Contents

Part I

Set

Chapter 1

Ring

1.1 morphism

Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \to B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
-

algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \to R$ is a morphism of unitary rings such that $\forall(b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \to R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n\in\mathbb{N}}$ couldn't be executed directly.
Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n\in\mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n\in\mathbb{N}} a_n T^n$ Such element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_nT^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]

$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$

let $F + G = (a_0 + b_0) + ...$

$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n$

Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n \right) (\sum\limits_{n\in\mathbb{N}} c_nT^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_pb_qc_l \right) T^n$ is a trick applied on integral

Derivative:

let $F(T) \in k[T]$

We denote by $F'(T)$ *or* $\mathcal{D}(F(T))$ the formal power series

$\mathcal{D}(F) = \sum\limits_{n\in\mathbb{N}} (n + 1)a_{n+1}T^n$

Properties:

- $\mathcal{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!}T^n$,which fulfil the differential equation $\mathcal{D}(exp(T)) = exp(T)$(interesting)

Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 2

Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i\in I} is\ a\ family\ of\ element\ in\ X, then\ \sup_{i\in I} x_i\ denotes\ \sup\{x_i \mid i \in I\}(inX)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i\in I,\mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N},k\geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

Chapter 3

Interval

We fix a totally ordered set $(X, \leq)$

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$,one has $[x, y] \subseteq I$ then we say that I is a interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X
- If $\bigcap\limits_{\lambda \in \Lambda} I_\lambda \neq \varnothing$, $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_] \mu$

- If $b \leq x \quad [a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a \quad [a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If a < x < b then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let $(X, \leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.

Prop.

Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x, b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let $(X, \leq)$ be a totally ordered set .If $\forall (x, z) \in X \times X$,such that $x < z \quad \exists y \in X$ such that $x < y < z$,than we say that $(X, \leq)$ is thick

Prop.

Let $(X, \leq)$ be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in]a, b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b[\subseteq]a, b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

Chapter 4

Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq$ *on* $\mathbb{R}$ *to* $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ become a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$ Obviously,this is a thick totally ordered set.
We define:

- $\forall x \in]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete
Example

Let Ω be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega), \sup \mathscr{F} = \bigcup_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$
$\mathcal{AXION}$:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$
Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ *and* $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R}\backslash\{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall(a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall(a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup\limits_{x \in I} f(x) \quad b = \sup\limits_{y \in J} g(y) \quad c = \sup\limits_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup\limits_{x \in I, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup\limits_{x \in I} f(x))(\sup\limits_{x \in I} g(x))$

inf takes the similar($\leq \to \geq$) (provided when the sum are defined)

Chapter 5

Vector space

In this section:
K denotes a unitary ring.
Let 0 be zero element of K
1 be the unity of K

5.1 K-module

5.1.1 Def

Let $(V, +)$ be a commutative group.We call left/right K-module structure:
any mapping $\Phi{:}K \times V \to V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x))/\Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

5.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a, b) \mapsto a + b$
- $K \times K \to K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
Any left $K^{op} - module$ is a right K-module
Any right $K^{op} - module$ is a left K-module
$(K^{op})^{op} = K$

5.1.3 Notation

When we talk about a left/right K-module $(V, +)$,we often write its left K-module structure as $K \times V \to V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned}
&\forall (a,b) \in K \times K, \forall x \in V \quad (ab)x = a(bx)/b(ax) \\
&\forall (a,b) \in K \times K, \forall x \in V \quad (a+b)x = ax + bx \\
&\forall a \in K, \forall (x,y) \in V \times V \quad a(x+y) = ax + ay \\
&\forall x \in V \quad 1x = x
\end{aligned}$$

5.1.4 K-vector space

If K is commutative,then $K^{op} = K$,so left K-module and right K-module structure are the same .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space

Let $\Phi : K \times V \to V$ be a left or right K-module structure

$$\forall x \in V, \Phi(\cdot, x) : K \to V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups.Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
$\forall a \in K, \Phi(a, \cdot) : V \to V$ is a morphism of groups.Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)(\cdot \ is \ a \ var)$

5.1.5 Association:

$\forall x \in K$

$$\begin{aligned}
(f(f+g)+h)(x) &= (f+g)(x) + h(x) = f(x) + g(x) + h(x) \\
(f + (g+h))(x) &= f(x) + ((g+h)(x)) = f(x) + g(x) + h(x)
\end{aligned}$$

Let $0 : I \to K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$
Let $-f : f + (-f) = 0$
The mapping $K \times K^I \to K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K-module structure
The mapping $K \times K^I \to K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K-module structure

5.1.6 Remark:

We can also write an element $\mu\ of\ K^I$ is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I\ by\ \mu$)
Then

$$(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I}$$
$$a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I}$$
$$(\mu_i)_{i\in I}a = (\mu_i a)_{i\in I}$$

5.2 sub K-module

5.2.1 Def

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub-K-module of V.

5.2.2 Example

Let I be a set .Let $K^{\bigoplus I}$ be the subset of K^I composed of mappings $f : I \to K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub-K-module of K^I

In fact,$\forall (f,g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$
Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of K^I
$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x\mapsto f(x)a)} \subseteq I_f$

5.3 morphism of K-modules

5.3.1 Def

Let V and W be left K-module, A morphism of groups $\phi : V \to W$ is called a morphism of left K-modules if $\forall (a,x) \in K \times V, \phi(ax) = a\phi(x)$

5.3.2 K-linear mapping

If K is commutative, a morphism of K-modules is also called a K-linear mapping. We denote by $\hom_{K-Mod}(V,W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of W^V

5.3.3 Theorem

Let V be a left K-module. Let I be a set.
The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I :\ \phi \to (\phi(e_i))_{i\in I}$ is a bijection where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

5.3.4 Remark:column

In the case where $I = 1, 2, 3, ..., n$ V^I is denoted as V^n, K^I is denoted as K^n For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n \phi(e_i) = x_i$

We write this ϕ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$

5.4 kernel

5.4.1 Prop

Let G and H be groups and $f : G \to H$ be a morphism of groups

- $I_m(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

5.4.2 Def

$\ker(f)$ is called the kernel of f

5.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

5.4.4 Proof

Let e_G and e_H be neutral element of G and H respectively

(1) Let x and y be element of G
$f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H

(2) Let x and y be element of $\ker(f)$ One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H \quad e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$ So $\ker(f)$ is a subgroup of G

(3) Suppose that f is injection.
Since $f(E_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ Suppose that $\ker(f) = \{e_G\}$ If $f(x) = f(y)$then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$
Hence $xy^{-1} = e_G \Rightarrow x = y$

5.4.5 Def

Let (V,+) be a commutative group, I be a set. We define a composition law + on V^I as follows

$$(x_i)_{i\in I} + (y_i)_{i\in I} \ := (x_i + y_i)_{i\in I}$$

Then V^I forms a commutative group

5.4.6 Remark

Let E and F be left K-modules
$\hom_{K=Mod}(E,F) := \{$morphisms of left K-modules from E to F$\} \subseteq F^E$ is a subgroup of F^E
In fact f and g are elements of $\hom_{K-Mod}(E,F)$, then $f-g$ is also a morphism of left K-module
$(f-g)(x+y) = f(x+y) - g(x+y) = (f(x)+f(y)) - (g(x)+g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f-g)(x) + (f-y)(x)$

5.4.7 Theorem

Let V be a left K-module,I be a set The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I \ : \ \phi \mapsto (\phi(e_i))_i \in I$ is an isomorphism of groups, where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

5.4.8 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_I) + \psi(e_i)$
$\forall (\phi, psi) \in \hom_{K-Mod}(K^{\bigoplus I}, V)^2$
Hence $\Psi(\phi,\psi) = (\phi(e_i) + \psi(e_i))_{i\in I} = \Psi(\phi) + \Psi(\psi)$
So Ψ is a morphism of groups

injectivity Let $\phi \in \hom_{K-Mod}(K^{\bigoplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$
Let $a = (a_i)_{i\in I} \in K^{\bigoplus I}$ One has $a = \sum\limits_{i\in I} a_i e_i$
If fact,$\forall j \in I, a_j = \sum\limits_{i\in I, a_i\neq 0} a_i e_i(j)$
Thus $\phi(a) = \sum\limits_{i\in I, a_i\neq 0} a - I\phi(e_i) = 0$
Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i\in I} \in V^I$ We define $\phi_x : K^{\bigoplus I} \to V$ such that $\forall a = (a_i)_{i\in I} \in K^{\bigoplus I}, \phi_x(a) = \sum\limits_{i\in I, a_i\neq 0} a_i x_i$
This is a morphism of left K-modules
$foralli \in I, \phi_x(e_i) = 1x_i = x_i$ So $\Psi(\phi_x) = x$

Suppose that K' is a unitary ring,and V is also equipped with a right K'-module structure, Then $\hom_{K-Mod}(K^{\bigoplus I}, V) \subseteq V^{K^{\bigoplus I}}$ is a right sub-k'-module ,and Ψ in the theorem is a right K'-module isomorphism

Chapter 6

Monotone mappings

6.1 Def

Let I and X be partially ordered sets,$f : I \to X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \le f(b)/f(a) < f(b)$,then we say that f is increasing/strictly increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

6.2 Prop.

Let X,Y,Z be partially ordered sets.$f : X \to Y, g : Y \to Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

6.3 Def

Let f be a function from a partially ordered set I to another partially ordered set .If $f \mid_{Dom(f)} \to X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

6.4 Prop.

Let I and X be partially ordered sets. f be function from I to X.

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing

- Assume that I is totally ordered and f is strictly monotone, then f is injection

6.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B
If f is increasing/decreasing ,then so is f^{-1}

6.6 Def

Let X and Y be partially ordered sets. $f : X \to Y$ be a bijection. If both f and f^{-1} are increasing ,then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \to Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

6.7 Prop.

Let I be a subset of $\mathbb{N}$ which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \to I$

6.8 Proof

6.8.1 bijection

We construct $f : \mathbb{N} \to I$ by induction as follows.
Let $f(0) = \min I$ Suppose that $f(0), ..., f(n)$ are constructed
then we take $f(n + 1) := \min(I\backslash\{f(0), ..., f(n)\})$
Since $I\backslash\{f(0), ..., f(n - 1)\} \supseteq I\backslash\{f(0), ..., f(n)\}$.Therefore $f(n) \leq f(n + 1)$
$Since f(n + 1) \notin \{f(0), ..., f(n)\}$,we have $f(n) < f(n + 1)$
Hence f is strictly increasing and ths is injective
If f is not surjective,then $I\backslash Im(f)$ has a element N.
Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.
Since $N \notin Im(f), N < f(m)$.
So $m \neq 0$.Hence $f(m - 1) < N < f(m) = \min(I\backslash\{f(0), ..., f(m - 1)\})$
By definition,$N \in I\backslash Im(f) \subseteq I\backslash\{f(0), ..., f(m - 1)\}$,
Hence $f(m) \leq N$,causing contradiction.

6.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from $\mathbb{N}$ to $\mathbb{N}$

Chapter 7

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

7.1 Def

Let X be a set.We call sequence in X parametrized by I a mapping from I to X.

7.2 Remark

If K is a unitary ring and E is a left K-module then the set of sequence E^I admits a left-K-module structure. If $x = (x_n)_{n \in I}$ is a sequence in E, we define a sequence $\sum(x) := (\sum\limits_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$,called the series associated with the sequence x.

7.3 Prop

$\sum : E^I \to E^{\mathbb{N}}$ is a morphism of left-K-module

7.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I
$\sum\limits_{i \in I, i \leq n} (x_i + y_i) = (\sum\limits_{i \in I, i \leq n} x_i) + (\sum\limits_{i \in I, i \leq n} y_i), \lambda \sum\limits_{i \in I, i \leq n} x_i = \sum\limits_{i \in I, i \leq n} \lambda x_i$

7.5 Prop

Let I be a totally ordered set . X be a partially ordered set,$f : I \to X$ be a mapping ,$J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

7.6 limit

7.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset.$\forall (x_i)_{n\in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$,we define:

$$\limsup_{n\in I, n\to +\infty} x_n := \inf_{n\in I}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to +\infty} x_n := \sup_{n\in I}(\inf_{i\in I, i\geq n} x_i)$$

If $\limsup_{n\in I, n\to +\infty} x_n = \liminf_{n\in I, n\to +\infty} x_n = l$, we then say that $(x_n)_{n\in I}$ tends to l and that l is the limit of $(x_n)_{n\in I}$. If in addition $(x_n)_{n\in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$,we say that $(x_n)_{n\in I}$ converges to l

7.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n\in I, n\to +\infty} = \inf_{n\in J}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to +\infty} x_n = \sup_{n\in J}(\inf_{i\in I, i\geq n} x_i)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i\in I}$ the limit superior and the limit inferior do not change.
In fact, if we take $J = \mathbb{N}\backslash\{0, ..., m\}$, then $\inf_{n\in J}(...)$ and $\sup_{n\in J}(...)$ only depends on the values of $x_i, i \in I, i \geq m$

7.6.3 Prop

$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$, $\liminf_{n\in I, n\to +\infty} x_n \leq \limsup_{n\in I, n\to +\infty} x_n$

7.6.4 Prop

Let $(x_n)_{n\in I} \in [-\infty, +\infty]^I$

$$\forall c \in \mathbb{R} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty} (x_n + c) &= (\limsup_{n\in I, n\to+\infty} x_n) + c \\ \liminf_{n\in I, n\to+\infty} (x_n + c) &= (\liminf_{n\in I, n\to+\infty} x_n) + c \end{aligned}$$

$$\forall c \in \mathbb{R}_{>0} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty} (\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty} (\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \end{aligned}$$

$$\forall c \in \mathbb{R}_{<0} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty} (\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty} (\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \end{aligned}$$

7.6.5 Prop

Let $(x_n)_{n\in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$,one has $x_n \leq y_n$ Then

$$\limsup_{n\in I, n\to+\infty} (x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

,

$$\liminf_{n\in I, n\to+\infty} (x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

7.6.6 Theorem

Let $(x_n)_{n\in I}, (y_n)_{n\in I}, (z_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$
Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n\in I}$ and $(z_n)_{n\in I}$ tend to the same limit l

Then $(y_n)_{n\in I}$ tends to l

7.6.7 Def

Let I be an infinite subset of $\mathbb{N}$, and $(x_n)_{n\in I}$ be a sequence in some set X. We call subsequence of $(x_n)_{n\in I}$ a sequence of the form $(x_n)_{n\in J}$,where J is an infinite subset of I

7.6.8 Prop

Let I and J be infinite subset of $\mathbb{N}$ such that $J \subseteq I$
$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$,one has

$$\liminf_{n\in I, n\to+\infty} (x_n) \leq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in I,n\to+\infty}(x_n)\geq\limsup_{n\in I,n\to+\infty}y_n$$

In particular, if $(x_n)_{n\in I}$ tends to $l\in[-\infty,+\infty]$,then $(x_n)_{n\in J}$ tends to l

7.6.9 Prop

$\forall n\in\mathbb{N}$,one has

$$\liminf_{n\in J,n\to+\infty}(x_n)\geq\liminf_{n\in I,n\to+\infty}y_n$$

$$\limsup_{n\in J,n\to+\infty}(x_n)\leq\limsup_{n\in I,n\to+\infty}y_n$$

7.6.10 Theorem

Let $I\subseteq\mathbb{N}$ be an infinite subset and $(x_N)_{n\in I}$ be a sequence in $[-\infty,+\infty]$

- If the mapping $(n\in I)\mapsto x_n$ is increasing,then $(x_N)_{i\in I}$ tends to $\sup\limits_{n\in I}x_n$
- If the mapping $(n\in I)\mapsto x_n$ is decreasing,then $(x_N)_{i\in I}$ tends to $\inf\limits_{n\in I}x_n$

7.6.11 Notation

If a sequence $(x_N)_{n\in I}\in[-\infty,+\infty]$ tends to some $l\in[-\infty,+\infty]$ the expression $\lim\limits_{n\in I,n\to}x_n$ denotes this limit l

7.6.12 Corollary

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{N}_{\geq0}$ Then the series $\sum\limits_{n\in I}x_n$(the sequence $(\sum\limits_{i\in I,i\leq n})_{n\in\mathbb{N}})$ tends to an element in $\mathbb{N}_{\geq0}\cup\{+\infty\}$ It converges in $\mathbb{R}$ iff it is bounded from above (namely has an upper bound in $\mathbb{R}$)

7.6.13 Notation

If a series $\sum\limits_{n\in I}x_n$ in $[-\infty,+\infty]$ tends to some limit, we use the expression $\sum\limits_{n\in I}x_n$ to denote the limit

7.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n\in I}$ be a sequence in $[-\infty,+\infty]$ There exists a subsequence of $(x_n)_{n\in I}$ that tends to $\limsup\limits_{n\in I,n\to+\infty}x_n$ There exists a subsequence of $(x_n)_{n\in I}$ that rends to $\liminf\limits_{n\in I,n\to+\infty}x_n$

7.6.15 Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_N)_{n\in J}$ is decreasing so it tends to $\inf\limits_{n\in J} x_n$

$\forall n \in J$ by definition $x_n = \sup\limits_{i\in I, i\geq n} x_i$ so $\limsup\limits_{n\in I, n\to+\infty} x_n = \inf\limits_{n\in J} \sup\limits_{i\in I, i\geq n} x_i = \inf\limits_{n\in J} x_n = \lim\limits_{n\in J, n\to+\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$.Denote by $l = \sup\limits_{n\in I, n\geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup\limits_{i\in I, i\geq n_0} x_i \leq l$. If the strict inequality $\sup\limits_{i\in I, i\geq N} x_i < l$ holds, then $\sup\limits_{i\in I, i\geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup\limits_{i\in I, i\geq N} x_i$ We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality. Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$ Thus $n \in J$ that contradicts the maximality of n_0 Therefore $l = \sup\limits_{i\in I, i\geq N} x_i$, which leads to $\limsup\limits_{n\in I, n\to+\infty} x_n = l$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$,so $x_m < l$(since otherwise $x_m = \sup\limits_{i\in I, i\geq m} x_i$ and hence $m \in J$)Hence,$\forall\ finite\ subset\ I'\ of \{m \in I \mid m \geq n_0\}$

$\max\limits_{i\in I} x_i < l$ and hence $\exists n \in I$,such that $n > \max I'$,and $\max\limits_{i\in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \to I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$
Hence the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is increasing, and $\sup\limits_{j\in\mathbb{N}} x_{n_j} \leq \sup\limits_{j\in\mathbb{N}} x_{f(j)} = \sup\limits_{n\in I, n\geq n_0} x_n = l$

$l = \sup\limits_{n\in I, n\geq n_0}$

So $(x_{n_j})_{j\in\mathbb{N}}$ tends to l

Chapter 8

Cauchy sequence

8.1 Def

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{R}$
If $\inf_{N\in\mathbb{N}} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = \lim_{N\to+\infty} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = 0$ then we say that $(x_n)_{n\in I}$ is a Cauchy sequence

8.2 Prop

- If $(x_n)_{i\in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_N)_{i\in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite,$(x_n)_{n\in I}$ is a Cauchy sequence.
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n\in I}$ converges to l, then $(x_n)_{n\in J}$ converges to l too.

8.3 Theorem: Completeness of real number

If $(x_n)_{n\in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in $\mathbb{R}$

8.3.1 Proof

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$ So $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n\in I}$ converges to the same limit.

8.4 Absolutely converge

We say that a series $\sum_{n\in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n\in I} |x_n| < +\infty$

8.4.1 Prop

If a series $\sum_{n\in I} x_n$ converges absolutely, then it converges in $\mathbb{R}$

Chapter 9

Comparison and Technics of Computation

9.1 Def

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be sequence in $\mathbb{R}$

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_N| \leq M|y_m|$ then we write $x_n = O(y_n), n \in I, n \to +\infty$
- If there exists $(\epsilon_n)_{n\in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim\limits_{n\in I, n\to+\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_N| \leq |\epsilon y_m|$, then we write $x_n = \circ(y_n), n \in I, n \to +\infty$
  Example:
  $$\lim_{n\to+\infty} \frac{1}{n} = 0$$

9.2 Prop.

Let I and X be partially ordered sets and $f : I \to X$ be an increasing/decreasing mapping. Let J ba a subset of I. Assume that any elements of I has an upper bound in J. Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

9.3 Theorem

Let I be a totally ordered set, $f : I \to [-\infty, +\infty]$ and $g : I \to [-\infty, +\infty]$ be two mappings that are both increasing/decreasing.Then the following equalities holds,provided that the sum on the right hand side of the equality is wel defined.

$$\sup_{x\in I, \{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\sup_{x\in I} f(x)) + (\sup_{y\in I} g(y))$$

$$\inf_{x\in I,\{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\inf_{x\in I} f(x)) + (\inf_{y\in I} g(y))$$

Proof

We can assume f and g increasing. Let $a = \sup f(I)$,$b = \sup g(I)$
Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$
We equip A with the following order relation.

$$(x, y) \leq (x', y') \; iff \; x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.
Consider

$$h : A \to [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

h is increasing.
Let $(x, y) \in A$. Assume that $x \leq y$
If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$
If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$
So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined,$(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if$x \geq y$, (x, y) has also an upper bound in B. Therefore: $\sup h(A) = \sup h(B)$

9.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I,n\to+\infty} (x_n + y_n) \leq (\limsup_{n\in I,n\to+\infty} x_n) + (\limsup_{n\in I,n\to+\infty} y_n)$$

$$\liminf_{n\in I,n\to+\infty} (x_n + y_n) \geq (\liminf_{n\in I,n\to+\infty} x_n) + (\liminf_{n\in I,n\to+\infty} y_n)$$

9.4.1 Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n\in I,n\geq N} x_n \quad B_N = \sup_{n\in I,n\geq N} y_n$. $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are decreasing, and $\limsup_{n\in I,n\to+\infty} x_n = \inf_{N\in\mathbb{N}} A_N \quad \limsup_{n\in I,n\to+\infty} y_n = \inf_{N\in\mathbb{N}} B_N$
By theorem:

$$\inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N = \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n\in I,n\geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.
Therefore

$$\inf_{N\in\mathbb{N}} C_N \leq \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N) = \inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N$$

9.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty,+\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) \geq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

9.5.1 Proof

a tricky proof?:

$$\limsup_{n\in I, n\to} x_n = \limsup_{n\in I, n\to}(x_n+y_n-y_n) \leq \limsup_{n\in I, n\to}(x_n+y_n) - \liminf_{n\in I, n\to} y_n$$

to have a true proof,only need to discuss conditions with ∞

9.6 Theorem

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n\in I}$ converges to some $i \in \mathbb{R}$.
Then:

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) = (\limsup_{n\in I, n\to+\infty} x_n) + l$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) = (\liminf_{n\in I, n\to+\infty} x_n) + l$$

9.7 Prop.

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Then:

$$\liminf_{n\in I, n\to+\infty} \max\{x_n, y_n\} = \max\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

$$\liminf_{n\in I, n\to+\infty} \min\{x_n, y_n\} = \min\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

Proof

About the first inequality.Since $\max\{x_n, y_n\} \geq x_n \quad \max\{x_n, y_N\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n\in J, n\to+\infty} = \limsup_{n\in J, n\to+\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $J_1 = \{n \in J \mid x_n \leq y_n\}$
$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite
Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \to} \max\{x_n, y_n\} = \lim_{n \in J_1, n \to} \max\{x_n, y_n\} = \lim_{n \in J, n \to} x_n \leq \limsup_{n \in I, n \to +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \to +\infty} = \lim_{n \in J_2, n \to +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \to +\infty} y_n$$

9.8 Theorem

Let $(a_N)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_N)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$

Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \to +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \to +\infty} a_n) - l, l - (\liminf_{n \in I, n \to +\infty} a_n)\}$$

(1) $\Rightarrow$ (2):
If $(a_n)_{n \in \mathbb{N}}$ converges to l, then $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$

(2) $\Rightarrow$ (1):
If $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$,then $\limsup_{n \in I, n \to +\infty} a_n \leq l \leq \liminf_{n \in I, n \to +\infty} a_n$
Therefore: $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$