

Contents

I	Set	5
1	Ring	7
1.1	morphism	7
II	Sequences	9
2	Supremum and infimum	11
3	Interval	13
4	Enhanced real line	15
5	Vector space	17
5.1	K-module	17
5.1.1	Def	17
5.1.2	Remark	17
5.1.3	Notation	18
5.1.4	K-vector space	18
5.1.5	Association:	18
5.1.6	Remark:	19
5.2	sub K-module	19
5.2.1	Def	19
5.2.2	Example	19
5.3	morphism of K-modules	19
5.3.1	Def	19
5.3.2	K-linear mapping	19
5.3.3	Theorem	19
5.3.4	Remark:column	20
5.4	kernel	20
5.4.1	Prop	20
5.4.2	Def	20
5.4.3	Theorem	20
5.4.4	Proof	20

5.4.5	Def	21
5.4.6	Remark	21
5.4.7	Theorem	21
5.4.8	Proof:	21
6	Monotone mappings	23
6.1	Def	23
6.2	Prop.	23
6.3	Def	23
6.4	Prop.	23
6.5	Prop	24
6.6	Def	24
6.7	Prop.	24
6.8	Proof	24
6.8.1	bijection	24
6.8.2	uniqueness	25
7	sequence and series	27
7.1	Def	27
7.2	Remark	27
7.3	Prop	27
7.4	proof	27
7.5	Prop	27
7.6	limit	28
7.6.1	Def	28
7.6.2	Remark	28
7.6.3	Prop	28
7.6.4	Prop	29
7.6.5	Prop	29
7.6.6	Theorem	29
7.6.7	Def	29
7.6.8	Prop	29
7.6.9	Prop	30
7.6.10	Theorem	30
7.6.11	Notation	30
7.6.12	Corollary	30
7.6.13	Notation	30
7.6.14	Theorem: Bolzano-Weierstrass	30
7.6.15	Proof	31
8	Cauchy sequence	33
8.1	Def	33
8.2	Prop	33
8.3	Theorem: Completeness of real number	33
8.3.1	Proof	33
8.4	Absolutely converge	34

<i>CONTENTS</i>	3
8.4.1 Prop	34
9 Comparison and Technics of Computation	35
9.1 Def	35

Part I

Set

Chapter 1

Ring

1.1 morphism

Def

Let A and B be unitary rings. We call morphism of unitary rings from A to B only mapping $A \rightarrow B$ is a morphism of group from $(A, +)$ to $(B, +)$, and a morphism of monoid from (A, \cdot) to (B, \cdot)

Properties

- Let R be a unitary ring. There is a unique morphism from \mathbb{Z} to R
-

algebra

we call k -algebra any pair (R, f) , when R is a unitary ring, and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R , the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of \mathbb{Z} -algebra on R (extra: \mathbb{Z} is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring, (A, f) be a k -algebra. If there is no ambiguity on f , for any $(\lambda, a) \in k \times A$, we denote $f(\lambda)a$ as λa

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite, so $\sum_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def: Let T be a formal symbol. We denote $k^{\mathbb{N}}$ as $k[T]$. If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$, when we denote $k^{\mathbb{N}}$ as $k[T]$ this element is denoted as $\sum_{n \in \mathbb{N}} a_n T^n$. Such

element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient 0
- write T as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$ is written as $1 + 2T + T^2$

Def Remind that $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$, define two composition laws on $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$ form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left(\sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left(\sum_{p+q+l=n} a_p b_q c_l \right) T^n$
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by $F'(T)$ or $\mathcal{D}(F(T))$ the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $\exp(T) \in k[T]$ as $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$, which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence: $(F_i(T))_{i \in \mathbb{N}}$ be a sequence of elements in $k[T]$, and $F(T) \in k[T]$ We say that $(F_i(T))_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$, there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$, $\text{ord}(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 2

Supremum and infimum

Def:

Let (X, \leq) be a partially ordered set A and Y be subsets of X , such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq y\}$ has a least element then we say that A has a Supremum in Y with respect to \leq denoted by $\sup_{(Y, \leq)} A$ this least element and called it the Supremum of A in Y (this respect to \leq)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has an infimum in Y with respect to \leq . We denote by $\inf_{(Y, \leq)} A$ this greatest element and call it the infimum of A in Y
- Observation: $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let (X, \leq) be a partially ordered set, I be a set.

- If f is a function from I to X $\sup f$ denotes the supremum of $f(I)$ is X . $\inf f$ takes the same
- If $(x_i)_{i \in I}$ is a family of element in X , then $\sup x_i$ denotes $\sup\{x_i \mid i \in I\}$ (in X)

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$ We equip \mathbb{R} with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$

Example: For $n \in \mathbb{N}$, let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let (X, \leq) be a partially ordered set, A, Y, Z be subset of X , such that $A \subseteq Z \subseteq Y$

- If $\max A$ exists, then it is also equal to $\sup_{(y, \leq)} A$
- If $\sup_{(y, \leq)} A$ exists and belongs to Z , then it is equal to $\sup A$

\inf takes the same Prop.

Let X, \leq be a partially ordered set, A, B, Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y, \leq)} A$ and $\sup_{(y, \leq)} B$ exists, then $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If $\inf_{(y, \leq)} A$ and $\inf_{(y, \leq)} B$ exists, then $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let (X, \leq) be a partially ordered set, I be a set and $f, g : I \rightarrow X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists, then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists, then $\sup f \leq \sup g$

Chapter 3

Interval

We fix a totally ordered set (X, \leq)

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$, $[a, b]$ denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$, one has $[x, y] \subseteq I$ then we say that I is an interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$. Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an interval in X
- If $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$, $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_\mu$

- If $b \leq x$ $[a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a$ $[a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If $a < x < b$ then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let (X, \leq) be a totally ordered set. I be a non-empty interval of X . If $\sup I$ exists in X , we call $\sup I$ the right endpoint; \inf takes the similar way.

Prop.

Let I be an interval in X .

- Suppose that $b = \sup I$ exists. $\forall x \in I, [x, b] \subseteq I$
- Suppose that $a = \inf I$ exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X . Suppose that I has supremum b and an infimum a in X . Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let (X, \leq) be a totally ordered set. If $\forall (x, z) \in X \times X$, such that $x < z \quad \exists y \in X$ such that $x < y < z$, then we say that (X, \leq) is thick

Prop.

Let (X, \leq) be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists $x_0 \in]a, b[$ By definition, b is an upper bound of I . If b is not the supremum of I , there exists an upper bound M of I such that $M \neq b$. Since X is thick, there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b] \subseteq I, a, b \in I$ Hence M and M' belong to I , which conflicts with the uniqueness of supremum.

Chapter 4

Enhanced real line

Def:

Let $+\infty$ and $-\infty$ be two symbols that are different and don't belong to \mathbb{R} . We extend the usual total order \leq on \mathbb{R} to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ becomes a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$. Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty[\quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in]-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in]0, +\infty[\quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in]-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
ARE NOT DEFINED

Def

Let (X, \leq) be a partially ordered set. If for any subset A of X , A has a supremum and an infimum in X , then we say that X is order complete.

Example

Let Ω be a set. $(\mathcal{P}(\Omega), \subseteq)$ is order complete. If \mathcal{F} is a subset of $\mathcal{P}(\Omega)$, $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$.

Interesting tip: $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

Axiom:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete.

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ and $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \setminus \{0\}$, λA denotes $\{\lambda a \mid a \in A\}$
- $-A$ denotes $(-1)A$

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$, $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ Def

We denote by (\mathbb{R}, \leq) a field \mathbb{R} equipped with a total order \leq , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, one has $\forall c \in \mathbb{R}$, $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, $ab > 0$
- $\forall A \subseteq \mathbb{R}$, if A has an upper bound in \mathbb{R} , then it has a supremum in \mathbb{R}

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

\inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$ then $c = a + b$

\inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set, $f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

\inf takes the similar ($\leq \rightarrow \geq$) (provided when the sum are defined)

Chapter 5

Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

5.1 K -module

5.1.1 Def

Let $(V, +)$ be a commutative group. We call left/right K -module structure:
any mapping $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group $(V, +)$ equipped with a left/right K -module structure is called a left/right K -module.

5.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
 Any left K^{op} - module is a right K -module
 Any right K^{op} - module is a left K -module
 $(K^{op})^{op} = K$

5.1.3 Notation

When we talk about a left/right K -module $(V, +)$, we often write its left K -module structure as $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

5.1.4 K -vector space

If K is commutative, then $K^{op} = K$, so left K -module and right K -module structure are the same. We simply call them K -module structure. A commutative group equipped with a K -module structure is called a K -module. If K is a field, a K -module is also called a K -vector space

Let $\Phi : K \times V \rightarrow V$ be a left or right K -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$ is a morphism of groups. Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$ (*is a var*)

5.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K -module structure

The mapping $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K -module structure

5.1.6 Remark:

We can also write an element μ of K^I in the form of a family $(\mu_i)_{i \in I}$ of elements in K (μ_i is the image of $i \in I$ by μ)
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

5.2 sub K-module**5.2.1 Def**

Let V be a left/right K -module. If W is a subgroup of V . Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub- K -module of V .

5.2.2 Example

Let I be a set. Let $K^{\oplus I}$ be the subset of K^I composed of mappings $f : I \rightarrow K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub- K -module of K^I

In fact, $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$
Hence $f - g \in K^{\oplus I}$ So $K^{\oplus I}$ is a subgroup of K^I
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

5.3 morphism of K-modules**5.3.1 Def**

Let V and W be left K -module, A morphism of groups $\phi : V \rightarrow W$ is called a morphism of left K -modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

5.3.2 K-linear mapping

If K is commutative, a morphism of K -modules is also called a K -linear mapping. We denote by $\text{hom}_{K\text{-Mod}}(V, W)$ the set of all morphism of left- K -module from V to W . This is a subgroup of W^V

5.3.3 Theorem

Let V be a left K -module. Let I be a set.
The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$ is a bijection where
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.3.4 Remark:column

In the case where $I = 1, 2, 3, \dots, n$ V^I is denoted as V^n , K^I is denoted as K^n . For any $(x_1, \dots, x_n) \in V^n$, by the theorem, there exists a unique morphism of left K -modules $\phi : K^n \rightarrow V$ such that $\forall i \in 1, \dots, n, \phi(e_i) = x_i$.

We write this ϕ as a column $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$. It sends $(a_1, \dots, a_n) \in K^n$ to $a_1x_1 + \dots + a_nx_n$.

5.4 kernel

5.4.1 Prop

Let G and H be groups and $f : G \rightarrow H$ be a morphism of groups

- $Im(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

5.4.2 Def

$\ker(f)$ is called the kernel of f

5.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

5.4.4 Proof

Let e_G and e_H be neutral element of G and H respectively

- (1) Let x and y be element of G
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H
- (2) Let x and y be element of $\ker(f)$. One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$. So $\ker(f)$ is a subgroup of G .
- (3) Suppose that f is injection.
 Since $f(e_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$. Suppose that $\ker(f) = \{e_G\}$. If $f(x) = f(y)$ then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$.
 Hence $xy^{-1} = e_G \Rightarrow x = y$

5.4.5 Def

Let $(V, +)$ be a commutative group, I be a set. We define a composition law $+$ on V^I as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then V^I forms a commutative group

5.4.6 Remark

Let E and F be left K -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$ is a subgroup of F^E

In fact f and g are elements of $\text{hom}_{K\text{-Mod}}(E, F)$, then $f - g$ is also a morphism of left K -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

5.4.7 Theorem

Let V be a left K -module, I be a set. The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$ is an isomorphism of groups, where $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.4.8 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So Ψ is a morphism of groups

injectivity Let $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i \in I} \in V^I$ We define $\phi_x : K^{\oplus I} \rightarrow V$ such that $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left K -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that K' is a unitary ring, and V is also equipped with a right K' -module structure, Then $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub- k' -module, and Ψ in the theorem is a right K' -module isomorphism

Chapter 6

Monotone mappings

6.1 Def

Let I and X be partially ordered sets, $f : I \rightarrow X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)$, then we say that f is increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

6.2 Prop.

Let X, Y, Z be partially ordered sets. $f : X \rightarrow Y, g : Y \rightarrow Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

6.3 Def

Let f be a function from a partially ordered set I to another partially ordered set X . If $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

6.4 Prop.

Let I and X be partially ordered sets. f be function from I to X .

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

6.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B

If f is increasing/decreasing, then so is f^{-1}

6.6 Def

Let X and Y be partially ordered sets. $f : X \rightarrow Y$ be a bijection. If both f and f^{-1} are increasing, then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \rightarrow Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

6.7 Prop.

Let I be a subset of \mathbb{N} which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \rightarrow I$

6.8 Proof

6.8.1 bijection

We construct $f : \mathbb{N} \rightarrow I$ by induction as follows.

Let $f(0) = \min I$ Suppose that $f(0), \dots, f(n)$ are constructed

then we take $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$. Therefore $f(n) \leq f(n+1)$

Since $f(n+1) \notin \{f(0), \dots, f(n)\}$, we have $f(n) < f(n+1)$

Hence f is strictly increasing and this is injective

If f is not surjective, then $I \setminus \text{Im}(f)$ has a element N .

Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.

Since $N \notin \text{Im}(f)$, $N < f(m)$.

So $m \neq 0$. Hence $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition, $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$,

Hence $f(m) \leq N$, causing contradiction.

6.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from \mathbb{N} to \mathbb{N}

Chapter 7

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

7.1 Def

Let X be a set. We call sequence in X parametrized by I a mapping from I to X .

7.2 Remark

If K is a unitary ring and E is a left K -module then the set of sequence E^I admits a left- K -module structure. If $x = (x_n)_{n \in I}$ is a sequence in E , we define a sequence $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$, called the series associated with the sequence x .

7.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$ is a morphism of left- K -module

7.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

7.5 Prop

Let I be a totally ordered set . X be a partially ordered set, $f : I \rightarrow X$ be a mapping , $J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

7.6 limit

7.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset. $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left(\inf_{i \in I, i \geq n} x_i \right)$$

If $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$, we then say that $(x_n)_{n \in I}$ tends to l and that l is the limit of $(x_n)_{n \in I}$. If in addition $(x_n)_{n \in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$, we say that $(x_n)_{n \in I}$ converges to l

7.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left(\inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i \in I}$ the limit superior and the limit inferior do not change.

In fact, if we take $J = \mathbb{N} \setminus \{0, \dots, m\}$, then $\inf_{n \in J}(\dots)$ and $\sup_{n \in J}(\dots)$ only depends on the values of $x_i, i \in I, i \geq m$

7.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

7.6.4 Prop

Let $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned} \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\ & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\ \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \end{aligned}$$

7.6.5 Prop

Let $(x_n)_{n \in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$, one has $x_n \leq y_n$. Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

7.6.6 Theorem

Let $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$ and $(z_n)_{n \in I}$ tend to the same limit l

Then $(y_n)_{n \in I}$ tends to l

7.6.7 Def

Let I be an infinite subset of \mathbb{N} , and $(x_n)_{n \in I}$ be a sequence in some set X . We call subsequence of $(x_n)_{n \in I}$ a sequence of the form $(x_n)_{n \in J}$, where J is an infinite subset of I

7.6.8 Prop

Let I and J be infinite subset of \mathbb{N} such that $J \subseteq I$. $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$, one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if $(x_n)_{n \in I}$ tends to $l \in [-\infty, +\infty]$, then $(x_n)_{n \in J}$ tends to l

7.6.9 Prop

$\forall n \in \mathbb{N}$, one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

7.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing, then $(x_n)_{n \in I}$ tends to $\sup_{n \in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing, then $(x_n)_{n \in I}$ tends to $\inf_{n \in I} x_n$

7.6.11 Notation

If a sequence $(x_n)_{n \in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim_{n \in I, n \rightarrow} x_n$ denotes this limit l

7.6.12 Corollary

Let $(x_n)_{n \in I}$ be a sequence in $\mathbb{N}_{\geq 0}$. Then the series $\sum_{n \in I} x_n$ (the sequence $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$. It converges in \mathbb{R} iff it is bounded from above (namely has an upper bound in \mathbb{R})

7.6.13 Notation

If a series $\sum_{n \in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum_{n \in I} x_n$ to denote the limit

7.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\limsup_{n \in I, n \rightarrow +\infty} x_n$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\liminf_{n \in I, n \rightarrow +\infty} x_n$.

7.6.15 Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_n)_{n \in J}$ is decreasing so it tends to $\inf_{n \in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i \in I, i \geq n} x_i$ so $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$. Denote by

$l = \sup_{n \in I, n \geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i \in I, i \geq n_0} x_i \leq l$. If the strict

inequality $\sup_{i \in I, i \geq N} x_i < l$ holds, then $\sup_{i \in I, i \geq N} x_i$ is NOT an upper bound of

$\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i \in I, i \geq N} x_i$. We

may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality. Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$. Thus $n \in J$ that contradicts the maximality of n_0 . Therefore $l = \sup_{i \in I, i \geq N} x_i$, which leads to $\limsup_{n \in I, n \rightarrow +\infty} x_n = l$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$, so $x_m < l$ (since otherwise $x_m = \sup_{i \in I, i \geq m} x_i$ and hence $m \in J$). Hence, \forall finite subset I' of $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$ and hence $\exists n \in I$, such that $n > \max I'$, and $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j \in \mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \rightarrow I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$

Hence the sequence $(x_{n_j})_{j \in \mathbb{N}}$ is increasing, and $\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n =$

l

$l = \sup_{n \in I, n \geq n_0}$

So $(x_{n_j})_{j \in \mathbb{N}}$ tends to l

Chapter 8

Cauchy sequence

8.1 Def

Let $(x_n)_{n \in I}$ be a sequence in \mathbb{R}
If $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$ then
we say that $(x_n)_{n \in I}$ is a Cauchy sequence

8.2 Prop

- If $(x_n)_{i \in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_n)_{i \in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite, $(x_n)_{n \in I}$ is a Cauchy sequence.
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n \in I}$ converges to l , then $(x_n)_{n \in J}$ converges to l too.

8.3 Theorem: Completeness of real number

If $(x_n)_{n \in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in \mathbb{R}

8.3.1 Proof

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$. So $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n \in I}$ converges to $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n \in I}$ converges to the same limit.

8.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

8.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in \mathbb{R}

Chapter 9

Comparison and Technics of Computation

9.1 Def

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be sequence in \mathbb{R}

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$ then we write $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$, then we write $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$