Chapter 1

λ-terms

1.1 Def

Let V be an infinite set (the elements of which are called variables)

We construct a set L which consists of finite sequences formed with the following symbols:

- elements of V
- left and right parenthesis()
- λ (Suppose that λ(,) are distinct and do not belongs to V,e.g.$\lambda x(t)$)

in a recursive way as follows:

- If $x \in V$ then $x \in L$
- If t and μ are elements of V, then $t(\mu) \in L$
- If $x \in V$ $t \in L$ then $\lambda xt \in L$

e.g.

$$I := \lambda xx \in L$$

some time we omit the parenthesis

$$t(\mu_1(\mu_2 \cdots (\mu_n) \cdots) \cdots)$$

can be written as

$$t\mu_1\mu_2 \cdots \mu_n \cdots$$

1.2 Def

Let $\alpha \in V$ and $t \in L$ We define the free occurrences of x in t as

- If $t = x$, the only occurrence of x in t is free

- If $t = \mu_1(\mu_2)$ the free occurrence of x in t are those of x in μ_1 and μ_2
- If $t = \lambda y \mu, y \neq x$, the free occurrence of x in t are those of x in μ
- If $t = \lambda x \mu$, no occurrence of x in t is free

If x has at least one free occurrence in t, we say that x is a free variable of t

If x occur in t just after λ, we say that x is a bound variable of t

Chapter 2

Substitutes

2.1 Def

Let $t, t_1, \cdots, t_k$ be elements of L and $x, x_1, \cdots, x_k$ be distinct variables in V. We define

$$t < t_1/x_1, \cdots, t_k/x_k > \in L$$

as follows:

- If $t = x_i$
$$t < t_1/x_1, \cdots, t_k/x_k >= t_i$$
- If $t \in V \setminus \{x_1, \cdots, x_k\}$
$$t < t_1/x_1, \cdots, t_k/x_k >= t$$
- If $t = \mu_1(\mu_2)$ then
$$t < t_1/x_1, \cdots, t_k/x_k >= \mu_1 < t_1/x_1, \cdots, t_k/x_k > (\mu_2 < t_1/x_1, \cdots, t_k/x_k >)$$
- If $t = \lambda x_i u$
$$t < t_1/x_1, \cdots, t_k/x_k >= \lambda x_i (t_1 < t_1/x_1, \cdots, t_{i-1}/x_{i-1}, t_{i+1}/x_{i+1}, \cdots, t_k/x_k >)$$
- If $t = \lambda x \mu, x \notin \{x_1, \cdots, x_k\}$
$$t < t_1/x_1, \cdots, t_k/x_k >= \lambda x \mu < t_1/x_1, \cdots, t_k/x_k >$$

<u>Reference</u>: Jean-Louis KrivineLambda-calculus, type and models.

Chapter 3

α-equivalence

3.1 Def

We define a binary relation $\equiv$ on L in a recursive way as follows:

- If $t \in V$ $t \equiv t'$ iff $t = t'$
- If $t = \mu_1(\mu_2)$ $t \equiv t'$ iff $\exists \mu_1'$ and μ_2' in L such that $\mu_1 \equiv \mu_1', \mu_2 \equiv \mu_2'$ and $t' = \mu_1'(\mu_2')$
- if $t = \lambda x \mu$ $t \equiv t'$ iff t' is of the form $t' = \lambda x' \mu'$ with $\mu < y/x > \equiv \mu' < y/x' >$ for all but finitely many $y \in V$

3.2 Facts

- $\equiv$ is an equivalence relation
- If $t = t'$ then t and t' have the same length and the same free variables.
- Let $t, t', t_1, t_1', \cdots, t_k, t_k'$ be elements of L $x_1, \cdots, x_k$ be distinct variables if $t \equiv t', t_i \equiv t_i', \forall i \in \{1, \cdots, k\}$, and no free variables of $t_1, \cdots, t_k$ is bound in t and t' then

$$t < t_1/x_1, \cdots, t_k/x_k > \equiv t' < t_1'/x_1, \cdots, t_k'/x_k >$$

- $\equiv$ is λ-compatible namely
  - if $\mu_1 \equiv \mu_1', \mu_2 \equiv \mu_2'$ then $\mu_1(\mu_2) \equiv \mu_1'(\mu_2')$
  - if $t \equiv t'$ then $\lambda x t \equiv \lambda x t'$

  Hence the constructions of L induces by taking equivalence classes the following constructions on $\Lambda := L/\equiv$

  - For any U_1 and U_2 in Λ with representation μ_1 and μ_2 respectively, we denote $U_1(U_2)$ as the equivalence class of $\mu_1(\mu_2)$
  - $\forall x \in V, \forall T \in \Lambda$ with representation t, we define $\lambda x T$ as the equivalence class of $\lambda x t$

- $\lambda x t \equiv \lambda y t < y/x >$ if y is a variable that does not occur on t
- Let $t \in L$ and $x_1, \cdots, x_k$ be elements of V. $\exists t' \in L, t' \equiv t$ such that none of $x_1, \cdots, x_k$ is bound on t'
- Let $T \in \Lambda$ All elements of T have the same set of free variables we call then free variables of T

3.3 Def

Let $T, T_1, \cdots, T_k$ be elements of Λ $t, t_1, \cdots, t_k$ be their representations such that no bound variables of t is free is $t_1, \cdots, t_k$. we define

$$T[T_1/x_1, \cdots, T_k/x_k] := \text{ the equivalence class of } t < t_1/x_1, \cdots, t_k/x_k >$$

3.3.1 Facts

- If x_1 is not free in T

$$T[T_1/x_1, \cdots, T_k/x_k] = T_1[T_2/x_2, \cdots, T_k/x_k]$$

- Let $x_1, \cdots, x_m, y_1, \cdots, y_m$ be variables such that $x_1 = y, \cdots, x_k = y_k$ and $x_1, \cdots, x_m, y_{k+1}, \cdots y_n$ are distinct.

  Let $T, T_1, \cdots, T_m, U_1, \cdots, U_n$ be elements of Λ

$$T_i' = T_i[\mu_1/y_1, \cdots, \mu_n/y_n]$$

  Then

$$T[T_1/x_1, \cdots, T_m/x_m][U_1/y_1, \cdots, U_n/y_n] = T[T_1'/x_1, \cdots, T_m'/x_m, U_{k+1}/y_{k+1}, \cdots, U_n/y_n]$$

- If $i \in \{1, \cdots, k\}$

$$x_i[T_1/x_1, \cdots, T_k/x_k] = T_i$$

  If $x \in V \setminus \{x_1, \cdots, x_k\}$

$$x[T_1/x_1, \cdots, T_k/x_k] = x$$

  (we still use x to represent its equivalence class $\{x\}$)

  If $T = \lambda x U$ x is not free in $T_1, \cdots, T_k$

$$x \notin \{x_1, \cdots, x_k\}, T[T_1/x_1, \cdots, T_k/x_k] = \lambda x U[T_1/x_1, \cdots, T_k/x_k]$$

Chapter 4

β-convention

4.1 Def

We define a binary relation β_0 on Λ as follows:

- If $x \in V$ there is no T' such that $x\beta_0 T'$
- If $T = U_1(U_2)$ $T\beta_0 T'$ iff either
  - $T' = U_1(U_2')$ with $U_2 \beta_0 U_2'$
  - $T' = U_1'(U_2)$ with $U_1 \beta_0 U_1'$
  - $U_1 = \lambda x W$ $T' = W[U_2/x]$

We denote by $\simeq_\beta$ the smallest equivalence relation that contains β_0

4.2 Def

Let $\mathcal{T}_0$ be a set, let $\mathcal{T}$ be the free magma generated by $\mathcal{T}_0$, the composition law of which is denoted as $\to$
$\forall(\alpha, \beta) \in \mathcal{T}^2,\ \alpha \to \beta$ is defined as an element of $\mathcal{T}$

- $\mathcal{T}$ is a set and $\to$ is a composition law of $\mathcal{T}$
- $\mathcal{T}_0 \subseteq \mathcal{T}$ and an element of $\mathcal{T}$ is obtained by successive composition of elements of $\mathcal{T}_0$

$$\alpha_0 \in \mathcal{T}_0, \beta_0 \in \mathcal{T},\ \alpha_0 \to \alpha_0 \in \mathcal{T}, \alpha_0 \to \beta_0 \in \mathcal{T}, (\alpha_0 \to \beta_0) \to \alpha_0$$

Suppose that $\mathcal{T}_n$ is constructed, we let

$$\mathcal{T}_{n+1} = \mathcal{T}_n \cup \{\alpha \to \beta \mid \alpha \in \mathcal{T}_n, \beta \in \mathcal{T}_n\}$$

let $\mathcal{T} = \bigcap_{n \in \mathbb{N}} \mathcal{T}_n$

4.3 Def

We call content any subset Γ of $\Lambda \times \mathcal{T}$(if $(T, \alpha) \in \Gamma$ we write $T : \alpha \quad \Gamma \vdash T : \alpha$) that satisfies the following conditions:

- If $x \in V, T \in \Lambda \quad x : \alpha, T : \beta$ then

$$\lambda x T : \alpha \to \beta$$

- If $T \in \Lambda, U \in \Lambda$ with $T : \alpha \to \beta$ $U : \alpha$ then

$$T(U) : \beta$$

Table 4.1: comparison

Type theory	Mathematic Logic	Set theory
α type	proposition	a set
$T : \alpha$	proof	$t \in A$ element
$\underline{0}$ $\underline{1}$	$\perp$ $\top$	$\varnothing$ $\{\varnothing\}$
$\alpha \to \beta$	$A \Rightarrow B$	set of mappings from A to B
$(Id_\alpha = \lambda x : \alpha x)\alpha \to \alpha$	$=$	$\{(x, x) \mid x \in \Lambda\}$
$\alpha + \beta$	A or B	$A \cup B$
$\alpha \times \beta$	A and B	Cartesian prod $A \times B$
$\sum\limits_{x:\alpha} \beta(x)$	$\forall x$	disjoint sum $\coprod\limits_{x \in A} B(x)$
$\prod\limits_{x:\alpha} \beta(x)$	$\exists x$	$\prod\limits_{x \in A} B(x)$