1 Def: Tensor

Let M and N be two R-modules. Then exists an R-module denoted by $M \otimes_R N$ and a bilinear mapping

$$t : M \times N \to M \otimes_R N$$

having the following properties:

(1) For any R-module P and any bilinear mapping $s : M \times N \to P$. There exists a unique linear mapping $f_s : M \otimes_R N \to P$ such that $s = f_s \circ t$

$$\begin{tikzcd} M \times N \arrow[r, "s"] \arrow[d, "t"] & P \\ M \otimes_R N \arrow[ur, "f_s"'] & \end{tikzcd}$$

(2) If T, t' is another couple that satisfies (1) with $s \mapsto g_s$ then there exists a unique isomorphism

$$T \cong M \otimes_R N$$

Let $\mathcal{F}$ be the free R-module generated by $M \times N$

$$\mathcal{F} = \{ \sum_{finite} a_{ij}(m_i, n_i) : a_{ij} \in R, m_i \in M, n_i \in N\}$$

let $\mathcal{G}$ be the R-submodule generated by the elements of the following shape $m, m' \in M \quad n, n' \in N \quad \mathfrak{r} \in R$

$$\begin{aligned} &(m + m', n) - (m, n) - (m', n) \\ &(m, n + n') - (m, n) - (m, n') \\ &(\mathfrak{r}m, n) - \mathfrak{r}(m, n) \\ &(m, \mathfrak{r}n) - \mathfrak{r}(m, n) \end{aligned}$$

$$M \otimes_R N := \mathcal{F}/\mathcal{G}$$

2 Def

$$f_s(\mathcal{G} + (m, n)) := s(m, n)$$

Extend this mapping to linearity. This makes the diagram commutative. It's clearly the unique mapping

3 Def

The R-module $M \otimes_R N$ constructed above is called the tensor product of M and N. An element of $M \otimes_R N$ is called tensor. We denote

$$t(m, n) := m \otimes n$$

and any elements of this form is called pure tensor.

4 Remark

Pure tensors generate $M \otimes_R N$. In particular any tensor can be written as sum of pure tensors.

5 tensor product and duality

5.1 product

Let $V_1, \cdots, V_n$ be vector spaces as above. Then

$$(V_1^\vee \otimes \cdots \otimes V_n^\vee) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

5.2 duality

Let V and W be vector spaces of finite dimension. Then

$$\mathscr{L}(V, W) \cong V^\vee \otimes W^\vee$$

6 Def

We went to define the tensor product of linear mappings. letM_1, M_2, N_1, N_2 be R-modules and let $f_i : M_i \to N_i$ be linear mappings. Then we define

$$\begin{aligned} f_1 \otimes f_2 : M_1 \otimes M_2 &\to N_1 \otimes N_2 \\ m_1 \otimes m_2 &\mapsto f_1(m_1) \otimes f_2(m_2) \end{aligned}$$

This is a linear mapping

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "f_1 \times f_2"] \arrow[d] & N_1 \times N_2 \arrow[d] \\ M_1 \otimes M_2 \arrow[r, dashed, "f_1 \otimes f_2"] & N_1 \otimes N_2 \end{tikzcd}$$

7 Extension of scalars

Let $\varphi : R \to S$ be a commutative unitary ring homomorphism. Let M be a R-module. Goal is to give to M also a structure of S-module "conveyed by φ"

Note that S has a structure of R-module $s \in S, \mathfrak{r} \in R$

$$\mathfrak{r}s := \varphi(\mathfrak{r})s$$

Now take thw tensor product $M \otimes_R S$. Now we give a structure of S-module to $M \otimes_R S$.

Take $s \in S$

$$s(\underbrace{m \otimes s'}_{\in M \otimes_R S}) := m \otimes ss'$$

note that ss' is a multi in S and we cannot product sm.

Notice we've a mapping

$$i : M \to M \otimes_R S$$
$$m \mapsto m \otimes s$$

Be careful, in general the mapping i is NOT injective.

8 Prop

Let $K \subseteq L$ be a field extension and let V be a K-vector space. Moreover let's denote $V_L = V \otimes_K L$. If $\{e_i\}_{i=1}^n$ is a basis of V then $\{e_i \otimes 1\}_{i=1}^n$ is a L-basis of V_L.(V_L has the same dim of V)

9 Def

We denote

$$\begin{aligned} T_p^q :=& (V^\vee)^{\otimes p} \otimes V^{\otimes q} \qquad p, q \in \mathbb{N} \\ =& \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{p \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \end{aligned}$$

An element of $T_p^q(V)$ is called a tensor of type (p, q) (or a mixed tensor which is p-covariant and q-contravariant)

Let's denote:

$$T(V) := \bigoplus_{q \in \mathbb{N}} T_0^q(V)$$

On $T(V)$ we have following operation:

$$T_0^l(V) \times T_0^q(V) \to T_0^{l+q}(V)$$
$$((x_1 \otimes \cdots \otimes x_l), (y_1 \otimes \cdots \otimes y_q)) \mapsto x_1 \otimes \cdots \otimes x_l \otimes y_1 \otimes \cdots \otimes y_q$$

With this operation $T(V)$ becomes a K-algebra. It called the tensor algebra associated to V

10 Def

The quotient algebra

$$\bigwedge(V) := T(V) / \left\{ \sum_{i(finite)} (y_1 \otimes \cdots \otimes y_{m_i}) \otimes (x_i \otimes x_i) \otimes (z_1 \otimes \cdots \otimes z_{n_i}) \right\}$$

is a K-algebra, which called the exterior algebra of V

$$\begin{aligned} \pi : &T(V) && \to \bigwedge(V) \\ &x_1 \otimes \cdots \otimes x_n && \mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

11 Notation

$$\bigwedge(V) = \bigoplus_{n \in \mathbb{N}} \bigwedge^n(V)$$

$$\bigwedge^n(V) := T_0^n(V) / (W \cap T_0^n(V))$$

this is called n-fold exterior product

12 Prop

FIx a vct space V. For any alternating multi-linear mapping

$$s : \underbrace{V \times \cdots \times V}_{n \text{ times}} \to W$$

when W is another vct space, there exists a unique linear mapping

$$g_s : \bigwedge^n(V) \to W$$

such that the following diagram commutes

$$\begin{tikzcd}
V^n \arrow[r, "s"] \arrow[d, "t"'] & W \\
T_0^n(V) \arrow[ur, dashed, "f_s"] \arrow[d] & \\
\bigwedge^n(V) \arrow[uur, "g_s"'] &
\end{tikzcd}$$

13 Prop

Let V be a vct space of dimension n with a basis $\{e_1, \cdots, e_n\}$. Then $\bigwedge^k(V)$ is a vct space with a basis given by

$$\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \big| \; 1 \leq i_1 < \cdots < i_k \leq n\}$$

In particular, $\bigwedge^k(V)$ has dimension $\begin{pmatrix} n \\ k \end{pmatrix}$

14 Def

Let V be a vct space of dimension n, then

$$\det(V) = \bigwedge^n(V)$$

is called the determinant of V. It is a vct space of dimension $1 = \binom{n}{n}$ and a basis is given by

$$e_1 \wedge \cdots \wedge e_n$$

when $\{e_1, \cdots, e_n\}$ is a basis of V.

15 Def

$$\begin{aligned} g_{\widetilde{f}} = \bigwedge^k f : \bigwedge^k(V) &\rightarrow \bigwedge^k(V) \\ v_1 \wedge \cdots \wedge v_k &\mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

16 Def

Let $F : V \rightarrow V$ be a linear mapping. A subspace $V_0 \subseteq V$ is said to be an invariant subspace of F is $F(V_0) \subseteq V_0$

17 Def

A linear mapping $f : V \rightarrow V$ (finite dim) is diagonalizable if the following equivalent conditions are satisfied

1 V decomposes as a direct sum of one-dimensional invariant subspace of f

2 There exists a basis of V, in which the matrix A_f is diagonal.

18 Def

V a vector space over K $dim(V) = n, f \in \mathscr{L}(V;V)$ let A_f be an associated matrix (in any basis) the mapping

$$\begin{aligned} P : K &\to K \\ t &\mapsto \det(tI_n - A_f) \end{aligned}$$

This is a polynomial in $K[t]$ (with degree n)

19 Def

Let $a_0 + a_1 t + \cdots + a_n t^n = Q(t) \in K[t]$, then for $f \in \mathscr{L}(V;V)$ we define

$$Q(f) := a_0 id_V + a_1 f + a_2 f^{\circ 2} + \cdots + a_n f^{\circ n}$$

Remark From now on we write

$$f^{\circ k} = f^k$$

these are operations in $\mathscr{L}(V;V), +, \circ$

we say that Q annihilates f if $Q(f) = 0$

20 Prop

Let $f \in \mathscr{L}(V;V)$. There exists a polynomial $Q \in K[t] \setminus \{0\}$ that annihilates f (i.e. $Q(f) = 0$)

Remark

The proof of this proposition also gives the degree of a polynomial that annihilates ($\leq n^2$)

21 Def

Let $m(t) \in K[t] \setminus \{0\}$ be a monic polynomial of minimal degree that annihilates $f \in \mathscr{L}(V;V)$. Then $m(t)$ is called minimal polynomial of f And by prop above (20), $m(t)$ exists.

22 Prop

If $m(t)$ is minimal polynomial of f, then $m(t)$ is unique.

23 Prop

Let $Q \in K[t] \setminus \{0\}$ be a polynomial that annihilates f. Then $m_f \mid Q$

24 Theorem: Cayley-Hamilton Theorem

The characteristic polynomial P_f annihilates f

25 Theorem

Let $f \in \mathscr{L}(V; V)$ when V is a vector space of dim n, over an algebraically closed field.
Then

(1) f can be represented by a Jordan matrix

(2) This above matrix is unique up to permutation of the Jordan blocks

26 Def

Let $f \in \mathscr{L}(V; V)$ and let $\lambda \in K$. A vector $w \in V \setminus \{0\}$ is called a root vector of f corresponding to λ, if there exists $\mathfrak{r} \in \mathbb{N}$ s.t.

$$(f - \lambda id_V)^{\mathfrak{r}}(w) = 0$$

Remark

Eigenvector are root vectors (corresponding to their eigenvalues) take $\mathfrak{r} = 1$

Remark

Let $J_{\mathfrak{r}}(\lambda)$ be a Jordan block. Then any $\sigma \in V$ is a root vector of f corresponding to λ. In fact:

$$(J_{\mathfrak{r}}(\lambda) - \lambda I_n)^m = 0 \quad \text{if } m \geq \mathfrak{r}$$

27 Prop

Let K be an algebraically closed field. Let $\lambda_1, \cdots, \lambda_k$ be all of distinct eigenvalues of $f (k \geq 1)$, then

$$V = \bigoplus_{i=1}^{k} V(\lambda_i)$$

28 Def

Let $f \in \mathscr{L}(V; V)$. Then f is said to be nilpotent if there exists $t \in \mathbb{N}$ that $f^t = 0$

29 Lemma

Let f be a nilpotent mapping, then $Ker(f) \neq \{0\}$

Proof

Let $\mathfrak{r}$ be the minimal integer s.t. $f^{\mathfrak{r}} = 0$ then

$$f^{\mathfrak{r}-1}(V) \subseteq Ker(f)$$

but $f^{\mathfrak{r}-1}(V) \neq \{0\}$ because of the minimality of $\mathfrak{r}$

30 Theorem

Let $f \in \mathscr{L}(V;V)$ be a nilpotent mapping, then there exists a Jordan basis for f that gives a Jordan matrix made of blocks of the type $J_{\mathfrak{r}}(0)$

31 Theorem

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. Then f admits a Jordan basis (namely there exists a basis s.t. A_f is a Jordan matrix).

32 Def

Let λ be an eigenvalue of $f \in \mathscr{L}(V)$

$$E(\lambda) := \ker(f - \lambda Id)$$

This $E(\lambda)$ is called the eigenspace of λ

$$mult(\lambda)_{geo} = dim(E(\lambda))$$

is called the geometric multiplicity of λ

Moreover

$$mult(\lambda)_{alg} = \max\left\{k \in \mathbb{N} \,\middle|\, (t-\lambda)^k \mid P_f(t)\right\}$$

is called the algebraic multiplicity of λ

33 Prop

Let K be algebraically closed. Then $\forall\lambda$ eigenvalues of f

$$mult(\lambda)_{geo} \leq mult(\lambda)_{alg}$$

34 Corollary

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. f is diagonalizable iff

$$\forall \lambda_i \quad mult(\lambda_i)_{geo} = mult(\lambda_i)_{alg}$$

35 Def

Two matrices $G, G' \in M_{n\times n}(K)$ are said conjugate if $\exists A \in \mathcal{Q}_{n\times n}(K)$ s.t. $G = G'^T$

36 Def

Let $p \in \mathbb{R}^n$ be a fixed point

$$\mathbb{R}^n_p := \{p\} \times \mathbb{R}^n$$

$(p, a) \in \mathbb{R}^n_p, a \in \mathbb{R}^n$

$$(p, a) + (p, b) = (p, a + b)$$
$$\alpha(p, a) = (p, \alpha a) \ \alpha \in \mathbb{R}$$

With these operation $\mathbb{R}^n_p$ is a vector space, which is called the tangent space of $\mathbb{R}^n$ at p.

The dual space is

$$(\mathbb{R}^n_p)^\vee = \{p\} \times (\mathbb{R}^n_p)^\vee$$

A basis of $\mathbb{R}^n_p$ is denoted by

$$(e_1 \mid_p, \cdots, e_n \mid_p)$$

$\bigsqcup\limits_p \mathbb{R}^n_p$ is called the tangent bundle of $\mathbb{R}^n$

36.1 Notation

$$a \mid_p := (p, a)$$

37 Def

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We have a projection mapping:

$$\begin{aligned} \bigsqcup_p \mathbb{R}^n_p &\xrightarrow{\pi} \mathbb{R}^n_p \\ (p, a) &\mapsto p \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n \cong \bigsqcup_p \mathbb{R}^n_p \\ (p, a) \leftarrow\!\!\dashv (p, a) \end{aligned}$$

Take $\{e_1 \mid_p, \cdots, e_n \mid_p\}$ as a basis of $\mathbb{R}^n_p$. The dual basis is denoted by

$$\{dx_1 \mid_p, \cdots, dx_n \mid_p\} = \{(e_1 \mid_p)^\vee, \cdots, (e_n \mid_p)^\vee\} \in (\mathbb{R}^n_p)^\vee$$

$$\begin{aligned} dx_i \mid_p : \mathbb{R}^n_p &\to \mathbb{R} \\ v = (\sum \alpha_i e_i \mid_p) &\mapsto \alpha_i \end{aligned}$$

$$\frac{\partial x_i}{\partial x_j} = dx_i \mid_p (e_j \mid_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recalled the wedge algebra:

$$\bigwedge (\mathbb{R}^n_p)^\vee := T(\mathbb{R}^n_p)^\vee / I = \bigoplus_{k \in \mathbb{N}} \bigwedge^k (\mathbb{R}^n_p)^\vee$$

Consider

$$\bigwedge^k (\mathbb{R}^n_p)^\vee$$

what's a basis of this vector space?

$$\left\{ dx_1 \mid_p \wedge \cdots \wedge dx_k \mid_p \,\middle|\, 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

and

$$\dim(\bigwedge^k (\mathbb{R}^n_p)^\vee) = \binom{n}{k}$$

Proved.

38 Do Carmo Differential forms

39 Def

An exterior k-form in $\mathbb{R}^n$ is a mapping:

$$\begin{aligned} \omega : \mathbb{R}^n &\to \bigsqcup_p \bigwedge^k (\mathbb{R}^n_p)^\vee \\ p &\mapsto \omega(p) \end{aligned}$$

that's a section of the projection π

$$(\pi \circ \omega = id_{\mathbb{R}}) = (\omega(p) \in \bigwedge^k (\mathbb{R}^n_p)^\vee)$$

$$\omega(p) = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}(p) dx_{i_1} \mid_p \wedge \cdots \wedge dx_{i_k} \mid_p \in \bigwedge^k (\mathbb{R}^n_p)^\vee$$

Note that

$$\begin{aligned} &\bigsqcup_p \bigwedge^k (\mathbb{R}^n_p)^\vee &\overset{\pi}{\twoheadrightarrow} \mathbb{R}^n \\ &f \mid_p &\mapsto p \\ &\omega \leftrightarrow \{a_{i_1}, \cdots, a_{i_k}\} \end{aligned}$$

if all a_{i_j} are of class $C^m(\mathbb{R})$ the ω is called a C^m-differential k-form. If $m = +\infty$ *omega* is called a smooth k-form.

40 Notation

$$\omega = \sum_I a_I dx_I$$

where $I = (i_1, \cdots, i_k)$

41 Notation

When $k = 0$ a 0-form of class C^m-differential 0-form is $f \in C^m(\mathbb{R}^n)$

$$C^m(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \text{ of class } C^m\}$$

42 Notation

$$\Omega^k_{(m)}(\mathbb{R}^n) := \{\text{set of } C^m\text{-diff } k\text{-forms}\}$$
$$\Omega^0_{(m)}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$$

m could be omitted if no confusion.

43 Def

Now we have

$$\Omega(\mathbb{R}^n) = \bigoplus_{k\in\mathbb{N}} \Omega^k(\mathbb{R}^n)$$

a $\mathbb{R}$-algebra with the $\wedge$-product

And it's also a $\Omega^0(\mathbb{R}^n)$ module and $\Omega^0(\mathbb{R}^n)$-algebra

44 Def: Pullback of forms

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of $C^{\mathfrak{r}}$, then it induces a mapping

$$f^* : \Omega^k_{(\mathfrak{r})}(\mathbb{R}^m) \to \Omega^k_{(\mathfrak{r})}(\mathbb{R}^n)$$
$$\omega \mapsto f^*\omega$$

and

$$f^*(\omega)(p)(v_1, \cdots, v_k) = \omega(f(p))(df\mid_p (v_1), \cdots, df\mid_p (v_k))$$

recalling

$$df\mid_p : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)} \;\Rightarrow\; df\mid_p (v_i) \in \mathbb{R}^n_{f(p)}$$

45 Remark

$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$

$$f \wedge \omega = f\omega$$

46 Prop

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable mapping. Then

(1) for any two forms in $\mathbb{R}^m$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*(\eta))$$

(2) for $g : \mathbb{R}^p \to \mathbb{R}^n$ differentiable

$$(f \circ g)^*\omega = g^*(f^*\omega)$$

47 Def: Path integral

Let γ and ω be as above.

$$\int_\gamma \omega := \sum_i \int_{t_k}^{t_{k+1}} \gamma_j^* \omega$$

this is the integral of ω along the parametric curve γ with

$$\gamma = t \mapsto (x_1(t), \cdots, x_n(t))$$

where $x_i(t) = \frac{\mathrm{d}x_i}{\mathrm{d}t}$

48 Def(σ-finite)

Let (X, Σ_X, μ) be a measure space. WE say that it's σ-finite if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets. (namely $E_n \in \Sigma_X$) such that

$$X = \bigcup_{n\in\mathbb{N}} E_n \text{ and } \mu(E_n) < +\infty, \forall n \in \mathbb{N}$$

49 Notation

Take sets $A \subseteq X \times Y$ For $x \in X$, we define

$$A_x := \{u \in Y \mid (x,y) \in A\}$$

called a **vertical section** of A or $x-$fiber of A

For $y \in Y$ we define

$$A_y := \{x \in X \mid (x,y) \in A\}$$

called a **horizontal section** of A, or y-fiber of A

50 Def

Let X be a set. then $\mathscr{D} \subseteq \wp(X)$ is a **Dynkin system** if

- $X \in \mathscr{D}$ and $\varnothing \in \mathscr{D}$
- $\forall D \in \mathscr{D} \quad X \setminus D \in \mathscr{D}$
- If $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{D}$ of pairwise disjoint sets, then

$$\bigsqcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

Remark

A σ-algebra is a Dynkin system

51 Def

Let $(\mathcal{G} \subseteq \wp(X))$ then $\delta(\mathcal{G}) \subseteq \wp(X)$ is called the Dynkin system generated by $\mathcal{G}$ if

- $\mathcal{G} \subseteq \delta(\mathcal{G})$
- If $\mathscr{D}$ is a Dynkin system containing $\mathcal{G}$, then $\delta(\mathcal{G}) \subseteq \mathscr{D}$

52 Prop

If $\mathscr{D}$ is a Dynkin system closed under the intersection, then it's a σ-algebra, namely

$$\forall (D,E) \in \mathscr{D}^2, D \cap E \in \mathscr{D} \;\Rightarrow\; \forall \{D_n\}_{n\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}} \quad \bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

53 Prop

Let X be a set and let $\mathcal{G} \subseteq \wp(X)$. Assume that $\mathcal{G}$ is closed under the finite intersection. Then

$$\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$$

54 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. Then $\forall E \in \Sigma_X \otimes \Sigma_Y$, the functions

$$\begin{aligned} f_E :& X &&\to \mathbb{R} \cup \{+\infty\} \\ & x &&\mapsto \nu(E_x) \\ g_E :& Y &&\to \mathbb{R} \cup \{+\infty\} \\ & y &&\mapsto \mu(E_y) \end{aligned}$$

are respectively Σ_X-measurable and Σ_Y-measurable

55 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. There exists a unique σ-finite measure $\mu \times \nu$ on $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ such that

$$\mu \times \nu(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

and moreover, we have

$$(\mu \times \nu)(E) = \int_X f_E \mathrm{d}\mu = \int_Y g_E \mathrm{d}\nu$$

56 Def: Push-forward measure

Let (X, Σ_X, μ) be a measure space, and let (Y, Σ_Y) be a measurable space. If $f : X \to Y$ is a measurable function, then define:

$$f_{*\mu}(E) = \mu(f^{-1}(E)) \quad \forall E \in \Sigma^Y$$

This is a measure on Y, called the push forward of μ through f

57 Fubini-Tobelli Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be two σ-finite measure spaces. Let $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$ be the product space. Let $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a measurable function. Then

$$\begin{aligned}\int_{X\times Y} |f| \,\mathrm{d}(\mu \times \nu) &= \int_X (\int_Y |f(x,y)| \,\mathrm{d}\nu(y))\mathrm{d}\mu(x) \\ &= \int_X (\int_Y |f(x,y)| \,\mathrm{d}\nu(y))\mathrm{d}\mu(x)\end{aligned}$$

58 Notation

For any mapping $\gamma : [a, b] \to U$

- γ is called a closed curve if $\gamma(a) = \gamma(b)$ and γ is a curve
- γ is called a path if γ is of class C^0
- γ is called a loop if γ is a closed path

59 Def: Lebesgue number

Let (X, ρ) be a metric space and $\mathcal{U} = \{U_i\}$ be an open covering X

A **Lebesgue number** $\delta = \delta_{\mathcal{U}}$ (of the open covering $\mathcal{U}$) is a non-negative number that:

If $Z \subseteq X$ is a subset with $diam(Z) < \delta$, then $Z \subseteq U_j$ for some $U_j \in \mathcal{U}$

Remark

- $\delta' < \delta$ is also a Lebesgue number
- In principle, a Lebesgue number δ can be 0

60 Lemma

If X is compact, then for any open covering there exists a positive Lebesgue number.

61 Theorem(homotopy invariance of the integrals)

Ler ω be a closed form on an open set U. Let γ_0, γ_1 be homotopy paths in U, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

62 Def: Free Homotopy

Let $\gamma_0, \gamma_1 : [a, b] \to U$ be two loops (namely $\gamma(a) = \gamma(b)$)
A **free homotopy** between γ_0 and γ_1 is a continuous mapping:

$$\begin{aligned} H : & [a, b] \times [0, 1] && \to U \\ & (s, t) && \mapsto H(s, t) \end{aligned}$$

such that

- $$H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$$
- For any fixed t_0
  $$H(\cdot, t_0)$$
  is a loop

63 Notation

A path $\gamma : [a, b] \to I$ is said simple if $\gamma \mid_{]a,b[}$ is injective (No self-cross this is)

64 Jordan Theorem

Let γ be a simple loop $\gamma : [a, b] \to U$, then $\mathbb{R}^2 \setminus \gamma([a, b])$ consists exactly of two connected components. One of this is bounded (interior), the other one unbounded (exterior). Moreover $\gamma([a, b])$ is the boundary of two components.

65 Def

Let $c : [a, b] \to S^1$ be a closed curve. Let φ be the angular function of c. We define the winding number of c as:

$$n(c) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

Since c us a closed curve, $n(c) \in \mathbb{Z}$

66 Def

Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\}$ be a closed curve. $(\gamma_p + \rho(t)c(t))$, when $c(t) \in S^1$

$$\gamma(t) = p + \rho(t)(\cos(\theta(t)) + \sin(\theta(t)))$$

Then we define the winding number of γ at p

$$n_p(\gamma) := n(c)$$

67 Prop

Let $\gamma = p + \rho(t)c(t)$ be a closed curve $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\}$ then

$$n_p(\gamma) = \frac{1}{2\pi i} \int_C \omega_0$$

where

$$\omega_0 = -\frac{y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y$$

68 Prop

Let $\gamma_0, \gamma_1 : [0, b] \to \mathbb{R}^2 \setminus \{p\}$ be two closed curves. Then they're freely homotopic iff

$$n_p(\gamma_0) = n_p(\gamma_1)$$

69 Def

Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differential mapping. We say that $p \in U$ is a zero of F if $F(p) = 0$. If then exists a neighborhood V of p such that V contains no zero of F other then p, then p is called isolated zero.

If p is a zero of F and $\mathrm{d}F \mid_p$ is non singular at p, then we say that p is a simple zero.

70 Def

The index of F in D, is defined as

$$n(F,D) := \frac{1}{2\pi}\int_C \theta$$

See that $\theta = F^*\omega_0$, $\omega_0 = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}$

$$\begin{aligned} n(F,D) &= \frac{1}{2\pi}\int_C \theta \\ &= \frac{1}{2\pi}\int_C F^*\omega_0 \\ &= \frac{1}{2\pi}\int_{F\circ} \omega_0 \\ &= \text{(winding number of } F\circ C \text{ at the center of } FD) \end{aligned}$$

71 Remark

$$n(F,D) = \frac{1}{2\pi}\int_C \theta = \frac{1}{2\pi}\int_{F\circ C} \omega_0$$

72 Prop

If $n(F,D) \neq 0$ then $\exists q \in D$ s.t. $F(q) = 0$

73 Def

A simple zero p of F is said **positive** if $\det(\mathrm{d}_p F) > 0$, otherwise is said **negative** ?(what's =0?)

74 Kronecker Index Theorem

Assume that $F; U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ has only finite simple zeros in a disk $D \subseteq U$ and none of them in ∂D. Then

$$n(F,D) = P - N$$

where P is the number of positive simple zeros and N is the number of negative simple zeros.

75 Def

Let $\mathscr{P} = \{t = t_a, t_1, \cdots, t_n = b\}$, $p_i = \gamma(t_i)$

$$l_{\mathscr{P}}(\gamma) = \sum_{i=0}^{n} \|p_{i+1} - p_i\|$$

The length of γ is

$$l(\gamma) := \sup_{p}\{l_p(\gamma)\}$$

If $l(\gamma) < +\infty$, then path γ is said rectifiable.

76 Prop

Let $\gamma : [a, b] \to \mathbb{R}^n$ be of class C^1, then γ is rectifiable and

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

moreover $l(\gamma)$ doesn't depend on the parametrization of γ

77 Corollary(exercise)

If γ is a curve (piecewise C^1), then γ is rectifiable and the length is the sum of the length of it's C^1 pieces.

78 Def

A C^1-curve is **regular** if $\gamma'(t) \neq 0$ for any $t \in [a, b]$ A piecewise C^1-path (curve) is regular if all its pieces are regular

79 Def

$$N := \frac{T}{\|T\|}$$

is **normal vector** of T

80 Def

Let $\gamma : [a, b] \to \mathbb{R}^n$ a C^1 curve; Let l be the length of γ (by theorem proved $l(\gamma) < +\infty$) Let's define the following function:

$$s(t) := \int_a^t \|\gamma'(t)\| \,\mathrm{d}u$$

$s(t)$ is the length of $\gamma \mid_{[a,t]}$ THe function $\|\gamma'(u)\|$ iss continuous, hence

$$s'(t) = \|\gamma'(t)\|$$

Now assume that γ is C^1 and **regucar**$(\gamma'(t) \neq 0, \forall t \in [a,b])$, then $s'(t) > 0$

So $s : [a,b] \to [0,l]$ is a C^1-differmorphism, the inverse is

$$t : [0,l] \to [a,b]$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = frac1\|\gamma'(t)\|$$

We reparameterize γ with t and get

$$\tilde{\gamma}(s) = (\gamma \circ t)(s)$$

$\tilde{\gamma} : [0,l] \to \mathbb{R}^n$ we say that $\tilde{\gamma}$ is the reparameterization of γ with respect to its **curvilinear coordinate** $s(t)$

81 Def

$f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ f is a $C^{(k)}$-differ if

- f is of class $C^{(k)}$
- f is bijection, and the inverse is $C^{(k)}$

82 Def

In general

$$\gamma : [a,b] \to \mathbb{R}^n \rightsquigarrow \tilde{\gamma} : [0,l] \to \mathbb{R}^n$$

regular and C^1

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$\left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}\right\| = 1$

$$T(t) := \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

tangent: (vector) $\to$ vector of norm 1

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\|T(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle T(t), T(t)\rangle = 2\langle T(t), T'(t)\rangle \Leftrightarrow T'(t) \perp T(t)$$

use the fact that in $\mathbb{R}^n$, $u, v : \mathbb{R} \to \mathbb{R}^n$ differentiable

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v(t)\rangle = \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v(t)\right\rangle + \left\langle u(t), \frac{\mathrm{d}v}{\mathrm{d}t}\right\rangle$$

then

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(T(t)) \\ &= \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \\ &= \frac{T'(t)}{\|\gamma'(t)\|}\end{aligned}$$

$N(t) = \frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} / \left\| \frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} \right\|$. If $n = 2$ Along the curve we have a 'moving' canonical basis of

$$\begin{aligned}\mathbb{R}^2_{\gamma(t)} =& \{T(t), N(t)\} \\ &= \{\alpha T(t) + \beta N(t) \mid \alpha, \beta \in \mathbb{R}\}\end{aligned}$$

$\{T(t), N(t)\}$ is a orthonormal basis of $\mathbb{R}^2_{\gamma(t)}$

83 Def:isometry

$$(V, g) \xrightarrow{f} (W, g')$$

a morphism f of vector space with inner product is **isometry** if

$$g(x, y) = g'(f(x), f(y))$$

84 Def:isometric

$V \xrightarrow{\cong} W$ up to isomorphism.
Then (V, g) and (W, g') are **isometric** if there are two isometry

$$\begin{aligned} f : (V, g) \to &\quad (W, g') \\ f' : (W, g') \to &\quad (V, g)\end{aligned}$$

such that

$$f \circ f' = f' \circ f = Id$$

85 Def: Semilinear

If V and W are two complex vector sapce, then a **semilinear mapping** is a mapping $f : V \to W$ such that

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha * f(v) = \overline{\alpha} f(v)$

So a semilinear mapping is a linear mapping: $f : V \to W$

For sesquilinear forms, the theory is similar to the theory of bilinear forms.

$$g \rightsquigarrow G(\text{fix a basis}) \quad g(x, y) = xG\overline{y}$$

If you change basis, then the Gram matrix changes in the following way:

$$G \rightsquigarrow A^T G\overline{A}$$

If g is bilinear

$$g \rightsquigarrow \tilde{g} : V \to V^\vee$$

and

$$g \rightsquigarrow \tilde{g} : V \to \overline{V^\vee}$$

linear if g is sesquilinear ($\tilde{g} : V \to V^\vee$ is semilinear)

86 Def

A sesquilinear form $g : V \times \overline{V} \to K$ is **hermitian** if

$$g(x, y) = \overline{g(y, x)}$$

And note that inner product is any of symmetric symplectic or hermitian.