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**Part I**

**Set**





# Chapter 1

## Ring

### 1.1 morphism

#### Def

Let  $A$  and  $B$  be unitary rings .We call morphism of unitary rings from  $A$  to  $B$  .only mapping  $A \rightarrow B$  is a morphism of group from  $(A,+)$  to  $(B,+)$ ,and a morphism of monoid from  $(A,\cdot)$  to  $(B,\cdot)$

#### Properties

- Let  $R$  be a unitary ring. There is a unique morphism from  $\mathbb{Z}$  to  $R$
- 

#### algebra

we call  $k$ -algebra any pair  $(R,f)$ ,when  $R$  is a unitary ring ,and  $f : k \rightarrow R$  is a morphism of unitary rings such that  $\forall (b,x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring  $R$ ,the unique morphism of unitary rings  $\mathbb{Z} \rightarrow R$  define a structure of  $\mathbb{Z}$ -algebra on  $R$  (extra:  $\mathbb{Z}$  is commutative despite  $R$  isn't guaranteed)

Notation: Let  $k$  be a commutative unitary ring , $(A,f)$  be a  $k$ -algebra. If there is no ambiguity on  $f$ ,for any  $(\lambda,a) \in k \times A$ ,we denote  $f(\lambda)a$  as  $\lambda a$

#### Formal power series

reminder:  $n \in \mathbb{N}$  is possible infinite ,so  $\sum_{n \in \mathbb{N}}$  couldn't be executed directly.

Def:

(extended polynomial actually) Let  $k$  be a commutative unitary ring. Def : Let  $T$  be a formal symbol.We denote  $k^{\mathbb{N}}$  as  $k[T]$  If  $(a_n)_{n \in \mathbb{N}}$  is an element of  $k^{\mathbb{N}}$ ,when we denote  $k^{\mathbb{N}}$  as  $k[T]$  this element is denote as  $\sum_{n \in \mathbb{N}} a_n T^n$  Such

element is called a formal power series over  $k$  and  $a_n$  is called the Coefficient of  $T^n$  of this formal power series Notation:

- omit terms with coefficient 0
- write  $T'$  as  $T$
- omit Coefficient those are 1;
- omit  $T^0$

Example  $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$  is written as  $1 + 2T + T^2$

Def Remind that  $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$ , define two composition laws on  $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$  form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$  is a morphism
- $(FG)H = \left( \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left( \sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left( \sum_{p,q,l=n} a_p b_q c_l \right) T^n$   
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by  $F'(T)$  or  $\mathcal{D}(F(T))$  the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$  is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote  $\exp(T) \in k[T]$  as  $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$ , which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence:  $(F_i(T))_{i \in \mathbb{N}}$  be a sequence of elements in  $k[T]$ , and  $F(T) \in k[T]$  We say that  $(F_i(T))_{i \in \mathbb{N}}$  is a Cauchy sequence if  $\forall l \in \mathbb{N}$ , there exists  $N(l) \in \mathbb{N}$  such that  $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$ ,  $\text{ord}(F_i(T) - F_j(T)) \geq l$

# Part II

## Sequences



## Chapter 2

# Supremum and infimum

Def:

Let  $(X, \leq)$  be a partially ordered set  $A$  and  $Y$  be subsets of  $X$ , such that  $A \subseteq Y$

- If the set  $\{y \in Y \mid \forall a \in A, a \leq y\}$  has a least element then we say that  $A$  has a Supremum in  $Y$  with respect to  $\leq$  denoted by  $\sup_{(Y, \leq)} A$  this least element and called it the Supremum of  $A$  in  $Y$  (this respect to  $\leq$ )
- If the set  $\{y \in Y \mid \forall a \in A, y \leq a\}$  has a greatest element, we say that  $A$  has an infimum in  $Y$  with respect to  $\leq$ . We denote by  $\inf_{(Y, \leq)} A$  this greatest element and call it the infimum of  $A$  in  $Y$
- Observation:  $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set.

- If  $f$  is a function from  $I$  to  $X$   $\sup f$  denotes the supremum of  $f(I)$  is  $X$ .  $\inf f$  takes the same
- If  $(x_i)_{i \in I}$  is a family of element in  $X$ , then  $\sup x_i$  denotes  $\sup\{x_i \mid i \in I\}$  (in  $X$ )

If moreover  $\mathbb{P}(\cdot)$  denotes a statement depending on a parameter in  $I$  then  $\sup_{i \in I, \mathbb{P}(i)} x_i$  denotes  $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let  $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$  We equip  $\mathbb{R}$  with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So  $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence  $\inf A = 0$

Example: For  $n \in \mathbb{N}$ , let  $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let  $(X, \leq)$  be a partially ordered set,  $A, Y, Z$  be subset of  $X$ , such that  $A \subseteq Z \subseteq Y$

- If  $\max A$  exists, then it is also equal to  $\sup_{(y, \leq)} A$
- If  $\sup_{(y, \leq)} A$  exists and belongs to  $Z$ , then it is equal to  $\sup A$

$\inf$  takes the same Prop.

Let  $X, \leq$  be a partially ordered set,  $A, B, Y$  be subsets of  $X$  such that  $A \subseteq B \subseteq Y$

- If  $\sup_{(y, \leq)} A$  and  $\sup_{(y, \leq)} B$  exists, then  $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If  $\inf_{(y, \leq)} A$  and  $\inf_{(y, \leq)} B$  exists, then  $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set and  $f, g : I \rightarrow X$  be mappings such that  $\forall t \in I, f(t) \leq g(t)$

- If  $\inf f$  and  $\inf g$  exists, then  $\inf f \leq \inf g$
- If  $\sup f$  and  $\sup g$  exists, then  $\sup f \leq \sup g$

## Chapter 3

# Interval

We fix a totally ordered set  $(X, \leq)$

Notation:

If  $(a, b) \in X \times X$  such that  $a \leq b$ ,  $[a, b]$  denotes  $\{x \in X \mid a \leq x \leq b\}$

Def:

Let  $I \subseteq X$ . If  $\forall (x, y) \in I \times I$  with  $x \leq y$ , one has  $[x, y] \subseteq I$  then we say that  $I$  is an interval in  $X$

Example:

Let  $(a, b) \in X \times X$ , such that  $a \leq b$ . Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let  $\Lambda$  be a non-empty set and  $(I_\lambda)_{\lambda \in \Lambda}$  be a family of intervals in  $X$ .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$
- If  $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ ,  $\bigcup_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$

We check that  $[a, b] \subseteq I_\lambda \cup I_\mu$

- If  $b \leq x$   $[a, b] \subseteq [a, x] \subseteq I_\lambda$  because  $\{a, x\} \subseteq I_\lambda$
- If  $x \leq a$   $[a, b] \subseteq [x, b] \subseteq I_\mu$  because  $\{b, x\} \subseteq I_\mu$
- If  $a < x < b$  then  $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let  $(X, \leq)$  be a totally ordered set.  $I$  be a non-empty interval of  $X$ . If  $\sup I$  exists in  $X$ , we call  $\sup I$  the right endpoint;  $\inf$  takes the similar way.

Prop.

Let  $I$  be an interval in  $X$ .

- Suppose that  $b = \sup I$  exists.  $\forall x \in I, [x, b] \subseteq I$
- Suppose that  $a = \inf I$  exists.  $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let  $I$  be an interval in  $X$ . Suppose that  $I$  has supremum  $b$  and an infimum  $a$  in  $X$ . Then  $I$  is equal to one of the following sets  $[a, b]$   $[a, b[$   $]a, b]$   $]a, b[$

Def

let  $(X, \leq)$  be a totally ordered set. If  $\forall (x, z) \in X \times X$ , such that  $x < z \quad \exists y \in X$  such that  $x < y < z$ , then we say that  $(X, \leq)$  is thick

Prop.

Let  $(X, \leq)$  be a thick totally ordered set.  $(a, b) \in X \times X, a < b$  If  $I$  is one of the following intervals  $[a, b]; [a, b[; ]a, b]; ]a, b[$  Then  $\inf I = a \quad \sup I = b$  (for it's thick empty set is impossible)

Proof:

Since  $X$  is thick, there exists  $x_0 \in ]a, b[$  By definition,  $b$  is an upper bound of  $I$ . If  $b$  is not the supremum of  $I$ , there exists an upper bound  $M$  of  $I$  such that  $M \neq b$ . Since  $X$  is thick, there is  $M' \in X$  such that  $x_0 \leq M, M' < b$  Since  $[x, b] \subseteq I, a, b \in I$  Hence  $M$  and  $M'$  belong to  $I$ , which conflicts with the uniqueness of supremum.



## Chapter 4

# Enhanced real line

Def:

Let  $+\infty$  and  $-\infty$  be two symbols that are different and don't belong to  $\mathbb{R}$ . We extend the usual total order  $\leq$  on  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus  $\mathbb{R} \cup \{-\infty, +\infty\}$  becomes a totally ordered set, and  $\mathbb{R} = ]-\infty, +\infty[$ . Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in ]-\infty, +\infty[ \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in ]0, +\infty[ \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$   
**ARE NOT DEFINED**

Def

Let  $(X, \leq)$  be a partially ordered set. If for any subset  $A$  of  $X$ ,  $A$  has a supremum and an infimum in  $X$ , then we say that  $X$  is order complete.

Example

Let  $\Omega$  be a set.  $(\mathcal{P}(\Omega), \subseteq)$  is order complete. If  $\mathcal{F}$  is a subset of  $\mathcal{P}(\Omega)$ ,  $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$ .

Interesting tip:  $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

**Axiom:**

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$  is order complete.

In  $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any  $A \subseteq \mathbb{R} \cup -\infty, +\infty$  and  $c \in \mathbb{R}$  We denote by  $A + c$  the set  $\{a + c \mid a \in A\}$
- If  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\lambda A$  denotes  $\{\lambda a \mid a \in A\}$
- $-A$  denotes  $(-1)A$

Prop.

For any  $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\sup(-A) = -\inf A$ ,  $\inf(-A) = -\sup A$  Def

We denote by  $(\mathbb{R}, \leq)$  a field  $\mathbb{R}$  equipped with a total order  $\leq$ , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$  such that  $a < b$ , one has  $\forall c \in \mathbb{R}$ ,  $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ ,  $ab > 0$
- $\forall A \subseteq \mathbb{R}$ , if  $A$  has an upper bound in  $\mathbb{R}$ , then it has a supremum in  $\mathbb{R}$

Prop.

Let  $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

$\inf$  takes the same

Theorem:

Let  $I$  and  $J$  be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If  $\{a, b\} \neq \{+\infty, -\infty\}$  then  $c = a + b$

$\inf$  takes the same if  $(-\infty) + (+\infty)$  doesn't happen

Corollary:

Let  $I$  be a non-empty set,  $f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

Then  $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

$\inf$  takes the similar ( $\leq \rightarrow \geq$ ) (provided when the sum are defined)

# Chapter 5

## Vector space

In this section:

$K$  denotes a unitary ring.

Let  $0$  be zero element of  $K$

$1$  be the unity of  $K$

### 5.1 $K$ -module

#### 5.1.1 Def

Let  $(V, +)$  be a commutative group. We call left/right  $K$ -module structure: any mapping  $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group  $(V, +)$  equipped with a left/right  $K$ -module structure is called a left/right  $K$ -module.

#### 5.1.2 Remark

Let  $K^{op}$  be the set  $K$  equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then  $K^{op}$  forms a unitary ring  
 Any left  $K^{op}$  - module is a right  $K$ -module  
 Any right  $K^{op}$  - module is a left  $K$ -module  
 $(K^{op})^{op} = K$

### 5.1.3 Notation

When we talk about a left/right  $K$ -module  $(V, +)$ , we often write its left  $K$ -module structure as  $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

### 5.1.4 $K$ -vector space

If  $K$  is commutative, then  $K^{op} = K$ , so left  $K$ -module and right  $K$ -module structure are the same. We simply call them  $K$ -module structure. A commutative group equipped with a  $K$ -module structure is called a  $K$ -module. If  $K$  is a field, a  $K$ -module is also called a  $K$ -vector space

Let  $\Phi : K \times V \rightarrow V$  be a left or right  $K$ -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence  $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$   
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$  is a morphism of groups. Hence  $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$  (*is a var*)

### 5.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping  $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$  is a left  $K$ -module structure

The mapping  $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$  is a right  $K$ -module structure

**5.1.6 Remark:**

We can also write an element  $\mu$  of  $K^I$  is the form of a family  $(\mu_i)_{i \in I}$  of elements in  $K$  ( $\mu_i$  is the image of  $i \in I$  by  $\mu$ )  
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

**5.2 sub K-module****5.2.1 Def**

Let  $V$  be a left/right  $K$ -module. If  $W$  is a subgroup of  $V$ . Such that  $\forall a \in K, \forall x \in W \quad ax/xa \in W$ , then we say that  $W$  is left/right sub- $K$ -module of  $V$ .

**5.2.2 Example**

Let  $I$  be a set. Let  $K^{\oplus I}$  be the subset of  $K^I$  composed of mappings  $f : I \rightarrow K$  such that  $I_f = \{x \in I \mid f(x) \neq 0\}$  is finite. It is a left and right sub- $K$ -module of  $K^I$

In fact,  $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$   
Hence  $f - g \in K^{\oplus I}$  So  $K^{\oplus I}$  is a subgroup of  $K^I$   
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

**5.3 morphism of K-modules****5.3.1 Def**

Let  $V$  and  $W$  be left  $K$ -module, A morphism of groups  $\phi : V \rightarrow W$  is called a morphism of left  $K$ -modules if  $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

**5.3.2 K-linear mapping**

If  $K$  is commutative, a morphism of  $K$ -modules is also called a  $K$ -linear mapping. We denote by  $\text{hom}_{K\text{-Mod}}(V, W)$  the set of all morphism of left- $K$ -module from  $V$  to  $W$ . This is a subgroup of  $W^V$

**5.3.3 Theorem**

Let  $V$  be a left  $K$ -module. Let  $I$  be a set.  
The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$  is a bijection where  
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 5.3.4 Remark:column

In the case where  $I = 1, 2, 3, \dots, n$   $V^I$  is denoted as  $V^n$ ,  $K^I$  is denoted as  $K^n$ . For any  $(x_1, \dots, x_n) \in V^n$ , by the theorem, there exists a unique morphism of left  $K$ -modules  $\phi : K^n \rightarrow V$  such that  $\forall i \in 1, \dots, n, \phi(e_i) = x_i$ .

We write this  $\phi$  as a column  $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ . It sends  $(a_1, \dots, a_n) \in K^n$  to  $a_1x_1 + \dots + a_nx_n$ .

## 5.4 kernel

### 5.4.1 Prop

Let  $G$  and  $H$  be groups and  $f : G \rightarrow H$  be a morphism of groups

- $Im(f) \subseteq H$  is a subgroup of  $H$
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- $f$  is injection iff  $\ker(f) = \{e_G\}$

### 5.4.2 Def

$\ker(f)$  is called the kernel of  $f$

### 5.4.3 Theorem

$f$  is injection iff  $\ker(f) = \{e_G\}$

### 5.4.4 Proof

Let  $e_G$  and  $e_H$  be neutral element of  $G$  and  $H$  respectively

- (1) Let  $x$  and  $y$  be element of  $G$   
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ . So  $Im(f)$  is a subgroup of  $H$
- (2) Let  $x$  and  $y$  be element of  $\ker(f)$ . One has  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$ . So  $xy^{-1} \in \ker(f)$ . So  $\ker(f)$  is a subgroup of  $G$ .
- (3) Suppose that  $f$  is injection.  
 Since  $f(e_G) = e_H$  one has  $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ . Suppose that  $\ker(f) = \{e_G\}$ . If  $f(x) = f(y)$  then  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$ .  
 Hence  $xy^{-1} = e_G \Rightarrow x = y$

### 5.4.5 Def

Let  $(V, +)$  be a commutative group,  $I$  be a set. We define a composition law  $+$  on  $V^I$  as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then  $V^I$  forms a commutative group

### 5.4.6 Remark

Let  $E$  and  $F$  be left  $K$ -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$  is a subgroup of  $F^E$

In fact  $f$  and  $g$  are elements of  $\text{hom}_{K\text{-Mod}}(E, F)$ , then  $f - g$  is also a morphism of left  $K$ -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

### 5.4.7 Theorem

Let  $V$  be a left  $K$ -module,  $I$  be a set The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$  is an isomorphism of groups, where  $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 5.4.8 Proof:

One has  $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So  $\Psi$  is a morphism of groups

injectivity Let  $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$  Such that  $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence  $\phi$  is the neutral element.

surjectivity Let  $x = (x_i)_{i \in I} \in V^I$  We define  $\phi_x : K^{\oplus I} \rightarrow V$  such that  $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left  $K$ -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that  $K'$  is a unitary ring, and  $V$  is also equipped with a right  $K'$ -module structure, Then  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$  is a right sub- $k'$ -module, and  $\Psi$  in the theorem is a right  $K'$ -module isomorphism





## Chapter 6

# Monotone mappings

### 6.1 Def

Let  $I$  and  $X$  be partially ordered sets,  $f : I \rightarrow X$  be a mapping.

- If  $\forall (a, b) \in I \times I$  such that  $a < b$ . One has  $f(a) \leq f(b)$ , then we say that  $f$  is increasing. decreasing takes similar way.
- If  $f$  is (strictly) increasing or decreasing, we say that  $f$  is (strictly) monotone.

### 6.2 Prop.

Let  $X, Y, Z$  be partially ordered sets.  $f : X \rightarrow Y, g : Y \rightarrow Z$  be mappings

- If  $f$  and  $g$  have the same monotonicity, then  $g \circ f$  is increasing
- If  $f$  and  $g$  have different monotonicities, then  $g \circ f$  is decreasing

strict monotonicities takes the same

### 6.3 Def

Let  $f$  be a function from a partially ordered set  $I$  to another partially ordered set  $X$ . If  $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$  is (strictly) increasing/decreasing then we say that  $f$  is (strictly) increasing/decreasing

### 6.4 Prop.

Let  $I$  and  $X$  be partially ordered sets.  $f$  be function from  $I$  to  $X$ .

- If  $f$  is increasing/decreasing and  $f$  is injection, then  $f$  is strictly increasing/decreasing
- Assume that  $I$  is totally ordered and  $f$  is strictly monotone, then  $f$  is injection

## 6.5 Prop

Let  $A$  be totally ordered set,  $B$  be a partially ordered set,  $f$  be an injective function from  $A$  to  $B$

If  $f$  is increasing/decreasing, then so is  $f^{-1}$

## 6.6 Def

Let  $X$  and  $Y$  be partially ordered sets.  $f : X \rightarrow Y$  be a bijection. If both  $f$  and  $f^{-1}$  are increasing, then we say that  $f$  is an isomorphism of partially ordered sets.

(If  $X$  is totally, then a mapping  $f : X \rightarrow Y$  is an isomorphism of partially ordered sets iff  $f$  is a bijection and  $f$  is increasing)

## 6.7 Prop.

Let  $I$  be a subset of  $\mathbb{N}$  which is infinite. Then there is a unique increasing bijection  $\lambda_I : \mathbb{N} \rightarrow I$

## 6.8 Proof

### 6.8.1 bijection

We construct  $f : \mathbb{N} \rightarrow I$  by induction as follows.

Let  $f(0) = \min I$  Suppose that  $f(0), \dots, f(n)$  are constructed

then we take  $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since  $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$ . Therefore  $f(n) \leq f(n+1)$

Since  $f(n+1) \notin \{f(0), \dots, f(n)\}$ , we have  $f(n) < f(n+1)$

Hence  $f$  is strictly increasing and this is injective

If  $f$  is not surjective, then  $I \setminus \text{Im}(f)$  has a element  $N$ .

Let  $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$ .

Since  $N \notin \text{Im}(f)$ ,  $N < f(m)$ .

So  $m \neq 0$ . Hence  $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition,  $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$ ,

Hence  $f(m) \leq N$ , causing contradiction.

**6.8.2 uniqueness**

exercise: Prove that  $Id_{\mathbb{N}}$  is the only isomorphism of partially ordered sets from  $\mathbb{N}$  to  $\mathbb{N}$



# Chapter 7

## sequence and series

Let  $I \subseteq \mathbb{N}$  be a infinite subset

### 7.1 Def

Let  $X$  be a set. We call sequence in  $X$  parametrized by  $I$  a mapping from  $I$  to  $X$ .

### 7.2 Remark

If  $K$  is a unitary ring and  $E$  is a left  $K$ -module then the set of sequence  $E^I$  admits a left- $K$ -module structure. If  $x = (x_n)_{n \in I}$  is a sequence in  $E$ , we define a sequence  $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ , called the series associated with the sequence  $x$ .

### 7.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$  is a morphism of left- $K$ -module

### 7.4 proof

Let  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$  be elements of  $E^I$

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

### 7.5 Prop

Let  $I$  be a totally ordered set.  $X$  be a partially ordered set,  $f : I \rightarrow X$  be a mapping,  $J \in I$ . Assume that  $J$  does not have any upper bound in  $I$

- If  $f$  is increasing ,then  $f(I)$  and  $f(J)$  have the same upper bounds in  $X$
- If  $f$  is decreasing ,then  $f(I)$  and  $f(J)$  have the same lower bounds in  $X$

## 7.6 limit

### 7.6.1 Def

Let  $i \subseteq \mathbb{N}$  be a infinite subset.  $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$  where  $[-\infty, +\infty]$  denotes  $\mathbb{R} \cup \{-\infty, +\infty\}$ , we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left( \inf_{i \in I, i \geq n} x_i \right)$$

If  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$ , we then say that  $(x_n)_{n \in I}$  tends to  $l$  and that  $l$  is the limit of  $(x_n)_{n \in I}$ . If in addition  $(x_n)_{n \in I} \in \mathbb{R}^I$  and  $l \in \mathbb{R}$ , we say that  $(x_n)_{n \in I}$  converges to  $l$

### 7.6.2 Remark

If  $J \subseteq \mathbb{N}$  is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left( \inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in  $(x_i)_{i \in I}$  the limit superior and the limit inferior do not change.

In fact, if we take  $J = \mathbb{N} \setminus \{0, \dots, m\}$ , then  $\inf_{n \in J}(\dots)$  and  $\sup_{n \in J}(\dots)$  only depends on the values of  $x_i, i \in I, i \geq m$

### 7.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

**7.6.4 Prop**

Let  $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

**7.6.5 Prop**

Let  $(x_n)_{n \in I}$  be elements in  $[-\infty, +\infty]^I$ . Suppose that there exists  $N_0 \in \mathbb{N}$  such that  $\forall n \in I, n \geq N_0$ , one has  $x_n \leq y_n$ . Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

**7.6.6 Theorem**

Let  $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$  one has  $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$  and  $(z_n)_{n \in I}$  tend to the same limit  $l$

Then  $(y_n)_{n \in I}$  tends to  $l$

**7.6.7 Def**

Let  $I$  be an infinite subset of  $\mathbb{N}$ , and  $(x_n)_{n \in I}$  be a sequence in some set  $X$ . We call subsequence of  $(x_n)_{n \in I}$  a sequence of the form  $(x_n)_{n \in J}$ , where  $J$  is an infinite subset of  $I$

**7.6.8 Prop**

Let  $I$  and  $J$  be infinite subset of  $\mathbb{N}$  such that  $J \subseteq I$ .  $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$ , one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if  $(x_n)_{n \in I}$  tends to  $l \in [-\infty, +\infty]$ , then  $(x_n)_{n \in J}$  tends to  $l$

### 7.6.9 Prop

$\forall n \in \mathbb{N}$ , one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

### 7.6.10 Theorem

Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$

- If the mapping  $(n \in I) \mapsto x_n$  is increasing, then  $(x_n)_{n \in I}$  tends to  $\sup_{n \in I} x_n$
- If the mapping  $(n \in I) \mapsto x_n$  is decreasing, then  $(x_n)_{n \in I}$  tends to  $\inf_{n \in I} x_n$

### 7.6.11 Notation

If a sequence  $(x_n)_{n \in I} \in [-\infty, +\infty]$  tends to some  $l \in [-\infty, +\infty]$  the expression  $\lim_{n \in I, n \rightarrow} x_n$  denotes this limit  $l$

### 7.6.12 Corollary

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{N}_{\geq 0}$ . Then the series  $\sum_{n \in I} x_n$  (the sequence  $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ ) tends to an element in  $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ . It converges in  $\mathbb{R}$  iff it is bounded from above (namely has an upper bound in  $\mathbb{R}$ )

### 7.6.13 Notation

If a series  $\sum_{n \in I} x_n$  in  $[-\infty, +\infty]$  tends to some limit, we use the expression  $\sum_{n \in I} x_n$  to denote the limit

### 7.6.14 Theorem: Bolzano-Weierstrass

Let  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\limsup_{n \in I, n \rightarrow +\infty} x_n$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\liminf_{n \in I, n \rightarrow +\infty} x_n$ .



## 7.6.15 Proof

Let  $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If  $J$  is infinite, the sequence  $(x_n)_{n \in J}$  is decreasing so it tends to  $\inf_{n \in J} x_n$

$\forall n \in J$  by definition  $x_n = \sup_{i \in I, i \geq n} x_i$  so  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that  $J$  is finite. Let  $n_0 \in I$  such that  $\forall n \in J, n < n_0$ . Denote by

$l = \sup_{n \in I, n \geq n_0}$

Let  $N \in \mathbb{N}$  such that  $N \geq n_0$ . By definition  $\sup_{i \in I, i \geq n_0} x_i \leq l$ . If the strict

inequality  $\sup_{i \in I, i \geq N} x_i < l$  holds, then  $\sup_{i \in I, i \geq N} x_i$  is NOT an upper bound of

$\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists  $n \in I$  such that  $n_0 \leq n < N$  such that  $x_n > \sup_{i \in I, i \geq N} x_i$ . We

may also assume that  $n$  is largest among elements of  $I \cap [n_0, N[$  that satisfies this inequality. Then  $\forall m \in I$  if  $m \geq n$  then  $x_m \leq x_n$ . Thus  $n \in J$  that contradicts the maximality of  $n_0$ . Therefore  $l = \sup_{i \in I, i \geq N} x_i$ , which leads to  $\limsup_{n \in I, n \rightarrow +\infty} x_n = l$

Moreover, if  $m \in I, m \geq n_0$  then  $m \notin J$ , so  $x_m < l$  (since otherwise  $x_m = \sup_{i \in I, i \geq m} x_i$  and hence  $m \in J$ ). Hence,  $\forall$  finite subset  $I'$  of  $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$  and hence  $\exists n \in I$ , such that  $n > \max I'$ , and  $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence  $(n_j)_{j \in \mathbb{N}}$  in  $I$

Let  $n_0$  be as above. Let  $f : \mathbb{N} \rightarrow I_{\geq n_0}$  be a surjective mapping.

If  $n_j$  is chosen, we choose  $n_{j+1} \in I$  such that  $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$

Hence the sequence  $(x_{n_j})_{j \in \mathbb{N}}$  is increasing, and  $\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n =$

$l$

$l = \sup_{n \in I, n \geq n_0}$

So  $(x_{n_j})_{j \in \mathbb{N}}$  tends to  $l$



## Chapter 8

# Cauchy sequence

### 8.1 Def

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{R}$   
If  $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$  then  
we say that  $(x_n)_{n \in I}$  is a Cauchy sequence

### 8.2 Prop

- If  $(x_n)_{i \in I} \in \mathbb{R}^I$  converges to some  $l \in \mathbb{R}$ , then it is a Cauchy sequence
- If  $(x_n)_{i \in I}$  is a Cauchy sequence, there exists  $M > 0$  such that  $\forall n \in I \quad |x_n| \leq M$
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite,  $(x_n)_{n \in I}$  is a Cauchy sequence.
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite and  $l \in \mathbb{R}$  such that  $(x_n)_{n \in I}$  converges to  $l$ , then  $(x_n)_{n \in J}$  converges to  $l$  too.

### 8.3 Theorem: Completeness of real number

If  $(x_n)_{n \in I} \in \mathbb{R}^I$  is a Cauchy sequence, then it converges in  $\mathbb{R}$

#### 8.3.1 Proof

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists M \in \mathbb{R}_{>0}$  such that  $-M \leq x_n \leq M \quad \forall x \in I$ . So  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . By Bolzano-Weierstrass theorem.  $\exists J \subseteq I$  infinite such that  $(x_n)_{n \in I}$  converges to  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . Therefore  $(x_n)_{n \in I}$  converges to the same limit.

## 8.4 Absolutely converge

We say that a series  $\sum_{n \in I} x_n \in \mathbb{R}$  converges absolutely if  $\sum_{n \in I} |x_n| < +\infty$

### 8.4.1 Prop

If a series  $\sum_{n \in I} x_n$  converges absolutely, then it converges in  $\mathbb{R}$

## Chapter 9

# Comparison and Technics of Computation

### 9.1 Def

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be sequence in  $\mathbb{R}$

- If there exists  $M \in \mathbb{R}_{>0}$  and  $N \in \mathbb{N}$  such that  $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$  then we write  $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists  $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$  and  $N \in \mathbb{N}$  such that  $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$  and  $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$ , then we write  $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

### 9.2 Prop.

Let  $I$  and  $X$  be partially ordered sets and  $f : I \rightarrow X$  be an increasing/decreasing mapping. Let  $J$  be a subset of  $I$ . Assume that any elements of  $I$  has an upper bound in  $J$ . Then  $f(I)$  and  $f(J)$  have the same upper/lower bounds in  $X$

### 9.3 Theorem

Let  $I$  be a totally ordered set,  $f : I \rightarrow [-\infty, +\infty]$  and  $g : I \rightarrow [-\infty, +\infty]$  be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

### 9.3.1 Proof

We can assume  $f$  and  $g$  increasing. Let  $a = \sup f(I), b = \sup g(I)$

Let  $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip  $A$  with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let  $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$ .

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

$h$  is increasing.

Let  $(x, y) \in A$ . Assume that  $x \leq y$

If  $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$  then  $(y, y) \in B$  and  $(x, y) \leq (y, y)$

If  $\{f(y), g(y)\} = \{-\infty, +\infty\}$  and for  $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$ . So  $a = +\infty$ , Hence  $b > -\infty$

So  $\exists z \in I$  such that  $g(z) > -\infty$ . We should have  $y \leq z$  Hence  $f(z) + g(z)$  is well defined,  $(z, z) \in B$  and  $(x, y) \leq (z, z)$  Similarly, if  $x \geq y$ ,  $(x, y)$  has also an upper bound in  $B$ . Therefore:  $\sup h(A) = \sup h(B)$

## 9.4 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

### 9.4.1 Proof

$\forall n \in \mathbb{N}$ , let  $A_N = \sup_{n \in I, n \geq N} x_n$   $B_N = \sup_{n \in I, n \geq N} y_n$ .  $(A_N)_{N \in \mathbb{N}}$  and  $(B_N)_{N \in \mathbb{N}}$  are decreasing, and  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$   $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let  $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$  if  $A_N + B_N$  is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

## 9.5 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n)$$

### 9.5.1 Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with  $\infty$

## 9.6 Theorem

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Assume that  $\forall n \in I, y_n \in \mathbb{R}$  and  $(y_n)_{n \in I}$  converges to some  $l \in \mathbb{R}$ . Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + l$$

## 9.7 Prop.

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

### 9.7.1 Proof

About the first inequality. Since  $\max\{x_n, y_n\} \geq x_n$  and  $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset  $J$  of  $I$  such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let  $J_1 = \{n \in J \mid x_n \geq y_n\}$   $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$  So either  $J_1$  or  $J_2$  is infinite

Suppose that  $J_1$  is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If  $J_2$  is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

## 9.8 Theorem

Let  $(a_n)_{n \in I} \in \mathbb{R}^I$   $l \in \mathbb{R}$ . The following statements are equivalent

- $(a_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

### 9.8.1 Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1)  $\Rightarrow$  (2):

If  $(a_n)_{n \in I}$  converges to  $l$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2)  $\Rightarrow$  (1):

If  $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore:  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

## 9.9 Remark

Let  $(a_n)_{n \in I}$  be a sequence in  $\mathbb{R}$ ,  $l \in \mathbb{R}$

The sequence  $(a_n)_{n \in I}$  converges to  $l$  iff  $a_n - l = o(1), n \in I, n \rightarrow +\infty$

## 9.10 Calculates on $O(), o()$

### 9.10.1 Plus

Let  $(a_n)_{n \in I}$   $(a'_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$



### 9.10.2 Transform

Let  $(a_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be two sequence in  $\mathbb{R}$  If  $a_n = o(b_n), n \in I, n \rightarrow +\infty$ , then  $a_n = O(b_n), n \in I, n \rightarrow +\infty$

### 9.10.3 Transition

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  and  $(c_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = O(b_n)$  and  $b_n = o(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$

### 9.10.4 Times

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$  be sequences in  $\mathbb{R}$

- If  $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If  $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

## 9.11 On the limit

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be elements of  $\mathbb{R}^I$  that converges to  $l \in \mathbb{R}$  and  $l' \in \mathbb{R}$  respectively. Then:

- $(a_n + b_n)_{n \in I}$  converges to  $l + l'$
- $(a_n b_n)_{n \in I}$  converges to  $ll'$

## 9.12 Prop

Let  $a \in \mathbb{R}$  Then  $a^n = o(n!) \quad n \rightarrow +\infty$

### 9.12.1 Proof

Let  $N \in \mathbb{N}$  such that  $|a| < N$  For  $n \in \mathbb{N}$  such that  $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{n!}{N!} \cdot \frac{|a|^{n-N}}{n!} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^{n-N} \rightarrow 0$$

And  $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$ . Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

### 9.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

#### 9.13.1 Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

### 9.14 Prop

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be the elements of  $\mathbb{R}^I$ . If the series  $\sum_{n \in I} b_n$  converges absolutely and if  $a_n = O(b_n) \quad n \rightarrow +\infty$  Then  $\sum_{n \in I} a_n$  converges absolutely

#### 9.14.1 Proof

By definition  $\sum_{n \in I} |b_n| < +\infty$ . If  $|a_n| \leq M|b_n|$  for  $n \in I, n \geq N$  where  $N \in \mathbb{N}$ . Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

### 9.15 Theorem: d'Alembert ratio test

Let  $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely
- If  $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  does not converge (diverges)

### 9.15.1 Proof

(1)

Let  $\alpha \in \mathbb{R}$  such that  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$ ,  $\alpha$  isn't a lower bound of  $\left( \sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$   
 So  $\exists N \in \mathbb{N}$  such that  $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$  Hence for  $n \geq N$   $|a_n| \leq \alpha^{n-N} |a_N|$  since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore  $a_n = O(\alpha^n)$  since  $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$ ,  $\sum_{n \in \mathbb{N}} a_n$  converge absolutely.

### 9.15.2 Lemma

If a series  $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$  converges, then  $\lim_{n \rightarrow +\infty} a_n = 0$

**Proof**

If  $\left( \sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$  converges to some  $l \in \mathbb{R}$ , then  $\left( \sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$  converges to  $l$ ,  
 too. Hence  $\left( a_n = \left( \sum_{i=0}^n a_i \right) - \left( \sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$  converges to  $l - l = 0$

### 9.15.3 (2)

Let  $\beta \in \mathbb{R}$  such that  $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 So there exists  $N \in \mathbb{N}$  such that  $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$   
 Hence  $(|a_n|)_{n \in \mathbb{N}}$  is not bounded since  $|a_n| \geq \beta^{n-N} |a_N|$   
 By the lemma:  $\sum_{n \in \mathbb{N}} a_n$  diverges.

## 9.16 Prop

Let  $a \in \mathbb{R}, a > 1$  Then  $n = o(a^n), n \rightarrow +\infty$

### 9.16.1 Proof

Let  $\epsilon > 0$  such that  $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

**9.16.2 Corollary**

Let  $a > 1, t \in \mathbb{R}_{\geq 0}$  Then  $n^t = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $d \in \mathbb{N}_{\geq 1}$  such that  $t \leq d$  Then  $n^{t-d} \leq 1$  So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let  $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence  $n^t = o(a^n)$

**9.16.3 Corollary**

There exists  $M \geq 1$  such that  $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

**Proof**

Let  $a \in \mathbb{R}$  such that  $1 < a < e$

**9.17 Theorem: Cauchy root test**

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Let  $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If  $\alpha < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.
- If  $\alpha > 1$  then  $\sum_{n \in \mathbb{N}} a_n$  diverges

**9.17.1 Proof**

(1)

Let  $\beta \in \mathbb{R}, \alpha < \beta < 1$ . There exists  $N \in \mathbb{N}$  such that  $|a_n|^{\frac{1}{n}} \leq \beta$  for  $n \geq N$ . That means  $|a_n| = O(\beta^n)$  since  $0 < \beta < 1$ ,  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.

(2)

If  $\alpha > 1$  then  $\forall N \in \mathbb{N} \exists n \geq N$  such that  $|a_n|^{\frac{1}{n}} \geq 1$ , since otherwise  $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$  contradiction  
Hence  $(|a_n|)_{n \in \mathbb{N}}$  cannot converge to 0.

**Part III**

**Topology**



# Chapter 10

## Absolute value and norms

### 10.1 Def

Let  $K$  be a field. By absolute value on  $K$ , we mean a mapping  $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- (1)  $\forall a \in K \quad |a| = 0$  iff  $a = 0$
- (2)  $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3)  $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$  (triangle inequality)

### 10.2 Notation

$\mathbb{Q}$  Take a prime num  $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$  there exists a integer  $ord_p(\alpha) \frac{a}{b}$ , where  
 $a \in \mathbb{Z} \setminus \{0\}$   
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

### 10.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on  $\mathbb{Q}$

#### 10.3.1 proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd$$

$$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d}$$

Assume  $ord_p(\alpha) \geq ord_p(\beta)$

$$\alpha + \beta$$

$$= p^{ord_p(\beta)} \left( p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right)$$

$$= p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd$$

So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$$

$$\text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$$

$$\max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$



## Chapter 11

# Quotient Structure

### 11.1 Def

Let  $X$  be a set and  $\sim$  be a binary relation on  $X$   
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$ , if  $x \sim y$  then  $y \sim x$
- $\forall (x, y, z) \in X^3$ , if  $x \sim y, y \sim z$  then  $x \sim z$

then we say that  $\sim$  is an equivalence relation

### 11.2 equivalence class

$\forall x \in X$  we denote by  $[x]$  the set  $\{y \in X \mid y \sim x\}$  and call it the equivalence class of  $x$  on  $X$ . Let  $X/\sim$  be the set  $\{[x] \mid x \in X\}$

### 11.3 Prop.

Let  $X$  be a set and  $\sim$  be an equivalence relation on  $X$

- (1)  $\forall x \in X, y \in [x]$  on has  $[x] = [y]$
- (2) If  $\alpha$  and  $\beta$  are elements of  $X/\sim$  such that  $\alpha \neq \beta$  then  $\alpha \cap \beta = \emptyset$
- (3)  $X = \bigcup_{\alpha \in X/\sim} \alpha$

### 11.3.1 Proof

- (1) Let  $z \in [y]$ . Then  $y \sim z$ . Since  $y \in [x]$  one has  $x \sim y$ .  
Therefore,  $x \sim z$  namely  $z \in [x]$ . This proves  $[y] \subseteq [x]$ . Moreover, since  $x \sim y$ , one has  $x \in [y]$ . Hence  $[x] \subseteq [y]$ . Thus we obtain  $[x] = [y]$ .
- (2) Suppose that  $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$   
By (1),  $\alpha = [y], \beta = [y]$ , Thus leads to a contradiction.
- (3)  $\forall x \in X \quad x \in [x]$  Hence  $x \in \bigcup_{\alpha \in X/\sim} \alpha$  Hence  $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$ . Conversely,  
 $\forall \alpha \in X/\sim, \alpha$  is a subset of  $X$ . Hence  $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$ . Then  $X = \bigcup_{\alpha \in X/\sim} \alpha$

## 11.4 Def

Let  $G$  be a group and  $X$  be a set  
We call left/right action of  $G$  on  $X$  ant mapping  $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$  that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

## 11.5 Remark

If we denote by  $G^{op}$  the set  $G$  equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of  $G$  on  $X$  is just a left action of  $G^{op}$  on  $X$ .

## 11.6 Prop

Let  $G$  be a group and  $X$  be a set . Assume given a left action of  $G$  on  $X$ . Then the binary relation  $\sim$  on  $X$  defined as  $x \sim y$  iff  $\exists g \in G \quad y = gx$  is an equivalence relation

## 11.7 Notation on Equivalence Class

We denote by  $G/X$  the set  $X/\sim \forall x \in X$  the equivalence class of  $x$  is denoted as  $Gx/xG$  or  $orb_G(x)$  call the orbit of  $x$  under the action of  $G$

## 11.8 Proof

- $\forall x \in X \quad x = 1x$  so  $x \sim x$
- $\forall (x, y) \in X^2$  if  $y = gx$  for same  $g \in G$  then  $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$ . ( $y \sim x$ )
- $\forall (x, y, z) \in X^3$ , if  $\exists (g, h) \in G^2$ , such that  $y = gx$  and then  $z = h(gx) = (hg)x$  So  $x \sim z$

## 11.9 Quotient set

Let  $X$  be a set and  $\sim$  be an equivalence relation, the mapping  $X \rightarrow X/\sim$ :  
 $(x \in X) \mapsto [x]$  is called the projection mapping.

$X/\sim$  is called the quotient set of  $X$  by equivalence relation  $\sim$

### 11.9.1 Example

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of  $H$  on  $G$ . Thus we obtain two quotient sets  $H/G$  and  $G/H$

## 11.10 Def

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . If  $\forall g \in G, h \in H \quad ghg^{-1} \in H$ ,  
 Then we say that  $H$  is a normal subgroup of  $G$

## 11.11 Remark

$\forall g \in G, gH = Hg$ , provided that  $H$  is a normal subgroup of  $G$ . In fact  $\forall h \in$ ,

- $\exists h' \in H$  such that  $ghg^{-1} = h'$  Hence  $gh = h'g$ . This shows  $gH \subseteq Hg$
- $\exists h'' \in H$  such that  $g^{-1}hg = h''$  Hence  $hg = gh''$ . This shows  $Hg \subseteq gH$

Thus  $gH = Hg$

## 11.12 Prop

If  $G$  is commutative, any subgroup of  $G$  is normal

### 11.13 Theorem

Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set  $G/H$ . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

#### 11.13.1 Proof

- If  $xH = x'H, yH = y'H$  then  $\exists h_1 \in H, h_2 \in H$  such that  $x' = xh_1, y' = yh_2$ . Hence  $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$ . For  $y^{-1}h_1y, h_2 \in H$  then  $(x'y')H = (xy)H$ . So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

### 11.14 Def

Let  $K$  be a unitary ring and  $E$  be a left  $K$ -module. We say that a subgroup  $F$  of  $(E, +)$  is a left sub- $K$ -module of  $E$  if  $\forall (a, x) \in K \times F, ax \in F$

### 11.15 Prop

Let  $K$  be a unitary ring,  $E$  be a left  $K$ -module and  $F$  be a sub- $K$ -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- $K$ -module structure on  $E/F$ . Moreover, the projection mapping  $\pi : E \rightarrow E/F$  is a morphism of left- $K$ -modules

### 11.15.1 Proof

Let  $x$  and  $x'$  be elements of  $E$  such that  $[x] = [x']$ , that means:  $x' - x \in F$   
Hence  $a(x' - x) = ax' - ax \in F$  So  $[ax] = [ax']$   
Let us check that  $E/F$  forms a left  $K$ -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition,  $\pi$  is a morphism of groups. Moreover  $\forall x \in E, a \in K$   $\pi(ax) = [ax] = a[x] = a\pi(x)$

## 11.16 Def

Let  $A$  be a unitary ring . We call two-sided ideal any subgroup  $I$  of  $(A, +)$  that satisfies :  $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I$  ( $I$  is a left and right sub- $K$ -module of  $A$ )

## 11.17 Theorem

Let  $A$  be a unitary ring and  $I$  be a two sided ideal of  $A$  . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover ,  $A/I$  becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely  $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$ )

### 11.17.1 Proof

If  $a' \sim a, b' \sim b$  that means  $a' - a \in I, b' - b \in I$  then  $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$ . For  $(a' - a), (b' - b) \in I$ , then  $a'b' - ab \in I$   
Therefore  $a'b' \sim ab$

### 11.17.2 Reside Class

Let  $d \in \mathbb{Z}$  and  $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$   $d\mathbb{Z}$  is a two sided ideal of  $\mathbb{Z}$   
 If  $m \in \mathbb{Z}$ , for any  $a \in \mathbb{Z}$   $adm = dma \in d\mathbb{Z}$

Denote by  $\mathbb{Z}/d\mathbb{Z}$  the quotient ring. The class of  $n \in \mathbb{Z}$  in  $\mathbb{Z}/d\mathbb{Z}$  is called the residue class of  $n$  modulo  $d$

If  $A$  is a commutative unitary ring, a two sided ideal of  $A$  is simply called an ideal of  $A$

### 11.18 Theorem

Let  $f : G \rightarrow H$  be a morphism of groups

- (1)  $Im(f)$  is a subgroup of  $H$
- (2)  $\ker(f) := \{x \in G \mid f(x) = 1_H\}$  is a normal subgroup of  $G$
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4)  $f$  is injective iff  $\ker(f) = \{1_G\}$

#### 11.18.1 Proof

- (1) Let  $\alpha$  and  $\beta$  be elements of  $Im(f)$ . Let  $(x, y) \in G^2$  such that  $\alpha = f(x), \beta = f(y)$  Then  $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$  So  $Im(f)$  is a subgroup
- (2) Let  $x$  and  $y$  be elements of  $\ker(f)$ .  
 One has  $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$   
 So  $xy^{-1} \in \ker f$ . Hence  $\ker f$  is a subgroup of  $G$   
 Let  $x \in \ker f, y \in G$ .  
 One has  $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$  Hence  $yxy^{-1} \in \ker f$ . So  $\ker f$  is a normal subgroup
- (3) If  $x \sim y$  then  $\exists z \in \ker f$  such that  $y = xz$  Hence  $f(y) = f(x)f(z) = f(x)1_H = f(x)$  So  $f$  is well defined.  
 Moreover  $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$  Hence  $\tilde{f}$  is a morphism of groups.  
 By definition  $Im(\tilde{f}) = Im(f)$  If  $x$  and  $y$  are elements of  $G$  such that  $f(x) = f(y)$  then  $f(xy^{-1}) = 1_H$   
 Hence  $xy^{-1} \in \ker f$  Since  $x = (xy^{-1})y$ ,  $x \sim y$  that means  $[x] = [y]$   
 Therefore  $\tilde{f}$  is injective.

- (4) If  $f$  is injective,  $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$ , so  $x = 1_G$ . Therefore  $\ker f = \{1_G\}$   
 Conversely, suppose that  $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$  if  $f(x) = f(y)$  then  $f(x)f(y)^{-1} = 1_H$ . Hence  $xy^{-1} = 1_G, x = y$

## 11.19 Theorem

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Then

- (1)  $\text{Im}(f)$  is a left-sub- $K$ -module of  $F$
- (2)  $\ker(f)$  is a left-sub- $K$ -module of  $E$
- (3)  $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$  is a isomorphism of left  $K$ -modules  
 $[x] \mapsto f(x)$

### 11.19.1 Proof

- (1)  $\forall x \in E, \quad f(ax) = af(x)$  So  $af(x) \in \text{Im}(f)$
- (2)
- (3)





# Chapter 12

## Topology

### 12.1 Def

Let  $X$  be a set. We call topology on  $X$  any subset  $\mathcal{J}$  of  $\wp(X)$  that satisfies:

- $\emptyset \in \mathcal{J}$  and  $X \in \mathcal{J}$
- If  $(u_i)_{i \in I}$  is an arbitrary family of elements in  $\mathcal{J}$ , then  $\bigcup_{i \in I} u_i \in \mathcal{J}$
- If  $u$  and  $v$  are elements of  $\mathcal{J}$ , then  $u \cap v \in \mathcal{J}$

### 12.2 Remark

If  $(u_i)_{i=1}^n$  is a finite family of elements of  $\mathcal{J}$ , then  $\bigcap_{i=1}^n u_i \in \mathcal{J}$  (by induction, this follows from (3))

#### 12.2.1 Example

$\{\emptyset, X\}$  is a topology. call the trivial topology on  $\wp(X)$  is a topology called the discrete topology.

### 12.3 Def

Let  $X$  be a set. We call metric on  $X$  any mapping  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , that satisfies

- $d(x, y) = 0$  iff  $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called a metric space

### 12.3.1 Example

Let  $X$  be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

## 12.4 Def

Let  $(X, d)$  be a metric space. For any  $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$ , let  $B(x, \epsilon) := \{y \in X \mid d(x, y) \leq \epsilon\}$ . We call the open ball of radius  $\epsilon$  centered at  $x$

### 12.4.1 Example

Consider  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$ , then  $B(x, \epsilon) = ]x - \epsilon, x + \epsilon[$

## 12.5 Prop.

Let  $(X, d)$  be a metric space. Let  $\mathcal{J}_d$  be the set of  $U \subseteq X$  such that  $\forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subseteq U$ . Then  $\mathcal{J}_d$  is a topology on  $X$

### 12.5.1 Proof

- $\emptyset \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let  $(u_i)_{i \in I}$  be a family of elements of  $\mathcal{J}_d$ . Let  $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$  such that  $x \in u_i$ . Since  $u_i \in \mathcal{J}_d, \exists \epsilon > 0$  such that  $B(x, \epsilon) \subseteq u_i \subseteq U$ . Hence  $U \in \mathcal{J}_d$
- Let  $U$  and  $V$  be elements of  $\mathcal{J}_d$ . Let  $x \in U \cap V, \exists a, b \in \mathbb{R}_{\geq 0}$  such that  $B(x, a) \subseteq U, B(x, b) \subseteq V$ . Taking  $\epsilon = \min\{a, b\}$ , Then  $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ . Therefore  $U \cap V \in \mathcal{J}_d$

## 12.6 Def

$\mathcal{J}_d$  is called the topology induced by the metric  $d$

## 12.7 Def

We call topology space any pair  $(X, \mathcal{J})$  where  $X$  is a set and  $\mathcal{J}$  is a topology on  $X$

Given a topological space  $(X, \mathcal{J})$  If  $U \in \mathcal{J}$  then we say that  $U$  is an open subset of  $X$ . If  $F \in \wp(X)$  such that  $X \setminus F \in \mathcal{J}$ , then we say that  $F$  is closed subset of  $X$

If there exists  $d$  a metric on  $X$  such that  $\mathcal{J} = \mathcal{J}_d$  then we say that  $\mathcal{J}$  is metrizable

### 12.7.1 Example

Let  $X$  be a set . The discrete topology on  $X$  is metrizable. In fact, if  $d$

denote the metric defined as  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$

$\forall x \in X \quad B(x, 1) = \{x\}$  So  $\{x\} \in \mathcal{J}_d$  Hence  $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{J}_d$

## 12.8 Axiom of choice

For any set  $I$  and any family  $(A_i)_{i \in I}$  of non-empty sets , there exists a mapping  $f : I \rightarrow \bigcup_{i \in I} A_i$  such that  $\forall i \in I, f(i) \in A_i$

## 12.9 Def

Let  $(X, \leq)$  be a partially ordered set If  $\forall A \subseteq X$   $A$  is non-empty , there exists a least element of  $A$  then we say that  $(X, \leq)$  is a well ordered set.

## 12.10 Theorem

For any set  $X$ , there exists an order relation  $\leq$  on  $X$  such that  $(X, \leq)$  forms a well ordered set.

## 12.11 Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set . If  $\forall A \subseteq X$  that is totally ordered with respect to  $\leq$ , there exists an upper bound of  $A$  inside  $X$ . Then , there exists a maximal element  $x_0$  of  $X$  ( $\forall y \in X, y > x_0$  does not hold)

## 12.12 Prop.

Let  $(X, \leq)$  be a well ordered set ,  $y \notin X$ . We extends  $\leq$  to  $X \cup \{y\}$ , such that  $\forall x \in X, x < y$ . Then  $(X \cup \{y\}, \leq)$  is well ordered.

### 12.13 Proof

Let  $A \subseteq X \cup \{y\}$ ,  $A \neq \emptyset$ . If  $A = \{y\}$  then  $y$  is the least element of  $A$ . If  $A \neq \{y\}$  then  $B = A \setminus \{y\}$  is non-empty. Let  $b$  be the least element of  $B$ . Since  $b < y$  it's also the least element of  $A$ .

### 12.14 Def

Let  $(X, \leq)$  be a well ordered set.  $S \subseteq X$ , If  $\forall s \in S, x \in X \quad x < s$  initial  $x \in S$  ( $X_{<s} \subseteq S$ ), then we say that  $S$  is an initial segment of  $X$ .

If  $S$  is a initial segment such that  $S = X$  then we say that  $S$  is a proper initial segment.

### 12.15 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$  Then  $X_{<x}$  is a proper initial segment of  $X$ .

### 12.16 Prop.

Let  $(X, \leq)$  be a well ordered set, If  $(S_i)_{i \in I}$  is a family of initial segment of  $X$ , then  $\bigcup_{i \in I} S_i$  is an initial segment of  $X$ .

### 12.17 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$  such that  $s \in S_i, i \in I$  Therefore  $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

### 12.18 Prop.

Let  $(X, < \leq)$  be a well ordered set.

- (1) Let  $S$  be a proper initial segment of  $X$ ,  $x = \min(X \setminus S)$  Then  $S = X_{<x}$
- (2)  $X \rightarrow \wp(X)$   
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of  $X$  forms a well ordered subset of  $(\wp(X), \subseteq)$

### 12.19 Proof

- (1)  $\forall s \in S$  if  $x \leq s$  then  $x \in S$  contradiction. Hence  $s < x$ , This shows  $S \subseteq X_{<x}$  Conversely, if  $t \in X, t \notin X_{<x}$  Hence  $t \in S$ . Hence  $X_{<x} \subseteq S$

- (2) Let  $x, y \in X, x < y$  By definition  $X_{<x} \subseteq X_{<y}$  Moreover  $x \in X_{<y} \setminus X_{<x}$  So  $X_{<x} \subsetneq X_{<y}$
- (3) Let  $\mathcal{F} \subseteq \wp(X)$  be a set of initial segments.  $\mathcal{F} \neq \emptyset$ . Then there exists  $A \subseteq X$  such that  $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$  If  $A = \emptyset$  then  $\mathcal{F} = \{X\}$ , and  $\{X\}$  is the least element of  $\mathcal{F}$ . Otherwise  $A \neq \emptyset$  and  $A$  has a least element  $a$ . Then by (2)  $X_{<a}$  is the least element of  $\mathcal{F}$

## 12.20 Lemma

Let  $(X, \leq)$  be a well ordered set,  $f : X \rightarrow X$  be a strictly increasing mapping. Then  $\forall x \in X, x \leq f(x)$

### 12.20.1 Proof

Let  $A = \{x \in X \mid f(x) < x\}$  If  $A \neq \emptyset$ , let  $a$  be the least element of  $A$ . By definition  $f(a) < a$ . Hence  $f(f(a)) < f(a)$  since  $f$  is strictly increasing. This shows  $f(a) \in A$ . But  $a$  is the least element of  $A$ ,  $f(a) < a$  cannot hold: contradiction.

## 12.21 Prop

Let  $(X, \leq)$  be a well ordered set,  $S$  and  $T$  be two initial segment of  $X$ . If  $f : S \rightarrow T$  is a bijection that's strictly increasing, then  $S = T, f = Id_S$

### 12.21.1 Proof

We may assume  $T \subseteq S$ . Let  $l : T \rightarrow S$  be the inclusion mapping and  $g = l \circ f : S \rightarrow S$ . Since  $g$  is strictly increasing, by the lemma,  $\forall s \in S, s \leq g(s) = f(s) \in T$ . Since  $T$  is an initial segment,  $s \in T$ . Hence  $S = T$ . Apply the lemma to  $f^{-1}$  we get  $\forall s \in S, s \leq f^{-1}(s)$  Hence  $f(s) \leq s$  Therefore  $f(s) = s$