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Part I

Set

Chapter 1

Ring

1.1 morphism

Def

Let A and B be unitary rings. We call morphism of unitary rings from A to B only mapping $A \rightarrow B$ is a morphism of group from $(A, +)$ to $(B, +)$, and a morphism of monoid from (A, \cdot) to (B, \cdot)

Properties

- Let R be a unitary ring. There is a unique morphism from \mathbb{Z} to R
-

algebra

we call k -algebra any pair (R, f) , when R is a unitary ring, and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R , the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of \mathbb{Z} -algebra on R (extra: \mathbb{Z} is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring, (A, f) be a k -algebra. If there is no ambiguity on f , for any $(\lambda, a) \in k \times A$, we denote $f(\lambda)a$ as λa

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite, so $\sum_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def: Let T be a formal symbol. We denote $k^{\mathbb{N}}$ as $k[T]$. If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$, when we denote $k^{\mathbb{N}}$ as $k[T]$ this element is denoted as $\sum_{n \in \mathbb{N}} a_n T^n$. Such

element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient 0
- write T^1 as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$ is written as $1 + 2T + T^2$

Def Remind that $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$, define two composition laws on $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$ form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left(\sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left(\sum_{p,q,l=n} a_p b_q c_l \right) T^n$
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by $F'(T)$ or $\mathcal{D}(F(T))$ the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $\exp(T) \in k[T]$ as $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$, which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence: $(F_i(T))_{i \in \mathbb{N}}$ be a sequence of elements in $k[T]$, and $F(T) \in k[T]$ We say that $(F_i(T))_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$, there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$, $\text{ord}(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 2

Supremum and infimum

Def:

Let (X, \leq) be a partially ordered set A and Y be subsets of X , such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq y\}$ has a least element then we say that A has a Supremum in Y with respect to \leq denoted by $\sup_{(Y, \leq)} A$ this least element and called it the Supremum of A in Y (this respect to \leq)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has an infimum in Y with respect to \leq . We denote by $\inf_{(Y, \leq)} A$ this greatest element and call it the infimum of A in Y
- Observation: $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let (X, \leq) be a partially ordered set, I be a set.

- If f is a function from I to X $\sup f$ denotes the supremum of $f(I)$ is X . $\inf f$ takes the same
- If $(x_i)_{i \in I}$ is a family of element in X , then $\sup x_i$ denotes $\sup\{x_i \mid i \in I\}$ (in X)

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$ We equip \mathbb{R} with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$

Example: For $n \in \mathbb{N}$, let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let (X, \leq) be a partially ordered set, A, Y, Z be subset of X , such that $A \subseteq Z \subseteq Y$

- If $\max A$ exists, then it is also equal to $\sup_{(y, \leq)} A$
- If $\sup_{(y, \leq)} A$ exists and belongs to Z , then it is equal to $\sup A$

\inf takes the same Prop.

Let X, \leq be a partially ordered set, A, B, Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y, \leq)} A$ and $\sup_{(y, \leq)} B$ exists, then $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If $\inf_{(y, \leq)} A$ and $\inf_{(y, \leq)} B$ exists, then $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let (X, \leq) be a partially ordered set, I be a set and $f, g : I \rightarrow X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists, then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists, then $\sup f \leq \sup g$

Chapter 3

Interval

We fix a totally ordered set (X, \leq)

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$, $[a, b]$ denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$, one has $[x, y] \subseteq I$ then we say that I is an interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$. Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an interval in X
- If $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$, $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_\mu$

- If $b \leq x$ $[a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a$ $[a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If $a < x < b$ then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let (X, \leq) be a totally ordered set. I be a non-empty interval of X . If $\sup I$ exists in X , we call $\sup I$ the right endpoint; \inf takes the similar way.

Prop.

Let I be an interval in X .

- Suppose that $b = \sup I$ exists. $\forall x \in I, [x, b] \subseteq I$
- Suppose that $a = \inf I$ exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X . Suppose that I has supremum b and an infimum a in X . Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let (X, \leq) be a totally ordered set. If $\forall (x, z) \in X \times X$, such that $x < z$ $\exists y \in X$ such that $x < y < z$, then we say that (X, \leq) is thick

Prop.

Let (X, \leq) be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a$ $\sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists $x_0 \in]a, b[$ By definition, b is an upper bound of I . If b is not the supremum of I , there exists an upper bound M of I such that $M \neq b$. Since X is thick, there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b] \subseteq I, a, b \in I$ Hence M and M' belong to I , which conflicts with the uniqueness of supremum.

Chapter 4

Enhanced real line

Def:

Let $+\infty$ and $-\infty$ be two symbols that are different and don't belong to \mathbb{R} . We extend the usual total order \leq on \mathbb{R} to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ becomes a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$. Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty[\quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in]0, +\infty[\quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
ARE NOT DEFINED

Def

Let (X, \leq) be a partially ordered set. If for any subset A of X , A has a supremum and an infimum in X , then we say that X is order complete.

Example

Let Ω be a set. $(\mathcal{P}(\Omega), \subseteq)$ is order complete. If \mathcal{F} is a subset of $\mathcal{P}(\Omega)$, $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$.

Interesting tip: $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

Axiom:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete.

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ and $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \setminus \{0\}$, λA denotes $\{\lambda a \mid a \in A\}$
- $-A$ denotes $(-1)A$

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$, $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ Def

We denote by (\mathbb{R}, \leq) a field \mathbb{R} equipped with a total order \leq , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, one has $\forall c \in \mathbb{R}$, $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, $ab > 0$
- $\forall A \subseteq \mathbb{R}$, if A has an upper bound in \mathbb{R} , then it has a supremum in \mathbb{R}

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

\inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$ then $c = a + b$

\inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set, $f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

\inf takes the similar ($\leq \rightarrow \geq$) (provided when the sum are defined)

Chapter 5

Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

5.1 K -module

5.1.1 Def

Let $(V, +)$ be a commutative group. We call left/right K -module structure: any mapping $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group $(V, +)$ equipped with a left/right K -module structure is called a left/right K -module.

5.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
 Any left K^{op} - module is a right K -module
 Any right K^{op} - module is a left K -module
 $(K^{op})^{op} = K$

5.1.3 Notation

When we talk about a left/right K -module $(V, +)$, we often write its left K -module structure as $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

5.1.4 K -vector space

If K is commutative, then $K^{op} = K$, so left K -module and right K -module structure are the same. We simply call them K -module structure. A commutative group equipped with a K -module structure is called a K -module. If K is a field, a K -module is also called a K -vector space

Let $\Phi : K \times V \rightarrow V$ be a left or right K -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$ is a morphism of groups. Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$ (*is a var*)

5.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K -module structure

The mapping $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K -module structure

5.1.6 Remark:

We can also write an element μ of K^I in the form of a family $(\mu_i)_{i \in I}$ of elements in K (μ_i is the image of $i \in I$ by μ)
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

5.2 sub K-module**5.2.1 Def**

Let V be a left/right K -module. If W is a subgroup of V . Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub- K -module of V .

5.2.2 Example

Let I be a set. Let $K^{\oplus I}$ be the subset of K^I composed of mappings $f : I \rightarrow K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub- K -module of K^I

In fact, $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$
Hence $f - g \in K^{\oplus I}$ So $K^{\oplus I}$ is a subgroup of K^I
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

5.3 morphism of K-modules**5.3.1 Def**

Let V and W be left K -module, A morphism of groups $\phi : V \rightarrow W$ is called a morphism of left K -modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

5.3.2 K-linear mapping

If K is commutative, a morphism of K -modules is also called a K -linear mapping. We denote by $\text{hom}_{K\text{-Mod}}(V, W)$ the set of all morphism of left- K -module from V to W . This is a subgroup of W^V

5.3.3 Theorem

Let V be a left K -module. Let I be a set.
The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$ is a bijection where
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.3.4 Remark:column

In the case where $I = 1, 2, 3, \dots, n$ V^I is denoted as V^n , K^I is denoted as K^n . For any $(x_1, \dots, x_n) \in V^n$, by the theorem, there exists a unique morphism of left K -modules $\phi : K^n \rightarrow V$ such that $\forall i \in 1, \dots, n, \phi(e_i) = x_i$.

We write this ϕ as a column $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$. It sends $(a_1, \dots, a_n) \in K^n$ to $a_1x_1 + \dots + a_nx_n$.

5.4 kernel

5.4.1 Prop

Let G and H be groups and $f : G \rightarrow H$ be a morphism of groups

- $Im(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

5.4.2 Def

$\ker(f)$ is called the kernel of f

5.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

5.4.4 Proof

Let e_G and e_H be neutral element of G and H respectively

- (1) Let x and y be element of G
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H
- (2) Let x and y be element of $\ker(f)$. One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$. So $\ker(f)$ is a subgroup of G .
- (3) Suppose that f is injection.
 Since $f(e_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$. Suppose that $\ker(f) = \{e_G\}$. If $f(x) = f(y)$ then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$.
 Hence $xy^{-1} = e_G \Rightarrow x = y$

5.4.5 Def

Let $(V, +)$ be a commutative group, I be a set. We define a composition law $+$ on V^I as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then V^I forms a commutative group

5.4.6 Remark

Let E and F be left K -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$ is a subgroup of F^E

In fact f and g are elements of $\text{hom}_{K\text{-Mod}}(E, F)$, then $f - g$ is also a morphism of left K -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

5.4.7 Theorem

Let V be a left K -module, I be a set. The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$ is an isomorphism of groups, where $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.4.8 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So Ψ is a morphism of groups

injectivity Let $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i \in I} \in V^I$ We define $\phi_x : K^{\oplus I} \rightarrow V$ such that $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left K -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that K' is a unitary ring, and V is also equipped with a right K' -module structure, Then $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub- k' -module, and Ψ in the theorem is a right K' -module isomorphism

Chapter 6

Monotone mappings

6.1 Def

Let I and X be partially ordered sets, $f : I \rightarrow X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)$, then we say that f is increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

6.2 Prop.

Let X, Y, Z be partially ordered sets. $f : X \rightarrow Y, g : Y \rightarrow Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

6.3 Def

Let f be a function from a partially ordered set I to another partially ordered set X . If $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

6.4 Prop.

Let I and X be partially ordered sets. f be function from I to X .

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

6.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B

If f is increasing/decreasing, then so is f^{-1}

6.6 Def

Let X and Y be partially ordered sets. $f : X \rightarrow Y$ be a bijection. If both f and f^{-1} are increasing, then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \rightarrow Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

6.7 Prop.

Let I be a subset of \mathbb{N} which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \rightarrow I$

6.8 Proof

6.8.1 bijection

We construct $f : \mathbb{N} \rightarrow I$ by induction as follows.

Let $f(0) = \min I$ Suppose that $f(0), \dots, f(n)$ are constructed

then we take $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$. Therefore $f(n) \leq f(n+1)$

Since $f(n+1) \notin \{f(0), \dots, f(n)\}$, we have $f(n) < f(n+1)$

Hence f is strictly increasing and this is injective

If f is not surjective, then $I \setminus \text{Im}(f)$ has a element N .

Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.

Since $N \notin \text{Im}(f)$, $N < f(m)$.

So $m \neq 0$. Hence $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition, $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$,

Hence $f(m) \leq N$, causing contradiction.

6.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from \mathbb{N} to \mathbb{N}

Chapter 7

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

7.1 Def

Let X be a set. We call sequence in X parametrized by I a mapping from I to X .

7.2 Remark

If K is a unitary ring and E is a left K -module then the set of sequence E^I admits a left- K -module structure. If $x = (x_n)_{n \in I}$ is a sequence in E , we define a sequence $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$, called the series associated with the sequence x .

7.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$ is a morphism of left- K -module

7.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

7.5 Prop

Let I be a totally ordered set . X be a partially ordered set, $f : I \rightarrow X$ be a mapping , $J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

7.6 limit

7.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset. $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left(\inf_{i \in I, i \geq n} x_i \right)$$

If $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$, we then say that $(x_n)_{n \in I}$ tends to l and that l is the limit of $(x_n)_{n \in I}$. If in addition $(x_n)_{n \in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$, we say that $(x_n)_{n \in I}$ converges to l

7.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left(\inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i \in I}$ the limit superior and the limit inferior do not change.

In fact, if we take $J = \mathbb{N} \setminus \{0, \dots, m\}$, then $\inf_{n \in J}(\dots)$ and $\sup_{n \in J}(\dots)$ only depends on the values of $x_i, i \in I, i \geq m$

7.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

7.6.4 Prop

Let $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned} \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + c \\ & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + c \\ \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \end{aligned}$$

7.6.5 Prop

Let $(x_n)_{n \in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$, one has $x_n \leq y_n$. Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

7.6.6 Theorem

Let $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$ and $(z_n)_{n \in I}$ tend to the same limit l

Then $(y_n)_{n \in I}$ tends to l

7.6.7 Def

Let I be an infinite subset of \mathbb{N} , and $(x_n)_{n \in I}$ be a sequence in some set X . We call subsequence of $(x_n)_{n \in I}$ a sequence of the form $(x_n)_{n \in J}$, where J is an infinite subset of I

7.6.8 Prop

Let I and J be infinite subset of \mathbb{N} such that $J \subseteq I$. $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$, one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if $(x_n)_{n \in I}$ tends to $l \in [-\infty, +\infty]$, then $(x_n)_{n \in J}$ tends to l

7.6.9 Prop

$\forall n \in \mathbb{N}$, one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

7.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing, then $(x_n)_{n \in I}$ tends to $\sup_{n \in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing, then $(x_n)_{n \in I}$ tends to $\inf_{n \in I} x_n$

7.6.11 Notation

If a sequence $(x_n)_{n \in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim_{n \in I, n \rightarrow} x_n$ denotes this limit l

7.6.12 Corollary

Let $(x_n)_{n \in I}$ be a sequence in $\mathbb{N}_{\geq 0}$. Then the series $\sum_{n \in I} x_n$ (the sequence $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$. It converges in \mathbb{R} iff it is bounded from above (namely has an upper bound in \mathbb{R})

7.6.13 Notation

If a series $\sum_{n \in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum_{n \in I} x_n$ to denote the limit

7.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\limsup_{n \in I, n \rightarrow +\infty} x_n$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\liminf_{n \in I, n \rightarrow +\infty} x_n$.

7.6.15 Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_n)_{n \in J}$ is decreasing so it tends to $\inf_{n \in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i \in I, i \geq n} x_i$ so $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$. Denote by

$$l = \sup_{n \in I, n \geq n_0}$$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i \in I, i \geq n_0} x_i \leq l$. If the strict

inequality $\sup_{i \in I, i \geq N} x_i < l$ holds, then $\sup_{i \in I, i \geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i \in I, i \geq N} x_i$. We

may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality. Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$. Thus $n \in J$ that contradicts the maximality of n_0 . Therefore $l = \sup_{i \in I, i \geq N} x_i$, which leads to $\limsup_{n \in I, n \rightarrow +\infty} x_n = l$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$, so $x_m < l$ (since otherwise $x_m = \sup_{i \in I, i \geq m} x_i$ and hence $m \in J$). Hence, \forall finite subset I' of $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$ and hence $\exists n \in I$, such that $n > \max I'$, and $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j \in \mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \rightarrow I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$

Hence the sequence $(x_{n_j})_{j \in \mathbb{N}}$ is increasing, and $\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n =$

l

$$l = \sup_{n \in I, n \geq n_0}$$

So $(x_{n_j})_{j \in \mathbb{N}}$ tends to l

Chapter 8

Cauchy sequence

8.1 Def

Let $(x_n)_{n \in I}$ be a sequence in \mathbb{R}
If $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$ then
we say that $(x_n)_{n \in I}$ is a Cauchy sequence

8.2 Prop

- If $(x_n)_{i \in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_n)_{i \in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite, $(x_n)_{n \in I}$ is a Cauchy sequence.
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n \in I}$ converges to l , then $(x_n)_{n \in J}$ converges to l too.

8.3 Theorem: Completeness of real number

If $(x_n)_{n \in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in \mathbb{R}

8.3.1 Proof

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$. So $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n \in I}$ converges to $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n \in I}$ converges to the same limit.

8.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

8.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in \mathbb{R}

Chapter 9

Comparison and Technics of Computation

9.1 Def

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be sequence in \mathbb{R}

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$ then we write $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$, then we write $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

9.2 Prop.

Let I and X be partially ordered sets and $f : I \rightarrow X$ be an increasing/decreasing mapping. Let J be a subset of I . Assume that any elements of I has an upper bound in J . Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

9.3 Theorem

Let I be a totally ordered set, $f : I \rightarrow [-\infty, +\infty]$ and $g : I \rightarrow [-\infty, +\infty]$ be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

9.3.1 Proof

We can assume f and g increasing. Let $a = \sup f(I), b = \sup g(I)$

Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip A with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

h is increasing.

Let $(x, y) \in A$. Assume that $x \leq y$

If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$

If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$

So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined, $(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if $x \geq y$, (x, y) has also an upper bound in B . Therefore: $\sup h(A) = \sup h(B)$

9.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

9.4.1 Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n \in I, n \geq N} x_n$ $B_N = \sup_{n \in I, n \geq N} y_n$. $(A_N)_{N \in \mathbb{N}}$ and $(B_N)_{N \in \mathbb{N}}$ are decreasing, and $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$ $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

9.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n)$$

9.5.1 Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with ∞

9.6 Theorem

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n \in I}$ converges to some $l \in \mathbb{R}$. Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + l$$

9.7 Prop.

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

9.7.1 Proof

About the first inequality. Since $\max\{x_n, y_n\} \geq x_n$ and $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite

Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

9.8 Theorem

Let $(a_n)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

9.8.1 Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1) \Rightarrow (2):

If $(a_n)_{n \in I}$ converges to l , then $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2) \Rightarrow (1):

If $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$, then $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore: $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

9.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in \mathbb{R} , $l \in \mathbb{R}$

The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \rightarrow +\infty$

9.10 Calculates on $O(), o()$

9.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$

9.10.2 Transform

Let $(a_n)_{n \in I}$ and $(b_n)_{n \in I}$ be two sequence in \mathbb{R} If $a_n = o(b_n), n \in I, n \rightarrow +\infty$, then $a_n = O(b_n), n \in I, n \rightarrow +\infty$

9.10.3 Transition

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ and $(c_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$

9.10.4 Times

Let $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$ be sequences in \mathbb{R}

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

9.11 On the limit

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be elements of \mathbb{R}^I that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n \in I}$ converges to $l + l'$
- $(a_n b_n)_{n \in I}$ converges to ll'

9.12 Prop

Let $a \in \mathbb{R}$ Then $a^n = o(n!) \quad n \rightarrow +\infty$

9.12.1 Proof

Let $N \in \mathbb{N}$ such that $|a| < N$ For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{n!}{N!} \cdot \frac{|a|^{n-N}}{n!} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^{n-N} - N$$

And $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$. Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

9.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

9.13.1 Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

9.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of \mathbb{R}^I . If the series $\sum_{n \in I} b_n$ converges absolutely and if $a_n = O(b_n) \quad n \rightarrow +\infty$ Then $\sum_{n \in I} a_n$ converges absolutely

9.14.1 Proof

By definition $\sum_{n \in I} |b_n| < +\infty$. If $|a_n| \leq M|b_n|$ for $n \in I, n \geq N$ where $N \in \mathbb{N}$. Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

9.15 Theorem: d'Alembert ratio test

Let $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

9.15.1 Proof

(1)

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, α isn't a lower bound of $\left(\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$
 So $\exists N \in \mathbb{N}$ such that $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$ Hence for $n \geq N$ $|a_n| \leq \alpha^{n-N} |a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$, $\sum_{n \in \mathbb{N}} a_n$ converge absolutely.

9.15.2 Lemma

If a series $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n \rightarrow +\infty} a_n = 0$

Proof

If $\left(\sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$ converges to some $l \in \mathbb{R}$, then $\left(\sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$ converges to l ,
 too. Hence $\left(a_n = \left(\sum_{i=0}^n a_i \right) - \left(\sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$ converges to $l - l = 0$

9.15.3 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$
 Hence $(|a_n|)_{n \in \mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N} |a_N|$
 By the lemma: $\sum_{n \in \mathbb{N}} a_n$ diverges.

9.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \rightarrow +\infty$

9.16.1 Proof

Let $\epsilon > 0$ such that $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

9.16.2 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \rightarrow +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$ Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

9.16.3 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

9.17 Theorem: Cauchy root test

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} . Let $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.
- If $\alpha > 1$ then $\sum_{n \in \mathbb{N}} a_n$ diverges

9.17.1 Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_n|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n \in \mathbb{N}}$ cannot converge to 0.

Part III

Topology

Chapter 10

Absolute value and norms

10.1 Def

Let K be a field. By absolute value on K , we mean a mapping $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- (1) $\forall a \in K \quad |a| = 0$ iff $a = 0$
- (2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$ (triangle inequality)

10.2 Notation

\mathbb{Q} Take a prime num $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$ there exists a integer $ord_p(\alpha) \frac{a}{b}$, where
 $a \in \mathbb{Z} \setminus \{0\}$
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

10.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on \mathbb{Q}

10.3.1 proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd \\ \alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d} \\ \text{Assume } ord_p(\alpha) \geq ord_p(\beta) \\ \alpha + \beta \\ = p^{ord_p(\beta)} \left(p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right) \\ = p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd \\ \text{So}$$

$$ord_p(\alpha + \beta) \geq ord_p(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\} \\ \text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} = \\ \max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

Chapter 11

Quotient Structure

11.1 Def

Let X be a set and \sim be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that \sim is an equivalence relation

11.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X . Let X/\sim be the set $\{[x] \mid x \in X\}$

11.3 Prop.

Let X be a set and \sim be an equivalence relation on X

- (1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$
- (2) If α and β are elements of X/\sim such that $\alpha \neq \beta$ then $\alpha \cap \beta = \emptyset$
- (3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

11.3.1 Proof

- (1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ one has $x \sim y$.
Therefore, $x \sim z$ namely $z \in [x]$. This proves $[y] \subseteq [x]$. Moreover, since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$.
- (2) Suppose that $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$
By (1), $\alpha = [y], \beta = [y]$, Thus leads to a contradiction.
- (3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup_{\alpha \in X/\sim} \alpha$ Hence $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$. Conversely,
 $\forall \alpha \in X/\sim, \alpha$ is a subset of X . Hence $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$. Then $X = \bigcup_{\alpha \in X/\sim} \alpha$

11.4 Def

Let G be a group and X be a set
We call left/right action of G on X ant mapping $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

11.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X .

11.6 Prop

Let G be a group and X be a set . Assume given a left action of G on X . Then the binary relation \sim on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

11.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim \forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

11.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$. ($y \sim x$)
- $\forall (x, y, z) \in X^3$, if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

11.9 Quotient set

Let X be a set and \sim be an equivalence relation, the mapping $X \rightarrow X/\sim$:
 $(x \in X) \mapsto [x]$ is called the projection mapping.

X/\sim is called the quotient set of X by equivalence relation \sim

11.9.1 Example

Let G be a group and H be a subgroup of G . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of H on G . Thus we obtain two quotient sets H/G and G/H

11.10 Def

Let G be a group and H be a subgroup of G . If $\forall g \in G, h \in H \quad ghg^{-1} \in H$,
 Then we say that H is a normal subgroup of G

11.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G . In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

11.12 Prop

If G is commutative, any subgroup of G is normal

11.13 Theorem

Let G be a group and H be a normal subgroup of G . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

11.13.1 Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$. Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

11.14 Def

Let K be a unitary ring and E be a left K -module. We say that a subgroup F of $(E, +)$ is a left sub- K -module of E if $\forall (a, x) \in K \times F, ax \in F$

11.15 Prop

Let K be a unitary ring, E be a left K -module and F be a sub- K -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- K -module structure on E/F . Moreover, the projection mapping $\pi : E \rightarrow E/F$ is a morphism of left- K -modules

11.15.1 Proof

Let x and x' be elements of E such that $[x] = [x']$, that means: $x' - x \in F$
Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K$ $\pi(ax) = [ax] = a[x] = a\pi(x)$

11.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$ (I is a left and right sub- K -module of A)

11.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

11.17.1 Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$
Therefore $a'b' \sim ab$

11.17.2 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of \mathbb{Z}
 If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z}$ $adm = dma \in d\mathbb{Z}$

Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the residue class of n modulo d

If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

11.18 Theorem

Let $f : G \rightarrow H$ be a morphism of groups

- (1) $Im(f)$ is a subgroup of H
- (2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4) f is injective iff $\ker(f) = \{1_G\}$

11.18.1 Proof

- (1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup
- (2) Let x and y be elements of $\ker(f)$.
 One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
 So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
 Let $x \in \ker f, y \in G$.
 One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup
- (3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
 Moreover $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence \tilde{f} is a morphism of groups.
 By definition $Im(\tilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
 Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y$, $x \sim y$ that means $[x] = [y]$
 Therefore \tilde{f} is injective.

- (4) If f is injective, $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f = \{1_G\}$.
 Conversely, suppose that $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$

11.19 Theorem

Let K be a unitary ring and $f : E \rightarrow F$ be a morphism of left K -modules. Then

- (1) $\text{Im}(f)$ is a left-sub- K -module of F
- (2) $\ker(f)$ is a left-sub- K -module of E
- (3) $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$ is a isomorphism of left K -modules
 $[x] \mapsto f(x)$

11.19.1 Proof

- (1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in \text{Im}(f)$
- (2)
- (3)

Chapter 12

Topology

12.1 Def

Let X be a set. We call topology on X any subset \mathcal{J} of $\wp(X)$ that satisfies:

- $\emptyset \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitrary family of elements in \mathcal{J} , then $\bigcup_{i \in I} u_i \in \mathcal{J}$
- If u and v are elements of \mathcal{J} , then $u \cap v \in \mathcal{J}$

12.2 Remark

If $(u_i)_i = 1$ is a finite family of elements of \mathcal{J} , then $\bigcap_{i=1}^n u_i \in \mathcal{J}$ (by induction, this follows from (3))

12.2.1 Example

$\{\emptyset, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

12.3 Def

Let X be a set. We call metric on X any mapping $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$, that satisfies
 $(x, y) \mapsto d(x, y)$

- $d(x, y) = 0$ iff $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

(X, d) is called a metric space

12.3.1 Example

Let X be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

12.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) \leq \epsilon\}$. We call the open ball of radius ϵ centered at x

12.4.1 Example

Consider (\mathbb{R}, d) with $d(x, y) = |x - y|$, then $B(x, \epsilon) =]x - \epsilon, x + \epsilon[$

12.5 Prop.

Let (X, d) be a metric space. Let \mathcal{J}_d be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subseteq U$. Then \mathcal{J}_d is a topology on X

12.5.1 Proof

- $\emptyset \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of \mathcal{J}_d . Let $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq u_i \subseteq U$. Hence $U \in \mathcal{J}_d$
- Let U and V be elements of \mathcal{J}_d . Let $x \in U \cap V, \exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$. Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$. Therefore $U \cap V \in \mathcal{J}_d$

12.6 Def

\mathcal{J}_d is called the topology induced by the metric d

12.7 Def

We call topology space any pair (X, \mathcal{J}) where X is a set and \mathcal{J} is a topology on X

Given a topological space (X, \mathcal{J}) If $U \in \mathcal{J}$ then we say that U is an open subset of X . If $F \in \wp(X)$ such that $X \setminus F \in \mathcal{J}$, then we say that F is closed subset of X

If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that \mathcal{J} is metrizable

12.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact, if d

denote the metric defined as $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$\forall x \in X \quad B(x, 1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{J}_d$

12.8 Axiom of choice

For any set I and any family $(A_i)_{i \in I}$ of non-empty sets , there exists a mapping $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

12.9 Def

Let (X, \leq) be a partially ordered set If $\forall A \subseteq X$ A is non-empty , there exists a least element of A then we say that (X, \leq) is a well ordered set.

12.10 Theorem

For any set X , there exists an order relation \leq on X such that (X, \leq) forms a well ordered set.

12.11 Zorn's lemma

Let (X, \leq) be a partially ordered set . If $\forall A \subseteq X$ that is totally ordered with respect to \leq , there exists an upper bound of A inside X . Then , there exists a maximal element x_0 of X ($\forall y \in X, y > x_0$ does not hold)

12.12 Prop.

Let (X, \leq) be a well ordered set , $y \notin X$. We extends \leq to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

12.13 Proof

Let $A \subseteq X \cup \{y\}$, $A \neq \emptyset$. If $A = \{y\}$ then Y is the least element of A . If $A \neq \{y\}$ then $B = A \setminus \{y\}$ is non-empty. Let b be the least element of B . Since $b < y$ it's also the least element of A

12.14 Def: Initial Segment

Let (X, \leq) be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S(X_{<s} \subseteq S)$, then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

12.15 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X .

12.16 Prop.

Let (X, \leq) be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X , then $\bigcup_{i \in I} S_i$ is an initial segment of X

12.17 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

12.18 Prop.

Let $(X, < \leq)$ be a well erodered set.

- (1) Let S be a proper initial segment of X , $x = \min(X \setminus S)$ Then $S = X_{<x}$
- (2) $X \rightarrow \wp(X)$
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of X forms a well ordered subset of $(\wp(X), \subseteq)$

12.19 Proof

- (1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction.Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \setminus S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

- (2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \setminus X_{<x}$ So $X_{<x} \subsetneq X_{<y}$
- (3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \emptyset$. Then there exists $A \subseteq X$ such that $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \emptyset$ then $\mathcal{F} = \{X\}$, and $\{X\}$ is the least element of \mathcal{F} . Otherwise $A \neq \emptyset$ and A has a least element a . Then by (2) $X_{<a}$ is the least element of \mathcal{F}

12.20 Lemma

Let (X, \leq) be a well ordered set, $f : X \rightarrow X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

12.20.1 Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \emptyset$, let a be the least element of A . By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing. This shows $f(a) \in A$. But a is the least element of A , $f(a) < a$ cannot hold: contradiction.

12.21 Prop

Let (X, \leq) be a well ordered set, S and T be two initial segment of X . If $f : S \rightarrow T$ is a bijection that's strictly increasing, then $S = T, f = Id_S$

12.21.1 Proof

We may assume $T \subseteq S$. Let $l : T \rightarrow S$ be the inclusion mapping and $g = l \circ f : S \rightarrow S$. Since g is strictly increasing, by the lemma, $\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$. Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

12.22 Def

Let (X, \leq) and (Y, \leq) be partially ordered sets. If $\exists f : X \rightarrow Y$ that's increasing and bijective, we say that (X, \leq) and (Y, \leq) are isomorphic

12.23 Def

Let (X, \leq) and (Y, \leq) be well ordered sets. If (X, \leq) is isomorphic to an initial segment of Y . We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y , we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \succ X$

12.24 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

12.24.1 Proof

We construct a correspondence f from X to Y , such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$
By the last proposition of Oct. 11, f is a function.

- If $a, b \in \text{Dom}(f)$, $a < b$, then $X_{<a} \subsetneq X_{<b}$
By definition, $Y_{<f(b)} \sim X_{<b}$ $Y_{<f(a)} \sim X_{<a}$
Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y_{<f(b)}$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_{<f(b)}$. We then get $f(a) < f(b)$. Thus f is strictly increasing.
 - Let $a \in \text{Dom}(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y$ $X_{<x} \sim Y_{<y}$ This shows that $x \in \text{Dom}(f)$. Hence $\text{Dom}(f)$ is an initial segment of X . Applying this to f^{-1} , we get : $\text{Im}(f) = \text{Dom}(f)$ is an initial segment of Y
 - Either $\text{Dom}(f) = X$ or $\text{Im}(f) = Y$.
Assume that $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$ are respectively the least elements of $X \setminus \text{Dom}(f)$ and $Y \setminus \text{Im}(f)$. Then we get $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$.
We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction
 -
- Case 1 $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$ $X \prec Y$
Case 2 $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$ $X \succ Y$
Case 3 $\text{Dom}(f) = X, \text{Im}(f) = Y$ $X \sim Y$

12.25 Lemma

Let (X, \leq) be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B , or B is an initial segment of A .

Let $Y = \bigcup_{A \in \mathfrak{S}} A$. Then (Y, \leq) is a well ordered set, and $\forall A \in \mathfrak{S}, A$ is an initial segment of Y .

12.25.1 Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$. Since $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B. Hence $y \in A$. Contradiction
- Let $Z \subseteq Y, Z \neq \emptyset$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$, then A is an initial segment of B.

Since B is well ordered, if $m \not\leq z$ then $z < m$. Since $m \in A$, we get $z \in A$. Contradiction.

Therefore, m is the least element of Z.

12.26 Theorem(Zorn's lemma)

Let (X, \leq) be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X, the X has a maximal element (a maximal element m of $\{x \mid x > m\} = \emptyset$)

12.26.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega. \exists f(A)$ such that $\forall a \in A, a < f(A)$

Let $\omega = \{\text{well ordered subset of } X\}$. (guaranteed by axiom of choice) Let $f : \omega \rightarrow X$ such that $f(A)$ is an upper bound of $A \in \omega$. If $A \in \omega$ satisfies $\forall a \in A, a < f(A)$, we say that A is a f -set

Let $\mathfrak{S} = \{f\text{-sets}\}$ Note that $\emptyset \in \mathfrak{S}$ If $A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$

In fact, if $a \in A$, then $A_{<a} = (A \cup \{f(A)\})_{<a}$. If $a = f(A) \notin A$, then $(A \cup \{f(A)\})_{<a} = A$

Let A and B be elements of \mathfrak{S} . Let I be the union of all common initial segments of A and B. This is also a common initial segment of A and B.

If $I \neq A$ and $I \neq B$, then $\exists(a, b) \in A \times B, I = A_{<a} = B_{<b}, f(I) = f(A_{<a}) = f(B_{<b})$. Hence $a = b$. Then $I \cup \{a\}$ is also a common initial segment of A and B, contradiction.

By the lemma, $Y := \bigcup_{A \in \mathfrak{S}} A$ is well-ordered, and any $A \in \mathfrak{S}$ is an initial segment of Y.

$\forall a \in Y, \exists A \in \mathfrak{S}, a \in A$. Since A is an initial segment of Y. $A_{<a} = Y_{<a}$. Hence $f(Y_{<a}) = f(A_{<a}) = a$. Hence $y \in \mathfrak{S}$. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However, $Y \cup \{f(Y)\} \in \mathfrak{S}$. Hence $f(y) \in Y$.

If $f(y)$ is not a maximal element of X, $\exists x \in X, f(y) < x$

Chapter 13

Filter

13.1 Def

Let X be a set. We call filter on X any $\mathcal{F} \subseteq \wp(X)$ that satisfies:

- (1) $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2) $\forall A \in \mathcal{F}, \forall B \in \wp(X)$, if $A \subseteq B$, then $B \in \mathcal{F}$
- (3) $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

13.1.1 Example

- (1) Let $Y \subseteq X, Y \neq \emptyset$. $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y .
- (2) Let X be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is finite}\}$$

is a filter called the Frechet filter of X .

- (3) Let (X, \mathcal{J}) be a topological space, $x \in X$. We call neighborhood of x any $V \in \wp(X)$ such that $\exists u \in \mathcal{J}$, satisfying $x \in U \subseteq V$. Then $\mathcal{V} = \{\text{neighborhoods of } x\}$ is a filter.

13.2