p) h^k \right\| \leq \frac{M}{(n+1)!} |h|^{h+1}$$

## Proof

Consider $\phi : [0,1] \to F$

$$\phi(t) = \sum_{k=0}^{n} \frac{(1-t)^k}{k!} D^k f(p+th)(h, \cdots, h)$$

$$\phi(1) = f(p+h)$$

$$\phi(0) = \sum_{k=0}^{n} \frac{1}{k!} D^k f(p)(h, \cdots, h)$$

$$\phi'(t) = \sum_{k=0}^{n} \frac{(1-t)^k}{k!} D^{k+1} f(p+th)(\underbrace{h, \cdots, h}_{\text{k+1 copies}}) - \sum_{k=1}^{n} \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p+th)(h, \cdots, h)$$

$$= \frac{(1-t)^n}{k!} D^{k+1} f(p+th)(h, \cdots, h)$$

then

$$\|\phi'(t)\| \leq M \frac{(1-t)^n}{n!} = (-M \frac{(1-t)^{n+1} \|h\|^{n+1}}{(n+1)!})'$$

By Gronwall inequality,

$$\|\phi(1) - \phi(0)\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

# 33.6 Def

Let $n \in \mathbb{N}$ $E_1, \cdots, E_n$ and $F$ be normed vector spaces over a complete non-trivial valued field $(K, |\cdot|)$ Let $U \in E_1 \times \cdots \times E_n$ be an open subset. $p = (p_1, ..., p_n) \in U$ $i \in \{1, \cdots, n\}$, $f : U \to F$ If there exists an open neighborhood $U_i$ of $p_i$ in $E_i$ such that

$$\begin{aligned} &U_i \to F \\ &x_i \mapsto f(p_1, \cdots, p_{i-1}, x_i, p_{i+1}, \cdots, p_n) \end{aligned}$$

is well defined and is differentiable at $p_i$

We denote by $\frac{\partial f}{\partial x_i}(p)$ the differential of this mapping $U_i \to F$ and say that $f$ admits the $i^{th}$ partial differentials at $p$

## 33.7 Prop

Suppose that $(K, |\cdot|)$ and $f$ has all partial differentials on $U$ and

$$\frac{\partial f}{\partial x_i} : U \to \mathscr{L}(E_i, F)$$

is continuous for any $i \in \{1, \cdots, n\}$ Then $f$ is of class $C^1$ and $\forall h = (h_1, \cdots, h_n) \in E_1 \times \cdots \times E_n$

$$\forall p \in U \quad d_p(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(h_i)$$

### Proof

By induction, it suffices to treat thr case where $n = 2$
$\forall \epsilon > 0 \; \exists \delta > 0$

$$\forall (h, k) \in E_1 \times E_2 \quad \max\{|h|, |k|\} \leq \delta$$

one has

$$\left\| \frac{\partial f}{\partial x_i}(a + h, b + k) - \frac{\partial f}{\partial x_2}(a, b) \right\| \leq \epsilon (\text{by continuity of } \frac{\partial f}{\partial x_2})$$

Consider the mapping $\phi : [0, 1] \to F$

$$\phi(t) = f(a + h, b + tk) - f(a + b, b) - t \underbrace{\frac{\partial f}{\partial x_2}(\overbrace{a + h, b}^{\in U})}_{\in \mathscr{L}(E_2, F)}(k)$$

$$\begin{aligned} \|\phi'(t)\| &= \left\| \frac{\partial f}{\partial x_2}(a + h, b + tk)(k) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \\ &\leq 2\epsilon \|k\| \\ \|\phi(1) - \phi(0)\| &\leq 2\epsilon \|k\| \end{aligned}$$

then

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \leq 2\epsilon \|k\|$$

So

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

$f$ has $1^{st}$ partial differential

$$\left\|f(a+h,b)-f(a,b)-\frac{\partial f}{\partial x_1}(a,b)(k)\right\|=o(\max\{\|h\|,\|k\|\})$$

by continuity of $\frac{\partial f}{\partial x_i}$

$$\left\|\frac{\partial f}{\partial x_2}(a+h,b)(k)-\frac{\partial f}{\partial x_2}(a,b)(k)\right\|=o(\max\{\|h\|,\|k\|\})$$

take the sum of above three statements, we get:

$$\left\|f(a+h,b+k)-f(a,b)-\frac{\partial f}{\partial x_1}(a,b)(h)-\frac{\partial f}{\partial x_2}(a,b)(k)\right\|=o(\max\{\|h\|,\|k\|\})$$

## 33.8 Theorem

Let $E, F$ be normed vector spaces over $\mathbb{R}$ $U \subseteq E$ open $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable mapping form $U$ to $F$ Let $g : U \to \mathscr{L}(E,F)$ Suppose that

(1) $(df_n)_{n\in\mathbb{N}}$ converges uniformly to $g$

(2) $(f_n)_{n\in\mathbb{N}}$ converges pointwisely to some mapping $f : U \to F$

Then $f$ is differentiable and $df = g$

### Proof

Let $p \in U, \forall (m,n) \in \mathbb{N}^2, \forall x \in \mathcal{B}(p,r) \in U (r>0)$

$\|f_n(x) - f(_m(x) - (f_n(p) - f_m(p)))\| \le (\sup_{\xi\in U} \|d_\xi f_m - d_\xi f_n\|) \cdot \|x-p\|$ (mean value inequality)

Take $\lim_{m\to+\infty}$ we get:

$$\|(f_n(x)-f(x))-(f_n(p)-f(p))\| \le \epsilon_n \|x-p\|$$

where $\epsilon_n = \sup_{\xi\in U} \|d_\xi f_m - g\|$.

So

$$\begin{aligned}\|f(x)-f(p)-g(p)(x-p)\| &\le \|(f(x)-f_n(x))-(f(p)-f_n(p))\| \\ &+ \|f_n(x)-f_n(p)-d_pf_n(x-p)\| \\ &+ \|d_pf_n(x-p)-g(p)(x-p)\| \\ &\le \epsilon_n\|x-p\| + \|f_n(x)-f_n(p)-d_pf_n(x-p)\| + \epsilon_n\|x-p\|\end{aligned}$$

$\limsup_{x\to p} \frac{\|f(x)-f(p)-g(p)(x-p)\|}{\|x-p\|} \le 2\epsilon_n$

Take $\lim_{n\to+\infty}$ we get:

$$\limsup_{x\to p} \frac{\|f(x)-f(p)-g(p)(x-p)\|}{\|x-p\|} = 0$$

# Chapter 34

# Permutations

## 34.1 Def

Let X be a set. We denote with $\mathfrak{S}_X$ the set of all bijections from X to itself. The elements of $\mathfrak{S}_X$ are called permutations if the set X is finite. If $x_1, \cdots, x_n \in X$ are distinct elements then

$$(x_1, \cdots, x_n) \in \mathfrak{S}_X$$

such that

$$x_i \mapsto x_{i+1}$$
$$x_n \to x_1$$

this is called an n-cycle. A 2-cycle is called a transposition.

### 34.1.1 Example

$X = \{1, \cdots, 7\}$

$$\begin{aligned} & 1 \mapsto 4 \\ & 2 \mapsto 1 \\ & 3 \mapsto 2 \\ (2\ 3)(4\ 2\ 1) = & 4 \mapsto 3 \\ & 5 \mapsto 5 \\ & 6 \mapsto 6 \\ & 7 \mapsto 7 \\ = & (1\ 4\ 3\ 2) \end{aligned}$$

## 34.2 Def

We denote with

$$orb_\sigma(x) = \{\underbrace{\sigma \circ \cdots \circ}_{\text{n-times}} \quad n \in \mathbb{N}\}$$

$x \in X, \sigma \in \mathfrak{S}_X$

## 34.3 Prop

If $orb_\sigma(x)$ is a finite set of d elements, then one has

$$\sigma^d(x) = x \quad orb_\sigma(x) = \{x, \sigma(x), \cdots, \sigma^{d-1}(x)\}$$

moreover

$$\sigma^{-1}(x) \in orb_\sigma(x)$$

### 34.3.1 Proof

The set

$$\{(n,m) \in \mathbb{N}^2, n < m, \sigma^n(X) = \sigma^m(x)\}$$

is not empty. Let

$$d' := \min\{m - n \quad (n,m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

therefore $x, \sigma(x), \cdots, \sigma^{d'-1}(x)$ are all distinct.

Now use the each deass division

$$h = qd' + r \quad r < d'$$

$$\sigma^h(x) = \sigma^r(x) \quad 0 \leq r < d'$$

then

$$d' \geq d$$

and for

$$\{x, \sigma(x), \cdots, \sigma^{d'-1}(x)\} \subseteq orb_\sigma(x)$$

$\Rightarrow$

$$d' \leq d$$

then

$$d' = d$$

## 34.4 Remark

Let $Y \subseteq X$, then we have a homomorphism of groups:

$$\begin{aligned} &\mathfrak{S}_Y \to \mathfrak{S}_X \\ &\sigma \mapsto \left( x \to \begin{cases} \sigma(x) & \text{if } x \in Y \\ x \text{ if } x \in X \setminus Y \end{cases} \right) \end{aligned}$$

If Y and Z are subset of X

$$Y \cap Z = \varnothing, \sigma \in \mathfrak{S}_Y, \tau \in \mathfrak{S}_Z$$

then

$$\sigma \circ \tau = \tau \circ \sigma$$

If X is finite with n elements $\mathfrak{S}_X = S_n$ permutation group of n elements.

# 34.5 Theorem

Let X be a finite set and let $\sigma \in \mathfrak{S}_X$ then exist $d \in \mathbb{N}$ and $(n_1, \cdots, n_d) \in \mathbb{N}^d_{\geq 2}$ and pairwise disjoint subsets $X_1, \cdots X_d$ of X of cardinalities $n_1, \cdots, n_d$, together with $n_i$-cycle $\tau_i$ of $X_i$ such that

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

In other words. Any permutation can be decomposed in composition of finitely many cycles on disjoint subsets.

## Proof

By induction on the cardinality of X.

The case $\sigma = id_X$ is trivial.$(d = 0)$ So the case when $N = 0, 1$ is clear.

Assume $N \geq 2$.Take $x \in X$ such that $\sigma(x) \neq x$ and let $X_1 = orb_\sigma(x)$ $Y = X \setminus X_1$ $\forall y \in Y$ we have $\sigma(y) \in Y$(because if $\sigma(y) \in X$ by the previous proposition $\sigma(y) \in X_1$ )

Let $\tau = \sigma \mid_Y \in \mathfrak{S}_Y$ Use the induction hypothesis, we get $X_2, \cdots, X_d$ of cardinalities $n_2, \cdots, n_d$ and $n_i$-cycle $\tau_1$ such that

$$\tau = \tau_2 \circ \cdots \circ \tau_d$$

Consider $\tau_1 = \sigma \mid_{X_1}$ then $\tau_1$ is a $n_1$-cycle of $X_1$
$\Rightarrow$

$$\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_d$$

### 34.5.1 Remark

This theorem say that the groups of permutation si generated by cycles.

# 34.6 Corollary

Let X be a finite set. Then $\mathfrak{S}_X$ is generated by transpositions.

### Proof

Note that

$$(x_1, \cdots, x_n) = (x_1, x_2) \circ (x_2, \cdots, x_n)$$

By induction

$$(x_1, \cdots, x_n) = (x_1, x_2) \circ \cdots \circ (x_{n-1}, x_n)$$

### 34.6.1 Remark

The decomposition of transposition in unique.

## 34.7 Def

Let $\tau \in \mathfrak{S}_n := G_{\{1,\cdots,n\}}$ is called adjacent if $\tau$ is of the form $(j, j+1)$ for $j = 1, \cdots, n-1$

## 34.8 Corollary

$\mathfrak{S}_n$ si generated by adjacent transposition.

### 34.8.1 Proof

Note that

$$(i, j) = (i, i+1) \circ (i+1, i+2) \circ \cdots \circ (j-1, j) \circ (j-2, j-1) \circ \cdots \circ (i+2, i+1)$$

**Some other information on $\mathfrak{S}_n$**

## 34.9 Caybey Theorem

Any finite group can be embedded (injective morphism) in a $\mathfrak{S}_n$ for some $n \in \mathbb{N}$

### Proof

Let G be a finite group and $n = card(G)$. Let

$$\begin{aligned} \varphi : G &\to \mathfrak{S} \\ g &\mapsto l_g \end{aligned}$$

be the mapping sends $g \in G$ to $l_g(x) = gx, \forall x \in G$

## 34.10 Theorem

Let X be a finite set. Assume that $\sigma \in \mathfrak{S}_X$ can be written as

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

where $\tau_1$ is transposition.
We put

$$sgn(\sigma) := (-1)^{\sigma}$$

This is a well-define function. Moreover $sgn$ is a morphism from $\mathfrak{S}_X$ to $(\{-1, 1\}, \times)$

### Proof

Let's define the mapping:

$$\begin{aligned} \phi : \mathfrak{S}_n &\to \mathbb{Q}^{\times} \\ \sigma &\mapsto \prod_{(i,j)\in\{1,\cdots,n\}^2, i<j} \frac{\sigma(i)-\sigma(j)}{i-j} \end{aligned}$$

To show that $\phi$ is a morphism of groups. Let

$$\theta = \{U \in \wp(\{1, \cdots, n\}) \mid \ card(U) = 2\}$$

$$\begin{aligned} \phi(\sigma\circ\tau) &= \prod_{(i,j)\in\theta} \frac{\sigma(\tau(i))-\sigma(\tau(j))}{i-j} \\ &= (\frac{\sigma(\tau(i))-\sigma(\tau(j))}{\tau(i)-\tau(j)}) \times (\prod_{(i,j)\in\theta} \frac{\tau(i)-\tau(j)}{i-j}) \\ &= \phi(\sigma)\phi(\tau) \end{aligned}$$

When $\tau$ is a transition, $\phi(\tau) = -1$. Therefore

$$\phi(\sigma) = \prod_{i=1}^{d} \phi(\tau_1)$$

since

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

## 34.11 Remark

Let $A_n \subsetneqq \mathfrak{S}_n$ such that

$$A_n = \{\sigma \in \mathfrak{S}_n \mid sgn(\sigma) = 1\}$$

is an alternating symmetric group.

## 34.12 Exercise

Let X be a set of cardinality $n$. Let $\sigma : X \to \{1, \cdots, n\}$ be a bijection. Prove that

$$\begin{aligned} \phi : \mathfrak{S}_X &\to \mathfrak{S}_n \\ \tau &\mapsto \sigma^{-1} \circ \tau \circ \sigma \end{aligned}$$

is an isomorphism.

## 34.13 Symmetric of multilinear mapping

We fix a commutative unitary ring $K$ and $K$-modules $E, F$

## 34.14 Def: Symmetric and Alternating

symmetric Let $n \in \mathbb{N}$ and $f \in Hom^{(n)}(E^n, F)$. If for any $\sigma \in \mathfrak{S}_n$ one has $\forall x \in E^n$

$$f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Then we say $f$ is symmetric

alternating If for any $(i, j) \in \{1, \cdots, n\}^2$ such that $i \neq j$ and any $(x_1, \cdots, x_n) \in E^n$ such that $x_i = x_j$

$$f(x_1, \cdots, x_n) = 0$$

then we say that $f$ is alternating.

## 34.15 Prop

Suppose that $f \in Hom^{(n)}(E^n, F)$ is alternating, then $\forall (x_1, \cdots, x_n) \in E^n$, $\sigma \in \mathfrak{S}_n$

$$f(x_1, \cdots, x_n) = sgn(\sigma) f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

### Proof

By corollary 34.8, it's enough to prove the proposition for adjacent transitions. Let $i \in \{1, , \cdots, n-1\}$ then

$$\begin{aligned} 0 &= f(x_1, \cdots, x_{i-1, x_i + x_{i+1}, x_{i+2}, \cdots, x_n}) \\ &= f(x_1, \cdots, x_{i-1}, x_i, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_{i+1}, x_{i+1}, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots, x_n) \end{aligned}$$

The adjacent transition $\sigma$ is $(i, i+1)$

## 34.16 Def:

$Hom_s$ and $Hom_a$

We denote with $Hom_s^{(n)}(E^n,F)$ and $Hom_a^{(n)}(E^n,F)$ the set of symmetric and alternating $n$-linear mappings from $E$ to $F$. $Hom_s^{(n)}(E^n,F)$ and $Hom_a^{(n)}(E^n,F)$ are sub-K-modules of $Hom^{(n)}(E^n,F)$ and when $n=1$, by convention

$$Hom_s^{(1)}(E,F) = Hom_a^{(1)}(E,F) = Hom(E,F)$$

## 34.17 Reminder

Let $E,F$ be two normed vector spaces over $\mathbb{R}$ $f : E \to F$ is differentiable(twice)

$$\begin{aligned} df : E &\to \mathscr{L}(E,F) \\ D^2 d : E &\to \mathscr{L}(E,\mathscr{L}(E,F)) \\ A &\mapsto ((x,y) \to A(x)(y)) \end{aligned}$$

## 34.18 Theorem(Schweiz)

$U \subseteq E$ is an open set, $f : U \to F$ is a function of class $C^n$. Then for any $p \in U$

$$D^n f(p) \in \mathscr{L}^n(E^n, F)$$

is symmetric

### Proof

By induction and by the fact that permutation are decomposed in transpositions, we can reduce to prove only the case $n=2$

$$d_{p+u}f - d_p f = D^2 f(p)(u,\cdot) + o(u)$$

$\forall \epsilon > 0, \exists \delta > 0$ such that $0 < \|u\| < \delta$,then

$$\left\| d_{p+u}f - d_p f - D^2 f(p)(u,\cdot) + o(u) \right\| \leq \epsilon \left\| u \right\|$$

For any $x \in \mathcal{B}(p, \frac{\epsilon}{2})$ let's introduce the following function

$$\varphi(x) = f(x+k) - f(x) - D^2 f(p)(k,x)$$

We use the mean value inequality on $\varphi$

$$\begin{aligned} &\|\varphi(p+h) - \varphi(p)\| \\ =& \left\| f(p+h+k) + f(p) - f(p+h) - D^2 f(p)(k,p+h) - f(p+k) - f(p) - D^2 f(p)(k,p) \right\| \\ =& \left\| f(p+h+k) + f(p) - f(p+h) - f(p+k) - D^2 f(p)(k,h) \right\| \\ \leq& (\sup_{t\in[0,1]} \|d_{p+th}\varphi\|) \left\| h \right\| \end{aligned}$$

$$\|d_{p+th}(\varphi)\| = \left\|d_{p+th+k}f - d_{p+th}f - D^2 f(p)(k,\cdot)\right\|$$

add and subtract $d_p f, D^2 f(p)(th,\cdot)$ then by triangle inequality

$$\begin{aligned}
&\left\|d_{p+th+k}f - d_{p+th}f - D^2 f(p)(k,\cdot)\right\| \\
\leq &\left\|d_{p+th+k}(f) - d_p f - D^2 f(p)(k+th,\cdot)\right\| \\
+ &\left\|d_{p+th}f - d_p f - D^2 f(p)(th,\cdot)\right\| \\
\leq &\epsilon \left\|th+k\right\| + \epsilon(th) \\
\leq &2\epsilon(\|h\| + \|k\|)
\end{aligned}$$

then

$$\begin{aligned}
&\left\|f(p+h+k) + f(p) - f(p+k) - f(p+h) - D^2 f(p)(k,h)\right\| \\
=&o(\max\{\|h\|,\|k\|\}^2)
\end{aligned}$$

exchange the role of $h, k$ then we get

$$\begin{aligned}
&\left\|f(p+h+k) + f(p) - f(p+k) - f(p+h) - f(p+k) - D^2 f(p)(h,k)\right\| \\
\leq &o(\max\{\|h\|,\|k\|\}^2)
\end{aligned}$$

then

$$\underbrace{\left\|D^2 f(p)(k,h) - D^2 f(p)(h,k)\right\|}_{\text{bilinear function}} = o(\underbrace{\max\{\|h\|,\|k\|\}^2}_{\text{quachetic}})$$

this implies that the LHS is 0

## 34.19 Def

Let E¡F be normed vector spaces over a complete value field $(K, |\cdot|)$ let $U \subseteq E, V \subseteq F$ be open subsets and $f : U \to V$ is a bijection.

(1) If $f$ and $f^{-1}$ are both continuous we say that $f$ is a homeomorphism

(2) If a $f$ and $f^{-1}$ are both of class $C^n$ we say that $f$ is a $e^n$-diffeomorphism

If (2) is true for any $n \in \mathbb{N}$ we say that $f$ is a $C^\infty$-diffeomorphism

## 34.20 Prop

Let E,F be two normed Banach spaces. Let $I(E,F) \in \mathscr{L}(E,F)$ be the set of linear continuous and invertible mappings such that $norm\varphi^{-1} \leq +\infty$. Then $I(E,F)$ is open in $\mathscr{L}(E,F)^\vee$ Moreover the mapping

$$\begin{aligned}
I(E,F) &\to I(F,E) \\
\phi &\mapsto \varphi^{-1}
\end{aligned}$$

is a $e^1$-diffeomorphism

## Proof

Let $\varphi \in I(E,F)$ we want to show that

$$\varphi - \psi \in I(E,F)$$

for $\psi \in \mathscr{E},\mathscr{F}$ such that $\|\psi\| < \frac{1}{\|\varphi^{-1}\|}$ Notice that

$$\varphi - \psi = \varphi \circ (Id_E - \varphi^{-1} \circ \psi)$$

Since

$$\left\|\varphi^{-1}\psi\right\| \leq \left\|\varphi^{-1}\right\| \|\psi\| < 1$$

This means that the series

$$\sum_{n\in\mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n}$$

is absolutely convergent in $\mathscr{L}(E,E)$ This series is the inverse of $(Id_E - \varphi^{-1}\psi)$

$$(Id_e - \varphi^{-1}\psi) \circ \sum_{n=0}^{N-1} (\varphi^{-1} \circ \psi)\overbrace{\circ n}^{\text{composite n times}} = Id_E - (\varphi^{-1} \circ \psi)^{\circ N}$$

take $\lim\limits_{N\to+\infty}$, then

$$(\varphi - \psi)^{-1} = \sum_{n\in\mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n} \circ \varphi^{-1}$$

and

$$(\varphi - \psi)^{-1} = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

replace the inverse with $i$

$$i(\varphi - \psi) - i(\varphi) = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

then

$$d_\varphi i(\psi) = i(\varphi) \circ (-\psi) \circ i(\varphi)$$

so $i$ is differentiable.Moreover $i$ and $i^{-1}$ are continuous.

## Remark

By induction we can show that $i$ is a $C^{+\infty}$-diffeomorphism

# 34.21 Prop

Let $n \in \mathbb{N} \cup \{\infty\}$ Let E F G be normed vector spaces over a complete valued field $(K, |\cdot|)$ $U \subseteq E, V \subseteq F$ be open sets. $f : U \to V$ $g : V \to G$ be mappings of class $C^n$, then $g \circ f$ also of class $C^n$

### 34.21.1 Proof

The case where $n = 0$ is known
Denote by

$$\Phi : \mathscr{L}(E,F) \times E \to F$$
$$(\beta, \alpha) \mapsto \beta \circ \alpha$$

$\Phi$ is a bounded bilinear mapping

$$\|\Phi(\beta,\alpha)\| \leq \|\beta\| \cdot \|\alpha\|$$

Suppose that $n \geq 1$ and the statement is true for mappings of class $C^{n-1}$ $g \circ f$ is differentiable.

$$\forall p \in U \quad d_p(g \circ f) = d_{f(p)} g \circ d_p f$$

$$D^1(g \circ f) : U \to \mathscr{L}(E,G)$$

$$D^1 = \Phi \circ (D^1 g \circ f, D^1 f)$$

$$(D^1 g \circ f, D^1 f) : U \to \mathscr{L}(F,G) \times \mathscr{L}(E,F)$$
$$p \mapsto (d_{f(p)} g, d_p f)$$

$$d_{\beta_0,\alpha_0} \Phi(\beta,\alpha) = \beta_0 \circ \alpha + \beta_0 \circ \alpha$$

$$D^1\Phi : \mathscr{L}(F,G) \times \mathscr{L}(E,F) \to \mathscr{L}(\mathscr{L}(F,G) \times \mathscr{L}(E,F), \mathscr{L}(E,G))$$
$$(\alpha_0, \beta_0) \mapsto ((\alpha,\beta) \mapsto \beta_0 = \alpha + \beta_0 \alpha_0)$$

Since $g, f$ are of class $C^n$ $D^1 f, D^1 g$ are of class $C^{n-1}$ Thus, by induction hypothesis,

$$(D^1 g \circ f, D^1 f)$$

is of class $C^{n-1}$ Since $\Phi$ is of class $C^\infty$, we obtain that

$$D^1(g \circ f)$$

is of class $C^{n-1}$ then

$$g \circ f$$

is of class $C^n$

## 34.22 Prop

Let E and F be Banach space over a complete valued field $(K, |\cdot|)$. U and V be open subsets of E and F respectively.$n \in \mathbb{N} \cup \{\infty\}$ and $f : U \to V$ be a bijection. If $f$ is of class $C^n$, then $f^{-1}$ is differentiable, then $f^{-1}$ is of class $C^n$

## Proof

$$f \circ f^{-1} = Id_V$$

$\forall y \in V$

$$d_y(f \circ f^{-1}) = d_{f^{-1}(p)} f \circ d_y f^{-1} = Id_F$$

For $x \in U, y = f(x)$

$$d_y(f \circ f^{-1}) = d_x f \circ d_y f^{-1} = Id_F$$

$$d_x(f^{-1} \circ f) = d_y f \circ d_x f^{-1} = Id_E$$

So

$$d_y f^{-1} - (d_x f)^{-1}$$

that is

$$D^1 f^{-1} = \iota \circ (D^1 f \circ f^{-1})$$

where

$$\iota : I(E, F) \to I(F, E)$$
$$\phi \mapsto \phi^{-1}$$

Suppose that $f^{-1}$ is of class $C^{n-1}$ then

$$D^1 f^{-1} = \iota D^1 f \circ f^{-1}$$

is of class $C^{n-1}$

# 34.23 Local Inversion Theorem

Let E and F be Banach space over $\mathbb{R}$ $U \in E$ open, $f : U \to F$ be a mapping of class $C^n$ and $a \in U$. Suppose that $d_a f \in I(E, F)(d_a f$ is invertible and of bounded inverse) Then there exists open neighborhoods V and W of $a$ and $f(a)$ respectively, such that

- $V \subseteq U$ and $f(V) \subseteq W$
- The restriction of $f$ to V defines a bijection from V to W
- $$(f\mid_V)^{-1} W \to V$$

  is of class $C^n$

### 34.23.1 Proof

For $y \in F$ consider the mapping:

$$\begin{aligned} \phi_y : U \to F \\ x \mapsto x - (d_a f)^{-1}(f(x) - y) \end{aligned}$$

$f(x) = y$ iff $\phi_y(x) = x$ i.e. $x$ is a fix point of $\phi_y$ $\phi_y$ is of class $C^1$ and

$$d_x\phi_y(v) = v - d_a f^{-1}(d_x f(v))$$

$\forall v$

$$d_a \phi y^{(v)} = 0$$

By the continuity of $D^1 f$ there exists $r > 0$ such that

$$\overline{\mathcal{B}}(a, r) \subseteq U$$

and $\forall y \in F, \forall x \in \overline{\mathcal{B}}(a, r)$

$$\|d_x\phi_y\| \leq \frac{1}{2}$$

By the mean value inequality. $\forall (x_1, x_2) \in \overline{\mathcal{B}}(a, r)$

$$\left\|\phi_y(x_1) - \phi_{=y(x_2)}\right\| \leq \frac{1}{2}\|x_1 - x_2\|$$

Hence $\phi_y$ is contraction.

By the boundedness of $(d_a f)^{-1}$ $\exists \delta > 0$ such that

$$\forall y \in \overline{\mathcal{B}}(f(a), \delta) \quad \left\|(d_a f)^{-1}(f(a) - y)\right\| \leq \frac{r}{2}$$

Then $\forall x \in \overline{\mathcal{B}}(a, r)$ $y \in \overline{\mathcal{B}}(f(a), \delta)$

$$\begin{aligned} \|\phi_y(x) - a\| &\leq \|\phi_y(x) - \phi_y(a)\| + \|\phi_y(a) - a\| \\ &\leq \frac{1}{2}\|x - a\| + \frac{r}{2} \\ &\leq \frac{r}{2} + \frac{r}{2} = r \end{aligned}$$

$\phi_y(\overline{\dashv, \nabla}) \in \overline{\mathcal{B}}(a, r)$. By the fixed point theorem

$$\exists g : \overline{\mathcal{B}}(f(a), \delta) \to \overline{\mathcal{B}}(a, r)$$

sending $y$ to the fixed point of $\phi_y$ Let $W = \mathcal{B}(f(a), g)$,then

$$g\mid_W : W \to V$$

is the inverse of $f\mid_V : V \to W$ Hence $f^{-1}(W) = V$ is open.

In the following, we prove that $g$ is of class $C^n$ on an open neighborhood of $f(a)$. By reducing V and W, we may assume that $\forall x \in V$

$$d_x f \in I(E, F)$$

Let $x_0 \in V$ $y_0 = f(x_0)$ $\quad x_0 = g(y_0)$

$$y - y_0 = f(g(y)) - f(g(y_0)) = d_{x_0} f(g(y) - g(y_0)) + o(\|g(y) - g(y_0)\|)$$

So

$$g(y) - g(y_0) = (d_x f)^{-1}(y - y_0) + o(\|g(y) - g(y_0)\|)$$

Thus leads to

$$g(y) - g(y_0) = O(\|y - y_0\|)$$

($\exists \epsilon > 0 \quad (1 - \epsilon) \|g(y) - g(y_0)\| \leq \|d_{x_0} f\|^{-1}$ when $\|y - y_0\|$ is sufficiently small)
So

$$d_{y_0} g = (d_x f)^{-1}$$

By the previous proposition, $g$ is of class $C^n$

# Part VII

# Integration

# Chapter 35

# Integral operators

We fix a set $\Omega$ and a vector subspace S of $\mathbb{R}^\Omega$ over $\mathbb{R}$ We suppose that $\forall (f,g) \in S^2$

$$\begin{aligned} f \wedge g : \Omega &\to \mathbb{R} \\ \omega &\mapsto \min\{f(\omega), g(\omega)\} \end{aligned}$$

belongs to S

## 35.1 Prop

(1) $\forall (f,g) \in S^2$

$$\begin{aligned} f \vee g : \Omega &\to \mathbb{R} \\ \omega &\mapsto \max\{f(\omega), g(\omega)\} \end{aligned}$$

$f \vee g \in S$

(2) $\forall f \in S$

$$\begin{aligned} |f| : \Omega &\to \mathbb{R} \\ \omega &\mapsto |f(\omega)| \end{aligned}$$

$|f| \in S$

### Proof

(1)

$$f \vee g = f + g - f \wedge g$$

(2)

$$|f| = f \vee (-f)$$

## 35.2 Def

We call integral operator on S any $\mathbb{R}$-linear mapping $I : S \to \mathbb{R}$ that satisfies the following conditions:

(1) If $f \in S$ is such that $\forall \omega \in \Omega, f(\omega) \geq 0$ then $I(f) \geq 0$

(2) If $(f_n)_{n\in\mathbb{N}}$ is a decreasing sequence of elements in $S$ such that $\forall \omega \in \Omega \lim\limits_{n\to+\infty} f_n(\omega) = 0$ then

$$\lim_{n\to+\infty} I(f_n) = 0$$

$(\forall \omega \in \Omega, n \in \mathbb{N}, f_n(\omega) \geq f_{n+1}(\omega))$

## 35.3 Example

(1) $\Omega = \mathbb{R}$ S=vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by mappings of the form $\mathbb{1}_{]a,b]}$ $(a, b) \in \mathbb{R}^2, a < b$

$$\mathbb{1}_{]a,b]} = \begin{cases} 1, x \in ]a, b] \\ \quad 0, else \end{cases}$$

Any element of S is of the form

$$\sum_{i=1}^{n} \mathbb{1}_{]a_i,b_i]}$$

$I : S \to R$ is defined as

$$I(\sum_{i=1}^{n} \lambda \mathbb{1}_{]a_i,b_I]}) = \sum_{i=1}^{n} \lambda_i(b_i - a_i)$$

More generally if $\varphi : \mathbb{R} \to R$ is increasing and right continuous $(\forall x \in \mathbb{R}, \lim_{\epsilon>0,\epsilon\to0} \varphi(x + \epsilon) = \varphi(x))$ We define

$$I_\varphi : S \to \mathbb{R}$$

$$I(\sum_{i=1}^{n} \lambda \mathbb{1}_{]a_i,b_I]}) = \sum_{i=1}^{n} \lambda_i(\varphi(b_i) - \varphi(a_i))$$

(2) (Radon measure)

Let $\Omega$ be a quasi-compact topological space

$$S = C^0(\Omega) := \{f\Omega \to \mathbb{R} \text{ continuous}\}$$

Let $I : S \to \mathbb{R}$ $\mathbb{R}$-linear, such that $\forall f \in S, f \geq 0$ one has $I(f) \geq 0$

## 35.4 Dini's theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $C^0(\Omega)$, that converges pointwisely to some $f \in C^0(\Omega)$ Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$

### Proof

Let $g_n = f_n - f > 0$ Fix $\epsilon > 0$ $\forall n \in \mathbb{N}$ let

$$U_n = \{\omega \in \Omega \mid g_n(\omega) < \epsilon\}$$

is open

Moreover

$$\bigcup_{n\in\mathbb{N}} U_n = \Omega \quad (U_0 \subseteq U_1 \subseteq \cdots)$$

Since $\Omega$ is quasi-compact, $\exists N \in \mathbb{N}, \Omega = U_N$ Therefore $\forall n \in \mathbb{N}, n \geq N, \forall \omega \in \Omega$

$$g_n(\omega) < \epsilon$$

Consequence. If $(f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is decreasing and converges pointwisely to 0, then

$$\|f_n\|_{\sup} := \sup_{\omega\in\Omega} |f_n(\omega)|$$

converges to 0 when $n \to +\infty$ $\forall n \in \mathbb{N}$

$$f_n \leq \|f_n\|_{\sup} \cdot \mathbb{1}_\Omega$$

So

$$0 \leq I(f_n) \leq \|f_n\|_{\sup} I(\mathbb{1}_\Omega) \to 0 \quad (n \to +\infty)$$

(If $f \leq g$ then $g - f \geq 0$ so $I(g-f) = I(g) - I(f) \geq 0$ $I(g) \geq I(f)$)

## 35.5 Def

We call $\sigma$-algebra any subset $\mathscr{A}$ of $\wp(\Omega)$ that satisfies the following conditions:

- $\varnothing \in \mathscr{A}$
- If $A \in \mathscr{A}$ then $\Omega \setminus A \in \mathscr{A}$
- If $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$

Given a $\sigma$-algebra $\mathscr{A}$ on $\Omega$ , we mean by measure on $(\Omega, \mathscr{A})$ any mapping $\mu : \mathscr{A} \to [0, +\infty]$ such that :

- $\mu(\varnothing) = 0$
- If $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ such that $A_i$ are pairwisely disjoint, then

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

# Chapter 36

# Riemann integral

## 36.1 Def

Let $\Omega$ be a non-empty set and S be a vector subspace of $\mathbb{R}^\Omega$ If $\forall (f,g) \in S^2, f \wedge g \in S$, we say that S is a Riesz space.

In this section, we fix a Riesz space and an integral operator $I : S \to \mathbb{R}$

## 36.2 Def

For any $f : \Omega \to \mathbb{R}$ let

$$I^*(f) := \inf_{\mu \in S, \mu \geq f} I(\mu)$$

$$I_*(f) := \sup_{l \in S, l \leq f} I(f)$$

If $I^*(f) = I_*(f)$ then we say that $f$ is I-Riemann integral, and denote by $I(f)$ the value $I^*(f)$ (or $I_*(f)$)

## 36.3 Theorem

The set $\mathscr{R}$ of all I-Riemann integral mappings form a vector space of $\mathbb{R}^\Omega$ that contains S. Moreover,$I : \mathscr{R} \to \mathbb{R}$ is an $\mathbb{R}$-linear mapping extending $I : S \to \mathbb{R}$

### Proof

$\forall h \in S$

$$I^*(h) = I_*(h) = I(h)$$

So $h \in \mathscr{R}$

Let $(f_1, f_2) \in \mathscr{R}$ If $(\mu_1, \mu_2) \in S^2, \mu_1 \geq f_1, \mu_2 \geq f_2$ then

$$\mu_1 + \mu_2 \in S, \mu_1 + \mu_2 \geq f_1 + f_2$$

Hence

$$I(\mu_1) + I(\mu_2) \geq I^*(f_1 + f_2)$$

Take the infimum with respect to $(\mu_1, \mu_2)$ we get

$$I^*(f_1) + I^*(f_2) \geq I^*(f_1 + f_2)$$

Similarly

$$I_*(f_1) + I_*(f_2) \leq I_*(f_1 + f_2)$$

Hence

$$I^*(f_1 + f_2) = I_*(f_1 + f_2) = I(f_1) + I(f_2)$$

Let $f : \Omega \to \mathbb{R}$ be a mapping ,$\lambda \in \mathbb{R}_{>0}$

$$I^*(\lambda f) = \inf_{\mu \in S, \mu \geq \lambda f} I(\mu) = \inf_{\nu \in S, \nu \geq f} I(\lambda \nu) = \lambda I^*(f)$$

Similarly

$$I_*(\lambda f) = \lambda I_*(f)$$

Hence if $f \in \mathscr{R}$ then $\lambda f \in \mathscr{R}$ and $I(\lambda f) - \lambda I(f)$

$$I^*(-f) = \inf_{\mu \in S, \mu \geq -f} I(\mu) = \inf_{l \in S, l \leq f} I(-l) = -\sup_{l \in S, l \leq f} I(l) = -I_*(f)$$

Similarly

$$I_*(-f) = -I^*(f)$$

Hence if $f \in \mathscr{R}$ then $-f \in \mathscr{R}$ and $I(-f) = -I(f)$

# Chapter 37

# Daniell integral

We fix an integral operator $I : S \to \mathbb{R}$

## 37.1 Prop

### 37.1.1

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in S that converges pointwisely to some $f \in S$. Then

$$\lim_{n\to+\infty} I(f_n) = I(f)$$

**Proof**

Let $g_n = f - f_n \in S$ $(g_n)_{n\in\mathbb{N}}$ is decreasing and converges pointwisely to 0. Then

$$\lim_{n\to+\infty} I(g_n) = 0$$

Hence

$$\lim_{n\to+\infty} I(f_n) = I(f)$$

### 37.1.2

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in S, $f \in S$ If $f \leq \lim_{n\to+\infty} f_n$, then

$$I(f) \leq \lim_{n\to+\infty} I(f_n)$$

**Proof**

$$f = \lim_{n\to+\infty} f \wedge f_n$$

So

$$I(f) = \lim_{n\to+\infty} I(f \wedge f_n) \leq \lim_{n\to+\infty} I(f_n)$$

## 37.2 Def

Let

$$S^{\uparrow} = \left\{ f : \Omega \to \mathbb{R} \cup \{+\infty\} \mid \begin{array}{r} \exists (f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}} \text{ increasing such that} \\ f = \lim\limits_{n\to+\infty} f_n \text{ pointwisely} \end{array} \right\}$$

## 37.3 Prop

Let $f, g$ be elements of $S^{\uparrow}$ such that $f \leq g$ Let $(f_n)_{n\in\mathbb{N}}$ and $(g_m)_{n\in\mathbb{N}}$ be increasing sequences in S such that $f = \lim\limits_{n\to+\infty} f_n, g = \lim\limits_{n\to+\infty} g_n$ THen

$$\lim_{n\to+\infty} I(f_n) \leq \lim_{n\to+\infty} I(g_n)$$

### Proof

For any $m \in \mathbb{N}$

$$f_m \leq f \leq g$$

Hence

$$I(f_m) \leq \lim_{n\to+\infty} I(g_n)$$

Taking $\lim\limits_{m\to+\infty}$ we get

$$\lim_{m\to+\infty} I(f_m) \leq \lim_{n\to+\infty} I(g_n)$$

## 37.4 Corollary

Let $f \in S^{\uparrow}$ If $(f_n)_{n\in\mathbb{N}}$ and $(\widetilde{f}_n)_{n\in\mathbb{N}}$ be increasing sequence in S such that

$$f = \lim_{n\to+\infty} f_n = \lim_{n\to+\infty} \widetilde{f}_n$$

then

$$\lim_{n\to+\infty} I(f_n) = \lim_{n\to+\infty} I(\widetilde{f}_n)$$

We denote by $I(f)$ the limit $\lim\limits_{n\to+\infty} I(f_n)$

Thus we obtain a mappign $I : S^{\uparrow} \to \mathbb{R} \cup \{+\infty\}$ such that

- If $(f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is increasing then

$$I(\lim_{n\to+\infty} f_n) = \lim_{n\to+\infty} I(f_n)$$

- If $(f, g) \in {S^{\uparrow}}^2$ $f \leq g$ then $I(f) \leq I(g)$
- If $(f, g) \in {S^{\uparrow}}^2$ then $f + g \in S^{\uparrow}$ and

$$I(f + g) = I(f) + I(g)$$

- If $f \in S^{\uparrow}, \lambda \geq 0$then $\lambda f \in S^{\uparrow}$ and $I(\lambda f) = \lambda I(f)$

## 37.5 Prop

Let $(f_n)_{n\in\mathbb{N}} \in (S^{\uparrow})^{\mathbb{N}}$ be an increasing sequence and $f = \lim\limits_{n\to+\infty} f_n$. Then

$$f \in S^{\uparrow}$$

and

$$I(f) = \lim_{n\to+\infty} I(f_n)$$

### Proof

For $k \in \mathbb{N}$ let $(g_{k,m})_{m\in\mathbb{N}} \in S^{\mathbb{N}}$ be an increasing sequence such that

$$f_k = \lim_{m\to+\infty} g_{k,m}$$

For $n \in \mathbb{N}$ let $h_n = g_{0,n} \vee \cdots \vee g_{n,n} \in S$ The sequence $(h_n)_{n\in\mathbb{N}}$ is increasing. Moreover

$$f_n \geq k_n \geq g_{k,n} \quad (k \leq n)$$

Hence

$$f_n \geq h_n$$

Taking $\lim\limits_{n\to+\infty}$ we get $\forall k \in \mathbb{N}$

$$f = \lim_{n\to+\infty} f_n \geq \lim_{n\to+\infty} h_n \geq \lim_{n\to+\infty} g_{k,n} = f_k$$

Taking $\lim\limits_{k\to+\infty}$ we get

$$f = \lim_{n\to+\infty} h_n$$

Hence $f \in S^{\uparrow}$ and

$$I(f) = \lim_{n\to+\infty} I(h_n) \leq \lim_{n\to+\infty} I(f_n)$$

Conversely, $\forall n \in \mathbb{N}, f \geq f_n$ Hence

$$I(f) \geq \lim_{n\to+\infty} I(f_n)$$

## 37.6 Def

Let $S^{\downarrow} = \{-f \mid f \in S^{\uparrow}\}$ We extend I to $I : S^{\downarrow} \to \mathbb{R}U{-\infty}$ by letting $I(-f) := -I(f)$ for $f \in S^{\uparrow}$

## 37.7 Prop

Let $(f,g) \in (S^{\uparrow} \cup S^{\downarrow})^2$ If $f \leq g$ then

$$I(f) \leq I(g)$$

### Proof

It suffices to treat the cases where $(f,g) \in S^{\uparrow} \times S^{\downarrow}$ and $(f,g) \in S^{\uparrow} \times S^{\downarrow}$
If $(f,g) \in S^{\uparrow} \times S^{\downarrow}$ then $-f \in S^{\downarrow}$ and hence $g - f \in S^{\uparrow}, g - f \geq 0$ In both cases,

$$0 \leq I(g-f) = I(g) + I(-f) = I(g) - I(f)$$

## 37.8 Def

Let $f : \Omega \to \mathbb{R}$ be a mapping. We define

$$\overline{I}(f) := \inf_{\mu \in S^{\uparrow}, \mu \geq f} I(\mu) \leq \inf_{\mu \in S, \mu \geq f} I(\mu) = I^*(f)$$

$$\underline{I}(f) := \sup_{\mu \in S^{\downarrow}, \mu \leq f} I(\mu) \geq \sup_{\mu \in S, \mu \leq f} I(\mu) = I_*(f)$$

If $\overline{I}(f) = \underline{I}(f)$ then we say that $f$ is $I$-integrable(in the sense of Daniell)

## 37.9 Remark

If $f$ is I-integrable in the sense of Riemann, then it is I-integrable in sense of Daniell

## 37.10 Daniell Theorem

The set $L^1(I)$ of all I-integrable mappings forms a vector subspace of $\mathbb{R}$. Moreover

- $\forall (f,g) \in L^1(I)$ $f \wedge g \in L^1(I)$
- $I : L^1(I) \to \mathbb{R}$ is an integral operator extending $I : S \to \mathbb{R}$

### Proof

Let $(f_1, f_2) \in L^1(I)^2$ let $(l_1, l_2) \in {S^{\downarrow}}^2, l_1 \leq f_1, l_2 \leq f_1$ Let $(\mu_1, \mu_2) \in {S^{\uparrow}}^2, f_1 \leq \mu_1, f_2 \leq \mu_2$
We have

$$l_1 + l_2 \leq f_1 + f_2 \leq \mu_1 + \mu_2$$

Taking the supremum with respect to $(l_1, l_2)$, we get

$$I(f_1) + I(f_2)(= \underline{I}(f_1) + \underline{I}(f_2)) \leq \underline{I}(f_1 + f_2)$$

Taking the infimum with respect to $(l_1, l_2)$, we get

$$\overline{I}(f_1 + f_2) \leq I(f_1) + I(f_2)$$

Then

$$\overline{I}(f_1 + f_2) = \underline{I}(f_1 + f_2)$$

So $f_1 + f_2 \in L^1(I)$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$

Similarly, if $f \in L^1(I), \lambda \geq 0$ then

$$\begin{aligned}\underline{I}(\lambda f) &= \sup_{l \leq \lambda f, l \in S^{\downarrow}} I(l) \\ &= \sup_{l \leq f, l \in S^{\downarrow}} I(\lambda l) \\ &= \lambda \underline{I}(f) = \lambda I(f)\end{aligned}$$

$$\overline{I}(\lambda f) = \lambda \overline{I}(f) = \lambda I(f)$$

So $\lambda f \in L^1(I)$ and $I(\lambda f) = \lambda I(f)$

Moreover, if $f \in L^1(I), \mu \in S^{\uparrow}, l \in S^{\downarrow}, l \leq f \leq \mu$ then

$$-\mu \in S^{\downarrow}, -l \in S^{\uparrow}, -\mu \leq -f \leq -l$$

Hence

$$\overline{I}(-f) = -\underline{I}(f) = -I(f) \quad \underline{I}(-f) = -\overline{I}(f) = -I(f)$$

So $-f \in L^1(I)$ and $I(-f) = -I(f)$

We proved that $\forall (f_1, f_2) \in L^1(I)^2$

$$f_1 \wedge f_2 \in L^1(I)$$

Let $(f_1, f_2) \in L^1(I)^2$, for any $\epsilon > 0$ $\exists (l_1, l_2) \in {S^{\downarrow}}^2, (\mu_1, \mu_2) \in {S^{\uparrow}}^2$ such that

$$l_1 \leq f_1 \leq \mu_1 \quad l_2 \leq f_2 \leq \mu_2$$

such that

$$I(\mu_1 - l_1) \leq \frac{\epsilon}{2} \quad I(\mu_2, l_2) \leq \frac{\epsilon}{2}$$

One has $l_1 \wedge l_2 \leq f_1 \wedge f_2 \leq \mu_1 \wedge \mu_2$

$$\mu_1 \wedge \mu_2 - l_1 \wedge l_2 \leq (\mu_1 - l_1) + (\mu_2 - l_2)$$

$$\begin{pmatrix} \text{If} \mu_1(\omega) \leq \mu_2(\omega), l_1 \leq l_1(\omega) \\ \quad LHS = \mu_1(\omega) - l_1(\omega) \\ \quad RHS = \mu_1(\omega) - l_2(\omega) + \mu_2(\omega) - l_1(\omega) \geq \mu_1(\omega) - l_2(\omega) \end{pmatrix}$$

## 37.11 Beppo Levi Theorem

Let $(f_n)_{n \in \mathbb{N}}$ be a monotone sequence of elements of $L_1(I)$, which converges pointwisely to some $f : \Omega \to \mathbb{R}$ If $(I(f_n))_{n \in \mathbb{N}}$ converges to a real number $\alpha$ Then $f \in L^1(I)$ and $I(f) = \alpha$

## Proof

Assume that $(f_n)_{n\in\mathbb{N}}$ is increasing. Moreover, by replacing $f_n$ by $f_n - f_0$ we may assume that $f_0 = 0$
Let $\epsilon > 0$ $\forall n \in \mathbb{N}$ let $\mu_n \in S^{\uparrow}$ such that $f_n - f_{n-1} \leq \mu_n$ and

$$I(f_n - f_{n-1}) \geq I(\mu_n) - \frac{\epsilon}{2}$$

the existence

$$I(f_n - f_{n-1}) = \inf_{\mu\in S^{\uparrow}, \mu \geq f_n - f_{n-1}} I(\mu)$$

If $\forall \mu \in S^{\uparrow}, \mu \geq f_n - f_{n-1}$ one has

$$I(\mu) > I(f_n - f_{n-1}) + \frac{\epsilon}{2}$$

then

$$I(f_n - f_{n-1}) + \frac{\epsilon}{2} \leq I(f_n - f_{n-1})$$

contraction.

Thus

$$f_n = \sum_{k=1}^{n}(f_k - f_{k-1}) \leq \mu_1 + \cdots + \mu_n$$

and

$$I(f_n) \geq \sum_{k=1}^{n}(I(\mu_k) - \frac{\epsilon}{2^k}) \geq I(\mu_1) + \cdots + I(\mu_n) - \epsilon$$

Let $\mu = \mu_1 + \cdots + \mu_n + \cdots \in S^{\uparrow}$

$$I(\mu) = \sum_{n\in\mathbb{N}} I(\mu)$$

One has $\mu \geq f$

$$\lim_{n\to+\infty} \geq I(\mu) - \epsilon \geq \overline{I}(f) - \epsilon$$

Similarly, one can choose $l_n \in S^{\downarrow}, l_n \leq f_n, I(l_n) \geq I(f_n) - \epsilon$

$$\liminf_{n\to+\infty} I(l_n) \geq \alpha - \epsilon$$

Note that $l_n \leq f_n \leq f$, so

$$\alpha - \epsilon \leq \liminf_{n\to+\infty} I(l_n) \leq \underline{I}(f)$$

Thus

$$\alpha - \epsilon \leq \underline{I}(f) \leq \overline{I}(f) \leq \alpha + \epsilon$$

Let $\epsilon \to 0$ we get

$$\overline{I}(f) = \underline{I}(f) = \alpha$$

## 37.12 Fatou's Lemma

Let $(f_n)_{n\in\mathbb{N}} \in L^1(I)^{\mathbb{N}}$ Assume tha there is $g \in L^1(I)$ such that

$$\forall n \in \mathbb{N} \quad f_n \geq g$$

If $\liminf_{n\to+\infty} f_n$ is a mapping from $\Omega$ to $\mathbb{R}$ and $\liminf_{n\to+\infty} I(f_n) < +\infty$, then $\liminf_{n\to+\infty} f_n \in L^1(I)$ and

$$I(\liminf_{n\to+\infty} f_n) \leq \liminf_{n\to+\infty} I(f_n)$$

### Proof

For any $n \in \mathbb{N}$, let

$$g_n = \lim_{k\to+\infty} (f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k})$$

Then

$$\liminf_{n\to+\infty} f_n = \lim_{n\to+\infty} g_n$$

For any $k$ one has

$$f_n \wedge \cdots \wedge f_{n+k} \geq g$$

Hence

$$I(f_n) \geq \lim_{n\to+\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \geq I(g)$$

By the theorem pf Beppo Levi,

$$g_n \in L^1(I) \text{ and } I(g_n) = \lim_{n\to+\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \leq I(f_n)$$

Note that $(g_n)_{n\in\mathbb{N}}$ is increasing and $\liminf_{n\to+\infty} I(f_n) < +\infty$ Hence

$$\lim_{n\to+\infty} I(g_n) = \liminf_{n\to+\infty} I(g_n) \leq \liminf_{n\to+\infty} I(f_n) < +\infty$$

By the theorem of Beppo Levi,

$$\lim_{n\to+\infty} g_n \in L^1(I)$$

and

$$I(\liminf_{n\to+\infty} f_n) = I(\lim_{n\to+\infty} g_n) = \lim_{n\to+\infty} I(g_n) \leq \liminf_{n\to+\infty} I(f_n)$$

## 37.13 Lebesgue dominated convergence theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(I)$ that converges pointwisely to some $f : \Omega \to \mathbb{R}$ Assume that there exists $g \in L^1(I)$ such that $\forall n \in \mathbb{N}, |f_n| \leq g$ Then $f \in L^1(I)$ and $I(f) = \lim_{n\to+\infty} I(f_n)$

### Proof

Apply Fatou's lemma to $(f_n)_{n\in\mathbb{N}}$ adn $(-f_n)_{n\in\mathbb{N}}$ to get

$$I(f) \leq \liminf_{n\to+\infty} I(f_n)$$

and

$$\begin{aligned} I(-f) &\leq \liminf_{n\to+\infty} I(-f_n) \\ &= \limsup_{n\to+\infty} I(f_n) \\ &\leq \limsup_{n\to+\infty} I(f_n) \leq I(f) \end{aligned}$$

## 37.14 Notation

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous mapping. Let $S$ be the vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by $\mathbb{1}_{]a,b]}$ with $(a,b) \in \mathbb{R}^2, a < b$ For any $f \in L^1(I_\varphi)$ $I_\varphi(f)$ is denoted as

$$\int_{\mathbb{R}} f(x)d\varphi(x)$$

For any subset $A$ of $\mathbb{R}$ if $\mathbb{1}_A f \in L^1(I)$ then

$$\int_A f(x)d\varphi(x) \text{ denotes } \int_{\mathbb{R}} \mathbb{1}_A(x)f(x)d\varphi(x) = I(\mathbb{1}_A f)$$

If $(a,b) \in \mathbb{R}^2, aa < b$

$$\int_a^b f(x)d\varphi(x) \text{ denotes } \int_{]a,b]} f(x)d\varphi(x)$$

$$\int_b^a f(x)d\varphi(x) \text{ denotes } -\int_{]a,b]} f(x)d\varphi(x)$$

If $\varphi(x) = x$ for any $x \in \mathbb{R}$ we replace $d\varphi(x)$ by $dx$.