Contents

Part I

Set

Chapter 1

product

1.1 direct sum

$\bigoplus$ is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i\in I} V_i \{(x_i)_{i\in I} \in \prod_{i\in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0\}$$

Chapter 2

Ring

2.1 morphism

Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \to B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
-

algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \to R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \to R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n \in \mathbb{N}}$ couldn't be executed directly.
Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n\in\mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n\in\mathbb{N}} a_n T^n$ Such element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_n T^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]
$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$
let $F + G = (a_0 + b_0) + ...$
$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n$
Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n \right) (\sum\limits_{n\in\mathbb{N}} c_n T^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_p b_q c_l \right) T^n$
  is a trick applied on integral

Derivative:
let $F(T) \in k[T]$
We denote by $F'(T)$ *or* $\mathcal{D}(F(T))$ the formal power series
$\mathcal{D}(F) = \sum\limits_{n\in\mathbb{N}} (n+1) a_{n+1} T^n$
Properties:

- $\mathcal{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!} T^n$,which fulfil the differential equation $\mathcal{D}(exp(T)) = exp(T)$(interesting)
Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall (i,j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 3

Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i \in I} is\ a\ family\ of\ element\ in\ X, then\ \sup\limits_{i \in I} x_i\ denotes\ \sup\{x_i \mid i \in I\}(inX)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup\limits_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N},k\geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

Chapter 4

Interval

We fix a totally ordered set $(X, \leq)$

Notation:

If $(a,b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x,y) \in I \times I$ with $x \leq y$,one has $[x,y] \subseteq I$ then we say that I is a interval in X

Example:

Let $(a,b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a,b[:= \{x \in X \mid a,x,b\}$
- $[a,b[:= \{x \in X \mid a,x,b\}$
- $]a,b] := \{x \in X \mid a,x,b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X
- If $\bigcap\limits_{\lambda \in \Lambda} I_\lambda \neq \varnothing$, $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X

We check that $[a,b] \subseteq I_\lambda \cup I_] \mu$

- If $b \leq x \quad [a,b] \subseteq [a,x] \subseteq I_\lambda$ because $\{a,x\} \subseteq I_\lambda$
- If $x \leq a \quad [a,b] \subseteq [x,b] \subseteq I_\mu$ because $\{b,x\} \subseteq I_\mu$
- If a < x < b then $[a,b] = [a,x] \cup [x,b] \subseteq I_\lambda \cup I_\mu$

Def:

Let $(X, \leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.

Prop.

Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x, b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let $(X, \leq)$ be a totally ordered set .If $\forall (x, z) \in X \times X$,such that $x < z$ $\exists y \in X$ such that $x < y < z$,than we say that $(X, \leq)$ is thick

Prop.

Let $(X, \leq)$ be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a$ $\sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in]a, b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b[\subseteq]a, b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

Chapter 5

Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq$ *on* $\mathbb{R}$ *to* $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ become a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$ Obviously,this is a thick totally ordered set.
We define:

- $\forall x \in]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete
Example

Let Ω be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega)$, $\sup \mathscr{F} = \bigcup_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$

$\mathcal{AXION}$:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ *and* $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R}\backslash\{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup\limits_{x \in I} f(x) \quad b = \sup\limits_{y \in J} g(y) \quad c = \sup\limits_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup\limits_{x \in I, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup\limits_{x \in I} f(x))(\sup\limits_{x \in I} g(x))$

inf takes the similar($\leq \to \geq$) (provided when the sum are defined)

Chapter 6

Vector space

In this section:
K denotes a unitary ring.
Let 0 be zero element of K
1 be the unity of K

6.1 K-module

6.1.1 Def

Let $(V, +)$ be a commutative group.We call left/right K-module structure:
any mapping Φ:$K \times V \to V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x))/\Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

6.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a, b) \mapsto a + b$
- $K \times K \to K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
Any left $K^{op}-module$ is a right K-module
Any right $K^{op}-module$ is a left K-module
$(K^{op})^{op} = K$

6.1.3 Notation

When we talk about a left/right K-module $(V,+)$,we often write its left K-module structure as $K \times V \to V \quad (a,x) \mapsto ax$
The axioms become:

$$\begin{aligned}
&\forall (a,b) \in K \times K, \forall x \in V \quad (ab)x = a(bx)/b(ax)\\
&\forall (a,b) \in K \times K, \forall x \in V \quad (a+b)x = ax + bx\\
&\forall a \in K, \forall (x,y) \in V \times V \quad a(x+y) = ax + ay\\
&\forall x \in V \quad 1x = x
\end{aligned}$$

6.1.4 K-vector space

If K is commutative,then $K^{op} = K$,so left K-module and right K-module structure are the same .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space
Let $\Phi : K \times V \to V$ be a left or right K-module structure

$$\forall x \in V, \Phi(\cdot, x) : K \to V \quad (a \in K) \mapsto \Phi(a,x)$$

is a morphism of addition groups.Hence $\Phi(0,x) = 0, \Phi(-a,x) = -\Phi(a,x)$
$\forall a \in K, \Phi(a,\cdot) : V \to V$ is a morphism of groups.Hence $\Phi(a,0) = 0, \Phi(a,-x) = -\Phi(a,x)(\cdot\ is\ a\ var)$

6.1.5 Association:

$\forall x \in K$

$$\begin{aligned}
(f(f+g)+h)(x) &= (f+g)(x) + h(x) = f(x) + g(x) + h(x)\\
(f+(g+h))(x) &= f(x) + ((g+h)(x)) = f(x) + g(x) + h(x)
\end{aligned}$$

Let $0 : I \to K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$
Let $-f : f + (-f) = 0$
The mapping $K \times K^I \to K^I : (a,f) \mapsto af \quad (af)(x) = af(x)$ is a left K-module structure
The mapping $K \times K^I \to K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K-module structure

6.1.6 Remark:

We can also write an element μ *of* K^I is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I$ *by* μ)
Then

$$(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I}$$
$$a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I}$$
$$(\mu_i)_{i\in I} a = (\mu_i a)_{i\in I}$$

6.2 sub K-module

6.2.1 Def

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub-K-module of V.

6.2.2 Example

Let I be a set .Let $K^{\bigoplus I}$ be the subset of K^I composed of mappings $f : I \to K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub-K-module of K^I

In fact,$\forall (f, g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$
Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of K^I
$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

6.3 morphism of K-modules

6.3.1 Def

Let V and W be left K-module, A morphism of groups $\phi : V \to W$ is called a morphism of left K-modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

6.3.2 K-linear mapping

If K is commutative, a morphism of K-modules is also called a K-linear mapping. We denote by $\hom_{K-Mod}(V, W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of W^V

6.3.3 Theorem

Let V be a left K-module. Let I be a set.
The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I : \ \phi \to (\phi(e_i))_{i\in I}$ is a bijection where
$e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

6.3.4 Remark:column

In the case where $I = 1, 2, 3, ..., n$ V^I is denoted as V^n, K^I is denoted as K^n For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n \phi(e_i) = x_i$

We write this ϕ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$

6.4 kernel

6.4.1 Prop

Let G and H be groups and $f : G \to H$ be a morphism of groups

- $I_m(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

6.4.2 Def

$\ker(f)$ is called the kernel of f

6.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

Proof

Let e_G and e_H be neutral element of G and H respectively

(1) Let x and y be element of G
$f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H

(2) Let x and y be element of $\ker(f)$ One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H \quad e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$ So $\ker(f)$ is a subgroup of G

(3) Suppose that f is injection.
Since $f(E_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ Suppose that $\ker(f) = \{e_G\}$ If $f(x) = f(y)$then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$
Hence $xy^{-1} = e_G \ \Rightarrow \ x = y$

6.4.4 Def

Let (V,+) be a commutative group, I be a set. We define a composition law + on V^I as follows

$$(x_i)_{i\in I} + (y_i)_{i\in I} \ := (x_i + y_i)_{i\in I}$$

Then V^I forms a commutative group

6.4.5 Remark

Let E and F be left K-modules
$\hom_{K=Mod}(E,F) := \{$morphisms of left K-modules from E to F$\} \subseteq F^E$ is a subgroup of F^E
In fact f and g are elements of $\hom_{K-Mod}(E,F)$, then $f-g$ is also a morphism of left K-module
$(f-g)(x+y) = f(x+y) - g(x+y) = (f(x)+f(y)) - (g(x)+g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f-g)(x) + (f-y)(x)$

6.4.6 Theorem

Let V be a left K-module,I be a set The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I \ : \ \phi \mapsto (\phi(e_i))_i \in I$ is an isomorphism of groups, where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

6.4.7 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_I) + \psi(e_i)$
$\forall(\phi, psi) \in \hom_{K-Mod}(K^{\bigoplus I}, V)^2$
Hence $\Psi(\phi,\psi) = (\phi(e_i) + \psi(e_i))_{i\in I} = \Psi(\phi) + \Psi(\psi)$
So Ψ is a morphism of groups

injectivity Let $\phi \in \hom_{K-Mod}(K^{\bigoplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$
Let $a = (a_i)_{i\in I} \in K^{\bigoplus I}$ One has $a = \sum\limits_{i\in I} a_i e_i$
If fact,$\forall j \in I, a_j = \sum\limits_{i\in I, a_i\neq 0} a_i e_i(j)$
Thus $\phi(a) = \sum\limits_{i\in I, a_i \neq 0} a - I\phi(e_i) = 0$
Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i\in I} \in V^I$ We define $\phi_x : K^{\bigoplus I} \to V$ such that $\forall a = (a_i)_{i\in I} \in K^{\bigoplus I}, \phi_x(a) = \sum\limits_{i\in I, a_i\neq 0} a_i x_i$
This is a morphism of left K-modules
$foralli \in I, \phi_x(e_i) = 1x_i = x_i$ So $\Psi(\phi_x) = x$

Suppose that K' is a unitary ring,and V is also equipped with a right K'-module structure, Then $\hom_{K-Mod}(K^{\bigoplus I}, V) \subseteq V^{K^{\bigoplus I}}$ is a right sub-k'-module ,and Ψ in the theorem is a right K'-module isomorphism

Chapter 7

Monotone mappings

7.1 Def

Let I and X be partially ordered sets,$f : I \to X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)/f(a) < f(b)$,then we say that f is increasing/strictly increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

7.2 Prop.

Let X,Y,Z be partially ordered sets.$f : X \to Y, g : Y \to Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

7.3 Def

Let f be a function from a partially ordered set I to another partially ordered set .If $f \mid_{Dom(f)} \to X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

7.4 Prop.

Let I and X be partially ordered sets. f be function from I to X.

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

7.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B
If f is increasing/decreasing ,then so is f^{-1}

7.6 Def

Let X and Y be partially ordered sets. $f : X \to Y$ be a bijection. If both f and f^{-1} are increasing ,then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \to Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

7.7 Prop.

Let I be a subset of $\mathbb{N}$ which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \to I$

7.8 Proof

7.8.1 bijection

We construct $f : \mathbb{N} \to I$ by induction as follows.
Let $f(0) = \min I$ Suppose that $f(0), ..., f(n)$ are constructed
then we take $f(n+1) := \min(I\backslash\{f(0), ..., f(n)\})$
Since $I\backslash\{f(0), ..., f(n-1)\} \supseteq I\backslash\{f(0), ..., f(n)\}$.Therefore $f(n) \leq f(n+1)$
$Since f(n+1) \notin \{f(0), ..., f(n)\}$,we have $f(n) < f(n+1)$
Hence f is strictly increasing and ths is injective
If f is not surjective,then $I\backslash Im(f)$ has a element N.
Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.
Since $N \notin Im(f), N < f(m)$.
So $m \neq 0$.Hence $f(m-1) < N < f(m) = \min(I\backslash\{f(0), ..., f(m-1)\})$
By definition,$N \in I\backslash Im(f) \subseteq I\backslash\{f(0), ..., f(m-1)\}$,
Hence $f(m) \leq N$,causing contradiction.

7.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from $\mathbb{N}$ to $\mathbb{N}$

Chapter 8

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

8.1 Def

Let X be a set.We call sequence in X parametrized by I a mapping from I to X.

8.2 Remark

If K is a unitary ring and E is a left K-module then the set of sequence E^I admits a left-K-module structure. If $x = (x_n)_{n\in I}$ is a sequence in E, we define a sequence $\sum(x) := (\sum\limits_{i\in I, i\leq n} x_i)_{n\in\mathbb{N}}$,called the series associated with the sequence x.

8.3 Prop

$\sum : E^I \to E^{\mathbb{N}}$ is a morphism of left-K-module

8.4 proof

Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of E^I
$\sum\limits_{i\in I, i\leq n}(x_i + y_i) = (\sum\limits_{i\in I, i\leq n} x_i) + (\sum\limits_{i\in I, i\leq n} y_i), \lambda \sum\limits_{i\in I, i\leq n} x_i = \sum\limits_{i\in I, i\leq n} \lambda x_i$

8.5 Prop

Let I be a totally ordered set . X be a partially ordered set,$f : I \to X$ be a mapping ,$J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

8.6 limit

8.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset.$\forall (x_i)_{n\in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$,we define:

$$\limsup_{n\in I, n\to+\infty} x_n := \inf_{n\in I}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n := \sup_{n\in I}(\inf_{i\in I, i\geq n} x_i)$$

If $\limsup_{n\in I, n\to+\infty} x_n = \liminf_{n\in I, n\to+\infty} x_n = l$, we then say that $(x_n)_{n\in I}$ tends to l and that l is the limit of $(x_n)_{n\in I}$. If in addition $(x_n)_{n\in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$,we say that $(x_n)_{n\in I}$ converges to l

8.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n\in I, n\to+\infty} = \inf_{n\in J}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n = \sup_{n\in J}(\inf_{i\in I, i\geq n} x_i)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i\in I}$ the limit superior and the limit inferior do not change.
In fact, if we take $J = \mathbb{N}\backslash\{0, ..., m\}$, then $\inf_{n\in J}(...)$ and $\sup_{n\in J}(...)$ only depends on the values of $x_i, i \in I, i \geq m$

8.6.3 Prop

$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$, $\liminf_{n\in I, n\to+\infty} x_n \leq \limsup_{n\in I, n\to+\infty} x_n$

8.6.4 Prop

Let $(x_n)_{n\in I} \in [-\infty, +\infty]^I$

$\forall c \in \mathbb{R}$

$$\limsup_{n\in I, n\to+\infty} (x_n + c) = (\limsup_{n\in I, n\to+\infty} x_n) + c$$

$$\liminf_{n\in I, n\to+\infty} (x_n + c) = (\liminf_{n\in I, n\to+\infty} x_n) + c$$

$\forall c \in \mathbb{R}_{>0}$

$$\limsup_{n\in I, n\to+\infty} (\lambda x_n) = \lambda \limsup_{n\in I, n\to+\infty} x_n$$

$$\liminf_{n\in I, n\to+\infty} (\lambda x_n) = \lambda \liminf_{n\in I, n\to+\infty} x_n$$

$\forall c \in \mathbb{R}_{<0}$

$$\limsup_{n\in I, n\to+\infty} (\lambda x_n) = \lambda \liminf_{n\in I, n\to+\infty} x_n$$

$$\liminf_{n\in I, n\to+\infty} (\lambda x_n) = \lambda \limsup_{n\in I, n\to+\infty} x_n$$

8.6.5 Prop

Let $(x_n)_{n\in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$,one has $x_n \leq y_n$ Then

$$\limsup_{n\in I, n\to+\infty} (x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

,

$$\liminf_{n\in I, n\to+\infty} (x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

8.6.6 Theorem

Let $(x_n)_{n\in I}, (y_n)_{n\in I}, (z_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$
Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n\in I}$ and $(z_n)_{n\in I}$ tend to the same limit l

Then $(y_n)_{n\in I}$ tends to l

8.6.7 Def

Let I be an infinite subset of $\mathbb{N}$, and $(x_n)_{n\in I}$ be a sequence in some set X. We call subsequence of $(x_n)_{n\in I}$ a sequence of the form $(x_n)_{n\in J}$,where J is an infinite subset of I

8.6.8 Prop

Let I and J be infinite subset of $\mathbb{N}$ such that $J \subseteq I$
$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$,one has

$$\liminf_{n\in I, n\to+\infty} (x_n) \leq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in I,n\to+\infty}(x_n) \geq \limsup_{n\in I,n\to+\infty} y_n$$

In particular, if $(x_n)_{n\in I}$ tends to $l \in [-\infty,+\infty]$,then $(x_n)_{n\in J}$ tends to l

8.6.9 Prop

$\forall n \in \mathbb{N}$,one has

$$\liminf_{n\in J,n\to+\infty}(x_n) \geq \liminf_{n\in I,n\to+\infty} y_n$$

$$\limsup_{n\in J,n\to+\infty}(x_n) \leq \limsup_{n\in I,n\to+\infty} y_n$$

8.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_N)_{n\in I}$ be a sequence in $[-\infty,+\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing,then $(x_N)_{i\in I}$ tends to $\sup\limits_{n\in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing,then $(x_N)_{i\in I}$ tends to $\inf\limits_{n\in I} x_n$

8.6.11 Notation

If a sequence $(x_N)_{n\in I} \in [-\infty,+\infty]$ tends to some $l \in [-\infty,+\infty]$ the expression $\lim\limits_{n\in I,n\to} x_n$ denotes this limit l

8.6.12 Corollary

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{N}_{\geq 0}$ Then the series $\sum\limits_{n\in I} x_n$(the sequence $(\sum\limits_{i\in I,i\leq n})_{n\in\mathbb{N}})$ tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ It converges in $\mathbb{R}$ iff it is bounded from above (namely has an upper bound in $\mathbb{R}$)

8.6.13 Notation

If a series $\sum\limits_{n\in I} x_n$ in $[-\infty,+\infty]$ tends to some limit, we use the expression $\sum\limits_{n\in I} x_n$ to denote the limit

8.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n\in I}$ be a sequence in $[-\infty,+\infty]$ There exists a subsequence of $(x_n)_{n\in I}$ that tends to $\limsup\limits_{n\in I,n\to+\infty} x_n$ There exists a subsequence of $(x_n)_{n\in I}$ that rends to $\liminf\limits_{n\in I,n\to+\infty} x_n$

Proof

Let $J = \{n \in I \mid \forall m \in I, \text{if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_N)_{n\in J}$ is decreasing so it tends to $\inf_{n\in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i\in I, i\geq n} x_i$ so $\limsup_{n\in I, n\to+\infty} x_n = \inf_{n\in J} \sup_{i\in I, i\geq n} x_i = \inf_{n\in J} x_n = \lim_{n\in J, n\to+\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$.Denote by $l = \sup_{n\in I, n\geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i\in I, i\geq n_0} x_i \leq l$. If the strict inequality $\sup_{i\in I, i\geq N} x_i < l$ holds, then $\sup_{i\in I, i\geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i\in I, i\geq N} x_i$ We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality.

Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$ Thus $n \in J$ that contradicts the maximality of n_0

Therefore

$$l = \sup_{i\in I, i\geq N} x_i$$

, which leads to

$$\limsup_{n\in I, n\to+\infty} x_n = l$$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$,so $x_m < l$(since otherwise $x_m = \sup_{i\in I, i\geq m} x_i$ and hence $m \in J$)Hence,$\forall$ *finite subset* I' *of*$\{m \in I \mid m \geq n_0\}$

$\max_{i\in I} x_i < l$ and hence $\exists n \in I$,such that $n > \max I'$,and $\max_{i\in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \to I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is increasing

And

$$\sup_{j\in\mathbb{N}} x_{n_j} \leq \sup_{j\in\mathbb{N}} x_{f(j)} = \sup_{n\in I, n\geq n_0} x_n = l$$

$$l = \sup_{n\in I, n\geq n_0}$$

So $(x_{n_j})_{j\in\mathbb{N}}$ tends to l

Chapter 9

Cauchy sequence

9.1 Def

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{R}$
If $\inf\limits_{N\in\mathbb{N}} \sup\limits_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = \lim\limits_{N\to+\infty} \sup\limits_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = 0$ then we say that $(x_n)_{n\in I}$ is a Cauchy sequence

9.2 Prop

- If $(x_n)_{i\in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_N)_{i\in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite,$(x_n)_{n\in I}$ is a Cauchy sequence.
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n\in I}$ converges to l, then $(x_n)_{n\in J}$ converges to l too.

9.3 Theorem: Completeness of real number

If $(x_n)_{n\in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in $\mathbb{R}$

Proof

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$ So $\limsup\limits_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup\limits_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n\in I}$ converges to the same limit.

9.4 Absolutely converge

We say that a series $\sum_{n\in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n\in I} |x_n| < +\infty$

9.4.1 Prop

If a series $\sum_{n\in I} x_n$ converges absolutely, then it converges in $\mathbb{R}$

Chapter 10

Comparison and Technics of Computation

10.1 Def

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be sequence in $\mathbb{R}$

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_N| \leq M|y_m|$ then we write $x_n = O(y_n), n \in I, n \to +\infty$
- If there exists $(\epsilon_n)_{n\in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim\limits_{n\in I, n\to+\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_N| \leq |\epsilon y_m|$, then we write $x_n = \circ(y_n), n \in I, n \to +\infty$
  Example:
  $$\lim_{n\to+\infty} \frac{1}{n} = 0$$

10.2 Prop.

Let I and X be partially ordered sets and $f : I \to X$ be an increasing/decreasing mapping. Let J ba a subset of I. Assume that any elements of I has an upper bound in J. Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

10.3 Theorem

Let I be a totally ordered set, $f : I \to [-\infty, +\infty]$ and $g : I \to [-\infty, +\infty]$ be two mappings that are both increasing/decreasing.Then the following equalities holds,provided that the sum on the right hand side of the equality is wel defined.

$$\sup_{x\in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x\in I} f(x)) + (\sup_{y\in I} g(y))$$

$$\inf_{x\in I,\{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\inf_{x\in I} f(x)) + (\inf_{y\in I} g(y))$$

Proof

We can assume f and g increasing. Let $a = \sup f(I)$,$b = \sup g(I)$
Let $A = \{(x,y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$
We equip A with the following order relation.

$$(x,y) \leq (x',y')\ iff\ x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x,y) \in A \mid x = y\}$.
Consider

$$h : A \to [-\infty, +\infty] \quad h(x,y) = f(x) + g(y)$$

h is increasing.
Let $(x,y) \in A$. Assume that $x \leq y$
If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y,y) \in B$ and $(x,y) \leq (y,y)$
If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x,y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$
So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined,$(z,z) \in B$ and $(x,y) \leq (z,z)$ Similarly, if$x \geq y$, (x,y) has also an upper bound in B. Therefore: $\sup h(A) = \sup h(B)$

10.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty} (x_n + y_n) \leq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty} (x_n + y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n\in I, n\geq N} x_n \quad B_N = \sup_{n\in I, n\geq N} y_n$. $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are decreasing, and $\limsup_{n\in I, n\to+\infty} x_n = \inf_{N\in\mathbb{N}} A_N \quad \limsup_{n\in I, n\to+\infty} y_n = \inf_{N\in\mathbb{N}} B_N$
By theorem:

$$\inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N = \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n\in I, n\geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.
Therefore

$$\inf_{N\in\mathbb{N}} C_N \leq \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N) = \inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N$$

10.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty,+\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) \geq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

Proof

a tricky proof ?:

$$\limsup_{n\in I, n\to} x_n = \limsup_{n\in I, n\to}(x_n+y_n-y_n) \leq \limsup_{n\in I, n\to}(x_n+y_n) - \liminf_{n\in I, n\to} y_n$$

to have a true proof,only need to discuss conditions with ∞

10.6 Theorem

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n\in I}$ converges to some $i \in \mathbb{R}$.
Then:

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) = (\limsup_{n\in I, n\to+\infty} x_n) + l$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) = (\liminf_{n\in I, n\to+\infty} x_n) + l$$

10.7 Prop.

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Then:

$$\liminf_{n\in I, n\to+\infty} \max\{x_n, y_n\} = \max\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

$$\liminf_{n\in I, n\to+\infty} \min\{x_n, y_n\} = \min\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

Proof

About the first inequality.Since $\max\{x_n, y_n\} \geq x_n \quad \max\{x_n, y_N\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n\in J, n\to+\infty} = \limsup_{n\in J, n\to+\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $\quad J_1 = \{n \in J \mid x_n \leq y_n\}$
$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite
Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \to} \max\{x_n, y_n\} = \lim_{n \in J_1, n \to} \max\{x_n, y_n\} = \lim_{n \in J, n \to} x_n \leq \limsup_{n \in I, n \to +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \to +\infty} = \lim_{n \in J_2, n \to +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \to +\infty} y_n$$

10.8 Theorem

Let $(a_N)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_N)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$

Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$
$$\limsup_{n \in I, n \to +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \to +\infty} a_n) - l, l - (\liminf_{n \in I, n \to +\infty} a_n)\}$$

(1) $\Rightarrow$ (2):
If $(a_n)_{n \in I}$ converges to l, then $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$
(2) $\Rightarrow$ (1):
If $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$,then $\limsup_{n \in I, n \to +\infty} a_n \leq l \leq \liminf_{n \in I, n \to +\infty} a_n$
Therefore: $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$

10.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in $\mathbb{R}$, $l \in \mathbb{R}$
The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \to +\infty$

10.10 Calculates on O(),o()

10.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \to +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \to +\infty$

10.10.2 Transform

Let $(a_n)_{n\in I}$ and $(b_n)_{n\in I}$ be two sequence in $\mathbb{R}$ If $a_n = o(b_n), n \in I, n \to +\infty$,then $a_n = O(b_n), n \in I, n \to +\infty$

10.10.3 Transition

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ and $(c_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = O(c_n), n \in I, n \to +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$

10.10.4 Times

Let $(a_n)_{n\in I}, (b_n)_{n\in I}, (c_n)_{n\in I}, (d_n)_{n\in I}$ be sequences in $\mathbb{R}$

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = O(b_n d_n), n \in I, n \to +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = o(b_n d_n), n \in I, n \to +\infty$

10.11 On the limit

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ be elements of $\mathbb{R}^I$ that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n\in I}$ converges to $l + l'$
- $(a_n b_n)_{n\in I}$ converges to ll'

10.12 Prop

Let $a \in \mathbb{R}$ then $a^n = o(n!) \quad n \to +\infty$

Proof

Let $N \in \mathbb{N}$ such that $|a| < N$
For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} (\frac{|a|}{N})^n - N$$

And $0 < \frac{|a|}{<} 1 \Rightarrow \lim_{n \to +\infty} (\frac{|a|}{N})^n = 0$. Therefore:

$$\lim_{n \to +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

10.13 Prop

$n! = o(n^n) \quad n \to +\infty$

Proof

Let $N \in \mathbb{N}_{\geq 1}$
$0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \to +\infty} \frac{n!}{n^n} = 0$

10.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of $\mathbb{R}^I$ If the series $\sum_{n \in I} b_n$ converges absolutely and if $on = O(b_n) \quad n \to +\infty$Then $\sum_{n \in I} a_n$ converges absolutely

Proof

By definition $\sum_{n \in I} |b_N| < +\infty$ If $|a_N| \leq M|b_N|$ fro $n \in I, n \geq N$ where $N \in \mathbb{N}$ Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

10.15 Theorem: d'Alembert ratio test

Let $(a_N)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If $\limsup_{n \to +\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \to +\infty} |\frac{a_{n+1}}{a_n}| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

Proof

(1)

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n\to+\infty} |\frac{a_{n+1}}{a_n}| < \alpha < 1$, *alpha* isn't a lower bound of $(\sup_{n\geq N} |\frac{a_{n+1}}{a_n}|)_{N\in\mathbb{N}}$
So $\exists N \in \mathbb{N}$ such that $\sup_{n\geq N} |\frac{a_{n+1}}{a_n}| < \alpha$Hence for $n \geq N \quad |a_n| \leq \alpha^{n-N}|a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N}\frac{a_{N+2}}{a_{N+1}}\cdots\frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n\in\mathbb{N}} = \frac{1}{1-\alpha} < +\infty$, $\sum_{n\in\mathbb{N}} a_n$ converge absolutely.

10.15.1 Lemma

If a series $\sum_{n\in\mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n\to+\infty} a_n = 0$

Proof

If $(\sum_{i=0}^{n} a_i)_{n\in\mathbb{N}}$ converges to some $l \in \mathbb{R}$,then $(\sum_{i=0}^{n-1} a_i)_{n\in\mathbb{N},n\geq 1}$ converges to l, too. Hence $\left(a_n = \left(\sum_{i=0}^{n} a_i\right) - \left(\sum_{i=0}^{n-1} a_i\right)\right)_{n\in\mathbb{N}}$ converges to $l - l = 0$

10.15.2 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n\to+\infty} |\frac{a_{n+1}}{a_n}| = \sup_{N\in\mathbb{N}} \inf_{n\geq N} |\frac{a_{n+1}}{a_n}|$
So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n\geq N} |\frac{a_{n+1}}{a_n}|$
$\forall n \in \mathbb{N}, n \geq N \quad |\frac{a_{n+1}}{a_n}| \geq \beta$
Hence $(|a_n|)_{n\in\mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N}|a_n|$
By the lemma: $\sum_{n\in\mathbb{N}} a_n$ diverges.

10.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \to +\infty$

Proof

Let $\epsilon > 0$ such that $a = (1+\epsilon)^2$

$$a^n = (1+\epsilon)^{2n} = (1+\epsilon)^n(1+\epsilon)^n \geq (1+n\epsilon)(1+n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

10.16.1 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \to +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

10.16.2 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

10.17 Theorem: Cauchy root test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $\alpha = \limsup_{n\to+\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n\in\mathbb{N}} a_n$ converges absolutely.
- If $a > 1$ then $\sum_{n\in\mathbb{N}} a_n$ diverges

Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_N|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n\in\mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \quad \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N}\ \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n\in\mathbb{N}}$ cannot converge to 0.

Part III

Axiom of choice

Chapter 11

Preparation

11.1 Statement of axiom of choice

For any set I and any family $(A_i)_{i\in I}$ of non-empty sets , there exists a mapping $f : I \to \bigcup_{i\in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

11.2 Def

Let $(X, \leq)$ be a partially ordered set If $\forall A \subseteq X$ A is non-empty , there exists a least element of A then we say that $(X, \leq)$ is a well ordered set.

11.3 Theorem

For any set X, there exists an order relation $\leq$ on such that $(X, \leq)$ forms a well ordered set.

11.4 Zorn's lemma

Let $(X, \leq)$ be a partially ordered set . If $\forall A \subseteq X$ that is totally ordered with respect to $\leq$, there exists an upper bound of A inside X. Then , there exists a maximal element x_0 of $X(\forall y \in X, y > x_0$ does not hold)

11.5 Prop.

Let $(X, \leq)$ be a well ordered set , $y \notin X$. We extends $\leq$ to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

11.6 Proof

Let $A \subseteq X \cup \{y\}, A \neq \varnothing$. If $A = \{y\}$ then Y is the least element of A. If $A \neq \{y\}$ then $B = A \backslash \{y\}$ is non-empty. Let b be the least element of B. Since $b < y$ it's also the least element of A

11.7 Def: Initial Segment

Let $(X, \leq)$ be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S(X_{<s} \subseteq S)$, then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X.

11.9 Prop.

Let $(X, \leq)$ be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X, then $\bigcup\limits_{i \in I} S_i$ is an initial segment of X

11.10 Proof

$\forall s \in \bigcup\limits_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup\limits_{i \in I} S_i$

11.11 Prop.

Let $(X, \leq)$ be a well ordered set.

(1) Let S be a proper initial segment of X, $x = \min(X \backslash S)$ Then $S = X_{<x}$

(2) $\begin{aligned} X &\to \wp(X) \\ x &\mapsto X_{<x} \end{aligned}$

(3) The set of all initial segments of X forms a well ordered subset of $(\wp(x), \subseteq)$

11.12 Proof

(1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction.Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \backslash S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

(2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \backslash X_{<x}$ So $X_{<x} \subsetneqq X_{<y}$

(3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \varnothing$. Then there exists $A \subseteq X$ such that $\mathcal{F} \backslash \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \varnothing$ then $\mathcal{F} = \{X\}$, and $\{X\}$is the least element of $\mathcal{F}$. Otherwise $A \neq \varnothing$ and A has a least element a. Then by(2) $X_{<}a$ is the least element of $\mathcal{F}$

11.13 Lemma

Let $(X, \leq)$ be a well ordered set, $f : X \to X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \varnothing$, let a be the least element of A. By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing . This shows $f(a) \in A$. But a is the least element of A, $f(a) < a$ cannot hold: contradiction.

11.14 Prop

Let $(X, \leq)$ be a well ordered set, S and T be two initial segment of X . If $f : S \to T$ is a bijection that's strictly increasing , then $S = T, f = Id_S$

Proof

We may assume $T \subseteq S$.Let $l : T \to S$ be the induction mapping and $g = l \circ f : S \to S$. Since g is strictly increasing , by the lemma ,$\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$
Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

11.15 Def

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets. If $\exists f : X \to Y$ that's increasing and bijective, we say that $(X, \leq)$ and $(Y, \leq)$ are isomorphic

11.16 Def

Let $(X, \leq)$ and $(Y, \leq)$ be well ordered sets. If $(X, \leq)$ is isomorphic to an initial segment of Y. We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y, we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \prec X$

11.17 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

Proof

We construct a correspondence f from X to Y, such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$

By the last proposition of Oct. 11, f is a function.

- If $a, b \in Dom(f)^2, a < b$,then $X_{<a} \subsetneqq X_{<b}$
  By definition, $Y_{<f(b)} \sim X_{<b} \quad Y_{<f(a)} \sim X_{<a}$
  Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y < f(b)$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_< f(B)$. We then get $f(a) < f(b)$.
  Thus f is strictly increasing.

- Let $a \in Dom(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y \quad X_{<x} \sim Y_{<y}$ This shows that $x \in Dom(f)$.
  Hence $Dom(f)$ is an initial segment of X. Applying this to f^{-1}, we get : $Im(f) = Dom(f)$ is an initial segment of Y

- Either $Dom(f) = X$ or $Im(f) = Y$.
  Assume that $x \in X \backslash Dom(f), y \in Y \backslash Im(f)$ are respectively the least elements of $X \backslash Dom(f)$ and $Y \backslash Im(f)$.
  Then we get $Dom(f) = X_{<x}, Im(f) = Y_{<y}$.
  We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction

-

Case 1 $Dom(f) = X, Im(f) \subsetneqq Y \quad X \prec Y$

Case 2 $Dom(f) \subsetneqq X, Im(f) = Y \quad X \succ Y$

Case 3 $Dom(f) = X, Im(f) = Y \quad X \sim Y$

11.18 Lemma

Let $(X, \leq)$ be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B, or B is a initial segment of A.

Let $Y = \bigcup\limits_{A \in \mathfrak{S}} A$. Then $(Y, \leq)$ is a well ordered set, and ,$\forall A \in \mathfrak{S}, A$ is an initial segment of Y.

Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$.Since $Y = \bigcup\limits_{B \in \mathfrak{S}} B$,$\exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B. Hence $y \in A$. Contradiction

- Let $Z \subseteq Y, Z \neq \varnothing$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \mathfrak{S}$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$, then A is an initial segment of B.

Since B is well ordered , if $m \not\leq z$ then $z < m$. Since $m \in A$, we het $z \in A$. Contradiction.

Therefore, m is the least element of Z.

Chapter 12

Zorn's lemma

Let $(X, \leq)$ be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X, the X has a maximal element(a maximal element m of $\{x \mid x > m\} = \varnothing$)

12.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega . \exists f(A)$ such that $\forall a \in A, a < f(A)$
Let

$$\omega = \{\text{well ordered subset of X}\}$$

. (guaranteed by axiom of choice)
Let $f : \omega \to X$ such that $f(A)$ is an upper bound of $A \in \omega$.
If $A \in \omega$ satisfies

$$\forall a \in A a = f(A_{<a})$$

, we say that A is a f-set
Let

$$\mathfrak{S} = \{f - sets\}$$

Note that

$$\varnothing \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cap \{f(A)\} \in \mathfrak{S}$$

In fact, if $a \in A$, then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If $a = f(A) \notin A$ then

$$(A \cup \{f(A)\})_{<a} = A$$

Let A and B be elements of $\mathfrak{S}$. Let I be the union of all common initial segments of A and B. This is also a common initial segment of A and B.
If $I \neq A$ and $I \neq B$, then

$$\exists (a,b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then $I \cup \{a\}$ is also a common initial segment of A and B, contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and $\forall A \in \mathfrak{S}$ is an initial segment of Y.

Since A is an initial segment of Y

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If $f(y)$ is not a maximal element of X

$$\exists x \in X, f(y) < x$$

Part IV

Topology

Chapter 13

Absolute value and norms

13.1 Def

Let K be a field . By absolute value on K, we mean a mapping $|\cdot| : K \to \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\forall a \in K \quad |a| = 0$ iff $a = 0$

(2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$

(3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$(triangle inequality)

13.2 Notation

$\mathbb{Q}$ Take a prime num p $\forall \alpha \in \mathbb{Q}\backslash\{0\}$ there exists a integer $ord_p(\alpha)\frac{a}{b}$, where $\begin{matrix} a \in \mathbb{Z}\backslash\{0\} \\ b \in \mathbb{N}\backslash\{0\} \end{matrix}, p \nmid a, p \nmid b$

13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \to \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & if \ \alpha \neq 0 \\ 0 & if \ \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on $\mathbb{Q}$

Proof

(1) Obviously

(2) If $\alpha = p^{ord_p(\alpha)}\frac{a}{b}, \beta = p^{ord_p(\beta)}\frac{c}{d} \quad p \nmid abcd$
$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)}\frac{ac}{bd} \quad p \nmid ac, p \nmid bd$

(3) $\alpha + \beta = p^{ord_p(\alpha)}\frac{a}{b} + p^{ord_p(\beta)}\frac{c}{d}$
Assume $ord_p(\alpha) \geq ord_p(\beta)$
$\alpha + \beta$
$= p^{ord_p(\beta)}\left(p^{ord_p(\alpha)-ord_p(\beta)}\frac{a}{b} + \frac{c}{d}\right)$
$= p^{ord_p(\beta)}\frac{p^{ord_p(\alpha)-ord_p(\beta)}ad+bc}{bd} \quad p \nmid bd$
So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

Hence $ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$
So $|\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$
$\max\{|\alpha|_p, |\alpha|_p\} \leq |\alpha|_p, |\alpha|_p$

13.4 Def

Let K be a filed and $|\cdot|$ be an absolute value. We call $(K, |\cdot|)$ a valued field.

Chapter 14

Quotient Structure

14.1 Def

Let X be a set and $\sim$ be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that $\sim$ is an equivalence relation

14.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X.Let $X/\sim$ be the set $\{[x] \mid x \in X\}$

14.3 Prop.

Let X be a set and $\sim$ be an equivalence relation on X

(1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$

(2) If α and β are elements of $X/\sim$ such that $\alpha \neq \beta$ then $\alpha \cap \beta = \varnothing$

(3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

Proof

(1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ on has $x \sim y$
Therefore ,$x \sim z$ namely $z \in [x]$. This proves $y[] \subseteq [x]$. Moreover ,since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$

(2) Suppose that $\alpha \cap \beta \neq \varnothing, y \in \alpha \cap \beta$
By (1),$\alpha = [y], \beta = [y]$, Thus leads to a contradiction.

(3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup\limits_{\alpha \in X/\sim} \alpha$Hence $X \subseteq \bigcup\limits_{\alpha \in X/\sim} \alpha$.Conversely, $\forall \alpha \in X/\sim, \alpha$ is a subset of X. Hence $\bigcup\limits_{\alpha \in X/\sim} \alpha \subseteq X$.Then $X = \bigcup\limits_{\alpha \in X/\sim} \alpha$

14.4 Def

Let G be a group and X be a set
We call left/right action of G on X ant mapping $G \times X \rightarrow X : (g, x) \mapsto gx/(g, x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x \ / \ x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x \ / \ (xg)h = x(gh)$

14.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X.

14.6 Prop

Let G be a group and X be a set . Assume given a left action of G on X. Then the binary relation $\sim$ on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

14.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim$ $\forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

14.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x.(y \sim x)$
- $\forall (x, y, z) \in X^3$,if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

14.9 Quotient set

Let X be a set and $\sim$ be an equivalence relation, the mapping $X \to X/\sim$: $(x \in X) \mapsto [x]$ is called the projection mapping.
$X/\sim$ is called the quotient set of X by equivalence relation $\sim$

14.9.1 Example

Let G be a group and H be a subgroup of G.Then the mapping

$$H \times G \to G$$

$$(h, g) \mapsto hg/(h, g) \mapsto gh$$

is a left/right action of H on G. Thus we obtain two quotient sets H/G and G/H

14.10 Def

Let G be a group and H be a subgroup of G. Ig $\forall g \in G, h \in H \quad ghg^{-1} \in H$, Then we say that H is a normal subgroup of G

14.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G. In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

14.12 Prop

If G is commutative, any subgroup of G is normal

14.13 Theorem

Let G be a group and H be a normal subgroup of G. Then the mapping

$$G/H \times H/G \to G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H Moreover the projection mapping

$$\pi : G \to G/H$$

$$x \mapsto xH$$

is a morphism of groups.

Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$ Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.

- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH)$

- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$

- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

14.14 Def

Let K be a unitary ring and E be a left K-module. We say that a subgroup F og $(E, +)$ is a left sub-K-module of E if $\forall (a, x) \in K \times F, ax \in F$

14.15 Prop

Let K be a unitary ring , E be a left K-module and F be a sub-K-module. Then the mapping

$$K \times (E/F) \to E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined , and defines a left-K-module structure on E/F. Moreover, the projection mapping $pi : E \to E/F$ is a morphism of left-K-modules

Proof

Let x and x' be elements of E such that $[x] = [x']$, that meas: $x' - x \in F$ Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K-module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K \quad \pi(ax) = [ax] = a[x] = a\pi(x)$

14.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$ (I is a left and right sub-K-module of A)

14.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \to A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$ Therefore $a'b' \sim ab$

14.17.1 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of $\mathbb{Z}$
If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z} \quad adm = dma \in d\mathbb{Z}$
Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the reside class of n modulo d
If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

14.18 Theorem

Let $f : G \to H$ be a morphism of groups

(1) $Im(f)$ is a subgroup of H

(2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G

(3) The mapping

$$\begin{aligned} \widetilde{f} :& G/Ker(f) \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

(4) f is injective iff $\ker(f) = \{1_G\}$

Proof

(1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup

(2) Let x and y be elements of $\ker(f)$.
One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
Let $x \in \ker f, y \in G$.
One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup

(3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
Moreover $\widetilde{f}([x][y]) = \widetilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence $\widetilde{f}$ is a morphism of groups.
By definition $Im(\widetilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y, x \sin y$ that means $[x] = [y]$
Therefore $\widetilde{f}$ is injective.

(4) If f is injective ,$\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f\{1_G\}$
Conversely , suppose that $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$

14.19 Theorem

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Then

(1) $Im(f)$ is a left-sub-K-module of F

(2) $\ker(f)$ is a left-sub-K-module of E

(3) $\begin{array}{l} \widetilde{f} : E/\ker f \to Im(f) \\ \quad [x] \mapsto f(x) \end{array}$ is a isomorphism of left K-modules

Proof

(1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in Im(f)$

(2)

(3)

Chapter 15

Topology

15.1 Def

Let X be a set. We call topology on X any subset $\mathcal{J}$ of $\wp(x)$ that satisfies:

- $\varnothing \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitrary family of elements in $\mathcal{J}$, then $\bigcup\limits_{i \in I} u_i \in \mathcal{J}$
- If u and v are elements of $\mathcal{J}$, then $u \cap v \in \mathcal{J}$

15.2 Remark

If $(u_i)_i^n = 1$ is a finite family of elements of $\mathcal{J}$, then $\bigcap\limits_{i=1}^{n} u_i \in \mathcal{J}$(by induction, this follows from (3))

15.2.1 Example

$\{\phi, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

15.3 Def

Let X be a set. We call metric on X any mapping $d : \begin{matrix} X \times X \to \mathbb{R}_{\geq 0} \\ (x,y) \mapsto d(x,y) \end{matrix}$,that satisfies

- $d(x,y) = 0$ iff x=y
- $\forall (x,y) \in X^2, d(x,y) = d(y,x)$
- $\forall (x,y,z) \in X^3 \quad d(x,z) \leq d(x,y) + d(y,z)$(triangle inequality)

(X, d) is called a metric space

15.3.1 Example

Let X be a set

$$d : X^2 \to \mathbb{R}_{\geq 0}$$
$$d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a metric

15.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ We call the open ball of radius ϵ centered at x

15.4.1 Example

Consider $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$,then $B(x, \epsilon) =]x - \epsilon, x + \epsilon[$

15.5 Prop.

Let (X,d) be a metric space . let $\mathcal{J}_d$ be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x, \epsilon) \subseteq U$ THen $\mathcal{J}_d$ is a topology on X

Proof

- $\varnothing \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of $\mathcal{J}_d$ Let $U = \bigcup\limits_{i \in I} u_i$, $\forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, y) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of $\mathcal{J}_d$ Let $x \in U \cap V$ $\exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$ Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

15.6 Def

$\mathcal{J}_d$ is called the topology induced by the metric d

15.7 Def

We call topology space any pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a topology on X

Given a topological space $(X, \mathcal{J})$ If $U \in \mathcal{J}$ then we say that U is an open subset of X. If $F \in \wp(X)$ such that $X \backslash F \in \mathcal{J}$, then we say that F is closed subset of X

If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that $\mathcal{J}$ is metrizable

15.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact,m if d denote the metric defined as $d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$

$\forall x \in X \quad B(x, 1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup\limits_{x \in A} \{x\} \in \mathcal{J}_d$

Chapter 16

Filter

16.1 Def

Let Xbe a set. We call filter if X any $\mathcal{F} \subseteq \wp(x)$ that satisfies:

(1) $\mathcal{F} \neq \varnothing, \varnothing \notin \mathcal{F}$

(2) $\forall A \in \mathcal{F}, \forall B \in \wp(X)$, if $A \subseteq B$, then $B \in \mathcal{F}$

(3) $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

16.1.1 Example

(1) Let $Y \subseteq X, Y \neq \varnothing$. $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y.

(2) Let X be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \backslash A \text{ is infinite}\}$$

is a filter called the Fréchet filter of X.

(3) Let $(X, \mathcal{J})$ be a topological space, $x \in X$ We call neighborhood of x any $V \in \wp(X)$ such that $\exists u \in \mathcal{J}$, satisfying $x \in U \subseteq V$. Then $\mathcal{V} = \{\text{neighborhoods of x}\}$ is a filter.

16.2 Def: Filter Basis

Let X ba a set. $\mathscr{B} \subseteq \wp(X)$. If $\varnothing \notin \mathscr{B}$ and $\forall (B_1, b_2) \in \mathscr{B}^2, \exists B \in \mathscr{B}$, such that $B \subseteq B_1 \cap B_2$. We say that $\mathscr{B}$ is a filter basis.

16.2.1 Remark

If $\mathscr{B}$ is a filter basis, then $\mathcal{F}(\mathscr{B}) = \{A \subseteq X \mid \exists B \in \mathscr{B} \quad B \subseteq A\}$ is a filter

Proof

$\varnothing \notin \mathcal{F}(\mathscr{B}), \mathcal{F}(\mathscr{B}) \neq \varnothing$ since $0 \neq B \subseteq \mathcal{F}(\mathscr{B})$. If $A \in \mathcal{F}(\mathscr{B}), A' \in \wp(X)$such that $A \subseteq A'$, then $\exists B \in \mathscr{B}$ such that $B \subseteq A \subseteq A'$. Hence $A' \in \mathcal{F}(\mathscr{B})$ If $A_1, A_2 \in \mathcal{F}(\mathscr{B})$, then $\exists (B_1, B_2) \in \mathscr{B}^2$ such that $B_1 \subseteq A_1, B_2 \subseteq A_2$. Since $\mathscr{B}$ is a filter basis, $\exists B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ Hence $A_1 \cap A_2 \in A_1 \cap A_2 \in \mathcal{F}(\mathscr{B})$

16.2.2 Example

- Let $Y \subseteq X, Y \neq \varnothing$
  $\mathscr{B} = \{Y\}$ is a filter basis. $\mathcal{F}(\mathscr{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let $(X, \mathcal{J})$ be a topological space $x \in X$. If $\mathscr{B}_x$ is a filter basis such that $\mathcal{F}(\mathscr{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$, then we say that $\mathscr{B}_x$ is a neighborhood basis of x

16.3 Remark

Let $\mathscr{B}_x$ is a neighborhood basis of x iff

- $\mathscr{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathscr{B}_x$ such that $U \subseteq V$
- Let (X, d) be a metric space , $x \in X \forall \epsilon > 0$, Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

  Then

  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x
  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x

16.3.1 Example

$\mathcal{V}_x \cap \mathcal{J}$ is a neighborhood basis of x

16.4 Def

$V \in \wp(X)$ is called a neighborhood of x if $\exists U \in \mathcal{J}$ such that $x \in U \subseteq V$

16.5 Remark

Let $(X, \mathcal{J})$ be a topological space, $x \in X$ and $\mathscr{B}_x$ a neighborhood basis os x. Suppose that $\mathscr{B}$ is countable . We choose a surjective mapping $(B_n)_{n\in\mathbb{N}}$ from $\mathbb{N}$ to$\mathscr{B}_x$. For any $n \in \mathbb{N}$, let $A_n = B_0 \cap B_1 \cap ... \cap B_n \in \mathcal{V}_x$ The sequence $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\{A_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of x.

16.6 Extra Episode

$\wp(\mathbb{N})$is NOT countable

Suppose that $f : \wp(\mathbb{N}) \to \mathbb{N}$ injective. Then $\exists g : \mathbb{N} \to \wp(\mathbb{N})$ surjective. Taking $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$. Since g is surjective, $\exists a \in \mathbb{N}$ such that $A = g(a)$.

If $a \in A$, then $a \in g(a)$, hence $a \notin A$

If $a \notin A$, then $a \in g(a) = A$

Contradiction

16.7 Prop.

Let Y and R be sets, $g : Y \to E$ be a mapping ,

- If $\mathcal{F}$ is a filter of Y, then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on E

- If $\mathscr{B}$ is a filter basis of Y, then

$$g(\mathscr{B}) := \{g(B) \mid B \in \mathscr{B}\}$$

is a filter of E, and $\mathcal{F}(g(\mathscr{B})) = g_*(\mathcal{F}(\mathscr{B}))$

Proof

(1) $E \in g_x(\mathcal{F})$ since $g^{-1}(E) = Y$
$\varnothing \notin g_x(\mathcal{F})$ since $g^{-1}(\varnothing) = \varnothing$

If $A \in g_x(\mathcal{F})$ and $A' \supseteq A$, then $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{J}$, so $g^{-1}(A') \in \mathcal{J}$, Hence $A' \in g_x(\mathcal{F})$

If $A_1, A_2 \in g_x(\mathcal{F})$. Then $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ Hence $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$. So $A_1 \cap A_2 \in g_x(\mathcal{F})$.

(2) Since g is a mapping , and $\varnothing \notin \mathscr{B}$, we get $\varnothing \notin g(\mathscr{B})$,since $\mathscr{B} \neq \varnothing, g(\mathscr{B}) \neq \varnothing$.

Let $B_1, B_2 \in \mathscr{B}$, there exists $C \in \mathscr{B}$ such that $C \subseteq B_1 \cap B_2$. Hence $g(C) \subseteq g(B_1) \cap g(B_2)$, namely $g(\mathscr{B})$ is a filter basis.

Chapter 17

Limit point and accumulation point

We fix a topological space $(X, \mathcal{J})$

17.1 Def

Let $\mathcal{F}$ be a filter of X and $x \in X$

- If $\mathcal{V}_x \subseteq \mathcal{F}$ then we say that x is an limit point of $\mathcal{F}$
- If $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \varnothing$, we say that x is an accumulation point of $\mathcal{F}$

So any limit point of $\mathcal{F}$ is necessarily a accumulation point of $mathcalF$

17.2 Prop

Let $\mathscr{B}$ be a filter basis of X, $x \in X, \mathscr{B}_x$ a neighborhood basis of x. Then x is an accumulation point of $\mathcal{F}(\mathscr{B})$ iff $\forall (B, U) \in \mathscr{B} \times \mathscr{B}_x, B \cap U \neq \varnothing$

Proof

Necessity

Since $\mathscr{B} \subseteq \mathcal{F}(\mathscr{B}), \mathscr{B} \subseteq \mathcal{V}_x$, the necessity is true.

Sufficiency

Let $(A, V) \in \mathcal{F}(\mathscr{B}) \times \mathcal{V}_x$. There exist $B \in \mathscr{B}, U \in \mathscr{B}_x$, such that $B \subseteq A, U \subseteq V$. Hence $\varnothing \neq B \cap U \subseteq A \cap V$

17.3 Def

Let $Y \subseteq X, Y \neq \varnothing$. W call accumulation point of Y any accumulation point of the principal filter $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$.

17.4 Def

We denote by $\overline{Y} = \{\text{accumulation points of } Y\}$.,called the closure of Y Note that $x \in \overline{Y}$ iff $\forall U \in \mathscr{B}_x, Y \cap U \neq \varnothing$

By convention $\overline{\varnothing} = \varnothing$

17.5 Prop

Let $Y \subseteq X$. Then $\overline{Y}$ is the smallest closed subset of X containing Y.

Proof

$\forall x \in X \backslash \overline{Y}$, then there exists $U_x = \mathcal{V} \cap \mathcal{J}$, such that $Y \cap U_x = \varnothing$. Moreover , $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{J}$. This shows that $\forall y \in U_x, y \notin \overline{Y}$. Therefore $X \backslash \overline{Y} = \bigcap_{x \in X \backslash \overline{Y}} U_x \in \mathcal{J}$

Let $Z \subseteq X$ be a closed subset that contain Y. Suppose that $\exists y \in \overline{Y} \backslash Z$. Then $U = X \backslash Z \in \mathcal{V}_y \cap \mathcal{J}$ and $U \cap Y \subseteq U \cap Z = \varnothing$. So $y \notin \overline{Y}$ contradiction. Hence $\overline{Y} \subseteq Z$.

17.6 Def: dense

Let $(X, \mathcal{J})$ be a topological space,Y a subset of X. We call Y is dense in X if

$$\overline{Y} = Y$$

Chapter 18

Limit of mappings

18.1 Def

Let $(E, \mathcal{J}_E)$ be a topological space . $f : Y \to E$ a mapping , and $\mathcal{F}$ eb a filter of Y. If $a \in E$ is a limit point of $F_*(\mathcal{F})$ namely , $\forall \text{neighborhood} V \text{of} a, f^{-1}(V) \in \mathcal{F}$, then we say that a is a limit of the filter $\mathcal{F}$ by f

18.2 Remark

Let $\mathscr{B}_a$ be a neighborhood basis of a. Then $\mathcal{V}_a \subseteq f_x(\mathcal{F})$, iff $\mathscr{B} \subseteq f_*(\mathcal{F})$ Therefore , a is a limit of $\mathcal{F}$ by f iff $\forall V \in \mathscr{B}_a, f^{-1}(V) \in \mathcal{F}$

18.2.1 Example

Let $(E, \mathcal{J}_E)$ be a topological space. $I \subseteq \mathbb{N}$ be an infinite subset, $x = (x_n)_{n \in I} \in E^I$. If the Fréchet filter $\mathcal{F}_{Fr}(I)$ has a limit $a \in E$ by the mapping $x : I \to E$, we say that $(x_n)_{n \in I}$ converges to a ,denote as

$$a = \lim_{n \in I, n \to +\infty} x_n$$

18.3 Remark

$a = \lim_{n \in I, n \to +\infty} x_n$ iff, $\forall U \in \mathscr{B}_a$(where $\mathscr{B}_a$ is a neighborhood basis of a), $\exists N \in \mathbb{N}$ such that $x_n \in U$ for any $n \in I_{\geq N}$

Suppose that $\mathcal{J}_E$ is induced by a metric d.$\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}\{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}\{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ are all neighborhood basis of a .There fore , the following are equivalent

- $a = \lim_{n \in i, n \to +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$

$(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\}$? unknown position)

18.4 Remark

We consider the metric d on $\mathbb{R}$ defined as

$$\forall (x, x) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of $\mathbb{R}$ defined by this metric is called the usual topology on $\mathbb{R}$

18.5 Prop

Let $(x_n)_{n\in I} \in \mathbb{R}^I$, where $I \subseteq \mathbb{N}$ is an infinite subset. Let $l \in \mathbb{R}$. The following statements are equivalent:

- The sequence $(x_n)_{n\in I}$ converges to l in the topological space $\mathbb{R}$
- $\liminf\limits_{n\in I, n\to+\infty} x_n = \limsup\limits_{n\in I, n\to+\infty} x_n = l$
- $\limsup\limits_{n\in I, n\to} |x_n - l| = 0$

18.6 Theorem

Let (X, d) be a metric space .Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n\in I}$ be an element of X^I. Let $l \in X$. The following statements are equivalent:

- $(x_n)_{n\in I}$ converges to l
- $\limsup\limits_{n\in I, n\to+\infty} d(x_n, l) = 0$(equivalent to $\lim\limits_{n\in I, n\to+\infty} d(x, l) = 0$)

Proof

(1) $\Rightarrow$ (2) The condition (1) is equivalent to $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$. We then get $\sup\limits_{n\in I_{geqN}} d(x, l) \leq \epsilon$. Therefore $\limsup\limits_{n\in I, n\to+\infty} d(x_n, l) \leq \epsilon$ We obtain that $\limsup\limits_{n\in I, n\to+\infty} = 0$

(2) $\Rightarrow$ (1) Let $\epsilon \in \mathbb{R}_{>0}$ If $\inf\limits_{N\in\mathbb{N}} \sup\limits_{n\in I_{\geq N}} d(x_n, l) = 0$. Then $\exists N \in \mathbb{N} \quad \sup\limits_{n\in I_{\leq N}} d(x_n, l) < \epsilon$. Hence ,$\forall n \in I_{\geq N} d(x_n, l) < \epsilon$. Since ϵ is arbitrary, $(*)$ is true, Hence (1) is also true .

18.7 Prop

Let $(X, \mathcal{J})$ be a topological space . $Y \subseteq X, p \in \overline{Y} \backslash Y$.Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of Y.

Proof

Y is not empty otherwise $\overline{Y} = \varnothing$.

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$
  $\varnothing \notin \mathcal{V}_{p,Y}$ since $p \in \overline{Y}$
- Let $V \in \mathcal{V}_p$ and $A \subseteq Y$ such that $V \cap Y \subseteq A$. Let $U = V \cup (A \backslash (V \cap Y)) \in \mathcal{V}_p$ and $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let U and V be elements of $\mathcal{V}_p$ Let $W = U \cap V \in \mathcal{V}_p$ Then $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

18.8 Def

Let $(X, \mathcal{J}_x)$ and $(E, \mathcal{J}_E)$ be topological spaces, $Y \subseteq X, p \in \overline{Y} \backslash Y$, and $f : Y \to E$ be a mapping . If a is a limit point of $(F_*(\mathcal{V}_{p,Y}))$, then we say that a is a limit of f when the variable $y \in Y$ tends to p, denoted as $a = \lim\limits_{y \in Y, y \to p} f(y)$

18.9 Remark

If $\mathscr{B}_a$ is a neighborhood basis of a. Then $a = \lim\limits_{y \in Y, y \to p} f(y)$ is equivalent to $\forall U \in \mathscr{B}_a \quad \exists V \in \mathcal{V}_p$ such that $Y \cap V \subseteq f_{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

18.10 Prop

Let X be a set, $\mathscr{B}$ be a filter basis, $\mathscr{G}$ be a filter. If $\mathscr{B} \subseteq \mathscr{G}$, then $\mathcal{F} \subseteq \mathscr{G}$.

Proof

Let $V \in \mathcal{F}(\mathscr{B})$ By definition $\exists U \in \mathscr{B}$ such that $U \subseteq V$, since $U \in \mathscr{G}($ for $\mathscr{B} \subseteq \mathscr{G})$ and since $\mathscr{G}$ is a filter, $V \in G$

18.11 Theorem

Let $(X, \mathcal{J}_x)$ and $(E < \mathcal{J}_E)$ be topological spaces. $Y \subseteq X$, $p \in \overline{T} \setminus Y, a \in E$. We consider the following conditions.

(i) $a = \lim_{y \in Y, y \to p} f(y)$

(ii) $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ if $\lim_{n \to +\infty} y_n = p$ then $\lim_{n \to \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that p has a countable neighborhood basis, then (i) and (ii) are equivalent.

Proof

(1) Let $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ such that $p = \lim_{n \to +\infty} y_n$. For any $U \in \mathcal{V}_p, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N} \quad y \in U \cap Y . y_n \in U \cap Y$ Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \to +\infty} f(y_n) = a$$

(2) Assume that p has a countable neighborhood basis . There exists a decreasing sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_P^{\mathbb{N}}$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of p.
Assume that (i) does not hold. Then there exists $U \in \mathcal{V}_a$ such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \to +\infty} y_n = \varnothing$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However $f(y_n) \notin U$ for any $n \in \mathbb{N}$,so $(f(y_n))_{n \in \mathbb{N}}$ cannot converges to a.

18.12 Prop.

Let X be a set. If $(\mathcal{J}_i)_{i\in I}$ is a family of topologies on X, then $\mathcal{J} = \bigcap_{i\in I} \mathcal{J}_i$ is a topology. In particular. for any $\mathcal{A} \subseteq \wp(X)$, there is a smallest topology on X that contain $\mathcal{A}$

18.12.1 Proof

- $\forall i \in I \quad \{\varnothing, X\} \subseteq \mathcal{J}_i$ So $\{\varnothing, X\} \subseteq \mathcal{J}$
- Let $(u_j)_{j\in J}$ be a family of elements of $\mathcal{J}$ $\forall j \in J, i \in I \quad u_i \in \mathcal{J}_i$ So $\bigcup_{j\in J} u_j \in \mathcal{J}_i$ We then get $\bigcup_{j\in J} u_j \in \mathcal{J}$
- Let U and V be elements of $\mathcal{J}$ $\forall i \in I, \{u, v\} \subseteq \mathcal{J}_i$ So $U \cap V \in \mathcal{J}_i$. THerefore we get $U \cap V \in \mathcal{J}$ Let $\mathcal{A} \subseteq \wp(X)$ Let $\mathcal{J}(\mathcal{A}) = \bigcap_{\mathcal{J} \subseteq \wp(X) \text{a topology} \mathcal{A} \subseteq \mathcal{J}} \mathcal{J}$ Then $\mathcal{J}(\mathcal{A})$ is a topology. By definition, if $\mathcal{J}$ is a topology containing $\mathcal{A}$, then $\mathcal{J}(\mathcal{A}) \subseteq \mathcal{J}$ Hence $\mathcal{J}(\mathcal{A})$ is the smallest topology containing $\mathcal{A}$

Chapter 19

Continuity

19.1 Def

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces f be a function from X to Y, $x \in Dom(f)$. If for any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$. Then we say that f is continuous at x. If f is continuous at any $x \in Dom(f)$ then we say f is continuous.

19.2 Remark

Let $\mathscr{B}_{f(x)}$ be a neighborhood basis of $f(x)$ If $\forall U \in \mathscr{B}_{f(x)}$ there exist $V \in \mathscr{B}_{f(x)} V_x$ such that $f(V) \subseteq U$, then f is continuous at x Suppose that X and Y are metric space. Then f is continuous at x iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in Dom(f) \quad d(y, x) < \delta \text{ implies } \ d(f(y), f(x)) < \epsilon$$

19.3 Theorem

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces, f be a function from X to Y $x \in Dom(f)$ Consider the following condition

- f is continuous at x
- $\forall (x_n)_{n\in\mathbb{N}} \in Dom(f)^{\mathbb{N}}$, if $\lim\limits_{n\to+\infty} x_n = x$, then $\lim\limits_{n\to+\infty} f(x_n) = f(x)$ THen (i) implies (ii) Moreover, if x has a countable neighborhood basis, then (i) and (ii) are equivalent.

19.4 Proof

$(i) \Rightarrow (ii)$ Let $(x_n)_{n\in\mathbb{N}} \in Dom(f)^{\mathbb{N}}$ that converges to x $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$ Since $\lim\limits_{n\to+\infty} x_n = x$,there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}$, $x_n \in V$.

Hence $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$. Thus $\lim_{n\to+\infty} f(x_n) = f(x)$

$(ii) \Rightarrow (i)$ under the hypothesis that x has countable neighborhood basis. actually we will prove $NOT(i) \Rightarrow NOT(ii)$
Let $(V_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{V}_x$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of x
If (i) does not hold, then $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \nsubseteq U$ Pick $x_n \in V_n$ such that $f(x_n) \notin U \quad \forall N \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$. Hence $(x_n)_{n\in\mathbb{N}}$ converges to x. However, $f(x_n) \notin U$ for any n So $(f(x_n))_{n\in\mathbb{N}}$ does not converges to $f(x)$. Therefore (ii) does not hold.

19.5 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$ be topological spaces. f be a function from X to Y, g be a function from Y to Z. Let $x \in Dom(g \circ f)$ If f and g are continuous at x. then $g \circ f$ is continuous at x sectionProof Let $U \in \mathcal{V}_{g(f(x))}$ Since g is continuous at $f(x)$:

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since f is continuous at x:

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore, $g(f(V)) \subseteq g(W) \subseteq U$ Hence $g \circ f$ is continuous at x

19.6 Def

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$, If any element of $\mathcal{J}$ can be written as the union of a family of sets in $\mathscr{B}$ we say that $\mathscr{B}$ is a topological basis of $\mathcal{J}$

19.7 Prop

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$ $\mathscr{B}$ is a topological basis iff

$$\forall x \in X, \mathscr{B}_x := \{V \in \mathscr{B} \mid x \in V\}$$

is a neighborhood basis of x

19.8 Proof

$\Rightarrow$:

$$\forall x \in X \mathscr{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since $\mathscr{B}$ is a topological basis of $\mathcal{J}$,

$$\exists W \in \mathscr{B}, x \in W \subseteq V \subseteq U$$

Hence $\mathcal{V}_x$ is generated by $\mathscr{B}_x$

$\Leftarrow$ Let $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathscr{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

19.9 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces. $\mathscr{B}_Y$ be a topological basis of $\mathcal{J}_Y$ $f : X \to Y$ be a mapping. The following conditions are equivalent:

(1) f is continuous

(2) $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$

(3) $\forall U \in \mathscr{B}, f^{-1}(U) \in \mathcal{J}_X$

Proof

(1) $\Rightarrow$ (2)

Lemma Let $(X, \mathcal{J})$ be a topological space, $V \in \wp(X)$, THen $V \in \mathcal{J}$ iff $\forall x \in V, V$ is a neighborhood of x

Proof of lemma $\Rightarrow$ is by definition
$Leftarrow$:

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W - x \in \mathcal{J}$$

Let $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood W of x such that $f(W) \subseteq U$
Since f is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since x is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2) $\Rightarrow$ (3) For (3) is a special situation of (2), it's natural.

(3) $\Rightarrow$ (1) Let $x \in X$

$$\forall U \in \mathscr{B}_Y \ s.t. \ f(x) \in U, f^{-1}(U)$$

is an open neighborhood of x, and

$$f(f^{-1}(U)) \subseteq U$$

Hence f is continuous at x

19.10 Def

LEt X be a set ,$((Y_i, \mathcal{J}_i))_{i\in I}$ be a family of topological spaces. $\forall i \in I$ let $f_i : X \to Y_i$ be a mapping. We call initial topology of $(f_i)_{i\in I}$ on X the smallest topology on X making all f_i continue

19.11 Remark

If $\mathcal{J}$ is the initial topology of $(f_i)_{i\in I}$, $\forall i \in I, U_i \in \mathcal{J}_i \quad f_i^{-1}(U_i) \in \mathcal{J}$ If $J \subseteq I$ is a finite subset, $(U_j)_{j\in J} \in \prod_{j\in J} \mathcal{J}_j$ then $\bigcap_{j\in J} f_j^{-1}(U_j) \in \mathcal{J}$

19.12 Prop

$$\mathscr{B} := \left\{ \bigcap_{j\in J} f_j^{-1}(U_j) \mid J \subseteq I \text{is finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology $\mathcal{J}$

Proof

First

$$\mathscr{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{$subset V of X that can be written as the union of a family of sets in$\mathscr{B}\}$

- $\varnothing \in \mathcal{J}' \quad X \in \mathscr{B} \subseteq \mathcal{J}'$
- $\mathcal{J}'$ is stable by taking the union of any family of elements in $\mathcal{J}'$
- If V_1, V_2 are elements of $\mathcal{J}'$, then

$$V_1 \cap V_2 \in \mathcal{J}'$$

In fact, V_1, V_2 are of the form of the union of some sets of $\mathscr{B}$

The intersection of two elements of $\mathscr{B}$ is still a element of $\mathscr{B}$

$$\left(\bigcap_{j\in J} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j\in J'} f_j^{-1}(U_j')\right)$$

$$= \bigcap_{j\in J\cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J\backslash J' \\ U_j' & j \in J'\backslash J \\ U_j \cap U_j' & j \in J \cap J' \end{cases}$$

$$\left(\bigcap_{j\in J\backslash J'} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j\in J\cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U_j')\right) \cap \left(\bigcap_{j\in J'\backslash J} f_j^{-1}(U_j')\right)$$

So $\mathcal{J}'$ is a topology making all f_i continuous.Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

Example

Let $((Y_i, \mathcal{J}_i))_{i\in I}$ be topological spaces. $Y = \prod\limits_{i\in I} Y_i$ and

$$\pi_i : \begin{matrix} Y \to Y_i \\ (y_j)_{j\in I} \mapsto y_i \end{matrix}$$

The product topology on Y is by definition the initial topology of $(\pi_i)_{i\in I}$

19.13 Theorem

Let X be a set , $((Y_i, \mathcal{J}_i))_{i\in I}$ be a family of topological spaces,

$$((f_i : X \to Y_i))_{i\in I}$$

be a family of mappings and we equip X with the initial topology $\mathcal{J}_X$ of $(f_i)_{i\in I}$ Let $(Z, \mathcal{J}_Z)$ be a topological space and

$$h : Z \to X$$

be a mapping. Then h is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{is continuous}$$

19.13.1 Proof

$\Rightarrow$ If h is continuous, since each f_i is continuous, $f_i \circ h$ is also continuous.

$\Leftarrow$ Suppose that $\forall i \in I, f_i \circ h$ is continuous .Hence

$$\forall U_i \in \mathcal{J}_i, (f_i \circ h)_{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Let

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_j^{-1}(U_j) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathcal{J}_j \right\}$$

$\forall U \in \mathscr{B}$

$$h^{-1}(U) = \bigcap_{j\in J} h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Therefore, h is continuous.

19.14 Remark

We keep the notation of the definition of initial topology If $\forall i \in I, \mathscr{B}_i$ is a topological basis of $\mathcal{J}_i$, then

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_i^{-1}(U_i) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathscr{B}_j \right\}$$

is also a topological basis of the initial topology,

19.14.1 Example

Let $((X_i, d_i))_{i\in\{1,\ldots,n\}}$ be a family of metric spaces.

$$X = \prod_{i\in\{1,\ldots,n\}} X_i$$

We define a mapping

$$d : \begin{array}{l} X \times X \to \mathbb{R}_{\geq 0} \\ ((x_i)_i \in \{1,...,n\}(y_i)_{i\in\{1,\ldots,n\}}) \mapsto \max_{i\in\{1,\ldots,n\}} d_i(x_i, y_i) \end{array}$$

d is a metric on X. If $x = (x_i)_{i\in\{1,\ldots,n\}}$ $y = (y_i)_{i\in\{1,\ldots,n\}}$ $z = (z_i)_{i\in\{1,\ldots,n\}}$ are elements of X, then

$$d(x, z) = \max_{i\in\{1,\ldots,n\}} d_i(x_i, z_i) \leq \max_{i\in\{1,\ldots,n\}} (d_i(x_i, y_i) + d(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i : \begin{array}{l} X \to X_i \\ (x_i)_{i\in\{1,\ldots,n\}} \mapsto x_i \end{array}$$

is continuous. Hence the product topology $\mathcal{J}$ is contained in $\mathcal{J}_d$
Let $x = (x_i)_{i\in\{1,\ldots,n\}} \in X, \epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i\in\{1,\ldots,n\}} \mid \max_{i\in\{1,\ldots,n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i\in\{1,\ldots,n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i\in\{1,\ldots,n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$

Chapter 20

Uniform continuity and convergency

20.1 Def

Let (X, d) be a metric space. $\forall A \subseteq X$, we define

$$diam(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A.By convention

$$diam(\varnothing) := 0$$

If $diam(A) < +\infty$, we say that A is bounded

20.2 Remark

- If A is finite, then it's bounded
- If $A \subseteq B$ then $diam(A) \leq diam(B)$

20.3 Prop

Let (X, d) be a metric space. $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$. Then

$$diam(A \cup B) \leq diam(A) + d(x_0, y_0) + diam(B)$$

In particular, if A,B are bounded, then $A \cup B$ is bounded.

Proof

Let $(x, y) \in (A \cup B)^2$. If $\{x, y\} \subseteq A$, then $d(x, y) \leq diam(A)$
If $\{x, y\} \subseteq B$ then $diam(B) \geq d(x, y)$
If $x \in A, y \in B$,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq diam(A) + d(x_0, y_0) + diam(B)$$

Similarly if $x \in B, y \in A$

$$d(x, y) \leq diam(A) + d(x_0, y_0) + diam(B)$$

20.4 Def

Let (X, d) be a metric space. $I \subseteq \mathbb{N}$ be an infinite subset, $(x_n)_{n\in I} \in X^I$. If

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \quad diam(\{x_n \mid n \in I_{\geq \mathbb{N}}\}) \leq \epsilon$$

then we say that $(x_n)_{n\in I}$ is a Cauchy sequence.

20.5 Prop

(1) If $(x_n)_{n\in I}$ converges, then it's a Cauchy sequence.

(2) If $(x_n)_{n\in I}$ is a Cauchy sequence, $\{x_n \mid n \in I\}$ is bounded

(3) Suppose that $(x_n)_{n\in I}$ is a Cauchy sequenceIf there exists an infinite subset J of I such that $(x_n)_{n\in J}$ converges to some $x \in X$,then $(x_n)_{n\in I}$ converges to x

20.5.1 Proof

(1) trivial

(2) trivial

(3) Let $\epsilon > 0, \exists N \in \mathbb{N}$

$$diam(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence $(x_n)_{n\in I}$ converges to x

20.6 Def

Let $(X,d_X),(Y,d_Y)$ be metric space. f be a function from X to Y. If $\forall\epsilon>0,\exists\delta>0$ such that

$$\forall(x,y)\in Dom(f)^2, d(x,y)\leq\delta$$

implies

$$d(f(x),f(y))\leq\epsilon$$

namely

$$\inf_{\delta>0}\sup_{(x,y)\in Dom(f)^2,d(x,y)\leq\delta} d(f(x),f(y))=0$$

we say that f is uniformly continuous.

20.7 Prop

Let $(X,d_X),(Y,d_Y)$ be metric spaces f be a function from X to Y which is uniformly continuous.

(1) If $I\subseteq\mathbb{N}$ is finite, and $(x_n)_{n\in I}$ is a Cauchy sequence in $Dom(f)^I$ then $(f(x_n))_{n\in I}$ is Cauchy sequence

(2) f is continuous

20.7.1 Proof

(1) $\forall\epsilon>0,\exists\delta>0$ such that

$$\forall(x,y)\in Dom(f)^2, d(x,y)\leq\delta\Rightarrow d(f(x),f(y))\leq\epsilon$$

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists N\in\mathbb{N}$ such that

$$\forall(n,m)\in I^2_{\geq N}, d_X(x_n,x_m)\leq\delta$$

Hence

$$d_Y(f(x_n),f(x_m))\leq\epsilon$$

THerefore $(f(x_n))_{n\in I}$ is a Cauchy sequence.

(2) Let $(x_n)_{n\in I}$ be a sequence in $Dom(f)^{\mathbb{N}}$ that converges to $x\in Dom(f)$ We define$(y_n)_{n\in\mathbb{N}}$ as

$$y_n=\begin{cases}x & \text{if } n \text{ is odd}\\ x_{\frac{1}{2}} & \text{if } n \text{ is even}\end{cases}$$

Then $(y_n)_{n\in\mathbb{N}}$ converges to x. Hence $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Since f is uniformly continuous, $(f(y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

$$(f(y_n))_{n\in\mathbb{N},n\text{ is odd}}=(f(x))_{n\in\mathbb{N},n\text{ is odd}}$$

converges to $f(x)$. Hence $(f(y_n))_{n\in\mathbb{N}}$ converges to $f(x)$

20.8 Def

Let X be a set , $Z \subseteq X, (Y, d)$ be a metric space, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$ and f be functions from X to Y, having Z as their common domain of definition.

- If $\forall x \in Z, (f_n(x))_{n \in I}$ converges to $f(x)$, we say that $(f_n)_{n \in I}$ converges pointwisely to f
- If
$$\lim_{n \in I, n \to +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$
we say that $(f_n)_{n \in I}$ converges uniformly to f

20.9 Theorem

Let X and Y be metric space, $Z \subseteq X, I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}, f$ be functions from X to Y, having Z as domain of definition. Suppose that

- $(f_n)_{n \in I}$ converges uniformly to f
- each f_n is uniformly continuous

Then f is uniformly continuous.

20.9.1 Proof

$\forall n \in I$ let
$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$
$$\lim_{n \in I, n \to +\infty} A_n = 0$$
$\forall (x, y) \in Z^2, n \in I$
$$\begin{aligned} & d(f(x), f(y)) \\ \leq & d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ \leq & 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$
$$\inf_{\delta > 0} \sup_{(x,y) \in Z^2, d(x,y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x,y) \in Z^2, d(x,y) \leq \delta} d(f_n(x), f_n(y)) = 0$$
Hence
$$0 \leq \inf_{\delta > 0} \sup_{(x,y) \in Z^2, d(x,y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$
Take $\lim\limits_{n \to +\infty}$, by squeeze theorem, we get
$$\inf_{\delta > 0} \sup_{(x,y) \in Z^2, d(x,y) \leq \delta} d(f(x), f(y)) = 0$$

20.10 Theorem

Let X be a topological space, Y be a metric space, $Z \subseteq X, p \in Z$, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}$ and f function from X to Y, having Z as domain of definition. Suppose that:

- $(f_n)_{n\in I}$ converges uniformly to f
- each f_n is continuous at p

Then f is continuous at p

20.10.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x\in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \; \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since f_n is continuous $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{\mathcal{B}}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x)f(p)) & \\ &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$f(U) \subseteq \overline{\mathcal{B}}(f(p), \epsilon)$

20.10.2 Def

Let X Y be metric spaces , f be a function from X to Y, $\epsilon > 0$. If

$$\forall (x, y) \in Dom(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that f is ϵ-Lipschitzian
If $\exists \epsilon > 0$ such that f is ϵ-Lipschitzian, then it's uniformly continuous.

20.11 Remark

If f is Lipschitzian, then it's uniformly continuous.

20.12 Example

- Let $((X_i, d_i))_{i\in I}$ be metric space. $X = \prod_{i\in I} X_i$ where I is finite

$$d : \begin{aligned} & X \times X \to \mathbb{R}_{\geq 0} \\ & d((x_i), (y_i)_{i\in I}) = \max_{i\in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \to X_i$$

is Lipschitzian.$(\forall x = (x_i)_{i \in I}, \forall x = (x_i)_{i \in I})$

- Let (X, d) be a metric space

$$d : X \times X \to \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

Part V

Normed Vector Space

Chapter 21

Linear Algebra

We fix a unitary ring K

21.1 Def

Let M be a left K-module ,and let $x = (x_i)_{i \in I}$ be a family of elements of M. We define a morphism of left K-module as following:

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} && \to M \\ & (a_i)_{i \in I} && \mapsto \sum_{i \in I} a_i x_i (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

21.1.1 Notation

$$K^{\bigoplus I} := \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{finite,such that} a_i = 0 \text{ for } i \in I \setminus J\}$$
$$\varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) = \varphi_x((a_i)_{i \in I})\varphi_x((b_i)_{i \in I})$$

21.2 Def

Ler M be a left K-module, I be a set, $x = (x_i)_{i \in I} \in M^I$ If

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} \to M \\ & (a_i)_{i \in I} \mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective	then we say $(x_i)_{i \in I}$ is K-linearly independent
surjective	then we say $(x_i)_{i \in I}$ is system of generator
a bijection	then we say $(x_i)_{i \in I}$ is a basis of M

Example

Let e_i be the element $(\delta_{ij})_{j\in I}$ with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i\in I} \in (K^{\bigoplus I})^I$$

is a basis of $K^{\bigoplus I}$

21.3 Def

Let M be a left K-module

- If M bas a basis, we say that M is a free K-module
- If M has finite system of generated
  ($\exists a$ finite set I and a family $(x_i)_{i\in I} \in M^I$ that forms a system of generator),
  then we say that M is of finite type.

21.4 Remark

Let $x = (x_i)_{i\in\{1,...,n\}} \in M^n$, where $n \in \mathbb{N}$

- x is linearly independent iff

  $$\forall a \in K^n \quad \sum a_i x_i = 0$$

  implies

  $$a = 0$$

- x is a system of generator iff for any element of M can be written in the form

  $$\sum b_i x_i \quad b \in K^n$$

  Such expression is called a K-linear combination of $x_1, ...x_n$

21.5 Theorem

Let K be a division ring ($0 \neq 1$ and $\forall k \in K \setminus \{0\}$ k is invertible)
Let V be a left K-module of finite type and $(x_i)_{i\in I}$ be a system of generators of V. Then ,there exists a subset I of $\{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V. (In particular, V is a free K-module)

Proof

(By induction on n)
If $n = 0$, then $V = \{0\}$
In this case $\varnothing$ is a basis of V

Induction hypothesis

True for a system of generators of $n-1$ elements. Let $(x_i)_{i\in\{1,\dots,n\}}$ be a system of generators of V. If $(x_i)_{i\in\{1,\dots,n\}}$ is linearly independent, it's a basis. Otherwise, $\exists (a_i)_{i\in I} \in K^n$ such that

$$(a_i, ...a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose $a_n \neq 0$. Then

$$x_n = -a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i)$$

Since $(x_i)_{i\in\{1,\dots,n\}}$ is a system of generators, any elements of V can be written as

$$\begin{aligned}\sum b_i x_i &= (\sum_{i=1}^{n-1} b_i x_i) - b_n a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i)\\ &= \sum_{i=1}^{n-1}(b_i - b_n a_n^{-1} a_i) x_i\end{aligned}$$

Thus $(x_i)_{i\in\{1,\dots n\}}$ forms a system of generators . By the induction hypothesis, there exists $I \subseteq \{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V.

21.6 Theorem

Let K be a unitary ring and B be a left K-module. W be a left K-submodule of V. Let $(x_i)_{i=1}^n$ be an element of W^n

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where $(n, l) \in \mathbb{N}^2$ $\forall j \in \{1, ...l\}$,let x_{n+j} be an element in the equivalence class α_j

- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are linearly independent, then $(x_i)_{i=1}^{n+l}$ is also linearly independent
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are system of generators of W and V/W respectively, then $(x_i)_{i=1}^{n+l}$ is also a system of generators
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are basis, then $(x_i)_{i=1}^{n+l}$ is also a basis

Proof

(1) Suppose that $(b_i)_{i=1}^{n+l}$ such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \to V/W$$

be the projection morphism($\pi(x) = [x]$)

$$0 = \pi(\sum_{i=1}^{n+l} b_i x_i) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^{l} b_{n+j}\pi(x_{n+j}) = \sum_{j=1}^{l} b_{n+j}\alpha_j$$

$\{x_1, ...x_n\} \subseteq W$ So$\forall i \in \{1, ..., n\}$

$$\pi(x_i) = 0$$

Since $(\alpha_j)_{j=1}^{l}$ is linearly independent,

$$b_{n+1} = ... = b_{n+j} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since $(x_i)_{i=1}^{n}$ is linearly independent,

$$b_1 = ...b_n = 0$$

(2) Let $y \in V$. Then $\pi(y) \in V/W$. So there exists

$$(c_{n+1}, ..., c_{n+l}) \in K^l$$

such that

$$\begin{aligned}\pi(y) &= \sum_{j=1}^{l} c_{n+j}\alpha_j \\ &= \sum_{j=1}^{l} c_{n+j}\pi(x_{n+j}) \quad = \pi(\sum_{j=1}^{l} c_{n+j}x_{n+j})\end{aligned}$$

So

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) \in W$$

$\exists c \in K^n$ such that

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) = (\sum_{i=1}^{n} c_i x_i)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

21.7 Corollary

Let K be a division ring and V be a left K-module of finite type. If $(x_i)_{i=1}^{n}$ is a linearly independent family of elements of V($n \in \mathbb{N}$), then

$$\exists\ l \in \mathbb{N} \quad \exists\ (x_{n+j})_{j=1}^{l} \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of V

Proof

Let W be the image of

$$\begin{aligned} \varphi_(x_i)_{i=1}^{n} :& K^n \to V \\ & (a_i)_{i=1}^{n} \mapsto \sum_{i=1}^{n} a_i x_i \end{aligned}$$

It's a left K-submodule of V.

Note that $(x_i)_{i=1}^{n}$ forms a basis of W.

$$\begin{aligned} \varphi_(x_i)_{i=1}^{n} :& K^n \to W \\ & \varphi_(x_i)_{i=1}^{n}(e_j) = x_j \in W \end{aligned}$$

Moreover ,since V is of finite type there exists $d \in \mathbb{N}$ and a surjective morphism of left K-modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \to V/W$$

is surjective.

Hence the composite morphism

$$\begin{tikzcd} K^d \arrow[r, "\psi"] \arrow[rr, bend right, "\pi\circ\psi"'] & V \arrow[r, "\pi"] & V/W \end{tikzcd}$$

is surjective. Thus V/W is of finite type. There exist then a basis

$$(a_j)_{j=1}^{l}$$

of V/W.

Taking $x_{n+j} \in \alpha_j$ for $j \in \{1, ..., l\}$, we get a basis of V:

$$(x_i)_{i=1}^{n+l}$$

21.8 Def

Let K be a division ring and V be a left K-module of finite type. We call rank of V the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If K is a field $rk(V)$ is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of V.

21.9 Theorem

Let K be a division ring and V be a left K-module of finite type. Let W be a left K-submodule of V.

(1) W and V/W are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of V has exactly $rk(V)$ elements

21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let $(x_i)_{i=1}^{n}$ be a basis of V. Let

$$\begin{aligned} \pi :&V \to V/W \\ &x \mapsto [x] \end{aligned}$$

In $(\pi(x_i))_{i=1}^n$ we extract a basis of V/W, say

$$(\pi(x_i))_{i=1}^l$$

For $j \in \{l+1, ..., n\}$,

$$\exists (b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^{l} b_{j,i}\pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^{l} b_{j,i}x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any $x \in W$,$\exists (a_i)_{i=1}^n \in K^n, x = \sum\limits_{i=1}^{n} a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^{l} a_i x_i + \sum_{j=l+1}^{n} a_j (y_j + \sum_{i=1}^{l} b_{j,i}x_i) \\ &= \sum_{j=l+1}^{n} a_j y_j + \sum_{i=1}^{l} (a_i + \sum_{j=l+1} a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^{l} (a_i + \sum_{j=l+1}^{n} a_j b_{j,i})\pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^{n} a_i y_i$$

Hence W is of finite type , and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let $(x_i)_{i=1}^n$ be a basis of V. Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of V/W.
We extract a subfamily , say $(x_i)_{i=1}^l$ such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of V/W.
For $j \in \{1, ..., l\}$, there exists:

$$(b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^{l} b_{j,i}\pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^{l} b_{j,i}x_i \in W$$

Let $x \in W$,$\exists (a_i)_{i=1}^n \in K^n$ let $x = \sum a_i x_i$,then

$$\begin{aligned} x &= (\sum_{i=1}^{l} a_i x_i) + (\sum_{j=l+1}^{n} a_j(y_j + \sum_{i=1}^{l} b_{j,i}x + i)) \\ &= (\sum_{i=1}^{l} a_i x_i) + (\sum_{i=1}^{l}\sum_{j=l+1}^{n} a_j b_{j,i} x_i) + (\sum_{j=l+1}^{n} a_j y_j) \\ &= \sum_{i=1}^{l}(a_i + \sum_{j=l+1}^{n} a_j b_{j,i})x_i + \sum_{j=l+1}^{n} a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^{l}(a_i + \sum_{j=l+1}^{n} a_j b_{j.i})\pi(x_i)$$

Therefore $(y_j)_{j=l+1}^n$ is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + tk(V/W)$$

(2) All basis of V have $rk(V)$ elements.

We reason by induction on $rk(V)$

(1)
$$rk(V) = 0$$
In this case $V = \{0\}$ The only basis of V is $\varnothing$. So the statement holds.

(2) Assume that there exists $e \in V \setminus \{0\}$ such that
$$V = \{\lambda e \mid \lambda \in K\}$$
Then any basis of V is of the form
$$ae$$
where $a \in K\{0\}$

Let $(e_i)_{i=1}^m$ be a basis of V. We reason by induction on m to prove that
$$m = rk(V)$$
The cases where $m = 0$ or 1 are proved in (1)(2) respectively. Induction hypothesis: true for a basis of $< m$ elements

Let
$$W = \{\lambda e_i \mid \lambda \in K\}$$
Let
$$\begin{aligned}\pi :&V \to V/W \\ &x \mapsto [x]\end{aligned}$$
Then
$$(\pi(e_i))_{i=1}^m$$
forms a system of generators of V/W.

If $(a_i)_{i=2}^m \in K^{m-1}$ such that
$$\sum_{i=2}^m a_i \pi(e_i) = 0$$
then
$$\sum_{i=2}^m a_i e_i \in W$$
Hence
$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$
And for $(e_i)_{i=1}^m$ a basis of V,
$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of V/W. We then obtain that

$$rk(V/W) \leq m-1 \leq n-1$$

By the induction hypothesis,

$$m-1 = rk(V/W)$$

By (2), $rk(W) = 1$. Hence

$$m = (m-1)+1 = rk(V/W) + rk(W) = rk(V)$$

21.11 Prop

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Let I be a set and $(x_i)_{i\in I} \in E^I$

- If $(x_i)_{i\in I}$ is linearly independent and f is injective, then $(f(x_i))_{i\in I}$ is linearly independent.
- If $(x_i)_{i\in I}$ is a system of generators and f is surjective, then $(f(x_i))_{i\in I}$ is a system of generators.
- If $(x_i)_{i\in I}$ is a basis and f is an isomorphism, then $(f(x_i))_{i\in I}$ is a basis.

21.11.1 Proof

$$\varphi_{(f(x_i))_{i\in I}} = f \circ \varphi_{(x)_{i\in I}}$$

Chapter 22

Matrices

We fix unitary ring K

22.1 Def

Let $n \in \mathbb{N}$ and V be a left K-module.

For any $(x_i)_{i=1}^n \in V^n$, we denote by $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ the morphism

$$\begin{aligned} \phi_{(x_i)_{i=1}^n} :& K^n \to V \\ (a_i)_{i=1}^n \mapsto & \sum_{i=1}^n a_i n_i \end{aligned}$$

22.1.1 Example

Suppose that $V = K^p (p \in \mathbb{N})$ Then each $x_i \in K^p$ is of the form $(x_{i,1}, ..., x_{i,p})$

Hence $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ can be written:

$$\begin{pmatrix} x_{1,1} & . & . & . & x_{1,p} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ x_{n,1} & . & . & . & x_{n,p} \end{pmatrix}$$

22.2 Def

Let $(n, p) \in \mathbb{N}^2$. We call n by p matrix of coefficient in K any morphism of left K-modules from K^n to K^p

22.2.1 Example

- Denote by I_n then identity mapping.Then $(e_i)_{i=1}^n$ is a basis of K^n called the canonical basis of K^n

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, ..., a_n)) = \sum_{i=1}^n a_i e_i = (a_1, ..., a_n)$$

- Let $(x_1, ..., x_n) \in K^n$, Denote by

$$\begin{aligned} diag(x_1, ...x_n)(= \varphi_{(x_i e_i)_{i=1}^n}) :& K^n \to K^n \\ & (a_1, ..., a_n) \mapsto (a_1 x_1, ..., a_n x_n) \end{aligned}$$

22.3 Def

We denote by $M_{n,p}(K)$ the set of all n by p matrices of coefficients in K. For $(n, p, r) \in \mathbb{N}^3$,we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) \to & \quad M_{n,r}(K) \\ (A, B) \mapsto & \quad AB := B \circ A \end{aligned}$$

22.4 Calculate Matrices

Let K be a unitary ring , and V be a left K-module. Let $n \in \mathbb{N}$ and

$$x = (x_1, ..., x_n) \in V^n$$

22.4.1 Remind

$$\begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \varphi : (a_1, ..., a_n) \mapsto a_1x_1, ..., a_nx_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1,...,p\} \times \{1,...,n\}} \in M_{p,n}(K)$$

A is a morphism of left K-modules from K^p to K^n Recall that

$$A \begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let $(b_1, ..., b_n) \in K^p$

$$\begin{aligned} A((b_1, ..., b_n)) &= \sum_{i=1}^{p} b_i(a_{i,1}, ..., a_{i,n}) \\ \varphi(A((b_1, ..., b_n))) &= \sum_{i=1}^{p} b_i \varphi_x((a_{i,1}, ..., a_{i,n})) \\ &= \sum_{i=1}^{p} b_i(a_{i,1}x_1, ..., a_{i,n}x_n) \end{aligned}$$

Let $B = \{b_{ij}\}_{(i,j) \in \{1,...,n\} \times \{1,...,r\}} : K^n \to K^r$

$$AB = \left\{ \sum_{j=1}^{n} a_{lj} b_{jm} \right\}_{(l,m) \in \{1,...,p\} \times \{1,...,r\}}$$

Chapter 23

Transpose

We fix a unitary ring K

23.1 Def

Let E be a left-K-module. Denote by

$$E^\vee := \{\text{morphisms of left K-modules } E \to K\}$$

$\forall (f,g) \in E^\vee$ let

$$\begin{aligned} f+g : & E \to K \\ & x \mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$ forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : & E \to K \\ & x \mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} & K \times E^\vee \to E^\vee \\ & (a, f) \mapsto fa : x \in E \to f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right K-module on $E^\vee$

23.2 Def

Let E and F be two left K-modules. $\varphi : E \to F$ be a morphism of left K-modules. We denote by

$$\varphi^\vee : F^\vee \to E^\vee$$

the morphism of right K-modules sending $g \in F^\vee$ to $g \circ \varphi \in E^\vee$
Actually $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

23.2.1 Example

Suppose that $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & . & . & . & b_{1,p} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ b_{n,1} & . & . & . & b_{n,p} \end{pmatrix}$$

φ sends $(a_1, ..., a_n)$ to $\{\sum_{i=1}^{n} a_i b_i j\}_{j \in \{1,...,p\}}$ Let $g \in F^\vee$ $\quad g : K^p \to K$, then g is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$ sends $(a_1, ..., a_n)$ to $\sum_{i=1}^{p}(\sum_{i=1}^{n} a_i b_{ij} y_j)$

Assume that K is commutative. We denote by

$$\iota_p : (K^p)^\vee \to K^p$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, ..., x_p)$$

$$\iota_n : (K^n)^\vee \to K^n$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, ..., x_n)$$

are isomorphisms of K-modules

For any morphism of K-modules $\varphi : K^n \to K^p$, we denote by φ^τ the morphism of K-modules $K^p \to K^n$ given by $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{tikzcd}
(K^p)^\vee \arrow[r, "\varphi^\vee"] \arrow[d, "\iota_p", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & (K^n)^\vee \arrow[d, "\cong"'] \\
K^p \arrow[r, "\varphi^\tau"'] & K^n
\end{tikzcd}$$

φ^τ is called the transpose of φ

23.3 Prop

Let E,F,G be left K-modules. $\varphi : E \to F, \psi : F \to G$ be morphisms of left K-modules. Then $(\psi \circ \varphi)^\vee$ is equal to $\varphi^\vee \circ \psi^\vee$

Proof

$\forall f \in G^\vee$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

23.4 Corollary

Assume that K is commutative. Let n, p, q be neutral numbers. $A \in M_{n,p}(K), B \in M_{p,q}(K)$. Then

$$(AB)^\tau = B^\tau A^\tau$$

Proof

$$A^tau = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^tau = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned}
B^\tau A^\tau &= A^\tau \circ B^\tau \\
&= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\
&= (AB)^tau
\end{aligned}$$

23.5 Remark

(1) For $A \in M_{n,p}(K)$, one has $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} &E \to (E^\vee)^\vee \\ &x \mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a K-linear mapping.

If K is a field and E is of finite dimension, this is a isomorphism of K-modules.

In fact, if $e = (e_i)_{i=1}^n$ is a basis of E over K.For $i \in \{1, ..., n\}$, let

$$\begin{aligned} &e_i^\vee : E \to K \\ &\lambda_1 e_1, ..., \lambda_n e_n \mapsto \lambda_i \end{aligned}$$

is called the dual basis of e

$$\begin{tikzcd} K^n \arrow[d, "\varphi_e"', "\cong"] \arrow[dr, "\varphi_{e^\vee}"] & (K^n)^\vee \arrow[l, "\cong"', "\iota_n"] \arrow[d, "\cong"', "\varphi_e^\vee"] \\ E \arrow[r, "\cong"'] & E^\vee \end{tikzcd}$$

$(e^\vee)^\vee$ gives a basis of $(E^\vee)^\vee$ Hence $E \to (E^\vee)^\vee$ is an isomorphism.

Chapter 24

Linear Equation

We fix a unitary ring K.

24.1 Def

For $a = (a_1, ..., a_n) \in K^n \setminus \{(0, ..., 0)\}$. Denote by $j(a)$ the first index $j \in \{1, ..., n\}$ such that $a_j \neq 0$.Let $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$. We write A as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, ..., a_n^{(i)}) \in K^p$$

We say that A is of row echelon form if,$\forall i \in \{1, ..., n-1\}$ one of following conditions is satisfied.

- $a^{(i+1)} = (0, ..., 0)$
- $a^{(i)}, a^{(i+1)}$ are non-zero, and $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, ..., n\}$ such that $a^{(i)} \neq (0, ..., 0)$,one has

  $$a^{(i)}_{j(a^{(i)})} = 1$$

  and

  $$\forall k \in \{1, ..., n\} \setminus \{i\} \quad a^{(k)}_{j(a^{(i)})} = 0$$

we say that A is of reduced row echelon form.

24.2 Prop

Suppose that $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$ is of row echelon form. Then $\{i \in \{1,...,n\} \mid a^{(i)} \neq (0,...,0)\}$ is of cardinal $\leq p$

Proof

Let $k = card\{i \in \{1,...,n\} \mid a^{(i)} \neq (0,...,0)\}$ $a^{(k+1)} = ... = a^{(n)} = (0,...,0)$ and $j(a^{(1)}) < j(a^{(2)}) < ... < j(a^{(k)})$ Hence

$$\{1,...,k\} \to \{1,...,p\}, i \mapsto j(a^{(i)})$$

is injection. So $k \leq p$

24.3 Linear Equation

Let $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$. Let V be a left K-module and $(b_1,...,b_n) \in V^n$. We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

The set of $(x_1,..,x_p) \in V^p$ that satisfies (*) is called the solution set of (*)

24.4 Prop

Suppose that A is of reduced row echelon form. Let

$$I(A) = \{i \in \{1,...,n\} \mid (a_{i,1},...,a_{i,p}) \neq (0,...,0)\}$$

$$J_0(A) = \{1,...,p\} \setminus \{j((a_{i,1},...,a_{i,p})) \mid i \in I(A)\}$$

- If $\exists i \in \{1,...,n\} \setminus I(A)$ such that $b_i \neq 0$ then (*) does not have any solution in K^n
- Suppose that $\forall i \in \{1,...,n\} \setminus I(A), b_i = 0$. Then (*) has at least one solution. Moreover
$$V^{J_0(A)} \to V^p$$
$$(z_k)_{k \in J_0(A)} \mapsto (x_1,...,x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum\limits_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, ..., a_{i,p})) \end{cases}$$

is an injective mapping , whose image is equal to the set of solution of (*)

24.5 Prop

Let $m \in \mathbb{N}, S \in M_{m,n}(K)$. If $(x_1, ..., x_p) \in V^p$ is a solution of (*), then $(x_1, ..., x_p)$ is a solution of $(*)_S$:

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

In the case where S is left invertible, namely there exist $R \in M_{n,m}(K)$ such that $RS = I_n \in M_{m,n}(K)$. Then (*) and $(*)_S$ have the same solution set.

24.6 Def

Let $G_n(K)$ be the set of $S \in M_{n,n}(K)$ that can be written as $U_1 ... U_N$(by convention $S = I_n$ where $N = 0$) where each U_i is of one of the following forms.

- P_σ where $\sigma \in \mathfrak{S}_n$
- $diag(r_1, ..., r_n)$ where each $r_i \in K$ is left invertible
- $S_{i,c}$ with $i \in \{1, ..., n\}$ $c = (c_1, ..., c_n) \in K^n, c_i = 0$

Let $p \in \mathbb{N}$,we say that $A \in M_{n,p}(K)$ is reducible by Gauss elimination if $\exists S \in G_n(K)$ such that SA is of reduced row echelon form

24.7 Theorem

Assume that K is a division ring $\forall (n,p) \in \mathbb{N}$ any $A \in M_{n,p}(K)$ is reducible by Gauss elimination

Proof

The case where $n = 0$ or $p = 0$ is trivial. We assume $n \geq 1, p \geq 1$ We write A as

$$\begin{pmatrix} \lambda_1 & \\ \vdots & B \\ \lambda_n & \end{pmatrix} \text{ where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If $\lambda_1 = ... = \lambda_n = 0$

  Applying the induction hypothesis to B, for $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 & \\ \vdots & SB \\ 0 & \end{pmatrix}$$

- Suppose that $(\lambda_1, ..., \lambda_n) \neq (0, ..., 0)$

  By permuting the rows we may assume $\lambda_1 \neq 0$. As K is division ring, by multiplying the first row by λ_1^{-1} , we amy assume $\lambda_1 = 1$. We add $(-\lambda_i)$ times the first row to the i^{th} row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 \dots & \mu_p \\ 0 & & \\ \vdots & C & \\ 0 & & \end{pmatrix} \qquad \begin{array}{l} C \in M_{n-1,p-1}(K) \\ (\mu_2, ..., \mu_p) \in K^{p-1} \end{array}$$

  Applying the induction hypothesis to C, we say assume that C is of reduced row echelon form . For $i \in \{2, ..., k\}$ we add $-\mu_{j(c_i)}$ times the i^{th} row of A to the first line to obtain a matrix of reduced row echelon form

Chapter 25

Normed Vector Space

25.1 Def

Let (X, d) be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is an element of $X^{\mathbb{N}}$ such that

$$\lim_{N\to+\infty} \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} d(x_n, x_m) = 0$$

we say that $(x_n)_{n\in N}$ is a Cauchy sequence. If any Cauchy sequence in X converges, then we say that (X, d) is complete.

Let $Cau(X, d)$ be the set of all Cauchy sequences in X. We define a binary relation $\sim$ on $Cau(X, d)$ as

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$$

iff

$$\lim_{n\to+\infty} d(x_n, y_n) = 0$$

25.2 Prop

$\sim$ is an equivalence relation.

25.2.1 Proof

$$\lim_{n\to+\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ be elements of $Cau(X, d)$. For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If
$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(y_n, z_n) = 0$$
then
$$\lim_{n\to+\infty} d(x_n, z_n) = 0$$

25.3 Def

$\hat{X} := Cau(X, d)\backslash \sim$

25.4 Def: The completion

The completion of (X, d) is defined as
$$Cau(X)/\sim$$
and is denoted as
$$\hat{X}$$

25.5 Theorem

The mapping
$$\begin{aligned} \hat{d} :&\hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n\to+\infty} d(x_n, y_n) \end{aligned}$$
is well defined, and it's a metric on $\hat{X}$

Proof

TO check that $\hat{d}$ is well defined , it suffices to prove that $\forall([x],[y]) \in \hat{X} \times \hat{X}$, $(d(x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy sequence and its limit doesn't depend on the choice of the representation x and y
For $N \in \mathbb{N}$ and $(n, m) \in \mathbb{N}_{\geq N}$ for
$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$
one has,
$$|dd(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$
then
$$\begin{aligned} \sup_{(n,m\in\mathbb{N}_{\geq N})} |d(x_n, y_n) - d(x_m, y_m)| \leq (&\sup_{(n,m\in\mathbb{N}_{\geq N})} d(x_n, x_m)) \\ +(&\sup_{(n,m\in\mathbb{N}_{\geq N})} d(y_n, y_m)) \end{aligned}$$

Taking $\lim_{N\to+\infty}$ we obtain that $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.
Hence it converges in $\mathbb{R}$. If $x' = (x'_n)_{n\in\mathbb{N}} \in [x], y' = (y'_n)_{n\in\mathbb{N}} \in [y]$, thus

$$\lim_{n\to+\infty} d(x_n, x'_n) = \lim_{n\to+\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking $\lim_{n\to+\infty}$ we get

$$\lim_{n\to+\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(x'_n, y'_n)$$

In the following, we check that $\hat{d}$ is a metric

- $\hat{d}([x], [y]) = 0$ iff $[x] = [y]$: trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$: trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$:

$$\begin{aligned} d([x], [y]) &= \lim_{n\to+\infty} \\ &\leq \lim_{n\to+\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

25.6 Remark

Let

$$\begin{aligned} i_X :& X \to \hat{X} \\ & a \mapsto [(a, a, ...)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular, i_x is injective(if $i_X(a) = i_X(b)$ then $d(a, b) = 0$ hence $a = b$)

25.7 Prop

$i_X(X)$ is dense in $\hat{X}$ (the closure of $i_X(X)$ in $\hat{X}$ is equal to $i_X(X)$ (or to say $\hat{X}$))

Proof

Let $[x]$ be an equivalence class in $\hat{X}$. We claim that $\forall (x_n)_{n\in\mathbb{N}} \in [x]$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} i_X(x_n)$$

For any $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n\to+\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} d(x_n, x_m) \end{aligned}$$

Taking $\lim\limits_{N\to+\infty}$ we get

$$\lim_{N\to+\infty} \hat{d}(i_X(x_N), [x]) = 0$$

25.8 Theorem

$(\hat{X}, \hat{d})$ is a complete metric space

Proof

Let $([x^{(N)}])_{N\in\mathbb{N}}$ be a Cauchy sequence in $\hat{X}$, where $\forall N \in \mathbb{N}$, $x^{(N)} = (x_n^{(N)})_{n\in\mathbb{N}}$ is a Cauchy sequence
$\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall (k,l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n\to+\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any $(\mu,\nu) \in \mathbb{N}_{\geq\alpha(N)}$
Let $y_N = x_{\alpha(N)}^{(N)}$. Without loss of generality , we assume that

$$\alpha(0) \leq \alpha(1) \leq ...$$

Let $\epsilon > 0$ Take $N_0 \in \mathbb{N}$ such that

(1) $\forall (k,l) \in \mathbb{N}$, $k,l \geq N_0$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

(2)

$$\frac{1}{N_0+1} \leq \frac{\epsilon}{3}$$

Let $(k,l) \in \mathbb{N}^2_{N_0}$,

$$d(y_k, y_l) = d(x^{(k)}_{\alpha(k)}, x^{(l)}_{\alpha(l)})$$

Since $\alpha(k) \geq N_0$, $\forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x^{(k)}_{\alpha(k)}, x^{(k)}_n) + d(x^{(k)}_n, x^{(k)}_n) + d(x^{(l)}_n, x^{(l)}_{\alpha(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x^{(k)}_n, x^{(l)}_n) \end{aligned}$$

Taking $\lim\limits_{n \to +\infty}$ get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So $y = (y_N)_{N \in \mathbb{N}}$ is a Cauchy sequence. We check that

$$\lim_{N \to +\infty} \hat{d}([x^{(N)}], [y]) = 0$$

$$\begin{aligned} 0 &\leq \limsup_{N \to +\infty} \lim_{n \to +\infty} d(x^{(N)}_n, x^{(N)}_{\alpha(n)}) \\ &\leq \lim_{N \to +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$d(x^{(N)}_n, y_n) \leq d(x^{(N)}_n, y_N) + d(y_n, y_N)$$

$$\limsup_{N \to +\infty} \lim_{n \to +\infty} d(x^{(N)}_n, y_n) \leq \limsup_{N \to +\infty} (\frac{1}{N+1} + \lim_{n \to +\infty} d(y_n, y_N))$$

Since y is Cauchy sequence

$$\leq \limsup_{N \to +\infty} \lim_{n \to +\infty} d(y_n, y_N) = 0$$

Example

Let $(K, |\cdot|)$ be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$ iff $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$Cau(K)$ forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \to +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in Cau_0(K)$$

where

$$Cau_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of $Cau(K)$
Hence

$$\hat{K} = Cau(K) \setminus Cau_0(K)$$

is a quotient ring of $Cau(K)$
$|\cdot|$ extend to $\hat{K}$:

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \to +\infty} |a_n|$$

that forms an absolute value.

Chapter 26

Norms

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\to \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

26.1 Def

Let V be a vector space over K (K-module). We call seminorm on V any mapping

$$\begin{aligned} \|\cdot\| : &V \to \mathbb{R}_{\geq 0} \\ s &\mapsto \|\cdot\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V,\ \|s\| = 0$ iff $s = 0$

We say that $\|\cdot\|$ is a norm and $(V, \|\cdot\|)$ is normed space over K.

26.2 Remark

If $\|\cdot\|$ is a norm then

$$\begin{aligned} d : &V \times V \to \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let $(V, \|\cdot\|)$ be a vector space over K equipped with a seminorm, and W be a vector space subspace of V (sub-K-module)

- The restriction of $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ to W forms a seminorm on W. It is a norm if $\|\cdot\|$ is a norm.
$$\begin{aligned} \|\cdot\|_W : &W \to \mathbb{R}_{\geq 0} \\ &x \mapsto \|x\| \end{aligned}$$
- The mapping
$$\begin{aligned} \|\cdot\|_{V/W} : &V/W \to \mathbb{R}_{\geq 0} \\ &\alpha \mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\| \end{aligned}$$
is a seminorm on V/W
Attention: Even if $\|\cdot\|$ is a norm, $\|\cdot\|_{V/W}$ **might only be a seminorm**

26.3 Def

$\|\cdot\|_{V/W}$ is called the quotient seminorm of $\|\cdot\|$

26.4 Prop

Let $(V, \|\cdot\|)$ be a vector space over K, equipped with a seminorm. Then
$$N = \{s \in V \mid \|s\| = 0\}$$
forms a vector subspace of V. Moreover, $\|\cdot\|_{V/N}$ is a norm

Proof

If $(a, s) \in K \times N$ then $\|as\| = |a| \cdot \|s\| = 0$ so $as \in N$
If $(s_1, s_2) \in N \times N$ then $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$ so $s_1 + s_2 \in N$

Proof

$$\begin{aligned} \|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W} \end{aligned}$$
Let $\alpha \in V/N$ such that $\|\alpha\|_{W/N} = 0$ Let $s \in \alpha, \forall t \in N$
$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$
$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$
Hence $\|\alpha\|_{V/N} = \|s\| = 0$ We obtain that $\alpha = N = [0]$

26.5 Def

Let $(V, \|\cdot\|)$ be a vector space over K,equipped with a seminorm. For any $x \in V$ and $r \geq 0$, we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$
$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

26.6 Remark

If $N = \{s \in V, \|s\| = 0\}$ then when $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

26.7 Def

We equip the topology such that $\forall U \subseteq V, U$ is open iff $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

26.8 Prop

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorms. Let $f : V_1 \to V_2$ be a K-linear mapping

- If f is continuous, $\forall s \in V_1$ if $\|s\|_1 = 0$ then $\|f(s)\|_2 = 0$
- If there exists $C > 0$ such that $\forall x \in V_1, \ \|f(x)\|_2 \leq C\|x\|_1$ then f is continuous.

  The converse is true

when $|\cdot|$ is non-trivial

or $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$ is of finite type

Proof

(1) Lemma If $(V, \|\cdot\|)$ is a vector space over K, equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let $s \in V \setminus N_{\|\cdot\|}$ Then $\|s\| > 0$.Let $\epsilon = \frac{\|s\|}{2}, \ \forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq |\|s\| - \|s - x\|| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subsetneqq V \setminus N_{\|\cdot\|}$$

- Then $f^{-1}(N_{\|\cdot\|_2})$ is closed.
  Note that
  $$0 \in f^{-1}(N_{\|\cdot\|_2})$$
  hence
  $$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$
  $\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$
  $$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$
  and
  $$0 \in \mathcal{B}(x, \epsilon)$$
  Therefore $x \in \overline{\{0\}}$

(2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of V_1 that converges to some $x \in V_1$
Hence
$$\begin{aligned}\limsup_{n\to+\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n\to+\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n\to+\infty} C\|x_n - x\|_1 \\ &= C \limsup_{n\to+\infty} \|x_n - x\| \\ &= 0\end{aligned}$$
So $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. Hence f is continuous at x
Assume that $|\cdot|$ is non-trivial and f is continuous.Then
$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
is an open subset of V_1 containing $0 \in V_1$
So there exists $\epsilon > 0$ such that
$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
namely $\forall x \in V_1$ if $\|x\|_1 < \epsilon$ then $\|f(x)\|_2 < 1$
Since $|\cdot|$ si nontrivial, $\exists a \in K,\ 0 < |a| < 1$ We prove that $\forall x \in V_1$
$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|}\|x\|_1$$
If $\|x\|_1 = 0$ by (1) we obtain
$$\|f(x)\|_2 = 0$$
Suppose that $\|x\|_1 > 0$ then $\exists n \in \mathbb{Z}$ such that
$$\|a^n x\|_1 = |a|^n \|x\|_1 < \epsilon \leq \|a^{n-1}x\|_1 = |a|^{n-1}\|x\|_1$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned}\|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}}\frac{1}{|a|} \\ &\leq \frac{1}{\epsilon}\|x\|_1\frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|}\end{aligned}$$

26.9 Def: Operator Seminorm

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorm. We say that a K-linear mapping $f : V_1 \to V_2$ is bounded if there exists $C > 0$ that

$$\forall x \in V_1 \quad \|f(x)\| \leq C\|x\|_1$$

For a general K-linear mapping $f : V_1 \to V_2$ we denote

$$\|f\| := \begin{cases} \sup\limits_{x\in V_1, \|x\|_1 > 0} (\frac{\|f(x)\|_2}{\|x\|_1}) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \nsubseteq N_{\|\cdot\|_2}) \end{cases}$$

f is bounded iff

$$\|f\| < +\infty$$

$\|f\|$ is called the operator seminorm of f

We denote by $\mathscr{L}(V_1, V_2)$ the set of all bounded K-linear mappings from V_1 to V_2

26.10 Prop

$\mathscr{L}(V_1, V_2)$ is a vector subspace of $Hom_K(V_1, V_2)$. Moreover $\|\cdot\|$ is a seminorm on $\mathscr{L}(V_1, V_2)$

Proof

Let f, g be elements of $\mathscr{L}(V_1, V_2)$

$$\begin{aligned}\|f+g\| &= \sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)+g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq (\sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1}) + (\sup_{x\in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1}) \\ &\leq +\infty\end{aligned}$$

Hence $f+g \in \mathscr{L}(V_1, V_2)$

Let $\lambda \in K,\ \lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

26.11 Remark

Let $f \in \mathscr{L}(V_1, V_2)$. Suppose that $\exists x \in V_1$ such that $f(x) \neq 0$. Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore $\|\cdot\|$ is a norm

26.12 Def

Let $(V, \|\cdot\|)$ be a normed vector space. If V is complete with respect to the metric

$$\begin{aligned}d :& V \times V \to \mathbb{R}_{\geq 0} \\ & (x, y) \mapsto \|x - y\|\end{aligned}$$

then we say that $(V, \|\cdot\|)$ is a Banach space.

26.13 Theorem

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorm. If $(V_2, \|\cdot\|_2)$ is a Banach space, then

$$(\mathscr{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

Proof

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(V_1, V_2)$.
$\forall x \in V_1$, the mapping

$$(f \in \mathscr{L}(V_1, V_2)) \mapsto f(x)$$

is $\|x\|_1$-Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f-g)(x)\|_2 \leq \|f-g\|\|x\|_1$$

So $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, for V_2 is complete, that converges to some $g(x) \in V_2$ Then we obtain a mapping $g : V_1 \to V_2$.We prove that g is an element of $\mathscr{L}(V_1, V_2)$

- $\forall (x,y) \in V_1^2$

$$g(x,y) = \lim_{n\to+\infty} f_n(x+y) = \lim_{n\to+\infty} f_n(x) + f_n(y)$$

$$\begin{aligned}\|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \to +\infty)\end{aligned}$$

So

$$\lim_{n\to+\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n\to+\infty} f_n(\lambda x) = \lim_{n\to+\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1)(n \to +\infty)$$

So $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n\to+\infty} \|f_n(x)\| \leq (\lim_{n\to+\infty} \|f_n\|) \cdot \|x\|$$

(because $\forall (a,b) \in V_2^2 \quad |\|a\| - \|b\|| \leq \|a-b\|$)Then

$$|\|f_n(x)\| - \|g_n(x)\|| \leq \|f_n(x) - g_n(x)\| = o(1) \ (n \to +\infty)$$

So $g \in \mathscr{L}(V_1, V_2)$

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall (n,m) \in \mathbb{N}_{\geq N}, \ \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking $\lim\limits_{n\to+\infty}$ we get

$$\|(f_n - g)(x)\| \leq \epsilon \, \|x\|$$

So $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$

Chapter 27

Differentiability

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$K \times K \to \mathbb{R}_{\geq 0}$$
$$(a, b) \mapsto |a - b|$$

27.1 Def

Let X be a topological space and $p \in X$. Let K be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K.

Let $f : X \to E$ be a mapping and $g : X \to \mathbb{R}_{\geq 0}$ be a non-negative mapping.

- We say that

  $$f(x) = O(g(x)) \ x \to p$$

  if there is a neighborhood V of p in X and a constant $C > 0$ such that $\forall x \in V$

  $$\|f(x)\| \leq Cg(x)$$

- We say that

  $$f(x) = o(g(x)) \ x \to p$$

  if there exists a neighborhood V of p in X and a mapping $\epsilon : V \to \mathbb{R}_{\geq 0}$ such that

  $$\lim_{x \in V, x \to p} \epsilon(x) = 0$$

  which is equivalent to

  $$\forall \delta > 0, \exists \text{ neighborhood U of } p \ U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$

  and $\forall x \in V$

  $$\|f(x)\| \leq \epsilon(x)g(x)$$

27.2 Def

Let E and F be normed vector space over K $U \subseteq E$ be an open subset, $f : U \to F$ be a mapping and $p \in U$ If there exists $\varphi \in \mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|) \quad x \to p$$

we say that f is differentiable at p, and φ is the differential of φ at p Suppose that $|\cdot|$ is not trivial.$\varphi(x-p)$ also written as

$$d_p f$$

Reminder

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|) \quad x \to p$$

means there exists an open neighborhood V of p with $V \subseteq U$ and a mapping $\epsilon V \to \mathbb{R}_{\geq 0}$ such that $\lim_{x \to p} \epsilon(x) = 0$ and that $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x-p)\| \leq \epsilon(x) \cdot \|x-p\|$$

27.3 Prop

If f is differentiable at p ,then its differential at p is unique

Proof

Suppose that there exists φ and ψ in $\mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|)$$

$$f(x) = f(p) + \psi(x-p) + o(\|x-p\|)$$

then

$$(\varphi - \psi)(x-p) = o(\|x-p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|(\| \varphi - \psi)}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y-p\| \leq \delta} \frac{\|\varphi - \psi\|(y-p)}{\|y-p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y-p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \to p} \epsilon(y) = 0 \end{aligned}$$

27.4 Example

27.4.1

$$f : U \to F : \; f(x) = y_0 \; \forall x \in U$$

$\forall p \in U$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence $\forall x \in E$

$$d_p(f(x)) = 0$$

27.4.2

Let $f \in \mathscr{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence $d_p f = f$

27.4.3

$$\begin{aligned} A :& E \times E \to E \\ & (x, y) \mapsto x + y \end{aligned}$$

Let E be a normed space. Then $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

27.4.4

$$\begin{aligned} m :& K \times E \to E \\ & (\lambda, x) \mapsto \lambda x \end{aligned}$$

Let $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when $(\lambda, x) \to (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|lambda - a| , \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

  is a K-linear mapping.

- $(\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2)$
- $b\mu p + a(by) = b(\mu p + ay)$
- $$\begin{aligned} \|\mu p + ay\| &\leq |\mu| \|p\| + |a| \|y\| \\ &\leq \max\{|\mu|, \|y\|\}(|a| + \|p\|) \end{aligned}$$

Hence m is differentiable and $\forall(\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

27.5 Theorem:Chain rule

Let E, F, G be normed vector spaces, $U \subseteq E, V \subseteq F$ be open subsets.

Let $f : U \to F,\ g : V \to G$ be mappings such that $f(U) \subseteq V$ Let $p \in U$ Assume that f is differentiable at p and g differentiable at $f(p)$ Then $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

Proof

Let $x \in U$ By definition

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ f(x) - f(p) &= O(\|x - p\|) \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|) \end{aligned}$$

So $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

27.6 Prop

Let n be a positive integer $E, (F_i)_{i \in \{1,\ldots,n\}}$ be normed vector spaces over K. $U \subseteq E$ an open subset, $p \in U$

$\forall i \in \{1, ..., n\}$ let $f_i : U \to F_i$ be a mapping. Let

$$f : U \to F = \prod F_i$$

be the mapping that sends $x \in U$ ot $(f_i(x))_{i \in \{1,\ldots,n\}}$ We equip F with the norm $\|\cdot\|$ defined as :

$$\left\|(y_i)_{i \in \{1,\ldots,n\}}\right\| = \max_{i \in \{1,\ldots,n\}} \|y_i\|$$

Then f is differentiable at p iff each f_i is differentiable at p. Moreover, when this happen, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i\in\{1,\dots,n\}}$$

Proof

$\Leftarrow$ Suppose that $(f_i)_{i\in\{1,\dots,n\}}$ are differentiable at p

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i\in\{1,\dots,n\}} \\ &= (d_p f_i(x-p))_{i\in\{1,\dots,n\}} + o(\|x-p\|) \end{aligned}$$

Therefore f is differentiable at p and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i\in\{1,\dots,n\}}$$

$\Rightarrow$ Let

$$\begin{aligned} \pi_i : & F \to F_i \\ & (x_i)_{i\in\{1,\dots,n\}} \mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has $\|\pi_i\| \leq 1$ because

$$\|x_i\| \leq \max_{i\in\{1,\dots,n\}} \|x_i\| = \left\|(x_i)_{i\in\{1,\dots,n\}}\right\|$$

π_i is differentiable at p then $\pi_i \circ f = f_i$ is differentiable at p

27.7 Def

Let U be an open subset of K and $(F, \|\cdot\|)$ be a normed vector space. If $f : U \to F$ is a mapping that is differentiable at some $p \in U$. We denote by $f'(p)$ the element

$$d_p f(1) \in F$$

called the derivative of f at p

27.8 Corollary

Let U and V be open subsets of K, $(F, \|\cdot\|)$ be a normed vector space over K. $f : U \to K,\ g : V \to F$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$ then

$$(g \circ f)'(p) = f'(p) g'(f(p))$$

Proof

By definition

$$\begin{aligned} d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p) \cdot 1) \\ &= f'(p) \cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

27.9 Corollary

Let E and F be normed vector spaces, $U \subseteq E$ an open subset. $f : U \to L$ and $g : U \to F$ be mappings and $p \in U$ If both f, g differentiable at p then

$$\begin{aligned} fg : &U \to F \\ &x \mapsto f(x)g(x) \end{aligned}$$

is also differentiable at p and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_pf(l) + g(p)d_pf(l)$$

Proof

Consider

$$\begin{aligned} m : &K \times F \to F \\ &(a, y) \to ay \end{aligned}$$

We have shown m is differentiable and

$$d_{a,y}m(b, z) = by = az$$

fg is the following composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "fg"'] & K \times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m \circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p),g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

27.10 Corollary

Let U be an open subset of K, f,g be mappings from U to K and to a normed space F respectively.If f,g are differentiable at $p \in U$ then

$$(fg)'(p) = d_p(fg)(1) = d_pf(1)g(p) + f(p)d_pg(1) = f'(p)g(p) + f(p)g'(p)$$

Example

$$\begin{aligned} f_n : & K \to K \\ & x \mapsto x^n \end{aligned}$$

is differentiable at any $x \in K$

$$f_n'(x) = nx^{n-1}$$

Proof

$f_1 : K \to K$ is differentiable $\forall x \in K$

$$d_xf_1 = f_1$$

If $f_n'(x) = nx^{n-1}$ then

$$\begin{aligned} f_{n+1}'(x) &= (f_nf_1)'(x) \\ &= f_n(x)f_1'(x) + f_n'(x)f_1(x) \\ &= x^n + x_n'(x) = x^n + xnx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_xf_n(l) &= ld_xf_n(1) \\ &= lnx^{n-1} \end{aligned}$$

27.11 Prop

Let E,F,G be normed vector spaces. $U \subseteq E$ be an open subset, $\varphi \in \mathscr{L}(F,G), p \in U$ if $f : U \to E$ is differentiable at p then so is $\varphi \circ f$. Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

Proof

φ is differentiable at $f(p)$ nad $d_{f(p)}\varphi = \varphi$

27.12 Corollary

Let E and F be normed vector spaces $U \in E$ be an open subset, $p \in U$. Let $f : U \to F$ and $g : U \to F$ be mappings that are differentiable at $p, (a, b) \in K \times K$. Then $af + bg$ is differentiable at p and

$$d_p(af_b g) = ad_p f + bd_p g$$

Proof

$af + bg$ is composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "ay+bz"'] & K \times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

27.13 Def: Equivalence of Norms

Let E be a vector space over K and $\|\cdot\|_1, \|\cdot\|_2$ be norms on E. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist constants $C_1, C_2 > 0$ such that $\forall s \in E$

$$C_1 \|S\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

27.14 Prop

If $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover $\|\cdot\|_1, \|\cdot\|_2$ defines the same topology on E.

Proof

$$\|s\|_2 \leq C_2 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are continuous. So $\forall$ open subset U of $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in $(E, \|\cdot\|_1)$. Conversely if V is open in $(E, \|\cdot\|_1)$ then

$$V = Id_E^{-1}(V)$$

is open in $(E, \|\cdot\|_2)$

27.15 Remark

If $\|\cdot\|_1, \|\cdot\|_2$ are two norms on E that define the same topology on E, then they are equivalent (under the assumption that $|\cdot|$ is not trivial)

27.16 Prop

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces $\|\cdot\|'_E$ and $\|\cdot\|'_F$ be norms on E and F that are equivalent to $\|\cdot\|_E, \|\cdot\|_F$ respectively. Let $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping.

Let $p \in U$ Then f is differentiable at p with respect to $\|\cdot\|_E$ and $\|\cdot\|_F$ iff it's differentiable with respect to $\|\cdot\|'_E$ and $\|\cdot\|'_F$

Moreover the differentiable of f at p is not changed in the change of norms from $(\|\cdot\|_E, \|\cdot\|_F)$ to $(\|\cdot\|'_E, \|\cdot\|'_F)$

Proof

$$\begin{tikzcd} U \arrow[r, "Id_U"] \arrow[rrr, bend right, "f"'] & U \arrow[r, "f"] & F \arrow[r, "Id_F"] & F \end{tikzcd}$$

$(E, \|\cdot\|'_E)$ $\quad$ $(E, \|\cdot\|_E)$ $\quad$ $\|\cdot\|_F$ $\quad$ $\|\cdot\|'_F$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$d'_p f : (E, \|\cdot\|'_E) \to (F, \|\cdot\|'_F)$

27.17 Theorem

Let V be a finite dimensional vector space over K. Then all norms on V are equivalent. Moreover V is complete with respect to any norm on V.

Proof

Let $(e_i)_{i=1}^n$ be a basis of V(linear independent system of generators)The the mapping:

$$V \to \mathbb{R}_{\geq 0}$$
$$\sum_{i\in\{1,\ldots,n\}} a_i e_i \mapsto \max_{i\in\{1,\ldots,n\}}\{|a_i|\}$$

is a norm on V

Let $\|\cdot\|$ be another norm on V. One has

$$\begin{aligned}\left\|\sum_{i\in\{1,\ldots,n\}} a_i e_i\right\| &\leq \sum_{1\in\{1,\ldots,n\}} |a_i|\,\|e_i\| \\ &\leq (\sum_{i\in\{1,\ldots,n\}} \|e_i\|)\max_{i\in\{1,\ldots,n\}}\{|a_i|\}\end{aligned}$$

We reason by induction that there exists $C > 0$ such that

$$\max_{i\in\{1,\ldots,n\}}\{|a_1|\} \leq C\left\|\sum_{i\in\{1,\ldots,n\}} a_i e_i\right\|$$

The case where $n = 0$ is trivial.

n=1
$$\|a_1e_1\| = |a_1|\,\|e_1\| \quad |a_1| = \|e_1\|^{-1}\cdot\|a_1e_1\|$$

Induction hypothesis true for vector spaces of dimension $< n$

Let
$$W = \{\sum_{i\in\{1,\ldots,n-1\}} a_i e_i \mid (a_i)_{i\in\{1,\ldots,n-1\}} \in K^{n-1}\}$$

equipped with $\|\cdot\|$ restricted to W

The induction hypothesis shows that W is complete. Hence it's closed in V. Let $Q = V/W$ and $\|\cdot\|_Q$ be the quotient norm on Q that's defined as

$$\forall\alpha\in Q \quad \|\alpha\|_Q = \inf_{s\in\alpha}\|s\|$$

- If $s \in V\setminus W, \exists\epsilon > 0$ such that

$$\overline{B}(s,\epsilon)\cap W = \phi$$

$\forall t \in W$,
$$s + t \notin \overline{B}(0,\epsilon)$$

since otherwise
$$-t \in W \cap \overline{B}(s,\epsilon)$$

Therefore
$$\|[s]\|_Q = \inf_{i\in W}\|s+t\| \geq \epsilon > 0$$

- $\forall \lambda \in K$
$$\|\lambda\alpha\|_Q = \inf_{s\in\alpha} \|\lambda s\| \quad = |\lambda|$$
$$\inf_{s\in\alpha} \|s\| = |\lambda| \cdot \|s\|_Q$$
-
$$\begin{aligned}\|\alpha+\beta\|_Q &= \inf_{s\in\alpha+\beta} \|s\| \\ &= \inf_{(x,y)\in\alpha\times\beta} \|x+y\| \\ &\le \inf_{(x,y)\in\alpha\times\beta} (\|x\|+\|y\|) \\ &= \inf_{x\in\alpha} \|x\| + \inf_{y\in\beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some $A > 0$ such that $\forall (a_i)_{i\in\{1,\dots,n-1\}} \in K^{n-1}$
$$\max_{i\in\{1,\dots,n-1\}} \{|a_i|\} \le A \left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\|$$

Take
$$s = \sum_{i\in\{1,\dots,n\}} a_i e_i \in V$$

Let $\alpha = [s] = a_n[e_n] \in Q$
$$\left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \le \|s\| + |a_n| \cdot \|e_n\| \le \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$
$$\|\alpha\|_Q = |a_n| \, \|[e_n]\|_Q = |a_n| \inf_{t\in W} \|e_n + t\|$$

Take $e'_n \in V$ such that $[e'_n] = [e_n]$ and $\|e'_n\| \le \|[e_n]\|_Q + \epsilon$
Note that $(e_1, ..., e_{n-1}, e'_n)$ forms also basis of V over K. Hence by replacing e_n by e'_n we may assume that $\|e_n\| \le \|[e_n]\|_Q + \epsilon$
$s = a_n e_n + t \in V$ with $t \in W$
$$\|s\| \ge \|[a_n e_n]\|_Q = |a_n| \, \|[e_n]\|_Q \ge B^{-1} |a_n| \cdot \|e_n\|$$

- If $\|a_n e_n\| < \frac{1}{2} \|t\|$
$$\|s\| \ge \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \ge \frac{1}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$
- If $\|a_n e_n\| \ge \frac{1}{2} \|t\|$
$$\|s\| \ge B^{-1} |a_n| \cdot \|e_n\| \ge \frac{B^{-1}}{2} \|t\| \ge \frac{B^{-1}A}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

We take $C = \max\{B^{-1} \|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$ Then

$$\|s\| \geq C \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Another proof

completeness Under the norm $\max\limits_{i\in\{1,...,n\}}$, a sequence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ is a Cauchy sequence iff $\forall i \in \{1, ..., n\} \quad (a_i^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence. Since K is complete each $(a_i^{(k)})_{k\in\mathbb{N}}$ converges to some $a_i \in K$ Hence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ converges.

27.18 Prop

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed vector spaces over K. Assume that E is finite dimensional. Then any K-linear mapping $\varphi : E \to F$ is bounded.

Proof

Let $(e_i)_{i=1}^n$ be a basis of E. For any two norms on E are equivalent.
$\forall (a_1, ..., a_n) \in K$

$$\left\|\sum_{i=1}^n a_ie_i\right\|_E = \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Then for any $s = \sum\limits_{i=1}^n a_ie_i$

$$\|\varphi(s)\|_F = \left\|\sum_{i=1}^n a_ie_i\right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq (\sum_{i=1}^n \|\varphi(e_i)\|_F) \|s\|_E$$

Chapter 28

Compactness

28.1 Def: cover

Let X be a topological space, $Y \subseteq X$ we call open cover of Y any family $(U_i)_{i \in I}$ open subset of X such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If I is finite set, we say that $(U_i)_{i \in I}$ iss a finite open cover. If $J \subseteq I$ such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that $(U_j)_{j \in J}$ is a sub cover of $(U_i)_{i \in I}$

28.2 Def: compact

If any open cover of Y has a finite subcover ,we say that Y is quasi-compact. If in addition X is Hausdorff, namely $\forall (x, y) \in X \times X$ with $x \neq y$ $\exists$ open neighborhoods U and V of x and y such that $U \cap V = \varnothing$, we say that Y is compact

28.3 Def

Let X be a set and $\mathcal{F}$ be a filter on X. If there does not exists any filter $\mathcal{F}'$ of X such that $\mathcal{F} \subsetneq \mathcal{F}'$, then we say that $\mathcal{F}$ is an ultrafilter.

Zorn's lemma implies that $\forall \mathcal{F}_,$ of X there exist an ultrafilter $\mathcal{F}$ if X containing $\mathcal{F}_0$

28.4 Prop

Let $\mathcal{F}$ be a filter on a set X. The following statements are equivalent.

(1) $\mathcal{F}$ is an ultrafilter

(2) $\forall A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$

(3) $\forall (A, B) \in \wp(X)^2$ if $A \cap B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$

Proof

(1) $\Rightarrow$ (2) Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$ $\forall B \in \mathcal{F}$ one has

$$B \cap A \neq \varnothing$$

since otherwise $B \subseteq X \setminus A$ and hence $X \setminus A \in \mathcal{F}$ contradiction.

(2) $\Rightarrow$ (3) Suppose that $B \notin \mathcal{F}$ then $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So $A \in \mathcal{F}$

(3) $\Rightarrow$ (1) Suppose that $\mathcal{F}'$ is a filter such that $\mathcal{F} \subsetneqq \mathcal{F}'$ Take $A \in \mathcal{F}' \setminus \mathcal{F}$ Then by $X = A \cup (X \setminus A) \in \mathcal{F}$ Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \varnothing = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

28.5 Theorem

Let $(X, \mathcal{J})$ be a topological space . The following are equivalent

(1) X is quasi-compact

(2) Any filter of X has an accumulation point

(3) Any ultrafilter of X is converges.

Proof

(1) $\Rightarrow$ (2) Assume that a filter $\mathcal{F}$ of X does not have any accumulation point. $\forall x \in X$ $\exists A_x \in \mathcal{F}$ $\exists$ open neighborhood V_x of x such that $A_x \cap V_x = \varnothing$ Since $X = \bigcup\limits_{x \in X} V_x$ there is

$$\{x_1, ..., x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^{n} V_{x_i}$$

Take $B = \bigcap_{i=1}^{n} A_{x_i} \in \mathcal{F}$

$$B \cap X = B = \varnothing$$

Since $\forall i \ B \cap V_x = \varnothing$ contradiction.

(2) $\Rightarrow$ (3) Let $\mathcal{F}$ be an ultrafilter of X. By (2) there exist $x \in X$ such that $\mathcal{F} \cup \mathcal{V}_x$ generates a filter $\mathcal{F}'$ Since $\mathcal{F}$ is an ultrafilter $\mathcal{F} = \mathcal{F}'$ and hence $\mathcal{V}_x \subseteq \mathcal{F}$

(3) $\Rightarrow$ (1) Let $(U_i)_{i \in I}$ be an open cover of X we suppose that this have no finite subcover. $\forall i \in I$ let

$$F_i = X \setminus U_i$$

For any $J \subseteq I$ finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \varnothing$$

Let $\mathcal{F}$ be the smallest filter on X that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let $\mathcal{F}'$ be ultrafilter containing $\mathcal{F}$. It has a limit point x There exist $i \in I$ such that $x \in U_i$.Since U_i is a neighborhood of x and $\mathcal{V}_x \subseteq \mathcal{F}'$ we get $U_i \in \mathcal{F}'$ This is impossible since $F_i \in \mathcal{F}'$

28.6 Theorem

Let (X, d) be a metric space. The following statements are equivalent:

(1) X is complete and $\forall \epsilon > 0 \ \exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

(2) X is compact

Proof

(1) $\Rightarrow$ (2) Let $\mathcal{F}$ be an ultrafilter Let $\epsilon > 0$ and $\{x_1, ..., x_n\} \subseteq X$ such that

$$X = \bigcup_{i=1}^{n} \mathcal{B}(x, \epsilon)$$

There exists some $i \in \{1, ..., n\}$ such that $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$ That means $\mathcal{F}$ is a Cauchy filter (namely $\forall \delta > 0 \ \exists A \in \mathcal{F}$ of diameter $\leq \delta$) Since X is complete $\mathcal{F}$ has a limit point. So $\mathcal{F}$ is compact.

(2) $\Rightarrow$ (1) Let $\epsilon > 0$ One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since X is compact $\exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$$\begin{aligned}
&\mathcal{F} \text{ is an ultrafilter} \\
\Leftrightarrow &\forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F} \\
\Leftrightarrow &\forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F} \\
\Leftrightarrow &\forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup ... \cup A_n \ \exists i \in \{1, ..., n\}, A_i \in \mathcal{F}
\end{aligned}$$

Let $\mathcal{F}$ be a Cauchy filter Let $x \in X$ be an accumulation point of $\mathcal{F}$ $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$ with diameter $\leq \frac{\epsilon}{2}$ Note that $A \cup \mathcal{B}x, \frac{\epsilon}{2} \neq \varnothing$ Take $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2})$ $\forall z \in A$

$$\begin{aligned}
d(x, z) &\leq d(x, y) + d(y, z) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore $A \subseteq \mathcal{B}(x, \epsilon)$ So $\mathcal{B}(x, \epsilon) \in \mathcal{F}$ This implies $\mathcal{V}_x \subseteq \mathcal{F}$

28.7 Lemma

Let (X, d) be a metric space

(1) Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point $\mathcal{F}$ a limit point of $\mathcal{F}$

(2) X is complete iff any Cauchy filter of X has a limit point

Proof

(1)

- Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point of $\mathcal{F}$ is a limit point of $\mathcal{F}$

(2) Suppose that X is complete.Let $\mathcal{F}$ be a Cauchy filter.$\forall n \in \mathbb{N}_{\geq 1}$ let $A_n \in \mathcal{F}$ such that $diam(A_n) \leq \frac{1}{n}$ Take $x_n \in \bigcap_{k=1}^{n} A_k \in \mathcal{F}$ Then $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ is a Cauchy sequence since $\forall \epsilon > 0$ if we take $N \in \mathbb{N}$ with $\frac{1}{N} \leq \epsilon$ then $\forall (n, m) \in \mathbb{N}_{\geq N}$ $d(x_n, x_m \leq \frac{1}{N})$ Hence $(x_n)_{N \in \mathbb{N}_{\geq 1}}$ converges to some $x \in X$ Note that x is an limit point of $\mathcal{F}$ since $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$ with $A_n \subseteq \mathcal{B}(x, \epsilon)$ It suffices to take n such that $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N},\ \{x_N, x_{N+1}, ...\} \subseteq A\}$$

This is a Cauchy filter on X since

$$\lim_{N\to+\infty} diam\{x_N, x_{N+1}, ...\} = 0$$

Hence $\mathcal{F}$ has a limit point $x \in X$ By definition $\forall U \in \mathcal{V}_x\ \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, ...\} \subseteq U$$

So $x = \lim_{n\to+\infty} x_n$

28.8 Prop

Let $f : X \to Y$ be a continuous mapping of topological spaces. If $A \subseteq X$ is quasi-compact then $f(A) \subseteq Y$ is also quasi-compact.

Proof

Let $(V_i)_{i\in I}$ be an open cover of $f(A)$ Then

$$(f^{-1}(V_i))_{i\in I}$$

is an open cover of A So $\exists J \subseteq I$ such that

$$A \subseteq \bigcup_{j\in J} f^{-1}(V_i)$$

This implies

$$f(A) \subseteq \bigcup_{j\in J} V_j$$

So $f(A)$ is quasi-compact.

28.9 Prop

Let X be a topological space and $A \subseteq X$ be a quasi-compact subset. For any closed subset F of X $A \cap F$ is quasi-compact.

Proof

Let $(U_i)_{i\in I}$ be an open cover of $A \cap F$.Then

$$A \subseteq (\bigcup_{i\in I} U_i) \cup (X \setminus F)$$

Since A is quasi-compact there exist $J \subseteq I$ finite such that

$$A \subseteq (\bigcup_{j \in J} U_j) \cup (X \setminus F)$$

Hence $A \cap F \subseteq \bigcup_{j \in J} U_j$

28.10 Prop

Let X be a Hausdorff topological space. Any compact subset A of X is closed.

Proof

Let $x \in X \setminus A$ $\forall y \in A, \exists$ open subsets U_y nad V_y such that $y \in U_y, x \in V_y$ and $U_y \cap V_y = \varnothing$ Since $A \subseteq \bigcup_{y \in A} U_y$ $\exists \{y_1, ..., y_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^{n} U_{y_i}$$

Let

$$U = \bigcup_{i=1}^{n} U_{y_i} \quad V = \bigcap_{i=1}^{n} V_{y_i}$$

These are open subset Moreover $A \subseteq U, x \in V$ and $U \cap V = \bigcup_{i=1}^{n} (U_{y_i} \cap V) = \varnothing$
In particular $x \in V \subseteq X \setminus A$ So $X \setminus A$ is open

28.11 Prop

Let X be a Hausdorff topological space and A and B be compact subsets of X such that $A \cap B = \varnothing$ Then there exists open subsets U and V such that

$$A \subseteq U, B \subseteq B and U \cap V = \varnothing$$

proof

We have seen in the proof of the previous proposition that $\forall x \in B, \exists U_x, V_x$ open such that $A \subseteq U_x, x \in V_x$ and $U_x \cap V_x = \varnothing$ Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, ..., x_m\} \subseteq B$ such that

$$B \subseteq \bigcup_{i=1}^{n} V_{x_i}$$

We take

$$U = \bigcap_{i=1}^{m} U_{x_i} \quad V = \bigcup_{i=1}^{m} U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq U \quad U \cap V = \varnothing$$

28.12 Theorem

Let $(X, \mathcal{J})$ be a Hausdorff topological space If $(A_n)_{n\in\mathbb{N}}$ is a sequence of non-empty compact subsets of X such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq ...$$

Then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

Proof

Suppose that

$$\bigcap_{n\in\mathbb{N}} A_n = \varnothing$$

then

$$A_0 \subseteq \bigcup_{n\in\mathbb{N}} (X \setminus A_n)$$

Since A_0 is compact, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^{N} (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^{N} A_n \\ &= X \setminus A_n \end{aligned}$$

So

$$A_n = \varnothing$$

28.13 Def

Let (X, τ) be a topological space if any sequence in X has a convergent subsequence, we say that X is sequentially compact.

Example

By Bolzano-Weierstrass, any bounded sequence in $\mathbb{R}$ has a convergent subsequence. So any bounded and closed subset of $\mathbb{R}$ is sequentially compact.

Note

bounded and closed together implies sequentially compact.

28.14 Theorem

Let (X, d) be a metric space. Then the following statements are equivalent:

(1) (X, d) is compact

(2) (X, d) is sequentially compact

Proof

(1) $\Rightarrow$ (2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Assume that no subsequence of $(x_n)_{n\in\mathbb{N}}$ converges in X. For any $p \in X$ there exists $\epsilon_p > 0$ such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For X is compact $\exists (p_i)_{i\in\{1,\ldots,n\}}$

$$X \subseteq \bigcup_{i=1}^{n} \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^{n} \{n \in \mathbb{N}\ d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

(2) $\Rightarrow$ (1)

prove (X, d) is complete Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences $(x_{k_n})_{n\in\mathbb{N}}$ must converges to the same limit.

So (X, d) is complete

If X is not covered by finitely many balls of radius ϵ we can construct a sequence $(x_{k_n})_{n\in\mathbb{N}}$ such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^{n} \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

28.15 Def

Let X be a Hausdorff topological space. If for any $x \in X$ there exist a compact neighborhood $\mathcal{C}_x$ we say that X is locally compact.

Example

$\mathbb{R}$ is locally compact.

28.16 Prop

Assume that $(K, |\cdot|)$ is a locally compact non-trivial valued field. field. Let $(E, \|\cdot\|)$ be a finite dimensional normed $K-vector$ space. A subset $Y \subseteq E$ is compact iff it's closed and bounded.

Proof

$\Rightarrow$ Let $Y \subseteq X$ be compact. Then for Y is Hausdorff, Y is closed. Moreover

$$Y \subseteq \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq ... \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^{n} \mathcal{B}(0, n_i)$$

$\Rightarrow$ Y is bounded.

$\Leftarrow$ We prove sequentially compact by a theorem proved before.
Let $(e_i)_{i=1}^{d}$ be a basis of E. Again we assume

$$\left\| \sum_{i=1}^{d} a_i e_i \right\| = \max_{i\in\{1,\ldots,d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n\in\mathbb{N}} = (\sum_{i=1}^{d} a_i^{(n)} e_i)_{n\in\mathbb{N}}$$

Since Y is bounded for any $i \in \{1, ..., d\}$ the sequence $(a_i^{(n)})$ is bounded. In particular we find $M > 0$ such that $\forall i \in \{1, ..., n\}$

$$\left| a_i^{(n)} \right| < M$$

Since $(K, |\cdot|)$ is locally compact, there exists a compact set $\mathcal{C} = \mathcal{C}_0 \subseteq K$ that's a neighborhood of 0. Let $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since K is not trivially valued, then exists $a \in K$ such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$ is compact. We have the K-linear mapping

$$\begin{aligned} K \to &\quad K \\ y \mapsto &\quad au \end{aligned}$$

is bounded, then continuous. Hence $a\mathcal{C}$ is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find $(I_i)_{i=1}^{d}$ are infinite subsets of $\mathbb{N}$ with

$$I_1 \supseteq ... \subseteq I_d$$

such that $(a_j)_{j\in I_i}^{(n)}$ converges to some $a_i \in K$. It follows that our original sequence has a convergent subsequence converges to $\sum_{i=1}^{d} a_i e_i$.

So Y is sequentially compact.