

Contents

I	Set	13
1	product	15
1.1	direct sum	15
2	Ring	17
2.1	morphism	17
II	Sequences	19
3	Supremum and infimum	21
4	Interval	23
5	Enhanced real line	25
6	Vector space	27
6.1	K-module	27
6.1.1	Def	27
6.1.2	Remark	27
6.1.3	Notation	28
6.1.4	K-vector space	28
6.1.5	Association:	28
6.1.6	Remark:	29
6.2	sub K-module	29
6.2.1	Def	29
6.2.2	Example	29
6.3	morphism of K-modules	29
6.3.1	Def	29
6.3.2	K-linear mapping	29
6.3.3	Theorem	29
6.3.4	Remark:column	30
6.4	kernel	30
6.4.1	Prop	30
6.4.2	Def	30

6.4.3	Theorem	30
6.4.4	Def	31
6.4.5	Remark	31
6.4.6	Theorem	31
6.4.7	Proof:	31
7	Monotone mappings	33
7.1	Def	33
7.2	Prop.	33
7.3	Def	33
7.4	Prop.	33
7.5	Prop	34
7.6	Def	34
7.7	Prop.	34
7.8	Proof	34
7.8.1	bijection	34
7.8.2	uniqueness	35
8	sequence and series	37
8.1	Def	37
8.2	Remark	37
8.3	Prop	37
8.4	proof	37
8.5	Prop	37
8.6	limit	38
8.6.1	Def	38
8.6.2	Remark	38
8.6.3	Prop	38
8.6.4	Prop	39
8.6.5	Prop	39
8.6.6	Theorem	39
8.6.7	Def	39
8.6.8	Prop	39
8.6.9	Prop	40
8.6.10	Theorem	40
8.6.11	Notation	40
8.6.12	Corollary	40
8.6.13	Notation	40
8.6.14	Theorem: Bolzano-Weierstrass	40
9	Cauchy sequence	43
9.1	Def	43
9.2	Prop	43
9.3	Theorem: Completeness of real number	43
9.4	Absolutely converge	44
9.4.1	Prop	44

10 Comparison and Technics of Computation	45
10.1 Def	45
10.2 Prop.	45
10.3 Theorem	45
10.4 Prop.	46
10.5 Prop.	47
10.6 Theorem	47
10.7 Prop.	47
10.8 Theorem	48
10.9 Remark	48
10.10 Calculates on $O(), o()$	48
10.10.1 Plus	48
10.10.2 Transform	49
10.10.3 Transition	49
10.10.4 Times	49
10.11 On the limit	49
10.12 Prop	49
10.13 Prop	50
10.14 Prop	50
10.15 Theorem: d'Alembert ratio test	50
10.15.1 Lemma	51
10.15.2 (2)	51
10.16 Prop	51
10.16.1 Corollary	52
10.16.2 Corollary	52
10.17 Theorem: Cauchy root test	52

III Axiom of choice 53

11 Preparation	55
11.1 Statement of axiom of choice	55
11.2 Def	55
11.3 Theorem	55
11.4 Zorn's lemma	55
11.5 Prop.	55
11.6 Proof	56
11.7 Def: Initial Segment	56
11.8 Example	56
11.9 Prop.	56
11.10 Proof	56
11.11 Prop.	56
11.12 Proof	56
11.13 Lemma	57
11.14 Prop	57
11.15 Def	57

11.16Def	57
11.17Prop.	58
11.18Lemma	58
12 Zorn's lemma	61
12.1 Proof	61
 IV Topology	 63
13 Absolute value and norms	65
13.1 Def	65
13.2 Notation	65
13.3 Prop	65
13.4 Def	66
14 Quotient Structure	67
14.1 Def	67
14.2 equivalence class	67
14.3 Prop.	67
14.4 Def	68
14.5 Remark	68
14.6 Prop	68
14.7 Notation on Equivalence Class	68
14.8 Proof	69
14.9 Quotient set	69
14.9.1 Example	69
14.10Def	69
14.11Remark	69
14.12Prop	69
14.13Theorem	70
14.14Def	70
14.15Prop	70
14.16Def	71
14.17Theorem	71
14.17.1 Reside Class	72
14.18Theorem	72
14.19Theorem	73
 15 Topology	 75
15.1 Def	75
15.2 Remark	75
15.2.1 Example	75
15.3 Def	75
15.3.1 Example	76
15.4 Def	76

15.4.1 Example	76
15.5 Prop.	76
15.6 Def	76
15.7 Def	77
15.7.1 Example	77
16 Filter	79
16.1 Def	79
16.1.1 Example	79
16.2 Def: Filter Basis	79
16.2.1 Remark	79
16.2.2 Example	80
16.3 Remark	80
16.3.1 Example	80
16.4 Def	80
16.5 Remark	81
16.6 Extra Episode	81
16.7 Prop.	81
17 Limit point and accumulation point	83
17.1 Def	83
17.2 Prop	83
17.3 Def	84
17.4 Def	84
17.5 Prop	84
17.6 Def: dense	84
18 Limit of mappings	85
18.1 Def	85
18.2 Remark	85
18.2.1 Example	85
18.3 Remark	85
18.4 Remark	86
18.5 Prop	86
18.6 Theorem	86
18.7 Prop	87
18.8 Def	87
18.9 Remark	87
18.10 Prop	87
18.11 Theorem	88
18.12 Prop.	89
18.12.1 Proof	89

19 Continuity	91
19.1 Def	91
19.2 Remark	91
19.3 Theorem	91
19.4 Proof	91
19.5 Prop	92
19.6 Def	92
19.7 Prop	92
19.8 Proof	92
19.9 Prop	93
19.10 Def	94
19.11 Remark	94
19.12 Prop	94
19.13 Theorem	96
19.13.1 Proof	96
19.14 Remark	96
19.14.1 Example	97
20 Uniform continuity and convergency	99
20.1 Def	99
20.2 Remark	99
20.3 Prop	99
20.4 Def	100
20.5 Prop	100
20.5.1 Proof	100
20.6 Def	101
20.7 Prop	101
20.7.1 Proof	101
20.8 Def	102
20.9 Theorem	102
20.9.1 Proof	102
20.10 Theorem	103
20.10.1 Proof	103
20.10.2 Def	103
20.11 Remark	103
20.12 Example	103
V Normed Vector Space	105
21 Linear Algebra	107
21.1 Def	107
21.1.1 Notation	107
21.2 Def	107
21.3 Def	108
21.4 Remark	108

21.5 Theorem	108
21.6 Theorem	109
21.7 Corollary	111
21.8 Def	112
21.9 Theorem	112
21.10 Proof	112
21.11 Prop	116
21.11.1 Proof	116
22 Matrices	117
22.1 Def	117
22.1.1 Example	118
22.2 Def	118
22.2.1 Example	118
22.3 Def	118
22.4 Calculate Matrices	119
22.4.1 Remind	119
23 Transpose	121
23.1 Def	121
23.2 Def	122
23.2.1 Example	122
23.3 Prop	123
23.4 Corollary	123
23.5 Remark	124
24 Linear Equation	125
24.1 Def	125
24.2 Prop	126
24.3 Linear Equation	126
24.4 Prop	126
24.5 Prop	127
24.6 Def	127
24.7 Theorem	127
25 Normed Vector Space	129
25.1 Def	129
25.2 Prop	129
25.2.1 Proof	129
25.3 Def	130
25.4 Def: The completion	130
25.5 Theorem	130
25.6 Remark	131
25.7 Prop	131
25.8 Theorem	132

26 Norms	135
26.1 Def	135
26.2 Remark	135
26.3 Def	136
26.4 Prop	136
26.5 Def	137
26.6 Remark	137
26.7 Def	137
26.8 Prop	137
26.9 Def: Operator Seminorm	139
26.10 Prop	139
26.11 Remark	140
26.12 Def	140
26.13 Theorem	140
27 Differentiability	143
27.1 Def	143
27.2 Def	144
27.3 Prop	144
27.4 Example	145
27.4.1	145
27.4.2	145
27.4.3	145
27.4.4	145
27.5 Theorem: Chain rule	146
27.6 Prop	146
27.7 Def	147
27.8 Corollary	147
27.9 Corollary	148
27.10 Corollary	149
27.11 Prop	149
27.12 Corollary	150
27.13 Def: Equivalence of Norms	150
27.14 Prop	150
27.15 Remark	151
27.16 Prop	151
27.17 Theorem	151
27.18 Prop	154
27.19 Theorem	154
28 Compactness	155
28.1 Def: cover	155
28.2 Def: compact	155
28.3 Def	155
28.4 Prop	156
28.5 Theorem	156

28.6 Theorem	157
28.7 Lemma	158
28.8 Prop	159
28.9 Prop	159
28.10 Prop	160
28.11 Prop	160
28.12 Theorem	161
28.13 Def	161
28.14 Theorem	162
28.15 Def	163
28.16 Prop	163
28.17 Theorem	164
29 Mean Value Theorems	167
29.1 Rolle Theorem	167
29.2 Mean value theorem(Lagrange)	167
29.3 Mean value inequality	168
29.4 Theorem	169
29.5 Theorem(Heine)	170
30 Fixed Point Theorem	171
30.1 Def	171
30.2 Def	171
30.3 Fixed Point Theorem	171
VI Higher differentials	173
31 Multilinear mapping	175
31.1 Def	175
31.2 Example	175
31.3 Remark	175
31.4 Prop	176
31.5 Remark	176
32 Operator norm of Multilinear field	177
32.1 Def	177
32.2 Theorem	177
32.3 Corollary	178
32.3.1 Proof	178
33 Higher differentials	181
33.1 Def	181
33.2 Remark	182
33.3 Theorem	182
33.4 Prop(Gronwall inequality)	182

33.5 Theorem	183
33.6 Def	184
33.7 Prop	185
33.8 Theorem	186
34 Permutations	187
34.1 Def	187
34.1.1 Example	187
34.2 Def	187
34.3 Prop	188
34.3.1 Proof	188
34.4 Remark	188
34.5 Theorem	189
34.5.1 Remark	189
34.6 Corollary	189
34.6.1 Remark	190
34.7 Def	190
34.8 Corollary	190
34.8.1 Proof	190
34.9 Caybey Theorem	190
34.10 Theorem	191
34.11 Remark	191
34.12 Exercise	192
34.13 Symmetric of multilinear mapping	192
34.14 Def: Symmetric and Alternating	192
34.15 Prop	192
34.16 Def:	193
34.17 Reminder	193
34.18 Theorem (Schweiz)	193
34.19 Def	194
34.20 Prop	194
34.21 Prop	195
34.21.1 Proof	196
34.22 Prop	196
34.23 Local Inversion Theorem	197
34.23.1 Proof	198
VII Integration	201
35 Integral operators	203
35.1 Prop	203
35.2 Def	204
35.3 Example	204
35.4 Dini's theorem	205
35.5 Def	205

36 Riemann integral	207
36.1 Def	207
36.2 Def	207
36.3 Theorem	207
37 Daniell integral	209
37.1 Prop	209
37.1.1	209
37.1.2	209
37.2 Def	210
37.3 Prop	210
37.4 Corollary	210
37.5 Prop	211
37.6 Def	211
37.7 Prop	211
37.8 Def	212
37.9 Remark	212
37.10 Daniell Theorem	212
37.11 Beppo Levi Theorem	213
37.12 Fatou's Lemma	215
37.13 Lebesgue dominated convergence theorem	215
37.14 Notation	216
38 Semialgebra	217
38.1 Notation	217
38.2 Def	217
38.2.1 Example	217
38.3 Def	218
38.4 Prop	218
38.5 Prop	218
38.5.1 Proof	219
38.5.2 Example	220
38.6 Theorem	220
38.7 Prop	221
39 Limit and Differential of Integrals with Parameters	223
39.1 Theorem	223
39.2 Theorem	224
40 Measure theory	227
40.1 Def	227
40.2 Prop	227
40.3 Def	228
40.4 Example	228
40.5 Def	229
40.6 Prop	229

40.7 Def	230
40.8 Prop	230
40.9 Def	230
40.10 Prop	230
40.11 Corollary	231
40.12 Example	231
40.13 Prop	231
40.14 Example	232
41 Measure	233
41.1 Def	233
41.2 Def	233
41.3 Def	234
41.4 Carathéodory Theorem	234
41.5 Example	234
41.6 Def	234
41.6.1 Particular case	235
41.7 Prop	235
41.8 Corollary	236
42 Fundamental theorem of calculus	237
42.1 Theorem	237
42.2 Corollary	237
42.2.1 Proof	237
43 L^p space	239
43.1 Def	239
43.2 Hölder inequality	239
43.3 Corollary	240

Part I

Set

Chapter 1

product

1.1 direct sum

\oplus is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0 \}$$

Chapter 2

Ring

2.1 morphism

Def

Let A and B be unitary rings. We call morphism of unitary rings from A to B only mapping $A \rightarrow B$ is a morphism of group from $(A, +)$ to $(B, +)$, and a morphism of monoid from (A, \cdot) to (B, \cdot)

Properties

- Let R be a unitary ring. There is a unique morphism from \mathbb{Z} to R
-

algebra

we call k -algebra any pair (R, f) , when R is a unitary ring, and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R , the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of \mathbb{Z} -algebra on R (extra: \mathbb{Z} is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring, (A, f) be a k -algebra. If there is no ambiguity on f , for any $(\lambda, a) \in k \times A$, we denote $f(\lambda)a$ as λa

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite, so $\sum_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def: Let T be a formal symbol. We denote $k^{\mathbb{N}}$ as $k[T]$. If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$, when we denote $k^{\mathbb{N}}$ as $k[T]$ this element is denoted as $\sum_{n \in \mathbb{N}} a_n T^n$. Such

element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient 0
- write T as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$ is written as $1 + 2T + T^2$

Def Remind that $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$, define two composition laws on $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$ form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left(\sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left(\sum_{p+q+l=n} a_p b_q c_l \right) T^n$
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by $F'(T)$ or $\mathcal{D}(F(T))$ the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $\exp(T) \in k[T]$ as $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$, which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence: $(F_i(T))_{i \in \mathbb{N}}$ be a sequence of elements in $k[T]$, and $F(T) \in k[T]$ We say that $(F_i(T))_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$, there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$, $\text{ord}(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 3

Supremum and infimum

Def:

Let (X, \leq) be a partially ordered set A and Y be subsets of X , such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq y\}$ has a least element then we say that A has a Supremum in Y with respect to \leq denoted by $\sup_{(Y, \leq)} A$ this least element and called it the Supremum of A in Y (this respect to \leq)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has an infimum in Y with respect to \leq . We denote by $\inf_{(Y, \leq)} A$ this greatest element and call it the infimum of A in Y
- Observation: $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let (X, \leq) be a partially ordered set, I be a set.

- If f is a function from I to X $\sup f$ denotes the supremum of $f(I)$ is X . $\inf f$ takes the same
- If $(x_i)_{i \in I}$ is a family of element in X , then $\sup x_i$ denotes $\sup\{x_i \mid i \in I\}$ (in X)

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$ We equip \mathbb{R} with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$

Example: For $n \in \mathbb{N}$, let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let (X, \leq) be a partially ordered set, A, Y, Z be subset of X , such that $A \subseteq Z \subseteq Y$

- If $\max A$ exists, then it is also equal to $\sup_{(y, \leq)} A$
- If $\sup_{(y, \leq)} A$ exists and belongs to Z , then it is equal to $\sup A$

\inf takes the same Prop.

Let X, \leq be a partially ordered set, A, B, Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y, \leq)} A$ and $\sup_{(y, \leq)} B$ exists, then $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If $\inf_{(y, \leq)} A$ and $\inf_{(y, \leq)} B$ exists, then $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let (X, \leq) be a partially ordered set, I be a set and $f, g : I \rightarrow X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists, then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists, then $\sup f \leq \sup g$

Chapter 4

Interval

We fix a totally ordered set (X, \leq)

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$, $[a, b]$ denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$, one has $[x, y] \subseteq I$ then we say that I is an interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$. Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an interval in X
- If $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$, $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_\mu$

- If $b \leq x$ $[a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a$ $[a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If $a < x < b$ then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let (X, \leq) be a totally ordered set. I be a non-empty interval of X . If $\sup I$ exists in X , we call $\sup I$ the right endpoint; \inf takes the similar way.

Prop.

Let I be an interval in X .

- Suppose that $b = \sup I$ exists. $\forall x \in I, [x, b] \subseteq I$
- Suppose that $a = \inf I$ exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X . Suppose that I has supremum b and an infimum a in X . Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let (X, \leq) be a totally ordered set. If $\forall (x, z) \in X \times X$, such that $x < z$ $\exists y \in X$ such that $x < y < z$, then we say that (X, \leq) is thick

Prop.

Let (X, \leq) be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a$ $\sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists $x_0 \in]a, b[$ By definition, b is an upper bound of I . If b is not the supremum of I , there exists an upper bound M of I such that $M \neq b$. Since X is thick, there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b] \subseteq I, a, b \in I$ Hence M and M' belong to I , which conflicts with the uniqueness of supremum.

Chapter 5

Enhanced real line

Def:

Let $+\infty$ and $-\infty$ be two symbols that are different and don't belong to \mathbb{R} . We extend the usual total order \leq on \mathbb{R} to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ becomes a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$. Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty[\quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in]-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in]0, +\infty[\quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in]-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
ARE NOT DEFINED

Def

Let (X, \leq) be a partially ordered set. If for any subset A of X , A has a supremum and an infimum in X , then we say that X is order complete.

Example

Let Ω be a set. $(\mathcal{P}(\Omega), \subseteq)$ is order complete. If \mathcal{F} is a subset of $\mathcal{P}(\Omega)$, $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$.

Interesting tip: $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

Axiom:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete.

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ and $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \setminus \{0\}$, λA denotes $\{\lambda a \mid a \in A\}$
- $-A$ denotes $(-1)A$

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$, $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ Def

We denote by (\mathbb{R}, \leq) a field \mathbb{R} equipped with a total order \leq , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, one has $\forall c \in \mathbb{R}$, $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, $ab > 0$
- $\forall A \subseteq \mathbb{R}$, if A has an upper bound in \mathbb{R} , then it has a supremum in \mathbb{R}

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

\inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$ then $c = a + b$

\inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set, $f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

\inf takes the similar ($\leq \rightarrow \geq$) (provided when the sum are defined)

Chapter 6

Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

6.1 K -module

6.1.1 Def

Let $(V, +)$ be a commutative group. We call left/right K -module structure: any mapping $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group $(V, +)$ equipped with a left/right K -module structure is called a left/right K -module.

6.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
 Any left K^{op} - module is a right K -module
 Any right K^{op} - module is a left K -module
 $(K^{op})^{op} = K$

6.1.3 Notation

When we talk about a left/right K -module $(V, +)$, we often write its left K -module structure as $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

6.1.4 K -vector space

If K is commutative, then $K^{op} = K$, so left K -module and right K -module structure are the same. We simply call them K -module structure. A commutative group equipped with a K -module structure is called a K -module. If K is a field, a K -module is also called a K -vector space

Let $\Phi : K \times V \rightarrow V$ be a left or right K -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$ is a morphism of groups. Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$ (*is a var*)

6.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K -module structure

The mapping $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K -module structure

6.1.6 Remark:

We can also write an element μ of K^I in the form of a family $(\mu_i)_{i \in I}$ of elements in K (μ_i is the image of $i \in I$ by μ)
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

6.2 sub K-module**6.2.1 Def**

Let V be a left/right K -module. If W is a subgroup of V . Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub- K -module of V .

6.2.2 Example

Let I be a set. Let $K^{\oplus I}$ be the subset of K^I composed of mappings $f : I \rightarrow K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub- K -module of K^I

In fact, $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$
Hence $f - g \in K^{\oplus I}$ So $K^{\oplus I}$ is a subgroup of K^I
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

6.3 morphism of K-modules**6.3.1 Def**

Let V and W be left K -module, A morphism of groups $\phi : V \rightarrow W$ is called a morphism of left K -modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

6.3.2 K-linear mapping

If K is commutative, a morphism of K -modules is also called a K -linear mapping. We denote by $\text{hom}_{K\text{-Mod}}(V, W)$ the set of all morphism of left- K -module from V to W . This is a subgroup of W^V

6.3.3 Theorem

Let V be a left K -module. Let I be a set.
The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$ is a bijection where
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

6.3.4 Remark:column

In the case where $I = 1, 2, 3, \dots, n$ V^I is denoted as V^n , K^I is denoted as K^n . For any $(x_1, \dots, x_n) \in V^n$, by the theorem, there exists a unique morphism of left K -modules $\phi : K^n \rightarrow V$ such that $\forall i \in 1, \dots, n, \phi(e_i) = x_i$.

We write this ϕ as a column $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$. It sends $(a_1, \dots, a_n) \in K^n$ to $a_1x_1 + \dots + a_nx_n$.

6.4 kernel

6.4.1 Prop

Let G and H be groups and $f : G \rightarrow H$ be a morphism of groups

- $Im(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

6.4.2 Def

$\ker(f)$ is called the kernel of f

6.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

Proof

Let e_G and e_H be neutral element of G and H respectively

- (1) Let x and y be element of G
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H
- (2) Let x and y be element of $\ker(f)$. One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$. So $\ker(f)$ is a subgroup of G .
- (3) Suppose that f is injection.
 Since $f(e_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$. Suppose that $\ker(f) = \{e_G\}$. If $f(x) = f(y)$ then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$.
 Hence $xy^{-1} = e_G \Rightarrow x = y$

6.4.4 Def

Let $(V, +)$ be a commutative group, I be a set. We define a composition law $+$ on V^I as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then V^I forms a commutative group

6.4.5 Remark

Let E and F be left K -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$ is a subgroup of F^E

In fact f and g are elements of $\text{hom}_{K\text{-Mod}}(E, F)$, then $f - g$ is also a morphism of left K -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

6.4.6 Theorem

Let V be a left K -module, I be a set The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$ is an isomorphism of groups, where $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

6.4.7 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So Ψ is a morphism of groups

injectivity Let $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i \in I} \in V^I$ We define $\phi_x : K^{\oplus I} \rightarrow V$ such that $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left K -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that K' is a unitary ring, and V is also equipped with a right K' -module structure, Then $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub- k' -module, and Ψ in the theorem is a right K' -module isomorphism

Chapter 7

Monotone mappings

7.1 Def

Let I and X be partially ordered sets, $f : I \rightarrow X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)$, then we say that f is increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

7.2 Prop.

Let X, Y, Z be partially ordered sets. $f : X \rightarrow Y, g : Y \rightarrow Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

7.3 Def

Let f be a function from a partially ordered set I to another partially ordered set X . If $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

7.4 Prop.

Let I and X be partially ordered sets. f be function from I to X .

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

7.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B

If f is increasing/decreasing, then so is f^{-1}

7.6 Def

Let X and Y be partially ordered sets. $f : X \rightarrow Y$ be a bijection. If both f and f^{-1} are increasing, then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \rightarrow Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

7.7 Prop.

Let I be a subset of \mathbb{N} which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \rightarrow I$

7.8 Proof

7.8.1 bijection

We construct $f : \mathbb{N} \rightarrow I$ by induction as follows.

Let $f(0) = \min I$ Suppose that $f(0), \dots, f(n)$ are constructed

then we take $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$. Therefore $f(n) \leq f(n+1)$

Since $f(n+1) \notin \{f(0), \dots, f(n)\}$, we have $f(n) < f(n+1)$

Hence f is strictly increasing and this is injective

If f is not surjective, then $I \setminus \text{Im}(f)$ has a element N .

Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.

Since $N \notin \text{Im}(f)$, $N < f(m)$.

So $m \neq 0$. Hence $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition, $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$,

Hence $f(m) \leq N$, causing contradiction.

7.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from \mathbb{N} to \mathbb{N}

Chapter 8

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

8.1 Def

Let X be a set. We call sequence in X parametrized by I a mapping from I to X .

8.2 Remark

If K is a unitary ring and E is a left K -module then the set of sequence E^I admits a left- K -module structure. If $x = (x_n)_{n \in I}$ is a sequence in E , we define a sequence $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$, called the series associated with the sequence x .

8.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$ is a morphism of left- K -module

8.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

8.5 Prop

Let I be a totally ordered set . X be a partially ordered set, $f : I \rightarrow X$ be a mapping , $J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

8.6 limit

8.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset. $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left(\inf_{i \in I, i \geq n} x_i \right)$$

If $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$, we then say that $(x_n)_{n \in I}$ tends to l and that l is the limit of $(x_n)_{n \in I}$. If in addition $(x_n)_{n \in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$, we say that $(x_n)_{n \in I}$ converges to l

8.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} = \inf_{n \in J} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left(\inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i \in I}$ the limit superior and the limit inferior do not change.

In fact, if we take $J = \mathbb{N} \setminus \{0, \dots, m\}$, then $\inf_{n \in J}(\dots)$ and $\sup_{n \in J}(\dots)$ only depends on the values of $x_i, i \in I, i \geq m$

8.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

8.6.4 Prop

Let $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

8.6.5 Prop

Let $(x_n)_{n \in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$, one has $x_n \leq y_n$. Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

8.6.6 Theorem

Let $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$ and $(z_n)_{n \in I}$ tend to the same limit l

Then $(y_n)_{n \in I}$ tends to l

8.6.7 Def

Let I be an infinite subset of \mathbb{N} , and $(x_n)_{n \in I}$ be a sequence in some set X . We call subsequence of $(x_n)_{n \in I}$ a sequence of the form $(x_n)_{n \in J}$, where J is an infinite subset of I

8.6.8 Prop

Let I and J be infinite subset of \mathbb{N} such that $J \subseteq I$. $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$, one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if $(x_n)_{n \in I}$ tends to $l \in [-\infty, +\infty]$, then $(x_n)_{n \in J}$ tends to l

8.6.9 Prop

$\forall n \in \mathbb{N}$, one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

8.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing, then $(x_n)_{n \in I}$ tends to $\sup_{n \in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing, then $(x_n)_{n \in I}$ tends to $\inf_{n \in I} x_n$

8.6.11 Notation

If a sequence $(x_n)_{n \in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim_{n \in I, n \rightarrow} x_n$ denotes this limit l

8.6.12 Corollary

Let $(x_n)_{n \in I}$ be a sequence in $\mathbb{N}_{\geq 0}$. Then the series $\sum_{n \in I} x_n$ (the sequence $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$. It converges in \mathbb{R} iff it is bounded from above (namely has an upper bound in \mathbb{R})

8.6.13 Notation

If a series $\sum_{n \in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum_{n \in I} x_n$ to denote the limit

8.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\limsup_{n \in I, n \rightarrow +\infty} x_n$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\liminf_{n \in I, n \rightarrow +\infty} x_n$.

Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_n)_{n \in J}$ is decreasing so it tends to $\inf_{n \in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i \in I, i \geq n} x_i$ so $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$. Denote by $l = \sup_{n \in I, n \geq n_0} x_n$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i \in I, i \geq n_0} x_i \leq l$. If the strict inequality $\sup_{i \in I, i \geq N} x_i < l$ holds, then $\sup_{i \in I, i \geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i \in I, i \geq N} x_i$. We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality.

Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$. Thus $n \in J$ that contradicts the maximality of n_0 .

Therefore

$$l = \sup_{i \in I, i \geq N} x_i$$

, which leads to

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = l$$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$, so $x_m < l$ (since otherwise $x_m = \sup_{i \in I, i \geq m} x_i$ and hence $m \in J$). Hence, \forall finite subset I' of $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$ and hence $\exists n \in I$, such that $n > \max I'$, and $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j \in \mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \rightarrow I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence $(x_{n_j})_{j \in \mathbb{N}}$ is increasing

And

$$\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n = l$$

$$l = \sup_{n \in I, n \geq n_0} x_n$$

So $(x_{n_j})_{j \in \mathbb{N}}$ tends to l

Chapter 9

Cauchy sequence

9.1 Def

Let $(x_n)_{n \in I}$ be a sequence in \mathbb{R}
If $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$ then
we say that $(x_n)_{n \in I}$ is a Cauchy sequence

9.2 Prop

- If $(x_n)_{i \in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_n)_{i \in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite, $(x_n)_{n \in I}$ is a Cauchy sequence.
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n \in I}$ converges to l , then $(x_n)_{n \in J}$ converges to l too.

9.3 Theorem: Completeness of real number

If $(x_n)_{n \in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in \mathbb{R}

Proof

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$. So $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n \in I}$ converges to $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n \in I}$ converges to the same limit.

9.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

9.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in \mathbb{R}

Chapter 10

Comparison and Technics of Computation

10.1 Def

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be sequence in \mathbb{R}

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$ then we write $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$, then we write $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

10.2 Prop.

Let I and X be partially ordered sets and $f : I \rightarrow X$ be an increasing/decreasing mapping. Let J be a subset of I . Assume that any elements of I has an upper bound in J . Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

10.3 Theorem

Let I be a totally ordered set, $f : I \rightarrow [-\infty, +\infty]$ and $g : I \rightarrow [-\infty, +\infty]$ be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

Proof

We can assume f and g increasing. Let $a = \sup f(I), b = \sup g(I)$

Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip A with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

h is increasing.

Let $(x, y) \in A$. Assume that $x \leq y$

If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$

If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$

So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined, $(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if $x \geq y$, (x, y) has also an upper bound in B . Therefore: $\sup h(A) = \sup h(B)$

10.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n \in I, n \geq N} x_n$ $B_N = \sup_{n \in I, n \geq N} y_n$. $(A_N)_{N \in \mathbb{N}}$ and $(B_N)_{N \in \mathbb{N}}$ are decreasing, and $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$ $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

10.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + \left(\limsup_{n \in I, n \rightarrow +\infty} y_n \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + \left(\liminf_{n \in I, n \rightarrow +\infty} y_n \right)$$

Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with ∞

10.6 Theorem

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n \in I}$ converges to some $l \in \mathbb{R}$.
Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

10.7 Prop.

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$.
Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

Proof

About the first inequality. Since $\max\{x_n, y_n\} \geq x_n$ and $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite

Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

10.8 Theorem

Let $(a_n)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1) \Rightarrow (2):

If $(a_n)_{n \in I}$ converges to l , then $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2) \Rightarrow (1):

If $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$, then $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore: $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

10.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in \mathbb{R} , $l \in \mathbb{R}$

The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \rightarrow +\infty$

10.10 Calculates on $O()$, $o()$

10.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$

10.10.2 Transform

Let $(a_n)_{n \in I}$ and $(b_n)_{n \in I}$ be two sequence in \mathbb{R} If $a_n = o(b_n), n \in I, n \rightarrow +\infty$, then $a_n = O(b_n), n \in I, n \rightarrow +\infty$

10.10.3 Transition

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ and $(c_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$

10.10.4 Times

Let $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$ be sequences in \mathbb{R}

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

10.11 On the limit

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be elements of \mathbb{R}^I that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n \in I}$ converges to $l + l'$
- $(a_n b_n)_{n \in I}$ converges to ll'

10.12 Prop

Let $a \in \mathbb{R}$ then $a^n = o(n!)$ $n \rightarrow +\infty$

Proof

Let $N \in \mathbb{N}$ such that $|a| < N$
For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^n - N$$

And $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$. Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

10.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

10.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of \mathbb{R}^I . If the series $\sum_{n \in I} b_n$ converges absolutely and if $a_n = O(b_n) \quad n \rightarrow +\infty$ Then $\sum_{n \in I} a_n$ converges absolutely

Proof

By definition $\sum_{n \in I} |b_n| < +\infty$. If $|a_n| \leq M|b_n|$ for $n \in I, n \geq N$ where $N \in \mathbb{N}$. Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

10.15 Theorem: d'Alembert ratio test

Let $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

Proof**(1)**

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, α isn't a lower bound of $\left(\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$
 So $\exists N \in \mathbb{N}$ such that $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$ Hence for $n \geq N$ $|a_n| \leq \alpha^{n-N} |a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$, $\sum_{n \in \mathbb{N}} a_n$ converge absolutely.

10.15.1 Lemma

If a series $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n \rightarrow +\infty} a_n = 0$

Proof

If $\left(\sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$ converges to some $l \in \mathbb{R}$, then $\left(\sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$ converges to l ,
 too. Hence $\left(a_n = \left(\sum_{i=0}^n a_i \right) - \left(\sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$ converges to $l - l = 0$

10.15.2 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$
 Hence $(|a_n|)_{n \in \mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N} |a_N|$
 By the lemma: $\sum_{n \in \mathbb{N}} a_n$ diverges.

10.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \rightarrow +\infty$

Proof

Let $\epsilon > 0$ such that $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

10.16.1 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \rightarrow +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$ Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

10.16.2 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

10.17 Theorem: Cauchy root test

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} . Let $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.
- If $\alpha > 1$ then $\sum_{n \in \mathbb{N}} a_n$ diverges

Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_n|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n \in \mathbb{N}}$ cannot converge to 0.

Part III

Axiom of choice

Chapter 11

Preparation

11.1 Statement of axiom of choice

For any set I and any family $(A_i)_{i \in I}$ of non-empty sets, there exists a mapping $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

11.2 Def

Let (X, \leq) be a partially ordered set. If $\forall A \subseteq X$ A is non-empty, there exists a least element of A then we say that (X, \leq) is a well ordered set.

11.3 Theorem

For any set X , there exists an order relation \leq on X such that (X, \leq) forms a well ordered set.

11.4 Zorn's lemma

Let (X, \leq) be a partially ordered set. If $\forall A \subseteq X$ that is totally ordered with respect to \leq , there exists an upper bound of A inside X . Then, there exists a maximal element x_0 of X ($\forall y \in X, y > x_0$ does not hold)

11.5 Prop.

Let (X, \leq) be a well ordered set, $y \notin X$. We extend \leq to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

11.6 Proof

Let $A \subseteq X \cup \{y\}$, $A \neq \emptyset$. If $A = \{y\}$ then Y is the least element of A . If $A \neq \{y\}$ then $B = A \setminus \{y\}$ is non-empty. Let b be the least element of B . Since $b < y$ it's also the least element of A

11.7 Def: Initial Segment

Let (X, \leq) be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S$ ($X_{<s} \subseteq S$), then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X .

11.9 Prop.

Let (X, \leq) be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X , then $\bigcup_{i \in I} S_i$ is an initial segment of X

11.10 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

11.11 Prop.

Let (X, \leq) be a well ordered set.

- (1) Let S be a proper initial segment of X , $x = \min(X \setminus S)$ Then $S = X_{<x}$
- (2) $X \rightarrow \wp(X)$
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of X forms a well ordered subset of $(\wp(X), \subseteq)$

11.12 Proof

- (1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction. Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \setminus S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

- (2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \setminus X_{<x}$ So $X_{<x} \subsetneq X_{<y}$
- (3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \emptyset$. Then there exists $A \subseteq X$ such that $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \emptyset$ then $\mathcal{F} = \{X\}$, and $\{X\}$ is the least element of \mathcal{F} . Otherwise $A \neq \emptyset$ and A has a least element a . Then by (2) $X_{<a}$ is the least element of \mathcal{F}

11.13 Lemma

Let (X, \leq) be a well ordered set, $f : X \rightarrow X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \emptyset$, let a be the least element of A . By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing. This shows $f(a) \in A$. But a is the least element of A , $f(a) < a$ cannot hold: contradiction.

11.14 Prop

Let (X, \leq) be a well ordered set, S and T be two initial segment of X . If $f : S \rightarrow T$ is a bijection that's strictly increasing, then $S = T, f = Id_S$

Proof

We may assume $T \subseteq S$. Let $l : T \rightarrow S$ be the inclusion mapping and $g = l \circ f : S \rightarrow S$. Since g is strictly increasing, by the lemma, $\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$. Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

11.15 Def

Let (X, \leq) and (Y, \leq) be partially ordered sets. If $\exists f : X \rightarrow Y$ that's increasing and bijective, we say that (X, \leq) and (Y, \leq) are isomorphic

11.16 Def

Let (X, \leq) and (Y, \leq) be well ordered sets. If (X, \leq) is isomorphic to an initial segment of Y . We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y , we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \succ X$

11.17 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

Proof

We construct a correspondence f from X to Y , such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$
By the last proposition of Oct. 11, f is a function.

- If $a, b \in \text{Dom}(f)$, $a < b$, then $X_{<a} \subsetneq X_{<b}$
By definition, $Y_{<f(b)} \sim X_{<b}$ $Y_{<f(a)} \sim X_{<a}$
Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y_{<f(b)}$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_{<f(b)}$. We then get $f(a) < f(b)$. Thus f is strictly increasing.
 - Let $a \in \text{Dom}(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y$ $X_{<x} \sim Y_{<y}$ This shows that $x \in \text{Dom}(f)$. Hence $\text{Dom}(f)$ is an initial segment of X . Applying this to f^{-1} , we get : $\text{Im}(f) = \text{Dom}(f)$ is an initial segment of Y
 - Either $\text{Dom}(f) = X$ or $\text{Im}(f) = Y$.
Assume that $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$ are respectively the least elements of $X \setminus \text{Dom}(f)$ and $Y \setminus \text{Im}(f)$.
Then we get $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$.
We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction
 -
- Case 1 $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$ $X \prec Y$
Case 2 $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$ $X \succ Y$
Case 3 $\text{Dom}(f) = X, \text{Im}(f) = Y$ $X \sim Y$

11.18 Lemma

Let (X, \leq) be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B , or B is an initial segment of A .

Let $Y = \bigcup_{A \in \mathfrak{S}} A$. Then (Y, \leq) is a well ordered set, and $\forall A \in \mathfrak{S}, A$ is an initial segment of Y .

Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$. Since $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B . Hence $y \in A$. Contradiction
- Let $Z \subseteq Y, Z \neq \emptyset$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$, then A is an initial segment of B .

Since B is well ordered, if $m \not\leq z$ then $z < m$. Since $m \in A$, we get $z \in A$. Contradiction.

Therefore, m is the least element of Z .

Chapter 12

Zorn's lemma

Let (X, \leq) be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X , the X has a maximal element (a maximal element m of $\{x \mid x > m\} = \emptyset$)

12.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega. \exists f(A)$ such that $\forall a \in A, a < f(A)$

Let

$$\omega = \{\text{well ordered subset of } X\}$$

. (guaranteed by axiom of choice)

Let $f : \omega \rightarrow X$ such that $f(A)$ is an upper bound of $A \in \omega$.

If $A \in \omega$ satisfies

$$\forall a \in A, a = f(A_{<a})$$

, we say that A is a f -set

Let

$$\mathfrak{S} = \{f\text{-sets}\}$$

Note that

$$\emptyset \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$$

In fact, if $a \in A$, then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If $a = f(A) \notin A$ then

$$(A \cup \{f(A)\})_{<a} = A$$

Let A and B be elements of \mathfrak{S} . Let I be the union of all common initial segments of A and B . This is also a common initial segment of A and B .

If $I \neq A$ and $I \neq B$, then

$$\exists(a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then $I \cup \{a\}$ is also a common initial segment of A and B , contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and $\forall A \in \mathfrak{S}$ is an initial segment of Y .

Since A is an initial segment of Y

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A \quad A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If $f(y)$ is not a maximal element of X

$$\exists x \in X, f(y) < x$$

Part IV

Topology

Chapter 13

Absolute value and norms

13.1 Def

Let K be a field. By absolute value on K , we mean a mapping $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- (1) $\forall a \in K \quad |a| = 0$ iff $a = 0$
- (2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$ (triangle inequality)

13.2 Notation

\mathbb{Q} Take a prime num $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$ there exists a integer $ord_p(\alpha) \frac{a}{b}$, where
 $a \in \mathbb{Z} \setminus \{0\}$
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on \mathbb{Q}

Proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd$$

$$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d}$$

Assume $ord_p(\alpha) \geq ord_p(\beta)$

$$\alpha + \beta$$

$$= p^{ord_p(\beta)} \left(p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right)$$

$$= p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd$$

So

$$ord_p(\alpha + \beta) \geq ord_p(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$$

$$\text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha + \beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$$

$$\max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

13.4 Def

Let K be a field and $|\cdot|$ be an absolute value. We call $(K, |\cdot|)$ a valued field.

Chapter 14

Quotient Structure

14.1 Def

Let X be a set and \sim be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that \sim is an equivalence relation

14.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X . Let X/\sim be the set $\{[x] \mid x \in X\}$

14.3 Prop.

Let X be a set and \sim be an equivalence relation on X

- (1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$
- (2) If α and β are elements of X/\sim such that $\alpha \neq \beta$ then $\alpha \cap \beta = \emptyset$
- (3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

Proof

- (1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ one has $x \sim y$. Therefore, $x \sim z$ namely $z \in [x]$. This proves $[y] \subseteq [x]$. Moreover, since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$.
- (2) Suppose that $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$.
By (1), $\alpha = [y], \beta = [y]$. Thus leads to a contradiction.
- (3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup_{\alpha \in X/\sim} \alpha$. Hence $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$. Conversely,
 $\forall \alpha \in X/\sim, \alpha$ is a subset of X . Hence $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$. Then $X = \bigcup_{\alpha \in X/\sim} \alpha$.

14.4 Def

Let G be a group and X be a set.
We call left/right action of G on X an mapping $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

14.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X .

14.6 Prop

Let G be a group and X be a set. Assume given a left action of G on X . Then the binary relation \sim on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

14.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim \forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

14.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$. ($y \sim x$)
- $\forall (x, y, z) \in X^3$, if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

14.9 Quotient set

Let X be a set and \sim be an equivalence relation, the mapping $X \rightarrow X/\sim$:
 $(x \in X) \mapsto [x]$ is called the projection mapping.

X/\sim is called the quotient set of X by equivalence relation \sim

14.9.1 Example

Let G be a group and H be a subgroup of G . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of H on G . Thus we obtain two quotient sets H/G and G/H

14.10 Def

Let G be a group and H be a subgroup of G . If $\forall g \in G, h \in H \quad ghg^{-1} \in H$,
 Then we say that H is a normal subgroup of G

14.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G . In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

14.12 Prop

If G is commutative, any subgroup of G is normal

14.13 Theorem

Let G be a group and H be a normal subgroup of G . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$. Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

14.14 Def

Let K be a unitary ring and E be a left K -module. We say that a subgroup F of $(E, +)$ is a left sub- K -module of E if $\forall (a, x) \in K \times F, ax \in F$

14.15 Prop

Let K be a unitary ring, E be a left K -module and F be a sub- K -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- K -module structure on E/F . Moreover, the projection mapping $\pi : E \rightarrow E/F$ is a morphism of left- K -modules

Proof

Let x and x' be elements of E such that $[x] = [x']$, that means: $x' - x \in F$
Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K$ $\pi(ax) = [ax] = a[x] = a\pi(x)$

14.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I$ (I is a left and right sub- K -module of A)

14.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$
Therefore $a'b' \sim ab$

14.17.1 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of \mathbb{Z}
 If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z}$ $adm = dma \in d\mathbb{Z}$

Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the residue class of n modulo d

If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

14.18 Theorem

Let $f : G \rightarrow H$ be a morphism of groups

- (1) $Im(f)$ is a subgroup of H
- (2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4) f is injective iff $\ker(f) = \{1_G\}$

Proof

- (1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup
- (2) Let x and y be elements of $\ker(f)$.
 One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
 So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
 Let $x \in \ker f, y \in G$.
 One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup
- (3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
 Moreover $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence \tilde{f} is a morphism of groups.
 By definition $Im(\tilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
 Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y$, $x \sim y$ that means $[x] = [y]$
 Therefore \tilde{f} is injective.

- (4) If f is injective, $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f = \{1_G\}$.
 Conversely, suppose that $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$.

14.19 Theorem

Let K be a unitary ring and $f : E \rightarrow F$ be a morphism of left K -modules. Then

- (1) $\text{Im}(f)$ is a left-sub- K -module of F
- (2) $\ker(f)$ is a left-sub- K -module of E
- (3) $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$ is a isomorphism of left K -modules
 $[x] \mapsto f(x)$

Proof

- (1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in \text{Im}(f)$
- (2)
- (3)

Chapter 15

Topology

15.1 Def

Let X be a set. We call topology on X any subset \mathcal{J} of $\wp(X)$ that satisfies:

- $\emptyset \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitrary family of elements in \mathcal{J} , then $\bigcup_{i \in I} u_i \in \mathcal{J}$
- If u and v are elements of \mathcal{J} , then $u \cap v \in \mathcal{J}$

15.2 Remark

If $(u_i)_{i=1}^n$ is a finite family of elements of \mathcal{J} , then $\bigcap_{i=1}^n u_i \in \mathcal{J}$ (by induction, this follows from (3))

15.2.1 Example

$\{\emptyset, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

15.3 Def

Let X be a set. We call metric on X any mapping $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$, that satisfies
 $(x, y) \mapsto d(x, y)$

- $d(x, y) = 0$ iff $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

(X, d) is called a metric space

15.3.1 Example

Let X be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

15.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ We call the open ball of radius ϵ centered at x

15.4.1 Example

Consider (\mathbb{R}, d) with $d(x, y) = |x - y|$, then $B(x, \epsilon) =]x - \epsilon, x + \epsilon[$

15.5 Prop.

Let (X, d) be a metric space. let \mathcal{J}_d be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x, \epsilon) \subseteq U$ Then \mathcal{J}_d is a topology on X

Proof

- $\emptyset \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of \mathcal{J}_d Let $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of \mathcal{J}_d Let $x \in U \cap V \exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$ Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

15.6 Def

\mathcal{J}_d is called the topology induced by the metric d

15.7 Def

We call topology space any pair (X, \mathcal{J}) where X is a set and \mathcal{J} is a topology on X

Given a topological space (X, \mathcal{J}) If $U \in \mathcal{J}$ then we say that U is an open subset of X . If $F \in \wp(X)$ such that $X \setminus F \in \mathcal{J}$, then we say that F is closed subset of X

If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that \mathcal{J} is metrizable

15.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact, if d denote the metric defined as $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
 $\forall x \in X \quad B(x, 1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{J}_d$

Chapter 16

Filter

16.1 Def

Let X be a set. We call filter on X any $\mathcal{F} \subseteq \wp(X)$ that satisfies:

- (1) $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2) $\forall A \in \mathcal{F}, \forall B \in \wp(X), \text{ if } A \subseteq B, \text{ then } B \in \mathcal{F}$
- (3) $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

16.1.1 Example

- (1) Let $Y \subseteq X, Y \neq \emptyset$. $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y .
- (2) Let X be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is infinite}\}$$

is a filter called the Fréchet filter of X .

- (3) Let (X, \mathcal{J}) be a topological space, $x \in X$. We call neighborhood of x any $V \in \wp(X)$ such that $\exists u \in \mathcal{J}$, satisfying $x \in U \subseteq V$. Then $\mathcal{V} = \{\text{neighborhoods of } x\}$ is a filter.

16.2 Def: Filter Basis

Let X be a set. $\mathcal{B} \subseteq \wp(X)$. If $\emptyset \notin \mathcal{B}$ and $\forall (B_1, B_2) \in \mathcal{B}^2, \exists B \in \mathcal{B}$, such that $B \subseteq B_1 \cap B_2$. We say that \mathcal{B} is a filter basis.

16.2.1 Remark

If \mathcal{B} is a filter basis, then $\mathcal{F}(\mathcal{B}) = \{A \subseteq X \mid \exists B \in \mathcal{B} \quad B \subseteq A\}$ is a filter

Proof

$\emptyset \notin \mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{B}) \neq \emptyset$ since $0 \neq B \subseteq \mathcal{F}(\mathcal{B})$. If $A \in \mathcal{F}(\mathcal{B}), A' \in \wp(X)$ such that $A \subseteq A'$, then $\exists B \in \mathcal{B}$ such that $B \subseteq A \subseteq A'$. Hence $A' \in \mathcal{F}(\mathcal{B})$. If $A_1, A_2 \in \mathcal{F}(\mathcal{B})$, then $\exists(B_1, B_2) \in \mathcal{B}^2$ such that $B_1 \subseteq A_1, B_2 \subseteq A_2$. Since \mathcal{B} is a filter basis, $\exists B \in \mathcal{B}$ such that $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$. Hence $A_1 \cap A_2 \in \mathcal{F}(\mathcal{B})$.

16.2.2 Example

- Let $Y \subseteq X, Y \neq \emptyset$
 $\mathcal{B} = \{Y\}$ is a filter basis. $\mathcal{F}(\mathcal{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let (X, \mathcal{J}) be a topological space $x \in X$. If \mathcal{B}_x is a filter basis such that $\mathcal{F}(\mathcal{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$, then we say that \mathcal{B}_x is a neighborhood basis of x .

16.3 Remark

Let \mathcal{B}_x is a neighborhood basis of x iff

- $\mathcal{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathcal{B}_x$ such that $U \subseteq V$
- Let (X, d) be a metric space, $x \in X \forall \epsilon > 0$, Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Then

- $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x
- $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x

16.3.1 Example

$\mathcal{V}_x \cap \mathcal{J}$ is a neighborhood basis of x

16.4 Def

$V \in \wp(X)$ is called a neighborhood of x if $\exists U \in \mathcal{J}$ such that $x \in U \subseteq V$

16.5 Remark

Let (X, \mathcal{J}) be a topological space, $x \in X$ and \mathcal{B}_x a neighborhood basis of x . Suppose that \mathcal{B} is countable. We choose a surjective mapping $(B_n)_{n \in \mathbb{N}}$ from \mathbb{N} to \mathcal{B}_x . For any $n \in \mathbb{N}$, let $A_n = B_0 \cap B_1 \cap \dots \cap B_n \in \mathcal{V}_x$. The sequence $(A_n)_{n \in \mathbb{N}}$ is decreasing and $\{A_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of x .

16.6 Extra Episode

$\wp(\mathbb{N})$ is NOT countable

Suppose that $f : \wp(\mathbb{N}) \rightarrow \mathbb{N}$ is injective. Then $\exists g : \mathbb{N} \rightarrow \wp(\mathbb{N})$ surjective. Taking $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$. Since g is surjective, $\exists a \in \mathbb{N}$ such that $A = g(a)$.

If $a \in A$, then $a \in g(a)$, hence $a \notin A$

If $a \notin A$, then $a \in g(a) = A$

Contradiction

16.7 Prop.

Let Y and E be sets, $g : Y \rightarrow E$ be a mapping,

- If \mathcal{F} is a filter of Y , then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on E

- If \mathcal{B} is a filter basis of Y , then

$$g(\mathcal{B}) := \{g(B) \mid B \in \mathcal{B}\}$$

is a filter of E , and $\mathcal{F}(g(\mathcal{B})) = g_*(\mathcal{F}(\mathcal{B}))$

Proof

- (1) $E \in g_*(\mathcal{F})$ since $g^{-1}(E) = Y$
 $\emptyset \notin g_*(\mathcal{F})$ since $g^{-1}(\emptyset) = \emptyset$

If $A \in g_*(\mathcal{F})$ and $A' \supseteq A$, then $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{F}$, so $g^{-1}(A') \in \mathcal{F}$,
Hence $A' \in g_*(\mathcal{F})$

If $A_1, A_2 \in g_*(\mathcal{F})$. Then $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$. Hence $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$. So $A_1 \cap A_2 \in g_*(\mathcal{F})$.

- (2) Since g is a mapping, and $\emptyset \notin \mathcal{B}$, we get $\emptyset \notin g(\mathcal{B})$, since $\mathcal{B} \neq \emptyset, g(\mathcal{B}) \neq \emptyset$.

Let $B_1, B_2 \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq B_1 \cap B_2$. Hence $g(C) \subseteq g(B_1) \cap g(B_2)$, namely $g(\mathcal{B})$ is a filter basis.

Chapter 17

Limit point and accumulation point

We fix a topological space (X, \mathcal{T})

17.1 Def

Let \mathcal{F} be a filter of X and $x \in X$

- If $\mathcal{V}_x \subseteq \mathcal{F}$ then we say that x is a limit point of \mathcal{F}
- If $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \emptyset$, we say that x is an accumulation point of \mathcal{F}

So any limit point of \mathcal{F} is necessarily a accumulation point of \mathcal{F}

17.2 Prop

Let \mathcal{B} be a filter basis of X , $x \in X$, \mathcal{B}_x a neighborhood basis of x . Then x is an accumulation point of $\mathcal{F}(\mathcal{B})$ iff $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \emptyset$

Proof

Necessity

Since $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{V}_x$, the necessity is true.

Sufficiency

Let $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$. There exist $B \in \mathcal{B}, U \in \mathcal{B}_x$, such that $B \subseteq A, U \subseteq V$. Hence $\emptyset \neq B \cap U \subseteq A \cap V$

17.3 Def

Let $Y \subseteq X, Y \neq \emptyset$. We call accumulation point of Y any accumulation point of the principal filter $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$.

17.4 Def

We denote by $\overline{Y} = \{\text{accumulation points of } Y\}$, called the closure of Y . Note that $x \in \overline{Y}$ iff $\forall U \in \mathcal{B}_x, Y \cap U \neq \emptyset$

By convention $\overline{\emptyset} = \emptyset$

17.5 Prop

Let $Y \subseteq X$. Then \overline{Y} is the smallest closed subset of X containing Y .

Proof

$\forall x \in X \setminus \overline{Y}$, then there exists $U_x = \mathcal{V} \cap \mathcal{J}$, such that $Y \cap U_x = \emptyset$. Moreover, $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{J}$. This shows that $\forall y \in U_x, y \notin \overline{Y}$. Therefore $X \setminus \overline{Y} = \bigcap_{x \in X \setminus \overline{Y}} U_x \in \mathcal{J}$

Let $Z \subseteq X$ be a closed subset that contain Y . Suppose that $\exists y \in \overline{Y} \setminus Z$. Then $U = X \setminus Z \in \mathcal{V}_y \cap \mathcal{J}$ and $U \cap Y \subseteq U \cap Z = \emptyset$. So $y \notin \overline{Y}$ contradiction. Hence $\overline{Y} \subseteq Z$.

17.6 Def: dense

Let (X, \mathcal{J}) be a topological space, Y a subset of X . We call Y is dense in X if

$$\overline{Y} = X$$

Chapter 18

Limit of mappings

18.1 Def

Let (E, \mathcal{J}_E) be a topological space. $f : Y \rightarrow E$ a mapping, and \mathcal{F} be a filter of Y . If $a \in E$ is a limit point of $F_*(\mathcal{F})$ namely, \forall neighborhood V of a , $f^{-1}(V) \in \mathcal{F}$, then we say that a is a limit of the filter \mathcal{F} by f

18.2 Remark

Let \mathcal{B}_a be a neighborhood basis of a . Then $\mathcal{V}_a \subseteq f_*(\mathcal{F})$, iff $\mathcal{B} \subseteq f_*(\mathcal{F})$. Therefore, a is a limit of \mathcal{F} by f iff $\forall V \in \mathcal{B}_a, f^{-1}(V) \in \mathcal{F}$

18.2.1 Example

Let (E, \mathcal{J}_E) be a topological space. $I \subseteq \mathbb{N}$ be an infinite subset, $x = (x_n)_{n \in I} \in E^I$. If the Fréchet filter $\mathcal{F}_{Fr}(I)$ has a limit $a \in E$ by the mapping $x : I \rightarrow E$, we say that $(x_n)_{n \in I}$ converges to a , denote as

$$a = \lim_{n \in I, n \rightarrow +\infty} x_n$$

18.3 Remark

$a = \lim_{n \in I, n \rightarrow +\infty} x_n$ iff, $\forall U \in \mathcal{B}_a$ (where \mathcal{B}_a is a neighborhood basis of a), $\exists N \in \mathbb{N}$ such that $x_n \in U$ for any $n \in I_{\geq N}$

Suppose that \mathcal{J}_E is induced by a metric d . $\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}, \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}, \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ are all neighborhood basis of a . Therefore, the following are equivalent

- $a = \lim_{n \in I, n \rightarrow +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
 - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
 - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$
- $(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\})$? unknown position)

18.4 Remark

We consider the metric d on \mathbb{R} defined as

$$\forall (x, y) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of \mathbb{R} defined by this metric is called the usual topology on \mathbb{R}

18.5 Prop

Let $(x_n)_{n \in I} \in \mathbb{R}^I$, where $I \subseteq \mathbb{N}$ is an infinite subset. Let $l \in \mathbb{R}$. The following statements are equivalent:

- The sequence $(x_n)_{n \in I}$ converges to l in the topological space \mathbb{R}
- $\liminf_{n \in I, n \rightarrow +\infty} x_n = \limsup_{n \in I, n \rightarrow +\infty} x_n = l$
- $\limsup_{n \in I, n \rightarrow +\infty} |x_n - l| = 0$

18.6 Theorem

Let (X, d) be a metric space. Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be an element of X^I . Let $l \in X$. The following statements are equivalent:

- $(x_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ (equivalent to $\lim_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$)

Proof

- (1) \Rightarrow (2) The condition (1) is equivalent to $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$. We then get $\sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$. Therefore $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) < \epsilon$. We obtain that $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$.
- (2) \Rightarrow (1) Let $\epsilon \in \mathbb{R}_{>0}$. If $\inf_{N \in \mathbb{N}} \sup_{n \in I_{\geq N}} d(x_n, l) = 0$. Then $\exists N \in \mathbb{N} \quad \sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$. Hence, $\forall n \in I_{\geq N} d(x_n, l) < \epsilon$. Since ϵ is arbitrary, (*) is true, Hence (1) is also true.

18.7 Prop

Let (X, \mathcal{J}) be a topological space . $Y \subseteq X, p \in \overline{Y} \setminus Y$. Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of Y .

Proof

Y is not empty otherwise $\overline{Y} = \emptyset$.

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$
 $\emptyset \notin \mathcal{V}_{p,Y}$ since $p \in \overline{Y}$
- Let $V \in \mathcal{V}_p$ and $A \subseteq Y$ such that $V \cap Y \subseteq A$. Let $U = V \cup (A \setminus (V \cap Y)) \in \mathcal{V}_p$
and $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let U and V be elements of \mathcal{V}_p Let $W = U \cap V \in \mathcal{V}_p$ Then $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

18.8 Def

Let (X, \mathcal{J}_x) and (E, \mathcal{J}_E) be topological spaces, $Y \subseteq X, p \in \overline{Y} \setminus Y$, and $f : Y \rightarrow E$ be a mapping . If a is a limit point of $(F_*(\mathcal{V}_{p,Y}))$, then we say that a is a limit of f when the variable $y \in Y$ tends to p , denoted as $a = \lim_{y \in Y, y \rightarrow p} f(y)$

18.9 Remark

If \mathcal{B}_a is a neighborhood basis of a . Then $a = \lim_{y \in Y, y \rightarrow p} f(y)$ is equivalent to
 $\forall U \in \mathcal{B}_a \quad \exists V \in \mathcal{V}_p$ such that $Y \cap V \subseteq f^{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

18.10 Prop

Let X be a set, \mathcal{B} be a filter basis, \mathcal{G} be a filter. If $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$.

Proof

Let $V \in \mathcal{F}(\mathcal{B})$ By definition $\exists U \in \mathcal{B}$ such that $U \subseteq V$, since $U \in \mathcal{G}$ (for $\mathcal{B} \subseteq \mathcal{G}$) and since \mathcal{G} is a filter, $V \in \mathcal{G}$

18.11 Theorem

Let (X, \mathcal{J}_x) and (E, \mathcal{J}_E) be topological spaces. $Y \subseteq X$, $p \in \overline{T} \setminus Y$, $a \in E$. We consider the following conditions.

- (i) $a = \lim_{y \in Y, y \rightarrow p} f(y)$
- (ii) $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ if $\lim_{n \rightarrow +\infty} y_n = p$ then $\lim_{n \rightarrow \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that p has a countable neighborhood basis, then (i) and (ii) are equivalent.

Proof

- (1) Let $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ such that $p = \lim_{n \rightarrow +\infty} y_n$. For any $U \in \mathcal{V}_p$, $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}$ $y_n \in U \cap Y$. Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \rightarrow +\infty} f(y_n) = a$$

- (2) Assume that p has a countable neighborhood basis. There exists a decreasing sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_p^{\mathbb{N}}$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of p .

Assume that (i) does not hold. Then there exists $U \in \mathcal{V}_a$ such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \rightarrow +\infty} y_n = \emptyset$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However $f(y_n) \notin U$ for any $n \in \mathbb{N}$, so $(f(y_n))_{n \in \mathbb{N}}$ cannot converges to a .

18.12 Prop.

Let X be a set. If $(\mathcal{J}_i)_{i \in I}$ is a family of topologies on X , then $\mathcal{J} = \bigcap_{i \in I} \mathcal{J}_i$ is a topology. In particular, for any $\mathcal{A} \subseteq \wp(X)$, there is a smallest topology on X that contains \mathcal{A} .

18.12.1 Proof

- $\forall i \in I \quad \{\emptyset, X\} \subseteq \mathcal{J}_i$ So $\{\emptyset, X\} \subseteq \mathcal{J}$
- Let $(u_j)_{j \in J}$ be a family of elements of $\mathcal{J} \quad \forall j \in J, i \in I \quad u_j \in \mathcal{J}_i$ So $\bigcup_{j \in J} u_j \in \mathcal{J}_i$ We then get $\bigcup_{j \in J} u_j \in \mathcal{J}$
- Let U and V be elements of $\mathcal{J} \quad \forall i \in I, \{u, v\} \subseteq \mathcal{J}_i$ So $U \cap V \in \mathcal{J}_i$. Therefore we get $U \cap V \in \mathcal{J}$ Let $\mathcal{A} \subseteq \wp(X)$ Let $\mathcal{J}(\mathcal{A}) = \bigcap_{\substack{\mathcal{J} \subseteq \wp(X) \text{ a topology} \\ \mathcal{A} \subseteq \mathcal{J}}} \mathcal{J}$ Then $\mathcal{J}(\mathcal{A})$ is a topology. By definition, if \mathcal{J} is a topology containing \mathcal{A} , then $\mathcal{J}(\mathcal{A}) \subseteq \mathcal{J}$ Hence $\mathcal{J}(\mathcal{A})$ is the smallest topology containing \mathcal{A} .

Chapter 19

Continuity

19.1 Def

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces f be a function from X to Y , $x \in \text{Dom}(f)$. If for any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$. Then we say that f is continuous at x . If f is continuous at any $x \in \text{Dom}(f)$ then we say f is continuous.

19.2 Remark

Let $\mathcal{B}_{f(x)}$ be a neighborhood basis of $f(x)$ If $\forall U \in \mathcal{B}_{f(x)}$ there exist $V \in \mathcal{B}_{f(x)} V_x$ such that $f(V) \subseteq U$, then f is continuous at x Suppose that X and Y are metric space. Then f is continuous at x iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in \text{Dom}(f) \quad d(y, x) < \delta \text{ implies } d(f(y), f(x)) < \epsilon$$

19.3 Theorem

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces, f be a function from X to Y $x \in \text{Dom}(f)$ Consider the following condition

- f is continuous at x
- $\forall (x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$, if $\lim_{n \rightarrow +\infty} x_n = x$, then $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$ THEN
(i) implies (ii) Moreover, if x has a countable neighborhood basis, then (i) and (ii) are equivalent.

19.4 Proof

(i) \Rightarrow (ii) Let $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ that converges to x $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$ Since $\lim_{n \rightarrow +\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}, x_n \in V$.

Hence $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$. Thus $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$

(ii) \Rightarrow (i) under the hypothesis that x has countable neighborhood basis. actually we will prove $NOT(i) \Rightarrow NOT(ii)$

Let $(V_n)_{n \in \mathbb{N}}$ be a decreasing sequence in \mathcal{V}_x such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of x

If (i) does not hold, then $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$ Pick $x_n \in V_n$ such that $f(x_n) \notin U \quad \forall n \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$. Hence $(x_n)_{n \in \mathbb{N}}$ converges to x . However, $f(x_n) \notin U$ for any n So $(f(x_n))_{n \in \mathbb{N}}$ does not converges to $f(x)$. Therefore (ii) does not hold.

19.5 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$ be topological spaces. f be a function from X to Y , g be a function from Y to Z . Let $x \in \text{Dom}(g \circ f)$ If f and g are continuous at x . then $g \circ f$ is continuous at x sectionProof Let $U \in \mathcal{V}_{g(f(x))}$ Since g is continuous at $f(x)$:

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since f is continuous at x :

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore, $g(f(V)) \subseteq g(W) \subseteq U$ Hence $g \circ f$ is continuous at x

19.6 Def

Let (X, \mathcal{J}) be a topological space, $\mathcal{B} \subseteq \mathcal{J}$, If any element of \mathcal{J} can be written as the union of a family of sets in \mathcal{B} we say that \mathcal{B} is a topological basis of \mathcal{J}

19.7 Prop

Let (X, \mathcal{J}) be a topological space, $\mathcal{B} \subseteq \mathcal{J}$ \mathcal{B} is a topological basis iff

$$\forall x \in X, \mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$$

is a neighborhood basis of x

19.8 Proof

\Rightarrow :

$$\forall x \in X \mathcal{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since \mathcal{B} is a topological basis of \mathcal{J} ,

$$\exists W \in \mathcal{B}, x \in W \subseteq V \subseteq U$$

Hence \mathcal{V}_x is generated by \mathcal{B}_x

\Leftarrow Let $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathcal{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

19.9 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces. \mathcal{B}_Y be a topological basis of \mathcal{J}_Y
 $f : X \rightarrow Y$ be a mapping. The following conditions are equivalent:

- (1) f is continuous
- (2) $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$
- (3) $\forall U \in \mathcal{B}_Y, f^{-1}(U) \in \mathcal{J}_X$

Proof

(1) \Rightarrow (2)

Lemma Let (X, \mathcal{J}) be a topological space, $V \in \wp(X)$, Then $V \in \mathcal{J}$ iff
 $\forall x \in V, V$ is a neighborhood of x

Proof of lemma \Rightarrow is by definition

Leftarrow:

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W_x - x \in \mathcal{J}$$

Let $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood W of x such that $f(W) \subseteq U$
 Since f is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since x is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2) \Rightarrow (3) For (3) is a special situation of (2), it's natural.

(3) \Rightarrow (1) Let $x \in X$

$$\forall U \in \mathcal{B}_Y \text{ s.t. } f(x) \in U, f^{-1}(U)$$

is an open neighborhood of x , and

$$f(f^{-1}(U)) \subseteq U$$

Hence f is continuous at x

19.10 Def

Let X be a set , $((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces. $\forall i \in I$ let $f_i : X \rightarrow Y_i$ be a mapping. We call initial topology of $(f_i)_{i \in I}$ on X the smallest topology on X making all f_i continue

19.11 Remark

If \mathcal{J} is the initial topology of $(f_i)_{i \in I}$, $\forall i \in I, U_i \in \mathcal{J}_i$ $f_i^{-1}(U_i) \in \mathcal{J}$ If $J \subseteq I$ is a finite subset, $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j$ then $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{J}$

19.12 Prop

$$\mathcal{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ is finite } (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology \mathcal{J}

Proof

First

$$\mathcal{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{\text{subset } V \text{ of } X \text{ that can be written as the union of a family of sets in } \mathcal{B}\}$

- $\emptyset \in \mathcal{J}' \quad X \in \mathcal{B} \subseteq \mathcal{J}'$
- \mathcal{J}' is stable by taking the union of any family of elements in \mathcal{J}'
- If V_1, V_2 are elements of \mathcal{J}' , then

$$V_1 \cap V_2 \in \mathcal{J}'$$

In fact, V_1, V_2 are of the form of the union of some sets of \mathcal{B}

The intersection of two elements of \mathcal{B} is still a element of \mathcal{B}

$$\begin{aligned} & \left(\bigcap_{j \in J} f_j^{-1}(U_j) \right) \cap \left(\bigcap_{j \in J'} f_j^{-1}(U'_j) \right) \\ &= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \setminus J' \\ U'_j & j \in J' \setminus J \\ U_j \cap U'_j & j \in J \cap J' \end{cases} \\ & \left(\bigcap_{j \in J \setminus J'} f_j^{-1}(U_j) \right) \cap \left(\bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U'_j) \right) \cap \left(\bigcap_{j \in J' \setminus J} f_j^{-1}(U'_j) \right) \end{aligned}$$

So \mathcal{J}' is a topology making all f_i continuous. Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

Example

Let $((Y_i, \mathcal{J}_i))_{i \in I}$ be topological spaces. $Y = \prod_{i \in I} Y_i$ and

$$\begin{aligned} \pi_i : Y &\rightarrow Y_i \\ (y_j)_{j \in I} &\mapsto y_i \end{aligned}$$

The product topology on Y is by definition the initial topology of $(\pi_i)_{i \in I}$

19.13 Theorem

Let X be a set, $((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces,

$$((f_i : X \rightarrow Y_i))_{i \in I}$$

be a family of mappings and we equip X with the initial topology \mathcal{J}_X of $(f_i)_{i \in I}$. Let (Z, \mathcal{J}_Z) be a topological space and

$$h : Z \rightarrow X$$

be a mapping. Then h is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{ is continuous}$$

19.13.1 Proof

\Rightarrow If h is continuous, since each f_i is continuous, $f_i \circ h$ is also continuous.

\Leftarrow Suppose that $\forall i \in I, f_i \circ h$ is continuous. Hence

$$\forall U_i \in \mathcal{J}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Let

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

$\forall U \in \mathcal{B}$

$$h^{-1}(U) = \bigcap_{j \in J} h^{-1}(f_j^{-1}(U_j)) \in \mathcal{J}_Z$$

Therefore, h is continuous.

19.14 Remark

We keep the notation of the definition of initial topology. If $\forall i \in I, \mathcal{B}_i$ is a topological basis of \mathcal{J}_i , then

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{B}_j \right\}$$

is also a topological basis of the initial topology,

19.14.1 Example

Let $((X_i, d_i))_{i \in \{1, \dots, n\}}$ be a family of metric spaces.

$$X = \prod_{i \in \{1, \dots, n\}} X_i$$

We define a mapping

$$d: (x, y) \in X \times X \rightarrow \mathbb{R}_{\geq 0}$$

$$d: ((x_i)_{i \in \{1, \dots, n\}}, (y_i)_{i \in \{1, \dots, n\}}) \mapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i)$$

d is a metric on X . If $x = (x_i)_{i \in \{1, \dots, n\}}$, $y = (y_i)_{i \in \{1, \dots, n\}}$, $z = (z_i)_{i \in \{1, \dots, n\}}$ are elements of X , then

$$d(x, z) = \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \leq \max_{i \in \{1, \dots, n\}} (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i: X \rightarrow X_i$$

$$(x_i)_{i \in \{1, \dots, n\}} \mapsto x_i$$

is continuous. Hence the product topology \mathcal{J} is contained in \mathcal{J}_d

Let $x = (x_i)_{i \in \{1, \dots, n\}} \in X$, $\epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i \in \{1, \dots, n\}} \mid \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i \in \{1, \dots, n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i \in \{1, \dots, n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$

Chapter 20

Uniform continuity and convergency

20.1 Def

Let (X, d) be a metric space. $\forall A \subseteq X$, we define

$$\text{diam}(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A. By convention

$$\text{diam}(\emptyset) := 0$$

If $\text{diam}(A) < +\infty$, we say that A is bounded

20.2 Remark

- If A is finite, then it's bounded
- If $A \subseteq B$ then $\text{diam}(A) \leq \text{diam}(B)$

20.3 Prop

Let (X, d) be a metric space. $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$. Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

In particular, if A, B are bounded, then $A \cup B$ is bounded.

Proof

Let $(x, y) \in (A \cup B)^2$. If $\{x, y\} \subseteq A$, then $d(x, y) \leq \text{diam}(A)$
 If $\{x, y\} \subseteq B$ then $\text{diam}(B) \geq d(x, y)$
 If $x \in A, y \in B$,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

Similarly if $x \in B, y \in A$

$$d(x, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

20.4 Def

Let (X, d) be a metric space. $I \subseteq \mathbb{N}$ be an infinite subset, $(x_n)_{n \in I} \in X^I$. If

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \quad \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \epsilon$$

then we say that $(x_n)_{n \in I}$ is a Cauchy sequence.

20.5 Prop

- (1) If $(x_n)_{n \in I}$ converges, then it's a Cauchy sequence.
- (2) If $(x_n)_{n \in I}$ is a Cauchy sequence, $\{x_n \mid n \in I\}$ is bounded
- (3) Suppose that $(x_n)_{n \in I}$ is a Cauchy sequence. If there exists an infinite subset J of I such that $(x_n)_{n \in J}$ converges to some $x \in X$, then $(x_n)_{n \in I}$ converges to x

20.5.1 Proof

- (1) trivial
- (2) trivial
- (3) Let $\epsilon > 0, \exists N \in \mathbb{N}$

$$\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence $(x_n)_{n \in I}$ converges to x

20.6 Def

Let $(X, d_X), (Y, d_Y)$ be metric space. f be a function from X to Y . If $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

we say that f is uniformly continuous.

20.7 Prop

Let $(X, d_X), (Y, d_Y)$ be metric spaces f be a function from X to Y which is uniformly continuous.

- (1) If $I \subseteq \mathbb{N}$ is finite, and $(x_n)_{n \in I}$ is a Cauchy sequence in $\text{Dom}(f)^I$ then $(f(x_n))_{n \in I}$ is Cauchy sequence
- (2) f is continuous

20.7.1 Proof

- (1) $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that

$$\forall (n, m) \in I_{\geq N}^2, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

Therefore $(f(x_n))_{n \in I}$ is a Cauchy sequence.

- (2) Let $(x_n)_{n \in I}$ be a sequence in $\text{Dom}(f)^{\mathbb{N}}$ that converges to $x \in \text{Dom}(f)$ We define $(y_n)_{n \in \mathbb{N}}$ as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then $(y_n)_{n \in \mathbb{N}}$ converges to x . Hence $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since f is uniformly continuous, $(f(y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y .

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to $f(x)$. Hence $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x)$

20.8 Def

Let X be a set, $Z \subseteq X$, (Y, d) be a metric space, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$ and f be functions from X to Y , having Z as their common domain of definition.

- If $\forall x \in Z, (f_n(x))_{n \in I}$ converges to $f(x)$, we say that $(f_n)_{n \in I}$ converges pointwisely to f
- If

$$\lim_{n \in I, n \rightarrow +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$

we say that $(f_n)_{n \in I}$ converges uniformly to f

20.9 Theorem

Let X and Y be metric space, $Z \subseteq X$, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$, f be functions from X to Y , having Z as domain of definition. Suppose that

- $(f_n)_{n \in I}$ converges uniformly to f
- each f_n is uniformly continuous

Then f is uniformly continuous.

20.9.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\lim_{n \in I, n \rightarrow +\infty} A_n = 0$$

$\forall (x, y) \in Z^2, n \in I$

$$\begin{aligned} & d(f(x), f(y)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & \leq 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f_n(x), f_n(y)) = 0$$

Hence

$$0 \leq \inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$

Take $\lim_{n \rightarrow +\infty}$, by squeeze theorem, we get

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

20.10 Theorem

Let X be a topological space, Y be a metric space, $Z \subseteq X, p \in Z, I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$ and f function from X to Y , having Z as domain of definition. Suppose that:

- $(f_n)_{n \in I}$ converges uniformly to f
- each f_n is continuous at p

Then f is continuous at p

20.10.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since f_n is continuous $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{B}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x), f(p)) & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ & \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$$f(U) \subseteq \overline{B}(f(p), \epsilon)$$

20.10.2 Def

Let X, Y be metric spaces, f be a function from X to Y , $\epsilon > 0$. If

$$\forall (x, y) \in \text{Dom}(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that f is ϵ -Lipschitzian

If $\exists \epsilon > 0$ such that f is ϵ -Lipschitzian, then it's uniformly continuous.

20.11 Remark

If f is Lipschitzian, then it's uniformly continuous.

20.12 Example

- Let $((X_i, d_i))_{i \in I}$ be metric space. $X = \prod_{i \in I} X_i$ where I is finite

$$\begin{aligned} X \times X & \rightarrow \mathbb{R}_{\geq 0} \\ d : d((x_i), (y_i)_{i \in I}) & = \max_{i \in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \rightarrow X_i$$

is Lipschitzian. ($\forall x = (x_i)_{i \in I}, \forall y = (y_i)_{i \in I}$)

- Let (X, d) be a metric space

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

Part V

Normed Vector Space

Chapter 21

Linear Algebra

We fix a unitary ring K

21.1 Def

Let M be a left K -module, and let $x = (x_i)_{i \in I}$ be a family of elements of M . We define a morphism of left K -module as following:

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \quad (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

21.1.1 Notation

$$\begin{aligned} K^{\oplus I} &:= \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{ finite, such that } a_i = 0 \text{ for } i \in I \setminus J\} \\ \varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) &= \varphi_x((a_i)_{i \in I}) + \varphi_x((b_i)_{i \in I}) \end{aligned}$$

21.2 Def

Let M be a left K -module, I be a set, $x = (x_i)_{i \in I} \in M^I$. If

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say $(x_i)_{i \in I}$ is K -linearly independent

surjective then we say $(x_i)_{i \in I}$ is system of generator

a bijection then we say $(x_i)_{i \in I}$ is a basis of M

Example

Let e_i be the element $(\delta_{ij})_{j \in I}$ with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i \in I} \in (K^{\oplus I})^I$$

is a basis of $K^{\oplus I}$

21.3 Def

Let M be a left K -module

- If M has a basis, we say that M is a free K -module
- If M has finite system of generated
(\exists a finite set I and a family $(x_i)_{i \in I} \in M^I$ that forms a system of generator),
then we say that M is of finite type.

21.4 Remark

Let $x = (x_i)_{i \in \{1, \dots, n\}} \in M^n$, where $n \in \mathbb{N}$

- x is linearly independent iff

$$\forall a \in K^n \quad \sum a_i x_i = 0$$

implies

$$a = 0$$

- x is a system of generator iff for any element of M can be written in the form

$$\sum b_i x_i \quad b \in K^n$$

Such expression is called a K -linear combination of x_1, \dots, x_n

21.5 Theorem

Let K be a division ring ($0 \neq 1$ and $\forall k \in K \setminus \{0\}$ k is invertible)

Let V be a left K -module of finite type and $(x_i)_{i \in I}$ be a system of generators of V . Then, there exists a subset I of $\{1, \dots, n\}$ such that $(x_i)_{i \in I}$ forms a basis of V . (In particular, V is a free K -module)

Proof

(By induction on n)

If $n = 0$, then $V = \{0\}$

In this case \emptyset is a basis of V

Induction hypothesis

True for a system of generators of $n - 1$ elements. Let $(x_i)_{i \in \{1, \dots, n\}}$ be a system of generators of V . If $(x_i)_{i \in \{1, \dots, n\}}$ is linearly independent, it's a basis. Otherwise, $\exists (a_i)_{i \in I} \in K^n$ such that

$$(a_i, \dots, a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose $a_n \neq 0$. Then

$$x_n = -a_n^{-1} \left(\sum_{i=1}^{n-1} a_i x_i \right)$$

Since $(x_i)_{i \in \{1, \dots, n\}}$ is a system of generators, any elements of V can be written as

$$\begin{aligned} \sum b_i x_i &= \left(\sum_{i=1}^{n-1} b_i x_i \right) - b_n a_n^{-1} \left(\sum_{i=1}^{n-1} a_i x_i \right) \\ &= \sum_{i=1}^{n-1} (b_i - b_n a_n^{-1} a_i) x_i \end{aligned}$$

Thus $(x_i)_{i \in \{1, \dots, n\}}$ forms a system of generators. By the induction hypothesis, there exists $I \subseteq \{1, \dots, n\}$ such that $(x_i)_{i \in I}$ forms a basis of V .

21.6 Theorem

Let K be a unitary ring and B be a left K -module. W be a left K -submodule of V . Let $(x_i)_{i=1}^n$ be an element of W^n

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where $(n, l) \in \mathbb{N}^2 \forall j \in \{1, \dots, l\}$, let x_{n+j} be an element in the equivalence class α_j

- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are linearly independent, then $(x_i)_{i=1}^{n+l}$ is also linearly independent
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are system of generators of W and V/W respectively, then $(x_i)_{i=1}^{n+l}$ is also a system of generators
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are basis, then $(x_i)_{i=1}^{n+l}$ is also a basis

Proof

(1) Suppose that $(b_i)_{i=1}^{n+l}$ such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \rightarrow V/W$$

be the projection morphism ($\pi(x) = [x]$)

$$0 = \pi\left(\sum_{i=1}^{n+l} b_i x_i\right) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^l b_{n+j} \pi(x_{n+j}) = \sum_{j=1}^l b_{n+j} \alpha_j$$

$$\{x_1, \dots, x_n\} \subseteq W \text{ So } \forall i \in \{1, \dots, n\}$$

$$\pi(x_i) = 0$$

Since $(\alpha_j)_{j=1}^l$ is linearly independent,

$$b_{n+1} = \dots = b_{n+l} = 0$$

Hence

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Since $(x_i)_{i=1}^n$ is linearly independent,

$$b_1 = \dots = b_n = 0$$

(2) Let $y \in V$. Then $\pi(y) \in V/W$. So there exists

$$(c_{n+1}, \dots, c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^l c_{n+j} \alpha_j \\ &= \sum_{j=1}^l c_{n+j} \pi(x_{n+j}) = \pi\left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \end{aligned}$$

So

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \in W$$

$\exists c \in K^n$ such that

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) = \left(\sum_{i=1}^n c_i x_i\right)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

21.7 Corollary

Let K be a division ring and V be a left K -module of finite type. If $(x_i)_{i=1}^n$ is a linearly independent family of elements of V ($n \in \mathbb{N}$), then

$$\exists l \in \mathbb{N} \quad \exists (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of V

Proof

Let W be the image of

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow V \\ (a_i)_{i=1}^n &\mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left K -submodule of V .

Note that $(x_i)_{i=1}^n$ forms a basis of W .

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow W \\ \varphi(x_i)_{i=1}^n(e_j) &= x_j \in W \end{aligned}$$

Moreover, since V is of finite type there exists $d \in \mathbb{N}$ and a surjective morphism of left K -modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \rightarrow V/W$$

is surjective.

Hence the composite morphism

$$K^d \begin{array}{c} \xrightarrow{\psi} \\ \searrow \pi \circ \psi \end{array} V \xrightarrow{\pi} V/W$$

is surjective. Thus V/W is of finite type. There exist then a basis

$$(a_j)_{j=1}^l$$

of V/W .

Taking $x_{n+j} \in \alpha_j$ for $j \in \{1, \dots, l\}$, we get a basis of V :

$$(x_i)_{i=1}^{n+l}$$

21.8 Def

Let K be a division ring and V be a left K -module of finite type. We call rank of V the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If K is a field $rk(V)$ is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of V .

21.9 Theorem

Let K be a division ring and V be a left K -module of finite type. Let W be a left K -submodule of V .

(1) W and V/W are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of V has exactly $rk(V)$ elements

21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let $(x_i)_{i=1}^n$ be a basis of V . Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

In $(\pi(x_i))_{i=1}^n$ we extract a basis of V/W , say

$$(\pi(x_i))_{i=1}^l$$

For $j \in \{l+1, \dots, n\}$,

$$\exists(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any $x \in W, \exists(a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^n a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^l a_i x_i + \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \\ &= \sum_{j=l+1}^n a_j y_j + \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^n a_j y_j$$

Hence W is of finite type, and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let $(x_i)_{i=1}^n$ be a basis of V . Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of V/W .

We extract a subfamily, say $(x_i)_{i=1}^l$ such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of V/W .

For $j \in \{1, \dots, l\}$, there exists:

$$(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Let $x \in W, \exists (a_i)_{i=1}^n \in K^n$ let $x = \sum a_i x_i$, then

$$\begin{aligned} x &= \left(\sum_{i=1}^l a_i x_i \right) + \left(\sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \right) \\ &= \left(\sum_{i=1}^l a_i x_i \right) + \left(\sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i \right) + \left(\sum_{j=l+1}^n a_j y_j \right) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i)$$

Therefore $(y_j)_{j=l+1}^n$ is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + rk(V/W)$$

(2) All basis of V have $rk(V)$ elements.

We reason by induction on $rk(V)$

(1)

$$rk(V) = 0$$

In this case $V = \{0\}$ The only basis of V is \emptyset . So the statement holds.

(2) Assume that there exists $e \in V \setminus \{0\}$ such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of V is of the form

$$ae$$

where $a \in K \setminus \{0\}$

Let $(e_i)_{i=1}^m$ be a basis of V . We reason by induction on m to prove that

$$m = rk(V)$$

The cases where $m = 0$ or 1 are proved in (1)(2) respectively. Induction hypothesis: true for a basis of $< m$ elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of V/W .

If $(a_i)_{i=2}^m \in K^{m-1}$ such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for $(e_i)_{i=1}^m$ a basis of V ,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of V/W . We then obtain that

$$rk(V/W) \leq m - 1 \leq n - 1$$

By the induction hypothesis,

$$m - 1 = rk(V/W)$$

By (2), $rk(W) = 1$. Hence

$$m = (m - 1) + 1 = rk(V/W) + rk(W) = rk(V)$$

21.11 Prop

Let K be a unitary ring and $f : E \rightarrow F$ be a morphism of left K -modules. Let I be a set and $(x_i)_{i \in I} \in E^I$

- If $(x_i)_{i \in I}$ is linearly independent and f is injective, then $(f(x_i))_{i \in I}$ is linearly independent.
- If $(x_i)_{i \in I}$ is a system of generators and f is surjective, then $(f(x_i))_{i \in I}$ is a system of generators.
- If $(x_i)_{i \in I}$ is a basis and f is an isomorphism, then $(f(x_i))_{i \in I}$ is a basis.

21.11.1 Proof

$$\varphi_{(f(x_i))_{i \in I}} = f \circ \varphi_{(x_i)_{i \in I}}$$

Chapter 22

Matrices

We fix unitary ring K

22.1 Def

Let $n \in \mathbb{N}$ and V be a left K -module.

For any $(x_i)_{i=1}^n \in V^n$, we denote by $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ the morphism

$$\begin{aligned} & \phi_{(x_i)_{i=1}^n} : K^n \rightarrow V \\ (a_i)_{i=1}^n & \mapsto \sum_{i=1}^n a_i n_i \end{aligned}$$

22.1.1 Example

Suppose that $V = K^p$ ($p \in \mathbb{N}$) Then each $x_i \in K^p$ is of the form $(x_{i,1}, \dots, x_{i,p})$

Hence $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be written:

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,p} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,p} \end{pmatrix}$$

22.2 Def

Let $(n, p) \in \mathbb{N}^2$. We call n by p matrix of coefficient in K any morphism of left K -modules from K^n to K^p

22.2.1 Example

- Denote by I_n then identity mapping. Then $(e_i)_{i=1}^n$ is a basis of K^n called the canonical basis of K^n

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, \dots, a_n)) = \sum_{i=1}^n a_i e_i = (a_1, \dots, a_n)$$

- Let $(x_1, \dots, x_n) \in K^n$, Denote by

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) (= \varphi_{(x_i e_i)_{i=1}^n}) : K^n &\rightarrow K^n \\ (a_1, \dots, a_n) &\mapsto (a_1 x_1, \dots, a_n x_n) \end{aligned}$$

22.3 Def

We denote by $M_{n,p}(K)$ the set of all n by p matrices of coefficients in K . For $(n, p, r) \in \mathbb{N}^3$, we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) &\rightarrow M_{n,r}(K) \\ (A, B) &\mapsto AB := B \circ A \end{aligned}$$

22.4 Calculate Matrices

Let K be a unitary ring, and V be a left K -module. Let $n \in \mathbb{N}$ and

$$x = (x_1, \dots, x_n) \in V^n$$

22.4.1 Remind

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \varphi : (a_1, \dots, a_n) \mapsto a_1 x_1, \dots, a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1, \dots, p\} \times \{1, \dots, n\}} \in M_{p,n}(K)$$

A is a morphism of left K -modules from K^p to K^n . Recall that

$$A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let $(b_1, \dots, b_n) \in K^p$

$$\begin{aligned} A((b_1, \dots, b_n)) &= \sum_{i=1}^p b_i(a_{i,1}, \dots, a_{i,n}) \\ \varphi(A((b_1, \dots, b_n))) &= \sum_{i=1}^p b_i \varphi_x((a_{i,1}, \dots, a_{i,n})) \\ &= \sum_{i=1}^p b_i(a_{i,1}x_1, \dots, a_{i,n}x_n) \end{aligned}$$

Let $B = \{b_{ij}\}_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\}} : K^n \rightarrow K^r$

$$AB = \left\{ \sum_{j=1}^n a_{lj} b_{jm} \right\}_{(l,m) \in \{1, \dots, p\} \times \{1, \dots, r\}}$$

Chapter 23

Transpose

We fix a unitary ring K

23.1 Def

Let E be a left- K -module. Denote by

$$E^\vee := \{\text{morphisms of left } K\text{-modules } E \rightarrow K\}$$

$\forall (f, g) \in E^\vee$ let

$$\begin{aligned} f + g : E &\rightarrow K \\ x &\mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$ forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : E &\rightarrow K \\ x &\mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} K \times E^\vee &\rightarrow E^\vee \\ (a, f) &\mapsto fa : x \in E \rightarrow f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right K -module on E^\vee

23.2 Def

Let E and F be two left K -modules. $\varphi : E \rightarrow F$ be a morphism of left K -modules. We denote by

$$\varphi^\vee : F^\vee \rightarrow E^\vee$$

the morphism of right K -modules sending $g \in F^\vee$ to $g \circ \varphi \in E^\vee$.
Actually $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

23.2.1 Example

Suppose that $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,p} \end{pmatrix}$$

φ sends (a_1, \dots, a_n) to $\{\sum_{i=1}^n a_i b_{ij}\}_{j \in \{1, \dots, p\}}$. Let $g \in F^\vee$ $g : K^p \rightarrow K$, then g is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$ sends (a_1, \dots, a_n) to $\sum_{i=1}^p (\sum_{j=1}^n a_j b_{ij} y_i)$

Assume that K is commutative. We denote by

$$\iota_p : (K^p)^\vee \rightarrow K^p$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, \dots, x_p)$$

$$\iota_n : (K^n)^\vee \rightarrow K^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, \dots, x_n)$$

are isomorphisms of K -modules

For any morphism of K-modules $\varphi : K^n \rightarrow K^p$, we denote by φ^τ the morphism of K-modules $K^p \rightarrow K^n$ given by $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{array}{ccc} (K^p)^\vee & \xrightarrow{\varphi^\vee} & (K^n)^\vee \\ \cong \downarrow \iota_p & \circlearrowleft & \cong \downarrow \iota_n \\ K^p & \xrightarrow{\varphi^\tau} & K^n \end{array}$$

φ^τ is called the transpose of φ

23.3 Prop

Let E,F,G be left K-modules. $\varphi : E \rightarrow F, \psi : F \rightarrow G$ be morphisms of left K-modules. Then $(\psi \circ \varphi)^\vee$ is equal to $\varphi^\vee \circ \psi^\vee$

Proof

$$\forall f \in G^\vee$$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

23.4 Corollary

Assume that K is commutative. Let n, p, q be neutral numbers. $A \in M_{n,p}(K), B \in M_{p,q}(K)$. Then

$$(AB)^\tau = B^\tau A^\tau$$

Proof

$$A^t a u = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^t a u = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned} B^\tau A^\tau &= A^\tau \circ B^\tau \\ &= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\ &= (AB)^t a u \end{aligned}$$

23.5 Remark

(1) For $A \in M_{n,p}(K)$, one has $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} E &\rightarrow (E^\vee)^\vee \\ x &\mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a K -linear mapping.

If K is a field and E is of finite dimension, this is an isomorphism of K -modules.

In fact, if $e = (e_i)_{i=1}^n$ is a basis of E over K . For $i \in \{1, \dots, n\}$, let

$$\begin{aligned} e_i^\vee : E &\rightarrow K \\ \lambda_1 e_1, \dots, \lambda_n e_n &\mapsto \lambda_i \end{aligned}$$

is called the dual basis of e

$$\begin{array}{ccc} K^n & \xleftarrow[\iota_n]{\cong} & (K^n)^\vee \\ \varphi_e \downarrow \cong & \searrow \varphi_{e^\vee} & \downarrow \varphi_e^\vee \\ E & \xrightarrow[\cong]{} & E^\vee \end{array}$$

$(e^\vee)^\vee$ gives a basis of $(E^\vee)^\vee$. Hence $E \rightarrow (E^\vee)^\vee$ is an isomorphism.

Chapter 24

Linear Equation

We fix a unitary ring K .

24.1 Def

For $a = (a_1, \dots, a_n) \in K^n \setminus \{(0, \dots, 0)\}$. Denote by $j(a)$ the first index $j \in \{1, \dots, n\}$ such that $a_j \neq 0$. Let $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$. We write A as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \in K^p$$

We say that A is of row echelon form if, $\forall i \in \{1, \dots, n-1\}$ one of following conditions is satisfied.

- $a^{(i+1)} = (0, \dots, 0)$
- $a^{(i)}, a^{(i+1)}$ are non-zero, and $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, \dots, n\}$ such that $a^{(i)} \neq (0, \dots, 0)$, one has

$$a_{j(a^{(i)})}^{(i)} = 1$$

and

$$\forall k \in \{1, \dots, n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that A is of reduced row echelon form.

24.2 Prop

Suppose that $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$ is of row echelon form. Then $\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$ is of cardinal $\leq p$

Proof

Let $k = \text{card}\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$ $a^{(k+1)} = \dots = a^{(n)} = (0, \dots, 0)$ and $j(a^{(1)}) < j(a^{(2)}) < \dots < j(a^{(k)})$ Hence

$$\{1, \dots, k\} \rightarrow \{1, \dots, p\}, i \mapsto j(a^{(i)})$$

is injection. So $k \leq p$

24.3 Linear Equation

Let $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$. Let V be a left K -module and $(b_1, \dots, b_n) \in V^n$. We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

The set of $(x_1, \dots, x_p) \in V^p$ that satisfies $(*)$ is called the solution set of $(*)$

24.4 Prop

Suppose that A is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, \dots, n\} \mid (a_{i,1}, \dots, a_{i,p}) \neq (0, \dots, 0)\}$$

$$J_0(A) = \{1, \dots, p\} \setminus \{j((a_{i,1}, \dots, a_{i,p})) \mid i \in I(A)\}$$

- If $\exists i \in \{1, \dots, n\} \setminus I(A)$ such that $b_i \neq 0$ then $(*)$ does not have any solution in K^n
- Suppose that $\forall i \in \{1, \dots, n\} \setminus I(A), b_i = 0$. Then $(*)$ has at least one solution. Moreover

$$V^{J_0(A)} \rightarrow V^p$$

$$(z_k)_{k \in J_0(A)} \mapsto (x_1, \dots, x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, \dots, a_{i,p})) \end{cases}$$

is an injective mapping, whose image is equal to the set of solution of (*)

24.5 Prop

Let $m \in \mathbb{N}, S \in M_{m,n}(K)$. If $(x_1, \dots, x_p) \in V^p$ is a solution of (*), then (x_1, \dots, x_p) is a solution of $(*)_S$:

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

In the case where S is left invertible, namely there exist $R \in M_{n,m}(K)$ such that $RS = I_n \in M_{m,n}(K)$. Then (*) and $(*)_S$ have the same solution set.

24.6 Def

Let $G_n(K)$ be the set of $S \in M_{n,n}(K)$ that can be written as $U_1 \dots U_N$ (by convention $S = I_n$ where $N = 0$) where each U_i is of one of the following forms.

- P_σ where $\sigma \in \mathfrak{S}_n$
- $\text{diag}(r_1, \dots, r_n)$ where each $r_i \in K$ is left invertible
- $S_{i,c}$ with $i \in \{1, \dots, n\}$ $c = (c_1, \dots, c_n) \in K^n, c_i = 0$

Let $p \in \mathbb{N}$, we say that $A \in M_{n,p}(K)$ is reducible by Gauss elimination if $\exists S \in G_n(K)$ such that SA is of reduced row echelon form

24.7 Theorem

Assume that K is a division ring $\forall (n, p) \in \mathbb{N}$ any $A \in M_{n,p}(K)$ is reducible by Gauss elimination

Proof

The case where $n = 0$ or $p = 0$ is trivial. We assume $n \geq 1, p \geq 1$ We write A as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} B \quad \text{where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If $\lambda_1 = \dots = \lambda_n = 0$

Applying the induction hypothesis to B, for $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} SB$$

- Suppose that $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

By permuting the rows we may assume $\lambda_1 \neq 0$. As K is division ring, by multiplying the first row by λ_1^{-1} , we may assume $\lambda_1 = 1$. We add $(-\lambda_i)$ times the first row to the i^{th} row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 & \dots & \mu_p \\ 0 & & & \\ \vdots & C & & \\ 0 & & & \end{pmatrix} \quad \begin{array}{l} C \in M_{n-1, p-1}(K) \\ (\mu_2, \dots, \mu_p) \in K^{p-1} \end{array}$$

Applying the induction hypothesis to C, we may assume that C is of reduced row echelon form. For $i \in \{2, \dots, k\}$ we add $-\mu_{j(c_i)}$ times the i^{th} row of A to the first line to obtain a matrix of reduced row echelon form

Chapter 25

Normed Vector Space

25.1 Def

Let (X, d) be a metric space. If $(x_n)_{n \in \mathbb{N}}$ is an element of $X^{\mathbb{N}}$ such that

$$\lim_{N \rightarrow +\infty} \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

we say that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. If any Cauchy sequence in X converges, then we say that (X, d) is complete.

Let $Cau(X, d)$ be the set of all Cauchy sequences in X . We define a binary relation \sim on $Cau(X, d)$ as

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$$

25.2 Prop

\sim is an equivalence relation.

25.2.1 Proof

$$\lim_{n \rightarrow +\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$ be elements of $Cau(X, d)$. For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n \rightarrow +\infty} d(x_n, z_n) = 0$$

25.3 Def

$$\hat{X} := \text{Cau}(X, d) \setminus \sim$$

25.4 Def: The completion

The completion of (X, d) is defined as

$$\text{Cau}(X) / \sim$$

and is denoted as

$$\hat{X}$$

25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} : \hat{X} \times \hat{X} &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n \rightarrow +\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on \hat{X}

Proof

TO check that \hat{d} is well defined, it suffices to prove that $\forall ([x], [y]) \in \hat{X} \times \hat{X}$, $(d(x_n, y_n))_{n \in \mathbb{N}}$ is Cauchy sequence and its limit doesn't depend on the choice of the representation x and y

For $N \in \mathbb{N}$ and $(n, m) \in \mathbb{N}_{\geq N}$ for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n, m) \in \mathbb{N}_{\geq N}} |d(x_n, y_n) - d(x_m, y_m)| &\leq \left(\sup_{(n, m) \in \mathbb{N}_{\geq N}} d(x_n, x_m) \right) \\ &\quad + \left(\sup_{(n, m) \in \mathbb{N}_{\geq N}} d(y_n, y_m) \right) \end{aligned}$$

Taking $\lim_{n \rightarrow +\infty}$ we obtain that $(d(x_n, y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence.

Hence it converges in \mathbb{R} . If $x' = (x'_n)_{n \in \mathbb{N}} \in [x], y' = (y'_n)_{n \in \mathbb{N}} \in [y]$, thus

$$\lim_{n \rightarrow +\infty} d(x_n, x'_n) = \lim_{n \rightarrow +\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking $\lim_{n \rightarrow +\infty}$ we get

$$\lim_{n \rightarrow +\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(x'_n, y'_n)$$

In the following, we check that \hat{d} is a metric

- $\hat{d}([x], [y]) = 0$ iff $[x] = [y]$: trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$: trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$:

$$\begin{aligned} d([x], [y]) &= \lim_{n \rightarrow +\infty} \\ &\leq \lim_{n \rightarrow +\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

25.6 Remark

Let

$$\begin{aligned} i_X : X &\rightarrow \hat{X} \\ a &\mapsto [(a, a, \dots)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular, i_x is injective (if $i_X(a) = i_X(b)$ then $d(a, b) = 0$ hence $a = b$)

25.7 Prop

$i_X(X)$ is dense in \hat{X} (the closure of $i_X(X)$ in \hat{X} is equal to $i_X(X)$ (or to say \hat{X}))

Proof

Let $[x]$ be an equivalence class in \hat{X} . We claim that $\forall (x_n)_{n \in \mathbb{N}} \in [x]$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} i_X(x_n)$$

For any $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n \rightarrow +\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \end{aligned}$$

Taking $\lim_{N \rightarrow +\infty}$ we get

$$\lim_{N \rightarrow +\infty} \hat{d}(i_X(x_N), [x]) = 0$$

25.8 Theorem

(\hat{X}, \hat{d}) is a complete metric space

Proof

Let $([x^{(N)}])_{N \in \mathbb{N}}$ be a Cauchy sequence in \hat{X} , where $\forall N \in \mathbb{N}$, $x^{(N)} = (x_n^{(N)})_{n \in \mathbb{N}}$ is a Cauchy sequence
 $\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall (k, l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n \rightarrow +\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any $(\mu, \nu) \in \mathbb{N}_{\geq \alpha(N)}$

Let $y_N = x_{\alpha(N)}^{(N)}$. Without loss of generality, we assume that

$$\alpha(0) \leq \alpha(1) \leq \dots$$

Let $\epsilon > 0$ Take $N_0 \in \mathbb{N}$ such that

$$(1) \quad \forall (k, l) \in \mathbb{N}, \quad k, l \geq N_0$$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

$$(2)$$

$$\frac{1}{N_0 + 1} \leq \frac{\epsilon}{3}$$

Let $(k, l) \in \mathbb{N}_{N_0}^2$,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since $\alpha(k) \geq N_0, \forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking $\lim_{n \rightarrow +\infty}$ get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So $y = (y_N)_{N \in \mathbb{N}}$ is a Cauchy sequence. We check that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \hat{d}([x^{(N)}], [y]) &= 0 \\ 0 &\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \rightarrow +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$\begin{aligned} d(x_n^{(N)}, y_n) &\leq d(x_n^{(N)}, y_N) + d(y_n, y_N) \\ \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, y_n) &\leq \limsup_{N \rightarrow +\infty} \left(\frac{1}{N+1} + \lim_{n \rightarrow +\infty} d(y_n, y_N) \right) \end{aligned}$$

Since y is Cauchy sequence

$$\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(y_n, y_N) = 0$$

Example

Let $(K, |\cdot|)$ be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$ iff $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$\text{Cau}(K)$ forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in \text{Cau}_0(K)$$

where

$$\text{Cau}_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of $\text{Cau}(K)$

Hence

$$\hat{K} = \text{Cau}(K) \setminus \text{Cau}_0(K)$$

is a quotient ring of $\text{Cau}(K)$

$|\cdot|$ extend to \hat{K} :

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \rightarrow +\infty} |a_n|$$

that forms an absolute value.

Chapter 26

Norms

In this chapter we fix a field K and an absolute value $|\cdot|$ on K . We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

26.1 Def

Let V be a vector space over K (K -module). We call seminorm on V any mapping

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}_{\geq 0} \\ s &\mapsto \|s\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V, \|s\| = 0$ iff $s = 0$

We say that $\|\cdot\|$ is a norm and $(V, \|\cdot\|)$ is normed space over K .

26.2 Remark

If $\|\cdot\|$ is a norm then

$$\begin{aligned} d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let $(V, \|\cdot\|)$ be a vector space over K equipped with a seminorm, and W be a vector space subspace of V (sub- K -module)

- The restriction of $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ to W forms a seminorm on W . It is a norm if $\|\cdot\|$ is a norm.

$$\begin{aligned}\|\cdot\|_W : W &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto \|x\|\end{aligned}$$

- The mapping

$$\begin{aligned}\|\cdot\|_{V/W} : V/W &\rightarrow \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\|\end{aligned}$$

is a seminorm on V/W

Attention: Even if $\|\cdot\|$ is a norm, $\|\cdot\|_{V/W}$ **might only be a seminorm**

26.3 Def

$\|\cdot\|_{V/W}$ is called the quotient seminorm of $\|\cdot\|$

26.4 Prop

Let $(V, \|\cdot\|)$ be a vector space over K , equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of V . Moreover, $\|\cdot\|_{V/N}$ is a norm

Proof

If $(a, s) \in K \times N$ then $\|as\| = |a| \cdot \|s\| = 0$ so $as \in N$

If $(s_1, s_2) \in N \times N$ then $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$ so $s_1 + s_2 \in N$

Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let $\alpha \in V/N$ such that $\|\alpha\|_{V/N} = 0$ Let $s \in \alpha, \forall t \in N$

$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$

$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$

Hence $\|\alpha\|_{V/N} = \|s\| = 0$ We obtain that $\alpha = N = [0]$

26.5 Def

Let $(V, \|\cdot\|)$ be a vector space over K , equipped with a seminorm. For any $x \in V$ and $r \geq 0$, we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$

$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

26.6 Remark

If $N = \{s \in V, \|s\| = 0\}$ then when $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

26.7 Def

We equip the topology such that $\forall U \subseteq V, U$ is open iff $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

26.8 Prop

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with seminorms. Let $f : V_1 \rightarrow V_2$ be a K -linear mapping

- If f is continuous, $\forall s \in V_1$ if $\|s\|_1 = 0$ then $\|f(s)\|_2 = 0$
- If there exists $C > 0$ such that $\forall x \in V_1, \|f(x)\|_2 \leq C\|x\|_1$ then f is continuous.

The converse is true

when $|\cdot|$ is non-trivial

or $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$ is of finite type

Proof

- (1) Lemma If $(V, \|\cdot\|)$ is a vector space over K , equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let $s \in V \setminus N_{\|\cdot\|}$ Then $\|s\| > 0$. Let $\epsilon = \frac{\|s\|}{2}$, $\forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq \|s\| - \|s - x\| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteq V \setminus N_{\|\cdot\|}$$

– Then $f^{-1}(N_{\|\cdot\|_2})$ is closed.

Note that

$$0 \in f^{-1}(N_{\|\cdot\|_2})$$

hence

$$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$

$$\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$$

$$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$

and

$$0 \in \mathcal{B}(x, \epsilon)$$

Therefore $x \in \overline{\{0\}}$

(2) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of V_1 that converges to some $x \in V_1$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n \rightarrow +\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n \rightarrow +\infty} C \|x_n - x\|_1 \\ &= C \limsup_{n \rightarrow +\infty} \|x_n - x\| \\ &= 0 \end{aligned}$$

So $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$. Hence f is continuous at x

Assume that $|\cdot|$ is non-trivial and f is continuous. Then

$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

is an open subset of V_1 containing $0 \in V_1$

So there exists $\epsilon > 0$ such that

$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

namely $\forall x \in V_1$ if $\|x\|_1 < \epsilon$ then $\|f(x)\|_2 < 1$

Since $|\cdot|$ is nontrivial, $\exists a \in K, 0 < |a| < 1$ We prove that $\forall x \in V_1$

$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|} \|x\|_1$$

If $\|x\|_1 = 0$ by (1) we obtain

$$\|f(x)\|_2 = 0$$

Suppose that $\|x\|_1 > 0$ then $\exists n \in \mathbb{Z}$ such that

$$\begin{aligned} \|a^n x\|_1 &= |a|^n \|x\|_1 \\ &< \epsilon \leq \\ &\|a^{n-1} x\|_1 = |a|^{n-1} \|x\|_1 \end{aligned}$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned} \|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}} \frac{1}{|a|} \\ &\leq \frac{1}{\epsilon} \|x\|_1 \frac{1}{|a|} = \frac{\|x\|_1}{\epsilon |a|} \end{aligned}$$

26.9 Def: Operator Seminorm

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with seminorm. We say that a K -linear mapping $f : V_1 \rightarrow V_2$ is bounded if there exists $C > 0$ that

$$\forall x \in V_1 \quad \|f(x)\|_2 \leq C \|x\|_1$$

For a general K -linear mapping $f : V_1 \rightarrow V_2$ we denote

$$\|f\| := \begin{cases} \sup_{x \in V_1, \|x\|_1 > 0} \left(\frac{\|f(x)\|_2}{\|x\|_1} \right) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

f is bounded iff

$$\|f\| < +\infty$$

$\|f\|$ is called the operator seminorm of f

We denote by $\mathcal{L}(V_1, V_2)$ the set of all bounded K -linear mappings from V_1 to V_2

26.10 Prop

$\mathcal{L}(V_1, V_2)$ is a vector subspace of $\text{Hom}_K(V_1, V_2)$. Moreover $\|\cdot\|$ is a seminorm on $\mathcal{L}(V_1, V_2)$

Proof

Let f, g be elements of $\mathcal{L}(V_1, V_2)$

$$\begin{aligned} \|f + g\| &= \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x) + g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq \left(\sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1} \right) + \left(\sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1} \right) \\ &\leq +\infty \end{aligned}$$

Hence $f + g \in \mathcal{L}(V_1, V_2)$

Let $\lambda \in K$, $\lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

26.11 Remark

Let $f \in \mathcal{L}(V_1, V_2)$. Suppose that $\exists x \in V_1$ such that $f(x) \neq 0$. Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore $\|\cdot\|$ is a norm

26.12 Def

Let $(V, \|\cdot\|)$ be a normed vector space. If V is complete with respect to the metric

$$\begin{aligned}d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|x - y\|\end{aligned}$$

then we say that $(V, \|\cdot\|)$ is a Banach space.

26.13 Theorem

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with semi-norm. If $(V_2, \|\cdot\|_2)$ is a Banach space, then

$$(\mathcal{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

Proof

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(V_1, V_2)$.
 $\forall x \in V_1$, the mapping

$$(f \in \mathcal{L}(V_1, V_2)) \mapsto f(x)$$

is $\|x\|_1$ -Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\| \|x\|_1$$

So $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence, for V_2 is complete, that converges to some $g(x) \in V_2$. Then we obtain a mapping $g : V_1 \rightarrow V_2$. We prove that g is an element of $\mathcal{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n \rightarrow +\infty} f_n(x + y) = \lim_{n \rightarrow +\infty} f_n(x) + f_n(y)$$

$$\begin{aligned} \|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \rightarrow +\infty) \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n \rightarrow +\infty} f_n(\lambda x) = \lim_{n \rightarrow +\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1) (n \rightarrow +\infty)$$

So $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n \rightarrow +\infty} \|f_n(x)\| \leq \left(\lim_{n \rightarrow +\infty} \|f_n\| \right) \cdot \|x\|$$

(because $\forall (a, b) \in V_2^2 \quad \|a\| - \|b\| \leq \|a - b\|$) Then

$$\|f_n(x)\| - \|g_n(x)\| \leq \|f_n(x) - g_n(x)\| = o(1) (n \rightarrow +\infty)$$

So $g \in \mathcal{L}(V_1, V_2)$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall (n, m) \in \mathbb{N}_{\geq N}, \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking $\lim_{n \rightarrow +\infty}$ we get

$$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

So $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$

Chapter 27

Differentiability

In this chapter we fix a field K and an absolute value $|\cdot|$ on K . We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

27.1 Def

Let X be a topological space and $p \in X$. Let K be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K .

Let $f : X \rightarrow E$ be a mapping and $g : X \rightarrow \mathbb{R}_{\geq 0}$ be a non-negative mapping.

- We say that

$$f(x) = O(g(x)) \text{ } x \rightarrow p$$

if there is a neighborhood V of p in X and a constant $C > 0$ such that $\forall x \in V$

$$\|f(x)\| \leq Cg(x)$$

- We say that

$$f(x) = o(g(x)) \text{ } x \rightarrow p$$

if there exists a neighborhood V of p in X and a mapping $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\lim_{x \in V, x \rightarrow p} \epsilon(x) = 0$$

which is equivalent to

$$\forall \delta > 0, \exists \text{ neighborhood } U \text{ of } p \text{ } U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$

and $\forall x \in V$

$$\|f(x)\| \leq \epsilon(x)g(x)$$

27.2 Def

Let E and F be normed vector space over K . $U \subseteq E$ be an open subset, $f : U \rightarrow F$ be a mapping and $p \in U$. If there exists $\varphi \in \mathcal{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

we say that f is differentiable at p , and φ is the differential of f at p . Suppose that $|\cdot|$ is not trivial. $\varphi(x - p)$ also written as

$$d_p f$$

Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

means there exists an open neighborhood V of p with $V \subseteq U$ and a mapping $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim_{x \rightarrow p} \epsilon(x) = 0$ and that $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

27.3 Prop

If f is differentiable at p , then its differential at p is unique

Proof

Suppose that there exists φ and ψ in $\mathcal{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|\varphi - \psi\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\| (y - p)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \rightarrow p} \epsilon(y) = 0 \end{aligned}$$

27.4 Example

27.4.1

$$f : U \rightarrow F : f(x) = y_0 \quad \forall x \in U$$

$$\forall p \in U$$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence $\forall x \in E$

$$d_p(f(x)) = 0$$

27.4.2

Let $f \in \mathcal{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence $d_p f = f$

27.4.3

$$A : E \times E \rightarrow E$$

$$(x, y) \mapsto x + y$$

Let E be a normed space. Then $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

27.4.4

$$m : K \times E \rightarrow E$$

$$(\lambda, x) \mapsto \lambda x$$

Let $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when $(\lambda, x) \rightarrow (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

is a K -linear mapping.

$$\begin{aligned}
- & (\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2) \\
- & b\mu p + a(by) = b(\mu p + ay) \\
- & \|\mu p + ay\| \leq |\mu| \|p\| + |a| \|y\| \\
& \leq \max\{|\mu|, \|y\|\}(|a| + \|p\|)
\end{aligned}$$

Hence m is differentiable and $\forall (\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

27.5 Theorem:Chain rule

Let E, F, G be normed vector spaces, $U \subseteq E, V \subseteq F$ be open subsets.

Let $f : U \rightarrow F, g : V \rightarrow G$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. Assume that f is differentiable at p and g differentiable at $f(p)$. Then $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

Proof

Let $x \in U$. By definition

$$\begin{aligned}
f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\
f(x) - f(p) &= O(\|x - p\|)
\end{aligned}$$

and

$$\begin{aligned}
(g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|)
\end{aligned}$$

So $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

27.6 Prop

Let n be a positive integer. Let $(F_i)_{i \in \{1, \dots, n\}}$ be normed vector spaces over K . Let $U \subseteq E$ be an open subset, $p \in U$.

$\forall i \in \{1, \dots, n\}$ let $f_i : U \rightarrow F_i$ be a mapping. Let

$$f : U \rightarrow F = \prod F_i$$

be the mapping that sends $x \in U$ to $(f_i(x))_{i \in \{1, \dots, n\}}$. We equip F with the norm $\|\cdot\|$ defined as :

$$\|(y_i)_{i \in \{1, \dots, n\}}\| = \max_{i \in \{1, \dots, n\}} \|y_i\|$$

Then f is differentiable at p iff each f_i is differentiable at p . Moreover, when this happens, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i \in \{1, \dots, n\}}$$

Proof

\Leftarrow Suppose that $(f_i)_{i \in \{1, \dots, n\}}$ are differentiable at p

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i \in \{1, \dots, n\}} \\ &= (d_p f_i(x - p))_{i \in \{1, \dots, n\}} + o(\|x - p\|) \end{aligned}$$

Therefore f is differentiable at p and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i \in \{1, \dots, n\}}$$

\Rightarrow Let

$$\begin{aligned} \pi_i : F &\rightarrow F_i \\ (x_i)_{i \in \{1, \dots, n\}} &\mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has $\|\pi_i\| \leq 1$ because

$$\|x_i\| \leq \max_{i \in \{1, \dots, n\}} \|x_i\| = \|(x_i)_{i \in \{1, \dots, n\}}\|$$

π_i is differentiable at p then $\pi_i \circ f = f_i$ is differentiable at p

27.7 Def

Let U be an open subset of K and $(F, \|\cdot\|)$ be a normed vector space. If $f : U \rightarrow F$ is a mapping that is differentiable at some $p \in U$. We denote by $f'(p)$ the element

$$d_p f(1) \in F$$

called the derivative of f at p

27.8 Corollary

Let U and V be open subsets of K , $(F, \|\cdot\|)$ be a normed vector space over K . $f : U \rightarrow K$, $g : V \rightarrow F$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$ then

$$(g \circ f)'(p) = f'(p)g'(f(p))$$

By definition

$$\begin{aligned} d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p) \cdot 1) \\ &= f'(p) \cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

Let E and F be normed vector spaces, $U \subseteq E$ an open subset. $f : U \rightarrow L$ and $g : U \rightarrow F$ be mappings and $p \in U$. If both f, g differentiable at p then

is also differentiable at p and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_p f(l) + g(p)d_p f(l)$$

Consider

$$\begin{aligned} m : K \times F &\rightarrow F \\ (a, y) &\rightarrow ay \end{aligned}$$

We have shown m is differentiable and

$$d_{a,y}m(b,z) = by = az$$

fg is the following composite:

$$U \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{\quad fg \quad} \end{array} K \times F \xrightarrow{m} F$$

$$x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m \circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p), g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

27.10 Corollary

Let U be an open subset of K , f, g be mappings from U to K and to a normed space F respectively. If f, g are differentiable at $p \in U$ then

$$(fg)'(p) = d_p(fg)(1) = d_p f(1)g(p) + f(p)d_p g(1) = f'(p)g(p) + f(p)g'(p)$$

Example

$$\begin{aligned} f_n : K &\rightarrow K \\ x &\mapsto x^n \end{aligned}$$

is differentiable at any $x \in K$

$$f'_n(x) = nx^{n-1}$$

Proof

$f_1 : K \rightarrow K$ is differentiable $\forall x \in K$

$$d_x f_1 = f_1$$

If $f'_n(x) = nx^{n-1}$ then

$$\begin{aligned} f'_{n+1}(x) &= (f_n f_1)'(x) \\ &= f_n(x)f'_1(x) + f'_n(x)f_1(x) \\ &= x^n + x'_n(x) = x^n + nx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(1) &= l d_x f_n(1) \\ &= nx^{n-1} \end{aligned}$$

27.11 Prop

Let E, F, G be normed vector spaces. $U \subseteq E$ be an open subset, $\varphi \in \mathcal{L}(F, G)$, $p \in U$ if $f : U \rightarrow E$ is differentiable at p then so is $\varphi \circ f$. Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

Proof

φ is differentiable at $f(p)$ nad $d_{f(p)}\varphi = \varphi$

27.12 Corollary

Let E and F be normed vector spaces $U \subseteq E$ be an open subset, $p \in U$. Let $f : U \rightarrow F$ and $g : U \rightarrow F$ be mappings that are differentiable at p , $(a, b) \in K \times K$. Then $af + bg$ is differentiable at p and

$$d_p(af + bg) = ad_p f + bd_p g$$

Proof

$af + bg$ is composite:

$$U \xrightarrow{h} K \times F \xrightarrow{m} F$$

$ay+bz$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

27.13 Def: Equivalence of Norms

Let E be a vector space over K and $\|\cdot\|_1, \|\cdot\|_2$ be norms on E . We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist constants $C_1, C_2 > 0$ such that $\forall s \in E$

$$C_1 \|s\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

27.14 Prop

If $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover $\|\cdot\|_1, \|\cdot\|_2$ defines the same topology on E .

Proof

$$\|s\|_2 \leq C_2 \|s\|_1 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are continuous. So \forall open subset U of $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in $(E, \|\cdot\|_1)$. Conversely if V is open in $(E, \|\cdot\|_1)$ then

$$V = Id_E^{-1}(V)$$

is open in $(E, \|\cdot\|_2)$

27.15 Remark

If $\|\cdot\|_1, \|\cdot\|_2$ are two norms on E that define the same topology on E , then they are equivalent (under the assumption that $|\cdot|$ is not trivial)

27.16 Prop

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces $\|\cdot\|'_E$ and $\|\cdot\|'_F$ be norms on E and F that are equivalent to $\|\cdot\|_E, \|\cdot\|_F$ respectively. Let $U \subseteq E$ be an open subset and $f : U \rightarrow F$ be a mapping.

Let $p \in U$ Then f is differentiable at p with respect to $\|\cdot\|_E$ and $\|\cdot\|_F$ iff it's differentiable with respect to $\|\cdot\|'_E$ and $\|\cdot\|'_F$

Moreover the differentiable of f at p is not changed in the change of norms from $(\|\cdot\|_E, \|\cdot\|_F)$ to $(\|\cdot\|'_E, \|\cdot\|'_F)$

Proof

$$U \xrightarrow{Id_U} U \xrightarrow{f} F \xrightarrow{Id_F} F$$

f

$$(E, \|\cdot\|'_E) \quad (E, \|\cdot\|_E) \quad \|\cdot\|_F \quad \|\cdot\|'_F$$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$$d'_p f : (E, \|\cdot\|'_E) \rightarrow (F, \|\cdot\|'_F)$$

27.17 Theorem

Let V be a finite dimensional vector space over K . Then all norms on V are equivalent. Moreover V is complete with respect to any norm on V .

Proof

Let $(e_i)_{i=1}^n$ be a basis of V (linear independent system of generators) The mapping:

$$V \rightarrow \mathbb{R}_{\geq 0}$$

$$\sum_{i \in \{1, \dots, n\}} a_i e_i \mapsto \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

is a norm on V

Let $\|\cdot\|$ be another norm on V . One has

$$\left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\| \leq \sum_{i \in \{1, \dots, n\}} |a_i| \|e_i\|$$

$$\leq \left(\sum_{i \in \{1, \dots, n\}} \|e_i\| \right) \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

We reason by induction that there exists $C > 0$ such that

$$\max_{i \in \{1, \dots, n\}} \{|a_i|\} \leq C \left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\|$$

The case where $n = 0$ is trivial.

$n=1$

$$\|a_1 e_1\| = |a_1| \|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension $< n$

Let

$$W = \left\{ \sum_{i \in \{1, \dots, n-1\}} a_i e_i \mid (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1} \right\}$$

equipped with $\|\cdot\|$ restricted to W

The induction hypothesis shows that W is complete. Hence it's closed in V . Let $Q = V/W$ and $\|\cdot\|_Q$ be the quotient norm on Q that's defined as

$$\forall \alpha \in Q \quad \|\alpha\|_Q = \inf_{s \in \alpha} \|s\|$$

– If $s \in V \setminus W$, $\exists \epsilon > 0$ such that

$$\overline{B}(s, \epsilon) \cap W = \emptyset$$

$\forall t \in W$,

$$s + t \notin \overline{B}(0, \epsilon)$$

since otherwise

$$-t \in W \cap \overline{B}(s, \epsilon)$$

Therefore

$$\|[s]\|_Q = \inf_{i \in W} \|s + t\| \geq \epsilon > 0$$

– $\forall \lambda \in K$

$$\begin{aligned}\|\lambda \alpha\|_Q &= \inf_{s \in \alpha} \|\lambda s\| = |\lambda| \\ \inf_{s \in \alpha} \|s\| &= |\lambda| \cdot \|\alpha\|_Q\end{aligned}$$

–

$$\begin{aligned}\|\alpha + \beta\|_Q &= \inf_{s \in \alpha + \beta} \|s\| \\ &= \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \inf_{x \in \alpha} \|x\| + \inf_{y \in \beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some $A > 0$ such that $\forall (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1}$

$$\max_{i \in \{1, \dots, n-1\}} \{|a_i|\} \leq A \left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\|$$

Take

$$s = \sum_{i \in \{1, \dots, n\}} a_i e_i \in V$$

Let $\alpha = [s] = a_n [e_n] \in Q$

$$\left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \leq \|s\| + |a_n| \cdot \|e_n\| \leq \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \|[e_n]\|_Q = |a_n| \inf_{t \in W} \|e_n + t\|$$

Take $e'_n \in V$ such that $[e'_n] = [e_n]$ and $\|e'_n\| \leq \|[e_n]\|_Q + \epsilon$

Note that $(e_1, \dots, e_{n-1}, e'_n)$ forms also basis of V over K . Hence by replacing e_n by e'_n we may assume that $\|e_n\| \leq \|[e_n]\|_Q + \epsilon$

$s = a_n e_n + t \in V$ with $t \in W$

$$\|s\| \geq \|a_n e_n\|_Q = |a_n| \|[e_n]\|_Q \geq B^{-1} |a_n| \cdot \|e_n\|$$

– If $\|a_n e_n\| < \frac{1}{2} \|t\|$

$$\|s\| \geq \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \geq \frac{1}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

– If $\|a_n e_n\| \geq \frac{1}{2} \|t\|$

$$\|s\| \geq B^{-1} |a_n| \cdot \|e_n\| \geq \frac{B^{-1}}{2} \|t\| \geq \frac{B^{-1}A}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

We take $C = \max\{B^{-1} \|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$ Then

$$\|s\| \geq C \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Another proof

completeness Under the norm $\max_{i \in \{1, \dots, n\}}$, a sequence $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$ is a Cauchy sequence iff $\forall i \in \{1, \dots, n\}$ $(a_i^{(k)})_{k \in \mathbb{N}}$ is a Cauchy sequence. Since K is complete each $(a_i^{(k)})_{k \in \mathbb{N}}$ converges to some $a_i \in K$ Hence $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$ converges.

27.18 Prop

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed vector spaces over K . Assume that E is finite dimensional. Then any K -linear mapping $\varphi : E \rightarrow F$ is bounded.

Proof

Let $(e_i)_{i=1}^n$ be a basis of E . For any two norms on E are equivalent.
 $\forall (a_1, \dots, a_n) \in K$

$$\left\| \sum_{i=1}^n a_i e_i \right\|_E = \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Then for any $s = \sum_{i=1}^n a_i e_i$

$$\|\varphi(s)\|_F = \left\| \sum_{i=1}^n a_i e_i \right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq \left(\sum_{i=1}^n \|\varphi(e_i)\|_F \right) \|s\|_E$$

27.19 Theorem

Let E, F be normed vector spaces over a complete valued field, $U \subseteq E$ be an open subset and $f : U \rightarrow F$ be a mapping. If f is differentiable at p then f is continuous at p

Proof

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ &= f(p) + O(\|x - p\|) \\ &= f(p) + o(1) \quad x \rightarrow p \\ &\Rightarrow \lim_{x \rightarrow p} f(x) = f(p) \end{aligned}$$

Chapter 28

Compactness

28.1 Def: cover

Let X be a topological space, $Y \subseteq X$ we call open cover of Y any family $(U_i)_{i \in I}$ open subset of X such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If I is finite set, we say that $(U_i)_{i \in I}$ is a finite open cover. If $J \subseteq I$ such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that $(U_j)_{j \in J}$ is a sub cover of $(U_i)_{i \in I}$

28.2 Def: compact

If any open cover of Y has a finite subcover, we say that Y is quasi-compact. If in addition X is Hausdorff, namely $\forall (x, y) \in X \times X$ with $x \neq y \exists$ open neighborhoods U and V of x and y such that $U \cap V = \emptyset$, we say that Y is compact

28.3 Def

Let X be a set and \mathcal{F} be a filter on X . If there does not exist any filter \mathcal{F}' of X such that $\mathcal{F} \subsetneq \mathcal{F}'$, then we say that \mathcal{F} is an ultrafilter.

Zorn's lemma implies that $\forall \mathcal{F}_0$ of X there exist an ultrafilter \mathcal{F} if X containing \mathcal{F}_0

28.4 Prop

Let \mathcal{F} be a filter on a set X . The following statements are equivalent.

- (1) \mathcal{F} is an ultrafilter
- (2) $\forall A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$
- (3) $\forall (A, B) \in \wp(X)^2$ if $A \cap B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$

Proof

- (1) \Rightarrow (2) Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F} \forall B \in \mathcal{F}$ one has

$$B \cap A \neq \emptyset$$

since otherwise $B \subseteq X \setminus A$ and hence $X \setminus A \in \mathcal{F}$ contradiction.

- (2) \Rightarrow (3) Suppose that $B \notin \mathcal{F}$ then $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So $A \in \mathcal{F}$

- (3) \Rightarrow (1) Suppose that \mathcal{F}' is a filter such that $\mathcal{F} \subsetneq \mathcal{F}'$ Take $A \in \mathcal{F}' \setminus \mathcal{F}$ Then by $X = A \cup (X \setminus A) \in \mathcal{F}$ Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \emptyset = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

28.5 Theorem

Let (X, \mathcal{J}) be a topological space . The following are equivalent

- (1) X is quasi-compact
- (2) Any filter of X has an accumulation point
- (3) Any ultrafilter of X is converges.

Proof

- (1) \Rightarrow (2) Assume that a filter \mathcal{F} of X does not have any accumulation point. $\forall x \in X \exists A_x \in \mathcal{F} \exists$ open neighborhood V_x of x such that $A_x \cap V_x = \emptyset$ Since $X = \bigcup_{x \in X} V_x$ there is

$$\{x_1, \dots, x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^n V_{x_i}$$

Take $B = \bigcap_{i=1}^n A_{x_i} \in \mathcal{F}$

$$B \cap X = B = \emptyset$$

Since $\forall i \ B \cap V_x = \emptyset$ contradiction.

- (2) \Rightarrow (3) Let \mathcal{F} be an ultrafilter of X . By (2) there exist $x \in X$ such that $\mathcal{F} \cup \mathcal{V}_x$ generates a filter \mathcal{F}' Since \mathcal{F} is an ultrafilter $\mathcal{F} = \mathcal{F}'$ and hence $\mathcal{V}_x \subseteq \mathcal{F}$
- (3) \Rightarrow (1) Let $(U_i)_{i \in I}$ be an open cover of X we suppose that this have no finite subcover. $\forall i \in I$ let

$$F_i = X \setminus U_i$$

For any $J \subseteq I$ finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \emptyset$$

Let \mathcal{F} be the smallest filter on X that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let \mathcal{F}' be ultrafilter containing \mathcal{F} . It has a limit point x There exist $i \in I$ such that $x \in U_i$. Since U_i is a neighborhood of x and $\mathcal{V}_x \subseteq \mathcal{F}'$ we get $U_i \in \mathcal{F}'$ This is impossible since $F_i \in \mathcal{F}'$

28.6 Theorem

Let (X, d) be a metric space. The following statements are equivalent:

- (1) X is complete and $\forall \epsilon > 0 \ \exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

- (2) X is compact

Proof

- (1) \Rightarrow (2) Let \mathcal{F} be an ultrafilter Let $\epsilon > 0$ and $\{x_1, \dots, x_n\} \subseteq X$ such that

$$X = \bigcup_{i=1}^n \mathcal{B}(x_i, \epsilon)$$

There exists some $i \in \{1, \dots, n\}$ such that $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$ That means \mathcal{F} is a Cauchy filter (namely $\forall \delta > 0 \ \exists A \in \mathcal{F}$ of diameter $\leq \delta$) Since X is complete \mathcal{F} has a limit point. So \mathcal{F} is compact.

(2) \Rightarrow (1) Let $\epsilon > 0$ One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since X is compact $\exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

\mathcal{F} is an ultrafilter

$$\Leftrightarrow \forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}$$

$$\Leftrightarrow \forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F}$$

$$\Leftrightarrow \forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup \dots \cup A_n \ \exists i \in \{1, \dots, n\}, A_i \in \mathcal{F}$$

Let \mathcal{F} be a Cauchy filter Let $x \in X$ be an accumulation point of \mathcal{F}
 $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$ with diameter $\leq \frac{\epsilon}{2}$ Note that $A \cup \mathcal{B}(x, \frac{\epsilon}{2}) \neq \emptyset$ Take
 $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2}) \ \forall z \in A$

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $A \subseteq \mathcal{B}(x, \epsilon)$ So $\mathcal{B}(x, \epsilon) \in \mathcal{F}$ This implies $\mathcal{V}_x \subseteq \mathcal{F}$

28.7 Lemma

Let (X, d) be a metric space

- (1) Let \mathcal{F} be a Cauchy filter on X . Any accumulation point of \mathcal{F} is a limit point of \mathcal{F}
- (2) X is complete iff any Cauchy filter of X has a limit point

Proof

(1)

- Let \mathcal{F} be a Cauchy filter on X . Any accumulation point of \mathcal{F} is a limit point of \mathcal{F}

- (2) Suppose that X is complete. Let \mathcal{F} be a Cauchy filter. $\forall n \in \mathbb{N}_{\geq 1}$ let $A_n \in \mathcal{F}$ such that $\text{diam}(A_n) \leq \frac{1}{n}$ Take $x_n \in \bigcap_{k=1}^n A_k \in \mathcal{F}$ Then $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ is a Cauchy sequence since $\forall \epsilon > 0$ if we take $N \in \mathbb{N}$ with $\frac{1}{N} \leq \epsilon$ then $\forall (n, m) \in \mathbb{N}_{\geq N} \ d(x_n, x_m) \leq \frac{1}{N}$ Hence $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ converges to some $x \in X$ Note that x is an accumulation point of \mathcal{F} since $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$ with $A_n \subseteq \mathcal{B}(x, \epsilon)$ It suffices to take n such that $\frac{1}{n} < \frac{\epsilon}{2}$

\Leftarrow Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in X . Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N}, \{x_N, x_{N+1}, \dots\} \subseteq A\}$$

This is a Cauchy filter on X since

$$\lim_{N \rightarrow +\infty} \text{diam}\{x_N, x_{N+1}, \dots\} = 0$$

Hence \mathcal{F} has a limit point $x \in X$ By definition $\forall U \in \mathcal{V}_x \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, \dots\} \subseteq U$$

$$\text{So } x = \lim_{n \rightarrow +\infty} x_n$$

28.8 Prop

Let $f : X \rightarrow Y$ be a continuous mapping of topological spaces. If $A \subseteq X$ is quasi-compact then $f(A) \subseteq Y$ is also quasi-compact.

Proof

Let $(V_i)_{i \in I}$ be an open cover of $f(A)$ Then

$$(f^{-1}(V_i))_{i \in I}$$

is an open cover of A So $\exists J \subseteq I$ such that

$$A \subseteq \bigcup_{j \in J} f^{-1}(V_j)$$

This implies

$$f(A) \subseteq \bigcup_{j \in J} V_j$$

So $f(A)$ is quasi-compact.

28.9 Prop

Let X be a topological space and $A \subseteq X$ be a quasi-compact subset. For any closed subset F of X $A \cap F$ is quasi-compact.

Proof

Let $(U_i)_{i \in I}$ be an open cover of $A \cap F$. Then

$$A \subseteq \left(\bigcup_{i \in I} U_i \right) \cup (X \setminus F)$$

Since A is quasi-compact there exist $J \subseteq I$ finite such that

$$A \subseteq \left(\bigcup_{j \in J} U_j \right) \cup (X \setminus F)$$

Hence $A \cap F \subseteq \bigcup_{j \in J} U_j$

28.10 Prop

Let X be a Hausdorff topological space. Any compact subset A of X is closed.

Proof

Let $x \in X \setminus A$. $\forall y \in A, \exists$ open subsets U_y and V_y such that $y \in U_y, x \in V_y$ and $U_y \cap V_y = \emptyset$. Since $A \subseteq \bigcup_{y \in A} U_y$, $\exists \{y_1, \dots, y_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^n U_{y_i}$$

Let

$$U = \bigcup_{i=1}^n U_{y_i} \quad V = \bigcap_{i=1}^n V_{y_i}$$

These are open subsets. Moreover $A \subseteq U, x \in V$ and $U \cap V = \bigcup_{i=1}^n (U_{y_i} \cap V) = \emptyset$. In particular $x \in V \subseteq X \setminus A$. So $X \setminus A$ is open.

28.11 Prop

Let X be a Hausdorff topological space and A and B be compact subsets of X such that $A \cap B = \emptyset$. Then there exist open subsets U and V such that

$$A \subseteq U, B \subseteq V \text{ and } U \cap V = \emptyset$$

proof

We have seen in the proof of the previous proposition that $\forall x \in B, \exists U_x, V_x$ open such that $A \subseteq U_x, x \in V_x$ and $U_x \cap V_x = \emptyset$. Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, \dots, x_m\} \subseteq B$ such that

$$B \subseteq \bigcup_{i=1}^m V_{x_i}$$

We take

$$U = \bigcap_{i=1}^m U_{x_i} \quad V = \bigcup_{i=1}^m U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq V \quad U \cap V = \emptyset$$

28.12 Theorem

Let (X, \mathcal{J}) be a Hausdorff topological space. If $(A_n)_{n \in \mathbb{N}}$ is a sequence of non-empty compact subsets of X such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

Then

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$$

Proof

Suppose that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

then

$$A_0 \subseteq \bigcup_{n \in \mathbb{N}} (X \setminus A_n)$$

Since A_0 is compact, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^N (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^N A_n \\ &= X \setminus A_N \end{aligned}$$

So

$$A_N = \emptyset$$

28.13 Def

Let (X, τ) be a topological space. If any sequence in X has a convergent subsequence, we say that X is sequentially compact.

Example

By Bolzano-Weierstrass, any bounded sequence in \mathbb{R} has a convergent subsequence. So any bounded and closed subset of \mathbb{R} is sequentially compact.

Note

bounded and closed together implies sequentially compact.

28.14 Theorem

Let (X, d) be a metric space. Then the following statements are equivalent:

- (1) (X, d) is compact
- (2) (X, d) is sequentially compact

Proof

- (1) \Rightarrow (2) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X . Assume that no subsequence of $(x_n)_{n \in \mathbb{N}}$ converges in X . For any $p \in X$ there exists $\epsilon_p > 0$ such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For X is compact $\exists (p_i)_{i \in \{1, \dots, n\}}$

$$X \subseteq \bigcup_{i=1}^n \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^n \{n \in \mathbb{N} : d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

- (2) \Rightarrow (1)

prove (X, d) is complete Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences $(x_{k_n})_{n \in \mathbb{N}}$ must converges to the same limit.

So (X, d) is complete

If X is not covered by finitely many balls of radius ϵ we can construct a sequence $(x_{k_n})_{n \in \mathbb{N}}$ such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^n \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

28.15 Def

Let X be a Hausdorff topological space. If for any $x \in X$ there exist a compact neighborhood \mathcal{C}_x we say that X is locally compact.

Example

\mathbb{R} is locally compact.

28.16 Prop

Assume that $(K, |\cdot|)$ is a locally compact non-trivial valued field. Let $(E, \|\cdot\|)$ be a finite dimensional normed K -vector space. A subset $Y \subseteq E$ is compact iff it's closed and bounded.

Proof

\Rightarrow Let $Y \subseteq X$ be compact. Then for Y is Hausdorff, Y is closed. Moreover

$$Y \subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq \dots \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^k \mathcal{B}(0, n_i)$$

$\Rightarrow Y$ is bounded.

\Leftarrow We prove sequentially compact by a theorem proved before.

Let $(e_i)_{i=1}^d$ be a basis of E . Again we assume

$$\left\| \sum_{i=1}^d a_i e_i \right\| = \max_{i \in \{1, \dots, d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n \in \mathbb{N}} = \left(\sum_{i=1}^d a_i^{(n)} e_i \right)_{n \in \mathbb{N}}$$

Since Y is bounded for any $i \in \{1, \dots, d\}$ the sequence $(a_i^{(n)})$ is bounded. In particular we find $M > 0$ such that $\forall i \in \{1, \dots, d\}$

$$|a_i^{(n)}| < M$$

Since $(K, |\cdot|)$ is locally compact, there exists a compact set $\mathcal{C} = \mathcal{C}_0 \subseteq K$ that's a neighborhood of 0. Let $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since K is not trivially valued, then exists $a \in K$ such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$ is compact. We have the K -linear mapping

$$\begin{aligned} K &\rightarrow K \\ y &\mapsto ay \end{aligned}$$

is bounded, then continuous. Hence $a\mathcal{C}$ is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find $(I_i)_{i=1}^d$ are infinite subsets of \mathbb{N} with

$$I_1 \supseteq \dots \supseteq I_d$$

such that $(a_j)_{j \in I_i}^{(n)}$ converges to some $a_i \in K$. It follows that our original

sequence has a convergent subsequence converges to $\sum_{i=1}^d a_i e_i$.

So Y is sequentially compact.

28.17 Theorem

Let X be a topological space and $f : X \rightarrow \mathbb{R}$ be a continuous mapping. If $Y \subseteq X$ is a quasi-compact subset, then there exists $a \in Y$ and $b \in Y$ such that $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of f to Y attains its maximum and minimum.

Proof

$f(Y) \subseteq \mathbb{R}$ is a non-empty compact subset since Y is quasi-compact and \mathbb{R} is Hausdorff. Moreover, since \mathbb{R} is locally compact. SO $f(Y)$ is bounded and closed.

Note that there exists sequences $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ in $f(Y)$ that tends to $\sup f(Y)$ and $\inf f(Y)$ respectively. Since $f(Y)$ is closed, $\sup f(Y), \inf f(Y)$ belongs to $f(Y)$. So $f(Y)$ has a greatest and a least element.

Chapter 29

Mean Value Theorems

29.1 Rolle Theorem

Let a, b be real numbers such that $a < b$. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$. If $f(a) = f(b)$ then $\exists t \in]a, b[$ such that

$$f'(t) = 0$$

Proof

Since $[a, b]$ is closed and bounded then it's compact, f attains its maximum and minimum. Let $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If $M \neq l \exists t \in]a, b[$ such that $f(t) = M$

$$f(t+x) = f(t) + f'(t)x + o(|X|)$$

$$f(t-x) = f(t) - f'(t)x + o(|X|)$$

$$0 \leq (f(t+x) - f(t))(f(t-x) - f(t))$$

$$= -f'(t)^2 x^2 + o(|x|^2)$$

$$0 \leq -f'(t)^2 + o(1) \quad x \rightarrow 0$$

Taking the limit when $x \rightarrow 0$ we get $f'(t)^2 = 0$

If $m \neq l$ then any $t \in]a, b[$ such that $f(t) = m$ verifies $f'(t) = 0$

If $m = l = M$ f is constant, so $\forall t \in]a, b[, f'(t) = 0$

29.2 Mean value theorem(Lagrange)

Let a, b be real numbers $a < b$, $f : [a, b] \rightarrow \mathbb{R}$ be a continuous mapping differentiable on $]a, b[$, then $\exists t \in]a, b[$ such that

$$f(b) - f(a) = f'(t)(b - a)$$

Proof

Let $g : [a, b] \rightarrow \mathbb{R}$ be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) = f(a)$ $g(b) = f(a)$ then apply Rolle Theorem to g we get the proof.

29.3 Mean value inequality

Let a, b be real numbers such that $a < b$ $(E, \|\cdot\|)$ be a normed vector space over \mathbb{R} $f : [a, b] \rightarrow E$ be a continuous mapping such that f is differentiable on $]a, b[$ Then

$$\|f(b) - f(a)\| \leq \left(\sup_{x \in]a, b[} \|f'(x)\| \right) (b - a)$$

Proof

Suppose that

$$\sup_{x \in]a, b[} \|f'(x)\| < +\infty$$

Let $M \in \mathbb{R}$ such that

$$M > \sup_{x \in]a, b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition J is an interval containing a , so J is of form $[a, c[$ or $[a, c]$ Since f is continuous by taking a sequence $(c_n)_{n \in \mathbb{N}}$ in $[a, b[$ that converges to c we obtain

$$\begin{aligned} \|f(c) - f(a)\| &= \lim_{n \rightarrow +\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n \rightarrow +\infty} M(c_n - a) \\ &= M(c - a) \end{aligned}$$

Hence $c \in J$ namely $J = [a, c]$

$c > a$ We will prove that $c = b$ by contradiction

Suppose that $c < b \forall h \in]0, b - c[$

$$\begin{aligned} \|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h) \end{aligned}$$

Since $M > \|f'(c)\|$, $\exists h_0 > 0$ such that $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$

Hence

$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h) - f(c)\| + \|f(c) - f(a)\| \\ &\leq M(c_h - c + c - a) \\ &= M(c + h - a)\end{aligned}$$

So $c + h_0 \in J$ Contradiction. Thus

$$\|f(b) - f(a)\| \leq M(b - a)$$

for any $M > \sup_{x \in]a, b[} \|f'(x)\|$ since M is arbitrary the expected inequality holds.

$c = a$ In general, we apply the particular case (fis-extendable to a differentiable mapping at a) to $[\frac{a+b}{2}, b]$ and $[a, \frac{a+b}{2}]$ to get

$$\begin{aligned}\left\|f(b) - f\left(\frac{a+b}{2}\right)\right\| &\leq C \frac{b-a}{2} \\ \left\|f\left(\frac{a+b}{2}\right) - f(a)\right\| &\leq C \frac{b-a}{2}\end{aligned}$$

with $C = \sup_{x \in]a, b[} \|f'(x)\|$

Remark If f is defined on an open neighborhood of a and is differentiable at a the the same arguments hold without the assumption

29.4 Theorem

Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ be a continuous mapping, then $f(I)$ is an interval.

Proof

Let $x \neq y$ be two elements of $f(I)$ Let a, b elements of I such that $x = f(a)$ $y = f(b)$ without loss of generality, we assume $a < b$
Let $z \in \mathbb{R}$ such

$$(z - x)(z - y) \leq 0$$

We construct by induction three sequences $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$ such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If a_n, b_n, c_n are constructed, satisfying

$$c_n = \frac{1}{2}(a_n + b_n)$$

$$(z - f(a_n))(z - f(b_n)) \leq 0$$

we let

$$\begin{aligned} (a_{n+1}, b_{n+1}) &= (a_n, c_n) & \text{if } (z - f(a_n))(z - f(c_n)) \leq 0 \\ (a_{n+1}, b_{n+1}) &= (c_n, b_n) & \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\ & & ((z - f(c_n))(z - f(b_n)) \leq 0) \end{aligned}$$

$$c_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

The sequence $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ are increasing and decreasing respectively and bounded, hence converges to some $l, m \in [a, b]$

Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \rightarrow 0 (n \rightarrow +\infty)$$

So $l = m$, by $(z - f(a_n))(z - f(b_n)) \leq 0$ we obtain by letting $n \rightarrow +\infty$

$$(z - f(l))^2 \leq 0$$

So $z = f(l)$

29.5 Theorem(Heine)

Let I be an open interval of \mathbb{R} and $f : I \rightarrow \mathbb{R}$ be a differentiable mapping. Then $f'(I)$ is an interval.

Proof

Let $(a, b) \in I^2$ such that $a < b$. Consider the following mappings:

$$\begin{aligned} g : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(x) - f(a)}{x - a} & x \neq a \\ f'(a) & x = a \end{cases} \\ h : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(b) - f(x)}{b - x} & x \neq b \\ f'(b) & x = b \end{cases} \end{aligned}$$

g, h are continuous $(\frac{f(x) - f(a)}{x - a} = f'(a) + o(1) \text{ as } x \rightarrow a)$

So $g([a, b])$ and $h([a, b])$ are intervals. Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$

$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that $g(b) = h(a)$ so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence $f'(I)$ is an interval.

Chapter 30

Fixed Point Theorem

30.1 Def

Let X be a set and $T : X \rightarrow X$ be a mapping. If $x \in X$ satisfies $T(x) = x$ we say that x is a fixed point of T .

30.2 Def

Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. If $\exists \epsilon \in [0, 1[$ such that T is ϵ -Lipschitzian then we say that T is a contraction.

30.3 Fixed Point Theorem

Let (X, d) be a COMPLETE non-empty metric space, and $T : X \rightarrow X$ be a contraction. Then T has a unique fixed point. Moreover, $\forall x_n \in X$ if we let

$$x_{n+1} = T(x_n), x_0 \in X$$

then $(x_n)_{n \in \mathbb{N}}$ converges to the fixed point.

Proof

If p and q are two fixed point of T , then

$$d(p, q) = d(T(p), T(q)) \leq \epsilon d(p, q)$$

So $d(p, q) = 0$.

Let

$$x_{n+1} = T(x_n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any $N \in \mathbb{N}$, $\forall (n, m) \in \mathbb{N}_{\geq N}^2$ $n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \end{aligned}$$

So

$$\lim_{N \rightarrow +\infty} \sup_{(n, m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges to some $p \in X$

$$d(T(p), p) = \lim_{n \rightarrow +\infty} d(T(x_n), x_n) = 0$$

since $d : X^2 \rightarrow \mathbb{R}_{\geq 0}$ is continuous.

Part VI

Higher differentials

Chapter 31

Multilinear mapping

Let K be a commutative cenitary ring.

31.1 Def

Let $n \in \mathbb{N}$, V_1, \dots, V_n, W be K -modules. We call n -linear mapping from $V_1 \times \dots \times V_n$ to W any mapping $f : V_1 \times \dots \times V_n \rightarrow W$ such that $\forall i \in \{1, \dots, n\} \forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in V_1 \times \dots \times V_{i-1} \times V_{i+1} \times \dots \times V_n$ the mapping

$$\begin{aligned} f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) : V_i &\rightarrow W \\ x_i &\mapsto f(x_i) \end{aligned}$$

is a morphism of K -modules

We denote by $Hom^{(n)}(V_1 \times \dots \times V_n, W)$ the set of all n -linear mappings from $V_1 \times \dots \times V_n$ to W .

31.2 Example

$$\begin{aligned} K \times K &\rightarrow K \\ (a, b) &\mapsto ab \end{aligned}$$

is a 2-linear mapping (bilinear mapping)

31.3 Remark

$$Hom^{(0)}(\{0\}, W) := W \text{ (by convention)}$$

$$Hom^{(1)}(V_1, W) = Home(V_1, W) = \{\text{morphism of } K\text{-module from } V_1 \text{ to } W\}$$

31.4 Prop

Suppose that $n \geq 2$ For any $i \in \{1, \dots, n-1\}$

$$\begin{aligned} \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) &\xrightarrow{\Phi} \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n)) \\ f &\mapsto ((x_1, \dots, x_i) \mapsto ((x_{i+1}, \dots, x_n) \mapsto f(x_1, \dots, x_n))) \end{aligned}$$

is a bijection

Proof

The inverse of Φ is given by

$$g \in \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \mapsto (((x_1, \dots, x_n) \in V_1 \times \dots \times V_n) \mapsto g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n))$$

31.5 Remark

$\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$ is a sub-K-module of $W^{V_1 \times \dots \times V_n}$ and Φ is an isomorphism of K-modules.

Chapter 32

Operator norm of Multilinear field

Let $(K, |\cdot|)$ be a complete valued field

32.1 Def

Let $V_1 \times \dots \times V_n$ and W be normed vector spaces over K . We define

$$\|\cdot\| : \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) \rightarrow [0, +\infty]$$

as

$$\|f\| := \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

If $\|f\| < \infty$ we say that f is bounded. We denote by $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$ the set of bounded n -linear mappings from $V_1 \times \dots \times V_n$ to W .

32.2 Theorem

For any $i \in \{1, \dots, n-1\}$, $\forall f \in \mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \forall (x_1, \dots, x_i) \in V_1 \times \dots \times V_i$ the $(n-i)$ -linear mapping

$$\begin{aligned} f(x_1, \dots, x_i, \cdot) : V_{i+1} \times \dots \times V_n &\rightarrow W \\ (x_{i+1}, \dots, x_n) &\mapsto f(x_1, \dots, x_n) \end{aligned}$$

belongs to $\mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)$. Moreover

$$\|f\| = \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

Proof

$$\forall (x_{i+1}, \dots, x_n) \in V_{i+1} \times \dots \times V_n$$

$$\begin{aligned} \|f(x_1, \dots, x_n)\| &\leq \|f\| \|x_1\| \dots \|x_n\| \\ &= (\|f\| \|x_1\| \dots \|x_i\|) \|x_{i+1}\| \dots \|x_n\| \end{aligned}$$

So

$$\|f(x_1, \dots, x_i, \cdot)\| \leq \|f\| \|x_1\|, \dots, \|x_i\|$$

If we define

$$\|f\|' := \sup_{(x_1, \dots, x_i) \in V_1 \times \dots \times V_i, x_1 \dots x_i \neq 0} \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|}$$

then

$$\|f\|' \leq \|f\|$$

32.3 Corollary

- (1) $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$ is a vector subspace of $\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$
- (2) $\|\cdot\|$ is a norm on $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$
- (3) $\forall i \in \{1, \dots, n\}$

$$\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \xrightarrow{\Phi} \mathcal{L}^{(n)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$$

is a K-linear isomorphism that preserves operator norms.

$$\|f\| = \|\Phi(f)\|$$

32.3.1 Proof

Conversely $\forall (x_1, \cdot, x_n) \in V_1 \times \dots \times V_n$ such that $x_1 \dots x_n \neq 0$

$$\|f(x_1, \dots, x_n)\| \leq \|f(x_1, \dots, x_i, \cdot)\| \|x_{i+1}\| \dots \|x_n\|$$

Hence

$$\frac{f(x_1, \dots, x_n)}{\|x_1\| \dots \|x_n\|} \leq \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|} \leq \|f\|'$$

Taking sup, we get

$$\|f\| \leq \|f\|'$$

We reason by induction on n

$n = 1$

$$\mathcal{L}^{(1)}(V_1, W) = \mathcal{L}(V_1, W)$$

$i \in \{1, \dots, n-1\}$ Suppose that the corollary is true for m -linear mappings with $m < n$. We consider the following diagram of mapping

To show that $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$ is a vector subspace, it suffices to check that $\forall g \in \mathcal{L}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$ one has $\|\Phi^{-1}(g)\| = \|g\| < +\infty$

$$\begin{aligned} \mathcal{L}^{(i)}(V_{i+1} \times \dots \times V_n, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \\ &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \end{aligned}$$

For any $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n$

$$\begin{aligned} \|\Phi^{-1}(g)(x_1, \dots, x_n)\| &= \|g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)\| \\ &\leq \|g(x_1, \dots, x_i)\| \|x_{i+1}\| \dots \|x_n\| \\ &\leq \|g\| \|x_1\| \dots \|x_i\| \|x_{i+1}\| \dots \|x_n\| \end{aligned}$$

Therefore

$$\|\Phi^{-1}(g)\| \leq \|g\| = \|\Phi^{-1}(g)\|$$

Chapter 33

Higher differentials

We fix a complete non-trivial valued field $(K, |\cdot|)$ and normed K -vector space E and F .

33.1 Def

Let $U \subseteq E$ be an open subset and $f : U \rightarrow F$ be a mapping

- (1) If f is continuous, we say that f is of class C^0 and f is 0-times differentiable
- (2) If f is differentiable on an open neighborhood $V \subseteq U$ of some point $p \in U$ and

$$\begin{aligned} df : V &\rightarrow \mathcal{L}(E, F) \\ x &\mapsto d_x f \end{aligned}$$

is n -times differentiable at p , then we say that f is $(n+1)$ -times differentiable at p . If f is $(n+1)$ -times differentiable at any point $p \in U$, we denote by

$$D^{n+1}f : U \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

the mapping that sends $x \in U$ to the image of $D^n(df)(x)$ by the K -linear bijection

$$\mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

$$df : U \rightarrow \mathcal{L}(E, F)$$

$$D^n(df) : U \rightarrow \mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \xrightarrow{\Phi} \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

If $D^{n+1}f$ is continuous, we say that f is of class C^{n+1} ($n \geq 0$) (Any mapping $f : U \rightarrow F$ is considered as 0-times differential $D^0f := f$)

33.2 Remark

If f is n -times differentiable $\forall i \in \{1, \dots, n-1\}$
 $\forall p \in U, (h_1, \dots, h_n) \in E^n$ one has

$$D^i(D^{n-i}f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n) = D^n f(p)(h_1, \dots, h_n)$$

$$D^{n-i}f : U \rightarrow \mathcal{L}^{(n-i)}(E^{n-i}, F)$$

$$D^i(D^{n-i}f) : \quad U \xrightarrow{\quad} \mathcal{L}^{(i)}(E^i, \mathcal{L}^{(n-i)}(E^{n-i}, F)) \quad U \rightarrow$$

$$\quad \quad \quad \searrow D^n f \quad \quad \quad \updownarrow \cong \quad \quad \quad \mathcal{L}^{(n)}(E^n, F)$$

33.3 Theorem

Assume that $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$
 Let $f : U \rightarrow F$ be a mapping that is $(n+1)$ -times differentiable on U . Let
 $p \in U$ and $h \in E$ such that $p + th \in U \forall t \in [0, 1]$ Then

$$\left\| f(p+h) - f(p) - \sum_{k=1}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq$$

$$\left(\sup_{t \in]0, 1[} \frac{(1-t)^n}{n!} \|D^{n+1} f(p+th)\| \right) \cdot \|h\|^{n+1}$$

(Taylor-Lagrange formula)

33.4 Prop(Gronwall inequality)

Let F be a normed vector space over \mathbb{R} $(a, b) \in \mathbb{R}^2, a < b$ Let $f : [a, b] \rightarrow F$
 and $g : [a, b] \rightarrow \mathbb{R}$ be continuous mappings that are differentiable on $]a, b[$

Suppose that $\forall t \in]a, b[$

$$\|f'(t)\| \leq g'(t)$$

then

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

Proof

Let $c \in]a, b[$ Let $\epsilon > 0$ Let

$$J = \{t \in [c, b] \mid \forall s \in [c, t], \|f(s) - f(c)\| \leq g(s) - g(c)\}$$

By definition J is an interval.

Since f, g are continuous, J is a closed interval, hence J is of the form $[c, t]$.
If $t < b$ then for $h > 0$ Sufficiently small.

$$f(t+h) - f(t) = hf'(t) + o(h)$$

$$g(t+h) - g(t) = hg'(t) + o(h)$$

$$\exists \delta > 0 \quad \forall h \in [0, \delta]$$

$$\|f(t+h)\| \leq \|f'(t)\| \cdot h + \frac{\epsilon}{2}h$$

$$g(t+h) - g(t) \geq g'(t)h - \frac{\epsilon}{2}h$$

So

$$\|f(t+h) - f(t)\| \leq g(t+h) - g(t) + \epsilon h$$

Moreover

$$\|f(t) - f(c)\| \leq g(t) - g(c) + \epsilon(t - c)$$

\Rightarrow

$$\|f(t+h) - f(c)\| \leq g(t+h) - g(c) + \epsilon(t+h-c)$$

\Rightarrow

$$J \supseteq [c, t + \delta]$$

Contradiction, hence

$$\|f(b) - f(c)\| \leq g(b) - g(c) + \epsilon(b - c)$$

For the same reason

$$\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c - a)$$

Hence

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b - a)$$

Since $\epsilon > 0$ is arbitrary

$$\|f(b) - f(c)\| \leq g(b) - g(c)$$

Mean value theorem:

$$g(t) = (\sup(\|f'(\cdot)\|))$$

33.5 Theorem

Let $n \in \mathbb{N}$, E, F be normed vector spaces over \mathbb{R} $U \subseteq E$ open and $f : U \rightarrow F$ be a mapping that is $(n+1)$ -times differentiable. Let $p \in U$ and $h \in E$. Assume that $\forall \epsilon \in [0, 1], p + th \in U$

Let

$$M = \sup_{t \in]0, 1[} \|D^{n+1}f(p + th)\|$$

Then

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

If $E = \mathbb{R}$ Then the formula become

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(p) h^k \right\| \leq \frac{M}{(n+1)!} |h|^{n+1}$$

Proof

Consider $\phi : [0, 1] \rightarrow F$

$$\phi(t) = \sum_{k=0}^n \frac{(1-t)^k}{k!} D^k f(p+th)(h, \dots, h)$$

$$\phi(1) = f(p+h)$$

$$\phi(0) = \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h)$$

$$\begin{aligned} \phi'(t) &= \sum_{k=0}^n \frac{(1-t)^k}{k!} D^{k+1} f(p+th)(\underbrace{h, \dots, h}_{k+1 \text{ copies}}) - \sum_{k=1}^n \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p+th)(h, \dots, h) \\ &= \frac{(1-t)^n}{n!} D^{n+1} f(p+th)(h, \dots, h) \end{aligned}$$

then

$$\|\phi'(t)\| \leq M \frac{(1-t)^n}{n!} = (-M \frac{(1-t)^{n+1}}{(n+1)!})'$$

By Gronwall inequality,

$$\|\phi(1) - \phi(0)\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

33.6 Def

Let $n \in \mathbb{N}$ E_1, \dots, E_n and F be normed vector spaces over a complete non-trivial valued field $(K, |\cdot|)$ Let $U \in E_1 \times \dots \times E_n$ be an open subset. $p = (p_1, \dots, p_n) \in U$ $i \in \{1, \dots, n\}$, $f : U \rightarrow F$ If there exists an open neighborhood U_i of p_i in E_i such that

$$\begin{aligned} U_i &\rightarrow F \\ x_i &\mapsto f(p_1, \dots, p_{i-1}, x_i, p_{i+1}, \dots, p_n) \end{aligned}$$

is well defined and is differentiable at p_i

We denote by $\frac{\partial f}{\partial x_i}(p)$ the differential of this mapping $U_i \rightarrow F$ and say that f admits the i^{th} partial differentials at p

33.7 Prop

Suppose that $(K, |\cdot|)$ and f has all partial differentials on U and

$$\frac{\partial f}{\partial x_i} : U \rightarrow \mathcal{L}(E_i, F)$$

is continuous for any $i \in \{1, \dots, n\}$ Then f is of class C^1 and $\forall h = (h_1, \dots, h_n) \in E_1 \times \dots \times E_n$

$$\forall p \in U \quad d_p(h) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(h_i)$$

Proof

By induction, it suffices to treat the case where $n = 2$
 $\forall \epsilon > 0 \exists \delta > 0$

$$\forall (h, k) \in E_1 \times E_2 \quad \max\{|h|, |k|\} \leq \delta$$

one has

$$\left\| \frac{\partial f}{\partial x_i}(a + h, b + k) - \frac{\partial f}{\partial x_i}(a, b) \right\| \leq \epsilon \text{ (by continuity of } \frac{\partial f}{\partial x_i} \text{)}$$

Consider the mapping $\phi : [0, 1] \rightarrow F$

$$\phi(t) = f(a + h, b + tk) - f(a + h, b) - t \underbrace{\frac{\partial f}{\partial x_2}(a + h, b)(k)}_{\in \mathcal{L}(E_2, F)}$$

$$\begin{aligned} \|\phi'(t)\| &= \left\| \frac{\partial f}{\partial x_2}(a + h, b + tk)(k) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \\ &\leq 2\epsilon \|k\| \end{aligned}$$

$$\|\phi(1) - \phi(0)\| \leq 2\epsilon \|k\|$$

then

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \leq 2\epsilon \|k\|$$

So

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

f has 1st partial differential

$$\left\| f(a+h, b) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

by continuity of $\frac{\partial f}{\partial x_i}$

$$\left\| \frac{\partial f}{\partial x_2}(a+h, b)(k) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

take the sum of above three statements, we get:

$$\left\| f(a+h, b+k) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(h) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

33.8 Theorem

Let E, F be normed vector spaces over \mathbb{R} $U \subseteq E$ open $(f_n)_{n \in \mathbb{N}}$ be a sequence of differentiable mapping from U to F Let $g : U \rightarrow \mathcal{L}(E, F)$ Suppose that

- (1) $(df_n)_{n \in \mathbb{N}}$ converges uniformly to g
- (2) $(f_n)_{n \in \mathbb{N}}$ converges pointwisely to some mapping $f : U \rightarrow F$

Then f is differentiable and $df = g$

Proof

Let $p \in U, \forall (m, n) \in \mathbb{N}^2, \forall x \in \mathcal{B}(p, r) \in U (r > 0)$

$$\|f_n(x) - f_m(x) - (f_n(p) - f_m(p))\| \leq (\sup_{\xi \in U} \|d_\xi f_m - d_\xi f_n\|) \cdot \|x - p\| \quad (\text{mean value inequality})$$

Take $\lim_{m \rightarrow +\infty}$ we get:

$$\|(f_n(x) - f(x)) - (f_n(p) - f(p))\| \leq \epsilon_n \|x - p\|$$

where $\epsilon_n = \sup_{\xi \in U} \|d_\xi f_m - g\|$.

So

$$\begin{aligned} \|f(x) - f(p) - g(p)(x-p)\| &\leq \|(f(x) - f_n(x)) - (f(p) - f_n(p))\| \\ &\quad + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| \\ &\quad + \|d_p f_n(x-p) - g(p)(x-p)\| \\ &\leq \epsilon_n \|x-p\| + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| + \epsilon_n \|x-p\| \\ \limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} &\leq 2\epsilon_n \end{aligned}$$

Take $\lim_{n \rightarrow +\infty}$ we get:

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} = 0$$

Chapter 34

Permutations

34.1 Def

Let X be a set. We denote with \mathfrak{S}_X the set of all bijections from X to itself. The elements of \mathfrak{S}_X are called permutations if the set X is finite. If $x_1, \dots, x_n \in X$ are distinct elements then

$$(x_1, \dots, x_n) \in \mathfrak{S}_X$$

such that

$$x_i \mapsto x_{i+1}$$

$$x_n \mapsto x_1$$

this is called an n -cycle. A 2-cycle is called a transposition.

34.1.1 Example

$$X = \{1, \dots, 7\}$$

$$1 \mapsto 4$$

$$2 \mapsto 1$$

$$3 \mapsto 2$$

$$(2\ 3)(4\ 2\ 1) = 4 \mapsto 3$$

$$5 \mapsto 5$$

$$6 \mapsto 6$$

$$7 \mapsto 7$$

$$= (1\ 4\ 3\ 2)$$

34.2 Def

We denote with

$$orb_\sigma(x) = \{\underbrace{\sigma \circ \dots \circ}_{n\text{-times}} \quad n \in \mathbb{N}\}$$

$$x \in X, \sigma \in \mathfrak{S}_X$$

34.3 Prop

If $\text{orb}_\sigma(x)$ is a finite set of d elements, then one has

$$\sigma^d(x) = x \quad \text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{d-1}(x)\}$$

moreover

$$\sigma^{-1}(x) \in \text{orb}_\sigma(x)$$

34.3.1 Proof

The set

$$\{(n, m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

is not empty. Let

$$d' := \min\{m - n \mid (n, m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

therefore $x, \sigma(x), \dots, \sigma^{d'-1}(x)$ are all distinct.

Now use the each deass division

$$h = qd' + r \quad r < d'$$

$$\sigma^h(x) = \sigma^r(x) \quad 0 \leq r < d'$$

then

$$d' \geq d$$

and for

$$\{x, \sigma(x), \dots, \sigma^{d'-1}(x)\} \subseteq \text{orb}_\sigma(x)$$

\Rightarrow

$$d' \leq d$$

then

$$d' = d$$

34.4 Remark

Let $Y \subseteq X$, then we have a homomorphism of groups:

$$\begin{aligned} \mathfrak{S}_Y &\rightarrow \mathfrak{S}_X \\ \sigma &\mapsto \left(x \mapsto \begin{cases} \sigma(x) & \text{if } x \in Y \\ x & \text{if } x \in X \setminus Y \end{cases} \right) \end{aligned}$$

If Y and Z are subset of X

$$Y \cap Z = \emptyset, \sigma \in \mathfrak{S}_Y, \tau \in \mathfrak{S}_Z$$

then

$$\sigma \circ \tau = \tau \circ \sigma$$

If X is finite with n elements $\mathfrak{S}_X = S_n$ permutation group of n elements.

34.5 Theorem

Let X be a finite set and let $\sigma \in \mathfrak{S}_X$ then exist $d \in \mathbb{N}$ and $(n_1, \dots, n_d) \in \mathbb{N}_{\geq 2}^d$ and pairwise disjoint subsets X_1, \dots, X_d of X of cardinalities n_1, \dots, n_d , together with n_i -cycle τ_i of X_i such that

$$\sigma = \tau_1 \circ \dots \circ \tau_d$$

In other words. Any permutation can be decomposed in composition of finitely many cycles on disjoint subsets.

Proof

By induction on the cardinality of X .

The case $\sigma = id_X$ is trivial. ($d = 0$) So the case when $N = 0, 1$ is clear.

Assume $N \geq 2$. Take $x \in X$ such that $\sigma(x) \neq x$ and let $X_1 = orb_\sigma(x)$
 $Y = X \setminus X_1 \forall y \in Y$ we have $\sigma(y) \in Y$ (because if $\sigma(y) \in X$ by the previous proposition $\sigma(y) \in X_1$)

Let $\tau = \sigma|_Y \in \mathfrak{S}_Y$ Use the induction hypothesis, we get X_2, \dots, X_d of cardinalities n_2, \dots, n_d and n_i -cycle τ_i such that

$$\tau = \tau_2 \circ \dots \circ \tau_d$$

Consider $\tau_1 = \sigma|_{X_1}$ then τ_1 is a n_1 -cycle of X_1

\Rightarrow

$$\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_d$$

34.5.1 Remark

This theorem say that the groups of permutation si generated by cycles.

34.6 Corollary

Let X be a finite set. Then \mathfrak{S}_X is generated by transpositions.

Proof

Note that

$$(x_1, \dots, x_n) = (x_1, x_2) \circ (x_2, \dots, x_n)$$

By induction

$$(x_1, \dots, x_n) = (x_1, x_2) \circ \dots \circ (x_{n-1}, x_n)$$

34.6.1 Remark

The decomposition of transposition is unique.

34.7 Def

Let $\tau \in \mathfrak{S}_n := G_{\{1, \dots, n\}}$ is called adjacent if τ is of the form $(j, j+1)$ for $j = 1, \dots, n-1$

34.8 Corollary

\mathfrak{S}_n is generated by adjacent transposition.

34.8.1 Proof

Note that

$$(i, j) = (i, i+1) \circ (i+1, i+2) \circ \dots \circ (j-1, j) \circ (j-2, j-1) \circ \dots \circ (i+2, i+1)$$

Some other information on \mathfrak{S}_n

34.9 Caybey Theorem

Any finite group can be embedded (injective morphism) in a \mathfrak{S}_n for some $n \in \mathbb{N}$

Proof

Let G be a finite group and $n = \text{card}(G)$. Let

$$\begin{aligned} \varphi : G &\rightarrow \mathfrak{S} \\ g &\mapsto l_g \end{aligned}$$

be the mapping sends $g \in G$ to $l_g(x) = gx, \forall x \in G$

34.10 Theorem

Let X be a finite set. Assume that $\sigma \in \mathfrak{S}_X$ can be written as

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

where τ_1 is transposition.

We put

$$\text{sgn}(\sigma) := (-1)^\sigma$$

This is a well-defined function. Moreover sgn is a morphism from \mathfrak{S}_X to $(\{-1, 1\}, \times)$

Proof

Let's define the mapping:

$$\begin{aligned} \phi : \mathfrak{S}_n &\rightarrow \mathbb{Q}^\times \\ \sigma &\mapsto \prod_{(i,j) \in \{1, \dots, n\}^2, i < j} \frac{\sigma(i) - \sigma(j)}{i - j} \end{aligned}$$

To show that ϕ is a morphism of groups. Let

$$\theta = \{U \in \wp(\{1, \dots, n\}) \mid \text{card}(U) = 2\}$$

$$\begin{aligned} \phi(\sigma \circ \tau) &= \prod_{(i,j) \in \theta} \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{i - j} \\ &= \left(\frac{\sigma(\tau(i)) - \sigma(\tau(j))}{\tau(i) - \tau(j)} \right) \times \left(\prod_{(i,j) \in \theta} \frac{\tau(i) - \tau(j)}{i - j} \right) \\ &= \phi(\sigma)\phi(\tau) \end{aligned}$$

When τ is a transposition, $\phi(\tau) = -1$. Therefore

$$\phi(\sigma) = \prod_{i=1}^d \phi(\tau_i)$$

since

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

34.11 Remark

Let $A_n \subsetneq \mathfrak{S}_n$ such that

$$A_n = \{\sigma \in \mathfrak{S}_n \mid \text{sgn}(\sigma) = 1\}$$

is an alternating symmetric group.

34.12 Exercise

Let X be a set of cardinality n . Let $\sigma : X \rightarrow \{1, \dots, n\}$ be a bijection. Prove that

$$\begin{aligned} \phi : \mathfrak{S}_X &\rightarrow \mathfrak{S}_n \\ \tau &\mapsto \sigma^{-1} \circ \tau \circ \sigma \end{aligned}$$

is an isomorphism.

34.13 Symmetric of multilinear mapping

We fix a commutative unitary ring K and K -modules E, F

34.14 Def: Symmetric and Alternating

symmetric Let $n \in \mathbb{N}$ and $f \in \text{Hom}^{(n)}(E^n, F)$. If for any $\sigma \in \mathfrak{S}_n$ one has $\forall x \in E^n$

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Then we say f is symmetric

alternating If for any $(i, j) \in \{1, \dots, n\}^2$ such that $i \neq j$ and any $(x_1, \dots, x_n) \in E^n$ such that $x_i = x_j$

$$f(x_1, \dots, x_n) = 0$$

then we say that f is alternating.

34.15 Prop

Suppose that $f \in \text{Hom}^{(n)}(E^n, F)$ is alternating, then $\forall (x_1, \dots, x_n) \in E^n$, $\sigma \in \mathfrak{S}_n$

$$f(x_1, \dots, x_n) = \text{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Proof

By corollary 34.8, it's enough to prove the proposition for adjacent transitions. Let $i \in \{1, \dots, n-1\}$ then

$$\begin{aligned} 0 &= f(x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_n) \\ &= f(x_1, \dots, x_{i-1}, x_i, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \end{aligned}$$

The adjacent transition σ is $(i, i+1)$

34.16 Def:

Hom_s and Hom_a

We denote with $Hom_s^{(n)}(E^n, F)$ and $Hom_a^{(n)}(E^n, F)$ the set of symmetric and alternating n -linear mappings from E to F . $Hom_s^{(n)}(E^n, F)$ and $Hom_a^{(n)}(E^n, F)$ are sub-K-modules of $Hom^{(n)}(E^n, F)$ and when $n = 1$, by convention

$$Hom_s^{(1)}(E, F) = Hom_a^{(1)}(E, F) = Hom(E, F)$$

34.17 Reminder

Let E, F be two normed vector spaces over \mathbb{R} $f : E \rightarrow F$ is differentiable(twice)

$$\begin{aligned} df : E &\rightarrow \mathcal{L}(E, F) \\ D^2d : E &\rightarrow \mathcal{L}(E, \mathcal{L}(E, F)) \\ A &\mapsto ((x, y) \rightarrow A(x)(y)) \end{aligned}$$

34.18 Theorem(Schweiz)

$U \subseteq E$ is an open set, $f : U \rightarrow F$ is a function of class C^n . Then for any $p \in U$

$$D^n f(p) \in \mathcal{L}^n(E^n, F)$$

is symmetric

Proof

By induction and by the fact that permutation are decomposed in transpositions, we can reduce to prove only the case $n = 2$

$$d_{p+u}f - d_p f = D^2 f(p)(u, \cdot) + o(u)$$

$\forall \epsilon > 0, \exists \delta > 0$ such that $0 < \|u\| < \delta$, then

$$\|d_{p+u}f - d_p f - D^2 f(p)(u, \cdot) + o(u)\| \leq \epsilon \|u\|$$

For any $x \in \mathcal{B}(p, \frac{\epsilon}{2})$ let's introduce the following function

$$\varphi(x) = f(x+k) - f(x) - D^2 f(p)(k, x)$$

We use the mean value inequality on φ

$$\begin{aligned} &\|\varphi(p+h) - \varphi(p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - D^2 f(p)(k, p+h) - f(p+k) - f(p) - D^2 f(p)(k, p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - f(p+k) - D^2 f(p)(k, h)\| \\ &\leq \left(\sup_{t \in [0,1]} \|d_{p+th} \varphi\| \right) \|h\| \end{aligned}$$

$$\|d_{p+th}(\varphi)\| = \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\|$$

add and subtract $d_p f, D^2f(p)(th, \cdot)$ then by triangle inequality

$$\begin{aligned} & \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\| \\ & \leq \|d_{p+th+k}(f) - d_p f - D^2f(p)(k + th, \cdot)\| \\ & \quad + \|d_{p+th}f - d_p f - D^2f(p)(th, \cdot)\| \\ & \leq \epsilon \|th + k\| + \epsilon(th) \\ & \leq 2\epsilon(\|h\| + \|k\|) \end{aligned}$$

then

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - D^2f(p)(k, h)\| \\ & = o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

exchange the role of h, k then we get

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - f(p + k) - D^2f(p)(h, k)\| \\ & \leq o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

then

$$\underbrace{\|D^2f(p)(k, h) - D^2f(p)(h, k)\|}_{\text{bilinear function}} = o(\underbrace{\max\{\|h\|, \|k\|\}^2}_{\text{quachetic}})$$

this implies that the LHS is 0

34.19 Def

Let E, F be normed vector spaces over a complete value field $(K, |\cdot|)$ let $U \subseteq E, V \subseteq F$ be open subsets and $f : U \rightarrow V$ is a bijection.

- (1) If f and f^{-1} are both continuous we say that f is a homeomorphism
- (2) If f and f^{-1} are both of class C^n we say that f is a e^n -diffeomorphism

If (2) is true for any $n \in \mathbb{N}$ we say that f is a C^∞ -diffeomorphism

34.20 Prop

Let E, F be two normed Banach spaces. Let $I(E, F) \in \mathcal{L}(E, F)$ be the set of linear continuous and invertible mappings such that $\text{norm} \varphi^{-1} \leq +\infty$. Then $I(E, F)$ is open in $\mathcal{L}(E, F)^\vee$ Moreover the mapping

$$\begin{aligned} I(E, F) & \rightarrow I(F, E) \\ \phi & \mapsto \varphi^{-1} \end{aligned}$$

is a e^1 -diffeomorphism

Proof

Let $\varphi \in I(E, F)$ we want to show that

$$\varphi - \psi \in I(E, F)$$

for $\psi \in \mathcal{E}, \mathcal{F}$ such that $\|\psi\| < \frac{1}{\|\varphi^{-1}\|}$ Notice that

$$\varphi - \psi = \varphi \circ (Id_E - \varphi^{-1} \circ \psi)$$

Since

$$\|\varphi^{-1}\psi\| \leq \|\varphi^{-1}\| \|\psi\| < 1$$

This means that the series

$$\sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n}$$

is absolutely convergent in $\mathcal{L}(E, E)$ This series is the inverse of $(Id_E - \varphi^{-1}\psi)$

$$(Id_E - \varphi^{-1}\psi) \circ \sum_{n=0}^{N-1} (\varphi^{-1} \circ \psi) \overset{\text{composite n times}}{\widehat{\circ n}} = Id_E - (\varphi^{-1} \circ \psi)^{\circ N}$$

take $\lim_{N \rightarrow +\infty}$, then

$$(\varphi - \psi)^{-1} = \sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n} \circ \varphi^{-1}$$

and

$$(\varphi - \psi)^{-1} = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

replace the inverse with i

$$i(\varphi - \psi) - i(\varphi) = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

then

$$d_\varphi i(\psi) = i(\varphi) \circ (-\psi) \circ i(\varphi)$$

so i is differentiable. Moreover i and i^{-1} are continuous.

Remark

By induction we can show that i is a $C^{+\infty}$ -diffeomorphism

34.21 Prop

Let $n \in \mathbb{N} \cup \{\infty\}$ Let E, F, G be normed vector spaces over a complete valued field $(K, |\cdot|)$ $U \subseteq E, V \subseteq F$ be open sets. $f : U \rightarrow V, g : V \rightarrow G$ be mappings of class C^n , then $g \circ f$ also of class C^n

34.21.1 Proof

The case where $n = 0$ is known

Denote by

$$\begin{aligned}\Phi : \mathcal{L}(E, F) \times E &\rightarrow F \\ (\beta, \alpha) &\mapsto \beta \circ \alpha\end{aligned}$$

Φ is a bounded bilinear mapping

$$\|\Phi(\beta, \alpha)\| \leq \|\beta\| \cdot \|\alpha\|$$

Suppose that $n \geq 1$ and the statement is true for mappings of class C^{n-1} $g \circ f$ is differentiable.

$$\forall p \in U \quad d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

$$D^1(g \circ f) : U \rightarrow \mathcal{L}(E, G)$$

$$D^1 = \Phi \circ (D^1 g \circ f, D^1 f)$$

$$\begin{aligned}(D^1 g \circ f, D^1 f) : U &\rightarrow \mathcal{L}(F, G) \times \mathcal{L}(E, F) \\ p &\mapsto (d_{f(p)}g, d_p f)\end{aligned}$$

$$d_{\beta_0, \alpha_0} \Phi(\beta, \alpha) = \beta_0 \circ \alpha + \beta \circ \alpha_0$$

$$\begin{aligned}D^1 \Phi : \mathcal{L}(F, G) \times \mathcal{L}(E, F) &\rightarrow \mathcal{L}(\mathcal{L}(F, G) \times \mathcal{L}(E, F), \mathcal{L}(E, G)) \\ (\alpha_0, \beta_0) &\mapsto ((\alpha, \beta) \mapsto \beta_0 \circ \alpha + \beta \circ \alpha_0)\end{aligned}$$

Since g, f are of class C^n $D^1 f, D^1 g$ are of class C^{n-1} Thus, by induction hypothesis,

$$(D^1 g \circ f, D^1 f)$$

is of class C^{n-1} Since Φ is of class C^∞ , we obtain that

$$D^1(g \circ f)$$

is of class C^{n-1} then

$$g \circ f$$

is of class C^n

34.22 Prop

Let E and F be Banach space over a complete valued field $(K, |\cdot|)$. U and V be open subsets of E and F respectively. $n \in \mathbb{N} \cup \{\infty\}$ and $f : U \rightarrow V$ be a bijection. If f is of class C^n , then f^{-1} is differentiable, then f^{-1} is of class C^n

Proof

$$f \circ f^{-1} = Id_V$$

$$\forall y \in V$$

$$d_y(f \circ f^{-1}) = d_{f^{-1}(p)}f \circ d_y f^{-1} = Id_F$$

For $x \in U, y = f(x)$

$$d_y(f \circ f^{-1}) = d_x f \circ d_y f^{-1} = Id_F$$

$$d_x(f^{-1} \circ f) = d_y f \circ d_x f^{-1} = Id_E$$

So

$$d_y f^{-1} - (d_x f)^{-1}$$

that is

$$D^1 f^{-1} = \iota \circ (D^1 f \circ f^{-1})$$

where

$$\begin{aligned} \iota : I(E, F) &\rightarrow I(F, E) \\ \phi &\mapsto \phi^{-1} \end{aligned}$$

Suppose that f^{-1} is of class C^{n-1} then

$$D^1 f^{-1} = \iota D^1 f \circ f^{-1}$$

is of class C^{n-1}

34.23 Local Inversion Theorem

Let E and F be Banach space over \mathbb{R} $U \in E$ open, $f : U \rightarrow F$ be a mapping of class C^n and $a \in U$. Suppose that $d_a f \in I(E, F)$ ($d_a f$ is invertible and of bounded inverse) Then there exists open neighborhoods V and W of a and $f(a)$ respectively, such that

- $V \subseteq U$ and $f(V) \subseteq W$
- The restriction of f to V defines a bijection from V to W
-

$$(f|_V)^{-1}W \rightarrow V$$

is of class C^n

34.23.1 Proof

For $y \in F$ consider the mapping:

$$\begin{aligned}\phi_y : U &\rightarrow F \\ x &\mapsto x - (d_a f)^{-1}(f(x) - y)\end{aligned}$$

$f(x) = y$ iff $\phi_y(x) = x$ i.e. x is a fix point of ϕ_y ϕ_y is of class C^1 and

$$d_x \phi_y(v) = v - d_a f^{-1}(d_x f(v))$$

$\forall v$

$$d_a \phi_y^{(v)} = 0$$

By the continuity of $D^1 f$ there exists $r > 0$ such that

$$\overline{\mathcal{B}}(a, r) \subseteq U$$

and $\forall y \in F, \forall x \in \overline{\mathcal{B}}(a, r)$

$$\|d_x \phi_y\| \leq \frac{1}{2}$$

By the mean value inequality. $\forall (x_1, x_2) \in \overline{\mathcal{B}}(a, r)$

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|$$

Hence ϕ_y is contraction.

By the boundedness of $(d_a f)^{-1} \exists \delta > 0$ such that

$$\forall y \in \overline{\mathcal{B}}(f(a), \delta) \quad \|(d_a f)^{-1}(f(a) - y)\| \leq \frac{r}{2}$$

Then $\forall x \in \overline{\mathcal{B}}(a, r) \quad y \in \overline{\mathcal{B}}(f(a), \delta)$

$$\begin{aligned}\|\phi_y(x) - a\| &\leq \|\phi_y(x) - \phi_y(a)\| + \|\phi_y(a) - a\| \\ &\leq \frac{1}{2} \|x - a\| + \frac{r}{2} \\ &\leq \frac{r}{2} + \frac{r}{2} = r\end{aligned}$$

$\phi_y(\overline{\mathcal{B}}(a, r)) \subseteq \overline{\mathcal{B}}(a, r)$. By the fixed point theorem

$$\exists g : \overline{\mathcal{B}}(f(a), \delta) \rightarrow \overline{\mathcal{B}}(a, r)$$

sending y to the fixed point of ϕ_y Let $W = \mathcal{B}(f(a), g)$, then

$$g|_W : W \rightarrow V$$

is the inverse of $f|_V : V \rightarrow W$ Hence $f^{-1}(W) = V$ is open.

In the following, we prove that g is of class C^n on an open neighborhood of $f(a)$. By reducing V and W , we may assume that $\forall x \in V$

$$d_x f \in I(E, F)$$

Let $x_0 \in V$ $y_0 = f(x_0)$ $x_0 = g(y_0)$

$$y - y_0 = f(g(y)) - f(g(y_0)) = d_{x_0} f(g(y) - g(y_0)) + o(\|g(y) - g(y_0)\|)$$

So

$$g(y) - g(y_0) = (d_x f)^{-1}(y - y_0) + o(\|g(y) - g(y_0)\|)$$

Thus leads to

$$g(y) - g(y_0) = O(\|y - y_0\|)$$

$(\exists \epsilon > 0 \quad (1 - \epsilon) \|g(y) - g(y_0)\| \leq \|d_{x_0} f\|^{-1} \text{ when } \|y - y_0\| \text{ is sufficiently small})$

So

$$d_{y_0} g = (d_x f)^{-1}$$

By the previous proposition, g is of class C^n

Part VII

Integration

Chapter 35

Integral operators

We fix a set Ω and a vector subspace S of \mathbb{R}^Ω over \mathbb{R} . We suppose that $\forall (f, g) \in S^2$

$$\begin{aligned} f \wedge g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \min\{f(\omega), g(\omega)\} \end{aligned}$$

belongs to S

35.1 Prop

(1) $\forall (f, g) \in S^2$

$$\begin{aligned} f \vee g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \max\{f(\omega), g(\omega)\} \end{aligned}$$

$$f \vee g \in S$$

(2) $\forall f \in S$

$$\begin{aligned} |f| : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto |f(\omega)| \end{aligned}$$

$$|f| \in S$$

Proof

(1)

$$f \vee g = f + g - f \wedge g$$

(2)

$$|f| = f \vee (-f)$$

35.2 Def

We call integral operator on S any \mathbb{R} -linear mapping $I : S \rightarrow \mathbb{R}$ that satisfies the following conditions:

- (1) If $f \in S$ is such that $\forall \omega \in \Omega, f(\omega) \geq 0$ then $I(f) \geq 0$
- (2) If $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence of elements in S such that $\forall \omega \in \Omega \lim_{n \rightarrow +\infty} f_n(\omega) = 0$ then

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

$$(\forall \omega \in \Omega, n \in \mathbb{N}, f_n(\omega) \geq f_{n+1}(\omega))$$

35.3 Example

- (1) $\Omega = \mathbb{R}$ $S =$ vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by mappings of the form $\mathbb{1}_{]a,b]}$ $(a, b) \in \mathbb{R}^2, a < b$

$$\mathbb{1}_{]a,b]} = \begin{cases} 1, x \in]a, b] \\ 0, else \end{cases}$$

Any element of S is of the form

$$\sum_{i=1}^n \mathbb{1}_{]a_i, b_i]}$$

$I : S \rightarrow \mathbb{R}$ is defined as

$$I\left(\sum_{i=1}^n \lambda \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (b_i - a_i)$$

More generally if $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is increasing and right continuous ($\forall x \in \mathbb{R}, \lim_{\epsilon > 0, \epsilon \rightarrow 0} \varphi(x + \epsilon) = \varphi(x)$) We define

$$I_\varphi : S \rightarrow \mathbb{R}$$

$$I\left(\sum_{i=1}^n \lambda \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (\varphi(b_i) - \varphi(a_i))$$

- (2) (Radon measure)

Let Ω be a quasi-compact topological space

$$S = C^0(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \text{ continuous}\}$$

Let $I : S \rightarrow \mathbb{R}$ \mathbb{R} -linear, such that $\forall f \in S, f \geq 0$ one has $I(f) \geq 0$

35.4 Dini's theorem

Let $(f_n)_{n \in \mathbb{N}}$ be a decreasing sequence in $C^0(\Omega)$, that converges pointwisely to some $f \in C^0(\Omega)$ Then $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f

Proof

Let $g_n = f_n - f > 0$ Fix $\epsilon > 0 \forall n \in \mathbb{N}$ let

$$U_n = \{\omega \in \Omega \mid g_n(\omega) < \epsilon\}$$

is open

Moreover

$$\bigcup_{n \in \mathbb{N}} U_n = \Omega \quad (U_0 \subseteq U_1 \subseteq \dots)$$

Since Ω is quasi-compact, $\exists N \in \mathbb{N}, \Omega = U_N$ Therefore $\forall n \in \mathbb{N}, n \geq N, \forall \omega \in \Omega$

$$g_n(\omega) < \epsilon$$

Consequence. If $(f_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$ is decreasing and converges pointwisely to 0, then

$$\|f_n\|_{\sup} := \sup_{\omega \in \Omega} |f_n(\omega)|$$

converges to 0 when $n \rightarrow +\infty \forall n \in \mathbb{N}$

$$f_n \leq \|f_n\|_{\sup} \cdot \mathbb{1}_{\Omega}$$

So

$$0 \leq I(f_n) \leq \|f_n\|_{\sup} I(\mathbb{1}_{\Omega}) \rightarrow 0 \quad (n \rightarrow +\infty)$$

(If $f \leq g$ then $g - f \geq 0$ so $I(g - f) = I(g) - I(f) \geq 0 \Rightarrow I(g) \geq I(f)$)

35.5 Def

We call σ -algebra any subset \mathcal{A} of $\wp(\Omega)$ that satisfies the following conditions:

- $\emptyset \in \mathcal{A}$
- If $A \in \mathcal{A}$ then $\Omega \setminus A \in \mathcal{A}$
- If $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ then $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Given a σ -algebra \mathcal{A} on Ω , we mean by measure on (Ω, \mathcal{A}) any mapping $\mu : \mathcal{A} \rightarrow [0, +\infty]$ such that :

- $\mu(\emptyset) = 0$
- If $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ such that A_i are pairwise disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

Chapter 36

Riemann integral

36.1 Def

Let Ω be a non-empty set and S be a vector subspace of \mathbb{R}^Ω . If $\forall (f, g) \in S^2, f \wedge g \in S$, we say that S is a Riesz space.

In this section, we fix a Riesz space and an integral operator $I : S \rightarrow \mathbb{R}$

36.2 Def

For any $f : \Omega \rightarrow \mathbb{R}$ let

$$I^*(f) := \inf_{\mu \in S, \mu \geq f} I(\mu)$$

$$I_*(f) := \sup_{l \in S, l \leq f} I(l)$$

If $I^*(f) = I_*(f)$ then we say that f is I-Riemann integral, and denote by $I(f)$ the value $I^*(f)$ (or $I_*(f)$)

36.3 Theorem

The set \mathcal{R} of all I-Riemann integral mappings form a vector space of \mathbb{R}^Ω that contains S . Moreover, $I : \mathcal{R} \rightarrow \mathbb{R}$ is an \mathbb{R} -linear mapping extending $I : S \rightarrow \mathbb{R}$

Proof

$$\forall h \in S$$

$$I^*(h) = I_*(h) = I(h)$$

So $h \in \mathcal{R}$

Let $(f_1, f_2) \in \mathcal{R}$. If $(\mu_1, \mu_2) \in S^2, \mu_1 \geq f_1, \mu_2 \geq f_2$ then

$$\mu_1 + \mu_2 \in S, \mu_1 + \mu_2 \geq f_1 + f_2$$

Hence

$$I(\mu_1) + I(\mu_2) \geq I^*(f_1 + f_2)$$

Take the infimum with respect to (μ_1, μ_2) we get

$$I^*(f_1) + I^*(f_2) \geq I^*(f_1 + f_2)$$

Similarly

$$I_*(f_1) + I_*(f_2) \leq I_*(f_1 + f_2)$$

Hence

$$I^*(f_1 + f_2) = I_*(f_1 + f_2) = I(f_1) + I(f_2)$$

Let $f : \Omega \rightarrow \mathbb{R}$ be a mapping, $\lambda \in \mathbb{R}_{>0}$

$$I^*(\lambda f) = \inf_{\mu \in S, \mu \geq \lambda f} I(\mu) = \inf_{\nu \in S, \nu \geq f} I(\lambda \nu) = \lambda I^*(f)$$

Similarly

$$I_*(\lambda f) = \lambda I_*(f)$$

Hence if $f \in \mathcal{R}$ then $\lambda f \in \mathcal{R}$ and $I(\lambda f) = \lambda I(f)$

$$I^*(-f) = \inf_{\mu \in S, \mu \geq -f} I(\mu) = \inf_{l \in S, l \leq f} I(-l) = - \sup_{l \in S, l \leq f} I(l) = -I_*(f)$$

Similarly

$$I_*(-f) = -I^*(f)$$

Hence if $f \in \mathcal{R}$ then $-f \in \mathcal{R}$ and $I(-f) = -I(f)$

Chapter 37

Daniell integral

We fix an integral operator $I : S \rightarrow \mathbb{R}$

37.1 Prop

37.1.1

Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence in S that converges pointwisely to some $f \in S$. Then

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

Proof

Let $g_n = f - f_n \in S$ $(g_n)_{n \in \mathbb{N}}$ is decreasing and converges pointwisely to 0. Then

$$\lim_{n \rightarrow +\infty} I(g_n) = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

37.1.2

Let $(f_n)_{n \in \mathbb{N}}$ be an increasing sequence in S , $f \in S$ If $f \leq \lim_{n \rightarrow +\infty} f_n$, then

$$I(f) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Proof

$$f = \lim_{n \rightarrow +\infty} f \wedge f_n$$

So

$$I(f) = \lim_{n \rightarrow +\infty} I(f \wedge f_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

37.2 Def

Let

$$S^\uparrow = \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\} \mid \begin{array}{l} \exists (f_n)_{n \in \mathbb{N}} \in S^\mathbb{N} \text{ increasing such that} \\ f = \lim_{n \rightarrow +\infty} f_n \text{ pointwisely} \end{array} \right\}$$

37.3 Prop

Let f, g be elements of S^\uparrow such that $f \leq g$. Let $(f_n)_{n \in \mathbb{N}}$ and $(g_m)_{m \in \mathbb{N}}$ be increasing sequences in S such that $f = \lim_{n \rightarrow +\infty} f_n, g = \lim_{m \rightarrow +\infty} g_m$. Then

$$\lim_{n \rightarrow +\infty} I(f_n) \leq \lim_{m \rightarrow +\infty} I(g_m)$$

Proof

For any $m \in \mathbb{N}$

$$f_m \leq f \leq g$$

Hence

$$I(f_m) \leq \lim_{n \rightarrow +\infty} I(g_n)$$

Taking $\lim_{m \rightarrow +\infty}$ we get

$$\lim_{m \rightarrow +\infty} I(f_m) \leq \lim_{n \rightarrow +\infty} I(g_n)$$

37.4 Corollary

Let $f \in S^\uparrow$. If $(f_n)_{n \in \mathbb{N}}$ and $(\tilde{f}_n)_{n \in \mathbb{N}}$ be increasing sequence in S such that

$$f = \lim_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} \tilde{f}_n$$

then

$$\lim_{n \rightarrow +\infty} I(f_n) = \lim_{n \rightarrow +\infty} I(\tilde{f}_n)$$

We denote by $I(f)$ the limit $\lim_{n \rightarrow +\infty} I(f_n)$

Thus we obtain a mapping $I : S^\uparrow \rightarrow \mathbb{R} \cup \{+\infty\}$ such that

- If $(f_n)_{n \in \mathbb{N}} \in S^\mathbb{N}$ is increasing then

$$I\left(\lim_{n \rightarrow +\infty} f_n\right) = \lim_{n \rightarrow +\infty} I(f_n)$$

- If $(f, g) \in S^{\uparrow 2}$ $f \leq g$ then $I(f) \leq I(g)$
- If $(f, g) \in S^{\uparrow 2}$ then $f + g \in S^\uparrow$ and

$$I(f + g) = I(f) + I(g)$$

- If $f \in S^\uparrow, \lambda \geq 0$ then $\lambda f \in S^\uparrow$ and $I(\lambda f) = \lambda I(f)$

37.5 Prop

Let $(f_n)_{n \in \mathbb{N}} \in (S^\uparrow)^\mathbb{N}$ be an increasing sequence and $f = \lim_{n \rightarrow +\infty} f_n$. Then

$$f \in S^\uparrow$$

and

$$I(f) = \lim_{n \rightarrow +\infty} I(f_n)$$

Proof

For $k \in \mathbb{N}$ let $(g_{k,m})_{m \in \mathbb{N}} \in S^\mathbb{N}$ be an increasing sequence such that

$$f_k = \lim_{m \rightarrow +\infty} g_{k,m}$$

For $n \in \mathbb{N}$ let $h_n = g_{0,n} \vee \cdots \vee g_{n,n} \in S$ The sequence $(h_n)_{n \in \mathbb{N}}$ is increasing. Moreover

$$f_n \geq k_n \geq g_{k,n} \quad (k \leq n)$$

Hence

$$f_n \geq h_n$$

Taking $\lim_{n \rightarrow +\infty}$ we get $\forall k \in \mathbb{N}$

$$f = \lim_{n \rightarrow +\infty} f_n \geq \lim_{n \rightarrow +\infty} h_n \geq \lim_{n \rightarrow +\infty} g_{k,n} = f_k$$

Taking $\lim_{k \rightarrow +\infty}$ we get

$$f = \lim_{n \rightarrow +\infty} h_n$$

Hence $f \in S^\uparrow$ and

$$I(f) = \lim_{n \rightarrow +\infty} I(h_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Conversely, $\forall n \in \mathbb{N}, f \geq f_n$ Hence

$$I(f) \geq \lim_{n \rightarrow +\infty} I(f_n)$$

37.6 Def

Let $S^\downarrow = \{-f \mid f \in S^\uparrow\}$ We extend I to $I : S^\downarrow \rightarrow \mathbb{R}U - \infty$ by letting $I(-f) := -I(f)$ for $f \in S^\uparrow$

37.7 Prop

Let $(f, g) \in (S^\uparrow \cup S^\downarrow)^2$ If $f \leq g$ then

$$I(f) \leq I(g)$$

Proof

It suffices to treat the cases where $(f, g) \in S^\uparrow \times S^\downarrow$ and $(f, g) \in S^\uparrow \times S^\downarrow$

If $(f, g) \in S^\uparrow \times S^\downarrow$ then $-f \in S^\downarrow$ and hence $g - f \in S^\uparrow, g - f \geq 0$ In both cases,

$$0 \leq I(g - f) = I(g) + I(-f) = I(g) - I(f)$$

37.8 Def

Let $f : \Omega \rightarrow \mathbb{R}$ be a mapping. We define

$$\bar{I}(f) := \inf_{\mu \in S^\uparrow, \mu \geq f} I(\mu) \leq \inf_{\mu \in S, \mu \geq f} I(\mu) = I^*(f)$$

$$\underline{I}(f) := \sup_{\mu \in S^\downarrow, \mu \leq f} I(\mu) \geq \sup_{\mu \in S, \mu \leq f} I(\mu) = I_*(f)$$

If $\bar{I}(f) = \underline{I}(f)$ then we say that f is I -integrable (in the sense of Daniell)

37.9 Remark

If f is I-integrable in the sense of Riemann, then it is I-integrable in sense of Daniell

37.10 Daniell Theorem

The set $L^1(I)$ of all I-integrable mappings forms a vector subspace of \mathbb{R} . Moreover

- $\forall (f, g) \in L^1(I) \ f \wedge g \in L^1(I)$
- $I : L^1(I) \rightarrow \mathbb{R}$ is an integral operator extending $I : S \rightarrow \mathbb{R}$

Proof

Let $(f_1, f_2) \in L^1(I)^2$ let $(l_1, l_2) \in S^{\downarrow 2}, l_1 \leq f_1, l_2 \leq f_1$ Let $(\mu_1, \mu_2) \in S^{\uparrow 2}, f_1 \leq \mu_1, f_2 \leq \mu_2$

We have

$$l_1 + l_2 \leq f_1 + f_2 \leq \mu_1 + \mu_2$$

Taking the supremum with respect to (l_1, l_2) , we get

$$I(f_1) + I(f_2) (= \underline{I}(f_1) + \underline{I}(f_2)) \leq \underline{I}(f_1 + f_2)$$

Taking the infimum with respect to (μ_1, μ_2) , we get

$$\bar{I}(f_1 + f_2) \leq I(f_1) + I(f_2)$$

Then

$$\bar{I}(f_1 + f_2) = \underline{I}(f_1 + f_2)$$

So $f_1 + f_2 \in L^1(I)$ and $I(f_1 + f_2) = I(f_1) + I(f_2)$

Similarly, if $f \in L^1(I), \lambda \geq 0$ then

$$\begin{aligned} \underline{I}(\lambda f) &= \sup_{l \leq \lambda f, l \in S^\downarrow} I(l) \\ &= \sup_{l \leq f, l \in S^\downarrow} I(\lambda l) \\ &= \lambda \underline{I}(f) = \lambda I(f) \end{aligned}$$

$$\bar{I}(\lambda f) = \lambda \bar{I}(f) = \lambda I(f)$$

So $\lambda f \in L^1(I)$ and $I(\lambda f) = \lambda I(f)$

Moreover, if $f \in L^1(I), \mu \in S^\uparrow, l \in S^\downarrow, l \leq f \leq \mu$ then

$$-\mu \in S^\downarrow, -l \in S^\uparrow, -\mu \leq -f \leq -l$$

Hence

$$\bar{I}(-f) = -\underline{I}(f) = -I(f) \quad \underline{I}(-f) = -\bar{I}(f) = -I(f)$$

So $-f \in L^1(I)$ and $I(-f) = -I(f)$

We proved that $\forall (f_1, f_2) \in L^1(I)^2$

$$f_1 \wedge f_2 \in L^1(I)$$

Let $(f_1, f_2) \in L^1(I)^2$, for any $\epsilon > 0 \exists (l_1, l_2) \in S^\downarrow{}^2, (\mu_1, \mu_2) \in S^\uparrow{}^2$ such that

$$l_1 \leq f_1 \leq \mu_1 \quad l_2 \leq f_2 \leq \mu_2$$

such that

$$I(\mu_1 - l_1) \leq \frac{\epsilon}{2} \quad I(\mu_2 - l_2) \leq \frac{\epsilon}{2}$$

One has $l_1 \wedge l_2 \leq f_1 \wedge f_2 \leq \mu_1 \wedge \mu_2$

$$\mu_1 \wedge \mu_2 - l_1 \wedge l_2 \leq (\mu_1 - l_1) + (\mu_2 - l_2)$$

$$\left(\begin{array}{l} \text{If } \mu_1(\omega) \leq \mu_2(\omega), l_1 \leq l_1(\omega) \\ LHS = \mu_1(\omega) - l_1(\omega) \\ RHS = \mu_1(\omega) - l_2(\omega) + \mu_2(\omega) - l_1(\omega) \geq \mu_1(\omega) - l_2(\omega) \end{array} \right)$$

37.11 Beppo Levi Theorem

Let $(f_n)_{n \in \mathbb{N}}$ be a monotone sequence of elements of $L_1(I)$, which converges pointwisely to some $f : \Omega \rightarrow \mathbb{R}$ If $(I(f_n))_{n \in \mathbb{N}}$ converges to a real number α Then $f \in L^1(I)$ and $I(f) = \alpha$

Proof

Assume that $(f_n)_{n \in \mathbb{N}}$ is increasing. Moreover, by replacing f_n by $f_n - f_0$ we may assume that $f_0 = 0$

Let $\epsilon > 0 \forall n \in \mathbb{N}$ let $\mu_n \in S^\uparrow$ such that $f_n - f_{n-1} \leq \mu_n$ and

$$I(f_n - f_{n-1}) \geq I(\mu_n) - \frac{\epsilon}{2}$$

the existence

$$I(f_n - f_{n-1}) = \inf_{\mu \in S^\uparrow, \mu \geq f_n - f_{n-1}} I(\mu)$$

If $\forall \mu \in S^\uparrow, \mu \geq f_n - f_{n-1}$ one has

$$I(\mu) > I(f_n - f_{n-1}) + \frac{\epsilon}{2}$$

then

$$I(f_n - f_{n-1}) + \frac{\epsilon}{2} \leq I(f_n - f_{n-1})$$

contradiction.

Thus

$$f_n = \sum_{k=1}^n (f_k - f_{k-1}) \leq \mu_1 + \cdots + \mu_n$$

and

$$I(f_n) \geq \sum_{k=1}^n (I(\mu_k) - \frac{\epsilon}{2^k}) \geq I(\mu_1) + \cdots + I(\mu_n) - \epsilon$$

Let $\mu = \mu_1 + \cdots + \mu_n + \cdots \in S^\uparrow$

$$I(\mu) = \sum_{n \in \mathbb{N}} I(\mu_n)$$

One has $\mu \geq f$

$$\lim_{n \rightarrow +\infty} \geq I(\mu) - \epsilon \geq \bar{I}(f) - \epsilon$$

Similarly, one can choose $l_n \in S^\downarrow, l_n \leq f_n, I(l_n) \geq I(f_n) - \epsilon$

$$\liminf_{n \rightarrow +\infty} I(l_n) \geq \alpha - \epsilon$$

Note that $l_n \leq f_n \leq f$, so

$$\alpha - \epsilon \leq \liminf_{n \rightarrow +\infty} I(l_n) \leq \underline{I}(f)$$

Thus

$$\alpha - \epsilon \leq \underline{I}(f) \leq \bar{I}(f) \leq \alpha + \epsilon$$

Let $\epsilon \rightarrow 0$ we get

$$\bar{I}(f) = \underline{I}(f) = \alpha$$

37.12 Fatou's Lemma

Let $(f_n)_{n \in \mathbb{N}} \in L^1(I)^\mathbb{N}$. Assume that there is $g \in L^1(I)$ such that

$$\forall n \in \mathbb{N} \quad f_n \geq g$$

If $\liminf_{n \rightarrow +\infty} f_n$ is a mapping from Ω to \mathbb{R} and $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$, then $\liminf_{n \rightarrow +\infty} f_n \in L^1(I)$ and

$$I(\liminf_{n \rightarrow +\infty} f_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

Proof

For any $n \in \mathbb{N}$, let

$$g_n = \lim_{k \rightarrow +\infty} (f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k})$$

Then

$$\liminf_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} g_n$$

For any k one has

$$f_n \wedge \cdots \wedge f_{n+k} \geq g$$

Hence

$$I(f_n) \geq \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \geq I(g)$$

By the theorem of Beppo Levi,

$$g_n \in L^1(I) \text{ and } I(g_n) = \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \leq I(f_n)$$

Note that $(g_n)_{n \in \mathbb{N}}$ is increasing and $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$. Hence

$$\lim_{n \rightarrow +\infty} I(g_n) = \liminf_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n) < +\infty$$

By the theorem of Beppo Levi,

$$\lim_{n \rightarrow +\infty} g_n \in L^1(I)$$

and

$$I(\liminf_{n \rightarrow +\infty} f_n) = I(\lim_{n \rightarrow +\infty} g_n) = \lim_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

37.13 Lebesgue dominated convergence theorem

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $L^1(I)$ that converges pointwisely to some $f : \Omega \rightarrow \mathbb{R}$. Assume that there exists $g \in L^1(I)$ such that $\forall n \in \mathbb{N}, |f_n| \leq g$. Then $f \in L^1(I)$ and $I(f) = \lim_{n \rightarrow +\infty} I(f_n)$.

Proof

Apply Fatou's lemma to $(f_n)_{n \in \mathbb{N}}$ and $(-f_n)_{n \in \mathbb{N}}$ to get

$$I(f) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

and

$$\begin{aligned} I(-f) &\leq \liminf_{n \rightarrow +\infty} I(-f_n) \\ &= \limsup_{n \rightarrow +\infty} I(f_n) \\ &\leq \limsup_{n \rightarrow +\infty} I(f_n) \leq I(f) \end{aligned}$$

37.14 Notation

Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing and right continuous mapping. Let S be the vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by $\mathbb{1}_{[a,b]}$ with $(a,b) \in \mathbb{R}^2, a < b$. For any $f \in L^1(I_\varphi)$ $I_\varphi(f)$ is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

For any subset A of \mathbb{R} if $\mathbb{1}_A f \in L^1(I)$ then

$$\int_A f(x) d\varphi(x) \text{ denotes } \int_{\mathbb{R}} \mathbb{1}_A(x) f(x) d\varphi(x) = I(\mathbb{1}_A f)$$

If $(a,b) \in \mathbb{R}^2, a < b$

$$\int_a^b f(x) d\varphi(x) \text{ denotes } \int_{[a,b]} f(x) d\varphi(x)$$

$$\int_b^a f(x) d\varphi(x) \text{ denotes } - \int_{[a,b]} f(x) d\varphi(x)$$

If $\varphi(x) = x$ for any $x \in \mathbb{R}$ we replace $d\varphi(x)$ by dx .

Chapter 38

Semialgebra

38.1 Notation

Let $A, (A_i)_{i \in I}$ be sets the notation.

$$A = \bigsqcup_{i \in I} A_i$$

denotes:

- $(A_i)_{i \in I}$ is a pairwise disjoint
- $A = \bigcup_{i \in I} A_i$

38.2 Def

Let Ω be a set. We call semialgebra on Ω any $\mathcal{C} \subseteq \wp(\Omega)$ that verifies:

- $\emptyset \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, A \cap B \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, \exists (C_i)_{i=1}^n$ a finite family of elements in \mathcal{C} such that $B \setminus A = \bigsqcup_{i=1}^n C_i$

38.2.1 Example

$$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$$

38.3 Def

Let \mathcal{C} be a semialgebra on Ω . The set

$$\{A \in \wp(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigcup_{i=1}^n A_i\}$$

is called the algebra generated by \mathcal{C}

38.4 Prop

Let \mathcal{C} be a semialgebra on Ω . \mathcal{A} be the algebra generated by \mathcal{C} . Then:

- $\emptyset \in \mathcal{A}$
- $\forall (A, B) \in \mathcal{A}^2, A \cap B \in \mathcal{A}, B \setminus A \in \mathcal{A}, A \cup B \in \mathcal{A}$

Proof

By definition, $\emptyset \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}$. Moreover, if A and B are elements of \mathcal{A} such that $A \cap B = \emptyset$ then $A \cup B \in \mathcal{A}$. Let $A = \bigcup_{i=1}^n A_i$ and $B = \bigcup_{i=1}^n B_i$ be elements of \mathcal{A} then

$$A \cap B = \bigcup_{(i,j) \in \{1, \dots, n\}^2} (A_i \cap B_j)$$

Hence $A \cap B \in \mathcal{A}$ Finally

$$A \cup B = (A \cap B) \sqcup (A \setminus B) \sqcup (B \setminus A) \in \mathcal{A}$$

38.5 Prop

Let \mathcal{C} be a semialgebra on Ω . \mathcal{A} be the algebra generated by \mathcal{C} . Let S be the \mathbb{R} -vector subspace of \mathbb{R}^Ω , $I : S \rightarrow \mathbb{R}$ be an \mathbb{R} -linear mapping generated by mappings of the form $\mathbb{1}_A, A \in \mathcal{C} (f \in S, f = \sum \lambda_i \mathbb{1}_{A_i})$
Assume that

$$\forall (f, g) \in S^2, f \leq g \text{ one has } I(f) \leq I(g)$$

Then I is an integral operator iff, for any decreasing sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{A}^\mathbb{N}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, one has

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

38.5.1 Proof

$$\forall A \in \mathcal{A}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigsqcup_{i=1}^n A_i \text{ so } \mathbb{1}_A = \sum_{i=1}^n \mathbb{1}_{A_i} \in S$$

Lemma $\forall (f, g) \in S^2, f \wedge g \in S$

\Rightarrow Suppose that I is an integral operator $(\mathbb{1}_{A_n})_{n \in \mathbb{N}}$ is a decreasing sequence in S and

$$\lim_{n \rightarrow +\infty} \mathbb{1}_{A_n} = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

\Leftarrow Let $(f_n)_{n \in \mathbb{N}}$ be a decreasing sequence in S that converges pointwisely to 0. Let

$$B = \{\omega \in \Omega \mid f_0(\omega) > 0\} \in \mathcal{A} \quad M = \max\{f_0(\omega) \mid \omega \in \Omega\}$$

- For any $\epsilon > 0$ let

$$A_n^\epsilon = \{\omega \in \Omega \mid f_n(\omega) \geq \epsilon\} \in \mathcal{A}$$

Moreover, since $\lim_{n \rightarrow +\infty} f_n(\omega) = 0, \bigcap_{n \in \mathbb{N}} A_n^\epsilon = \emptyset$

$$f(\omega) = \begin{cases} \lambda_i & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \in \Omega \setminus \bigcap_{i=1}^n A_i \end{cases}$$

$(\forall f \in S, \exists (A_i)_{i=1}^n \text{ pairwise disjoint and } (\lambda_i)_{i=1}^n \in \mathbb{R} \text{ } f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i})$

Note that

$$0 \leq f_n \leq \epsilon \mathbb{1}_B + M \mathbb{1}_{A_n^\epsilon}$$

So

$$0 \leq I(f_n) \leq \epsilon I(\mathbb{1}_B) + M I(\mathbb{1}_{A_n^\epsilon})$$

which leads to

$$\limsup_{n \rightarrow +\infty} I(f_n) \leq \epsilon I(\mathbb{1}_B) \quad \forall \epsilon > 0$$

So

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

38.5.2 Example

Let $\Omega = \mathbb{R}$ and $\mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$
 \mathcal{A} be algebra generated by \mathcal{C} . $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ increasing, right continuous. S be
 \mathbb{R} -vector subspace generated by $\mathbb{1}_{]a, b]}, (a, b) \in \mathbb{R}, a \leq b$

$$\begin{aligned} I_\varphi : S &\rightarrow \mathbb{R} \\ \mathbb{1}_{]a, b]} &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

Lemma $\forall \epsilon > 0, A \in \mathcal{A}, A \neq \emptyset, \exists B \in \mathcal{A},$

$$\emptyset \neq \overline{B} \subseteq A \text{ and } I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) \leq \epsilon$$

Proof

We first consider the the case where $A \in \mathcal{C}, A =]a, b], a < b$ By the right continuous of $\varphi, \exists]a', b[$ such that $\varphi(a') - \varphi(a) \leq \epsilon$. Let $B =]a', b], \overline{B} = [a', b] \subseteq]a, b]$.

$$\begin{aligned} I_\varphi(\mathbb{1}_B) &= \varphi(b) - \varphi(a') \\ I_\varphi(\mathbb{1}_A) &= \varphi(b) - \varphi(a) \\ I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) &= \varphi(a') - \varphi(a) \leq \epsilon \end{aligned}$$

In general

$$A = \bigsqcup_{i=1}^n A_i$$

with $A_i \in \mathcal{C} \forall i \in \{1, \dots, n\}, \exists B_i \in \mathcal{C}$

$$\emptyset \neq \overline{B_i} \subseteq A_i \quad I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \frac{\epsilon}{n}$$

Let $B = \bigsqcup_{i=1}^n B_i$ then

$$I(\mathbb{1}_A) - I(\mathbb{1}_B) = \sum_{i=1}^n (I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i})) \leq \epsilon$$

38.6 Theorem

I_φ is an integral operator

Proof

Let $(A_n)_{n \in \mathbb{N}}$ be a decreasing sequence in \mathcal{A} such that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

Let $\epsilon > 0$ For any $n \in \mathbb{N}$ let $B_n \in \mathcal{A}$ such that

$$\overline{B_n} \subseteq A_n \text{ and } I_\varphi(A_n) - I_\varphi(B_n) \leq \frac{\epsilon}{2^n}$$

Note that $\overline{B_n}$ is compact. For any $n \in \mathbb{N}$ let

$$\begin{aligned} C_n &= \bigcap_{i=1}^n B_i \\ &\subseteq \bigcap_{i=1}^n \overline{B_i} \end{aligned}$$

Since $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$, $\bigcap_{n \in \mathbb{N}} \overline{B_n} = \emptyset$

So

$$I_\varphi(\mathbb{1}_{A_n}) \leq \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}} \cdots \frac{\epsilon}{2} \leq \epsilon$$

Thus

$$\lim_{n \rightarrow +\infty} I_\varphi(\mathbb{1}_{A_n}) = 0$$

Let Ω be a set \mathcal{C} be a semialgebra on Ω and \mathcal{A} be the algebra generated by \mathcal{C} . Let S be the \mathbb{R} -vector subspace of \mathbb{R}^Ω generated by mappings of the form $\mathbb{1}_A, A \in \mathcal{C}$

38.7 Prop

For any $f \in S, \exists (A_i)_{i=1}^n \in \mathcal{A}^n$ pairwise disjoint and $(\lambda_i)_{i=1}^n$ such that

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$$

Proof

f is of the form

$$f = \sum_{j=1}^m a_j \mathbb{1}_{B_j} \quad B_j \in \mathcal{C}$$

We reason by induction on m . For any $I \subseteq \{1, \dots, m\}$ let

$$B_I = \bigcap_{j \in I} B_j \cap \bigcap_{j \in \{1, \dots, m\} \setminus I} (\Omega \setminus B_j)$$

Then $(B_I)_{I \subseteq \{1, \dots, m\}}$ are pairwise disjoint.

Chapter 39

Limit and Differential of Integrals with Parameters

Let Ω be a set. $S \subseteq \mathbb{R}^\Omega$ be \mathbb{R} -vector subspace such that $\forall (f, g) \in S^2, f \wedge g \in S$. Let $I : S \rightarrow \mathbb{R}$ be an integral operator.

39.1 Theorem

Let X be a topological space, $p \in X$, $f : \Omega \times X \rightarrow \mathbb{R}$ be a mapping, $g \in L^1(I)$. Suppose that

(1) $\forall \omega \in \Omega$

$$\begin{aligned} f(\omega, \cdot) : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto f(\omega, x) \end{aligned}$$

is continuous at p

(2) $\forall x \in X$

$$\begin{aligned} f(\cdot, x) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, x) \end{aligned}$$

belongs to $L^1(I)$ and $\forall \omega \in \Omega \quad |f(\omega, x)| \leq g(\omega)$

(3) p has a countable neighborhood basis in X

Then

$$(x \in X) \mapsto I(f(\cdot, x))$$

is continuous at p

Proof

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X that converges to p . For any $n \in \mathbb{N}$ let $f_n : \Omega \rightarrow \mathbb{R}$, $f_n(\omega) := f(\omega, x_n)$. One has $|f_n| \leq g$. Moreover $\forall \omega \in \Omega$

$$\lim_{n \rightarrow +\infty} f_n(\omega) = \lim_{n \rightarrow +\infty} f(\omega, x_n) = f(\omega, p)$$

Hence, by dominate convergence theorem

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f(\cdot, p))$$

39.2 Theorem

Let J be an open interval in \mathbb{R} . $f : \Omega \times J \rightarrow \mathbb{R}$ be a mapping. $g \in L^1(I)$. Assume that

$$(1) \quad \forall \omega \in \Omega$$

$$f(\omega, \cdot) : J \rightarrow \mathbb{R}$$

is differentiable (we denote by $\frac{\partial f}{\partial t}(\omega, t)$ its derivative at t)
and $\forall t \in J$

$$\left| \frac{\partial f}{\partial t}(\omega, t) \right| \leq g(\omega)$$

$$(2) \quad \forall t \in J$$

$$\begin{aligned} f(\cdot, t) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, t) \end{aligned}$$

belongs to $L^1(I)$

Then

$$\begin{aligned} \varphi : J &\rightarrow \mathbb{R} \\ t &\mapsto I(f(\cdot, t)) \end{aligned}$$

is differentiable and

$$\varphi'(t) = I\left(\frac{\partial f}{\partial t}(\cdot, t)\right) = \frac{d}{dt} I(f(\cdot, t))$$

Proof

Let $a \in J$ and $(t_n)_{n \in \mathbb{N}}$ be a sequence in $J \setminus \{a\}$ such that

$$\lim_{n \rightarrow +\infty} t_n = a$$

. Then

$$\frac{\varphi(t_n) - \varphi(a)}{t_n - a} = I\left(\frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a}\right)$$

$\forall \omega \in \Omega$

$$\left| \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} \right| \leq g(\omega) \text{ (by mean value theorem)}$$

and

$$\lim_{n \rightarrow +\infty} \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} = \frac{\partial f}{\partial t}(\omega, a)$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{\varphi(t_n) - \varphi(a)}{t_n - a} = \frac{d}{dt} I(f(\cdot, t))$$

Chapter 40

Measure theory

40.1 Def

We call measure space any pair (E, \mathcal{E}) , where E is a set and \mathcal{E} is a σ -algebra on E .

40.2 Prop

Let Ω be a set. And $(\mathcal{G}_i)_{i \in I}$ be a family of σ -algebras on Ω . Then $\bigcap_{i \in I} \mathcal{G}_i$ is a σ -algebra

Proof

- $\forall i \in I$

$$\emptyset \in \mathcal{G}_i$$

Hence

$$\emptyset \in \bigcap_{i \in I} \mathcal{G}_i$$

- If $A \in \bigcap_{i \in I} \mathcal{G}_i$ then $\forall i \in I$ $A \in \mathcal{G}_i$ Hence $\forall i \in I$

$$\Omega \setminus A \in \mathcal{G}_i$$

So

$$\Omega \setminus A \in \bigcap_{i \in I} \mathcal{G}_i$$

- Let $(A_n)_{n \in \mathbb{N}} \in (\bigcap_{i \in I} \mathcal{G}_i)^{\mathbb{N}}$. For any $i \in I$

$$(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_i^{\mathbb{N}}$$

So

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \mathcal{G}_i$$

so

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \bigcap_{i \in I} \mathcal{G}_i$$

40.3 Def

Let $\mathcal{C} \subseteq \wp(\Omega)$. We denote by $\sigma(\mathcal{C})$ the intersection of all σ -algebras on Ω containing \mathcal{C} . It's the smallest σ -algebra containing \mathcal{C}

40.4 Example

- Let (X, \mathcal{J}) be a topological space. $\sigma(\mathcal{J})$ is called the Borel σ -algebra of X
- On $[-\infty, +\infty]$ the following σ -algebras are the same:

$$\begin{aligned} g_1 &= \sigma(\{[a, +\infty] \mid a \in \mathbb{R}\}) \\ g_2 &= \sigma(\{]a, +\infty[\mid a \in \mathbb{R}\}) \\ g_3 &= \sigma(\{[-\infty, a] \mid a \in \mathbb{R}\}) \\ g_4 &= \sigma(\{[-\infty, a[\mid a \in \mathbb{R}\}) \end{aligned}$$

Moreover

$$\mathcal{B} = \{A \subseteq \mathbb{R} \mid A \in g_1\}$$

is equal to the Borel σ -algebra of \mathbb{R}

proof $\forall a \in \mathbb{R}$

$$[a, +\infty] = \bigcap_{n \in \mathbb{N}_{\geq 1}}]a - \frac{1}{n}, +\infty[\in g_2 \quad \Rightarrow g_1 \in g_2$$

$$]a, +\infty[= [-\infty, +\infty] \setminus [-\infty, a] \in g_3 \quad \Rightarrow g_2 \in g_3$$

$$[-\infty, a] = \bigcap_{n \in \mathbb{N}_{\geq 1}} [-\infty, a + \frac{1}{n}[\in g_4 \quad \Rightarrow g_3 \in g_4$$

$$[-\infty, a[= [-\infty, +\infty] \setminus [a, +\infty] \in g_1 \quad \Rightarrow g_4 \in g_1$$

$$\sigma(\{]a, b[\mid a < b, (a, b) \in \mathbb{R}^2\}) = \text{Borel } \sigma - \text{algebra of } \mathbb{R}$$

$J \subseteq \mathbb{R}$ open We define \sim a binary relation on J such that $x \sim y \Leftrightarrow$ there exists an interval I such that $\{x, y\} \subseteq I \subseteq J$

Any equivalence class of \sim is a non-empty open interval.

$$]a, b[= [a, +\infty] \cup [-\infty, b[$$

Hence Borel σ -algebra of $\mathbb{R} \subseteq \{A \subseteq \mathbb{R} \mid A \in g_1\}$

40.5 Def

Let $f : X \rightarrow Y$ be a mapping of sets.

- For any $\mathcal{C}_Y \subseteq \wp(Y)$ we denote by

$$f^{-1}(\mathcal{C}_Y) := \{f^{-1}(B) \mid B \in \mathcal{C}_Y\}$$

- For any $\mathcal{C}_X \subseteq \wp(X)$ we denote by

$$f_*(\mathcal{C}_X) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{C}_X\}$$

40.6 Prop

Let $f : X \rightarrow Y$ be a mapping.

- (1) If \mathcal{G}_Y is a σ -algebra on Y then $f^{-1}(\mathcal{G}_Y)$ is a σ -algebra on X
- (2) If \mathcal{G}_X is a σ -algebra on X then $f_*(\mathcal{G}_X)$ is a σ -algebra on Y

Proof

(1)

$$\emptyset = f^{-1}(\emptyset) \in f^{-1}(\mathcal{G}_Y)$$

$$\forall B \in \mathcal{G}_Y$$

$$X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$$

If $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_Y^{\mathbb{N}}$, $A = \bigcup_{n \in \mathbb{N}} A_n$, $A_n \in \mathcal{G}_Y$, then

$$\bigcup_{n \in \mathbb{N}} f^{-1}(A_n) = f^{-1}(A) \in f^{-1}(\mathcal{G}_Y)$$

(2)

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{G}_X$$

so

$$\emptyset \in f_*(\mathcal{G}_X)$$

$$\forall B \in f_*(\mathcal{G}_X)$$

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \in \mathcal{G}_X$$

so

$$Y \setminus B \in f_*(\mathcal{G}_X)$$

$$\forall (B_n)_{n \in \mathbb{N}} \in f_*(\mathcal{G}_X)^{\mathbb{N}}, B = \bigcup_{n \in \mathbb{N}} B_n$$

$$f^{-1}(B) = \bigcap_{n \in \mathbb{N}} f^{-1}(B_n)$$

So $B \in f_*(\mathcal{G}_X)$

40.7 Def

Let (X, \mathcal{G}_X) and (Y, g_Y) be measurable spaces, $f : X \rightarrow Y$ be a mapping. If $f^{-1}(g_Y) \subseteq \mathcal{G}_X$ or equivalently $g_Y \subseteq f_*(\mathcal{G}_X)$ (or $\forall B \in g_Y, f^{-1}(B) \in \mathcal{G}_X$) then we say that f is measurable.

40.8 Prop

Let $(X, \mathcal{G}_X), (Y, g_Y), (Z, g_Z)$ be measurable spaces. $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be measurable mappings. Then $g \circ f$ is measurable.

Proof

$$\forall B \in g_Z$$

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

and

$$g^{-1}(B) \in g_Y$$

so

$$f^{-1}(g^{-1}(B)) \in \mathcal{G}_X$$

40.9 Def

Let Ω be a set $((E_i, \mathcal{E}))_{i \in I}$ be a family of measurable spaces. $f = (f_i)_{i \in I}$ where $f_i : \Omega \rightarrow E_i$ is a mapping. We denote by $\sigma(f)$ the σ -algebra $\sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i))$. It's the smallest σ -algebra on Ω making all f_i measurable.

40.10 Prop

We keep the notation of the above definition. For any $i \in I$, let $\mathcal{C} \subseteq \wp(E_i)$ such that $\sigma(\mathcal{C}_i) = \mathcal{E}_i$. Then

$$\sigma(f) = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$$

Proof

Let $g = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$. By definition

$$g \subseteq \sigma(f)$$

For any $i \in I$, $f_{i,*}(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$ is a σ -algebra on Ω containing \mathcal{C}_i . So

$$\mathcal{E} \subseteq f_{i,*}(\sigma(f_i^{-1}(\mathcal{C}_i)))$$

which leads to

$$f_i^{-1}(\mathcal{C}_i) \subseteq \sigma(f_i^{-1}(\mathcal{C}_i)) \subseteq g$$

Hence

$$\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i) \subseteq g$$

\Rightarrow

$$\sigma(f) \subseteq g$$

$$(f_{i,*}(\mathcal{A}) = \{B \subseteq E_i \mid f_i(B) \in \mathcal{A}\})$$

40.11 Corollary

Let $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$ be measurable spaces. $f : X \rightarrow Y$ be a mapping. $\mathcal{C}_Y \subseteq \mathcal{G}_Y$ such that $\mathcal{G}_Y = \sigma(\mathcal{C}_Y)$ Then f is measurable iff

$$\forall B \in \mathcal{C}_Y \quad f^{-1}(B) \in \mathcal{G}_X$$

Proof

$$\sigma(f) = \sigma(f^{-1}(\mathcal{C}_Y))$$

f is measurable iff $\sigma(f) \subseteq \mathcal{G}_X$

40.12 Example

Let $((E_i, \mathcal{E}_i))_{i \in I}$ be a family of measurable spaces.

$$E = \prod_{i \in I} E_i$$

$\forall i \in I$

$$\begin{aligned} \pi_i : E &\rightarrow E_i \\ (x_j)_{j \in I} &\mapsto x_i \end{aligned}$$

We denote by $\bigotimes_{i \in I} \mathcal{E}_i$ the σ -algebra $\sigma((\pi_i)_{i \in I})$

40.13 Prop

Let X be a set $((E_i, \mathcal{E}_i))_{i \in I}$ be a family of measurable spaces. (Ω, g) be a measurable space. $f = (f_i : X \rightarrow E_i)_{i \in I}$ be a mappings, $\varphi : \Omega \rightarrow X$ be a mapping. Then

$$\varphi : (\Omega, g) \rightarrow (X, \sigma(f))$$

is measurable iff

$$\forall i \in I \quad f_i \circ \varphi : (\Omega, g) \rightarrow (E_i, \mathcal{E}_i) \text{ is measurable.}$$

Proof

\Rightarrow If φ is measurable, since each f_i is measurable, one has $f_i \circ \varphi$ is measurable.

\Leftarrow If $f_i \circ \varphi$ is measurable, $\forall B \in \mathcal{E}_i$

$$(f_i \circ \varphi)^{-1}(B) = \varphi^{-1}(f_i^{-1}(B)) \in g$$

Hence

$$\varphi^{-1}\left(\bigcup_{i \in I} f_i^{-1}(B)\right) \subseteq g$$

Since

$$\sigma(f) = \sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i)\right)$$

φ is measurable.

40.14 Example

Let (Ω, \mathcal{G}) be a measurable space

- $\forall A \in \mathcal{G} \mathbb{1}_A : \Omega \rightarrow \mathbb{R}$ is measurable. For any $U \subseteq \mathbb{R}$

$$\mathbb{1}_A^{-1}(U) = A \text{ or } \Omega \setminus A \text{ or } \Omega \text{ or } \emptyset$$

- If X and Y be topological spaces. $f : X \rightarrow Y$ be a continuous mapping, then f is measurable with respect to Borel σ -algebra. In fact, $\forall V \subseteq Y$ open $f^{-1}(V) \subseteq X$ open.
- Let (Ω, \mathcal{G}) be a measurable space If $f : \Omega \rightarrow \mathbb{R}, g : \Omega \rightarrow \mathbb{R}$ are measurable then $f + g, fg, f \wedge g, f \vee g, |f|$ are measurable.
- Let $(f_n)_{n \in \mathbb{N}}$ be a family of measurable mappings from Ω to $[-\infty, +\infty]$

$$f = \sup_{n \in \mathbb{N}} f_n \quad (f(\omega) = \sup_{n \in \mathbb{N}} f_n(\omega))$$

Then f is measurable.

Similarly $\inf_{n \in \mathbb{N}} f_n$ is measurable.

In fact, for any $a \in \mathbb{R}$

$$\{\omega \in \Omega \mid f(\omega) > a\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid f_n(\omega) > a\}$$

Chapter 41

Measure

41.1 Def

Let Ω be a set. \mathcal{C} be a semi-algebra on Ω . $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$ be a mapping. If $\forall n \in \mathbb{N}, \forall (A_i)_{i=1}^n \in \mathcal{C}^n$ pairwise disjoint, with $A = \bigcup_{i=1}^n A_i$ one has

$$\mu(A) = \sum_{i=1}^n \mu(A_i)$$

we say that μ is additive.

Let

$S =$ vector subspace of \mathbb{R}^Ω generated by $(\mathbb{1}_A)_{A \in \mathcal{C}}$

Then

$$I_\mu : S \rightarrow \mathbb{R}$$
$$\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} \mapsto \sum_{i=1}^n \lambda_i \mu(A_i)$$

is well defined. If I_μ is an integral operator, we say that μ is σ -additive.

41.2 Def

Let (Ω, \mathcal{G}) be a measurable space. $\mu : \mathcal{G} \rightarrow [0, +\infty]$ be a mapping. If $\mu(\emptyset) = 0$ and if for any $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ pairwise disjoint.

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

we say that μ is a measure.

$(\Omega, \mathcal{G}, \mu)$ is called a measure space.

41.3 Def

If $\exists (A_n)_{n \in \mathbb{N}}$ such that $\Omega = \bigcup_{n \in \mathbb{N}} A_n$ and $\mu(A_n) < +\infty$ then μ is said to be σ -finite.

41.4 Carathéodory Theorem

Let Ω be a set, \mathcal{C} be a semi-algebra on Ω , $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$ be a σ -additive mapping. Assume that there is a sequence $(A_n)_{n \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ such that $\Omega = \bigcup_{n \in \mathbb{N}} A_n$. Then μ extends to a σ -finite measure on $\sigma(\mathcal{C})$.

Proof

Let $S \subseteq \mathbb{R}^{\Omega}$ be the vector subspace generated by $(\mathbb{1}_A), A \in \mathcal{C}$. Let $\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I_{\mu})^{\uparrow}\}$ then \mathcal{G} is a σ -algebra containing \mathcal{C} . Hence $\sigma(\mathcal{C}) = \mathcal{G}$. Moreover, $(A \in \mathcal{G}) \mapsto I_{\mu}(\mathbb{1}_A)$ is a measure on \mathcal{G} which is σ -finite.

41.5 Example

$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a < b\}$ $\sigma(\mathcal{C}) = \text{Borel } \sigma\text{-algebra}$ $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ increasing and right continuous.

$$\begin{aligned} \mu_{\varphi} : \mathcal{C} &\rightarrow \mathbb{R}_{\geq 0} \\]a, b] &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

is σ -additive.

Hence μ_{φ} extends to a measure:

$$\sigma(\mathcal{C}) \rightarrow [0, +\infty]$$

called the Stieltjes measure. In the particular case where $\varphi(x) = x$ ($\forall x \in \mathbb{R}$) μ_{φ} is called a Lebesgue measure.

41.6 Def

Let $(\Omega, \mathcal{G}, \mu)$ be a measure space. Then

$$\{A \in \mathcal{G} \mid \mu(A) < +\infty\}$$

is a semialgebra. $\sigma(\mathcal{C}) = \mathcal{G}$ and $\mu|_{\mathcal{C}}$ is σ -additive.

•

$$\begin{aligned} \mu(A_0) &= \sum_{n \in \mathbb{N}} \mu(B_n) < +\infty \\ \sum_{k \geq n} \mu(B_k) &= \mu(A_n) \rightarrow 0 \quad (n \rightarrow +\infty) \end{aligned}$$

We denote by $L^1(\Omega, \mathcal{G}, \mu)$ the set of measurable mappings $f : \Omega \rightarrow \mathbb{R}$ that belongs to $L^1(I_\mu)$. For $f \in L^1(\Omega, \mathcal{G}, \mu)$

$$I_\mu(f)$$

is denoted as

$$\int_{\Omega} f(\omega) \mu(d\omega)$$

41.6.1 Particular case

If $\Omega = \mathbb{R}$ $\mu = \mu_\varphi$ Stieftjes measure.

$$\int_{\mathbb{R}} f(x) \mu_\varphi(dx)$$

is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

41.7 Prop

Let $(\Omega, \mathcal{G}, \mu)$ be a σ -finite measure space. $f : \Omega \rightarrow \mathbb{R}$ is measurable. If

$$\exists g \in L^1(\Omega, \mathcal{G}, \mu), g \leq f$$

then

$$f \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

Proof

By replacing f by $f - g$, we may assume that $g = 0$ Consider first the case where

$$f = \mathbb{1}_B, B \in \mathcal{G}$$

Let $(A_n)_{n \in \mathbb{N}}$ be a increasing sequence in \mathcal{G} , $\mu(A_n) < +\infty$, $\bigcup_{n \in \mathbb{N}} A_n = \Omega$ Then

$$\mathbb{1}_B = \lim_{n \in \mathbb{N}} \mathbb{1}_{B \cap A_n} \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

For general $f \geq 0$

$$f = \lim_{n \rightarrow +\infty} f_n \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

where

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega \in \Omega \mid \frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}\}} + n \mathbb{1}_{\{\omega \in \Omega \mid f(\omega) \geq n\}}$$

41.8 Corollary

Let $f : \Omega \rightarrow \mathbb{R}$ be a measurable mapping. Then

$$f \in L^1(\Omega, \mathcal{G}, \mu)$$

iff

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

Proof

\Rightarrow One has $f \in L^1(I_{\mu})$. Hence $|f| \in L^1(I_{\mu})$ So $I_{\mu}(|f|) < +\infty$

\Leftarrow Suppose that

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

Since $f \vee 0$ and $-(f \wedge 0)$ belongs to $L^1(\Omega, \mathcal{G}, \mu)^+$ and $f \vee 0 \leq |f|$, $-(f \wedge 0) \leq |f|$ so

$$f \vee 0 \text{ and } -(f \wedge 0) \in L^1(\Omega, \mathcal{G}, \mu)$$

Hence

$$f = f \vee 0 + f \wedge 0 \in L^1(\Omega, \mathcal{G}, \mu)$$

Chapter 42

Fundamental theorem of calculus

42.1 Theorem

Let J be an open interval in \mathbb{R} $x_0 \in J$ $f : J \rightarrow \mathbb{R}$ be a continuous mapping.

(1) $\forall (a, b) \in J^2, a < b$

$$\begin{aligned} \mathbb{1}_{]a, b]} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto f(x) \quad \text{if } x \in]a, b] \\ &\mapsto 0 \quad \text{if } x \notin]a, b] \end{aligned}$$

belongs to $L^1(\mathbb{R}, \mathcal{B}, \mu)$ (\mathcal{B} is Borel σ -algebra, μ is Lebesgue measure)

(2) Let $F : J \rightarrow \mathbb{R}$ $F(x) := \int_{x_0}^x f(t)dt$. Then F is differentiable on J with $F'(x) = f(x), \forall x \in J$

42.2 Corollary

If $G : J \rightarrow \mathbb{R}$ is a mapping such that $G' = f$ then $\forall (a, b) \in J^2, a < b$

$$G(b) - G(a) = \int_a^b f(t)dt$$

42.2.1 Proof

(1) f is bounded on $[a, b]$ Hence

$$\int_{\mathbb{R}} \mathbb{1}_{]a, b]}^{(x)} |f(t)^{(x)}| dx < +\infty$$

- (2) Let $x \in J, h > 0$ such that $[x, x+h] \subseteq J$, f is uniformly continuous on $[x, x+h]$
For $0 < t \leq h$

$$\inf_{[x, x+t]} f \leq \frac{F(x+t) - F(x)}{t} = \frac{1}{t} \int_x^{x+t} f(s) ds \leq \sup_{[x, x+t]} f$$

Since f is continuous

$$\liminf_{t \rightarrow 0} \inf_{[x, x+t]} f = \limsup_{t \rightarrow 0} \sup_{[x, x+t]} f = f(x)$$

So

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Similarly

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Hence

$$F'(x) = f(x)$$

Application

- Let F and G be two mapping of class C^1 from J to \mathbb{R} . Then FG is of class C^1 and

$$FG'(x) = F'(x)G(x) + F(x)G'(x)$$

Let $f = F', g = G'$, then $\forall (a, b) \in J^2, a, b$

$$\int_a^b f(t)G(t)dt = F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t)dt$$

- Let $\varphi : I \rightarrow J$ be a mapping of class C^1 , where I is open interval. Let $F : J \rightarrow \mathbb{R}$ be a mapping of class C^1 .

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x)$$

- Hence $\forall (\alpha, \beta) \in I^2, \alpha < \beta$

$$\int_{\alpha}^{\beta} F(\varphi(t))\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha))$$

Chapter 43

L^p space

43.1 Def

We fix a measure space $(\Omega, \mathcal{G}, \mu)$ the set of measurable mappings $f : \Omega \rightarrow \mathbb{R}$ such that

$$\|f\|_{L^p} := \left(\int_{\Omega} |f(\omega)|^p \mu(dx) \right)^{\frac{1}{p}} < +\infty$$

Lemma Let $(p, q) \in \mathbb{R}_{\geq 1}^2$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

for any $(a, b) \in \mathbb{R}_{\geq 0}^2$

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

Proof We may assume that $(a, b) \in \mathbb{R}_{\geq 0}$

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{1}{p} \exp(p \ln a) + \frac{1}{q} \exp(q \ln b) \geq \exp(\ln a + \ln b) = ab$$

$$\begin{aligned} \int_{\Omega} |\varphi(x)\psi(x)| \mu(dx) &\leq \frac{\int_{\Omega} |\varphi(x)|^p \mu(dx)}{p \|f\|_{L^p}^p} + \frac{\int_{\Omega} |\psi(x)|^q \mu(dx)}{q \|g\|_{L^q}^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

43.2 Hölder inequality

Let $f : \Omega \rightarrow \mathbb{R}$ and $g : \Omega \rightarrow \mathbb{R}$ be measurable mappings. One has

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

Proof

Take

$$\varphi = \frac{f}{\|f\|_{L^p}}, \psi = \frac{g}{\|g\|_{L^q}}$$

then

$$|\varphi(x)\psi(x)| \leq \frac{|\varphi(x)|^p}{p} + \frac{|\psi(x)|^q}{q}$$

43.3 Corollary

Let $p \geq 1$ $\forall (f, g) \in L^p(\Omega, \mathcal{G}, \mu)$

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Proof

Apply Hölder inequality to $f(f + g)^{p-1}$ and $g(f + g)^{p-1}$