

# Contents

<b>I</b>	<b>Set</b>	<b>21</b>
<b>1</b>	<b>product</b>	<b>23</b>
1.1	direct sum . . . . .	23
<b>2</b>	<b>Ring</b>	<b>25</b>
2.1	morphism . . . . .	25
<b>II</b>	<b>Sequences</b>	<b>27</b>
<b>3</b>	<b>Supremum and infimum</b>	<b>29</b>
<b>4</b>	<b>Interval</b>	<b>31</b>
<b>5</b>	<b>Enhanced real line</b>	<b>33</b>
<b>6</b>	<b>Vector space</b>	<b>35</b>
6.1	K-module . . . . .	35
6.1.1	Def . . . . .	35
6.1.2	Remark . . . . .	35
6.1.3	Notation . . . . .	36
6.1.4	K-vector space . . . . .	36
6.1.5	Association: . . . . .	36
6.1.6	Remark: . . . . .	37
6.2	sub K-module . . . . .	37
6.2.1	Def . . . . .	37
6.2.2	Example . . . . .	37
6.3	morphism of K-modules . . . . .	37
6.3.1	Def . . . . .	37
6.3.2	K-linear mapping . . . . .	37
6.3.3	Theorem . . . . .	37
6.3.4	Remark:column . . . . .	38
6.4	kernel . . . . .	38
6.4.1	Prop . . . . .	38
6.4.2	Def . . . . .	38

6.4.3	Theorem . . . . .	38
6.4.4	Def . . . . .	39
6.4.5	Remark . . . . .	39
6.4.6	Theorem . . . . .	39
6.4.7	Proof: . . . . .	39
<b>7</b>	<b>Monotone mappings</b>	<b>41</b>
7.1	Def . . . . .	41
7.2	Prop. . . . .	41
7.3	Def . . . . .	41
7.4	Prop. . . . .	41
7.5	Prop . . . . .	42
7.6	Def . . . . .	42
7.7	Prop. . . . .	42
7.8	Proof . . . . .	42
7.8.1	bijection . . . . .	42
7.8.2	uniqueness . . . . .	43
<b>8</b>	<b>sequence and series</b>	<b>45</b>
8.1	Def . . . . .	45
8.2	Remark . . . . .	45
8.3	Prop . . . . .	45
8.4	proof . . . . .	45
8.5	Prop . . . . .	45
8.6	limit . . . . .	46
8.6.1	Def . . . . .	46
8.6.2	Remark . . . . .	46
8.6.3	Prop . . . . .	46
8.6.4	Prop . . . . .	47
8.6.5	Prop . . . . .	47
8.6.6	Theorem . . . . .	47
8.6.7	Def . . . . .	47
8.6.8	Prop . . . . .	47
8.6.9	Prop . . . . .	48
8.6.10	Theorem . . . . .	48
8.6.11	Notation . . . . .	48
8.6.12	Corollary . . . . .	48
8.6.13	Notation . . . . .	48
8.6.14	Theorem: Bolzano-Weierstrass . . . . .	48
<b>9</b>	<b>Cauchy sequence</b>	<b>51</b>
9.1	Def . . . . .	51
9.2	Prop . . . . .	51
9.3	Theorem: Completeness of real number . . . . .	51
9.4	Absolutely converge . . . . .	52
9.4.1	Prop . . . . .	52

<b>10 Comparison and Technics of Computation</b>	<b>53</b>
10.1 Def . . . . .	53
10.2 Prop. . . . .	53
10.3 Theorem . . . . .	53
10.4 Prop. . . . .	54
10.5 Prop. . . . .	55
10.6 Theorem . . . . .	55
10.7 Prop. . . . .	55
10.8 Theorem . . . . .	56
10.9 Remark . . . . .	56
10.10 Calculates on $O(), o()$ . . . . .	56
10.10.1 Plus . . . . .	56
10.10.2 Transform . . . . .	57
10.10.3 Transition . . . . .	57
10.10.4 Times . . . . .	57
10.11 On the limit . . . . .	57
10.12 Prop . . . . .	57
10.13 Prop . . . . .	58
10.14 Prop . . . . .	58
10.15 Theorem: d'Alembert ratio test . . . . .	58
10.15.1 Lemma . . . . .	59
10.15.2 (2) . . . . .	59
10.16 Prop . . . . .	59
10.16.1 Corollary . . . . .	60
10.16.2 Corollary . . . . .	60
10.17 Theorem: Cauchy root test . . . . .	60
 <b>III Axiom of choice</b>	 <b>61</b>
<b>11 Preparation</b>	<b>63</b>
11.1 Statement of axiom of choice . . . . .	63
11.2 Def . . . . .	63
11.3 Theorem . . . . .	63
11.4 Zorn's lemma . . . . .	63
11.5 Prop. . . . .	63
11.6 Proof . . . . .	64
11.7 Def: Initial Segment . . . . .	64
11.8 Example . . . . .	64
11.9 Prop. . . . .	64
11.10 Proof . . . . .	64
11.11 Prop . . . . .	64
11.12 Proof . . . . .	64
11.13 Lemma . . . . .	65
11.14 Prop . . . . .	65
11.15 Def . . . . .	65

11.16Def . . . . .	65
11.17Prop. . . . .	66
11.18Lemma . . . . .	66
<b>12 Zorn's lemma</b>	<b>69</b>
12.1 Proof . . . . .	69
 <b>IV Topology</b>	 <b>71</b>
<b>13 Absolute value and norms</b>	<b>73</b>
13.1 Def . . . . .	73
13.2 Notation . . . . .	73
13.3 Prop . . . . .	73
13.4 Def . . . . .	74
<b>14 Quotient Structure</b>	<b>75</b>
14.1 Def . . . . .	75
14.2 equivalence class . . . . .	75
14.3 Prop. . . . .	75
14.4 Def . . . . .	76
14.5 Remark . . . . .	76
14.6 Prop . . . . .	76
14.7 Notation on Equivalence Class . . . . .	76
14.8 Proof . . . . .	77
14.9 Quotient set . . . . .	77
14.9.1 Example . . . . .	77
14.10Def . . . . .	77
14.11Remark . . . . .	77
14.12Prop . . . . .	77
14.13Theorem . . . . .	78
14.14Def . . . . .	78
14.15Prop . . . . .	78
14.16Def . . . . .	79
14.17Theorem . . . . .	79
14.17.1 Reside Class . . . . .	80
14.18Theorem . . . . .	80
14.19Theorem . . . . .	81
<b>15 Topology</b>	<b>83</b>
15.1 Def . . . . .	83
15.2 Remark . . . . .	83
15.2.1 Example . . . . .	83
15.3 Def . . . . .	83
15.3.1 Example . . . . .	84
15.4 Def . . . . .	84

15.4.1 Example . . . . .	84
15.5 Prop. . . . .	84
15.6 Def . . . . .	84
15.7 Def . . . . .	85
15.7.1 Example . . . . .	85
<b>16 Filter</b>	<b>87</b>
16.1 Def . . . . .	87
16.1.1 Example . . . . .	87
16.2 Def: Filter Basis . . . . .	87
16.2.1 Remark . . . . .	87
16.2.2 Example . . . . .	88
16.3 Remark . . . . .	88
16.3.1 Example . . . . .	88
16.4 Def . . . . .	88
16.5 Remark . . . . .	89
16.6 Extra Episode . . . . .	89
16.7 Prop. . . . .	89
<b>17 Limit point and accumulation point</b>	<b>91</b>
17.1 Def . . . . .	91
17.2 Prop . . . . .	91
17.3 Def . . . . .	92
17.4 Def . . . . .	92
17.5 Prop . . . . .	92
17.6 Def: dense . . . . .	92
<b>18 Limit of mappings</b>	<b>93</b>
18.1 Def . . . . .	93
18.2 Remark . . . . .	93
18.2.1 Example . . . . .	93
18.3 Remark . . . . .	93
18.4 Remark . . . . .	94
18.5 Prop . . . . .	94
18.6 Theorem . . . . .	94
18.7 Prop . . . . .	95
18.8 Def . . . . .	95
18.9 Remark . . . . .	95
18.10 Prop . . . . .	95
18.11 Theorem . . . . .	96
18.12 Prop. . . . .	97
18.12.1 Proof . . . . .	97

<b>19 Continuity</b>	<b>99</b>
19.1 Def . . . . .	99
19.2 Remark . . . . .	99
19.3 Theorem . . . . .	99
19.4 Proof . . . . .	99
19.5 Prop . . . . .	100
19.6 Def . . . . .	100
19.7 Prop . . . . .	100
19.8 Proof . . . . .	100
19.9 Prop . . . . .	101
19.10 Def . . . . .	102
19.11 Remark . . . . .	102
19.12 Prop . . . . .	102
19.13 Theorem . . . . .	104
19.13.1 Proof . . . . .	104
19.14 Remark . . . . .	104
19.14.1 Example . . . . .	105
<b>20 Uniform continuity and convergency</b>	<b>107</b>
20.1 Def . . . . .	107
20.2 Remark . . . . .	107
20.3 Prop . . . . .	107
20.4 Def . . . . .	108
20.5 Prop . . . . .	108
20.5.1 Proof . . . . .	108
20.6 Def . . . . .	109
20.7 Prop . . . . .	109
20.7.1 Proof . . . . .	109
20.8 Def . . . . .	110
20.9 Theorem . . . . .	110
20.9.1 Proof . . . . .	110
20.10 Theorem . . . . .	111
20.10.1 Proof . . . . .	111
20.10.2 Def . . . . .	111
20.11 Remark . . . . .	111
20.12 Example . . . . .	111
<b>V Normed Vector Space</b>	<b>113</b>
<b>21 Linear Algebra</b>	<b>115</b>
21.1 Def . . . . .	115
21.1.1 Notation . . . . .	115
21.2 Def . . . . .	115
21.3 Def . . . . .	116
21.4 Remark . . . . .	116

21.5 Theorem . . . . .	116
21.6 Theorem . . . . .	117
21.7 Corollary . . . . .	119
21.8 Def . . . . .	120
21.9 Theorem . . . . .	120
21.10 Proof . . . . .	120
21.11 Prop . . . . .	124
21.11.1 Proof . . . . .	124
<b>22 Matrices</b>	<b>125</b>
22.1 Def . . . . .	125
22.1.1 Example . . . . .	126
22.2 Def . . . . .	126
22.2.1 Example . . . . .	126
22.3 Def . . . . .	126
22.4 Calculate Matrices . . . . .	127
22.4.1 Remind . . . . .	127
<b>23 Transpose</b>	<b>129</b>
23.1 Def . . . . .	129
23.2 Def . . . . .	130
23.2.1 Example . . . . .	130
23.3 Prop . . . . .	131
23.4 Corollary . . . . .	131
23.5 Remark . . . . .	132
<b>24 Linear Equation</b>	<b>133</b>
24.1 Def . . . . .	133
24.2 Prop . . . . .	134
24.3 Linear Equation . . . . .	134
24.4 Prop . . . . .	134
24.5 Prop . . . . .	135
24.6 Def . . . . .	135
24.7 Theorem . . . . .	135
<b>25 Normed Vector Space</b>	<b>137</b>
25.1 Def . . . . .	137
25.2 Prop . . . . .	137
25.2.1 Proof . . . . .	137
25.3 Def . . . . .	138
25.4 Def: The completion . . . . .	138
25.5 Theorem . . . . .	138
25.6 Remark . . . . .	139
25.7 Prop . . . . .	139
25.8 Theorem . . . . .	140

<b>26 Norms</b>	<b>143</b>
26.1 Def . . . . .	143
26.2 Remark . . . . .	143
26.3 Def . . . . .	144
26.4 Prop . . . . .	144
26.5 Def . . . . .	145
26.6 Remark . . . . .	145
26.7 Def . . . . .	145
26.8 Prop . . . . .	145
26.9 Def: Operator Seminorm . . . . .	147
26.10 Prop . . . . .	147
26.11 Remark . . . . .	148
26.12 Def . . . . .	148
26.13 Theorem . . . . .	148
<b>27 Differentiability</b>	<b>151</b>
27.1 Def . . . . .	151
27.2 Def . . . . .	152
27.3 Prop . . . . .	152
27.4 Example . . . . .	153
27.4.1 . . . . .	153
27.4.2 . . . . .	153
27.4.3 . . . . .	153
27.4.4 . . . . .	153
27.5 Theorem: Chain rule . . . . .	154
27.6 Prop . . . . .	154
27.7 Def . . . . .	155
27.8 Corollary . . . . .	155
27.9 Corollary . . . . .	156
27.10 Corollary . . . . .	157
27.11 Prop . . . . .	157
27.12 Corollary . . . . .	158
27.13 Def: Equivalence of Norms . . . . .	158
27.14 Prop . . . . .	158
27.15 Remark . . . . .	159
27.16 Prop . . . . .	159
27.17 Theorem . . . . .	159
27.18 Prop . . . . .	162
27.19 Theorem . . . . .	162
<b>28 Compactness</b>	<b>163</b>
28.1 Def: cover . . . . .	163
28.2 Def: compact . . . . .	163
28.3 Def . . . . .	163
28.4 Prop . . . . .	164
28.5 Theorem . . . . .	164



28.6 Theorem . . . . .	165
28.7 Lemma . . . . .	166
28.8 Prop . . . . .	167
28.9 Prop . . . . .	167
28.10 Prop . . . . .	168
28.11 Prop . . . . .	168
28.12 Theorem . . . . .	169
28.13 Def . . . . .	169
28.14 Theorem . . . . .	170
28.15 Def . . . . .	171
28.16 Prop . . . . .	171
28.17 Theorem . . . . .	172
<b>29 Mean Value Theorems</b>	<b>175</b>
29.1 Rolle Theorem . . . . .	175
29.2 Mean value theorem(Lagrange) . . . . .	175
29.3 Mean value inequality . . . . .	176
29.4 Theorem . . . . .	177
29.5 Theorem(Heine) . . . . .	178
<b>30 Fixed Point Theorem</b>	<b>179</b>
30.1 Def . . . . .	179
30.2 Def . . . . .	179
30.3 Fixed Point Theorem . . . . .	179
<b>VI Higher differentials</b>	<b>181</b>
<b>31 Multilinear mapping</b>	<b>183</b>
31.1 Def . . . . .	183
31.2 Example . . . . .	183
31.3 Remark . . . . .	183
31.4 Prop . . . . .	184
31.5 Remark . . . . .	184
<b>32 Operator norm of Multilinear field</b>	<b>185</b>
32.1 Def . . . . .	185
32.2 Theorem . . . . .	185
32.3 Corollary . . . . .	186
32.3.1 Proof . . . . .	186
<b>33 Higher differentials</b>	<b>189</b>
33.1 Def . . . . .	189
33.2 Remark . . . . .	190
33.3 Theorem . . . . .	190
33.4 Prop(Gronwall inequality) . . . . .	190

33.5 Theorem . . . . .	191
33.6 Def . . . . .	192
33.7 Prop . . . . .	193
33.8 Theorem . . . . .	194
<b>34 Permutations</b>	<b>195</b>
34.1 Def . . . . .	195
34.1.1 Example . . . . .	195
34.2 Def . . . . .	195
34.3 Prop . . . . .	196
34.3.1 Proof . . . . .	196
34.4 Remark . . . . .	196
34.5 Theorem . . . . .	197
34.5.1 Remark . . . . .	197
34.6 Corollary . . . . .	197
34.6.1 Remark . . . . .	198
34.7 Def . . . . .	198
34.8 Corollary . . . . .	198
34.8.1 Proof . . . . .	198
34.9 Caybey Theorem . . . . .	198
34.10 Theorem . . . . .	199
34.11 Remark . . . . .	199
34.12 Exercise . . . . .	200
34.13 Symmetric of multilinear mapping . . . . .	200
34.14 Def: Symmetric and Alternating . . . . .	200
34.15 Prop . . . . .	200
34.16 Def: . . . . .	201
34.17 Reminder . . . . .	201
34.18 Theorem (Schwarz) . . . . .	201
34.19 Def . . . . .	202
34.20 Prop . . . . .	202
34.21 Prop . . . . .	203
34.21.1 Proof . . . . .	204
34.22 Prop . . . . .	204
34.23 Local Inversion Theorem . . . . .	205
34.23.1 Proof . . . . .	206
<b>VII Integration</b>	<b>209</b>
<b>35 Integral operators</b>	<b>211</b>
35.1 Prop . . . . .	211
35.2 Def . . . . .	212
35.3 Example . . . . .	212
35.4 Dini's theorem . . . . .	213
35.5 Def . . . . .	213

<b>36 Riemann integral</b>	<b>215</b>
36.1 Def . . . . .	215
36.2 Def . . . . .	215
36.3 Theorem . . . . .	215
<b>37 Daniell integral</b>	<b>217</b>
37.1 Prop . . . . .	217
37.1.1 . . . . .	217
37.1.2 . . . . .	217
37.2 Def . . . . .	218
37.3 Prop . . . . .	218
37.4 Corollary . . . . .	218
37.5 Prop . . . . .	219
37.6 Def . . . . .	219
37.7 Prop . . . . .	219
37.8 Def . . . . .	220
37.9 Remark . . . . .	220
37.10 Daniell Theorem . . . . .	220
37.11 Beppo Levi Theorem . . . . .	221
37.12 Fatou's Lemma . . . . .	223
37.13 Lebesgue dominated convergence theorem . . . . .	223
37.14 Notation . . . . .	224
<b>38 Semialgebra</b>	<b>225</b>
38.1 Notation . . . . .	225
38.2 Def . . . . .	225
38.2.1 Example . . . . .	225
38.3 Def . . . . .	226
38.4 Prop . . . . .	226
38.5 Prop . . . . .	226
38.5.1 Proof . . . . .	227
38.5.2 Example . . . . .	228
38.6 Theorem . . . . .	228
38.7 Prop . . . . .	229
38.8 Corollary . . . . .	230
38.9 Lemma . . . . .	231
38.9.1 Proof . . . . .	231
<b>39 Integral function</b>	<b>233</b>
39.1 Setting . . . . .	233
39.2 Prop . . . . .	233
39.2.1 Proof . . . . .	233
<b>40 Limit and Differential of Integrals with Parameters</b>	<b>235</b>
40.1 Theorem . . . . .	235
40.2 Theorem . . . . .	236

<b>41 Measure theory</b>	<b>239</b>
41.1 Def . . . . .	239
41.2 Prop . . . . .	239
41.3 Def . . . . .	240
41.4 Example . . . . .	240
41.5 Def . . . . .	241
41.6 Prop . . . . .	241
41.7 Def . . . . .	242
41.8 Prop . . . . .	242
41.9 Def . . . . .	242
41.10 Prop . . . . .	242
41.11 Corollary . . . . .	243
41.12 Example . . . . .	243
41.13 Prop . . . . .	243
41.14 Example . . . . .	244
<b>42 Measure</b>	<b>245</b>
42.1 Def . . . . .	245
42.2 Def . . . . .	245
42.3 Def . . . . .	246
42.4 Carathéodory Theorem . . . . .	246
42.5 Example . . . . .	246
42.6 Def . . . . .	246
42.6.1 Particular case . . . . .	247
42.7 Prop . . . . .	247
42.8 Corollary . . . . .	248
<b>43 Fundamental theorem of calculus</b>	<b>249</b>
43.1 Theorem . . . . .	249
43.2 Corollary . . . . .	249
43.2.1 Proof . . . . .	249
<b>44 <math>L^p</math> space</b>	<b>251</b>
44.1 Def . . . . .	251
44.2 Hölder inequality . . . . .	251
44.3 Corollary . . . . .	252
<b>VIII tensor</b>	<b>253</b>
<b>45 tensor product</b>	<b>255</b>
45.1 Theorem . . . . .	255
45.2 Def . . . . .	256
45.3 Def . . . . .	256
45.4 Remark . . . . .	256
45.5 Corollary . . . . .	257

45.6 exercise . . . . .	257
45.6.1 . . . . .	257
45.6.2 . . . . .	257
45.6.3 . . . . .	257
45.7 Lemma . . . . .	258
45.7.1 Proof . . . . .	258
45.8 Prop . . . . .	259
45.9 tensor product and duality . . . . .	259
45.9.1 product . . . . .	259
45.9.2 duality . . . . .	260
45.9.3 Exercise . . . . .	261
45.10Def . . . . .	261
45.11Extension of scalars . . . . .	261
45.12Prop . . . . .	262
45.13Remark . . . . .	263
45.14Exercise . . . . .	263
45.15Exactness of the tensor product . . . . .	263
45.16Def . . . . .	263
45.17Prop . . . . .	264
45.17.1 Example . . . . .	265
45.18Exercise(important) . . . . .	265
<b>46 Tensor algebra</b>	<b>267</b>
46.1 Def . . . . .	267
46.2 exterior product . . . . .	268
46.3 Def . . . . .	268
46.4 Notation . . . . .	268
46.5 Prop . . . . .	268
46.6 Def . . . . .	269
46.7 Def . . . . .	269
46.8 Prop . . . . .	269
46.9 Remark/exercise . . . . .	270
46.10Prop . . . . .	270
46.10.1 Proof . . . . .	271
<b>47 Determinant</b>	<b>273</b>
47.1 Def . . . . .	273
47.1.1 Proof . . . . .	273
47.2 Prop . . . . .	274
47.2.1 Proof . . . . .	274
47.3 Prop . . . . .	274
47.4 Prop . . . . .	274
47.5 Prop . . . . .	275
47.6 Prop . . . . .	276
47.7 Corollary . . . . .	276
47.8 Prop . . . . .	276

47.9 ? . . . . .	277
47.10Def . . . . .	277
47.11Laplace expansion of the determinant . . . . .	277
<b>48 The Structure of Linear Mappings</b>	<b>281</b>
48.1 Theorem . . . . .	281
48.2 Def . . . . .	281
48.3 Def . . . . .	281
48.3.1 Example . . . . .	282
48.4 Def . . . . .	282
48.5 Remark . . . . .	282
48.6 Remark/exercise . . . . .	283
48.7 Def . . . . .	283
48.8 Lemma . . . . .	283
48.9 Theorem . . . . .	283
48.10Def . . . . .	284
48.11Corollary . . . . .	284
48.12Remark . . . . .	284
48.13Def: Jordan block . . . . .	284
48.14Def: Jordan matrix . . . . .	285
48.15Example . . . . .	285
48.16Def . . . . .	285
48.17Prop . . . . .	286
48.18Def . . . . .	286
48.19Prop . . . . .	286
48.20Prop . . . . .	287
48.21Def . . . . .	287
48.22Prop . . . . .	287
48.23Theorem: Cayley-Hamilton Theorem . . . . .	288
48.24Example . . . . .	289
48.25Theorem . . . . .	290
48.26Def . . . . .	290
48.27Prop . . . . .	290
48.28Def . . . . .	292
48.29Lemma . . . . .	293
48.30Lemma . . . . .	293
48.31Jordan matrix of form $J_t(0)$ . . . . .	293
48.32Theorem . . . . .	294
48.33Prop . . . . .	296
48.34Lemma . . . . .	297
48.35Theorem . . . . .	298
<b>49 Jordan Matrix</b>	<b>299</b>
49.1 Def . . . . .	299
49.2 Prop . . . . .	299
49.3 Corollary . . . . .	300

<b>50 Inner Product</b>	<b>301</b>
50.1 Def . . . . .	301
50.2 Def . . . . .	301
50.3 Prop . . . . .	302
50.4 Def . . . . .	302
50.5 Def . . . . .	303
50.5.1 Example . . . . .	303
50.6 Def . . . . .	303
50.7 Def . . . . .	303
50.8 Def . . . . .	304
50.9 Remark . . . . .	304
<b>51 Differential Forms in <math>\mathbb{R}^n</math></b>	<b>305</b>
51.0.1 Notation . . . . .	305
51.1 Def . . . . .	305
51.2 Do Carmo Differential forms . . . . .	306
51.3 Def . . . . .	306
51.4 Notation . . . . .	307
51.5 Notation . . . . .	307
51.6 Notation . . . . .	307
51.7 Prop . . . . .	308
51.8 Def . . . . .	308
51.9 Prop . . . . .	310
51.10 Def . . . . .	310
51.11 Remark . . . . .	310
51.12 Def: Pullback of forms . . . . .	310
51.13 Prop . . . . .	311
51.14 Remark . . . . .	312
51.15 . . . . .	312
51.16 Prop . . . . .	313
51.17 . . . . .	314
51.18 Example . . . . .	314
51.19 Prop . . . . .	314
51.20 . . . . .	316
51.21 Def? . . . . .	316
<b>52 Line integral</b>	<b>317</b>
52.1 Def . . . . .	317
52.2 Def: Path integral . . . . .	317
52.3 What's this in physics? . . . . .	318
<b>53 Complement of measure theory</b>	<b>319</b>
53.1 Def( $\sigma$ -finite) . . . . .	319
53.2 Example( $\mathbb{R}$ , Norel $\sigma$ -algebra, Lebesgue measure) . . . . .	319
53.3 Notation . . . . .	319
53.4 Def . . . . .	320

53.5 Def . . . . .	320
53.6 Prop . . . . .	320
53.7 Prop . . . . .	321
53.8 Lemma . . . . .	321
53.9 Theorem . . . . .	322
53.10 Prop . . . . .	324
53.11 Prop . . . . .	325
53.12 Prop . . . . .	325
53.13 Theorem . . . . .	327
53.14 Corollary . . . . .	327
53.15 Monotone convergence theorem . . . . .	328
53.16 Recall . . . . .	328
53.17 Def . . . . .	328
53.18 Prop . . . . .	328
53.19 Recall . . . . .	329
53.20 Corollary . . . . .	329
53.21 Def: Push-forward measure . . . . .	329
53.22 Prop . . . . .	329
53.23 Lemma . . . . .	330
53.24 Fubini-Tobelli Theorem . . . . .	330
53.25 Corollary . . . . .	333
53.25.1 Proof . . . . .	333
53.26 Remark . . . . .	333
53.27 Remark . . . . .	333
53.28 Notation . . . . .	334
53.29 Remark . . . . .	334
53.30 Theorem (Change of variables for the Lebesgue integral) . . . . .	335
53.31 Remark . . . . .	335
53.32 Compute integrals in $\mathbb{R}^n$ . . . . .	335
53.32.1 Example . . . . .	335
53.33 Def . . . . .	335
53.34 Def . . . . .	336
53.35 Def . . . . .	337
53.36 Lemma . . . . .	337
53.37 Notation . . . . .	337
53.38 Def . . . . .	337
53.39 Theorem . . . . .	338
53.40 Poincare Lemma . . . . .	339
53.41 Notation . . . . .	340
53.42 Def . . . . .	340
53.43 Prop . . . . .	340
53.44 Def . . . . .	341
53.45 Def: Lebesgue number . . . . .	341
53.46 Lemma . . . . .	341
53.47 Theorem (homotopy invariance of the integrals) . . . . .	342



<b>54 Winding Numbers</b>	<b>345</b>
54.1 Def: Free Homotopy . . . . .	345
54.2 Notation . . . . .	345
54.3 Jordan Theorem . . . . .	345
54.4 Def . . . . .	346
54.5 Def . . . . .	346
54.6 Prop . . . . .	347
54.7 Prop . . . . .	347
54.8 Def . . . . .	348
54.9 Remark . . . . .	348
54.10Def . . . . .	348
54.11Remark . . . . .	349
54.12Prop . . . . .	349
54.13Def . . . . .	349
54.14Kronecker Index Theorem . . . . .	349
54.15Lemma . . . . .	351
54.16Def . . . . .	352
54.16.1 Example . . . . .	352
54.17Prop . . . . .	353
54.18Corollary(exercise) . . . . .	355
54.19Def . . . . .	355
54.20Def . . . . .	355
54.21Def . . . . .	355
54.22Def . . . . .	356
54.23Def . . . . .	356
<b>55 Curvilinear integral</b>	<b>359</b>
55.1 Def . . . . .	359
55.2 Prop . . . . .	359
55.3 Geometry . . . . .	360
55.4 Def . . . . .	360
55.5 Def . . . . .	361
55.6 Def . . . . .	361
55.7 Def . . . . .	361
55.8 Remark . . . . .	361
55.9 Prop . . . . .	361
55.10Prop . . . . .	362
55.11Prop . . . . .	362
55.12Def:isometry . . . . .	363
55.13Def:isometric . . . . .	363
<b>56 Complex conjugate vector space</b>	<b>365</b>
56.1 Def:complex conjugate vector space . . . . .	365
56.2 Def: Semilinear . . . . .	365
56.3 Def . . . . .	366

<b>57 Classification (up to isometry) of vector spaces of small dim</b>	<b>367</b>
57.1 $\dim V = 1$ and $g$ is symmetric . . . . .	367
57.1.1 Prop . . . . .	367
57.1.2 Theorem . . . . .	368
57.2 $\dim V = 1$ $g$ is hermitian . . . . .	368
57.2.1 Theorem . . . . .	369
57.3 $\dim V = 1$ $g$ symplectic . . . . .	369
57.3.1 Theorem . . . . .	369
57.4 $\dim V = 2$ $g$ symplectic . . . . .	369
57.4.1 Theorem . . . . .	370
<b>58 Compliment</b>	<b>371</b>
58.1 Def: non-degenerate . . . . .	371
58.2 Def . . . . .	371
58.3 Prop . . . . .	372
58.4 Theorem . . . . .	372
58.5 Theorem . . . . .	373
<b>59 Signature</b>	<b>375</b>
59.1 Def . . . . .	375
59.2 Notation . . . . .	375
59.3 Def . . . . .	375
59.4 Theorem . . . . .	376
59.5 Theorem . . . . .	377
<b>60 Orthonormal</b>	<b>379</b>
60.1 Def . . . . .	379
60.2 Def . . . . .	379
60.3 Gram-Schmidt algorithm . . . . .	380
<b>61 Euclidean and Unitary Spaces</b>	<b>381</b>
61.1 Def:Euclidean vector space . . . . .	381
61.2 Prop . . . . .	382
61.3 Remark . . . . .	382
61.4 Pythagona's Theorem . . . . .	382
61.5 Def:Angles . . . . .	382
61.6 Notation . . . . .	383
61.7 Def . . . . .	383
61.8 Prop . . . . .	383
61.9 Prop . . . . .	384
61.10 Relationship with calculus . . . . .	384
61.11 Prop . . . . .	385

<b>62 Unitary Space</b>	<b>387</b>
62.1 Def . . . . .	387
62.2 Decomplexification . . . . .	387
62.3 Theorem . . . . .	387
62.4 Corollary . . . . .	389
<b>63 Complexification</b>	<b>391</b>
63.1 Def: Complex structure . . . . .	391
63.2 Theorem . . . . .	391
63.3 Corollary . . . . .	391
63.4 Notation . . . . .	392
63.5 Prop . . . . .	392
63.6 Complex Cauchy-Schwartz inequality . . . . .	393
63.7 Corollary:Complex triangle inequality . . . . .	394
63.8 Def:Angle for unitary space . . . . .	394
63.9 Prop . . . . .	394
63.10Def . . . . .	396
63.11Corollary . . . . .	396



**Part I**

**Set**



# Chapter 1

## product

### 1.1 direct sum

$\oplus$  is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0 \}$$





## Chapter 2

# Ring

### 2.1 morphism

#### Def

Let  $A$  and  $B$  be unitary rings. We call morphism of unitary rings from  $A$  to  $B$  only mapping  $A \rightarrow B$  is a morphism of group from  $(A, +)$  to  $(B, +)$ , and a morphism of monoid from  $(A, \cdot)$  to  $(B, \cdot)$

#### Properties

- Let  $R$  be a unitary ring. There is a unique morphism from  $\mathbb{Z}$  to  $R$
- 

#### algebra

we call  $k$ -algebra any pair  $(R, f)$ , when  $R$  is a unitary ring, and  $f : k \rightarrow R$  is a morphism of unitary rings such that  $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring  $R$ , the unique morphism of unitary rings  $\mathbb{Z} \rightarrow R$  define a structure of  $\mathbb{Z}$ -algebra on  $R$  (extra:  $\mathbb{Z}$  is commutative despite  $R$  isn't guaranteed)

Notation: Let  $k$  be a commutative unitary ring,  $(A, f)$  be a  $k$ -algebra. If there is no ambiguity on  $f$ , for any  $(\lambda, a) \in k \times A$ , we denote  $f(\lambda)a$  as  $\lambda a$

#### Formal power series

reminder:  $n \in \mathbb{N}$  is possible infinite, so  $\sum_{n \in \mathbb{N}}$  couldn't be executed directly.

Def:

(extended polynomial actually) Let  $k$  be a commutative unitary ring. Def : Let  $T$  be a formal symbol. We denote  $k^{\mathbb{N}}$  as  $k[T]$ . If  $(a_n)_{n \in \mathbb{N}}$  is an element of  $k^{\mathbb{N}}$ , when we denote  $k^{\mathbb{N}}$  as  $k[T]$  this element is denoted as  $\sum_{n \in \mathbb{N}} a_n T^n$ . Such

element is called a formal power series over  $k$  and  $a_n$  is called the Coefficient of  $T^n$  of this formal power series Notation:

- omit terms with coefficient 0
- write  $T^0$  as 1
- omit Coefficient those are 1;
- omit  $T^0$

Example  $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$  is written as  $1 + 2T + T^2$

Def Remind that  $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$ , define two composition laws on  $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$  form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$  is a morphism
- $(FG)H = \left( \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left( \sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left( \sum_{p+q+l=n} a_p b_q c_l \right) T^n$   
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by  $F'(T)$  or  $\mathcal{D}(F(T))$  the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$  is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote  $\exp(T) \in k[T]$  as  $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$ , which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence:  $(F_i(T))_{i \in \mathbb{N}}$  be a sequence of elements in  $k[T]$ , and  $F(T) \in k[T]$  We say that  $(F_i(T))_{i \in \mathbb{N}}$  is a Cauchy sequence if  $\forall l \in \mathbb{N}$ , there exists  $N(l) \in \mathbb{N}$  such that  $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$ ,  $\text{ord}(F_i(T) - F_j(T)) \geq l$

# Part II

## Sequences



## Chapter 3

# Supremum and infimum

Def:

Let  $(X, \leq)$  be a partially ordered set  $A$  and  $Y$  be subsets of  $X$ , such that  $A \subseteq Y$

- If the set  $\{y \in Y \mid \forall a \in A, a \leq y\}$  has a least element then we say that  $A$  has a Supremum in  $Y$  with respect to  $\leq$  denoted by  $\sup_{(Y, \leq)} A$  this least element and called it the Supremum of  $A$  in  $Y$  (this respect to  $\leq$ )
- If the set  $\{y \in Y \mid \forall a \in A, y \leq a\}$  has a greatest element, we say that  $A$  has an infimum in  $Y$  with respect to  $\leq$ . We denote by  $\inf_{(Y, \leq)} A$  this greatest element and call it the infimum of  $A$  in  $Y$
- Observation:  $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set.

- If  $f$  is a function from  $I$  to  $X$   $\sup f$  denotes the supremum of  $f(I)$  is  $X$ .  $\inf f$  takes the same
- If  $(x_i)_{i \in I}$  is a family of element in  $X$ , then  $\sup x_i$  denotes  $\sup\{x_i \mid i \in I\}$  (in  $X$ )

If moreover  $\mathbb{P}(\cdot)$  denotes a statement depending on a parameter in  $I$  then  $\sup_{i \in I, \mathbb{P}(i)} x_i$  denotes  $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let  $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$  We equip  $\mathbb{R}$  with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So  $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence  $\inf A = 0$

Example: For  $n \in \mathbb{N}$ , let  $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let  $(X, \leq)$  be a partially ordered set,  $A, Y, Z$  be subset of  $X$ , such that  $A \subseteq Z \subseteq Y$

- If  $\max A$  exists, then it is also equal to  $\sup_{(y, \leq)} A$
- If  $\sup_{(y, \leq)} A$  exists and belongs to  $Z$ , then it is equal to  $\sup A$

$\inf$  takes the same Prop.

Let  $X, \leq$  be a partially ordered set,  $A, B, Y$  be subsets of  $X$  such that  $A \subseteq B \subseteq Y$

- If  $\sup_{(y, \leq)} A$  and  $\sup_{(y, \leq)} B$  exists, then  $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If  $\inf_{(y, \leq)} A$  and  $\inf_{(y, \leq)} B$  exists, then  $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set and  $f, g : I \rightarrow X$  be mappings such that  $\forall t \in I, f(t) \leq g(t)$

- If  $\inf f$  and  $\inf g$  exists, then  $\inf f \leq \inf g$
- If  $\sup f$  and  $\sup g$  exists, then  $\sup f \leq \sup g$

## Chapter 4

# Interval

We fix a totally ordered set  $(X, \leq)$

Notation:

If  $(a, b) \in X \times X$  such that  $a \leq b$ ,  $[a, b]$  denotes  $\{x \in X \mid a \leq x \leq b\}$

Def:

Let  $I \subseteq X$ . If  $\forall (x, y) \in I \times I$  with  $x \leq y$ , one has  $[x, y] \subseteq I$  then we say that  $I$  is an interval in  $X$

Example:

Let  $(a, b) \in X \times X$ , such that  $a \leq b$ . Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let  $\Lambda$  be a non-empty set and  $(I_\lambda)_{\lambda \in \Lambda}$  be a family of intervals in  $X$ .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$
- If  $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ ,  $\bigcup_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$

We check that  $[a, b] \subseteq I_\lambda \cup I_\mu$

- If  $b \leq x$   $[a, b] \subseteq [a, x] \subseteq I_\lambda$  because  $\{a, x\} \subseteq I_\lambda$
- If  $x \leq a$   $[a, b] \subseteq [x, b] \subseteq I_\mu$  because  $\{b, x\} \subseteq I_\mu$
- If  $a < x < b$  then  $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let  $(X, \leq)$  be a totally ordered set.  $I$  be a non-empty interval of  $X$ . If  $\sup I$  exists in  $X$ , we call  $\sup I$  the right endpoint;  $\inf$  takes the similar way.

Prop.

Let  $I$  be an interval in  $X$ .

- Suppose that  $b = \sup I$  exists.  $\forall x \in I, [x, b] \subseteq I$
- Suppose that  $a = \inf I$  exists.  $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let  $I$  be an interval in  $X$ . Suppose that  $I$  has supremum  $b$  and an infimum  $a$  in  $X$ . Then  $I$  is equal to one of the following sets  $[a, b]$   $[a, b[$   $]a, b]$   $]a, b[$

Def

let  $(X, \leq)$  be a totally ordered set. If  $\forall (x, z) \in X \times X$ , such that  $x < z \quad \exists y \in X$  such that  $x < y < z$ , then we say that  $(X, \leq)$  is thick

Prop.

Let  $(X, \leq)$  be a thick totally ordered set.  $(a, b) \in X \times X, a < b$  If  $I$  is one of the following intervals  $[a, b]; [a, b[; ]a, b]; ]a, b[$  Then  $\inf I = a \quad \sup I = b$  (for it's thick empty set is impossible)

Proof:

Since  $X$  is thick, there exists  $x_0 \in ]a, b[$  By definition,  $b$  is an upper bound of  $I$ . If  $b$  is not the supremum of  $I$ , there exists an upper bound  $M$  of  $I$  such that  $M \neq b$ . Since  $X$  is thick, there is  $M' \in X$  such that  $x_0 \leq M, M' < b$  Since  $[x_0, b] \subseteq I$  Hence  $M$  and  $M'$  belong to  $I$ , which conflicts with the uniqueness of supremum.



## Chapter 5

# Enhanced real line

Def:

Let  $+\infty$  and  $-\infty$  be two symbols that are different and don't belong to  $\mathbb{R}$ . We extend the usual total order  $\leq$  on  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus  $\mathbb{R} \cup \{-\infty, +\infty\}$  becomes a totally ordered set, and  $\mathbb{R} = ]-\infty, +\infty[$ . Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in ]-\infty, +\infty[ \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in ]0, +\infty[ \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$   
**ARE NOT DEFINED**

Def

Let  $(X, \leq)$  be a partially ordered set. If for any subset  $A$  of  $X$ ,  $A$  has a supremum and an infimum in  $X$ , then we say that  $X$  is order complete.

Example

Let  $\Omega$  be a set  $(\mathcal{P}(\Omega), \subseteq)$  is order complete. If  $\mathcal{F}$  is a subset of  $\mathcal{P}(\Omega)$ ,  $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$

Interesting tip:  $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

**AXIOM :**

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$  is order complete

In  $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any  $A \subseteq \mathbb{R} \cup -\infty, +\infty$  and  $c \in \mathbb{R}$  We denote by  $A + c$  the set  $\{a + c \mid a \in A\}$
- If  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\lambda A$  denotes  $\{\lambda a \mid a \in A\}$
- $-A$  denotes  $(-1)A$

Prop.

For any  $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\sup(-A) = -\inf A$ ,  $\inf(-A) = -\sup A$  Def

We denote by  $(\mathbb{R}, \leq)$  a field  $\mathbb{R}$  equipped with a total order  $\leq$ , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$  such that  $a < b$ , one has  $\forall c \in \mathbb{R}$ ,  $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ ,  $ab > 0$
- $\forall A \subseteq \mathbb{R}$ , if  $A$  has an upper bound in  $\mathbb{R}$ , then it has a supremum in  $\mathbb{R}$

Prop.

Let  $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

$\inf$  takes the same

Theorem:

Let  $I$  and  $J$  be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If  $\{a, b\} \neq \{+\infty, -\infty\}$  then  $c = a + b$

$\inf$  takes the same if  $(-\infty) + (+\infty)$  doesn't happen

Corollary:

Let  $I$  be a non-empty set,  $f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

Then  $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

$\inf$  takes the similar ( $\leq \rightarrow \geq$ ) (provided when the sum are defined)

# Chapter 6

## Vector space

In this section:

$K$  denotes a unitary ring.

Let  $0$  be zero element of  $K$

$1$  be the unity of  $K$

### 6.1 $K$ -module

#### 6.1.1 Def

Let  $(V, +)$  be a commutative group. We call left/right  $K$ -module structure: any mapping  $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group  $(V, +)$  equipped with a left/right  $K$ -module structure is called a left/right  $K$ -module.

#### 6.1.2 Remark

Let  $K^{op}$  be the set  $K$  equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then  $K^{op}$  forms a unitary ring  
 Any left  $K^{op}$  - module is a right  $K$ -module  
 Any right  $K^{op}$  - module is a left  $K$ -module  
 $(K^{op})^{op} = K$

### 6.1.3 Notation

When we talk about a left/right  $K$ -module  $(V, +)$ , we often write its left  $K$ -module structure as  $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

### 6.1.4 $K$ -vector space

If  $K$  is commutative, then  $K^{op} = K$ , so left  $K$ -module and right  $K$ -module structure are the same. We simply call them  $K$ -module structure. A commutative group equipped with a  $K$ -module structure is called a  $K$ -module. If  $K$  is a field, a  $K$ -module is also called a  $K$ -vector space

Let  $\Phi : K \times V \rightarrow V$  be a left or right  $K$ -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence  $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$   
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$  is a morphism of groups. Hence  $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$  (*is a var*)

### 6.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping  $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$  is a left  $K$ -module structure

The mapping  $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$  is a right  $K$ -module structure

**6.1.6 Remark:**

We can also write an element  $\mu$  of  $K^I$  in the form of a family  $(\mu_i)_{i \in I}$  of elements in  $K$  ( $\mu_i$  is the image of  $i \in I$  by  $\mu$ )  
Then

$$\begin{aligned} (\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I} \end{aligned}$$

**6.2 sub K-module****6.2.1 Def**

Let  $V$  be a left/right  $K$ -module. If  $W$  is a subgroup of  $V$ . Such that  $\forall a \in K, \forall x \in W \quad ax/xa \in W$ , then we say that  $W$  is left/right sub- $K$ -module of  $V$ .

**6.2.2 Example**

Let  $I$  be a set. Let  $K^{\oplus I}$  be the subset of  $K^I$  composed of mappings  $f : I \rightarrow K$  such that  $I_f = \{x \in I \mid f(x) \neq 0\}$  is finite. It is a left and right sub- $K$ -module of  $K^I$

In fact,  $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$   
Hence  $f - g \in K^{\oplus I}$  So  $K^{\oplus I}$  is a subgroup of  $K^I$   
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

**6.3 morphism of K-modules****6.3.1 Def**

Let  $V$  and  $W$  be left  $K$ -module, A morphism of groups  $\phi : V \rightarrow W$  is called a morphism of left  $K$ -modules if  $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

**6.3.2 K-linear mapping**

If  $K$  is commutative, a morphism of  $K$ -modules is also called a  $K$ -linear mapping. We denote by  $\text{hom}_{K\text{-Mod}}(V, W)$  the set of all morphism of left- $K$ -module from  $V$  to  $W$ . This is a subgroup of  $W^V$

**6.3.3 Theorem**

Let  $V$  be a left  $K$ -module. Let  $I$  be a set.  
The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$  is a bijection where  
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.3.4 Remark:column

In the case where  $I = 1, 2, 3, \dots, n$   $V^I$  is denoted as  $V^n$ ,  $K^I$  is denoted as  $K^n$ . For any  $(x_1, \dots, x_n) \in V^n$ , by the theorem, there exists a unique morphism of left  $K$ -modules  $\phi : K^n \rightarrow V$  such that  $\forall i \in 1, \dots, n, \phi(e_i) = x_i$ .

We write this  $\phi$  as a column  $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ . It sends  $(a_1, \dots, a_n) \in K^n$  to  $a_1x_1 + \dots + a_nx_n$ .

## 6.4 kernel

### 6.4.1 Prop

Let  $G$  and  $H$  be groups and  $f : G \rightarrow H$  be a morphism of groups

- $Im(f) \subseteq H$  is a subgroup of  $H$
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- $f$  is injection iff  $\ker(f) = \{e_G\}$

### 6.4.2 Def

$\ker(f)$  is called the kernel of  $f$

### 6.4.3 Theorem

$f$  is injection iff  $\ker(f) = \{e_G\}$

### Proof

Let  $e_G$  and  $e_H$  be neutral element of  $G$  and  $H$  respectively

- (1) Let  $x$  and  $y$  be element of  $G$   
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ . So  $Im(f)$  is a subgroup of  $H$
- (2) Let  $x$  and  $y$  be element of  $\ker(f)$ . One has  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$ . So  $xy^{-1} \in \ker(f)$ . So  $\ker(f)$  is a subgroup of  $G$ .
- (3) Suppose that  $f$  is injection.  
 Since  $f(e_G) = e_H$  one has  $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ . Suppose that  $\ker(f) = \{e_G\}$ . If  $f(x) = f(y)$  then  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$ .  
 Hence  $xy^{-1} = e_G \Rightarrow x = y$

### 6.4.4 Def

Let  $(V, +)$  be a commutative group,  $I$  be a set. We define a composition law  $+$  on  $V^I$  as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then  $V^I$  forms a commutative group

### 6.4.5 Remark

Let  $E$  and  $F$  be left  $K$ -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$  is a subgroup of  $F^E$

In fact  $f$  and  $g$  are elements of  $\text{hom}_{K\text{-Mod}}(E, F)$ , then  $f - g$  is also a morphism of left  $K$ -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

### 6.4.6 Theorem

Let  $V$  be a left  $K$ -module,  $I$  be a set. The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$  is an isomorphism of groups, where  $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.4.7 Proof:

One has  $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So  $\Psi$  is a morphism of groups

injectivity Let  $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$  Such that  $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence  $\phi$  is the neutral element.

surjectivity Let  $x = (x_i)_{i \in I} \in V^I$  We define  $\phi_x : K^{\oplus I} \rightarrow V$  such that  $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left  $K$ -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that  $K'$  is a unitary ring, and  $V$  is also equipped with a right  $K'$ -module structure, Then  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$  is a right sub- $k'$ -module, and  $\Psi$  in the theorem is a right  $K'$ -module isomorphism





## Chapter 7

# Monotone mappings

### 7.1 Def

Let  $I$  and  $X$  be partially ordered sets,  $f : I \rightarrow X$  be a mapping.

- If  $\forall (a, b) \in I \times I$  such that  $a < b$ . One has  $f(a) \leq f(b)$ , then we say that  $f$  is increasing. decreasing takes similar way.
- If  $f$  is (strictly) increasing or decreasing, we say that  $f$  is (strictly) monotone.

### 7.2 Prop.

Let  $X, Y, Z$  be partially ordered sets.  $f : X \rightarrow Y, g : Y \rightarrow Z$  be mappings

- If  $f$  and  $g$  have the same monotonicity, then  $g \circ f$  is increasing
- If  $f$  and  $g$  have different monotonicities, then  $g \circ f$  is decreasing

strict monotonicities takes the same

### 7.3 Def

Let  $f$  be a function from a partially ordered set  $I$  to another partially ordered set  $X$ . If  $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$  is (strictly) increasing/decreasing then we say that  $f$  is (strictly) increasing/decreasing

### 7.4 Prop.

Let  $I$  and  $X$  be partially ordered sets.  $f$  be function from  $I$  to  $X$ .

- If  $f$  is increasing/decreasing and  $f$  is injection, then  $f$  is strictly increasing/decreasing
- Assume that  $I$  is totally ordered and  $f$  is strictly monotone, then  $f$  is injection

## 7.5 Prop

Let  $A$  be totally ordered set,  $B$  be a partially ordered set,  $f$  be an injective function from  $A$  to  $B$

If  $f$  is increasing/decreasing, then so is  $f^{-1}$

## 7.6 Def

Let  $X$  and  $Y$  be partially ordered sets.  $f : X \rightarrow Y$  be a bijection. If both  $f$  and  $f^{-1}$  are increasing, then we say that  $f$  is an isomorphism of partially ordered sets.

(If  $X$  is totally, then a mapping  $f : X \rightarrow Y$  is an isomorphism of partially ordered sets iff  $f$  is a bijection and  $f$  is increasing)

## 7.7 Prop.

Let  $I$  be a subset of  $\mathbb{N}$  which is infinite. Then there is a unique increasing bijection  $\lambda_I : \mathbb{N} \rightarrow I$

## 7.8 Proof

### 7.8.1 bijection

We construct  $f : \mathbb{N} \rightarrow I$  by induction as follows.

Let  $f(0) = \min I$  Suppose that  $f(0), \dots, f(n)$  are constructed

then we take  $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since  $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$ . Therefore  $f(n) \leq f(n+1)$

Since  $f(n+1) \notin \{f(0), \dots, f(n)\}$ , we have  $f(n) < f(n+1)$

Hence  $f$  is strictly increasing and this is injective

If  $f$  is not surjective, then  $I \setminus \text{Im}(f)$  has a element  $N$ .

Let  $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$ .

Since  $N \notin \text{Im}(f)$ ,  $N < f(m)$ .

So  $m \neq 0$ . Hence  $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition,  $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$ ,

Hence  $f(m) \leq N$ , causing contradiction.

**7.8.2 uniqueness**

exercise: Prove that  $Id_{\mathbb{N}}$  is the only isomorphism of partially ordered sets from  $\mathbb{N}$  to  $\mathbb{N}$



# Chapter 8

## sequence and series

Let  $I \subseteq \mathbb{N}$  be a infinite subset

### 8.1 Def

Let  $X$  be a set. We call sequence in  $X$  parametrized by  $I$  a mapping from  $I$  to  $X$ .

### 8.2 Remark

If  $K$  is a unitary ring and  $E$  is a left  $K$ -module then the set of sequence  $E^I$  admits a left- $K$ -module structure. If  $x = (x_n)_{n \in I}$  is a sequence in  $E$ , we define a sequence  $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ , called the series associated with the sequence  $x$ .

### 8.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$  is a morphism of left- $K$ -module

### 8.4 proof

Let  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$  be elements of  $E^I$

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

### 8.5 Prop

Let  $I$  be a totally ordered set .  $X$  be a partially ordered set,  $f : I \rightarrow X$  be a mapping ,  $J \in I$  Assume that  $J$  does not have any upper bound in  $I$

- If  $f$  is increasing ,then  $f(I)$  and  $f(J)$  have the same upper bounds in  $X$
- If  $f$  is decreasing ,then  $f(I)$  and  $f(J)$  have the same lower bounds in  $X$

## 8.6 limit

### 8.6.1 Def

Let  $i \subseteq \mathbb{N}$  be a infinite subset.  $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$  where  $[-\infty, +\infty]$  denotes  $\mathbb{R} \cup \{-\infty, +\infty\}$ , we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left( \inf_{i \in I, i \geq n} x_i \right)$$

If  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$ , we then say that  $(x_n)_{n \in I}$  tends to  $l$  and that  $l$  is the limit of  $(x_n)_{n \in I}$ . If in addition  $(x_n)_{n \in I} \in \mathbb{R}^I$  and  $l \in \mathbb{R}$ , we say that  $(x_n)_{n \in I}$  converges to  $l$

### 8.6.2 Remark

If  $J \subseteq \mathbb{N}$  is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} = \inf_{n \in J} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left( \inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in  $(x_i)_{i \in I}$  the limit superior and the limit inferior do not change.

In fact, if we take  $J = \mathbb{N} \setminus \{0, \dots, m\}$ , then  $\inf_{n \in J}(\dots)$  and  $\sup_{n \in J}(\dots)$  only depends on the values of  $x_i, i \in I, i \geq m$

### 8.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

**8.6.4 Prop**

Let  $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

**8.6.5 Prop**

Let  $(x_n)_{n \in I}$  be elements in  $[-\infty, +\infty]^I$ . Suppose that there exists  $N_0 \in \mathbb{N}$  such that  $\forall n \in I, n \geq N_0$ , one has  $x_n \leq y_n$ . Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

**8.6.6 Theorem**

Let  $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$  one has  $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$  and  $(z_n)_{n \in I}$  tend to the same limit  $l$

Then  $(y_n)_{n \in I}$  tends to  $l$

**8.6.7 Def**

Let  $I$  be an infinite subset of  $\mathbb{N}$ , and  $(x_n)_{n \in I}$  be a sequence in some set  $X$ . We call subsequence of  $(x_n)_{n \in I}$  a sequence of the form  $(x_n)_{n \in J}$ , where  $J$  is an infinite subset of  $I$

**8.6.8 Prop**

Let  $I$  and  $J$  be infinite subset of  $\mathbb{N}$  such that  $J \subseteq I$ .  $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$ , one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if  $(x_n)_{n \in I}$  tends to  $l \in [-\infty, +\infty]$ , then  $(x_n)_{n \in J}$  tends to  $l$

### 8.6.9 Prop

$\forall n \in \mathbb{N}$ , one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

### 8.6.10 Theorem

Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$

- If the mapping  $(n \in I) \mapsto x_n$  is increasing, then  $(x_n)_{n \in I}$  tends to  $\sup_{n \in I} x_n$
- If the mapping  $(n \in I) \mapsto x_n$  is decreasing, then  $(x_n)_{n \in I}$  tends to  $\inf_{n \in I} x_n$

### 8.6.11 Notation

If a sequence  $(x_n)_{n \in I} \in [-\infty, +\infty]$  tends to some  $l \in [-\infty, +\infty]$  the expression  $\lim_{n \in I, n \rightarrow} x_n$  denotes this limit  $l$

### 8.6.12 Corollary

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{N}_{\geq 0}$ . Then the series  $\sum_{n \in I} x_n$  (the sequence  $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ ) tends to an element in  $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ . It converges in  $\mathbb{R}$  iff it is bounded from above (namely has an upper bound in  $\mathbb{R}$ )

### 8.6.13 Notation

If a series  $\sum_{n \in I} x_n$  in  $[-\infty, +\infty]$  tends to some limit, we use the expression  $\sum_{n \in I} x_n$  to denote the limit

### 8.6.14 Theorem: Bolzano-Weierstrass

Let  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\limsup_{n \in I, n \rightarrow +\infty} x_n$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\liminf_{n \in I, n \rightarrow +\infty} x_n$ .



**Proof**

Let  $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If  $J$  is infinite, the sequence  $(x_n)_{n \in J}$  is decreasing so it tends to  $\inf_{n \in J} x_n$

$\forall n \in J$  by definition  $x_n = \sup_{i \in I, i \geq n} x_i$  so  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that  $J$  is finite. Let  $n_0 \in I$  such that  $\forall n \in J, n < n_0$ . Denote by

$l = \sup_{n \in I, n \geq n_0}$

Let  $N \in \mathbb{N}$  such that  $N \geq n_0$ . By definition  $\sup_{i \in I, i \geq n_0} x_i \leq l$ . If the strict inequality  $\sup_{i \in I, i \geq N} x_i < l$  holds, then  $\sup_{i \in I, i \geq N} x_i$  is NOT an upper bound of  $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists  $n \in I$  such that  $n_0 \leq n < N$  such that  $x_n > \sup_{i \in I, i \geq N} x_i$ . We may also assume that  $n$  is largest among elements of  $I \cap [n_0, N[$  that satisfies this inequality.

Then  $\forall m \in I$  if  $m \geq n$  then  $x_m \leq x_n$ . Thus  $n \in J$  that contradicts the maximality of  $n_0$ .

Therefore

$$l = \sup_{i \in I, i \geq N} x_i$$

, which leads to

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = l$$

Moreover, if  $m \in I, m \geq n_0$  then  $m \notin J$ , so  $x_m < l$  (since otherwise  $x_m = \sup_{i \in I, i \geq m} x_i$  and hence  $m \in J$ ). Hence,  $\forall$  finite subset  $I'$  of  $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$  and hence  $\exists n \in I$ , such that  $n > \max I'$ , and  $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence  $(n_j)_{j \in \mathbb{N}}$  in  $I$

Let  $n_0$  be as above. Let  $f : \mathbb{N} \rightarrow I_{\geq n_0}$  be a surjective mapping.

If  $n_j$  is chosen, we choose  $n_{j+1} \in I$  such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence  $(x_{n_j})_{j \in \mathbb{N}}$  is increasing

And

$$\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n = l$$

$$l = \sup_{n \in I, n \geq n_0} x_n$$

So  $(x_{n_j})_{j \in \mathbb{N}}$  tends to  $l$



## Chapter 9

# Cauchy sequence

### 9.1 Def

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{R}$   
If  $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$  then  
we say that  $(x_n)_{n \in I}$  is a Cauchy sequence

### 9.2 Prop

- If  $(x_n)_{i \in I} \in \mathbb{R}^I$  converges to some  $l \in \mathbb{R}$ , then it is a Cauchy sequence
- If  $(x_n)_{i \in I}$  is a Cauchy sequence, there exists  $M > 0$  such that  $\forall n \in I \quad |x_n| \leq M$
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite,  $(x_n)_{n \in I}$  is a Cauchy sequence.
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite and  $l \in \mathbb{R}$  such that  $(x_n)_{n \in I}$  converges to  $l$ , then  $(x_n)_{n \in J}$  converges to  $l$  too.

### 9.3 Theorem: Completeness of real number

If  $(x_n)_{n \in I} \in \mathbb{R}^I$  is a Cauchy sequence, then it converges in  $\mathbb{R}$

#### Proof

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists M \in \mathbb{R}_{>0}$  such that  $-M \leq x_n \leq M \quad \forall x \in I$ . So  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . By Bolzano-Weierstrass theorem.  $\exists J \subseteq I$  infinite such that  $(x_n)_{n \in I}$  converges to  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . Therefore  $(x_n)_{n \in I}$  converges to the same limit.

## 9.4 Absolutely converge

We say that a series  $\sum_{n \in I} x_n \in \mathbb{R}$  converges absolutely if  $\sum_{n \in I} |x_n| < +\infty$

### 9.4.1 Prop

If a series  $\sum_{n \in I} x_n$  converges absolutely, then it converges in  $\mathbb{R}$

## Chapter 10

# Comparison and Technics of Computation

### 10.1 Def

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be sequence in  $\mathbb{R}$

- If there exists  $M \in \mathbb{R}_{>0}$  and  $N \in \mathbb{N}$  such that  $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$  then we write  $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists  $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$  and  $N \in \mathbb{N}$  such that  $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$  and  $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$ , then we write  $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

### 10.2 Prop.

Let  $I$  and  $X$  be partially ordered sets and  $f : I \rightarrow X$  be an increasing/decreasing mapping. Let  $J$  be a subset of  $I$ . Assume that any elements of  $I$  has an upper bound in  $J$ . Then  $f(I)$  and  $f(J)$  have the same upper/lower bounds in  $X$

### 10.3 Theorem

Let  $I$  be a totally ordered set,  $f : I \rightarrow [-\infty, +\infty]$  and  $g : I \rightarrow [-\infty, +\infty]$  be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

### Proof

We can assume  $f$  and  $g$  increasing. Let  $a = \sup f(I), b = \sup g(I)$

Let  $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip  $A$  with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let  $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$ .

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

$h$  is increasing.

Let  $(x, y) \in A$ . Assume that  $x \leq y$

If  $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$  then  $(y, y) \in B$  and  $(x, y) \leq (y, y)$

If  $\{f(y), g(y)\} = \{-\infty, +\infty\}$  and for  $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$ . So  $a = +\infty$ , Hence  $b > -\infty$

So  $\exists z \in I$  such that  $g(z) > -\infty$ . We should have  $y \leq z$  Hence  $f(z) + g(z)$  is well defined,  $(z, z) \in B$  and  $(x, y) \leq (z, z)$  Similarly, if  $x \geq y$ ,  $(x, y)$  has also an upper bound in  $B$ . Therefore:  $\sup h(A) = \sup h(B)$

## 10.4 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

### Proof

$\forall n \in \mathbb{N}$ , let  $A_N = \sup_{n \in I, n \geq N} x_n$   $B_N = \sup_{n \in I, n \geq N} y_n$ .  $(A_N)_{N \in \mathbb{N}}$  and  $(B_N)_{N \in \mathbb{N}}$  are decreasing, and  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$   $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let  $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$  if  $A_N + B_N$  is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

## 10.5 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + \left( \limsup_{n \in I, n \rightarrow +\infty} y_n \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + \left( \liminf_{n \in I, n \rightarrow +\infty} y_n \right)$$

### Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with  $\infty$

## 10.6 Theorem

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Assume that  $\forall n \in I, y_n \in \mathbb{R}$  and  $(y_n)_{n \in I}$  converges to some  $l \in \mathbb{R}$ .  
Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

## 10.7 Prop.

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ .  
Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

### Proof

About the first inequality. Since  $\max\{x_n, y_n\} \geq x_n$  and  $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset  $J$  of  $I$  such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let  $J_1 = \{n \in J \mid x_n \geq y_n\}$   $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$  So either  $J_1$  or  $J_2$  is infinite

Suppose that  $J_1$  is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If  $J_2$  is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

## 10.8 Theorem

Let  $(a_n)_{n \in I} \in \mathbb{R}^I$   $l \in \mathbb{R}$ . The following statements are equivalent

- $(a_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

### Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1)  $\Rightarrow$  (2):

If  $(a_n)_{n \in I}$  converges to  $l$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2)  $\Rightarrow$  (1):

If  $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore:  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

## 10.9 Remark

Let  $(a_n)_{n \in I}$  be a sequence in  $\mathbb{R}$ ,  $l \in \mathbb{R}$

The sequence  $(a_n)_{n \in I}$  converges to  $l$  iff  $a_n - l = o(1), n \in I, n \rightarrow +\infty$

## 10.10 Calculates on $O()$ , $o()$

### 10.10.1 Plus

Let  $(a_n)_{n \in I}$   $(a'_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$



**10.10.2 Transform**

Let  $(a_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be two sequence in  $\mathbb{R}$  If  $a_n = o(b_n), n \in I, n \rightarrow +\infty$ , then  $a_n = O(b_n), n \in I, n \rightarrow +\infty$

**10.10.3 Transition**

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  and  $(c_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = O(b_n)$  and  $b_n = o(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$

**10.10.4 Times**

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$  be sequences in  $\mathbb{R}$

- If  $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If  $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

**10.11 On the limit**

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be elements of  $\mathbb{R}^I$  that converges to  $l \in \mathbb{R}$  and  $l' \in \mathbb{R}$  respectively. Then:

- $(a_n + b_n)_{n \in I}$  converges to  $l + l'$
- $(a_n b_n)_{n \in I}$  converges to  $ll'$

**10.12 Prop**

Let  $a \in \mathbb{R}$  then  $a^n = o(n!)$   $n \rightarrow +\infty$

**Proof**

Let  $N \in \mathbb{N}$  such that  $|a| < N$   
For  $n \in \mathbb{N}$  such that  $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^n - N$$

And  $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$ . Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

### 10.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

#### Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

### 10.14 Prop

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be the elements of  $\mathbb{R}^I$ . If the series  $\sum_{n \in I} b_n$  converges absolutely and if  $a_n = O(b_n) \quad n \rightarrow +\infty$  Then  $\sum_{n \in I} a_n$  converges absolutely

#### Proof

By definition  $\sum_{n \in I} |b_n| < +\infty$ . If  $|a_n| \leq M|b_n|$  for  $n \in I, n \geq N$  where  $N \in \mathbb{N}$ . Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

### 10.15 Theorem: d'Alembert ratio test

Let  $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely
- If  $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  does not converge (diverges)

**Proof****(1)**

Let  $\alpha \in \mathbb{R}$  such that  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$ ,  $\alpha$  isn't a lower bound of  $\left( \sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$   
 So  $\exists N \in \mathbb{N}$  such that  $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$  Hence for  $n \geq N$   $|a_n| \leq \alpha^{n-N} |a_N|$  since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore  $a_n = O(\alpha^n)$  since  $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$ ,  $\sum_{n \in \mathbb{N}} a_n$  converge absolutely.

**10.15.1 Lemma**

If a series  $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$  converges, then  $\lim_{n \rightarrow +\infty} a_n = 0$

**Proof**

If  $\left( \sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$  converges to some  $l \in \mathbb{R}$ , then  $\left( \sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$  converges to  $l$ ,  
 too. Hence  $\left( a_n = \left( \sum_{i=0}^n a_i \right) - \left( \sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$  converges to  $l - l = 0$

**10.15.2 (2)**

Let  $\beta \in \mathbb{R}$  such that  $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 So there exists  $N \in \mathbb{N}$  such that  $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$   
 Hence  $(|a_n|)_{n \in \mathbb{N}}$  is not bounded since  $|a_n| \geq \beta^{n-N} |a_N|$   
 By the lemma:  $\sum_{n \in \mathbb{N}} a_n$  diverges.

**10.16 Prop**

Let  $a \in \mathbb{R}, a > 1$  Then  $n = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $\epsilon > 0$  such that  $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

**10.16.1 Corollary**

Let  $a > 1, t \in \mathbb{R}_{\geq 0}$  Then  $n^t = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $d \in \mathbb{N}_{\geq 1}$  such that  $t \leq d$  Then  $n^{t-d} \leq 1$  So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let  $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence  $n^t = o(a^n)$

**10.16.2 Corollary**

There exists  $M \geq 1$  such that  $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

**Proof**

Let  $a \in \mathbb{R}$  such that  $1 < a < e$

**10.17 Theorem: Cauchy root test**

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Let  $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If  $\alpha < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.
- If  $\alpha > 1$  then  $\sum_{n \in \mathbb{N}} a_n$  diverges

**Proof**

(1)

Let  $\beta \in \mathbb{R}, \alpha < \beta < 1$ . There exists  $N \in \mathbb{N}$  such that  $|a_n|^{\frac{1}{n}} \leq \beta$  for  $n \geq N$ . That means  $|a_n| = O(\beta^n)$  since  $0 < \beta < 1$ ,  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.

(2)

If  $\alpha > 1$  then  $\forall N \in \mathbb{N} \exists n \geq N$  such that  $|a_n|^{\frac{1}{n}} \geq 1$ , since otherwise  $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$  contradiction  
Hence  $(|a_n|)_{n \in \mathbb{N}}$  cannot converge to 0.

## Part III

# Axiom of choice



# Chapter 11

## Preparation

### 11.1 Statement of axiom of choice

For any set  $I$  and any family  $(A_i)_{i \in I}$  of non-empty sets, there exists a mapping  $f : I \rightarrow \bigcup_{i \in I} A_i$  such that  $\forall i \in I, f(i) \in A_i$

### 11.2 Def

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$   $A$  is non-empty, there exists a least element of  $A$  then we say that  $(X, \leq)$  is a well ordered set.

### 11.3 Theorem

For any set  $X$ , there exists an order relation  $\leq$  on  $X$  such that  $(X, \leq)$  forms a well ordered set.

### 11.4 Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$  that is totally ordered with respect to  $\leq$ , there exists an upper bound of  $A$  inside  $X$ . Then, there exists a maximal element  $x_0$  of  $X$  ( $\forall y \in X, y > x_0$  does not hold)

### 11.5 Prop.

Let  $(X, \leq)$  be a well ordered set,  $y \notin X$ . We extend  $\leq$  to  $X \cup \{y\}$ , such that  $\forall x \in X, x < y$ . Then  $(X \cup \{y\}, \leq)$  is well ordered.

## 11.6 Proof

Let  $A \subseteq X \cup \{y\}$ ,  $A \neq \emptyset$ . If  $A = \{y\}$  then  $Y$  is the least element of  $A$ . If  $A \neq \{y\}$  then  $B = A \setminus \{y\}$  is non-empty. Let  $b$  be the least element of  $B$ . Since  $b < y$  it's also the least element of  $A$

## 11.7 Def: Initial Segment

Let  $(X, \leq)$  be a well ordered set.  $S \subseteq X$ , If  $\forall s \in S, x \in X \quad x < s$  initial  $x \in S$  ( $X_{<s} \subseteq S$ ), then we say that  $S$  is an initial segment of  $X$

If  $S$  is a initial segment such that  $S \neq X$  then we sat that  $S$  is a proper initial segment.

## 11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$  Then  $X_{<x}$  is a proper initial segment of  $X$ .

## 11.9 Prop.

Let  $(X, \leq)$  be a well ordered set , If  $(S_i)_{i \in I}$  is a family of initial segment of  $X$ , then  $\bigcup_{i \in I} S_i$  is an initial segment of  $X$

## 11.10 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$  such that  $s \in S_i, i \in I$  Therefore  $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

## 11.11 Prop

Let  $(X, \leq)$  be a well ordered set.

- (1) Let  $S$  be a proper initial segment of  $X$ ,  $x = \min(X \setminus S)$  Then  $S = X_{<x}$
- (2)  $X \rightarrow \wp(X)$   
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of  $X$  forms a well ordered subset of  $(\wp(X), \subseteq)$

## 11.12 Proof

- (1)  $\forall s \in S$  if  $x \leq s$  then  $x \in S$  contradiction. Hence  $s < x$ , This shows  $S \subseteq X_{<x}$  Conversely , if  $t \in X, t \notin X \setminus S$  Hence  $t \in S$ . Hence  $X_{<x} \subseteq S$



- (2) Let  $x, y \in X, x < y$  By definition  $X_{<x} \subseteq X_{<y}$  Moreover  $x \in X_{<y} \setminus X_{<x}$  So  $X_{<x} \subsetneq X_{<y}$
- (3) Let  $\mathcal{F} \subseteq \wp(X)$  be a set of initial segments.  $\mathcal{F} \neq \emptyset$ . Then there exists  $A \subseteq X$  such that  $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$  If  $A = \emptyset$  then  $\mathcal{F} = \{X\}$ , and  $\{X\}$  is the least element of  $\mathcal{F}$ . Otherwise  $A \neq \emptyset$  and  $A$  has a least element  $a$ . Then by (2)  $X_{<a}$  is the least element of  $\mathcal{F}$

### 11.13 Lemma

Let  $(X, \leq)$  be a well ordered set,  $f : X \rightarrow X$  be a strictly increasing mapping. Then  $\forall x \in X, x \leq f(x)$

#### Proof

Let  $A = \{x \in X \mid f(x) < x\}$  If  $A \neq \emptyset$ , let  $a$  be the least element of  $A$ . By definition  $f(a) < a$ . Hence  $f(f(a)) < f(a)$  since  $f$  is strictly increasing. This shows  $f(a) \in A$ . But  $a$  is the least element of  $A$ ,  $f(a) < a$  cannot hold: contradiction.

### 11.14 Prop

Let  $(X, \leq)$  be a well ordered set,  $S$  and  $T$  be two initial segment of  $X$ . If  $f : S \rightarrow T$  is a bijection that's strictly increasing, then  $S = T, f = Id_S$

#### Proof

We may assume  $T \subseteq S$ . Let  $l : T \rightarrow S$  be the inclusion mapping and  $g = l \circ f : S \rightarrow S$ . Since  $g$  is strictly increasing, by the lemma,  $\forall s \in S, s \leq g(s) = f(s) \in T$ . Since  $T$  is an initial segment,  $s \in T$ . Hence  $S = T$ . Apply the lemma to  $f^{-1}$  we get  $\forall s \in S, s \leq f^{-1}(s)$  Hence  $f(s) \leq s$  Therefore  $f(s) = s$

### 11.15 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be partially ordered sets. If  $\exists f : X \rightarrow Y$  that's increasing and bijective, we say that  $(X, \leq)$  and  $(Y, \leq)$  are isomorphic

### 11.16 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be well ordered sets. If  $(X, \leq)$  is isomorphic to an initial segment of  $Y$ . We note  $X \preceq Y$  or  $Y \succeq X$ . If  $X$  is isomorphic to  $Y$ , we note  $X \sim Y$ . If  $X \preceq Y$  but  $X \not\sim Y$ , we note  $X \prec Y$  or  $Y \succ X$

### 11.17 Prop.

Let  $X$  and  $Y$  be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

#### Proof

We construct a correspondence  $f$  from  $X$  to  $Y$ , such that  $(x, y) \in \Gamma_f$ , iff  $X_{<x} \sim Y_{<y}$   
By the last proposition of Oct. 11,  $f$  is a function.

- If  $a, b \in \text{Dom}(f)^2$ ,  $a < b$ , then  $X_{<a} \subsetneq X_{<b}$   
By definition,  $Y_{<f(b)} \sim X_{<b}$   $Y_{<f(a)} \sim X_{<a}$   
Hence  $Y_{<f(a)}$  is isomorphic to a proper initial segment of  $Y_{<f(b)}$ . Therefore  $Y_{f(a)}$  is a proper initial segment of  $Y_{<f(b)}$ . We then get  $f(a) < f(b)$ . Thus  $f$  is strictly increasing.
  - Let  $a \in \text{Dom}(f)$  Let  $x \in X, x < a$  Then  $X_{<x}$  is a initial segment of  $X_{<a} \sim Y_{<f(a)}$  Hence  $\exists y \in Y$   $X_{<x} \sim Y_{<y}$  This shows that  $x \in \text{Dom}(f)$ . Hence  $\text{Dom}(f)$  is an initial segment of  $X$ . Applying this to  $f^{-1}$ , we get :  $\text{Im}(f) = \text{Dom}(f)$  is an initial segment of  $Y$
  - Either  $\text{Dom}(f) = X$  or  $\text{Im}(f) = Y$ .  
Assume that  $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$  are respectively the least elements of  $X \setminus \text{Dom}(f)$  and  $Y \setminus \text{Im}(f)$ .  
Then we get  $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$ .  
We obtain  $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$ . Contradiction
  -
- Case 1  $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$   $X \prec Y$   
Case 2  $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$   $X \succ Y$   
Case 3  $\text{Dom}(f) = X, \text{Im}(f) = Y$   $X \sim Y$

### 11.18 Lemma

Let  $(X, \leq)$  be a partially ordered set .  $\mathfrak{S} \subseteq \wp(X)$ . Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$  is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$ , either  $A$  is an initial segment of  $B$ , or  $B$  is an initial segment of  $A$ .

Let  $Y = \bigcup_{A \in \mathfrak{S}} A$ . Then  $(Y, \leq)$  is a well ordered set, and  $\forall A \in \mathfrak{S}, A$  is an initial segment of  $Y$ .

**Proof**

- Let  $A \in \mathfrak{S}, x \in A, y \in Y, y < x$ . Since  $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$ , such that  $y \in B$ . If  $y \notin A$  then  $B \not\subseteq A$ . Hence  $A$  is an initial segment of  $B$ . Hence  $y \in A$ . Contradiction
- Let  $Z \subseteq Y, Z \neq \emptyset$ . Then  $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$ . Let  $m$  be the least element of  $A \cap Z$ . Let  $z \in Z, B \in \mathfrak{S}$ , such that  $z \in B$ . If  $z \in A$ , then  $m \leq z$ . If  $z \notin A$ , then  $A$  is an initial segment of  $B$ .

Since  $B$  is well ordered, if  $m \not\leq z$  then  $z < m$ . Since  $m \in A$ , we get  $z \in A$ . Contradiction.

Therefore,  $m$  is the least element of  $Z$ .



## Chapter 12

# Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. Suppose that any well-ordered subset of  $X$  has an upper bound on  $X$ , the  $X$  has a maximal element (a maximal element  $m$  of  $\{x \mid x > m\} = \emptyset$ )

### 12.1 Proof

Suppose that  $X$  doesn't have any maximal element.  $\forall A \in \omega. \exists f(A)$  such that  $\forall a \in A, a < f(A)$

Let

$$\omega = \{\text{well ordered subset of } X\}$$

. (guaranteed by axiom of choice)

Let  $f : \omega \rightarrow X$  such that  $f(A)$  is an upper bound of  $A \in \omega$ .

If  $A \in \omega$  satisfies

$$\forall a \in A, a = f(A_{<a})$$

, we say that  $A$  is a  $f$ -set

Let

$$\mathfrak{S} = \{f\text{-sets}\}$$

Note that

$$\emptyset \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$$

In fact, if  $a \in A$ , then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If  $a = f(A) \notin A$  then

$$(A \cup \{f(A)\})_{<a} = A$$

Let  $A$  and  $B$  be elements of  $\mathfrak{S}$ . Let  $I$  be the union of all common initial segments of  $A$  and  $B$ . This is also a common initial segment of  $A$  and  $B$ .

If  $I \neq A$  and  $I \neq B$ , then

$$\exists(a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then  $I \cup \{a\}$  is also a common initial segment of  $A$  and  $B$ , contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and  $\forall A \in \mathfrak{S}$  is an initial segment of  $Y$ .

Since  $A$  is an initial segment of  $Y$

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A \quad A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus  $Y$  is the greatest element of  $(\mathfrak{S}, \subseteq)$ . However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If  $f(y)$  is not a maximal element of  $X$

$$\exists x \in X, f(y) < x$$

**Part IV**

**Topology**





## Chapter 13

# Absolute value and norms

### 13.1 Def

Let  $K$  be a field. By absolute value on  $K$ , we mean a mapping  $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- (1)  $\forall a \in K \quad |a| = 0$  iff  $a = 0$
- (2)  $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3)  $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$  (triangle inequality)

### 13.2 Notation

$\mathbb{Q}$  Take a prime num  $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$  there exists a integer  $ord_p(\alpha) \frac{a}{b}$ , where  
 $a \in \mathbb{Z} \setminus \{0\}$   
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

### 13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on  $\mathbb{Q}$

### Proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd \\ \alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d} \\ \text{Assume } ord_p(\alpha) \geq ord_p(\beta) \\ \alpha + \beta \\ = p^{ord_p(\beta)} \left( p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right) \\ = p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd \\ \text{So}$$

$$ord_p(\alpha + \beta) \geq ord_p(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\} \\ \text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} = \\ \max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

### 13.4 Def

Let  $K$  be a field and  $|\cdot|$  be an absolute value. We call  $(K, |\cdot|)$  a valued field.

## Chapter 14

# Quotient Structure

### 14.1 Def

Let  $X$  be a set and  $\sim$  be a binary relation on  $X$   
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$ , if  $x \sim y$  then  $y \sim x$
- $\forall (x, y, z) \in X^3$ , if  $x \sim y, y \sim z$  then  $x \sim z$

then we say that  $\sim$  is an equivalence relation

### 14.2 equivalence class

$\forall x \in X$  we denote by  $[x]$  the set  $\{y \in X \mid y \sim x\}$  and call it the equivalence class of  $x$  on  $X$ . Let  $X/\sim$  be the set  $\{[x] \mid x \in X\}$

### 14.3 Prop.

Let  $X$  be a set and  $\sim$  be an equivalence relation on  $X$

- (1)  $\forall x \in X, y \in [x]$  on has  $[x] = [y]$
- (2) If  $\alpha$  and  $\beta$  are elements of  $X/\sim$  such that  $\alpha \neq \beta$  then  $\alpha \cap \beta = \emptyset$
- (3)  $X = \bigcup_{\alpha \in X/\sim} \alpha$

**Proof**

- (1) Let  $z \in [y]$ . Then  $y \sim z$ . Since  $y \in [x]$  one has  $x \sim y$ . Therefore,  $x \sim z$  namely  $z \in [x]$ . This proves  $[y] \subseteq [x]$ . Moreover, since  $x \sim y$ , one has  $x \in [y]$ . Hence  $[x] \subseteq [y]$ . Thus we obtain  $[x] = [y]$ .
- (2) Suppose that  $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$ . By (1),  $\alpha = [y], \beta = [y]$ . Thus leads to a contradiction.
- (3)  $\forall x \in X \quad x \in [x]$  Hence  $x \in \bigcup_{\alpha \in X/\sim} \alpha$ . Hence  $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$ . Conversely,  $\forall \alpha \in X/\sim, \alpha$  is a subset of  $X$ . Hence  $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$ . Then  $X = \bigcup_{\alpha \in X/\sim} \alpha$ .

**14.4 Def**

Let  $G$  be a group and  $X$  be a set. We call left/right action of  $G$  on  $X$  an mapping  $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$  that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

**14.5 Remark**

If we denote by  $G^{op}$  the set  $G$  equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of  $G$  on  $X$  is just a left action of  $G^{op}$  on  $X$ .

**14.6 Prop**

Let  $G$  be a group and  $X$  be a set. Assume given a left action of  $G$  on  $X$ . Then the binary relation  $\sim$  on  $X$  defined as  $x \sim y$  iff  $\exists g \in G \quad y = gx$  is an equivalence relation

**14.7 Notation on Equivalence Class**

We denote by  $G/X$  the set  $X/\sim \forall x \in X$  the equivalence class of  $x$  is denoted as  $Gx/xG$  or  $orb_G(x)$  call the orbit of  $x$  under the action of  $G$

## 14.8 Proof

- $\forall x \in X \quad x = 1x$  so  $x \sim x$
- $\forall (x, y) \in X^2$  if  $y = gx$  for same  $g \in G$  then  $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$ . ( $y \sim x$ )
- $\forall (x, y, z) \in X^3$ , if  $\exists (g, h) \in G^2$ , such that  $y = gx$  and then  $z = h(gx) = (hg)x$  So  $x \sim z$

## 14.9 Quotient set

Let  $X$  be a set and  $\sim$  be an equivalence relation, the mapping  $X \rightarrow X/\sim$ :  
 $(x \in X) \mapsto [x]$  is called the projection mapping.

$X/\sim$  is called the quotient set of  $X$  by equivalence relation  $\sim$

### 14.9.1 Example

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of  $H$  on  $G$ . Thus we obtain two quotient sets  $H/G$  and  $G/H$

## 14.10 Def

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . If  $\forall g \in G, h \in H \quad ghg^{-1} \in H$ ,  
 Then we say that  $H$  is a normal subgroup of  $G$

## 14.11 Remark

$\forall g \in G, gH = Hg$ , provided that  $H$  is a normal subgroup of  $G$ . In fact  $\forall h \in$ ,

- $\exists h' \in H$  such that  $ghg^{-1} = h'$  Hence  $gh = h'g$ . This shows  $gH \subseteq Hg$
- $\exists h'' \in H$  such that  $g^{-1}hg = h''$  Hence  $hg = gh''$ . This shows  $Hg \subseteq gH$

Thus  $gH = Hg$

## 14.12 Prop

If  $G$  is commutative, any subgroup of  $G$  is normal

### 14.13 Theorem

Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set  $G/H$ . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

#### Proof

- If  $xH = x'H, yH = y'H$  then  $\exists h_1 \in H, h_2 \in H$  such that  $x' = xh_1, y' = yh_2$ . Hence  $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$ . For  $y^{-1}h_1y, h_2 \in H$  then  $(x'y')H = (xy)H$ . So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

### 14.14 Def

Let  $K$  be a unitary ring and  $E$  be a left  $K$ -module. We say that a subgroup  $F$  of  $(E, +)$  is a left sub- $K$ -module of  $E$  if  $\forall (a, x) \in K \times F, ax \in F$

### 14.15 Prop

Let  $K$  be a unitary ring,  $E$  be a left  $K$ -module and  $F$  be a sub- $K$ -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- $K$ -module structure on  $E/F$ . Moreover, the projection mapping  $\pi : E \rightarrow E/F$  is a morphism of left- $K$ -modules

**Proof**

Let  $x$  and  $x'$  be elements of  $E$  such that  $[x] = [x']$ , that means:  $x' - x \in F$   
Hence  $a(x' - x) = ax' - ax \in F$  So  $[ax] = [ax']$   
Let us check that  $E/F$  forms a left  $K$ -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition,  $\pi$  is a morphism of groups. Moreover  $\forall x \in E, a \in K$   $\pi(ax) = [ax] = a[x] = a\pi(x)$

**14.16 Def**

Let  $A$  be a unitary ring . We call two-sided ideal any subgroup  $I$  of  $(A, +)$  that satisfies :  $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I$  ( $I$  is a left and right sub- $K$ -module of  $A$ )

**14.17 Theorem**

Let  $A$  be a unitary ring and  $I$  be a two sided ideal of  $A$  . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover ,  $A/I$  becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely  $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$ )

**Proof**

If  $a' \sim a, b' \sim b$  that means  $a' - a \in I, b' - b \in I$  then  $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$ . For  $(a' - a), (b' - b) \in I$ , then  $a'b' - ab \in I$   
Therefore  $a'b' \sim ab$

### 14.17.1 Reside Class

Let  $d \in \mathbb{Z}$  and  $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$   $d\mathbb{Z}$  is a two sided ideal of  $\mathbb{Z}$   
 If  $m \in \mathbb{Z}$ , for any  $a \in \mathbb{Z}$   $adm = dma \in d\mathbb{Z}$

Denote by  $\mathbb{Z}/d\mathbb{Z}$  the quotient ring. The class of  $n \in \mathbb{Z}$  in  $\mathbb{Z}/d\mathbb{Z}$  is called the residue class of  $n$  modulo  $d$

If  $A$  is a commutative unitary ring, a two sided ideal of  $A$  is simply called an ideal of  $A$

### 14.18 Theorem

Let  $f : G \rightarrow H$  be a morphism of groups

- (1)  $Im(f)$  is a subgroup of  $H$
- (2)  $\ker(f) := \{x \in G \mid f(x) = 1_H\}$  is a normal subgroup of  $G$
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4)  $f$  is injective iff  $\ker(f) = \{1_G\}$

### Proof

- (1) Let  $\alpha$  and  $\beta$  be elements of  $Im(f)$ . Let  $(x, y) \in G^2$  such that  $\alpha = f(x), \beta = f(y)$  Then  $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$  So  $Im(f)$  is a subgroup
- (2) Let  $x$  and  $y$  be elements of  $\ker(f)$ .  
 One has  $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$   
 So  $xy^{-1} \in \ker f$ . Hence  $\ker f$  is a subgroup of  $G$   
 Let  $x \in \ker f, y \in G$ .  
 One has  $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$  Hence  $yxy^{-1} \in \ker f$ . So  $\ker f$  is a normal subgroup
- (3) If  $x \sim y$  then  $\exists z \in \ker f$  such that  $y = xz$  Hence  $f(y) = f(x)f(z) = f(x)1_H = f(x)$  So  $f$  is well defined.  
 Moreover  $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$  Hence  $\tilde{f}$  is a morphism of groups.  
 By definition  $Im(\tilde{f}) = Im(f)$  If  $x$  and  $y$  are elements of  $G$  such that  $f(x) = f(y)$  then  $f(xy^{-1}) = 1_H$   
 Hence  $xy^{-1} \in \ker f$  Since  $x = (xy^{-1})y$ ,  $x \sim y$  that means  $[x] = [y]$   
 Therefore  $\tilde{f}$  is injective.



- (4) If  $f$  is injective,  $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$ , so  $x = 1_G$ . Therefore  $\ker f = \{1_G\}$ .  
 Conversely, suppose that  $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$  if  $f(x) = f(y)$  then  $f(x)f(y)^{-1} = 1_H$ . Hence  $xy^{-1} = 1_G, x = y$ .

### 14.19 Theorem

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Then

- (1)  $\text{Im}(f)$  is a left-sub- $K$ -module of  $F$
- (2)  $\ker(f)$  is a left-sub- $K$ -module of  $E$
- (3)  $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$  is a isomorphism of left  $K$ -modules  
 $[x] \mapsto f(x)$

#### Proof

- (1)  $\forall x \in E, \quad f(ax) = af(x)$  So  $af(x) \in \text{Im}(f)$
- (2)
- (3)



# Chapter 15

## Topology

### 15.1 Def

Let  $X$  be a set. We call topology on  $X$  any subset  $\mathcal{G}$  of  $\wp(X)$  that satisfies:

- $\emptyset \in \mathcal{G}$  and  $X \in \mathcal{G}$
- If  $(u_i)_{i \in I}$  is an arbitrary family of elements in  $\mathcal{G}$ , then  $\bigcup_{i \in I} u_i \in \mathcal{G}$
- If  $u$  and  $v$  are elements of  $\mathcal{G}$ , then  $u \cap v \in \mathcal{G}$

### 15.2 Remark

If  $(u_i)_{i=1}^n$  is a finite family of elements of  $\mathcal{G}$ , then  $\bigcap_{i=1}^n u_i \in \mathcal{G}$  (by induction, this follows from (3))

#### 15.2.1 Example

$\{\emptyset, X\}$  is a topology. call the trivial topology on  $\wp(X)$  is a topology called the discrete topology.

### 15.3 Def

Let  $X$  be a set. We call metric on  $X$  any mapping  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , that satisfies

- $d(x, y) = 0$  iff  $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called a metric space

### 15.3.1 Example

Let  $X$  be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

## 15.4 Def

Let  $(X, d)$  be a metric space. For any  $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$ , let  $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ . We call the open ball of radius  $\epsilon$  centered at  $x$

### 15.4.1 Example

Consider  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$ , then  $B(x, \epsilon) = ]x - \epsilon, x + \epsilon[$

## 15.5 Prop.

Let  $(X, d)$  be a metric space. Let  $\mathcal{G}_d$  be the set of  $U \subseteq X$  such that  $\forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subseteq U$ . Then  $\mathcal{G}_d$  is a topology on  $X$ .

### Proof

- $\emptyset \in \mathcal{G}_d \quad X \in \mathcal{G}_d$
- Let  $(u_i)_{i \in I}$  be a family of elements of  $\mathcal{G}_d$ . Let  $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$  such that  $x \in u_i$ . Since  $u_i \in \mathcal{G}_d, \exists \epsilon > 0$  such that  $B(x, \epsilon) \subseteq u_i \subseteq U$ . Hence  $U \in \mathcal{G}_d$ .
- Let  $U$  and  $V$  be elements of  $\mathcal{G}_d$ . Let  $x \in U \cap V, \exists a, b \in \mathbb{R}_{\geq 0}$  such that  $B(x, a) \subseteq U, B(x, b) \subseteq V$ . Taking  $\epsilon = \min\{a, b\}$ , Then  $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ . Therefore  $U \cap V \in \mathcal{G}_d$ .

## 15.6 Def

$\mathcal{G}_d$  is called the topology induced by the metric  $d$

## 15.7 Def

We call topology space any pair  $(X, \mathcal{G})$  where  $X$  is a set and  $\mathcal{G}$  is a topology on  $X$

Given a topological space  $(X, \mathcal{G})$  If  $U \in \mathcal{G}$  then we say that  $U$  is an open subset of  $X$ . If  $F \in \wp(X)$  such that  $X \setminus F \in \mathcal{G}$ , then we say that  $F$  is closed subset of  $X$

If there exists  $d$  a metric on  $X$  such that  $\mathcal{G} = \mathcal{G}_d$  then we say that  $\mathcal{G}$  is metrizable

### 15.7.1 Example

Let  $X$  be a set . The discrete topology on  $X$  is metrizable. In fact, if  $d$  denote the metric defined as  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$   
 $\forall x \in X \quad B(x, 1) = \{x\}$  So  $\{x\} \in \mathcal{G}_d$  Hence  $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{G}_d$



# Chapter 16

## Filter

### 16.1 Def

Let  $X$  be a set. We call filter if  $\mathcal{F} \subseteq \wp(X)$  that satisfies:

- (1)  $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2)  $\forall A \in \mathcal{F}, \forall B \in \wp(X), \text{ if } A \subseteq B, \text{ then } B \in \mathcal{F}$
- (3)  $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

#### 16.1.1 Example

- (1) Let  $Y \subseteq X, Y \neq \emptyset$ .  $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$  is a filter, called the principal filter of  $Y$ .
- (2) Let  $X$  be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is infinite}\}$$

is a filter called the Fréchet filter of  $X$ .

- (3) Let  $(X, \mathcal{G})$  be a topological space,  $x \in X$ . We call neighborhood of  $x$  any  $V \in \wp(X)$  such that  $\exists u \in \mathcal{G}$ , satisfying  $x \in U \subseteq V$ . Then  $\mathcal{V} = \{\text{neighborhoods of } x\}$  is a filter.

### 16.2 Def: Filter Basis

Let  $X$  be a set.  $\mathcal{B} \subseteq \wp(X)$ . If  $\emptyset \notin \mathcal{B}$  and  $\forall (B_1, B_2) \in \mathcal{B}^2, \exists B \in \mathcal{B}$ , such that  $B \subseteq B_1 \cap B_2$ . We say that  $\mathcal{B}$  is a filter basis.

#### 16.2.1 Remark

If  $\mathcal{B}$  is a filter basis, then  $\mathcal{F}(\mathcal{B}) = \{A \subseteq X \mid \exists B \in \mathcal{B} \quad B \subseteq A\}$  is a filter

**Proof**

$\emptyset \notin \mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{B}) \neq \emptyset$  since  $0 \neq B \subseteq \mathcal{F}(\mathcal{B})$ . If  $A \in \mathcal{F}(\mathcal{B}), A' \in \wp(X)$  such that  $A \subseteq A'$ , then  $\exists B \in \mathcal{B}$  such that  $B \subseteq A \subseteq A'$ . Hence  $A' \in \mathcal{F}(\mathcal{B})$ . If  $A_1, A_2 \in \mathcal{F}(\mathcal{B})$ , then  $\exists (B_1, B_2) \in \mathcal{B}^2$  such that  $B_1 \subseteq A_1, B_2 \subseteq A_2$ . Since  $\mathcal{B}$  is a filter basis,  $\exists B \in \mathcal{B}$  such that  $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ . Hence  $A_1 \cap A_2 \in \mathcal{F}(\mathcal{B})$ .

**16.2.2 Example**

- Let  $Y \subseteq X, Y \neq \emptyset$   
 $\mathcal{B} = \{Y\}$  is a filter basis.  $\mathcal{F}(\mathcal{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let  $(X, \mathcal{G})$  be a topological space  $x \in X$ . If  $\mathcal{B}_x$  is a filter basis such that  $\mathcal{F}(\mathcal{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$ , then we say that  $\mathcal{B}_x$  is a neighborhood basis of  $x$ .

**16.3 Remark**

Let  $\mathcal{B}_x$  is a neighborhood basis of  $x$  iff

- $\mathcal{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathcal{B}_x$  such that  $U \subseteq V$
- Let  $(X, d)$  be a metric space,  $x \in X \forall \epsilon > 0$ , Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Then

- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$
- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$

**16.3.1 Example**

$\mathcal{V}_x \cap \mathcal{G}$  is a neighborhood basis of  $x$

**16.4 Def**

$V \in \wp(X)$  is called a neighborhood of  $x$  if  $\exists U \in \mathcal{G}$  such that  $x \in U \subseteq V$



## 16.5 Remark

Let  $(X, \mathcal{G})$  be a topological space,  $x \in X$  and  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Suppose that  $\mathcal{B}$  is countable. We choose a surjective mapping  $(B_n)_{n \in \mathbb{N}}$  from  $\mathbb{N}$  to  $\mathcal{B}_x$ . For any  $n \in \mathbb{N}$ , let  $A_n = B_0 \cap B_1 \cap \dots \cap B_n \in \mathcal{V}_x$ . The sequence  $(A_n)_{n \in \mathbb{N}}$  is decreasing and  $\{A_n \mid n \in \mathbb{N}\}$  is a neighborhood basis of  $x$ .

## 16.6 Extra Episode

$\wp(\mathbb{N})$  is NOT countable

Suppose that  $f : \wp(\mathbb{N}) \rightarrow \mathbb{N}$  is injective. Then  $\exists g : \mathbb{N} \rightarrow \wp(\mathbb{N})$  surjective. Taking  $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$ . Since  $g$  is surjective,  $\exists a \in \mathbb{N}$  such that  $A = g(a)$ .

If  $a \in A$ , then  $a \in g(a)$ , hence  $a \notin A$

If  $a \notin A$ , then  $a \in g(a) = A$

Contradiction

## 16.7 Prop.

Let  $Y$  and  $E$  be sets,  $g : Y \rightarrow E$  be a mapping,

- If  $\mathcal{F}$  is a filter of  $Y$ , then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on  $E$

- If  $\mathcal{B}$  is a filter basis of  $Y$ , then

$$g(\mathcal{B}) := \{g(B) \mid B \in \mathcal{B}\}$$

is a filter of  $E$ , and  $\mathcal{F}(g(\mathcal{B})) = g_*(\mathcal{F}(\mathcal{B}))$

### Proof

- (1)  $E \in g_*(\mathcal{F})$  since  $g^{-1}(E) = Y$   
 $\emptyset \notin g_*(\mathcal{F})$  since  $g^{-1}(\emptyset) = \emptyset$

If  $A \in g_*(\mathcal{F})$  and  $A' \supseteq A$ , then  $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{F}$ , so  $g^{-1}(A') \in \mathcal{F}$ ,  
Hence  $A' \in g_*(\mathcal{F})$

If  $A_1, A_2 \in g_*(\mathcal{F})$ . Then  $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ . Hence  $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$ . So  $A_1 \cap A_2 \in g_*(\mathcal{F})$ .

- (2) Since  $g$  is a mapping, and  $\emptyset \notin \mathcal{B}$ , we get  $\emptyset \notin g(\mathcal{B})$ , since  $\mathcal{B} \neq \emptyset, g(\mathcal{B}) \neq \emptyset$ .

Let  $B_1, B_2 \in \mathcal{B}$ , there exists  $C \in \mathcal{B}$  such that  $C \subseteq B_1 \cap B_2$ . Hence  $g(C) \subseteq g(B_1) \cap g(B_2)$ , namely  $g(\mathcal{B})$  is a filter basis.



## Chapter 17

# Limit point and accumulation point

We fix a topological space  $(X, \mathcal{G})$

### 17.1 Def

Let  $\mathcal{F}$  be a filter of  $X$  and  $x \in X$

- If  $\mathcal{V}_x \subseteq \mathcal{F}$  then we say that  $x$  is a limit point of  $\mathcal{F}$
- If  $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \emptyset$ , we say that  $x$  is an accumulation point of  $\mathcal{F}$

So any limit point of  $\mathcal{F}$  is necessarily a accumulation point of  $\mathcal{F}$

### 17.2 Prop

Let  $\mathcal{B}$  be a filter basis of  $X$ ,  $x \in X$ ,  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Then  $x$  is an accumulation point of  $\mathcal{F}(\mathcal{B})$  iff  $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \emptyset$

#### Proof

##### Necessity

Since  $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B})$ ,  $\mathcal{B} \subseteq \mathcal{V}_x$ , the necessity is true.

##### Sufficiency

Let  $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$ . There exist  $B \in \mathcal{B}, U \in \mathcal{B}_x$ , such that  $B \subseteq A, U \subseteq V$ . Hence  $\emptyset \neq B \cap U \subseteq A \cap V$

### 17.3 Def

Let  $Y \subseteq X, Y \neq \emptyset$ . We call accumulation point of  $Y$  any accumulation point of the principal filter  $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$ .

### 17.4 Def

We denote by  $\overline{Y} = \{\text{accumulation points of } Y\}$ , called the closure of  $Y$ . Note that  $x \in \overline{Y}$  iff  $\forall U \in \mathcal{B}_x, Y \cap U \neq \emptyset$

By convention  $\overline{\emptyset} = \emptyset$

### 17.5 Prop

Let  $Y \subseteq X$ . Then  $\overline{Y}$  is the smallest closed subset of  $X$  containing  $Y$ .

#### Proof

$\forall x \in X \setminus \overline{Y}$ , then there exists  $U_x = \mathcal{V} \cap \mathcal{G}$ , such that  $Y \cap U_x = \emptyset$ . Moreover,  $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{G}$ . This shows that  $\forall y \in U_x, y \notin \overline{Y}$ . Therefore  $X \setminus \overline{Y} = \bigcap_{x \in X \setminus \overline{Y}} U_x \in \mathcal{G}$

Let  $Z \subseteq X$  be a closed subset that contain  $Y$ . Suppose that  $\exists y \in \overline{Y} \setminus Z$ . Then  $U = X \setminus Z \in \mathcal{V}_y \cap \mathcal{G}$  and  $U \cap Y \subseteq U \cap Z = \emptyset$ . So  $y \notin \overline{Y}$  contradiction. Hence  $\overline{Y} \subseteq Z$ .

### 17.6 Def: dense

Let  $(X, \mathcal{G})$  be a topological space,  $Y$  a subset of  $X$ . We call  $Y$  is dense in  $X$  if

$$\overline{Y} = X$$

## Chapter 18

# Limit of mappings

### 18.1 Def

Let  $(E, \mathcal{G}_E)$  be a topological space.  $f : Y \rightarrow E$  a mapping, and  $\mathcal{F}$  be a filter of  $Y$ . If  $a \in E$  is a limit point of  $F_*(\mathcal{F})$  namely,  $\forall$  neighborhood  $V$  of  $a$ ,  $f^{-1}(V) \in \mathcal{F}$ , then we say that  $a$  is a limit of the filter  $\mathcal{F}$  by  $f$ .

### 18.2 Remark

Let  $\mathcal{B}_a$  be a neighborhood basis of  $a$ . Then  $\mathcal{V}_a \subseteq f_*(\mathcal{F})$ , iff  $\mathcal{B} \subseteq f_*(\mathcal{F})$ . Therefore,  $a$  is a limit of  $\mathcal{F}$  by  $f$  iff  $\forall V \in \mathcal{B}_a, f^{-1}(V) \in \mathcal{F}$ .

#### 18.2.1 Example

Let  $(E, \mathcal{G}_E)$  be a topological space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $x = (x_n)_{n \in I} \in E^I$ . If the Fréchet filter  $\mathcal{F}_{Fr}(I)$  has a limit  $a \in E$  by the mapping  $x : I \rightarrow E$ , we say that  $(x_n)_{n \in I}$  converges to  $a$ , denote as

$$a = \lim_{n \in I, n \rightarrow +\infty} x_n$$

### 18.3 Remark

$a = \lim_{n \in I, n \rightarrow +\infty} x_n$  iff,  $\forall U \in \mathcal{B}_a$  (where  $\mathcal{B}_a$  is a neighborhood basis of  $a$ ),  $\exists N \in \mathbb{N}$  such that  $x_n \in U$  for any  $n \in I_{\geq N}$ .

Suppose that  $\mathcal{G}_E$  is induced by a metric  $d$ .  $\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}, \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}, \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  are all neighborhood basis of  $a$ . Therefore, the following are equivalent

- $a = \lim_{n \in I, n \rightarrow +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$
- $(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\})$ ? unknown position )

## 18.4 Remark

We consider the metric  $d$  on  $\mathbb{R}$  defined as

$$\forall (x, y) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of  $\mathbb{R}$  defined by this metric is called the usual topology on  $\mathbb{R}$

## 18.5 Prop

Let  $(x_n)_{n \in I} \in \mathbb{R}^I$ , where  $I \subseteq \mathbb{N}$  is an infinite subset. Let  $l \in \mathbb{R}$ . The following statements are equivalent:

- The sequence  $(x_n)_{n \in I}$  converges to  $l$  in the topological space  $\mathbb{R}$
- $\liminf_{n \in I, n \rightarrow +\infty} x_n = \limsup_{n \in I, n \rightarrow +\infty} x_n = l$
- $\limsup_{n \in I, n \rightarrow +\infty} |x_n - l| = 0$

## 18.6 Theorem

Let  $(X, d)$  be a metric space. Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be an element of  $X^I$ . Let  $l \in X$ . The following statements are equivalent:

- $(x_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$  (equivalent to  $\lim_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ )

### Proof

- (1)  $\Rightarrow$  (2) The condition (1) is equivalent to  $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$ .  
 We then get  $\sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ . Therefore  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) < \epsilon$ . We obtain that  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ .
- (2)  $\Rightarrow$  (1) Let  $\epsilon \in \mathbb{R}_{>0}$ . If  $\inf_{N \in \mathbb{N}} \sup_{n \in I_{\geq N}} d(x_n, l) = 0$ . Then  $\exists N \in \mathbb{N} \quad \sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ .  
 Hence,  $\forall n \in I_{\geq N} d(x_n, l) < \epsilon$ . Since  $\epsilon$  is arbitrary, (\*) is true, Hence (1) is also true .

## 18.7 Prop

Let  $(X, \mathcal{G})$  be a topological space .  $Y \subseteq X, p \in \overline{Y} \setminus Y$ . Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of  $Y$ .

### Proof

$Y$  is not empty otherwise  $\overline{Y} = \emptyset$ .

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$   
 $\emptyset \notin \mathcal{V}_{p,Y}$  since  $p \in \overline{Y}$
- Let  $V \in \mathcal{V}_p$  and  $A \subseteq Y$  such that  $V \cap Y \subseteq A$ . Let  $U = V \cup (A \setminus (V \cap Y)) \in \mathcal{V}_p$   
and  $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let  $U$  and  $V$  be elements of  $\mathcal{V}_p$  Let  $W = U \cap V \in \mathcal{V}_p$  Then  $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

## 18.8 Def

Let  $(X, \mathcal{G}_x)$  and  $(E, \mathcal{G}_E)$  be topological spaces,  $Y \subseteq X, p \in \overline{Y} \setminus Y$ , and  $f : Y \rightarrow E$  be a mapping . If  $a$  is a limit point of  $(F_*(\mathcal{V}_{p,Y}))$ , then we say that  $a$  is a limit of  $f$  when the variable  $y \in Y$  tends to  $p$ , denoted as  $a = \lim_{y \in Y, y \rightarrow p} f(y)$

## 18.9 Remark

If  $\mathcal{B}_a$  is a neighborhood basis of  $a$ . Then  $a = \lim_{y \in Y, y \rightarrow p} f(y)$  is equivalent to  
 $\forall U \in \mathcal{B}_a \quad \exists V \in \mathcal{V}_p$  such that  $Y \cap V \subseteq f^{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

## 18.10 Prop

Let  $X$  be a set,  $\mathcal{B}$  be a filter basis,  $\mathcal{G}$  be a filter. If  $\mathcal{B} \subseteq \mathcal{G}$ , then  $\mathcal{F} \subseteq \mathcal{G}$ .

### Proof

Let  $V \in \mathcal{F}(\mathcal{B})$  By definition  $\exists U \in \mathcal{B}$  such that  $U \subseteq V$ , since  $U \in \mathcal{G}$  ( for  $\mathcal{B} \subseteq \mathcal{G}$ ) and since  $\mathcal{G}$  is a filter,  $V \in \mathcal{G}$

### 18.11 Theorem

Let  $(X, \mathcal{G}_x)$  and  $(E, \mathcal{G}_E)$  be topological spaces.  $Y \subseteq X$ ,  $p \in \overline{T} \setminus Y$ ,  $a \in E$ . We consider the following conditions.

- (i)  $a = \lim_{y \in Y, y \rightarrow p} f(y)$
- (ii)  $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  if  $\lim_{n \rightarrow +\infty} y_n = p$  then  $\lim_{n \rightarrow \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that  $p$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

#### Proof

- (1) Let  $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  such that  $p = \lim_{n \rightarrow +\infty} y_n$ . For any  $U \in \mathcal{V}_p$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}$   $y_n \in U \cap Y$ . Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \rightarrow +\infty} f(y_n) = a$$

- (2) Assume that  $p$  has a countable neighborhood basis. There exists a decreasing sequence  $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_p^{\mathbb{N}}$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $p$ .

Assume that (i) does not hold. Then there exists  $U \in \mathcal{V}_a$  such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \rightarrow +\infty} y_n = \emptyset$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However  $f(y_n) \notin U$  for any  $n \in \mathbb{N}$ , so  $(f(y_n))_{n \in \mathbb{N}}$  cannot converges to  $a$ .



## 18.12 Prop.

Let  $X$  be a set. If  $(\mathcal{G}_i)_{i \in I}$  is a family of topologies on  $X$ , then  $\mathcal{G} = \bigcap_{i \in I} \mathcal{G}_i$  is a topology. In particular, for any  $\mathcal{A} \subseteq \wp(X)$ , there is a smallest topology on  $X$  that contains  $\mathcal{A}$ .

### 18.12.1 Proof

- $\forall i \in I \quad \{\emptyset, X\} \subseteq \mathcal{G}_i$  So  $\{\emptyset, X\} \subseteq \mathcal{G}$
- Let  $(u_j)_{j \in J}$  be a family of elements of  $\mathcal{G} \quad \forall j \in J, i \in I \quad u_j \in \mathcal{G}_i$  So  $\bigcup_{j \in J} u_j \in \mathcal{G}_i$  We then get  $\bigcup_{j \in J} u_j \in \mathcal{G}$
- Let  $U$  and  $V$  be elements of  $\mathcal{G} \quad \forall i \in I, \{u, v\} \subseteq \mathcal{G}_i$  So  $U \cap V \in \mathcal{G}_i$ . Therefore we get  $U \cap V \in \mathcal{G}$  Let  $\mathcal{A} \subseteq \wp(X)$  Let  $\mathcal{G}(\mathcal{A}) = \bigcap_{\mathcal{G} \subseteq \wp(X) \text{ a topology } \mathcal{A} \subseteq \mathcal{G}} \mathcal{G}$  Then  $\mathcal{G}(\mathcal{A})$  is a topology. By definition, if  $\mathcal{G}$  is a topology containing  $\mathcal{A}$ , then  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}$  Hence  $\mathcal{G}(\mathcal{A})$  is the smallest topology containing  $\mathcal{A}$



## Chapter 19

# Continuity

### 19.1 Def

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces  $f$  be a function from  $X$  to  $Y$ ,  $x \in \text{Dom}(f)$ . If for any neighborhood  $U$  of  $f(x)$ , there exists a neighborhood  $V$  of  $x$  such that  $f(V) \subseteq U$ . Then we say that  $f$  is continuous at  $x$ . If  $f$  is continuous at any  $x \in \text{Dom}(f)$  then we say  $f$  is continuous.

### 19.2 Remark

Let  $\mathcal{B}_{f(x)}$  be a neighborhood basis of  $f(x)$  If  $\forall U \in \mathcal{B}_{f(x)}$  there exist  $V \in \mathcal{B}_{f(x)} V_x$  such that  $f(V) \subseteq U$ , then  $f$  is continuous at  $x$  Suppose that  $X$  and  $Y$  are metric space. Then  $f$  is continuous at  $x$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in \text{Dom}(f) \quad d(y, x) < \delta \text{ implies } d(f(y), f(x)) < \epsilon$$

### 19.3 Theorem

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces,  $f$  be a function from  $X$  to  $Y$   $x \in \text{Dom}(f)$  Consider the following condition

- $f$  is continuous at  $x$
- $\forall (x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ , if  $\lim_{n \rightarrow +\infty} x_n = x$ , then  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$  THEN  
(i) implies (ii) Moreover, if  $x$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

### 19.4 Proof

(i)  $\Rightarrow$  (ii) Let  $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$  that converges to  $x \forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$  Since  $\lim_{n \rightarrow +\infty} x_n = x$ , there exists  $N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}, x_n \in V$ .

Hence  $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$ . Thus  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$

(ii)  $\Rightarrow$  (i) under the hypothesis that  $x$  has countable neighborhood basis. actually we will prove  $NOT(i) \Rightarrow NOT(ii)$

Let  $(V_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $\mathcal{V}_x$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $x$

If (i) does not hold, then  $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$  Pick  $x_n \in V_n$  such that  $f(x_n) \notin U \quad \forall n \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$ . Hence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ . However,  $f(x_n) \notin U$  for any  $n$  So  $(f(x_n))_{n \in \mathbb{N}}$  does not converges to  $f(x)$ . Therefore (ii) does not hold.

## 19.5 Prop

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y), (Z, \mathcal{G}_Z)$  be topological spaces.  $f$  be a function from  $X$  to  $Y$ ,  $g$  be a function from  $Y$  to  $Z$ . Let  $x \in \text{Dom}(g \circ f)$  If  $f$  and  $g$  are continuous at  $x$ . then  $g \circ f$  is continuous at  $x$  sectionProof Let  $U \in \mathcal{V}_{g(f(x))}$  Since  $g$  is continuous at  $f(x)$ :

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since  $f$  is continuous at  $x$ :

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore,  $g(f(V)) \subseteq g(W) \subseteq U$  Hence  $g \circ f$  is continuous at  $x$

## 19.6 Def

Let  $(X, \mathcal{G})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{G}$ , If any element of  $\mathcal{G}$  can be written as the union of a family of sets in  $\mathcal{B}$  we say that  $\mathcal{B}$  is a topological basis of  $\mathcal{G}$

## 19.7 Prop

Let  $(X, \mathcal{G})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{G}$   $\mathcal{B}$  is a topological basis iff

$$\forall x \in X, \mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$$

is a neighborhood basis of  $x$

## 19.8 Proof

$\Rightarrow$ :

$$\forall x \in X \mathcal{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since  $\mathcal{B}$  is a topological basis of  $\mathcal{G}$ ,

$$\exists W \in \mathcal{B}, x \in W \subseteq V \subseteq U$$

Hence  $\mathcal{V}_x$  is generated by  $\mathcal{B}_x$

$\Leftarrow$  Let  $U \in \mathcal{G}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathcal{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{G}$$

## 19.9 Prop

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces.  $\mathcal{B}_Y$  be a topological basis of  $\mathcal{G}_Y$   
 $f : X \rightarrow Y$  be a mapping. The following conditions are equivalent:

- (1)  $f$  is continuous
- (2)  $\forall U \in \mathcal{G}_Y, f^{-1}(U) \in \mathcal{G}_X$
- (3)  $\forall U \in \mathcal{B}_Y, f^{-1}(U) \in \mathcal{G}_X$

### Proof

(1)  $\Rightarrow$  (2)

Lemma Let  $(X, \mathcal{G})$  be a topological space,  $V \in \wp(X)$ , Then  $V \in \mathcal{G}$  iff  $\forall x \in V, V$  is a neighborhood of  $x$

Proof of lemma  $\Rightarrow$  is by definition

*Leftarrow:*

$$\forall x \in V, \exists W_x \in \mathcal{G}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W_x \quad x \in V$$

Let  $U \in \mathcal{G}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood  $W$  of  $x$  such that  $f(W) \subseteq U$   
 Since  $f$  is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since  $x$  is arbitrary,

$$f^{-1}(U) \in \mathcal{G}_X$$

(2)  $\Rightarrow$  (3) For (3) is a special situation of (2), it's natural.

(3)  $\Rightarrow$  (1) Let  $x \in X$

$$\forall U \in \mathcal{B}_Y \text{ s.t. } f(x) \in U, f^{-1}(U)$$

is an open neighborhood of  $x$ , and

$$f(f^{-1}(U)) \subseteq U$$

Hence  $f$  is continuous at  $x$

## 19.10 Def

Let  $X$  be a set,  $((Y_i, \mathcal{G}_i))_{i \in I}$  be a family of topological spaces.  $\forall i \in I$  let  $f_i : X \rightarrow Y_i$  be a mapping. We call initial topology of  $(f_i)_{i \in I}$  on  $X$  the smallest topology on  $X$  making all  $f_i$  continue

## 19.11 Remark

If  $\mathcal{G}$  is the initial topology of  $(f_i)_{i \in I}$ ,  $\forall i \in I, U_i \in \mathcal{G}_i$   $f_i^{-1}(U_i) \in \mathcal{G}$  If  $J \subseteq I$  is a finite subset,  $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j$  then  $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{G}$

## 19.12 Prop

$$\mathcal{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ is finite } (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j \right\}$$

is a topological basis of the initial topology  $\mathcal{G}$

**Proof**

First

$$\mathcal{B} \subseteq \mathcal{G}$$

Let

$\mathcal{G}' = \{\text{subset } V \text{ of } X \text{ that can be written as the union of a family of sets in } \mathcal{B}\}$

- $\emptyset \in \mathcal{G}' \quad X \in \mathcal{B} \subseteq \mathcal{G}'$
- $\mathcal{G}'$  is stable by taking the union of any family of elements in  $\mathcal{G}'$
- If  $V_1, V_2$  are elements of  $\mathcal{G}'$ , then

$$V_1 \cap V_2 \in \mathcal{G}'$$

In fact,  $V_1, V_2$  are of the form of the union of some sets of  $\mathcal{B}$

The intersection of two elements of  $\mathcal{B}$  is still a element of  $\mathcal{B}$

$$\begin{aligned} & \left( \bigcap_{j \in J} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J'} f_j^{-1}(U'_j) \right) \\ &= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \setminus J' \\ U'_j & j \in J' \setminus J \\ U_j \cap U'_j & j \in J \cap J' \end{cases} \\ & \left( \bigcap_{j \in J \setminus J'} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U'_j) \right) \cap \left( \bigcap_{j \in J' \setminus J} f_j^{-1}(U'_j) \right) \end{aligned}$$

So  $\mathcal{G}'$  is a topology making all  $f_i$  continuous. Hence

$$\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{G} \Rightarrow \mathcal{G}' = \mathcal{G}$$

**Example**

Let  $((Y_i, \mathcal{G}_i))_{i \in I}$  be topological spaces.  $Y = \prod_{i \in I} Y_i$  and

$$\begin{aligned} \pi_i : Y &\rightarrow Y_i \\ (y_j)_{j \in I} &\mapsto y_i \end{aligned}$$

The product topology on  $Y$  is by definition the initial topology of  $(\pi_i)_{i \in I}$

### 19.13 Theorem

Let  $X$  be a set,  $((Y_i, \mathcal{G}_i))_{i \in I}$  be a family of topological spaces,

$$((f_i : X \rightarrow Y_i))_{i \in I}$$

be a family of mappings and we equip  $X$  with the initial topology  $\mathcal{G}_X$  of  $(f_i)_{i \in I}$ . Let  $(Z, \mathcal{G}_Z)$  be a topological space and

$$h : Z \rightarrow X$$

be a mapping. Then  $h$  is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{ is continuous}$$

#### 19.13.1 Proof

$\Rightarrow$  If  $h$  is continuous, since each  $f_i$  is continuous,  $f_i \circ h$  is also continuous.

$\Leftarrow$  Suppose that  $\forall i \in I, f_i \circ h$  is continuous. Hence

$$\forall U_i \in \mathcal{G}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{G}_Z$$

Let

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j \right\}$$

$\forall U \in \mathcal{B}$

$$h^{-1}(U) = \bigcap_{j \in J} h^{-1}(f_j^{-1}(U_j)) \in \mathcal{G}_Z$$

Therefore,  $h$  is continuous.

### 19.14 Remark

We keep the notation of the definition of initial topology. If  $\forall i \in I, \mathcal{B}_i$  is a topological basis of  $\mathcal{G}_i$ , then

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{B}_j \right\}$$

is also a topological basis of the initial topology,



**19.14.1 Example**

Let  $((X_i, d_i))_{i \in \{1, \dots, n\}}$  be a family of metric spaces.

$$X = \prod_{i \in \{1, \dots, n\}} X_i$$

We define a mapping

$$d: (X \times X \rightarrow \mathbb{R}_{\geq 0}) \\ d: ((x_i)_{i \in \{1, \dots, n\}}, (y_i)_{i \in \{1, \dots, n\}}) \mapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i)$$

$d$  is a metric on  $X$ . If  $x = (x_i)_{i \in \{1, \dots, n\}}$ ,  $y = (y_i)_{i \in \{1, \dots, n\}}$ ,  $z = (z_i)_{i \in \{1, \dots, n\}}$  are elements of  $X$ , then

$$d(x, z) = \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \leq \max_{i \in \{1, \dots, n\}} (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i: X \rightarrow X_i \\ \pi_i: (x_i)_{i \in \{1, \dots, n\}} \mapsto x_i$$

is continuous. Hence the product topology  $\mathcal{G}$  is contained in  $\mathcal{G}_d$

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in X$ ,  $\epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i \in \{1, \dots, n\}} \mid \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i \in \{1, \dots, n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i \in \{1, \dots, n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{G} \end{aligned}$$



## Chapter 20

# Uniform continuity and convergency

### 20.1 Def

Let  $(X, d)$  be a metric space.  $\forall A \subseteq X$ , we define

$$\text{diam}(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A. By convention

$$\text{diam}(\emptyset) := 0$$

If  $\text{diam}(A) < +\infty$ , we say that A is bounded

### 20.2 Remark

- If A is finite, then it's bounded
- If  $A \subseteq B$  then  $\text{diam}(A) \leq \text{diam}(B)$

### 20.3 Prop

Let  $(X, d)$  be a metric space.  $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$ . Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

In particular, if A, B are bounded, then  $A \cup B$  is bounded.

**Proof**

Let  $(x, y) \in (A \cup B)^2$ . If  $\{x, y\} \subseteq A$ , then  $d(x, y) \leq \text{diam}(A)$   
 If  $\{x, y\} \subseteq B$  then  $\text{diam}(B) \geq d(x, y)$   
 If  $x \in A, y \in B$ ,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

Similarly if  $x \in B, y \in A$

$$d(x, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

**20.4 Def**

Let  $(X, d)$  be a metric space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $(x_n)_{n \in I} \in X^I$ . If

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \quad \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \epsilon$$

then we say that  $(x_n)_{n \in I}$  is a Cauchy sequence.

**20.5 Prop**

- (1) If  $(x_n)_{n \in I}$  converges, then it's a Cauchy sequence.
- (2) If  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\{x_n \mid n \in I\}$  is bounded
- (3) Suppose that  $(x_n)_{n \in I}$  is a Cauchy sequence. If there exists an infinite subset  $J$  of  $I$  such that  $(x_n)_{n \in J}$  converges to some  $x \in X$ , then  $(x_n)_{n \in I}$  converges to  $x$

**20.5.1 Proof**

- (1) trivial
- (2) trivial
- (3) Let  $\epsilon > 0, \exists N \in \mathbb{N}$

$$\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take  $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence  $(x_n)_{n \in I}$  converges to  $x$

## 20.6 Def

Let  $(X, d_X), (Y, d_Y)$  be metric space.  $f$  be a function from  $X$  to  $Y$ . If  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

we say that  $f$  is uniformly continuous.

## 20.7 Prop

Let  $(X, d_X), (Y, d_Y)$  be metric spaces  $f$  be a function from  $X$  to  $Y$  which is uniformly continuous.

- (1) If  $I \subseteq \mathbb{N}$  is finite, and  $(x_n)_{n \in I}$  is a Cauchy sequence in  $\text{Dom}(f)^I$  then  $(f(x_n))_{n \in I}$  is Cauchy sequence
- (2)  $f$  is continuous

### 20.7.1 Proof

- (1)  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists N \in \mathbb{N}$  such that

$$\forall (n, m) \in I_{\geq N}^2, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

Therefore  $(f(x_n))_{n \in I}$  is a Cauchy sequence.

- (2) Let  $(x_n)_{n \in I}$  be a sequence in  $\text{Dom}(f)^{\mathbb{N}}$  that converges to  $x \in \text{Dom}(f)$  We define  $(y_n)_{n \in \mathbb{N}}$  as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then  $(y_n)_{n \in \mathbb{N}}$  converges to  $x$ . Hence  $(y_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Since  $f$  is uniformly continuous,  $(f(y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $Y$ .

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to  $f(x)$ . Hence  $(f(y_n))_{n \in \mathbb{N}}$  converges to  $f(x)$

## 20.8 Def

Let  $X$  be a set,  $Z \subseteq X$ ,  $(Y, d)$  be a metric space,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as their common domain of definition.

- If  $\forall x \in Z, (f_n(x))_{n \in I}$  converges to  $f(x)$ , we say that  $(f_n)_{n \in I}$  converges pointwisely to  $f$
- If

$$\lim_{n \in I, n \rightarrow +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$

we say that  $(f_n)_{n \in I}$  converges uniformly to  $f$

## 20.9 Theorem

Let  $X$  and  $Y$  be metric space,  $Z \subseteq X$ ,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$ ,  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is uniformly continuous

Then  $f$  is uniformly continuous.

### 20.9.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\lim_{n \in I, n \rightarrow +\infty} A_n = 0$$

$\forall (x, y) \in Z^2, n \in I$

$$\begin{aligned} & d(f(x), f(y)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & \leq 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f_n(x), f_n(y)) = 0$$

Hence

$$0 \leq \inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$

Take  $\lim_{n \rightarrow +\infty}$ , by squeeze theorem, we get

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

## 20.10 Theorem

Let  $X$  be a topological space,  $Y$  be a metric space,  $Z \subseteq X, p \in Z, I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  function from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that:

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is continuous at  $p$

Then  $f$  is continuous at  $p$

### 20.10.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since  $f_n$  is continuous  $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{\mathcal{B}}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x), f(p)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$$f(U) \subseteq \overline{\mathcal{B}}(f(p), \epsilon)$$

### 20.10.2 Def

Let  $X, Y$  be metric spaces,  $f$  be a function from  $X$  to  $Y$ ,  $\epsilon > 0$ . If

$$\forall (x, y) \in \text{Dom}(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that  $f$  is  $\epsilon$ -Lipschitzian

If  $\exists \epsilon > 0$  such that  $f$  is  $\epsilon$ -Lipschitzian, then it's uniformly continuous.

## 20.11 Remark

If  $f$  is Lipschitzian, then it's uniformly continuous.

## 20.12 Example

- Let  $((X_i, d_i))_{i \in I}$  be metric space.  $X = \prod_{i \in I} X_i$  where  $I$  is finite

$$\begin{aligned} X \times X &\rightarrow \mathbb{R}_{\geq 0} \\ d : d((x_i), (y_i)_{i \in I}) &= \max_{i \in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \rightarrow X_i$$

is Lipschitzian. ( $\forall x = (x_i)_{i \in I}, \forall y = (y_i)_{i \in I}$ )

- Let  $(X, d)$  be a metric space

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$



Part V

**Normed Vector Space**



# Chapter 21

## Linear Algebra

We fix a unitary ring  $K$

### 21.1 Def

Let  $M$  be a left  $K$ -module, and let  $x = (x_i)_{i \in I}$  be a family of elements of  $M$ . We define a morphism of left  $K$ -module as following:

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \quad (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

#### 21.1.1 Notation

$$\begin{aligned} K^{\oplus I} &:= \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{ finite, such that } a_i = 0 \text{ for } i \in I \setminus J\} \\ \varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) &= \varphi_x((a_i)_{i \in I}) + \varphi_x((b_i)_{i \in I}) \end{aligned}$$

### 21.2 Def

Let  $M$  be a left  $K$ -module,  $I$  be a set,  $x = (x_i)_{i \in I} \in M^I$ . If

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say  $(x_i)_{i \in I}$  is  $K$ -linearly independent

surjective then we say  $(x_i)_{i \in I}$  is system of generator

a bijection then we say  $(x_i)_{i \in I}$  is a basis of  $M$

**Example**

Let  $e_i$  be the element  $(\delta_{ij})_{j \in I}$  with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i \in I} \in (K^{\oplus I})^I$$

is a basis of  $K^{\oplus I}$

**21.3 Def**

Let  $M$  be a left  $K$ -module

- If  $M$  has a basis, we say that  $M$  is a free  $K$ -module
- If  $M$  has finite system of generated  
( $\exists$  a finite set  $I$  and a family  $(x_i)_{i \in I} \in M^I$  that forms a system of generator),  
then we say that  $M$  is of finite type.

**21.4 Remark**

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in M^n$ , where  $n \in \mathbb{N}$

- $x$  is linearly independent iff

$$\forall a \in K^n \quad \sum a_i x_i = 0$$

implies

$$a = 0$$

- $x$  is a system of generator iff for any element of  $M$  can be written in the form

$$\sum b_i x_i \quad b \in K^n$$

Such expression is called a  $K$ -linear combination of  $x_1, \dots, x_n$

**21.5 Theorem**

Let  $K$  be a division ring ( $0 \neq 1$  and  $\forall k \in K \setminus \{0\}$   $k$  is invertible)

Let  $V$  be a left  $K$ -module of finite type and  $(x_i)_{i \in I}$  be a system of generators of  $V$ . Then, there exists a subset  $I$  of  $\{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ . (In particular,  $V$  is a free  $K$ -module)

**Proof**

(By induction on  $n$ )

If  $n = 0$ , then  $V = \{0\}$

In this case  $\emptyset$  is a basis of  $V$

**Induction hypothesis**

True for a system of generators of  $n - 1$  elements. Let  $(x_i)_{i \in \{1, \dots, n\}}$  be a system of generators of  $V$ . If  $(x_i)_{i \in \{1, \dots, n\}}$  is linearly independent, it's a basis. Otherwise,  $\exists (a_i)_{i \in I} \in K^n$  such that

$$(a_i, \dots, a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose  $a_n \neq 0$ . Then

$$x_n = -a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right)$$

Since  $(x_i)_{i \in \{1, \dots, n\}}$  is a system of generators, any elements of  $V$  can be written as

$$\begin{aligned} \sum b_i x_i &= \left( \sum_{i=1}^{n-1} b_i x_i \right) - b_n a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right) \\ &= \sum_{i=1}^{n-1} (b_i - b_n a_n^{-1} a_i) x_i \end{aligned}$$

Thus  $(x_i)_{i \in \{1, \dots, n\}}$  forms a system of generators. By the induction hypothesis, there exists  $I \subseteq \{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ .

**21.6 Theorem**

Let  $K$  be a unitary ring and  $B$  be a left  $K$ -module.  $W$  be a left  $K$ -submodule of  $V$ . Let  $(x_i)_{i=1}^n$  be an element of  $W^n$

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where  $(n, l) \in \mathbb{N}^2 \forall j \in \{1, \dots, l\}$ , let  $x_{n+j}$  be an element in the equivalence class  $\alpha_j$

- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are linearly independent, then  $(x_i)_{i=1}^{n+l}$  is also linearly independent
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are system of generators of  $W$  and  $V/W$  respectively, then  $(x_i)_{i=1}^{n+l}$  is also a system of generators
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are basis, then  $(x_i)_{i=1}^{n+l}$  is also a basis

**Proof**

(1) Suppose that  $(b_i)_{i=1}^{n+l}$  such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \rightarrow V/W$$

be the projection morphism ( $\pi(x) = [x]$ )

$$0 = \pi\left(\sum_{i=1}^{n+l} b_i x_i\right) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^l b_{n+j} \pi(x_{n+j}) = \sum_{j=1}^l b_{n+j} \alpha_j$$

$$\{x_1, \dots, x_n\} \subseteq W \text{ So } \forall i \in \{1, \dots, n\}$$

$$\pi(x_i) = 0$$

Since  $(\alpha_j)_{j=1}^l$  is linearly independent,

$$b_{n+1} = \dots = b_{n+l} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since  $(x_i)_{i=1}^n$  is linearly independent,

$$b_1 = \dots = b_n = 0$$

(2) Let  $y \in V$ . Then  $\pi(y) \in V/W$ . So there exists

$$(c_{n+1}, \dots, c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^l c_{n+j} \alpha_j \\ &= \sum_{j=1}^l c_{n+j} \pi(x_{n+j}) = \pi\left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \end{aligned}$$

So

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \in W$$

$\exists c \in K^n$  such that

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) = \left(\sum_{i=1}^n c_i x_i\right)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

## 21.7 Corollary

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. If  $(x_i)_{i=1}^n$  is a linearly independent family of elements of  $V$  ( $n \in \mathbb{N}$ ), then

$$\exists l \in \mathbb{N} \quad \exists (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of  $V$

### Proof

Let  $W$  be the image of

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow V \\ (a_i)_{i=1}^n &\mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left  $K$ -submodule of  $V$ .

Note that  $(x_i)_{i=1}^n$  forms a basis of  $W$ .

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow W \\ \varphi(x_i)_{i=1}^n(e_j) &= x_j \in W \end{aligned}$$

Moreover, since  $V$  is of finite type there exists  $d \in \mathbb{N}$  and a surjective morphism of left  $K$ -modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \rightarrow V/W$$

is surjective.

Hence the composite morphism

$$K^d \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\pi \circ \psi} \end{array} V \xrightarrow{\pi} V/W$$

is surjective. Thus  $V/W$  is of finite type. There exist then a basis

$$(a_j)_{j=1}^l$$

of  $V/W$ .

Taking  $x_{n+j} \in \alpha_j$  for  $j \in \{1, \dots, l\}$ , we get a basis of  $V$ :

$$(x_i)_{i=1}^{n+l}$$

## 21.8 Def

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. We call rank of  $V$  the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If  $K$  is a field  $rk(V)$  is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of  $V$ .

## 21.9 Theorem

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. Let  $W$  be a left  $K$ -submodule of  $V$ .

(1)  $W$  and  $V/W$  are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of  $V$  has exactly  $rk(V)$  elements

## 21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$



In  $(\pi(x_i))_{i=1}^n$  we extract a basis of  $V/W$ , say

$$(\pi(x_i))_{i=1}^l$$

For  $j \in \{l+1, \dots, n\}$ ,

$$\exists(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any  $x \in W, \exists(a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^n a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^l a_i x_i + \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \\ &= \sum_{j=l+1}^n a_j y_j + \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^n a_j y_j$$

Hence  $W$  is of finite type, and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of  $V/W$ .

We extract a subfamily, say  $(x_i)_{i=1}^l$  such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of  $V/W$ .

For  $j \in \{1, \dots, l\}$ , there exists:

$$(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Let  $x \in W, \exists (a_i)_{i=1}^n \in K^n$  let  $x = \sum a_i x_i$ , then

$$\begin{aligned} x &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \right) \\ &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i \right) + \left( \sum_{j=l+1}^n a_j y_j \right) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i)$$

Therefore  $(y_j)_{j=l+1}^n$  is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + rk(V/W)$$

(2) All basis of  $V$  have  $rk(V)$  elements.

We reason by induction on  $rk(V)$

(1)

$$rk(V) = 0$$

In this case  $V = \{0\}$  The only basis of  $V$  is  $\emptyset$ . So the statement holds.

(2) Assume that there exists  $e \in V \setminus \{0\}$  such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of  $V$  is of the form

$$ae$$

where  $a \in K \setminus \{0\}$

Let  $(e_i)_{i=1}^m$  be a basis of  $V$ . We reason by induction on  $m$  to prove that

$$m = rk(V)$$

The cases where  $m = 0$  or  $1$  are proved in (1)(2) respectively. Induction hypothesis: true for a basis of  $< m$  elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of  $V/W$ .

If  $(a_i)_{i=2}^m \in K^{m-1}$  such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for  $(e_i)_{i=1}^m$  a basis of  $V$ ,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of  $V/W$ . We then obtain that

$$rk(V/W) \leq m - 1 \leq n - 1$$

By the induction hypothesis,

$$m - 1 = rk(V/W)$$

By (2),  $rk(W) = 1$ . Hence

$$m = (m - 1) + 1 = rk(V/W) + rk(W) = rk(V)$$

## 21.11 Prop

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Let  $I$  be a set and  $(x_i)_{i \in I} \in E^I$

- If  $(x_i)_{i \in I}$  is linearly independent and  $f$  is injective, then  $(f(x_i))_{i \in I}$  is linearly independent.
- If  $(x_i)_{i \in I}$  is a system of generators and  $f$  is surjective, then  $(f(x_i))_{i \in I}$  is a system of generators.
- If  $(x_i)_{i \in I}$  is a basis and  $f$  is an isomorphism, then  $(f(x_i))_{i \in I}$  is a basis.

### 21.11.1 Proof

$$\varphi_{(f(x_i))_{i \in I}} = f \circ \varphi_{(x_i)_{i \in I}}$$

## Chapter 22

# Matrices

We fix unitary ring  $K$

### 22.1 Def

Let  $n \in \mathbb{N}$  and  $V$  be a left  $K$ -module.

For any  $(x_i)_{i=1}^n \in V^n$ , we denote by  $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$  the morphism

$$\begin{aligned} & \phi_{(x_i)_{i=1}^n} : K^n \rightarrow V \\ (a_i)_{i=1}^n & \mapsto \sum_{i=1}^n a_i n_i \end{aligned}$$

### 22.1.1 Example

Suppose that  $V = K^p$  ( $p \in \mathbb{N}$ ) Then each  $x_i \in K^p$  is of the form  $(x_{i,1}, \dots, x_{i,p})$

Hence  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  can be written:

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,p} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,p} \end{pmatrix}$$

## 22.2 Def

Let  $(n, p) \in \mathbb{N}^2$ . We call  $n$  by  $p$  matrix of coefficient in  $K$  any morphism of left  $K$ -modules from  $K^n$  to  $K^p$

### 22.2.1 Example

- Denote by  $I_n$  then identity mapping. Then  $(e_i)_{i=1}^n$  is a basis of  $K^n$  called the canonical basis of  $K^n$

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, \dots, a_n)) = \sum_{i=1}^n a_i e_i = (a_1, \dots, a_n)$$

- Let  $(x_1, \dots, x_n) \in K^n$ , Denote by

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) (= \varphi_{(x_i e_i)_{i=1}^n}) : K^n &\rightarrow K^n \\ (a_1, \dots, a_n) &\mapsto (a_1 x_1, \dots, a_n x_n) \end{aligned}$$

## 22.3 Def

We denote by  $M_{n,p}(K)$  the set of all  $n$  by  $p$  matrices of coefficients in  $K$ . For  $(n, p, r) \in \mathbb{N}^3$ , we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) &\rightarrow M_{n,r}(K) \\ (A, B) &\mapsto AB := B \circ A \end{aligned}$$

## 22.4 Calculate Matrices

Let  $K$  be a unitary ring, and  $V$  be a left  $K$ -module. Let  $n \in \mathbb{N}$  and

$$x = (x_1, \dots, x_n) \in V^n$$

### 22.4.1 Remind

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \varphi : (a_1, \dots, a_n) \mapsto a_1 x_1, \dots, a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1, \dots, p\} \times \{1, \dots, n\}} \in M_{p,n}(K)$$

$A$  is a morphism of left  $K$ -modules from  $K^p$  to  $K^n$ . Recall that

$$A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let  $(b_1, \dots, b_n) \in K^p$

$$\begin{aligned} A((b_1, \dots, b_n)) &= \sum_{i=1}^p b_i(a_{i,1}, \dots, a_{i,n}) \\ \varphi(A((b_1, \dots, b_n))) &= \sum_{i=1}^p b_i \varphi_x((a_{i,1}, \dots, a_{i,n})) \\ &= \sum_{i=1}^p b_i(a_{i,1}x_1, \dots, a_{i,n}x_n) \end{aligned}$$

Let  $B = \{b_{ij}\}_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\}} : K^n \rightarrow K^r$

$$AB = \left\{ \sum_{j=1}^n a_{lj} b_{jm} \right\}_{(l,m) \in \{1, \dots, p\} \times \{1, \dots, r\}}$$





## Chapter 23

# Transpose

We fix a unitary ring  $K$

### 23.1 Def

Let  $E$  be a left- $K$ -module. Denote by

$$E^\vee := \{\text{morphisms of left } K\text{-modules } E \rightarrow K\}$$

$\forall (f, g) \in E^\vee$  let

$$\begin{aligned} f + g : E &\rightarrow K \\ x &\mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$  forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : E &\rightarrow K \\ x &\mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} K \times E^\vee &\rightarrow E^\vee \\ (a, f) &\mapsto fa : x \in E \rightarrow f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right  $K$ -module on  $E^\vee$

## 23.2 Def

Let  $E$  and  $F$  be two left  $K$ -modules.  $\varphi : E \rightarrow F$  be a morphism of left  $K$ -modules. We denote by

$$\varphi^\vee : F^\vee \rightarrow E^\vee$$

the morphism of right  $K$ -modules sending  $g \in F^\vee$  to  $g \circ \varphi \in E^\vee$ .  
Actually  $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

### 23.2.1 Example

Suppose that  $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,p} \end{pmatrix}$$

$\varphi$  sends  $(a_1, \dots, a_n)$  to  $\{\sum_{i=1}^n a_i b_{ij}\}_{j \in \{1, \dots, p\}}$ . Let  $g \in F^\vee$   $g : K^p \rightarrow K$ , then  $g$  is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$  sends  $(a_1, \dots, a_n)$  to  $\sum_{i=1}^p (\sum_{j=1}^n a_j b_{ij} y_i)$

Assume that  $K$  is commutative. We denote by

$$\iota_p : (K^p)^\vee \rightarrow K^p$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, \dots, x_p)$$

$$\iota_n : (K^n)^\vee \rightarrow K^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, \dots, x_n)$$

are isomorphisms of  $K$ -modules

For any morphism of K-modules  $\varphi : K^n \rightarrow K^p$ , we denote by  $\varphi^\tau$  the morphism of K-modules  $K^p \rightarrow K^n$  given by  $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{array}{ccc} (K^p)^\vee & \xrightarrow{\varphi^\vee} & (K^n)^\vee \\ \cong \downarrow \iota_p & \circlearrowleft & \cong \downarrow \iota_n \\ K^p & \xrightarrow{\varphi^\tau} & K^n \end{array}$$

$\varphi^\tau$  is called the transpose of  $\varphi$

### 23.3 Prop

Let E,F,G be left K-modules.  $\varphi : E \rightarrow F, \psi : F \rightarrow G$  be morphisms of left K-modules. Then  $(\psi \circ \varphi)^\vee$  is equal to  $\varphi^\vee \circ \psi^\vee$

#### Proof

$$\forall f \in G^\vee$$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

### 23.4 Corollary

Assume that K is commutative. Let  $n, p, q$  be neutral numbers.  $A \in M_{n,p}(K), B \in M_{p,q}(K)$ . Then

$$(AB)^\tau = B^\tau A^\tau$$

#### Proof

$$A^t a u = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^t a u = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned} B^\tau A^\tau &= A^\tau \circ B^\tau \\ &= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\ &= (AB)^t a u \end{aligned}$$

### 23.5 Remark

(1) For  $A \in M_{n,p}(K)$ , one has  $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} E &\rightarrow (E^\vee)^\vee \\ x &\mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a  $K$ -linear mapping.

If  $K$  is a field and  $E$  is of finite dimension, this is an isomorphism of  $K$ -modules.

In fact, if  $e = (e_i)_{i=1}^n$  is a basis of  $E$  over  $K$ . For  $i \in \{1, \dots, n\}$ , let

$$\begin{aligned} e_i^\vee : E &\rightarrow K \\ \lambda_1 e_1, \dots, \lambda_n e_n &\mapsto \lambda_i \end{aligned}$$

is called the dual basis of  $e$

$$\begin{array}{ccc} K^n & \xleftarrow[\iota_n]{\cong} & (K^n)^\vee \\ \varphi_e \downarrow \cong & \searrow \varphi_{e^\vee} & \downarrow \varphi_e^\vee \\ E & \xrightarrow[\cong]{} & E^\vee \end{array}$$

$(e^\vee)^\vee$  gives a basis of  $(E^\vee)^\vee$ . Hence  $E \rightarrow (E^\vee)^\vee$  is an isomorphism.

## Chapter 24

# Linear Equation

We fix a unitary ring  $K$ .

### 24.1 Def

For  $a = (a_1, \dots, a_n) \in K^n \setminus \{(0, \dots, 0)\}$ . Denote by  $j(a)$  the first index  $j \in \{1, \dots, n\}$  such that  $a_j \neq 0$ . Let  $(n, p) \in \mathbb{N}^2$ ,  $A \in M_{n,p}(K)$ . We write  $A$  as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \in K^p$$

We say that  $A$  is of row echelon form if,  $\forall i \in \{1, \dots, n-1\}$  one of following conditions is satisfied.

- $a^{(i+1)} = (0, \dots, 0)$
- $a^{(i)}, a^{(i+1)}$  are non-zero, and  $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, \dots, n\}$  such that  $a^{(i)} \neq (0, \dots, 0)$ , one has

$$a_{j(a^{(i)})}^{(i)} = 1$$

and

$$\forall k \in \{1, \dots, n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that  $A$  is of reduced row echelon form.

## 24.2 Prop

Suppose that  $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$  is of row echelon form. Then  $\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$  is of cardinal  $\leq p$

### Proof

Let  $k = \text{card}\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$   $a^{(k+1)} = \dots = a^{(n)} = (0, \dots, 0)$  and  $j(a^{(1)}) < j(a^{(2)}) < \dots < j(a^{(k)})$  Hence

$$\{1, \dots, k\} \rightarrow \{1, \dots, p\}, i \mapsto j(a^{(i)})$$

is injection. So  $k \leq p$

## 24.3 Linear Equation

Let  $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$ . Let  $V$  be a left  $K$ -module and  $(b_1, \dots, b_n) \in V^n$ . We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

The set of  $(x_1, \dots, x_p) \in V^p$  that satisfies  $(*)$  is called the solution set of  $(*)$

## 24.4 Prop

Suppose that  $A$  is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, \dots, n\} \mid (a_{i,1}, \dots, a_{i,p}) \neq (0, \dots, 0)\}$$

$$J_0(A) = \{1, \dots, p\} \setminus \{j((a_{i,1}, \dots, a_{i,p})) \mid i \in I(A)\}$$

- If  $\exists i \in \{1, \dots, n\} \setminus I(A)$  such that  $b_i \neq 0$  then  $(*)$  does not have any solution in  $K^n$
- Suppose that  $\forall i \in \{1, \dots, n\} \setminus I(A), b_i = 0$ . Then  $(*)$  has at least one solution. Moreover

$$V^{J_0(A)} \rightarrow V^p$$

$$(z_k)_{k \in J_0(A)} \mapsto (x_1, \dots, x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, \dots, a_{i,p})) \end{cases}$$

is an injective mapping, whose image is equal to the set of solution of (\*)

## 24.5 Prop

Let  $m \in \mathbb{N}, S \in M_{m,n}(K)$ . If  $(x_1, \dots, x_p) \in V^p$  is a solution of (\*), then  $(x_1, \dots, x_p)$  is a solution of  $(*)_S$ :

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

In the case where S is left invertible, namely there exist  $R \in M_{n,m}(K)$  such that  $RS = I_n \in M_{m,n}(K)$ . Then (\*) and  $(*)_S$  have the same solution set.

## 24.6 Def

Let  $G_n(K)$  be the set of  $S \in M_{n,n}(K)$  that can be written as  $U_1 \dots U_N$  (by convention  $S = I_n$  where  $N = 0$ ) where each  $U_i$  is of one of the following forms.

- $P_\sigma$  where  $\sigma \in \mathfrak{S}_n$
- $\text{diag}(r_1, \dots, r_n)$  where each  $r_i \in K$  is left invertible
- $S_{i,c}$  with  $i \in \{1, \dots, n\}$   $c = (c_1, \dots, c_n) \in K^n, c_i = 0$

Let  $p \in \mathbb{N}$ , we say that  $A \in M_{n,p}(K)$  is reducible by Gauss elimination if  $\exists S \in G_n(K)$  such that  $SA$  is of reduced row echelon form

## 24.7 Theorem

Assume that K is a division ring  $\forall (n, p) \in \mathbb{N}$  any  $A \in M_{n,p}(K)$  is reducible by Gauss elimination

### Proof

The case where  $n = 0$  or  $p = 0$  is trivial. We assume  $n \geq 1, p \geq 1$  We write A as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} B \quad \text{where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If  $\lambda_1 = \dots = \lambda_n = 0$

Applying the induction hypothesis to B, for  $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} SB$$

- Suppose that  $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

By permuting the rows we may assume  $\lambda_1 \neq 0$ . As K is division ring, by multiplying the first row by  $\lambda_1^{-1}$ , we may assume  $\lambda_1 = 1$ . We add  $(-\lambda_i)$  times the first row to the  $i^{th}$  row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 & \dots & \mu_p \\ 0 & & & \\ \vdots & C & & \\ 0 & & & \end{pmatrix} \quad \begin{array}{l} C \in M_{n-1, p-1}(K) \\ (\mu_2, \dots, \mu_p) \in K^{p-1} \end{array}$$

Applying the induction hypothesis to C, we may assume that C is of reduced row echelon form. For  $i \in \{2, \dots, k\}$  we add  $-\mu_{j(c_i)}$  times the  $i^{th}$  row of A to the first line to obtain a matrix of reduced row echelon form



## Chapter 25

# Normed Vector Space

### 25.1 Def

Let  $(X, d)$  be a metric space. If  $(x_n)_{n \in \mathbb{N}}$  is an element of  $X^{\mathbb{N}}$  such that

$$\lim_{N \rightarrow +\infty} \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

we say that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. If any Cauchy sequence in  $X$  converges, then we say that  $(X, d)$  is complete.

Let  $Cau(X, d)$  be the set of all Cauchy sequences in  $X$ . We define a binary relation  $\sim$  on  $Cau(X, d)$  as

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$$

### 25.2 Prop

$\sim$  is an equivalence relation.

#### 25.2.1 Proof

$$\lim_{n \rightarrow +\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$  be elements of  $Cau(X, d)$ . For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n \rightarrow +\infty} d(x_n, z_n) = 0$$

### 25.3 Def

$$\hat{X} := \text{Cau}(X, d) \setminus \sim$$

### 25.4 Def: The completion

The completion of  $(X, d)$  is defined as

$$\text{Cau}(X) / \sim$$

and is denoted as

$$\hat{X}$$

### 25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} : \hat{X} \times \hat{X} &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n \rightarrow +\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on  $\hat{X}$

#### Proof

TO check that  $\hat{d}$  is well defined, it suffices to prove that  $\forall ([x], [y]) \in \hat{X} \times \hat{X}$ ,  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is Cauchy sequence and its limit doesn't depend on the choice of the representation  $x$  and  $y$

For  $N \in \mathbb{N}$  and  $(n, m) \in \mathbb{N}_{\geq N}$  for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n, m) \in \mathbb{N}_{\geq N}} |d(x_n, y_n) - d(x_m, y_m)| &\leq \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(x_n, x_m) \right) \\ &\quad + \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(y_n, y_m) \right) \end{aligned}$$

Taking  $\lim_{n \rightarrow +\infty}$  we obtain that  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence.

Hence it converges in  $\mathbb{R}$ . If  $x' = (x'_n)_{n \in \mathbb{N}} \in [x], y' = (y'_n)_{n \in \mathbb{N}} \in [y]$ , thus

$$\lim_{n \rightarrow +\infty} d(x_n, x'_n) = \lim_{n \rightarrow +\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\lim_{n \rightarrow +\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(x'_n, y'_n)$$

In the following, we check that  $\hat{d}$  is a metric

- $\hat{d}([x], [y]) = 0$  iff  $[x] = [y]$ : trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$ : trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$ :

$$\begin{aligned} d([x], [y]) &= \lim_{n \rightarrow +\infty} \\ &\leq \lim_{n \rightarrow +\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

## 25.6 Remark

Let

$$\begin{aligned} i_X : X &\rightarrow \hat{X} \\ a &\mapsto [(a, a, \dots)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular,  $i_x$  is injective (if  $i_X(a) = i_X(b)$  then  $d(a, b) = 0$  hence  $a = b$ )

## 25.7 Prop

$i_X(X)$  is dense in  $\hat{X}$  (the closure of  $i_X(X)$  in  $\hat{X}$  is equal to  $i_X(X)$  (or to say  $\hat{X}$ ))

**Proof**

Let  $[x]$  be an equivalence class in  $\hat{X}$ . We claim that  $\forall (x_n)_{n \in \mathbb{N}} \in [x]$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} i_X(x_n)$$

For any  $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n \rightarrow +\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \end{aligned}$$

Taking  $\lim_{N \rightarrow +\infty}$  we get

$$\lim_{N \rightarrow +\infty} \hat{d}(i_X(x_N), [x]) = 0$$

**25.8 Theorem**

$(\hat{X}, \hat{d})$  is a complete metric space

**Proof**

Let  $([x^{(N)}])_{N \in \mathbb{N}}$  be a Cauchy sequence in  $\hat{X}$ , where  $\forall N \in \mathbb{N}$ ,  $x^{(N)} = (x_n^{(N)})_{n \in \mathbb{N}}$  is a Cauchy sequence  
 $\forall \epsilon > 0$ ,  $\exists N_0 \in \mathbb{N}$  such that  $\forall (k, l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n \rightarrow +\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any  $(\mu, \nu) \in \mathbb{N}_{\geq \alpha(N)}$

Let  $y_N = x_{\alpha(N)}^{(N)}$ . Without loss of generality, we assume that

$$\alpha(0) \leq \alpha(1) \leq \dots$$

Let  $\epsilon > 0$  Take  $N_0 \in \mathbb{N}$  such that

$$(1) \quad \forall (k, l) \in \mathbb{N}, \quad k, l \geq N_0$$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

$$(2)$$

$$\frac{1}{N_0 + 1} \leq \frac{\epsilon}{3}$$

Let  $(k, l) \in \mathbb{N}_{N_0}^2$ ,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since  $\alpha(k) \geq N_0, \forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking  $\lim_{n \rightarrow +\infty}$  get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So  $y = (y_N)_{N \in \mathbb{N}}$  is a Cauchy sequence. We check that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \hat{d}([x^{(N)}], [y]) &= 0 \\ 0 &\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \rightarrow +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$\begin{aligned} d(x_n^{(N)}, y_n) &\leq d(x_n^{(N)}, y_N) + d(y_n, y_N) \\ \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, y_n) &\leq \limsup_{N \rightarrow +\infty} \left( \frac{1}{N+1} + \lim_{n \rightarrow +\infty} d(y_n, y_N) \right) \end{aligned}$$

Since  $y$  is Cauchy sequence

$$\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(y_n, y_N) = 0$$

### Example

Let  $(K, |\cdot|)$  be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$  iff  $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$\text{Cau}(K)$  forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in \text{Cau}_0(K)$$

where

$$\text{Cau}_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of  $\text{Cau}(K)$

Hence

$$\hat{K} = \text{Cau}(K) \setminus \text{Cau}_0(K)$$

is a quotient ring of  $\text{Cau}(K)$

$|\cdot|$  extend to  $\hat{K}$ :

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \rightarrow +\infty} |a_n|$$

that forms an absolute value.

# Chapter 26

## Norms

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 26.1 Def

Let  $V$  be a vector space over  $K$  ( $K$ -module). We call seminorm on  $V$  any mapping

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}_{\geq 0} \\ s &\mapsto \|s\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V, \|s\| = 0$  iff  $s = 0$

We say that  $\|\cdot\|$  is a norm and  $(V, \|\cdot\|)$  is normed space over  $K$ .

### 26.2 Remark

If  $\|\cdot\|$  is a norm then

$$\begin{aligned} d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let  $(V, \|\cdot\|)$  be a vector space over  $K$  equipped with a seminorm, and  $W$  be a vector space subspace of  $V$  (sub- $K$ -module)

- The restriction of  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  to  $W$  forms a seminorm on  $W$ . It is a norm if  $\|\cdot\|$  is a norm.

$$\begin{aligned}\|\cdot\|_W : W &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto \|x\|\end{aligned}$$

- The mapping

$$\begin{aligned}\|\cdot\|_{V/W} : V/W &\rightarrow \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\|\end{aligned}$$

is a seminorm on  $V/W$

**Attention:** Even if  $\|\cdot\|$  is a norm,  $\|\cdot\|_{V/W}$  **might only be a seminorm**

### 26.3 Def

$\|\cdot\|_{V/W}$  is called the quotient seminorm of  $\|\cdot\|$

### 26.4 Prop

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of  $V$ . Moreover,  $\|\cdot\|_{V/N}$  is a norm

#### Proof

If  $(a, s) \in K \times N$  then  $\|as\| = |a| \cdot \|s\| = 0$  so  $as \in N$

If  $(s_1, s_2) \in N \times N$  then  $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$  so  $s_1 + s_2 \in N$

#### Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let  $\alpha \in V/N$  such that  $\|\alpha\|_{V/N} = 0$  Let  $s \in \alpha, \forall t \in N$

$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$

$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$

Hence  $\|\alpha\|_{V/N} = \|s\| = 0$  We obtain that  $\alpha = N = [0]$



## 26.5 Def

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. For any  $x \in V$  and  $r \geq 0$ , we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$

$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

## 26.6 Remark

If  $N = \{s \in V, \|s\| = 0\}$  then when  $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

## 26.7 Def

We equip the topology such that  $\forall U \subseteq V, U$  is open iff  $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

## 26.8 Prop

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorms. Let  $f : V_1 \rightarrow V_2$  be a  $K$ -linear mapping

- If  $f$  is continuous,  $\forall s \in V_1$  if  $\|s\|_1 = 0$  then  $\|f(s)\|_2 = 0$
- If there exists  $C > 0$  such that  $\forall x \in V_1, \|f(x)\|_2 \leq C\|x\|_1$  then  $f$  is continuous.

The converse is true

when  $|\cdot|$  is non-trivial

or  $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$  is of finite type

### Proof

- (1) Lemma If  $(V, \|\cdot\|)$  is a vector space over  $K$ , equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let  $s \in V \setminus N_{\|\cdot\|}$  Then  $\|s\| > 0$ . Let  $\epsilon = \frac{\|s\|}{2}$ ,  $\forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq \|s\| - \|s - x\| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteq V \setminus N_{\|\cdot\|}$$

– Then  $f^{-1}(N_{\|\cdot\|_2})$  is closed.

Note that

$$0 \in f^{-1}(N_{\|\cdot\|_2})$$

hence

$$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$

$$\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$$

$$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$

and

$$0 \in \mathcal{B}(x, \epsilon)$$

Therefore  $x \in \overline{\{0\}}$

(2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $V_1$  that converges to some  $x \in V_1$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n \rightarrow +\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n \rightarrow +\infty} C \|x_n - x\|_1 \\ &= C \limsup_{n \rightarrow +\infty} \|x_n - x\| \\ &= 0 \end{aligned}$$

So  $(f(x_n))_{n \in \mathbb{N}}$  converges to  $f(x)$ . Hence  $f$  is continuous at  $x$

Assume that  $|\cdot|$  is non-trivial and  $f$  is continuous. Then

$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

is an open subset of  $V_1$  containing  $0 \in V_1$

So there exists  $\epsilon > 0$  such that

$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

namely  $\forall x \in V_1$  if  $\|x\|_1 < \epsilon$  then  $\|f(x)\|_2 < 1$

Since  $|\cdot|$  is nontrivial,  $\exists a \in K, 0 < |a| < 1$  We prove that  $\forall x \in V_1$

$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|} \|x\|_1$$

If  $\|x\|_1 = 0$  by (1) we obtain

$$\|f(x)\|_2 = 0$$

Suppose that  $\|x\|_1 > 0$  then  $\exists n \in \mathbb{Z}$  such that

$$\begin{aligned} \|a^n x\|_1 &= |a|^n \|x\|_1 \\ &< \epsilon \leq \\ &\|a^{n-1} x\|_1 = |a|^{n-1} \|x\|_1 \end{aligned}$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned} \|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}} \frac{1}{|a|} \\ &\leq \frac{1}{\epsilon} \|x\|_1 \frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|} \end{aligned}$$

## 26.9 Def: Operator Seminorm

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorm. We say that a  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  is bounded if there exists  $C > 0$  that

$$\forall x \in V_1 \quad \|f(x)\|_2 \leq C\|x\|_1$$

For a general  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  we denote

$$\|f\| := \begin{cases} \sup_{x \in V_1, \|x\|_1 > 0} \left( \frac{\|f(x)\|_2}{\|x\|_1} \right) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

$f$  is bounded iff

$$\|f\| < +\infty$$

$\|f\|$  is called the operator seminorm of  $f$

We denote by  $\mathcal{L}(V_1, V_2)$  the set of all bounded  $K$ -linear mappings from  $V_1$  to  $V_2$

## 26.10 Prop

$\mathcal{L}(V_1, V_2)$  is a vector subspace of  $\text{Hom}_K(V_1, V_2)$ . Moreover  $\|\cdot\|$  is a seminorm on  $\mathcal{L}(V_1, V_2)$

### Proof

Let  $f, g$  be elements of  $\mathcal{L}(V_1, V_2)$

$$\begin{aligned} \|f + g\| &= \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x) + g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1} \right) + \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1} \right) \\ &\leq +\infty \end{aligned}$$

Hence  $f + g \in \mathcal{L}(V_1, V_2)$

Let  $\lambda \in K$ ,  $\lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

### 26.11 Remark

Let  $f \in \mathcal{L}(V_1, V_2)$ . Suppose that  $\exists x \in V_1$  such that  $f(x) \neq 0$ . Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore  $\|\cdot\|$  is a norm

### 26.12 Def

Let  $(V, \|\cdot\|)$  be a normed vector space. If  $V$  is complete with respect to the metric

$$\begin{aligned}d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|x - y\|\end{aligned}$$

then we say that  $(V, \|\cdot\|)$  is a Banach space.

### 26.13 Theorem

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with semi-norm. If  $(V_2, \|\cdot\|_2)$  is a Banach space, then

$$(\mathcal{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

**Proof**

Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{L}(V_1, V_2)$ .  
 $\forall x \in V_1$ , the mapping

$$(f \in \mathcal{L}(V_1, V_2)) \mapsto f(x)$$

is  $\|x\|_1$ -Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\| \|x\|_1$$

So  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence, for  $V_2$  is complete, that converges to some  $g(x) \in V_2$ . Then we obtain a mapping  $g : V_1 \rightarrow V_2$ . We prove that  $g$  is an element of  $\mathcal{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n \rightarrow +\infty} f_n(x + y) = \lim_{n \rightarrow +\infty} f_n(x) + f_n(y)$$

$$\begin{aligned} \|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \rightarrow +\infty) \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n \rightarrow +\infty} f_n(\lambda x) = \lim_{n \rightarrow +\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n \rightarrow +\infty} \|f_n(x)\| \leq (\lim_{n \rightarrow +\infty} \|f_n\|) \cdot \|x\|$$

(because  $\forall (a, b) \in V_2^2 \quad \|a\| - \|b\| \leq \|a - b\|$ ) Then

$$\|f_n(x)\| - \|g_n(x)\| \leq \|f_n(x) - g_n(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g \in \mathcal{L}(V_1, V_2)$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall (n, m) \in \mathbb{N}_{\geq N}, \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

So  $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$



## Chapter 27

# Differentiability

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 27.1 Def

Let  $X$  be a topological space and  $p \in X$ . Let  $K$  be a complete valued field and  $(E, \|\cdot\|)$  be a normed vector space over  $K$ .

Let  $f : X \rightarrow E$  be a mapping and  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative mapping.

- We say that

$$f(x) = O(g(x)) \text{ } x \rightarrow p$$

if there is a neighborhood  $V$  of  $p$  in  $X$  and a constant  $C > 0$  such that  $\forall x \in V$

$$\|f(x)\| \leq Cg(x)$$

- We say that

$$f(x) = o(g(x)) \text{ } x \rightarrow p$$

if there exists a neighborhood  $V$  of  $p$  in  $X$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\lim_{x \in V, x \rightarrow p} \epsilon(x) = 0$$

which is equivalent to

$$\forall \delta > 0, \exists \text{ neighborhood } U \text{ of } p \text{ } U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$

and  $\forall x \in V$

$$\|f(x)\| \leq \epsilon(x)g(x)$$

## 27.2 Def

Let  $E$  and  $F$  be normed vector space over  $K$   $U \subseteq E$  be an open subset,  $f : U \rightarrow F$  be a mapping and  $p \in U$  If there exists  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

we say that  $f$  is differentiable at  $p$ , and  $\varphi$  is the differential of  $f$  at  $p$  Suppose that  $|\cdot|$  is not trivial.  $\varphi(x - p)$  also written as

$$d_p f$$

## Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

means there exists an open neighborhood  $V$  of  $p$  with  $V \subseteq U$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that  $\lim_{x \rightarrow p} \epsilon(x) = 0$  and that  $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

## 27.3 Prop

If  $f$  is differentiable at  $p$ , then its differential at  $p$  is unique

## Proof

Suppose that there exists  $\varphi$  and  $\psi$  in  $\mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|\varphi - \psi\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\| (y - p)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \rightarrow p} \epsilon(y) = 0 \end{aligned}$$



## 27.4 Example

### 27.4.1

$$f : U \rightarrow F : f(x) = y_0 \quad \forall x \in U$$

$$\forall p \in U$$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence  $\forall x \in E$

$$d_p(f(x)) = 0$$

### 27.4.2

Let  $f \in \mathcal{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence  $d_p f = f$

### 27.4.3

$$A : E \times E \rightarrow E$$

$$(x, y) \mapsto x + y$$

Let  $E$  be a normed space. Then  $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

### 27.4.4

$$m : K \times E \rightarrow E$$

$$(\lambda, x) \mapsto \lambda x$$

Let  $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when  $(\lambda, x) \rightarrow (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

is a  $K$ -linear mapping.

$$\begin{aligned}
- & (\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2) \\
- & b\mu p + a(by) = b(\mu p + ay) \\
- & \|\mu p + ay\| \leq |\mu| \|p\| + |a| \|y\| \\
& \leq \max\{|\mu|, \|y\|\}(|a| + \|p\|)
\end{aligned}$$

Hence  $m$  is differentiable and  $\forall (\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

## 27.5 Theorem:Chain rule

Let  $E, F, G$  be normed vector spaces,  $U \subseteq E, V \subseteq F$  be open subsets.

Let  $f : U \rightarrow F, g : V \rightarrow G$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . Assume that  $f$  is differentiable at  $p$  and  $g$  differentiable at  $f(p)$ . Then  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

### Proof

Let  $x \in U$ . By definition

$$\begin{aligned}
f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\
f(x) - f(p) &= O(\|x - p\|)
\end{aligned}$$

and

$$\begin{aligned}
(g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|)
\end{aligned}$$

So  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

## 27.6 Prop

Let  $n$  be a positive integer. Let  $(F_i)_{i \in \{1, \dots, n\}}$  be normed vector spaces over  $K$ . Let  $U \subseteq E$  be an open subset,  $p \in U$ .

$\forall i \in \{1, \dots, n\}$  let  $f_i : U \rightarrow F_i$  be a mapping. Let

$$f : U \rightarrow F = \prod F_i$$

be the mapping that sends  $x \in U$  to  $(f_i(x))_{i \in \{1, \dots, n\}}$ . We equip  $F$  with the norm  $\|\cdot\|$  defined as :

$$\|(y_i)_{i \in \{1, \dots, n\}}\| = \max_{i \in \{1, \dots, n\}} \|y_i\|$$

Then  $f$  is differentiable at  $p$  iff each  $f_i$  is differentiable at  $p$ . Moreover, when this happen, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i \in \{1, \dots, n\}}$$

### Proof

$\Leftarrow$  Suppose that  $(f_i)_{i \in \{1, \dots, n\}}$  are differentiable at  $p$

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i \in \{1, \dots, n\}} \\ &= (d_p f_i(x - p))_{i \in \{1, \dots, n\}} + o(\|x - p\|) \end{aligned}$$

Therefore  $f$  is differentiable at  $p$  and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i \in \{1, \dots, n\}}$$

$\Rightarrow$  Let

$$\begin{aligned} \pi_i : F &\rightarrow F_i \\ (x_i)_{i \in \{1, \dots, n\}} &\mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has  $\|\pi_i\| \leq 1$  because

$$\|x_i\| \leq \max_{i \in \{1, \dots, n\}} \|x_i\| = \|(x_i)_{i \in \{1, \dots, n\}}\|$$

$\pi_i$  is differentiable at  $p$  then  $\pi_i \circ f = f_i$  is differentiable at  $p$

## 27.7 Def

Let  $U$  be an open subset of  $K$  and  $(F, \|\cdot\|)$  be a normed vector space. If  $f : U \rightarrow F$  is a mapping that is differentiable at some  $p \in U$ . We denote by  $f'(p)$  the element

$$d_p f(1) \in F$$

called the derivative of  $f$  at  $p$

## 27.8 Corollary

Let  $U$  and  $V$  be open subsets of  $K$ ,  $(F, \|\cdot\|)$  be a normed vector space over  $K$ .  $f : U \rightarrow K$ ,  $g : V \rightarrow F$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$  then

$$(g \circ f)'(p) = f'(p)g'(f(p))$$

**Proof**

By definition

$$\begin{aligned}
 d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\
 &= d_{f(p)}g(f'(p)) \\
 &= d_{f(p)}g(f'(p) \cdot 1) \\
 &= f'(p) \cdot d_{f(p)}g(1) \\
 &= f'(p)g'(f(p))
 \end{aligned}$$

**27.9 Corollary**

Let  $E$  and  $F$  be normed vector spaces,  $U \subseteq E$  an open subset.  $f : U \rightarrow L$  and  $g : U \rightarrow F$  be mappings and  $p \in U$ . If both  $f, g$  differentiable at  $p$  then

$$\begin{aligned}
 fg : U &\rightarrow F \\
 x &\mapsto f(x)g(x)
 \end{aligned}$$

is also differentiable at  $p$  and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_p f(l) + g(p)d_p f(l)$$

**Proof**

Consider

$$\begin{aligned}
 m : K \times F &\rightarrow F \\
 (a, y) &\rightarrow ay
 \end{aligned}$$

We have shown  $m$  is differentiable and

$$d_{a,y}m(b, z) = by = az$$

$fg$  is the following composite:

$$U \begin{array}{c} \xrightarrow{h} \\ \searrow fg \\ \xrightarrow{m} \end{array} K \times F \xrightarrow{\quad} F$$

$$x \longmapsto (f(x), g(x)) \longmapsto f(x)g(x)$$

$$\begin{aligned}
 d_p(fg)(l) &= d_p(m \circ h)(l) \\
 &= d_{h(p)}m(d_p h(l)) \\
 &= d_{(f(p), g(p))}m(d_p f(l), d_p g(l)) \\
 &= f(p)d_p g(l) + d_p f(l)g(p)
 \end{aligned}$$

## 27.10 Corollary

Let  $U$  be an open subset of  $K$ ,  $f, g$  be mappings from  $U$  to  $K$  and to a normed space  $F$  respectively. If  $f, g$  are differentiable at  $p \in U$  then

$$(fg)'(p) = d_p(fg)(1) = d_p f(1)g(p) + f(p)d_p g(1) = f'(p)g(p) + f(p)g'(p)$$

### Example

$$\begin{aligned} f_n : K &\rightarrow K \\ x &\mapsto x^n \end{aligned}$$

is differentiable at any  $x \in K$

$$f'_n(x) = nx^{n-1}$$

### Proof

$f_1 : K \rightarrow K$  is differentiable  $\forall x \in K$

$$d_x f_1 = f_1$$

If  $f'_n(x) = nx^{n-1}$  then

$$\begin{aligned} f'_{n+1}(x) &= (f_n f_1)'(x) \\ &= f_n(x)f'_1(x) + f'_n(x)f_1(x) \\ &= x^n + x'_n(x) = x^n + nx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(1) &= l d_x f_n(1) \\ &= nx^{n-1} \end{aligned}$$

## 27.11 Prop

Let  $E, F, G$  be normed vector spaces.  $U \subseteq E$  be an open subset,  $\varphi \in \mathcal{L}(F, G)$ ,  $p \in U$  if  $f : U \rightarrow E$  is differentiable at  $p$  then so is  $\varphi \circ f$ . Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

### Proof

$\varphi$  is differentiable at  $f(p)$  nad  $d_{f(p)}\varphi = \varphi$

### 27.12 Corollary

Let  $E$  and  $F$  be normed vector spaces  $U \subseteq E$  be an open subset,  $p \in U$ . Let  $f : U \rightarrow F$  and  $g : U \rightarrow F$  be mappings that are differentiable at  $p$ ,  $(a, b) \in K \times K$ . Then  $af + bg$  is differentiable at  $p$  and

$$d_p(af + bg) = ad_p f + bd_p g$$

#### Proof

$af + bg$  is composite:

$$U \xrightarrow{h} K \times F \xrightarrow{m} F$$

$ay+bz$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

### 27.13 Def: Equivalence of Norms

Let  $E$  be a vector space over  $K$  and  $\|\cdot\|_1, \|\cdot\|_2$  be norms on  $E$ . We say that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if there exist constants  $C_1, C_2 > 0$  such that  $\forall s \in E$

$$C_1 \|s\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

### 27.14 Prop

If  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover  $\|\cdot\|_1, \|\cdot\|_2$  defines the same topology on  $E$ .

#### Proof

$$\|s\|_2 \leq C_2 \|s\|_1 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are continuous. So  $\forall$  open subset  $U$  of  $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in  $(E, \|\cdot\|_1)$ . Conversely if  $V$  is open in  $(E, \|\cdot\|_1)$  then

$$V = Id_E^{-1}(V)$$

is open in  $(E, \|\cdot\|_2)$

## 27.15 Remark

If  $\|\cdot\|_1, \|\cdot\|_2$  are two norms on  $E$  that define the same topology on  $E$ , then they are equivalent (under the assumption that  $|\cdot|$  is not trivial)

## 27.16 Prop

Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$  be norms on  $E$  and  $F$  that are equivalent to  $\|\cdot\|_E, \|\cdot\|_F$  respectively. Let  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping.

Let  $p \in U$  Then  $f$  is differentiable at  $p$  with respect to  $\|\cdot\|_E$  and  $\|\cdot\|_F$  iff it's differentiable with respect to  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$

Moreover the differentiable of  $f$  at  $p$  is not changed in the change of norms from  $(\|\cdot\|_E, \|\cdot\|_F)$  to  $(\|\cdot\|'_E, \|\cdot\|'_F)$

### Proof

$$U \xrightarrow{Id_U} U \xrightarrow{f} F \xrightarrow{Id_F} F$$

$f$

$$(E, \|\cdot\|'_E) \quad (E, \|\cdot\|_E) \quad \|\cdot\|_F \quad \|\cdot\|'_F$$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$$d'_p f : (E, \|\cdot\|'_E) \rightarrow (F, \|\cdot\|'_F)$$

## 27.17 Theorem

Let  $V$  be a finite dimensional vector space over  $K$ . Then all norms on  $V$  are equivalent. Moreover  $V$  is complete with respect to any norm on  $V$ .

**Proof**

Let  $(e_i)_{i=1}^n$  be a basis of  $V$  (linear independent system of generators) The mapping:

$$V \rightarrow \mathbb{R}_{\geq 0}$$

$$\sum_{i \in \{1, \dots, n\}} a_i e_i \mapsto \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

is a norm on  $V$

Let  $\|\cdot\|$  be another norm on  $V$ . One has

$$\left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\| \leq \sum_{i \in \{1, \dots, n\}} |a_i| \|e_i\|$$

$$\leq \left( \sum_{i \in \{1, \dots, n\}} \|e_i\| \right) \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

We reason by induction that there exists  $C > 0$  such that

$$\max_{i \in \{1, \dots, n\}} \{|a_i|\} \leq C \left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\|$$

The case where  $n = 0$  is trivial.

$n=1$

$$\|a_1 e_1\| = |a_1| \|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension  $< n$

Let

$$W = \left\{ \sum_{i \in \{1, \dots, n-1\}} a_i e_i \mid (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1} \right\}$$

equipped with  $\|\cdot\|$  restricted to  $W$

The induction hypothesis shows that  $W$  is complete. Hence it's closed in  $V$ . Let  $Q = V/W$  and  $\|\cdot\|_Q$  be the quotient norm on  $Q$  that's defined as

$$\forall \alpha \in Q \quad \|\alpha\|_Q = \inf_{s \in \alpha} \|s\|$$

– If  $s \in V \setminus W$ ,  $\exists \epsilon > 0$  such that

$$\overline{B}(s, \epsilon) \cap W = \emptyset$$

$\forall t \in W$ ,

$$s + t \notin \overline{B}(0, \epsilon)$$

since otherwise

$$-t \in W \cap \overline{B}(s, \epsilon)$$

Therefore

$$\|[s]\|_Q = \inf_{i \in W} \|s + t\| \geq \epsilon > 0$$



–  $\forall \lambda \in K$

$$\begin{aligned}\|\lambda \alpha\|_Q &= \inf_{s \in \alpha} \|\lambda s\| = |\lambda| \\ \inf_{s \in \alpha} \|s\| &= |\lambda| \cdot \|\alpha\|_Q\end{aligned}$$

–

$$\begin{aligned}\|\alpha + \beta\|_Q &= \inf_{s \in \alpha + \beta} \|s\| \\ &= \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \inf_{x \in \alpha} \|x\| + \inf_{y \in \beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some  $A > 0$  such that  $\forall (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1}$

$$\max_{i \in \{1, \dots, n-1\}} \{|a_i|\} \leq A \left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\|$$

Take

$$s = \sum_{i \in \{1, \dots, n\}} a_i e_i \in V$$

Let  $\alpha = [s] = a_n [e_n] \in Q$

$$\left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \leq \|s\| + |a_n| \cdot \|e_n\| \leq \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \|[e_n]\|_Q = |a_n| \inf_{t \in W} \|e_n + t\|$$

Take  $e'_n \in V$  such that  $[e'_n] = [e_n]$  and  $\|e'_n\| \leq \|[e_n]\|_Q + \epsilon$

Note that  $(e_1, \dots, e_{n-1}, e'_n)$  forms also basis of  $V$  over  $K$ . Hence by replacing  $e_n$  by  $e'_n$  we may assume that  $\|e_n\| \leq \|[e_n]\|_Q + \epsilon$

$s = a_n e_n + t \in V$  with  $t \in W$

$$\|s\| \geq \|a_n e_n\|_Q = |a_n| \|[e_n]\|_Q \geq B^{-1} |a_n| \cdot \|e_n\|$$

– If  $\|a_n e_n\| < \frac{1}{2} \|t\|$

$$\|s\| \geq \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \geq \frac{1}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

– If  $\|a_n e_n\| \geq \frac{1}{2} \|t\|$

$$\|s\| \geq B^{-1} |a_n| \cdot \|e_n\| \geq \frac{B^{-1}}{2} \|t\| \geq \frac{B^{-1}A}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

We take  $C = \max\{B^{-1} \|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$  Then

$$\|s\| \geq C \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Another proof

completeness Under the norm  $\max_{i \in \{1, \dots, n\}}$ , a sequence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  is a Cauchy sequence iff  $\forall i \in \{1, \dots, n\}$   $(a_i^{(k)})_{k \in \mathbb{N}}$  is a Cauchy sequence. Since  $K$  is complete each  $(a_i^{(k)})_{k \in \mathbb{N}}$  converges to some  $a_i \in K$  Hence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  converges.

## 27.18 Prop

Let  $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . Assume that  $E$  is finite dimensional. Then any  $K$ -linear mapping  $\varphi : E \rightarrow F$  is bounded.

### Proof

Let  $(e_i)_{i=1}^n$  be a basis of  $E$ . For any two norms on  $E$  are equivalent.  
 $\forall (a_1, \dots, a_n) \in K$

$$\left\| \sum_{i=1}^n a_i e_i \right\|_E = \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Then for any  $s = \sum_{i=1}^n a_i e_i$

$$\|\varphi(s)\|_F = \left\| \sum_{i=1}^n a_i e_i \right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq \left( \sum_{i=1}^n \|\varphi(e_i)\|_F \right) \|s\|_E$$

## 27.19 Theorem

Let  $E, F$  be normed vector spaces over a complete valued field,  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping. If  $f$  is differentiable at  $p$  then  $f$  is continuous at  $p$

### Proof

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ &= f(p) + O(\|x - p\|) \\ &= f(p) + o(1) \quad x \rightarrow p \\ &\Rightarrow \lim_{x \rightarrow p} f(x) = f(p) \end{aligned}$$

## Chapter 28

# Compactness

### 28.1 Def: cover

Let  $X$  be a topological space,  $Y \subseteq X$  we call open cover of  $Y$  any family  $(U_i)_{i \in I}$  open subset of  $X$  such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If  $I$  is finite set, we say that  $(U_i)_{i \in I}$  is a finite open cover. If  $J \subseteq I$  such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that  $(U_j)_{j \in J}$  is a sub cover of  $(U_i)_{i \in I}$

### 28.2 Def: compact

If any open cover of  $Y$  has a finite subcover, we say that  $Y$  is quasi-compact. If in addition  $X$  is Hausdorff, namely  $\forall (x, y) \in X \times X$  with  $x \neq y \exists$  open neighborhoods  $U$  and  $V$  of  $x$  and  $y$  such that  $U \cap V = \emptyset$ , we say that  $Y$  is compact

### 28.3 Def

Let  $X$  be a set and  $\mathcal{F}$  be a filter on  $X$ . If there does not exist any filter  $\mathcal{F}'$  of  $X$  such that  $\mathcal{F} \subsetneq \mathcal{F}'$ , then we say that  $\mathcal{F}$  is an ultrafilter.

**Zorn's lemma** implies that  $\forall \mathcal{F}_0$  of  $X$  there exist an ultrafilter  $\mathcal{F}$  if  $X$  containing  $\mathcal{F}_0$

## 28.4 Prop

Let  $\mathcal{F}$  be a filter on a set  $X$ . The following statements are equivalent.

- (1)  $\mathcal{F}$  is an ultrafilter
- (2)  $\forall A \subseteq X$  either  $A \in \mathcal{F}$  or  $X \setminus A \in \mathcal{F}$
- (3)  $\forall (A, B) \in \wp(X)^2$  if  $A \cap B \in \mathcal{F}$  then  $A \in \mathcal{F}$  or  $B \in \mathcal{F}$

### Proof

- (1)  $\Rightarrow$  (2) Suppose that  $A \in \wp(X)$  such that  $A \notin \mathcal{F}$  and  $X \setminus A \notin \mathcal{F} \forall B \in \mathcal{F}$  one has

$$B \cap A \neq \emptyset$$

since otherwise  $B \subseteq X \setminus A$  and hence  $X \setminus A \in \mathcal{F}$  contradiction.

- (2)  $\Rightarrow$  (3) Suppose that  $B \notin \mathcal{F}$  then  $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So  $A \in \mathcal{F}$

- (3)  $\Rightarrow$  (1) Suppose that  $\mathcal{F}'$  is a filter such that  $\mathcal{F} \subsetneq \mathcal{F}'$  Take  $A \in \mathcal{F}' \setminus \mathcal{F}$  Then by  $X = A \cup (X \setminus A) \in \mathcal{F}$  Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \emptyset = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

## 28.5 Theorem

Let  $(X, \mathcal{G})$  be a topological space . The following are equivalent

- (1)  $X$  is quasi-compact
- (2) Any filter of  $X$  has an accumulation point
- (3) Any ultrafilter of  $X$  is converges.

### Proof

- (1)  $\Rightarrow$  (2) Assume that a filter  $\mathcal{F}$  of  $X$  does not have any accumulation point.  $\forall x \in X \exists A_x \in \mathcal{F} \exists$  open neighborhood  $V_x$  of  $x$  such that  $A_x \cap V_x = \emptyset$  Since  $X = \bigcup_{x \in X} V_x$  there is

$$\{x_1, \dots, x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^n V_{x_i}$$

$$\text{Take } B = \bigcap_{i=1}^n A_{x_i} \in \mathcal{F}$$

$$B \cap X = B = \emptyset$$

Since  $\forall i \ B \cap V_x = \emptyset$  contradiction.

- (2)  $\Rightarrow$  (3) Let  $\mathcal{F}$  be an ultrafilter of  $X$ . By (2) there exist  $x \in X$  such that  $\mathcal{F} \cup \mathcal{V}_x$  generates a filter  $\mathcal{F}'$  Since  $\mathcal{F}$  is an ultrafilter  $\mathcal{F} = \mathcal{F}'$  and hence  $\mathcal{V}_x \subseteq \mathcal{F}$
- (3)  $\Rightarrow$  (1) Let  $(U_i)_{i \in I}$  be an open cover of  $X$  we suppose that this have no finite subcover.  $\forall i \in I$  let

$$F_i = X \setminus U_i$$

For any  $J \subseteq I$  finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \emptyset$$

Let  $\mathcal{F}$  be the smallest filter on  $X$  that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let  $\mathcal{F}'$  be ultrafilter containing  $\mathcal{F}$ . It has a limit point  $x$  There exist  $i \in I$  such that  $x \in U_i$ . Since  $U_i$  is a neighborhood of  $x$  and  $\mathcal{V}_x \subseteq \mathcal{F}'$  we get  $U_i \in \mathcal{F}'$  This is impossible since  $F_i \in \mathcal{F}'$

## 28.6 Theorem

Let  $(X, d)$  be a metric space. The following statements are equivalent:

- (1)  $X$  is complete and  $\forall \epsilon > 0 \ \exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

- (2)  $X$  is compact

### Proof

- (1)  $\Rightarrow$  (2) Let  $\mathcal{F}$  be an ultrafilter Let  $\epsilon > 0$  and  $\{x_1, \dots, x_n\} \subseteq X$  such that

$$X = \bigcup_{i=1}^n \mathcal{B}(x_i, \epsilon)$$

There exists some  $i \in \{1, \dots, n\}$  such that  $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$  That means  $\mathcal{F}$  is a Cauchy filter (namely  $\forall \delta > 0 \ \exists A \in \mathcal{F}$  of diameter  $\leq \delta$ ) Since  $X$  is complete  $\mathcal{F}$  has a limit point. So  $\mathcal{F}$  is compact.

(2)  $\Rightarrow$  (1) Let  $\epsilon > 0$  One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since  $X$  is compact  $\exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$\mathcal{F}$  is an ultrafilter

$$\Leftrightarrow \forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}$$

$$\Leftrightarrow \forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F}$$

$$\Leftrightarrow \forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup \dots \cup A_n \ \exists i \in \{1, \dots, n\}, A_i \in \mathcal{F}$$

Let  $\mathcal{F}$  be a Cauchy filter Let  $x \in X$  be an accumulation point of  $\mathcal{F}$   
 $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$  with diameter  $\leq \frac{\epsilon}{2}$  Note that  $A \cap \mathcal{B}(x, \frac{\epsilon}{2}) \neq \emptyset$  Take  
 $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2}) \ \forall z \in A$

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore  $A \subseteq \mathcal{B}(x, \epsilon)$  So  $\mathcal{B}(x, \epsilon) \in \mathcal{F}$  This implies  $\nu_x \subseteq \mathcal{F}$

## 28.7 Lemma

Let  $(X, d)$  be a metric space

- (1) Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$
- (2)  $X$  is complete iff any Cauchy filter of  $X$  has a limit point

### Proof

(1)

- Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$

- (2) Suppose that  $X$  is complete. Let  $\mathcal{F}$  be a Cauchy filter.  $\forall n \in \mathbb{N}_{\geq 1}$  let  $A_n \in \mathcal{F}$  such that  $\text{diam}(A_n) \leq \frac{1}{n}$  Take  $x_n \in \bigcap_{k=1}^n A_k \in \mathcal{F}$  Then  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  is a Cauchy sequence since  $\forall \epsilon > 0$  if we take  $N \in \mathbb{N}$  with  $\frac{1}{N} \leq \epsilon$  then  $\forall (n, m) \in \mathbb{N}_{\geq N} \ d(x_n, x_m) \leq \frac{1}{N}$  Hence  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  converges to some  $x \in X$  Note that  $x$  is an limit point of  $\mathcal{F}$  since  $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$  with  $A_n \subseteq \mathcal{B}(x, \epsilon)$  It suffices to take  $n$  such that  $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$  Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $X$ . Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N}, \{x_N, x_{N+1}, \dots\} \subseteq A\}$$

This is a Cauchy filter on  $X$  since

$$\lim_{N \rightarrow +\infty} \text{diam}\{x_N, x_{N+1}, \dots\} = 0$$

Hence  $\mathcal{F}$  has a limit point  $x \in X$  By definition  $\forall U \in \mathcal{V}_x \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, \dots\} \subseteq U$$

$$\text{So } x = \lim_{n \rightarrow +\infty} x_n$$

## 28.8 Prop

Let  $f : X \rightarrow Y$  be a continuous mapping of topological spaces. If  $A \subseteq X$  is quasi-compact then  $f(A) \subseteq Y$  is also quasi-compact.

### Proof

Let  $(V_i)_{i \in I}$  be an open cover of  $f(A)$  Then

$$(f^{-1}(V_i))_{i \in I}$$

is an open cover of  $A$  So  $\exists J \subseteq I$  such that

$$A \subseteq \bigcup_{j \in J} f^{-1}(V_j)$$

This implies

$$f(A) \subseteq \bigcup_{j \in J} V_j$$

So  $f(A)$  is quasi-compact.

## 28.9 Prop

Let  $X$  be a topological space and  $A \subseteq X$  be a quasi-compact subset. For any closed subset  $F$  of  $X$   $A \cap F$  is quasi-compact.

### Proof

Let  $(U_i)_{i \in I}$  be an open cover of  $A \cap F$ . Then

$$A \subseteq \left( \bigcup_{i \in I} U_i \right) \cup (X \setminus F)$$

Since  $A$  is quasi-compact there exist  $J \subseteq I$  finite such that

$$A \subseteq \left( \bigcup_{j \in J} U_j \right) \cup (X \setminus F)$$

Hence  $A \cap F \subseteq \bigcup_{j \in J} U_j$

### 28.10 Prop

Let  $X$  be a Hausdorff topological space. Any compact subset  $A$  of  $X$  is closed.

#### Proof

Let  $x \in X \setminus A$ .  $\forall y \in A, \exists$  open subsets  $U_y$  and  $V_y$  such that  $y \in U_y, x \in V_y$  and  $U_y \cap V_y = \emptyset$ . Since  $A \subseteq \bigcup_{y \in A} U_y$ ,  $\exists \{y_1, \dots, y_n\} \subseteq A$  such that

$$A \subseteq \bigcup_{i=1}^n U_{y_i}$$

Let

$$U = \bigcup_{i=1}^n U_{y_i} \quad V = \bigcap_{i=1}^n V_{y_i}$$

These are open subsets. Moreover  $A \subseteq U, x \in V$  and  $U \cap V = \bigcup_{i=1}^n (U_{y_i} \cap V) = \emptyset$ . In particular  $x \in V \subseteq X \setminus A$ . So  $X \setminus A$  is open.

### 28.11 Prop

Let  $X$  be a Hausdorff topological space and  $A$  and  $B$  be compact subsets of  $X$  such that  $A \cap B = \emptyset$ . Then there exist open subsets  $U$  and  $V$  such that

$$A \subseteq U, B \subseteq V \text{ and } U \cap V = \emptyset$$

#### proof

We have seen in the proof of the previous proposition that  $\forall x \in B, \exists U_x, V_x$  open such that  $A \subseteq U_x, x \in V_x$  and  $U_x \cap V_x = \emptyset$ . Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, \dots, x_m\} \subseteq B$  such that

$$B \subseteq \bigcup_{i=1}^m V_{x_i}$$



We take

$$U = \bigcap_{i=1}^m U_{x_i} \quad V = \bigcup_{i=1}^m U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq V \quad U \cap V = \emptyset$$

## 28.12 Theorem

Let  $(X, \mathcal{G})$  be a Hausdorff topological space. If  $(A_n)_{n \in \mathbb{N}}$  is a sequence of non-empty compact subsets of  $X$  such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

Then

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$$

### Proof

Suppose that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

then

$$A_0 \subseteq \bigcup_{n \in \mathbb{N}} (X \setminus A_n)$$

Since  $A_0$  is compact,  $\exists N \in \mathbb{N}$  such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^N (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^N A_n \\ &= X \setminus A_N \end{aligned}$$

So

$$A_N = \emptyset$$

## 28.13 Def

Let  $(X, \tau)$  be a topological space. If any sequence in  $X$  has a convergent subsequence, we say that  $X$  is sequentially compact.

**Example**

By Bolzano-Weierstrass, any bounded sequence in  $\mathbb{R}$  has a convergent subsequence. So any bounded and closed subset of  $\mathbb{R}$  is sequentially compact.

**Note**

bounded and closed together implies sequentially compact.

**28.14 Theorem**

Let  $(X, d)$  be a metric space. Then the following statements are equivalent:

- (1)  $(X, d)$  is compact
- (2)  $(X, d)$  is sequentially compact

**Proof**

- (1)  $\Rightarrow$  (2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Assume that no subsequence of  $(x_n)_{n \in \mathbb{N}}$  converges in  $X$ . For any  $p \in X$  there exists  $\epsilon_p > 0$  such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For  $X$  is compact  $\exists (p_i)_{i \in \{1, \dots, n\}}$

$$X \subseteq \bigcup_{i=1}^n \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^n \{n \in \mathbb{N} : d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

- (2)  $\Rightarrow$  (1)

prove  $(X, d)$  is complete Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences  $(x_{k_n})_{n \in \mathbb{N}}$  must converges to the same limit.

So  $(X, d)$  is complete

If  $X$  is not covered by finitely many balls of radius  $\epsilon$  we can construct a sequence  $(x_{k_n})_{n \in \mathbb{N}}$  such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^n \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

## 28.15 Def

Let  $X$  be a Hausdorff topological space. If for any  $x \in X$  there exist a compact neighborhood  $\mathcal{C}_x$  we say that  $X$  is locally compact.

### Example

$\mathbb{R}$  is locally compact.

## 28.16 Prop

Assume that  $(K, |\cdot|)$  is a locally compact non-trivial valued field. Let  $(E, \|\cdot\|)$  be a finite dimensional normed  $K$ -vector space. A subset  $Y \subseteq E$  is compact iff it's closed and bounded.

### Proof

$\Rightarrow$  Let  $Y \subseteq X$  be compact. Then for  $Y$  is Hausdorff,  $Y$  is closed. Moreover

$$Y \subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq \dots \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^k \mathcal{B}(0, n_i)$$

$\Rightarrow Y$  is bounded.

$\Leftarrow$  We prove sequentially compact by a theorem proved before.

Let  $(e_i)_{i=1}^d$  be a basis of  $E$ . Again we assume

$$\left\| \sum_{i=1}^d a_i e_i \right\| = \max_{i \in \{1, \dots, d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n \in \mathbb{N}} = \left( \sum_{i=1}^d a_i^{(n)} e_i \right)_{n \in \mathbb{N}}$$

Since  $Y$  is bounded for any  $i \in \{1, \dots, d\}$  the sequence  $(a_i^{(n)})$  is bounded. In particular we find  $M > 0$  such that  $\forall i \in \{1, \dots, n\}$

$$|a_i^{(n)}| < M$$

Since  $(K, |\cdot|)$  is locally compact, there exists a compact set  $\mathcal{C} = \mathcal{C}_0 \subseteq K$  that's a neighborhood of 0. Let  $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since  $K$  is not trivially valued, then exists  $a \in K$  such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$  is compact. We have the  $K$ -linear mapping

$$\begin{aligned} K &\rightarrow K \\ y &\mapsto ay \end{aligned}$$

is bounded, then continuous. Hence  $a\mathcal{C}$  is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find  $(I_i)_{i=1}^d$  are infinite subsets of  $\mathbb{N}$  with

$$I_1 \supseteq \dots \supseteq I_d$$

such that  $(a_j)_{j \in I_i}^{(n)}$  converges to some  $a_i \in K$ . It follows that our original

sequence has a convergent subsequence converges to  $\sum_{i=1}^d a_i e_i$ .

So  $Y$  is sequentially compact.

## 28.17 Theorem

Let  $X$  be a topological space and  $f : X \rightarrow \mathbb{R}$  be a continuous mapping. If  $Y \subseteq X$  is a quasi-compact subset, then there exists  $a \in Y$  and  $b \in Y$  such that  $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of  $f$  to  $Y$  attains its maximum and minimum.

**Proof**

$f(Y) \subseteq \mathbb{R}$  is a non-empty compact subset since  $Y$  is quasi-compact and  $\mathbb{R}$  is Hausdorff. Moreover, since  $\mathbb{R}$  is locally compact. SO  $f(Y)$  is bounded and closed.

Note that there exists sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  in  $f(Y)$  that tends to  $\sup f(Y)$  and  $\inf f(Y)$  respectively. Since  $f(Y)$  is closed,  $\sup f(Y), \inf f(Y)$  belongs to  $f(Y)$ . So  $f(Y)$  has a greatest and a least element.



## Chapter 29

# Mean Value Theorems

### 29.1 Rolle Theorem

Let  $a, b$  be real numbers such that  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping that is differentiable on  $]a, b[$ . If  $f(a) = f(b)$  then  $\exists t \in ]a, b[$  such that

$$f'(t) = 0$$

#### Proof

Since  $[a, b]$  is closed and bounded then it's compact,  $f$  attains its maximum and minimum. Let  $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If  $M \neq l \exists t \in ]a, b[$  such that  $f(t) = M$

$$f(t+x) = f(t) + f'(t)x + o(|X|)$$

$$f(t-x) = f(t) - f'(t)x + o(|X|)$$

$$0 \leq (f(t+x) - f(t))(f(t-x) - f(t))$$

$$= -f'(t)^2 x^2 + o(|x|^2)$$

$$0 \leq -f'(t)^2 + o(1) \quad x \rightarrow 0$$

Taking the limit when  $x \rightarrow 0$  we get  $f'(t)^2 = 0$

If  $m \neq l$  then any  $t \in ]a, b[$  such that  $f(t) = m$  verifies  $f'(t) = 0$

If  $m = l = M$   $f$  is constant, so  $\forall t \in ]a, b[, f'(t) = 0$

### 29.2 Mean value theorem(Lagrange)

Let  $a, b$  be real numbers  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping differentiable on  $]a, b[$ , then  $\exists t \in ]a, b[$  such that

$$f(b) - f(a) = f'(t)(b - a)$$

**Proof**

Let  $g : [a, b] \rightarrow \mathbb{R}$  be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then  $g(a) = f(a)$   $g(b) = f(a)$  then apply Rolle Theorem to  $g$  we get the proof.

**29.3 Mean value inequality**

Let  $a, b$  be real numbers such that  $a < b$   $(E, \|\cdot\|)$  be a normed vector space over  $\mathbb{R}$   $f : [a, b] \rightarrow E$  be a continuous mapping such that  $f$  is differentiable on  $]a, b[$  Then

$$\|f(b) - f(a)\| \leq \left( \sup_{x \in ]a, b[} \|f'(x)\| \right) (b - a)$$

**Proof**

Suppose that

$$\sup_{x \in ]a, b[} \|f'(x)\| < +\infty$$

Let  $M \in \mathbb{R}$  such that

$$M > \sup_{x \in ]a, b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition  $J$  is an interval containing  $a$ , so  $J$  is of form  $[a, c[$  or  $[a, c]$  Since  $f$  is continuous by taking a sequence  $(c_n)_{n \in \mathbb{N}}$  in  $[a, b[$  that converges to  $c$  we obtain

$$\begin{aligned} \|f(c) - f(a)\| &= \lim_{n \rightarrow +\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n \rightarrow +\infty} M(c_n - a) \\ &= M(c - a) \end{aligned}$$

Hence  $c \in J$  namely  $J = [a, c]$

$c > a$  We will prove that  $c = b$  by contradiction

Suppose that  $c < b$   $\forall h \in ]0, b - c[$

$$\begin{aligned} \|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h) \end{aligned}$$

Since  $M > \|f'(c)\|$ ,  $\exists h_0 > 0$  such that  $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$



Hence

$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h) - f(c)\| + \|f(c) - f(a)\| \\ &\leq M(c_h - c + c - a) \\ &= M(c + h - a)\end{aligned}$$

So  $c + h_0 \in J$  Contradiction. Thus

$$\|f(b) - f(a)\| \leq M(b - a)$$

for any  $M > \sup_{x \in ]a, b[} \|f'(x)\|$  since  $M$  is arbitrary the expected inequality holds.

$c = a$  In general, we apply the particular case (fis-extendable to a differentiable mapping at  $a$ ) to  $[\frac{a+b}{2}, b]$  and  $[a, \frac{a+b}{2}]$  to get

$$\begin{aligned}\left\|f(b) - f\left(\frac{a+b}{2}\right)\right\| &\leq C \frac{b-a}{2} \\ \left\|f\left(\frac{a+b}{2}\right) - f(a)\right\| &\leq C \frac{b-a}{2}\end{aligned}$$

with  $C = \sup_{x \in ]a, b[} \|f'(x)\|$

**Remark** If  $f$  is defined on an open neighborhood of  $a$  and is differentiable at  $a$  the the same arguments hold without the assumption

## 29.4 Theorem

Let  $I$  be an interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a continuous mapping, then  $f(I)$  is an interval.

### Proof

Let  $x \neq y$  be two elements of  $f(I)$  Let  $a, b$  elements of  $I$  such that  $x = f(a)$   $y = f(b)$  without loss of generality, we assume  $a < b$   
Let  $z \in \mathbb{R}$  such

$$(z - x)(z - y) \leq 0$$

We construct by induction three sequences  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$  such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If  $a_n, b_n, c_n$  are constructed, satisfying

$$c_n = \frac{1}{2}(a_n + b_n)$$

$$(z - f(a_n))(z - f(b_n)) \leq 0$$

we let

$$\begin{aligned} (a_{n+1}, b_{n+1}) &= (a_n, c_n) & \text{if } (z - f(a_n))(z - f(c_n)) \leq 0 \\ (a_{n+1}, b_{n+1}) &= (c_n, b_n) & \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\ & & ((z - f(c_n))(z - f(b_n)) \leq 0) \end{aligned}$$

$$c_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

The sequence  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  are increasing and decreasing respectively and bounded, hence converges to some  $l, m \in [a, b]$

Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \rightarrow 0 (n \rightarrow +\infty)$$

So  $l = m$ , by  $(z - f(a_n))(z - f(b_n)) \leq 0$  we obtain by letting  $n \rightarrow +\infty$

$$(z - f(l))^2 \leq 0$$

So  $z = f(l)$

## 29.5 Theorem(Heine)

Let  $I$  be an open interval of  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a differentiable mapping. Then  $f'(I)$  is an interval.

### Proof

Let  $(a, b) \in I^2$  such that  $a < b$ . Consider the following mappings:

$$\begin{aligned} g : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(x) - f(a)}{x - a} & x \neq a \\ f'(a) & x = a \end{cases} \\ h : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(b) - f(x)}{b - x} & x \neq b \\ f'(b) & x = b \end{cases} \end{aligned}$$

$g, h$  are continuous  $(\frac{f(x) - f(a)}{x - a} = f'(a) + o(1) \text{ } x \rightarrow a)$

So  $g([a, b])$  and  $h([a, b])$  are intervals. Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$

$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that  $g(b) = h(a)$  so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence  $f'(I)$  is an interval.

## Chapter 30

# Fixed Point Theorem

### 30.1 Def

Let  $X$  be a set and  $T : X \rightarrow X$  be a mapping. If  $x \in X$  satisfies  $T(x) = x$  we say that  $x$  is a fixed point of  $T$ .

### 30.2 Def

Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be a mapping. If  $\exists \epsilon \in [0, 1[$  such that  $T$  is  $\epsilon$ -Lipschitzian then we say that  $T$  is a contraction.

### 30.3 Fixed Point Theorem

Let  $(X, d)$  be a COMPLETE non-empty metric space, and  $T : X \rightarrow X$  be a contraction. Then  $T$  has a unique fixed point. Moreover,  $\forall x_n \in X$  if we let

$$x_{n+1} = T(x_n), x_0 \in X$$

then  $(x_n)_{n \in \mathbb{N}}$  converges to the fixed point.

#### Proof

If  $p$  and  $q$  are two fixed point of  $T$ , then

$$d(p, q) = d(T(p), T(q)) \leq \epsilon d(p, q)$$

So  $d(p, q) = 0$ .

Let

$$x_{n+1} = T(x_n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any  $N \in \mathbb{N}$ ,  $\forall (n, m) \in \mathbb{N}_{\geq N}^2$   $n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \end{aligned}$$

So

$$\lim_{N \rightarrow +\infty} \sup_{(n, m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, hence converges to some  $p \in X$

$$d(T(p), p) = \lim_{n \rightarrow +\infty} d(T(x_n), x_n) = 0$$

since  $d : X^2 \rightarrow \mathbb{R}_{\geq 0}$  is continuous.

Part VI

Higher differentials



## Chapter 31

# Multilinear mapping

Let  $K$  be a commutative cenitary ring.

### 31.1 Def

Let  $n \in \mathbb{N}$ ,  $V_1, \dots, V_n, W$  be  $K$ -modules. We call  $n$ -linear mapping from  $V_1 \times \dots \times V_n$  to  $W$  any mapping  $f : V_1 \times \dots \times V_n \rightarrow W$  such that  $\forall i \in \{1, \dots, n\} \forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in V_1 \times \dots \times V_{i-1} \times V_{i+1} \times \dots \times V_n$  the mapping

$$\begin{aligned} f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) : V_i &\rightarrow W \\ x_i &\mapsto f(x_i) \end{aligned}$$

is a morphism of  $K$ -modules

We denote by  $Hom^{(n)}(V_1 \times \dots \times V_n, W)$  the set of all  $n$ -linear mappings from  $V_1 \times \dots \times V_n$  to  $W$ .

### 31.2 Example

$$\begin{aligned} K \times K &\rightarrow K \\ (a, b) &\mapsto ab \end{aligned}$$

is a 2-linear mapping (bilinear mapping)

### 31.3 Remark

$$Hom^{(0)}(\{0\}, W) := W \text{ (by convention)}$$

$$Hom^{(1)}(V_1, W) = Home(V_1, W) = \{\text{morphism of } K\text{-module from } V_1 \text{ to } W\}$$

### 31.4 Prop

Suppose that  $n \geq 2$  For any  $i \in \{1, \dots, n-1\}$

$$\begin{aligned} \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) &\xrightarrow{\Phi} \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n)) \\ f &\mapsto ((x_1, \dots, x_i) \mapsto ((x_{i+1}, \dots, x_n) \mapsto f(x_1, \dots, x_n))) \end{aligned}$$

is a bijection

### Proof

The inverse of  $\Phi$  is given by

$$g \in \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \mapsto ((x_1, \dots, x_n) \in V_1 \times \dots \times V_n) \mapsto g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)$$

### 31.5 Remark

$\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$  is a sub-K-module of  $W^{V_1 \times \dots \times V_n}$  and  $\Phi$  is an isomorphism of K-modules.



## Chapter 32

# Operator norm of Multilinear field

Let  $(K, |\cdot|)$  be a complete valued field

### 32.1 Def

Let  $V_1 \times \dots \times V_n$  and  $W$  be normed vector spaces over  $K$ . We define

$$\|\cdot\| : \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) \rightarrow [0, +\infty]$$

as

$$\|f\| := \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

If  $\|f\| < \infty$  we say that  $f$  is bounded. We denote by  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  the set of bounded  $n$ -linear mappings from  $V_1 \times \dots \times V_n$  to  $W$ .

### 32.2 Theorem

For any  $i \in \{1, \dots, n-1\}$ ,  $\forall f \in \mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \forall (x_1, \dots, x_i) \in V_1 \times \dots \times V_i$  the  $(n-i)$ -linear mapping

$$\begin{aligned} f(x_1, \dots, x_i, \cdot) : V_{i+1} \times \dots \times V_n &\rightarrow W \\ (x_{i+1}, \dots, x_n) &\mapsto f(x_1, \dots, x_n) \end{aligned}$$

belongs to  $\mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)$ . Moreover

$$\|f\| = \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

**Proof**

$$\forall (x_{i+1}, \dots, x_n) \in V_{i+1} \times \dots \times V_n$$

$$\begin{aligned} \|f(x_1, \dots, x_n)\| &\leq \|f\| \|x_1\| \dots \|x_n\| \\ &= (\|f\| \|x_1\| \dots \|x_i\|) \|x_{i+1}\| \dots \|x_n\| \end{aligned}$$

So

$$\|f(x_1, \dots, x_i, \cdot)\| \leq \|f\| \|x_1\|, \dots, \|x_i\|$$

If we define

$$\|f\|' := \sup_{(x_1, \dots, x_i) \in V_1 \times \dots \times V_i, x_1 \dots x_i \neq 0} \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|}$$

then

$$\|f\|' \leq \|f\|$$

**32.3 Corollary**

- (1)  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  is a vector subspace of  $\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$
- (2)  $\|\cdot\|$  is a norm on  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$
- (3)  $\forall i \in \{1, \dots, n\}$

$$\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \xrightarrow{\Phi} \mathcal{L}^{(n)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$$

is a K-linear isomorphism that preserves operator norms.

$$\|f\| = \|\Phi(f)\|$$

**32.3.1 Proof**

Conversely  $\forall (x_1, \cdot, x_n) \in V_1 \times \dots \times V_n$  such that  $x_1 \dots x_n \neq 0$

$$\|f(x_1, \dots, x_n)\| \leq \|f(x_1, \dots, x_i, \cdot)\| \|x_{i+1}\| \dots \|x_n\|$$

Hence

$$\frac{f(x_1, \dots, x_n)}{\|x_1\| \dots \|x_n\|} \leq \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|} \leq \|f\|'$$

Taking sup, we get

$$\|f\| \leq \|f\|'$$

We reason by induction on  $n$

$n = 1$

$$\mathcal{L}^{(1)}(V_1, W) = \mathcal{L}(V_1, W)$$

$i \in \{1, \dots, n-1\}$  Suppose that the corollary is true for  $m$ -linear mappings with  $m < n$ . We consider the following diagram of mapping

To show that  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  is a vector subspace, it suffices to check that  $\forall g \in \mathcal{L}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$  one has  $\|\Phi^{-1}(g)\| = \|g\| < +\infty$

$$\begin{aligned} \mathcal{L}^{(i)}(V_{i+1} \times \dots \times V_n, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \\ &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \end{aligned}$$

For any  $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n$

$$\begin{aligned} \|\Phi^{-1}(g)(x_1, \dots, x_n)\| &= \|g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)\| \\ &\leq \|g(x_1, \dots, x_i)\| \|x_{i+1}\| \dots \|x_n\| \\ &\leq \|g\| \|x_1\| \dots \|x_i\| \|x_{i+1}\| \dots \|x_n\| \end{aligned}$$

Therefore

$$\|\Phi^{-1}(g)\| \leq \|g\| = \|\Phi^{-1}(g)\|$$



## Chapter 33

# Higher differentials

We fix a complete non-trivial valued field  $(K, |\cdot|)$  and normed  $K$ -vector space  $E$  and  $F$ .

### 33.1 Def

Let  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping

- (1) If  $f$  is continuous, we say that  $f$  is of class  $C^0$  and  $f$  is 0-times differentiable
- (2) If  $f$  is differentiable on an open neighborhood  $V \subseteq U$  of some point  $p \in U$  and

$$\begin{aligned} df : V &\rightarrow \mathcal{L}(E, F) \\ x &\mapsto d_x f \end{aligned}$$

is  $n$ -times differentiable at  $p$ , then we say that  $f$  is  $(n+1)$ -times differentiable at  $p$ . If  $f$  is  $(n+1)$ -times differentiable at any point  $p \in U$ , we denote by

$$D^{n+1}f : U \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

the mapping that sends  $x \in U$  to the image of  $D^n(df)(x)$  by the  $K$ -linear bijection

$$\mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

$$df : U \rightarrow \mathcal{L}(E, F)$$

$$D^n(df) : U \rightarrow \mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \xrightarrow{\Phi} \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

If  $D^{n+1}f$  is continuous, we say that  $f$  is of class  $C^{n+1}$  ( $n \geq 0$ ) (Any mapping  $f : U \rightarrow F$  is considered as 0-times differential  $D^0f := f$ )

### 33.2 Remark

If  $f$  is  $n$ -times differentiable  $\forall i \in \{1, \dots, n-1\}$   
 $\forall p \in U, (h_1, \dots, h_n) \in E^n$  one has

$$D^i(D^{n-i}f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n) = D^n f(p)(h_1, \dots, h_n)$$

$$D^{n-i}f : U \rightarrow \mathcal{L}^{(n-i)}(E^{n-i}, F)$$

$$D^i(D^{n-i}f) : \quad U \xrightarrow{\quad} \mathcal{L}^{(i)}(E^i, \mathcal{L}^{(n-i)}(E^{n-i}, F)) \quad U \rightarrow$$

$$\quad \quad \quad \searrow D^n f \quad \quad \quad \updownarrow \cong \quad \quad \quad \mathcal{L}^{(n)}(E^n, F)$$

### 33.3 Theorem

Assume that  $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$   
 Let  $f : U \rightarrow F$  be a mapping that is  $(n+1)$ -times differentiable on  $U$ . Let  
 $p \in U$  and  $h \in E$  such that  $p + th \in U \forall t \in [0, 1]$  Then

$$\left\| f(p+h) - f(p) - \sum_{k=1}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq$$

$$\left( \sup_{t \in ]0, 1[} \frac{(1-t)^n}{n!} \|D^{n+1} f(p+th)\| \right) \cdot \|h\|^{n+1}$$

(Taylor-Lagrange formula)

### 33.4 Prop(Gronwall inequality)

Let  $F$  be a normed vector space over  $\mathbb{R}$   $(a, b) \in \mathbb{R}^2, a < b$  Let  $f : [a, b] \rightarrow F$   
 and  $g : [a, b] \rightarrow \mathbb{R}$  be continuous mappings that are differentiable on  $]a, b[$

Suppose that  $\forall t \in ]a, b[$

$$\|f'(t)\| \leq g'(t)$$

then

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

#### Proof

Let  $c \in ]a, b[$  Let  $\epsilon > 0$  Let

$$J = \{t \in [c, b] \mid \forall s \in [c, t], \|f(s) - f(c)\| \leq g(s) - g(c)\}$$

By definition  $J$  is an interval.

Since  $f, g$  are continuous,  $J$  is a closed interval, hence  $J$  is of the form  $[c, t]$ .  
If  $t < b$  then for  $h > 0$  Sufficiently small.

$$f(t+h) - f(t) = hf'(t) + o(h)$$

$$g(t+h) - g(t) = hg'(t) + o(h)$$

$$\exists \delta > 0 \forall h \in [0, \delta]$$

$$\|f(t+h)\| \leq \|f'(t)\| \cdot h + \frac{\epsilon}{2}h$$

$$g(t+h) - g(t) \geq g'(t)h - \frac{\epsilon}{2}h$$

So

$$\|f(t+h) - f(t)\| \leq g(t+h) - g(t) + \epsilon h$$

Moreover

$$\|f(t) - f(c)\| \leq g(t) - g(c) + \epsilon(t - c)$$

$\Rightarrow$

$$\|f(t+h) - f(c)\| \leq g(t+h) - g(c) + \epsilon(t+h-c)$$

$\Rightarrow$

$$J \supseteq [c, t+\delta]$$

Contradiction, hence

$$\|f(b) - f(c)\| \leq g(b) - g(c) + \epsilon(b - c)$$

For the same reason

$$\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c - a)$$

Hence

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b - a)$$

Since  $\epsilon > 0$  is arbitrary

$$\|f(b) - f(c)\| \leq g(b) - g(c)$$

Mean value theorem:

$$g(t) = (\sup(\|f'(\cdot \cdot \cdot)\|))$$

### 33.5 Theorem

Let  $n \in \mathbb{N}$ ,  $E, F$  be normed vector spaces over  $\mathbb{R}$   $U \subseteq E$  open and  $f : U \rightarrow F$  be a mapping that is  $(n+1)$ -times differentiable. Let  $p \in U$  and  $h \in E$ . Assume that  $\forall \epsilon \in [0, 1], p + th \in U$

Let

$$M = \sup_{t \in ]0, 1[} \|D^{n+1}f(p + th)\|$$

Then

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

If  $E = \mathbb{R}$  Then the formula become

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(p) h^k \right\| \leq \frac{M}{(n+1)!} |h|^{n+1}$$

### Proof

Consider  $\phi : [0, 1] \rightarrow F$

$$\phi(t) = \sum_{k=0}^n \frac{(1-t)^k}{k!} D^k f(p+th)(h, \dots, h)$$

$$\phi(1) = f(p+h)$$

$$\phi(0) = \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h)$$

$$\begin{aligned} \phi'(t) &= \sum_{k=0}^n \frac{(1-t)^k}{k!} D^{k+1} f(p+th)(\underbrace{h, \dots, h}_{k+1 \text{ copies}}) - \sum_{k=1}^n \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p+th)(h, \dots, h) \\ &= \frac{(1-t)^n}{n!} D^{n+1} f(p+th)(h, \dots, h) \end{aligned}$$

then

$$\|\phi'(t)\| \leq M \frac{(1-t)^n}{n!} = (-M \frac{(1-t)^{n+1}}{(n+1)!})'$$

By Gronwall inequality,

$$\|\phi(1) - \phi(0)\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

### 33.6 Def

Let  $n \in \mathbb{N}$   $E_1, \dots, E_n$  and  $F$  be normed vector spaces over a complete non-trivial valued field  $(K, |\cdot|)$  Let  $U \in E_1 \times \dots \times E_n$  be an open subset.  $p = (p_1, \dots, p_n) \in U$   $i \in \{1, \dots, n\}$ ,  $f : U \rightarrow F$  If there exists an open neighborhood  $U_i$  of  $p_i$  in  $E_i$  such that

$$\begin{aligned} U_i &\rightarrow F \\ x_i &\mapsto f(p_1, \dots, p_{i-1}, x_i, p_{i+1}, \dots, p_n) \end{aligned}$$

is well defined and is differentiable at  $p_i$

We denote by  $\frac{\partial f}{\partial x_i}(p)$  the differential of this mapping  $U_i \rightarrow F$  and say that  $f$  admits the  $i^{th}$  partial differentials at  $p$



### 33.7 Prop

Suppose that  $(K, |\cdot|)$  and  $f$  has all partial differentials on  $U$  and

$$\frac{\partial f}{\partial x_i} : U \rightarrow \mathcal{L}(E_i, F)$$

is continuous for any  $i \in \{1, \dots, n\}$  Then  $f$  is of class  $C^1$  and  $\forall h = (h_1, \dots, h_n) \in E_1 \times \dots \times E_n$

$$\forall p \in U \quad d_p(h) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(h_i)$$

#### Proof

By induction, it suffices to treat the case where  $n = 2$   
 $\forall \epsilon > 0 \exists \delta > 0$

$$\forall (h, k) \in E_1 \times E_2 \quad \max\{|h|, |k|\} \leq \delta$$

one has

$$\left\| \frac{\partial f}{\partial x_i}(a + h, b + k) - \frac{\partial f}{\partial x_i}(a, b) \right\| \leq \epsilon \text{ (by continuity of } \frac{\partial f}{\partial x_i} \text{)}$$

Consider the mapping  $\phi : [0, 1] \rightarrow F$

$$\phi(t) = f(a + h, b + tk) - f(a + h, b) - t \underbrace{\frac{\partial f}{\partial x_2}(a + h, b)(k)}_{\in \mathcal{L}(E_2, F)}$$

$$\begin{aligned} \|\phi'(t)\| &= \left\| \frac{\partial f}{\partial x_2}(a + h, b + tk)(k) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \\ &\leq 2\epsilon \|k\| \end{aligned}$$

$$\|\phi(1) - \phi(0)\| \leq 2\epsilon \|k\|$$

then

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \leq 2\epsilon \|k\|$$

So

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

$f$  has 1<sup>st</sup> partial differential

$$\left\| f(a+h, b) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

by continuity of  $\frac{\partial f}{\partial x_i}$

$$\left\| \frac{\partial f}{\partial x_2}(a+h, b)(k) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

take the sum of above three statements, we get:

$$\left\| f(a+h, b+k) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(h) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

### 33.8 Theorem

Let  $E, F$  be normed vector spaces over  $\mathbb{R}$   $U \subseteq E$  open  $(f_n)_{n \in \mathbb{N}}$  be a sequence of differentiable mapping from  $U$  to  $F$  Let  $g : U \rightarrow \mathcal{L}(E, F)$  Suppose that

- (1)  $(df_n)_{n \in \mathbb{N}}$  converges uniformly to  $g$
- (2)  $(f_n)_{n \in \mathbb{N}}$  converges pointwisely to some mapping  $f : U \rightarrow F$

Then  $f$  is differentiable and  $df = g$

#### Proof

Let  $p \in U, \forall (m, n) \in \mathbb{N}^2, \forall x \in \mathcal{B}(p, r) \in U (r > 0)$

$$\|f_n(x) - f_m(x) - (f_n(p) - f_m(p))\| \leq (\sup_{\xi \in U} \|d_\xi f_m - d_\xi f_n\|) \cdot \|x - p\| \quad (\text{mean value inequality})$$

Take  $\lim_{m \rightarrow +\infty}$  we get:

$$\|(f_n(x) - f(x)) - (f_n(p) - f(p))\| \leq \epsilon_n \|x - p\|$$

where  $\epsilon_n = \sup_{\xi \in U} \|d_\xi f_m - g\|$ .

So

$$\begin{aligned} \|f(x) - f(p) - g(p)(x-p)\| &\leq \|(f(x) - f_n(x)) - (f(p) - f_n(p))\| \\ &\quad + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| \\ &\quad + \|d_p f_n(x-p) - g(p)(x-p)\| \\ &\leq \epsilon_n \|x-p\| + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| + \epsilon_n \|x-p\| \\ \limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} &\leq 2\epsilon_n \end{aligned}$$

Take  $\lim_{n \rightarrow +\infty}$  we get:

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} = 0$$

# Chapter 34

## Permutations

### 34.1 Def

Let  $X$  be a set. We denote with  $\mathfrak{S}_X$  the set of all bijections from  $X$  to itself. The elements of  $\mathfrak{S}_X$  are called permutations if the set  $X$  is finite. If  $x_1, \dots, x_n \in X$  are distinct elements then

$$(x_1, \dots, x_n) \in \mathfrak{S}_X$$

such that

$$x_i \mapsto x_{i+1}$$

$$x_n \mapsto x_1$$

this is called an  $n$ -cycle. A 2-cycle is called a transposition.

#### 34.1.1 Example

$$X = \{1, \dots, 7\}$$

$$1 \mapsto 4$$

$$2 \mapsto 1$$

$$3 \mapsto 2$$

$$(2\ 3)(4\ 2\ 1) = 4 \mapsto 3$$

$$5 \mapsto 5$$

$$6 \mapsto 6$$

$$7 \mapsto 7$$

$$= (1\ 4\ 3\ 2)$$

### 34.2 Def

We denote with

$$orb_\sigma(x) = \{\underbrace{\sigma \circ \dots \circ}_{n\text{-times}} \quad n \in \mathbb{N}\}$$

$$x \in X, \sigma \in \mathfrak{S}_X$$

### 34.3 Prop

If  $\text{orb}_\sigma(x)$  is a finite set of  $d$  elements, then one has

$$\sigma^d(x) = x \quad \text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{d-1}(x)\}$$

moreover

$$\sigma^{-1}(x) \in \text{orb}_\sigma(x)$$

#### 34.3.1 Proof

The set

$$\{(n, m) \in \mathbb{N}^2, n < m, \sigma^n(X) = \sigma^m(x)\}$$

is not empty. Let

$$d' := \min\{m - n \mid (n, m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

therefore  $x, \sigma(x), \dots, \sigma^{d'-1}(x)$  are all distinct.

Now use the each deass division

$$h = qd' + r \quad r < d'$$

$$\sigma^h(x) = \sigma^r(x) \quad 0 \leq r < d'$$

then

$$d' \geq d$$

and for

$$\{x, \sigma(x), \dots, \sigma^{d'-1}(x)\} \subseteq \text{orb}_\sigma(x)$$

$\Rightarrow$

$$d' \leq d$$

then

$$d' = d$$

### 34.4 Remark

Let  $Y \subseteq X$ , then we have a homomorphism of groups:

$$\begin{aligned} \mathfrak{S}_Y &\rightarrow \mathfrak{S}_X \\ \sigma &\mapsto \left( x \mapsto \begin{cases} \sigma(x) & \text{if } x \in Y \\ x & \text{if } x \in X \setminus Y \end{cases} \right) \end{aligned}$$

If  $Y$  and  $Z$  are subset of  $X$

$$Y \cap Z = \emptyset, \sigma \in \mathfrak{S}_Y, \tau \in \mathfrak{S}_Z$$

then

$$\sigma \circ \tau = \tau \circ \sigma$$

If  $X$  is finite with  $n$  elements  $\mathfrak{S}_X = S_n$  permutation group of  $n$  elements.

### 34.5 Theorem

Let  $X$  be a finite set and let  $\sigma \in \mathfrak{S}_X$  then exist  $d \in \mathbb{N}$  and  $(n_1, \dots, n_d) \in \mathbb{N}_{\geq 2}^d$  and pairwise disjoint subsets  $X_1, \dots, X_d$  of  $X$  of cardinalities  $n_1, \dots, n_d$ , together with  $n_i$ -cycle  $\tau_i$  of  $X_i$  such that

$$\sigma = \tau_1 \circ \dots \circ \tau_d$$

In other words. Any permutation can be decomposed in composition of finitely many cycles on disjoint subsets.

#### Proof

By induction on the cardinality of  $X$ .

The case  $\sigma = id_X$  is trivial. ( $d = 0$ ) So the case when  $N = 0, 1$  is clear.

Assume  $N \geq 2$ . Take  $x \in X$  such that  $\sigma(x) \neq x$  and let  $X_1 = orb_\sigma(x)$   
 $Y = X \setminus X_1 \forall y \in Y$  we have  $\sigma(y) \in Y$  (because if  $\sigma(y) \in X$  by the previous proposition  $\sigma(y) \in X_1$ )

Let  $\tau = \sigma|_Y \in \mathfrak{S}_Y$  Use the induction hypothesis, we get  $X_2, \dots, X_d$  of cardinalities  $n_2, \dots, n_d$  and  $n_i$ -cycle  $\tau_i$  such that

$$\tau = \tau_2 \circ \dots \circ \tau_d$$

Consider  $\tau_1 = \sigma|_{X_1}$  then  $\tau_1$  is a  $n_1$ -cycle of  $X_1$

$\Rightarrow$

$$\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_d$$

#### 34.5.1 Remark

This theorem say that the groups of permutation si generated by cycles.

### 34.6 Corollary

Let  $X$  be a finite set. Then  $\mathfrak{S}_X$  is generated by transpositions.

**Proof**

Note that

$$(x_1, \dots, x_n) = (x_1, x_2) \circ (x_2, \dots, x_n)$$

By induction

$$(x_1, \dots, x_n) = (x_1, x_2) \circ \dots \circ (x_{n-1}, x_n)$$

**34.6.1 Remark**

The decomposition of transposition is unique.

**34.7 Def**

Let  $\tau \in \mathfrak{S}_n := G_{\{1, \dots, n\}}$  is called adjacent if  $\tau$  is of the form  $(j, j+1)$  for  $j = 1, \dots, n-1$

**34.8 Corollary**

$\mathfrak{S}_n$  is generated by adjacent transposition.

**34.8.1 Proof**

Note that

$$(i, j) = (i, i+1) \circ (i+1, i+2) \circ \dots \circ (j-1, j) \circ (j-2, j-1) \circ \dots \circ (i+2, i+1)$$

**Some other information on  $\mathfrak{S}_n$**

**34.9 Caybey Theorem**

Any finite group can be embedded (injective morphism) in a  $\mathfrak{S}_n$  for some  $n \in \mathbb{N}$

**Proof**

Let  $G$  be a finite group and  $n = \text{card}(G)$ . Let

$$\begin{aligned} \varphi : G &\rightarrow \mathfrak{S} \\ g &\mapsto l_g \end{aligned}$$

be the mapping sends  $g \in G$  to  $l_g(x) = gx, \forall x \in G$

### 34.10 Theorem

Let  $X$  be a finite set. Assume that  $\sigma \in \mathfrak{S}_X$  can be written as

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

where  $\tau_1$  is transposition.

We put

$$\text{sgn}(\sigma) := (-1)^\sigma$$

This is a well-defined function. Moreover  $\text{sgn}$  is a morphism from  $\mathfrak{S}_X$  to  $(\{-1, 1\}, \times)$

### Proof

Let's define the mapping:

$$\begin{aligned} \phi : \mathfrak{S}_n &\rightarrow \mathbb{Q}^\times \\ \sigma &\mapsto \prod_{(i,j) \in \{1, \dots, n\}^2, i < j} \frac{\sigma(i) - \sigma(j)}{i - j} \end{aligned}$$

To show that  $\phi$  is a morphism of groups. Let

$$\theta = \{U \in \wp(\{1, \dots, n\}) \mid \text{card}(U) = 2\}$$

$$\begin{aligned} \phi(\sigma \circ \tau) &= \prod_{(i,j) \in \theta} \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{i - j} \\ &= \left( \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{\tau(i) - \tau(j)} \right) \times \left( \prod_{(i,j) \in \theta} \frac{\tau(i) - \tau(j)}{i - j} \right) \\ &= \phi(\sigma)\phi(\tau) \end{aligned}$$

When  $\tau$  is a transposition,  $\phi(\tau) = -1$ . Therefore

$$\phi(\sigma) = \prod_{i=1}^d \phi(\tau_i)$$

since

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

### 34.11 Remark

Let  $A_n \subsetneq \mathfrak{S}_n$  such that

$$A_n = \{\sigma \in \mathfrak{S}_n \mid \text{sgn}(\sigma) = 1\}$$

is an alternating symmetric group.

### 34.12 Exercise

Let  $X$  be a set of cardinality  $n$ . Let  $\sigma : X \rightarrow \{1, \dots, n\}$  be a bijection. Prove that

$$\begin{aligned} \phi : \mathfrak{S}_X &\rightarrow \mathfrak{S}_n \\ \tau &\mapsto \sigma^{-1} \circ \tau \circ \sigma \end{aligned}$$

is an isomorphism.

### 34.13 Symmetric of multilinear mapping

We fix a commutative unitary ring  $K$  and  $K$ -modules  $E, F$

### 34.14 Def: Symmetric and Alternating

symmetric Let  $n \in \mathbb{N}$  and  $f \in \text{Hom}^{(n)}(E^n, F)$ . If for any  $\sigma \in \mathfrak{S}_n$  one has  $\forall x \in E^n$

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Then we say  $f$  is symmetric

alternating If for any  $(i, j) \in \{1, \dots, n\}^2$  such that  $i \neq j$  and any  $(x_1, \dots, x_n) \in E^n$  such that  $x_i = x_j$

$$f(x_1, \dots, x_n) = 0$$

then we say that  $f$  is alternating.

### 34.15 Prop

Suppose that  $f \in \text{Hom}^{(n)}(E^n, F)$  is alternating, then  $\forall (x_1, \dots, x_n) \in E^n$ ,  $\sigma \in \mathfrak{S}_n$

$$f(x_1, \dots, x_n) = \text{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

#### Proof

By corollary 34.8, it's enough to prove the proposition for adjacent transitions. Let  $i \in \{1, \dots, n-1\}$  then

$$\begin{aligned} 0 &= f(x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_n) \\ &= f(x_1, \dots, x_{i-1}, x_i, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \end{aligned}$$

The adjacent transition  $\sigma$  is  $(i, i+1)$



### 34.16 Def:

$Hom_s$  and  $Hom_a$

We denote with  $Hom_s^{(n)}(E^n, F)$  and  $Hom_a^{(n)}(E^n, F)$  the set of symmetric and alternating  $n$ -linear mappings from  $E$  to  $F$ .

$Hom_s^{(n)}(E^n, F)$  and  $Hom_a^{(n)}(E^n, F)$  are sub-K-modules of  $Hom^{(n)}(E^n, F)$  and when  $n = 1$ , by convention

$$Hom_s^{(1)}(E, F) = Hom_a^{(1)}(E, F) = Hom(E, F)$$

### 34.17 Reminder

Let  $E, F$  be two normed vector spaces over  $\mathbb{R}$   $f : E \rightarrow F$  is differentiable (twice)

$$\begin{aligned} df : E &\rightarrow \mathcal{L}(E, F) \\ D^2d : E &\rightarrow \mathcal{L}(E, \mathcal{L}(E, F)) \\ A &\mapsto ((x, y) \rightarrow A(x)(y)) \end{aligned}$$

### 34.18 Theorem(Schwarz)

$U \subseteq E$  is an open set,  $f : U \rightarrow F$  is a function of class  $C^n$ . Then for any  $p \in U$

$$D^n f(p) \in \mathcal{L}^n(E^n, F)$$

is symmetric

#### Proof

By induction and by the fact that permutation are decomposed in transpositions, we can reduce to prove only the case  $n = 2$

$$d_{p+u}f - d_p f = D^2 f(p)(u, \cdot) + o(u)$$

$\forall \epsilon > 0, \exists \delta > 0$  such that  $0 < \|u\| < \delta$ , then

$$\|d_{p+u}f - d_p f - D^2 f(p)(u, \cdot) + o(u)\| \leq \epsilon \|u\|$$

For any  $x \in \mathcal{B}(p, \frac{\epsilon}{2})$  let's introduce the following function

$$\varphi(x) = f(x+k) - f(x) - D^2 f(p)(k, x)$$

We use the mean value inequality on  $\varphi$

$$\begin{aligned} &\|\varphi(p+h) - \varphi(p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - D^2 f(p)(k, p+h) - f(p+k) - f(p) - D^2 f(p)(k, p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - f(p+k) - D^2 f(p)(k, h)\| \\ &\leq \left( \sup_{t \in [0,1]} \|d_{p+th} \varphi\| \right) \|h\| \end{aligned}$$

$$\|d_{p+th}(\varphi)\| = \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\|$$

add and subtract  $d_p f, D^2f(p)(th, \cdot)$  then by triangle inequality

$$\begin{aligned} & \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\| \\ & \leq \|d_{p+th+k}(f) - d_p f - D^2f(p)(k + th, \cdot)\| \\ & + \|d_{p+th}f - d_p f - D^2f(p)(th, \cdot)\| \\ & \leq \epsilon \|th + k\| + \epsilon(th) \\ & \leq 2\epsilon(\|h\| + \|k\|) \end{aligned}$$

then

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - D^2f(p)(k, h)\| \\ & = o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

exchange the role of  $h, k$  then we get

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - f(p + k) - D^2f(p)(h, k)\| \\ & \leq o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

then

$$\underbrace{\|D^2f(p)(k, h) - D^2f(p)(h, k)\|}_{\text{bilinear function}} = o(\underbrace{\max\{\|h\|, \|k\|\}^2}_{\text{quachetic}})$$

this implies that the LHS is 0

### 34.19 Def

Let  $E, F$  be normed vector spaces over a complete value field  $(K, |\cdot|)$  let  $U \subseteq E, V \subseteq F$  be open subsets and  $f : U \rightarrow V$  is a bijection.

- (1) If  $f$  and  $f^{-1}$  are both continuous we say that  $f$  is a homeomorphism
- (2) If  $f$  and  $f^{-1}$  are both of class  $C^n$  we say that  $f$  is a  $e^n$ -diffeomorphism

If (2) is true for any  $n \in \mathbb{N}$  we say that  $f$  is a  $C^\infty$ -diffeomorphism

### 34.20 Prop

Let  $E, F$  be two normed Banach spaces. Let  $I(E, F) \in \mathcal{L}(E, F)$  be the set of linear continuous and invertible mappings such that  $\text{norm}\varphi^{-1} \leq +\infty$ . Then  $I(E, F)$  is open in  $\mathcal{L}(E, F)^\vee$  Moreover the mapping

$$\begin{aligned} I(E, F) & \rightarrow I(F, E) \\ \phi & \mapsto \varphi^{-1} \end{aligned}$$

is a  $e^1$ -diffeomorphism

**Proof**

Let  $\varphi \in I(E, F)$  we want to show that

$$\varphi - \psi \in I(E, F)$$

for  $\psi \in \mathcal{E}, \mathcal{F}$  such that  $\|\psi\| < \frac{1}{\|\varphi^{-1}\|}$  Notice that

$$\varphi - \psi = \varphi \circ (Id_E - \varphi^{-1} \circ \psi)$$

Since

$$\|\varphi^{-1}\psi\| \leq \|\varphi^{-1}\| \|\psi\| < 1$$

This means that the series

$$\sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n}$$

is absolutely convergent in  $\mathcal{L}(E, E)$  This series is the inverse of  $(Id_E - \varphi^{-1}\psi)$

$$(Id_E - \varphi^{-1}\psi) \circ \sum_{n=0}^{N-1} (\varphi^{-1} \circ \psi) \overset{\text{composite n times}}{\widehat{\circ n}} = Id_E - (\varphi^{-1} \circ \psi)^{\circ N}$$

take  $\lim_{N \rightarrow +\infty}$ , then

$$(\varphi - \psi)^{-1} = \sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n} \circ \varphi^{-1}$$

and

$$(\varphi - \psi)^{-1} = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

replace the inverse with  $i$

$$i(\varphi - \psi) - i(\varphi) = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

then

$$d_\varphi i(\psi) = i(\varphi) \circ (-\psi) \circ i(\varphi)$$

so  $i$  is differentiable. Moreover  $i$  and  $i^{-1}$  are continuous.

**Remark**

By induction we can show that  $i$  is a  $C^{+\infty}$ -diffeomorphism

**34.21 Prop**

Let  $n \in \mathbb{N} \cup \{\infty\}$  Let  $E, F, G$  be normed vector spaces over a complete valued field  $(K, |\cdot|)$   $U \subseteq E, V \subseteq F$  be open sets.  $f : U \rightarrow V, g : V \rightarrow G$  be mappings of class  $C^n$ , then  $g \circ f$  also of class  $C^n$

### 34.21.1 Proof

The case where  $n = 0$  is known

Denote by

$$\begin{aligned}\Phi : \mathcal{L}(E, F) \times E &\rightarrow F \\ (\beta, \alpha) &\mapsto \beta \circ \alpha\end{aligned}$$

$\Phi$  is a bounded bilinear mapping

$$\|\Phi(\beta, \alpha)\| \leq \|\beta\| \cdot \|\alpha\|$$

Suppose that  $n \geq 1$  and the statement is true for mappings of class  $C^{n-1}$   $g \circ f$  is differentiable.

$$\forall p \in U \quad d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

$$D^1(g \circ f) : U \rightarrow \mathcal{L}(E, G)$$

$$D^1 = \Phi \circ (D^1 g \circ f, D^1 f)$$

$$\begin{aligned}(D^1 g \circ f, D^1 f) : U &\rightarrow \mathcal{L}(F, G) \times \mathcal{L}(E, F) \\ p &\mapsto (d_{f(p)}g, d_p f)\end{aligned}$$

$$d_{\beta_0, \alpha_0} \Phi(\beta, \alpha) = \beta_0 \circ \alpha + \beta \circ \alpha_0$$

$$\begin{aligned}D^1 \Phi : \mathcal{L}(F, G) \times \mathcal{L}(E, F) &\rightarrow \mathcal{L}(\mathcal{L}(F, G) \times \mathcal{L}(E, F), \mathcal{L}(E, G)) \\ (\alpha_0, \beta_0) &\mapsto ((\alpha, \beta) \mapsto \beta_0 \circ \alpha + \beta \circ \alpha_0)\end{aligned}$$

Since  $g, f$  are of class  $C^n$   $D^1 f, D^1 g$  are of class  $C^{n-1}$  Thus, by induction hypothesis,

$$(D^1 g \circ f, D^1 f)$$

is of class  $C^{n-1}$  Since  $\Phi$  is of class  $C^\infty$ , we obtain that

$$D^1(g \circ f)$$

is of class  $C^{n-1}$  then

$$g \circ f$$

is of class  $C^n$

### 34.22 Prop

Let  $E$  and  $F$  be Banach space over a complete valued field  $(K, |\cdot|)$ .  $U$  and  $V$  be open subsets of  $E$  and  $F$  respectively.  $n \in \mathbb{N} \cup \{\infty\}$  and  $f : U \rightarrow V$  be a bijection. If  $f$  is of class  $C^n$ , then  $f^{-1}$  is differentiable, then  $f^{-1}$  is of class  $C^n$

**Proof**

$$f \circ f^{-1} = Id_V$$

$$\forall y \in V$$

$$d_y(f \circ f^{-1}) = d_{f^{-1}(p)}f \circ d_y f^{-1} = Id_F$$

$$\text{For } x \in U, y = f(x)$$

$$d_y(f \circ f^{-1}) = d_x f \circ d_y f^{-1} = Id_F$$

$$d_x(f^{-1} \circ f) = d_y f \circ d_x f^{-1} = Id_E$$

So

$$d_y f^{-1} - (d_x f)^{-1}$$

that is

$$D^1 f^{-1} = \iota \circ (D^1 f \circ f^{-1})$$

where

$$\begin{aligned} \iota : I(E, F) &\rightarrow I(F, E) \\ \phi &\mapsto \phi^{-1} \end{aligned}$$

Suppose that  $f^{-1}$  is of class  $C^{n-1}$  then

$$D^1 f^{-1} = \iota D^1 f \circ f^{-1}$$

is of class  $C^{n-1}$

**34.23 Local Inversion Theorem**

Let  $E$  and  $F$  be Banach space over  $\mathbb{R}$ .  $U \subseteq E$  open,  $f : U \rightarrow F$  be a mapping of class  $C^n$  and  $a \in U$ . Suppose that  $d_a f \in I(E, F)$  ( $d_a f$  is invertible and of bounded inverse). Then there exists open neighborhoods  $V$  and  $W$  of  $a$  and  $f(a)$  respectively, such that

- $V \subseteq U$  and  $f(V) \subseteq W$
- The restriction of  $f$  to  $V$  defines a bijection from  $V$  to  $W$
- 

$$(f|_V)^{-1} : W \rightarrow V$$

is of class  $C^n$

**34.23.1 Proof**

For  $y \in F$  consider the mapping:

$$\begin{aligned}\phi_y : U &\rightarrow F \\ x &\mapsto x - (d_a f)^{-1}(f(x) - y)\end{aligned}$$

$f(x) = y$  iff  $\phi_y(x) = x$  i.e.  $x$  is a fix point of  $\phi_y$   $\phi_y$  is of class  $C^1$  and

$$d_x \phi_y(v) = v - d_a f^{-1}(d_x f(v))$$

$\forall v$

$$d_a \phi_y^{(v)} = 0$$

By the continuity of  $D^1 f$  there exists  $r > 0$  such that

$$\overline{\mathcal{B}}(a, r) \subseteq U$$

and  $\forall y \in F, \forall x \in \overline{\mathcal{B}}(a, r)$

$$\|d_x \phi_y\| \leq \frac{1}{2}$$

By the mean value inequality.  $\forall (x_1, x_2) \in \overline{\mathcal{B}}(a, r)$

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|$$

Hence  $\phi_y$  is contraction.

By the boundedness of  $(d_a f)^{-1} \exists \delta > 0$  such that

$$\forall y \in \overline{\mathcal{B}}(f(a), \delta) \quad \|(d_a f)^{-1}(f(a) - y)\| \leq \frac{r}{2}$$

Then  $\forall x \in \overline{\mathcal{B}}(a, r) \quad y \in \overline{\mathcal{B}}(f(a), \delta)$

$$\begin{aligned}\|\phi_y(x) - a\| &\leq \|\phi_y(x) - \phi_y(a)\| + \|\phi_y(a) - a\| \\ &\leq \frac{1}{2} \|x - a\| + \frac{r}{2} \\ &\leq \frac{r}{2} + \frac{r}{2} = r\end{aligned}$$

$\phi_y(\overline{a}, \overline{r}) \in \overline{\mathcal{B}}(a, r)$ . By the fixed point theorem

$$\exists g : \overline{\mathcal{B}}(f(a), \delta) \rightarrow \overline{\mathcal{B}}(a, r)$$

sending  $y$  to the fixed point of  $\phi_y$  Let  $W = \mathcal{B}(f(a), g)$ , then

$$g|_W : W \rightarrow V$$

is the inverse of  $f|_V : V \rightarrow W$  Hence  $f^{-1}(W) = V$  is open.

In the following, we prove that  $g$  is of class  $C^n$  on an open neighborhood of  $f(a)$ . By reducing  $V$  and  $W$ , we may assume that  $\forall x \in V$

$$d_x f \in I(E, F)$$

Let  $x_0 \in V$   $y_0 = f(x_0)$   $x_0 = g(y_0)$

$$y - y_0 = f(g(y)) - f(g(y_0)) = d_{x_0} f(g(y) - g(y_0)) + o(\|g(y) - g(y_0)\|)$$

So

$$g(y) - g(y_0) = (d_x f)^{-1}(y - y_0) + o(\|g(y) - g(y_0)\|)$$

Thus leads to

$$g(y) - g(y_0) = O(\|y - y_0\|)$$

$(\exists \epsilon > 0 \quad (1 - \epsilon) \|g(y) - g(y_0)\| \leq \|d_{x_0} f\|^{-1}$  when  $\|y - y_0\|$  is sufficiently small)

So

$$d_{y_0} g = (d_x f)^{-1}$$

By the previous proposition,  $g$  is of class  $C^n$





**Part VII**

**Integration**



## Chapter 35

# Integral operators

We fix a set  $\Omega$  and a vector subspace  $S$  of  $\mathbb{R}^\Omega$  over  $\mathbb{R}$ . We suppose that  $\forall (f, g) \in S^2$

$$\begin{aligned} f \wedge g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \min\{f(\omega), g(\omega)\} \end{aligned}$$

belongs to  $S$

### 35.1 Prop

(1)  $\forall (f, g) \in S^2$

$$\begin{aligned} f \vee g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \max\{f(\omega), g(\omega)\} \end{aligned}$$

$$f \vee g \in S$$

(2)  $\forall f \in S$

$$\begin{aligned} |f| : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto |f(\omega)| \end{aligned}$$

$$|f| \in S$$

### Proof

(1)

$$f \vee g = f + g - f \wedge g$$

(2)

$$|f| = f \vee (-f)$$

### 35.2 Def

We call integral operator on  $S$  any  $\mathbb{R}$ -linear mapping  $I : S \rightarrow \mathbb{R}$  that satisfies the following conditions:

- (1) If  $f \in S$  is such that  $\forall \omega \in \Omega, f(\omega) \geq 0$  then  $I(f) \geq 0$
- (2) If  $(f_n)_{n \in \mathbb{N}}$  is a decreasing sequence of elements in  $S$  such that  $\forall \omega \in \Omega \lim_{n \rightarrow +\infty} f_n(\omega) = 0$  then

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

$$(\forall \omega \in \Omega, n \in \mathbb{N}, f_n(\omega) \geq f_{n+1}(\omega))$$

### 35.3 Example

- (1)  $\Omega = \mathbb{R}$   $S$ =vector subspace of  $\mathbb{R}^{\mathbb{R}}$  generated by mappings of the form  $\mathbb{1}_{]a,b]}$   $(a, b) \in \mathbb{R}^2, a < b$

$$\mathbb{1}_{]a,b]} = \begin{cases} 1, x \in ]a, b] \\ 0, else \end{cases}$$

Any element of  $S$  is of the form

$$\sum_{i=1}^n \lambda_i \mathbb{1}_{]a_i, b_i]}$$

$I : S \rightarrow \mathbb{R}$  is defined as

$$I\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (b_i - a_i)$$

More generally if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and right continuous ( $\forall x \in \mathbb{R}, \lim_{\epsilon > 0, \epsilon \rightarrow 0} \varphi(x + \epsilon) = \varphi(x)$ ) We define

$$I_\varphi : S \rightarrow \mathbb{R}$$

$$I\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (\varphi(b_i) - \varphi(a_i))$$

- (2) (Radon measure)

Let  $\Omega$  be a quasi-compact topological space

$$S = C^0(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \text{ continuous}\}$$

Let  $I : S \rightarrow \mathbb{R}$   $\mathbb{R}$ -linear, such that  $\forall f \in S, f \geq 0$  one has  $I(f) \geq 0$

### 35.4 Dini's theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $C^0(\Omega)$ , that converges pointwisely to some  $f \in C^0(\Omega)$  Then  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$

#### Proof

Let  $g_n = f_n - f > 0$  Fix  $\epsilon > 0 \forall n \in \mathbb{N}$  let

$$U_n = \{\omega \in \Omega \mid g_n(\omega) < \epsilon\}$$

is open

Moreover

$$\bigcup_{n \in \mathbb{N}} U_n = \Omega \quad (U_0 \subseteq U_1 \subseteq \dots)$$

Since  $\Omega$  is quasi-compact,  $\exists N \in \mathbb{N}, \Omega = U_N$  Therefore  $\forall n \in \mathbb{N}, n \geq N, \forall \omega \in \Omega$

$$g_n(\omega) < \epsilon$$

Consequence. If  $(f_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$  is decreasing and converges pointwisely to 0, then

$$\|f_n\|_{\sup} := \sup_{\omega \in \Omega} |f_n(\omega)|$$

converges to 0 when  $n \rightarrow +\infty \forall n \in \mathbb{N}$

$$f_n \leq \|f_n\|_{\sup} \cdot \mathbb{1}_{\Omega}$$

So

$$0 \leq I(f_n) \leq \|f_n\|_{\sup} I(\mathbb{1}_{\Omega}) \rightarrow 0 \quad (n \rightarrow +\infty)$$

(If  $f \leq g$  then  $g - f \geq 0$  so  $I(g - f) = I(g) - I(f) \geq 0 \Rightarrow I(g) \geq I(f)$ )

### 35.5 Def

We call  $\sigma$ -algebra any subset  $\mathcal{A}$  of  $\wp(\Omega)$  that satisfies the following conditions:

- $\emptyset \in \mathcal{A}$
- If  $A \in \mathcal{A}$  then  $\Omega \setminus A \in \mathcal{A}$
- If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Given a  $\sigma$ -algebra  $\mathcal{A}$  on  $\Omega$ , we mean by measure on  $(\Omega, \mathcal{A})$  any mapping  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that :

- $\mu(\emptyset) = 0$
- If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  such that  $A_i$  are pairwise disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$



## Chapter 36

# Riemann integral

### 36.1 Def

Let  $\Omega$  be a non-empty set and  $S$  be a vector subspace of  $\mathbb{R}^\Omega$ . If  $\forall (f, g) \in S^2, f \wedge g \in S$ , we say that  $S$  is a Riesz space.

In this section, we fix a Riesz space and an integral operator  $I : S \rightarrow \mathbb{R}$

### 36.2 Def

For any  $f : \Omega \rightarrow \mathbb{R}$  let

$$I^*(f) := \inf_{\mu \in S, \mu \geq f} I(\mu)$$

$$I_*(f) := \sup_{l \in S, l \leq f} I(l)$$

If  $I^*(f) = I_*(f)$  then we say that  $f$  is I-Riemann integral, and denote by  $I(f)$  the value  $I^*(f)$  (or  $I_*(f)$ )

### 36.3 Theorem

The set  $\mathcal{R}$  of all I-Riemann integral mappings form a vector space of  $\mathbb{R}^\Omega$  that contains  $S$ . Moreover,  $I : \mathcal{R} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear mapping extending  $I : S \rightarrow \mathbb{R}$

#### Proof

$$\forall h \in S$$

$$I^*(h) = I_*(h) = I(h)$$

So  $h \in \mathcal{R}$

Let  $(f_1, f_2) \in \mathcal{R}$ . If  $(\mu_1, \mu_2) \in S^2, \mu_1 \geq f_1, \mu_2 \geq f_2$  then

$$\mu_1 + \mu_2 \in S, \mu_1 + \mu_2 \geq f_1 + f_2$$

Hence

$$I(\mu_1) + I(\mu_2) \geq I^*(f_1 + f_2)$$

Take the infimum with respect to  $(\mu_1, \mu_2)$  we get

$$I^*(f_1) + I^*(f_2) \geq I^*(f_1 + f_2)$$

Similarly

$$I_*(f_1) + I_*(f_2) \leq I_*(f_1 + f_2)$$

Hence

$$I^*(f_1 + f_2) = I_*(f_1 + f_2) = I(f_1) + I(f_2)$$

Let  $f : \Omega \rightarrow \mathbb{R}$  be a mapping,  $\lambda \in \mathbb{R}_{>0}$

$$I^*(\lambda f) = \inf_{\mu \in S, \mu \geq \lambda f} I(\mu) = \inf_{\nu \in S, \nu \geq f} I(\lambda \nu) = \lambda I^*(f)$$

Similarly

$$I_*(\lambda f) = \lambda I_*(f)$$

Hence if  $f \in \mathcal{R}$  then  $\lambda f \in \mathcal{R}$  and  $I(\lambda f) = \lambda I(f)$

$$I^*(-f) = \inf_{\mu \in S, \mu \geq -f} I(\mu) = \inf_{l \in S, l \leq f} I(-l) = - \sup_{l \in S, l \leq f} I(l) = -I_*(f)$$

Similarly

$$I_*(-f) = -I^*(f)$$

Hence if  $f \in \mathcal{R}$  then  $-f \in \mathcal{R}$  and  $I(-f) = -I(f)$



## Chapter 37

# Daniell integral

We fix an integral operator  $I : S \rightarrow \mathbb{R}$

### 37.1 Prop

#### 37.1.1

Let  $(f_n)_{n \in \mathbb{N}}$  be an increasing sequence in  $S$  that converges pointwisely to some  $f \in S$ . Then

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

**Proof**

Let  $g_n = f - f_n \in S$   $(g_n)_{n \in \mathbb{N}}$  is decreasing and converges pointwisely to 0. Then

$$\lim_{n \rightarrow +\infty} I(g_n) = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

#### 37.1.2

Let  $(f_n)_{n \in \mathbb{N}}$  be an increasing sequence in  $S$ ,  $f \in S$  If  $f \leq \lim_{n \rightarrow +\infty} f_n$ , then

$$I(f) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

**Proof**

$$f = \lim_{n \rightarrow +\infty} f \wedge f_n$$

So

$$I(f) = \lim_{n \rightarrow +\infty} I(f \wedge f_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.2 Def

Let

$$S^\uparrow = \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\} \mid \begin{array}{l} \exists (f_n)_{n \in \mathbb{N}} \in S^\mathbb{N} \text{ increasing such that} \\ f = \lim_{n \rightarrow +\infty} f_n \text{ pointwisely} \end{array} \right\}$$

### 37.3 Prop

Let  $f, g$  be elements of  $S^\uparrow$  such that  $f \leq g$ . Let  $(f_n)_{n \in \mathbb{N}}$  and  $(g_m)_{m \in \mathbb{N}}$  be increasing sequences in  $S$  such that  $f = \lim_{n \rightarrow +\infty} f_n, g = \lim_{m \rightarrow +\infty} g_m$ . Then

$$\lim_{n \rightarrow +\infty} I(f_n) \leq \lim_{m \rightarrow +\infty} I(g_m)$$

#### Proof

For any  $m \in \mathbb{N}$

$$f_m \leq f \leq g$$

Hence

$$I(f_m) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Taking  $\lim_{m \rightarrow +\infty}$  we get

$$\lim_{m \rightarrow +\infty} I(f_m) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.4 Corollary

Let  $f \in S^\uparrow$ . If  $(f_n)_{n \in \mathbb{N}}$  and  $(\tilde{f}_n)_{n \in \mathbb{N}}$  be increasing sequence in  $S$  such that

$$f = \lim_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} \tilde{f}_n$$

then

$$\lim_{n \rightarrow +\infty} I(f_n) = \lim_{n \rightarrow +\infty} I(\tilde{f}_n)$$

We denote by  $I(f)$  the limit  $\lim_{n \rightarrow +\infty} I(f_n)$

Thus we obtain a mapping  $I : S^\uparrow \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

- If  $(f_n)_{n \in \mathbb{N}} \in S^\mathbb{N}$  is increasing then

$$I\left(\lim_{n \rightarrow +\infty} f_n\right) = \lim_{n \rightarrow +\infty} I(f_n)$$

- If  $(f, g) \in S^{\uparrow 2}$   $f \leq g$  then  $I(f) \leq I(g)$
- If  $(f, g) \in S^{\uparrow 2}$  then  $f + g \in S^\uparrow$  and

$$I(f + g) = I(f) + I(g)$$

- If  $f \in S^\uparrow, \lambda \geq 0$  then  $\lambda f \in S^\uparrow$  and  $I(\lambda f) = \lambda I(f)$

### 37.5 Prop

Let  $(f_n)_{n \in \mathbb{N}} \in (S^\uparrow)^\mathbb{N}$  be an increasing sequence and  $f = \lim_{n \rightarrow +\infty} f_n$ . Then

$$f \in S^\uparrow$$

and

$$I(f) = \lim_{n \rightarrow +\infty} I(f_n)$$

#### Proof

For  $k \in \mathbb{N}$  let  $(g_{k,m})_{m \in \mathbb{N}} \in S^\mathbb{N}$  be an increasing sequence such that

$$f_k = \lim_{m \rightarrow +\infty} g_{k,m}$$

For  $n \in \mathbb{N}$  let  $h_n = g_{0,n} \vee \cdots \vee g_{n,n} \in S$  The sequence  $(h_n)_{n \in \mathbb{N}}$  is increasing. Moreover

$$f_n \geq k_n \geq g_{k,n} \quad (k \leq n)$$

Hence

$$f_n \geq h_n$$

Taking  $\lim_{n \rightarrow +\infty}$  we get  $\forall k \in \mathbb{N}$

$$f = \lim_{n \rightarrow +\infty} f_n \geq \lim_{n \rightarrow +\infty} h_n \geq \lim_{n \rightarrow +\infty} g_{k,n} = f_k$$

Taking  $\lim_{k \rightarrow +\infty}$  we get

$$f = \lim_{n \rightarrow +\infty} h_n$$

Hence  $f \in S^\uparrow$  and

$$I(f) = \lim_{n \rightarrow +\infty} I(h_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Conversely,  $\forall n \in \mathbb{N}, f \geq f_n$  Hence

$$I(f) \geq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.6 Def

Let  $S^\downarrow = \{-f \mid f \in S^\uparrow\}$  We extend  $I$  to  $I : S^\downarrow \rightarrow \mathbb{R}U - \infty$  by letting  $I(-f) := -I(f)$  for  $f \in S^\uparrow$

### 37.7 Prop

Let  $(f, g) \in (S^\uparrow \cup S^\downarrow)^2$  If  $f \leq g$  then

$$I(f) \leq I(g)$$

**Proof**

It suffices to treat the cases where  $(f, g) \in S^\uparrow \times S^\downarrow$  and  $(f, g) \in S^\uparrow \times S^\downarrow$

If  $(f, g) \in S^\uparrow \times S^\downarrow$  then  $-f \in S^\downarrow$  and hence  $g - f \in S^\uparrow, g - f \geq 0$  In both cases,

$$0 \leq I(g - f) = I(g) + I(-f) = I(g) - I(f)$$

**37.8 Def**

Let  $f : \Omega \rightarrow \mathbb{R}$  be a mapping. We define

$$\bar{I}(f) := \inf_{\mu \in S^\uparrow, \mu \geq f} I(\mu) \leq \inf_{\mu \in S, \mu \geq f} I(\mu) = I^*(f)$$

$$\underline{I}(f) := \sup_{\mu \in S^\downarrow, \mu \leq f} I(\mu) \geq \sup_{\mu \in S, \mu \leq f} I(\mu) = I_*(f)$$

If  $\bar{I}(f) = \underline{I}(f)$  then we say that  $f$  is  $I$ -integrable (in the sense of Daniell)

**37.9 Remark**

If  $f$  is I-integrable in the sense of Riemann, then it is I-integrable in sense of Daniell

**37.10 Daniell Theorem**

The set  $L^1(I)$  of all I-integrable mappings forms a vector subspace of  $\mathbb{R}$ . Moreover

- $\forall (f, g) \in L^1(I) \ f \wedge g \in L^1(I)$
- $I : L^1(I) \rightarrow \mathbb{R}$  is an integral operator extending  $I : S \rightarrow \mathbb{R}$

**Proof**

Let  $(f_1, f_2) \in L^1(I)^2$  let  $(l_1, l_2) \in S^{\downarrow 2}, l_1 \leq f_1, l_2 \leq f_1$  Let  $(\mu_1, \mu_2) \in S^{\uparrow 2}, f_1 \leq \mu_1, f_2 \leq \mu_2$

We have

$$l_1 + l_2 \leq f_1 + f_2 \leq \mu_1 + \mu_2$$

Taking the supremum with respect to  $(l_1, l_2)$ , we get

$$I(f_1) + I(f_2) (= \underline{I}(f_1) + \underline{I}(f_2)) \leq \underline{I}(f_1 + f_2)$$

Taking the infimum with respect to  $(\mu_1, \mu_2)$ , we get

$$\bar{I}(f_1 + f_2) \leq I(f_1) + I(f_2)$$

Then

$$\bar{I}(f_1 + f_2) = \underline{I}(f_1 + f_2)$$

So  $f_1 + f_2 \in L^1(I)$  and  $I(f_1 + f_2) = I(f_1) + I(f_2)$

Similarly, if  $f \in L^1(I), \lambda \geq 0$  then

$$\begin{aligned} \underline{I}(\lambda f) &= \sup_{l \leq \lambda f, l \in S^\downarrow} I(l) \\ &= \sup_{l \leq f, l \in S^\downarrow} I(\lambda l) \\ &= \lambda \underline{I}(f) = \lambda I(f) \end{aligned}$$

$$\bar{I}(\lambda f) = \lambda \bar{I}(f) = \lambda I(f)$$

So  $\lambda f \in L^1(I)$  and  $I(\lambda f) = \lambda I(f)$

Moreover, if  $f \in L^1(I), \mu \in S^\uparrow, l \in S^\downarrow, l \leq f \leq \mu$  then

$$-\mu \in S^\downarrow, -l \in S^\uparrow, -\mu \leq -f \leq -l$$

Hence

$$\bar{I}(-f) = -\underline{I}(f) = -I(f) \quad \underline{I}(-f) = -\bar{I}(f) = -I(f)$$

So  $-f \in L^1(I)$  and  $I(-f) = -I(f)$

We proved that  $\forall (f_1, f_2) \in L^1(I)^2$

$$f_1 \wedge f_2 \in L^1(I)$$

Let  $(f_1, f_2) \in L^1(I)^2$ , for any  $\epsilon > 0 \exists (l_1, l_2) \in S^\downarrow{}^2, (\mu_1, \mu_2) \in S^\uparrow{}^2$  such that

$$l_1 \leq f_1 \leq \mu_1 \quad l_2 \leq f_2 \leq \mu_2$$

such that

$$I(\mu_1 - l_1) \leq \frac{\epsilon}{2} \quad I(\mu_2 - l_2) \leq \frac{\epsilon}{2}$$

One has  $l_1 \wedge l_2 \leq f_1 \wedge f_2 \leq \mu_1 \wedge \mu_2$

$$\mu_1 \wedge \mu_2 - l_1 \wedge l_2 \leq (\mu_1 - l_1) + (\mu_2 - l_2)$$

$$\left( \begin{array}{l} \text{If } \mu_1(\omega) \leq \mu_2(\omega), l_1 \leq l_1(\omega) \\ LHS = \mu_1(\omega) - l_1(\omega) \\ RHS = \mu_1(\omega) - l_2(\omega) + \mu_2(\omega) - l_1(\omega) \geq \mu_1(\omega) - l_2(\omega) \end{array} \right)$$

### 37.11 Beppo Levi Theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a monotone sequence of elements of  $L_1(I)$ , which converges pointwisely to some  $f : \Omega \rightarrow \mathbb{R}$  If  $(I(f_n))_{n \in \mathbb{N}}$  converges to a real number  $\alpha$  Then  $f \in L^1(I)$  and  $I(f) = \alpha$

**Proof**

Assume that  $(f_n)_{n \in \mathbb{N}}$  is increasing. Moreover, by replacing  $f_n$  by  $f_n - f_0$  we may assume that  $f_0 = 0$

Let  $\epsilon > 0 \forall n \in \mathbb{N}$  let  $\mu_n \in S^\uparrow$  such that  $f_n - f_{n-1} \leq \mu_n$  and

$$I(f_n - f_{n-1}) \geq I(\mu_n) - \frac{\epsilon}{2}$$

the existence

$$I(f_n - f_{n-1}) = \inf_{\mu \in S^\uparrow, \mu \geq f_n - f_{n-1}} I(\mu)$$

If  $\forall \mu \in S^\uparrow, \mu \geq f_n - f_{n-1}$  one has

$$I(\mu) > I(f_n - f_{n-1}) + \frac{\epsilon}{2}$$

then

$$I(f_n - f_{n-1}) + \frac{\epsilon}{2} \leq I(f_n - f_{n-1})$$

contradiction.

Thus

$$f_n = \sum_{k=1}^n (f_k - f_{k-1}) \leq \mu_1 + \cdots + \mu_n$$

and

$$I(f_n) \geq \sum_{k=1}^n (I(\mu_k) - \frac{\epsilon}{2^k}) \geq I(\mu_1) + \cdots + I(\mu_n) - \epsilon$$

Let  $\mu = \mu_1 + \cdots + \mu_n + \cdots \in S^\uparrow$

$$I(\mu) = \sum_{n \in \mathbb{N}} I(\mu_n)$$

One has  $\mu \geq f$

$$\lim_{n \rightarrow +\infty} \geq I(\mu) - \epsilon \geq \bar{I}(f) - \epsilon$$

Similarly, one can choose  $l_n \in S^\downarrow, l_n \leq f_n, I(l_n) \geq I(f_n) - \epsilon$

$$\liminf_{n \rightarrow +\infty} I(l_n) \geq \alpha - \epsilon$$

Note that  $l_n \leq f_n \leq f$ , so

$$\alpha - \epsilon \leq \liminf_{n \rightarrow +\infty} I(l_n) \leq \underline{I}(f)$$

Thus

$$\alpha - \epsilon \leq \underline{I}(f) \leq \bar{I}(f) \leq \alpha + \epsilon$$

Let  $\epsilon \rightarrow 0$  we get

$$\bar{I}(f) = \underline{I}(f) = \alpha$$

### 37.12 Fatou's Lemma

Let  $(f_n)_{n \in \mathbb{N}} \in L^1(I)^\mathbb{N}$ . Assume that there is  $g \in L^1(I)$  such that

$$\forall n \in \mathbb{N} \quad f_n \geq g$$

If  $\liminf_{n \rightarrow +\infty} f_n$  is a mapping from  $\Omega$  to  $\mathbb{R}$  and  $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$ , then  $\liminf_{n \rightarrow +\infty} f_n \in L^1(I)$  and

$$I(\liminf_{n \rightarrow +\infty} f_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

#### Proof

For any  $n \in \mathbb{N}$ , let

$$g_n = \lim_{k \rightarrow +\infty} (f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k})$$

Then

$$\liminf_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} g_n$$

For any  $k$  one has

$$f_n \wedge \cdots \wedge f_{n+k} \geq g$$

Hence

$$I(f_n) \geq \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \geq I(g)$$

By the theorem of Beppo Levi,

$$g_n \in L^1(I) \text{ and } I(g_n) = \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \leq I(f_n)$$

Note that  $(g_n)_{n \in \mathbb{N}}$  is increasing and  $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$ . Hence

$$\lim_{n \rightarrow +\infty} I(g_n) = \liminf_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n) < +\infty$$

By the theorem of Beppo Levi,

$$\lim_{n \rightarrow +\infty} g_n \in L^1(I)$$

and

$$I(\liminf_{n \rightarrow +\infty} f_n) = I(\lim_{n \rightarrow +\infty} g_n) = \lim_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

### 37.13 Lebesgue dominated convergence theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $L^1(I)$  that converges pointwisely to some  $f : \Omega \rightarrow \mathbb{R}$ . Assume that there exists  $g \in L^1(I)$  such that  $\forall n \in \mathbb{N}, |f_n| \leq g$ . Then  $f \in L^1(I)$  and  $I(f) = \lim_{n \rightarrow +\infty} I(f_n)$ .

**Proof**

Apply Fatou's lemma to  $(f_n)_{n \in \mathbb{N}}$  and  $(-f_n)_{n \in \mathbb{N}}$  to get

$$I(f) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

and

$$\begin{aligned} I(-f) &\leq \liminf_{n \rightarrow +\infty} I(-f_n) \\ &= \limsup_{n \rightarrow +\infty} I(f_n) \\ &\leq \limsup_{n \rightarrow +\infty} I(f_n) \leq I(f) \end{aligned}$$

**37.14 Notation**

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing and right continuous mapping. Let  $S$  be the vector subspace of  $\mathbb{R}^{\mathbb{R}}$  generated by  $\mathbb{1}_{[a,b]}$  with  $(a,b) \in \mathbb{R}^2, a < b$ . For any  $f \in L^1(I_\varphi)$   $I_\varphi(f)$  is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

For any subset  $A$  of  $\mathbb{R}$  if  $\mathbb{1}_A f \in L^1(I)$  then

$$\int_A f(x) d\varphi(x) \text{ denotes } \int_{\mathbb{R}} \mathbb{1}_A(x) f(x) d\varphi(x) = I(\mathbb{1}_A f)$$

If  $(a,b) \in \mathbb{R}^2, a < b$

$$\int_a^b f(x) d\varphi(x) \text{ denotes } \int_{[a,b]} f(x) d\varphi(x)$$

$$\int_b^a f(x) d\varphi(x) \text{ denotes } - \int_{[a,b]} f(x) d\varphi(x)$$

If  $\varphi(x) = x$  for any  $x \in \mathbb{R}$  we replace  $d\varphi(x)$  by  $dx$ .



## Chapter 38

# Semialgebra

### 38.1 Notation

Let  $A, (A_i)_{i \in I}$  be sets the notation.

$$A = \bigsqcup_{i \in I} A_i$$

denotes:

- $(A_i)_{i \in I}$  is a pairwise disjoint
- $A = \bigcup_{i \in I} A_i$

### 38.2 Def

Let  $\Omega$  be a set. We call semialgebra on  $\Omega$  any  $\mathcal{C} \subseteq \wp(\Omega)$  that verifies:

- $\emptyset \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, A \cap B \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, \exists (C_i)_{i=1}^n$  a finite family of elements in  $\mathcal{C}$  such that  $B \setminus A = \bigsqcup_{i=1}^n C_i$

#### 38.2.1 Example

$$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$$

### 38.3 Def

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ . The set

$$\{A \in \wp(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigsqcup_{i=1}^n A_i\}$$

is called the algebra generated by  $\mathcal{C}$

### 38.4 Prop

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ .  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Then:

- $\emptyset \in \mathcal{A}$
- $\forall (A, B) \in \mathcal{A}^2, A \cap B \in \mathcal{A}, B \setminus A \in \mathcal{A}, A \cup B \in \mathcal{A}$

#### Proof

By definition,  $\emptyset \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}$ . Moreover, if  $A$  and  $B$  are elements of  $\mathcal{A}$  such that  $A \cap B = \emptyset$  then  $A \cup B \in \mathcal{A}$ . Let  $A = \bigsqcup_{i=1}^n A_i$  and  $B = \bigsqcup_{i=1}^n B_i$  be elements of  $\mathcal{A}$  then

$$A \cap B = \bigsqcup_{(i,j) \in \{1, \dots, n\}^2} (A_i \cap B_j)$$

Hence  $A \cap B \in \mathcal{A}$  Finally

$$A \cup B = (A \cap B) \sqcup (A \setminus B) \sqcup (B \setminus A) \in \mathcal{A}$$

### 38.5 Prop

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ .  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Let  $S$  be the  $\mathbb{R}$ -vector subspace of  $\mathbb{R}^\Omega$ ,  $I : S \rightarrow \mathbb{R}$  be an  $\mathbb{R}$ -linear mapping generated by mappings of the form  $\mathbb{1}_A, A \in \mathcal{C} (f \in S, f = \sum \lambda_i \mathbb{1}_{A_i})$   
Assume that

$$\forall (f, g) \in S^2, f \leq g \text{ one has } I(f) \leq I(g)$$

Then  $I$  is an integral operator iff, for any decreasing sequence  $(A_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}^\mathbb{N}$  such that  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ , one has

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

### 38.5.1 Proof

$$\forall A \in \mathcal{A}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigcup_{i=1}^n A_i \text{ so } \mathbb{1}_A = \sum_{i=1}^n \mathbb{1}_{A_i} \in S$$

Lemma  $\forall (f, g) \in S^2, f \wedge g \in S$

$\Rightarrow$  Suppose that  $I$  is an integral operator  $(\mathbb{1}_{A_n})_{n \in \mathbb{N}}$  is a decreasing sequence in  $S$  and

$$\lim_{n \rightarrow +\infty} \mathbb{1}_{A_n} = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

$\Leftarrow$  Let  $(f_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $S$  that converges pointwisely to 0. Let

$$B = \{\omega \in \Omega \mid f_0(\omega) > 0\} \in \mathcal{A} \quad M = \max\{f_0(\omega) \mid \omega \in \Omega\}$$

- For any  $\epsilon > 0$  let

$$A_n^\epsilon = \{\omega \in \Omega \mid f_n(\omega) \geq \epsilon\} \in \mathcal{A}$$

Moreover, since  $\lim_{n \rightarrow +\infty} f_n(\omega) = 0, \bigcap_{n \in \mathbb{N}} A_n^\epsilon = \emptyset$

$$f(\omega) = \begin{cases} \lambda_i & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \in \Omega \setminus \bigcap_{i=1}^n A_i \end{cases}$$

$(\forall f \in S, \exists (A_i)_{i=1}^n \text{ pairwise disjoint and } (\lambda_i)_{i=1}^n \in \mathbb{R} \text{ } f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i})$

Note that

$$0 \leq f_n \leq \epsilon \mathbb{1}_B + M \mathbb{1}_{A_n^\epsilon}$$

So

$$0 \leq I(f_n) \leq \epsilon I(\mathbb{1}_B) + M I(\mathbb{1}_{A_n^\epsilon})$$

which leads to

$$\limsup_{n \rightarrow +\infty} I(f_n) \leq \epsilon I(\mathbb{1}_B) \quad \forall \epsilon > 0$$

So

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

### 38.5.2 Example

Let  $\Omega = \mathbb{R}$  and  $\mathcal{C} = \{]a, b[ \mid (a, b) \in \mathbb{R}^2, a \leq b\}$   
 $\mathcal{A}$  be algebra generated by  $\mathcal{C}$ .  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  increasing, right continuous.  $S$  be  $\mathbb{R}$ -vector subspace generated by  $\mathbb{1}_{]a, b[}, (a, b) \in \mathbb{R}, a \leq b$

$$\begin{aligned} I_\varphi : S &\rightarrow \mathbb{R} \\ \mathbb{1}_{]a, b[} &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

Lemma  $\forall \epsilon > 0, A \in \mathcal{A}, A \neq \emptyset, \exists B \in \mathcal{A},$

$$\emptyset \neq \overline{B} \subseteq A \text{ and } I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) \leq \epsilon$$

#### Proof

We first consider the the case where  $A \in \mathcal{C}, A = ]a, b[, a < b$  By the right continuous of  $\varphi, \exists ]a', b[$  such that  $\varphi(a') - \varphi(a) \leq \epsilon$ . Let  $B = ]a', b[, \overline{B} = [a', b] \subseteq ]a, b[$ .

$$\begin{aligned} I_\varphi(\mathbb{1}_B) &= \varphi(b) - \varphi(a') \\ I_\varphi(\mathbb{1}_A) &= \varphi(b) - \varphi(a) \\ I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) &= \varphi(a') - \varphi(a) \leq \epsilon \end{aligned}$$

In general

$$A = \bigsqcup_{i=1}^n A_i$$

with  $A_i \in \mathcal{C} \forall i \in \{1, \dots, n\}, \exists B_i \in \mathcal{C}$

$$\emptyset \neq \overline{B_i} \subseteq A_i \quad I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \frac{\epsilon}{n}$$

Let  $B = \bigsqcup_{i=1}^n B_i$  then

$$I(\mathbb{1}_A) - I(\mathbb{1}_B) = \sum_{i=1}^n I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \epsilon$$

### 38.6 Theorem

$I_\varphi$  is an integral operator

#### Proof

Let  $(A_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $\mathcal{A}$  such that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

Let  $\epsilon > 0$  For any  $n \in \mathbb{N}$  let  $B_n \in \mathcal{A}$  such that

$$\overline{B_n} \subseteq A_n \text{ and } I_\varphi(A_n) - I_\varphi(B_n) \leq \frac{\epsilon}{2^n}$$

Note that  $\overline{B_n}$  is compact. For any  $n \in \mathbb{N}$  let

$$\begin{aligned} C_n &= \bigcap_{i=1}^n B_i \\ &\subseteq \bigcap_{i=1}^n \overline{B_i} \end{aligned}$$

Since  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ ,  $\bigcap_{n \in \mathbb{N}} \overline{B_n} = \emptyset$

So

$$I_\varphi(\mathbb{1}_{A_n}) \leq \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}} \cdots \frac{\epsilon}{2} \leq \epsilon$$

Thus

$$\lim_{n \rightarrow +\infty} I_\varphi(\mathbb{1}_{A_n}) = 0$$

Let  $\Omega$  be a set  $\mathcal{C}$  be a semialgebra on  $\Omega$  and  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Let  $S$  be the  $\mathbb{R}$ -vector subspace of  $\mathbb{R}^\Omega$  generated by mappings of the form  $\mathbb{1}_A, A \in \mathcal{C}$

## 38.7 Prop

For any  $f \in S, \exists (A_i)_{i=1}^n \in \mathcal{A}^n$  pairwise disjoint and  $(\lambda_i)_{i=1}^n$  such that

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$$

### Proof

$f$  is of the form

$$f = \sum_{j=1}^m a_j \mathbb{1}_{B_j} \quad B_j \in \mathcal{C}$$

We reason by induction on  $m$ . For any  $I \subseteq \{1, \dots, m\}$  let

$$B_I = \bigcap_{j \in I} B_j \cap \bigcap_{j \in \{1, \dots, m\} \setminus I} (\Omega \setminus B_j)$$

Then  $(B_I)_{I \subseteq \{1, \dots, m\}}$  are pairwise disjoint.

Moreover, if  $I = \emptyset, B_I \in \mathcal{A}$

$$B_i = \bigsqcup_{I \subseteq \{1, \dots, m\}, i \in I} B_I$$

Hence

$$f = \sum_{U \subseteq \{1, \dots, m\}} \left( \sum_{j \in U} a_j \mathbb{1}_{B_j} \right)$$

### 38.8 Corollary

(1) If  $f \in S$  then

$$f \wedge 0 \in S$$

(2) If  $(f, g) \in S^2$  then

$$f \wedge g = (f - g) \wedge 0 + g \in S$$

#### Proof

We intend to define

$$I_\mu\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}\right)$$

as

$$\sum_{i=1}^n \lambda_i I_\mu(\mathbb{1}_{A_i})$$

for  $A_i \in \mathcal{C}$ . We need to check that if  $f \in S$  is written as

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} = \sum_{j=1}^m \xi_j \mathbb{1}_{B_j}$$

then

$$\sum_{i=1}^n \xi_i I_\mu(\mathbb{1}_{A_i}) = \sum_{j=1}^m \xi_j I_\mu(\mathbb{1}_{B_j})$$

so

$$0 = \sum_{i=1}^n \xi_i \mathbb{1}_{A_i} - \sum_{j=1}^m \xi_j \mathbb{1}_{B_j}$$

It suffices to prove that if

$$\sum_{i=1}^n \xi_i \mathbb{1}_{A_i} = 0$$

then

$$\sum_{j=1}^m \xi_j \mathbb{1}_{B_j} = 0$$

For  $I \subseteq \{1, \dots, n\}$  let

$$A_I = \{\omega \in \Omega \mid \forall i \in I, \omega \in A_i, \forall i \in \{1, \dots, n\} \setminus I, \omega \in \Omega \setminus A_i\}$$

$A_I \in \mathcal{A}$  when  $I \neq \emptyset$

### 38.9 Lemma

Let  $B \in \mathcal{A}$  If

$$B = \bigsqcup_{i=1}^n B_i = \bigsqcup_{j=1}^m C_j$$

with  $B_i, C_j \in \mathcal{C}$ , then

$$\sum_{i=1}^n \mu(B_i) = \sum_{j=1}^m \mu(C_j)$$

In particular, we can extend  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  to  $\mu : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$  such that  $\forall D_1, \dots, D_n$  in  $\mathcal{A}$  disjoint

$$\mu(D_1 \cup \dots \cup D_n) = \sum_{i=1}^n \mu(D_i)$$

#### 38.9.1 Proof

$$\begin{aligned} B_i &= \bigsqcup_{j=1}^m (B_i \cap C_j) & \mu(B_i) &= \sum_{j=1}^m \mu(B_i \cap C_j) \\ \sum_{i=1}^n \mu(B_i) &= \sum_{i=1}^n \sum_{j=1}^m \mu(B_i \cap C_j) \\ &= \sum_{j=1}^m \sum_{i=1}^n \mu(B_i \cap C_j) \\ &= \sum_{j=1}^m \mu(C_j) \end{aligned}$$

Back to the proof

$$0 = \sum_{i=1}^n a_i \mathbb{1}_{A_i} = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \left( \sum_{i \in I} a_i \right) \mathbb{1}_{A_i}$$

hence

$$\begin{aligned} \sum_{i \in I} a_i &= 0 \\ 0 &= \sum_{i=1}^n a_i \mu(A_i) \\ &= \sum_{i=1}^n a_i \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \mu(A_i) \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \mu(A_i) \sum_{i \in I} a_i \end{aligned}$$





## Chapter 39

# Integral function

### 39.1 Setting

Let  $\Omega$  be a set.  $S \subseteq \mathbb{R}^\Omega$  be  $\mathbb{R}$ -vector subspace,  $\forall (f, g) \in S^2, f \wedge g \in S$   
 $I : S \rightarrow \mathbb{R}$  integral operator.

### 39.2 Prop

Suppose that  $\mathbb{1}_\Omega \in L^1(I)^\uparrow$  The set

$$\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I)^\uparrow\}$$

is a  $\sigma$ -algebra on  $\Omega$

Moreover, if we denote by  $\mu : \mathcal{G} \rightarrow \mathbb{R}_{\geq 0}$  the mapping define as

$$\mu(A) := I(\mathbb{1}_A)$$

then  $\mu$  satisfies :

$\forall (A_n)_{n \in \mathbb{N}} \in \mathcal{G}^\mathbb{N}$  that's is pairwise disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

#### 39.2.1 Proof

(1)

$$\emptyset \in \mathcal{G}$$

since

$$0 = \mathbb{1} \in L^1(I)^\uparrow$$

(2) If A and B are elements of  $\mathcal{G}$ ,  $A \subseteq B$ , then

$$\mu(A) \leq \mu(B)$$

so

$$\mathbb{1}_B - \mathbb{1}_{B \setminus A} \in L^1(I)^\uparrow \Rightarrow B \setminus A \in \mathcal{G}$$

(3) If  $(A, B) \in \mathcal{G}^2$

$$\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B} \in L^1(I)^\uparrow$$

So  $A \cup B \in \mathcal{G}$

If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}$ ,  $A = \bigcup_{n \in \mathbb{N}} A_n$  then

$$\mathbb{1}_A = \lim_{n \rightarrow +\infty} \mathbb{1}_{A_1 \cup \dots \cup A_n} \in L^1(I)^\uparrow \Rightarrow A \in \mathcal{G}$$

$$\begin{aligned} \underbrace{I(\mathbb{1}_{\sum_{n \in \mathbb{N}} A_n})}_{\mu(\bigcup_{n \in \mathbb{N}} A_n)} &= \lim_{n \rightarrow +\infty} \mathbb{1}_{A_0 \cup \dots \cup A_n} \\ &= \lim_{n \rightarrow +\infty} \sum_{i=0}^n \mathbb{1}_{A_i} \in L^1(I)^\uparrow \\ &= - \sum_{n \in \mathbb{N}} \mu(A_n) \end{aligned}$$

## Chapter 40

# Limit and Differential of Integrals with Parameters

Let  $\Omega$  be a set.  $S \subseteq \mathbb{R}^\Omega$  be  $\mathbb{R}$ -vector subspace such that  $\forall (f, g) \in S^2, f \wedge g \in S$ . Let  $I : S \rightarrow \mathbb{R}$  be an integral operator.

### 40.1 Theorem

Let  $X$  be a topological space,  $p \in X$ ,  $f : \Omega \times X \rightarrow \mathbb{R}$  be a mapping,  $g \in L^1(I)$ . Suppose that

(1)  $\forall \omega \in \Omega$

$$\begin{aligned} f(\omega, \cdot) : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto f(\omega, x) \end{aligned}$$

is continuous at  $p$

(2)  $\forall x \in X$

$$\begin{aligned} f(\cdot, x) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, x) \end{aligned}$$

belongs to  $L^1(I)$  and  $\forall \omega \in \Omega \quad |f(\omega, x)| \leq g(\omega)$

(3)  $p$  has a countable neighborhood basis in  $X$

Then

$$(x \in X) \mapsto I(f(\cdot, x))$$

is continuous at  $p$

**Proof**

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  that converges to  $p$ . For any  $n \in \mathbb{N}$  let  $f_n : \Omega \rightarrow \mathbb{R}$ ,  $f_n(\omega) := f(\omega, x_n)$ . One has  $|f_n| \leq g$ . Moreover  $\forall \omega \in \Omega$

$$\lim_{n \rightarrow +\infty} f_n(\omega) = \lim_{n \rightarrow +\infty} f(\omega, x_n) = f(\omega, p)$$

Hence, by dominate convergence theorem

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f(\cdot, p))$$

**40.2 Theorem**

Let  $J$  be an open interval in  $\mathbb{R}$ .  $f : \Omega \times J \rightarrow \mathbb{R}$  be a mapping.  $g \in L^1(I)$ . Assume that

$$(1) \quad \forall \omega \in \Omega$$

$$f(\omega, \cdot) : J \rightarrow \mathbb{R}$$

is differentiable (we denote by  $\frac{\partial f}{\partial t}(\omega, t)$  its derivative at  $t$ )  
and  $\forall t \in J$

$$\left| \frac{\partial f}{\partial t}(\omega, t) \right| \leq g(\omega)$$

$$(2) \quad \forall t \in J$$

$$\begin{aligned} f(\cdot, t) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, t) \end{aligned}$$

belongs to  $L^1(I)$

Then

$$\begin{aligned} \varphi : J &\rightarrow \mathbb{R} \\ t &\mapsto I(f(\cdot, t)) \end{aligned}$$

is differentiable and

$$\varphi'(t) = I\left(\frac{\partial f}{\partial t}(\cdot, t)\right) = \frac{d}{dt} I(f(\cdot, t))$$

**Proof**

Let  $a \in J$  and  $(t_n)_{n \in \mathbb{N}}$  be a sequence in  $J \setminus \{a\}$  such that

$$\lim_{n \rightarrow +\infty} t_n = a$$

. Then

$$\frac{\varphi(t_n) - \varphi(a)}{t_n - a} = I\left(\frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a}\right)$$

$\forall \omega \in \Omega$

$$\left| \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} \right| \leq g(\omega) \text{ (by mean value theorem)}$$

and

$$\lim_{n \rightarrow +\infty} \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} = \frac{\partial f}{\partial t}(\omega, a)$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{\varphi(t_n) - \varphi(a)}{t_n - a} = \frac{d}{dt} I(f(\cdot, t))$$



# Chapter 41

## Measure theory

### 41.1 Def

We call measure space any pair  $(E, \mathcal{E})$ , where  $E$  is a set and  $\mathcal{E}$  is a  $\sigma$ -algebra on  $E$ .

### 41.2 Prop

Let  $\Omega$  be a set. And  $(\mathcal{G}_i)_{i \in I}$  be a family of  $\sigma$ -algebras on  $\Omega$ . Then  $\bigcap_{i \in I} \mathcal{G}_i$  is a  $\sigma$ -algebra

#### Proof

- $\forall i \in I$

$$\emptyset \in \mathcal{G}_i$$

Hence

$$\emptyset \in \bigcap_{i \in I} \mathcal{G}_i$$

- If  $A \in \bigcap_{i \in I} \mathcal{G}_i$  then  $\forall i \in I$   $A \in \mathcal{G}_i$  Hence  $\forall i \in I$

$$\Omega \setminus A \in \mathcal{G}_i$$

So

$$\Omega \setminus A \in \bigcap_{i \in I} \mathcal{G}_i$$

- Let  $(A_n)_{n \in \mathbb{N}} \in (\bigcap_{i \in I} \mathcal{G}_i)^{\mathbb{N}}$ . For any  $i \in I$

$$(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_i^{\mathbb{N}}$$

So

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \mathcal{G}_i$$

so

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \bigcap_{i \in I} \mathcal{G}_i$$

### 41.3 Def

Let  $\mathcal{C} \subseteq \wp(\Omega)$ . We denote by  $\sigma(\mathcal{C})$  the intersection of all  $\sigma$ -algebras on  $\Omega$  containing  $\mathcal{C}$ . It's the smallest  $\sigma$ -algebra containing  $\mathcal{C}$

### 41.4 Example

- Let  $(X, \mathcal{G})$  be a topological space.  $\sigma(\mathcal{G})$  is called the Borel  $\sigma$ -algebra of  $X$
- On  $[-\infty, +\infty]$  the following  $\sigma$ -algebras are the same:

$$g_1 = \sigma(\{[a, +\infty] \mid a \in \mathbb{R}\})$$

$$g_2 = \sigma(\{]a, +\infty[ \mid a \in \mathbb{R}\})$$

$$g_3 = \sigma(\{[-\infty, a] \mid a \in \mathbb{R}\})$$

$$g_4 = \sigma(\{[-\infty, a[ \mid a \in \mathbb{R}\})$$

Moreover

$$\mathcal{B} = \{A \subseteq \mathbb{R} \mid A \in g_1\}$$

is equal to the Borel  $\sigma$ -algebra of  $\mathbb{R}$

proof  $\forall a \in \mathbb{R}$

$$[a, +\infty] = \bigcap_{n \in \mathbb{N}_{\geq 1}} ]a - \frac{1}{n}, +\infty[ \in g_2 \quad \Rightarrow g_1 \in g_2$$

$$]a, +\infty[ = [-\infty, +\infty] \setminus [-\infty, a] \in g_3 \quad \Rightarrow g_2 \in g_3$$

$$[-\infty, a] = \bigcap_{n \in \mathbb{N}_{\geq 1}} [-\infty, a + \frac{1}{n}[ \in g_4 \quad \Rightarrow g_3 \in g_4$$

$$[-\infty, a[ = [-\infty, +\infty] \setminus [a, +\infty] \in g_1 \quad \Rightarrow g_4 \in g_1$$

$$\sigma(\{]a, b[ \mid a < b, (a, b) \in \mathbb{R}^2\}) = \text{Borel } \sigma\text{-algebra of } \mathbb{R}$$

$J \subseteq \mathbb{R}$  open We define  $\sim$  a binary relation on  $J$  such that  $x \sim y \Leftrightarrow$  there exists an interval  $I$  such that  $\{x, y\} \subseteq I \subseteq J$

Any equivalence class of  $\sim$  is a non-empty open interval.

$$]a, b[ = [a, +\infty] \cup [-\infty, b[$$

Hence Borel  $\sigma$ -algebra of  $\mathbb{R} \subseteq \{A \subseteq \mathbb{R} \mid A \in g_1\}$



## 41.5 Def

Let  $f : X \rightarrow Y$  be a mapping of sets.

- For any  $\mathcal{C}_Y \subseteq \wp(Y)$  we denote by

$$f^{-1}(\mathcal{C}_Y) := \{f^{-1}(B) \mid B \in \mathcal{C}_Y\}$$

- For any  $\mathcal{C}_X \subseteq \wp(X)$  we denote by

$$f_*(\mathcal{C}_X) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{C}_X\}$$

## 41.6 Prop

Let  $f : X \rightarrow Y$  be a mapping.

- (1) If  $\mathcal{G}_Y$  is a  $\sigma$ -algebra on  $Y$  then  $f^{-1}(\mathcal{G}_Y)$  is a  $\sigma$ -algebra on  $X$
- (2) If  $\mathcal{G}_X$  is a  $\sigma$ -algebra on  $X$  then  $f_*(\mathcal{G}_X)$  is a  $\sigma$ -algebra on  $Y$

### Proof

(1)

$$\emptyset = f^{-1}(\emptyset) \in f^{-1}(\mathcal{G}_Y)$$

$$\forall B \in \mathcal{G}_Y$$

$$X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$$

If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_Y^{\mathbb{N}}$ ,  $A = \bigcup_{n \in \mathbb{N}} A_n$ ,  $A_n \in \mathcal{G}_Y$ , then

$$\bigcup_{n \in \mathbb{N}} f^{-1}(A_n) = f^{-1}(A) \in f^{-1}(\mathcal{G}_Y)$$

(2)

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{G}_X$$

so

$$\emptyset \in f_*(\mathcal{G}_X)$$

$$\forall B \in f_*(\mathcal{G}_X)$$

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \in \mathcal{G}_X$$

so

$$Y \setminus B \in f_*(\mathcal{G}_X)$$

$$\forall (B_n)_{n \in \mathbb{N}} \in f_*(\mathcal{G}_X)^{\mathbb{N}}, B = \bigcup_{n \in \mathbb{N}} B_n$$

$$f^{-1}(B) = \bigcap_{n \in \mathbb{N}} f^{-1}(B_n)$$

So  $B \in f_*(\mathcal{G}_X)$

### 41.7 Def

Let  $(X, \mathcal{G}_X)$  and  $(Y, g_Y)$  be measurable spaces,  $f : X \rightarrow Y$  be a mapping. If  $f^{-1}(g_Y) \subseteq \mathcal{G}_X$  or equivalently  $g_Y \subseteq f_s(\mathcal{G}_X)$  (or  $\forall B \in g_Y, f^{-1}(B) \in \mathcal{G}_X$ ) then we say that  $f$  is measurable.

### 41.8 Prop

Let  $(X, \mathcal{G}_X), (Y, g_Y), (Z, g_Z)$  be measurable spaces.  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be measurable mappings. Then  $g \circ f$  is measurable.

#### Proof

$$\forall B \in g_Z$$

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

and

$$g^{-1}(B) \in g_Y$$

so

$$f^{-1}(g^{-1}(B)) \in \mathcal{G}_X$$

### 41.9 Def

Let  $\Omega$  be a set  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.  $f = (f_i)_{i \in I}$  where  $f_i : \Omega \rightarrow E_i$  is a mapping. We denote by  $\sigma(f)$  the  $\sigma$ -algebra  $\sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i))$ . It's the smallest  $\sigma$ -algebra on  $\Omega$  making all  $f_i$  measurable.

### 41.10 Prop

We keep the notation of the above definition. For any  $i \in I$ , let  $\mathcal{C} \subseteq \wp(E_i)$  such that  $\sigma(\mathcal{C}_i) = \mathcal{E}_i$ . Then

$$\sigma(f) = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$$

#### Proof

Let  $g = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$ . By definition

$$g \subseteq \sigma(f)$$

For any  $i \in I$ ,  $f_{i,*}(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$  is a  $\sigma$ -algebra on  $\Omega$  containing  $\mathcal{C}_i$ . So

$$\mathcal{E} \subseteq f_{i,*}(\sigma(f_i^{-1}(\mathcal{C}_i)))$$

which leads to

$$f_i^{-1}(C_i) \subseteq \sigma(f_i^{-1}(C_i)) \subseteq g$$

Hence

$$\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i) \subseteq g$$

$\Rightarrow$

$$\sigma(f) \subseteq g$$

$$(f_{i,*}(\mathcal{A}) = \{B \subseteq E_i \mid f_i(B) \in \mathcal{A}\})$$

### 41.11 Corollary

Let  $(X, \mathcal{G}_X), (Y, g_Y)$  be measurable spaces.  $f : X \rightarrow Y$  be a mapping.  $\mathcal{C}_Y \subseteq g_Y$  such that  $g_Y = \sigma(\mathcal{Y})$  Then  $f$  is measurable iff

$$\forall B \in \mathcal{C}_Y \quad f^{-1}(B) \in \mathcal{G}_X$$

**Proof**

$$\sigma(f) = \sigma(f^{-1}(\mathcal{C}_Y))$$

$f$  is measurable iff  $\sigma(f) \subseteq \mathcal{G}_X$

### 41.12 Example

Let  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.

$$E = \prod_{i \in I} E_i$$

$\forall i \in I$

$$\begin{aligned} \pi_i : E &\rightarrow E_i \\ (x_j)_{j \in I} &\mapsto x_i \end{aligned}$$

We denote by  $\bigotimes_{i \in I} \mathcal{E}_i$  the  $\sigma$ -algebra  $\sigma((\pi_i)_{i \in I})$

### 41.13 Prop

Let  $X$  be a set  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.  $(\Omega, g)$  be a measurable space.  $f = (f_i : X \rightarrow E_i)_{i \in I}$  be a mappings,  $\varphi : \Omega \rightarrow X$  be a mapping. Then

$$\varphi : (\Omega, g) \rightarrow (X, \sigma(f))$$

is measurable iff

$$\forall i \in I \quad f_i \circ \varphi : (\Omega, g) \rightarrow (E_i, \mathcal{E}_i) \text{ is measurable.}$$

**Proof**

$\Rightarrow$  If  $\varphi$  is measurable, since each  $f_i$  is measurable, one has  $f_i \circ \varphi$  is measurable.

$\Leftarrow$  If  $f_i \circ \varphi$  is measurable,  $\forall B \in \mathcal{E}_i$

$$(f_i \circ \varphi)^{-1}(B) = \varphi^{-1}(f_i^{-1}(B)) \in g$$

Hence

$$\varphi^{-1}\left(\bigcup_{i \in I} f_i^{-1}(B)\right) \subseteq g$$

Since

$$\sigma(f) = \sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i)\right)$$

$\varphi$  is measurable.

**41.14 Example**

Let  $(\Omega, \mathcal{G})$  be a measurable space

- $\forall A \in \mathcal{G} \quad \mathbb{1}_A : \Omega \rightarrow \mathbb{R}$  is measurable. For any  $U \subseteq \mathbb{R}$

$$\mathbb{1}_A^{-1}(U) = A \text{ or } \Omega \setminus A \text{ or } \Omega \text{ or } \emptyset$$

- If  $X$  and  $Y$  be topological spaces.  $f : X \rightarrow Y$  be a continuous mapping, then  $f$  is measurable with respect to Borel  $\sigma$ -algebra. In fact,  $\forall V \subseteq Y$  open  $f^{-1}(V) \subseteq X$  open.
- Let  $(\Omega, \mathcal{G})$  be a measurable space. If  $f : \Omega \rightarrow \mathbb{R}, g : \Omega \rightarrow \mathbb{R}$  are measurable then  $f + g, fg, f \wedge g, f \vee g, |f|$  are measurable.
- Let  $(f_n)_{n \in \mathbb{N}}$  be a family of measurable mappings from  $\Omega$  to  $[-\infty, +\infty]$

$$f = \sup_{n \in \mathbb{N}} f_n \quad (f(\omega) = \sup_{n \in \mathbb{N}} f_n(\omega))$$

Then  $f$  is measurable.

Similarly  $\inf_{n \in \mathbb{N}} f_n$  is measurable.

In fact, for any  $a \in \mathbb{R}$

$$\{\omega \in \Omega \mid f(\omega) > a\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid f_n(\omega) > a\}$$

## Chapter 42

# Measure

### 42.1 Def

Let  $\Omega$  be a set.  $\mathcal{C}$  be a semi-algebra on  $\Omega$ .  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  be a mapping. If  $\forall n \in \mathbb{N}, \forall (A_i)_{i=1}^n \in \mathcal{C}^n$  pairwise disjoint, with  $A = \bigcup_{i=1}^n A_i$  one has

$$\mu(A) = \sum_{i=1}^n \mu(A_i)$$

we say that  $\mu$  is additive.

Let

$S =$  vector subspace of  $\mathbb{R}^\Omega$  generated by  $(\mathbb{1}_A)_{A \in \mathcal{C}}$

Then

$$I_\mu : S \rightarrow \mathbb{R}$$
$$\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} \mapsto \sum_{i=1}^n \lambda_i \mu(A_i)$$

is well defined. If  $I_\mu$  is an integral operator, we say that  $\mu$  is  $\sigma$ -additive.

### 42.2 Def

Let  $(\Omega, \mathcal{G})$  be a measurable space.  $\mu : \mathcal{G} \rightarrow [0, +\infty]$  be a mapping. If  $\mu(\emptyset) = 0$  and if for any  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$  pairwise disjoint.

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

we say that  $\mu$  is a measure.

$(\Omega, \mathcal{G}, \mu)$  is called a measure space.

### 42.3 Def

If  $\exists (A_n)_{n \in \mathbb{N}}$  such that  $\Omega = \bigcup_{n \in \mathbb{N}} A_n$  and  $\mu(A_n) < +\infty$  then  $\mu$  is said to be  $\sigma$ -finite.

### 42.4 Carathéodory Theorem

Let  $\Omega$  be a set,  $\mathcal{C}$  be a semi-algebra on  $\Omega$ ,  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  be a  $\sigma$ -additive mapping. Assume that there is a sequence  $(A_n)_{n \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$  such that  $\Omega = \bigcup_{n \in \mathbb{N}} A_n$ . Then  $\mu$  extends to a  $\sigma$ -finite measure on  $\sigma(\mathcal{C})$

#### Proof

Let  $S \subseteq \mathbb{R}^{\Omega}$  be the vector subspace generated by  $(\mathbb{1}_A), A \in \mathcal{C}$ . Let  $\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I_{\mu})^{\uparrow}\}$  then  $\mathcal{G}$  is a  $\sigma$ -algebra containing  $\mathcal{C}$ . Hence  $\sigma(\mathcal{C}) = \mathcal{G}$ . Moreover,  $(A \in \mathcal{G}) \mapsto I_{\mu}(\mathbb{1}_A)$  is a measure on  $\mathcal{G}$  which is  $\sigma$ -finite.

### 42.5 Example

$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a < b\}$   $\sigma(\mathcal{C}) = \text{Borel } \sigma\text{-algebra}$   $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  increasing and right continuous.

$$\begin{aligned} \mu_{\varphi} : \mathcal{C} &\rightarrow \mathbb{R}_{\geq 0} \\ ]a, b] &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

is  $\sigma$ -additive.

Hence  $\mu_{\varphi}$  extends to a measure:

$$\sigma(\mathcal{C}) \rightarrow [0, +\infty]$$

called the Stieltjes measure. In the particular case where  $\varphi(x) = x$  ( $\forall x \in \mathbb{R}$ )  $\mu_{\varphi}$  is called a Lebesgue measure.

### 42.6 Def

Let  $(\Omega, \mathcal{G}, \mu)$  be a measure space. Then

$$\{A \in \mathcal{G} \mid \mu(A) < +\infty\}$$

is a semialgebra.  $\sigma(\mathcal{C}) = \mathcal{G}$  and  $\mu|_{\mathcal{C}}$  is  $\sigma$ -additive.

•

$$\mu(A_0) = \sum_{n \in \mathbb{N}} \mu(B_n) < +\infty$$

$$\sum_{k \geq n} \mu(B_k) = \mu(A_n) \rightarrow 0 \quad (n \rightarrow +\infty)$$

We denote by  $L^1(\Omega, \mathcal{G}, \mu)$  the set of measurable mappings  $f : \Omega \rightarrow \mathbb{R}$  that belongs to  $L^1(I_\mu)$ . For  $f \in L^1(\Omega, \mathcal{G}, \mu)$

$$I_\mu(f)$$

is denoted as

$$\int_{\Omega} f(\omega) \mu(d\omega)$$

### 42.6.1 Particular case

If  $\Omega = \mathbb{R}$   $\mu = \mu_\varphi$  Stieltjes measure.

$$\int_{\mathbb{R}} f(x) \mu_\varphi(dx)$$

is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

## 42.7 Prop

Let  $(\Omega, \mathcal{G}, \mu)$  be a  $\sigma$ -finite measure space.  $f : \Omega \rightarrow \mathbb{R}$  is measurable. If

$$\exists g \in L^1(\Omega, \mathcal{G}, \mu), g \leq f$$

then

$$f \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

### Proof

By replacing  $f$  by  $f - g$ , we may assume that  $g = 0$  Consider first the case where

$$f = \mathbb{1}_B, B \in \mathcal{G}$$

Let  $(A_n)_{n \in \mathbb{N}}$  be a increasing sequence in  $\mathcal{G}$ ,  $\mu(A_n) < +\infty$ ,  $\bigcup_{n \in \mathbb{N}} A_n = \Omega$  Then

$$\mathbb{1}_B = \lim_{n \in \mathbb{N}} \mathbb{1}_{B \cap A_n} \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

For general  $f \geq 0$

$$f = \lim_{n \rightarrow +\infty} f_n \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

where

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega \in \Omega \mid \frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}\}} + n \mathbb{1}_{\{\omega \in \Omega \mid f(\omega) \geq n\}}$$

### 42.8 Corollary

Let  $f : \Omega \rightarrow \mathbb{R}$  be a measurable mapping. Then

$$f \in L^1(\Omega, \mathcal{G}, \mu)$$

iff

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

#### Proof

$\Rightarrow$  One has  $f \in L^1(I_{\mu})$ . Hence  $|f| \in L^1(I_{\mu})$  So  $I_{\mu}(|f|) < +\infty$

$\Leftarrow$  Suppose that

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

Since  $f \vee 0$  and  $-(f \wedge 0)$  belongs to  $L^1(\Omega, \mathcal{G}, \mu)^+$  and  $f \vee 0 \leq |f|$ ,  $-(f \wedge 0) \leq |f|$  so

$$f \vee 0 \text{ and } -(f \wedge 0) \in L^1(\Omega, \mathcal{G}, \mu)$$

Hence

$$f = f \vee 0 + f \wedge 0 \in L^1(\Omega, \mathcal{G}, \mu)$$



## Chapter 43

# Fundamental theorem of calculus

### 43.1 Theorem

Let  $J$  be an open interval in  $\mathbb{R}$   $x_0 \in J$   $f : J \rightarrow \mathbb{R}$  be a continuous mapping.

(1)  $\forall (a, b) \in J^2, a < b$

$$\begin{aligned} \mathbb{1}_{]a, b]} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto f(x) \quad \text{if } x \in ]a, b] \\ &\mapsto 0 \quad \text{if } x \notin ]a, b] \end{aligned}$$

belongs to  $L^1(\mathbb{R}, \mathcal{B}, \mu)$  ( $\mathcal{B}$  is Borel  $\sigma$ -algebra,  $\mu$  is Lebesgue measure)

(2) Let  $F : J \rightarrow \mathbb{R}$   $F(x) := \int_{x_0}^x f(t)dt$ . Then  $F$  is differentiable on  $J$  with  $F'(x) = f(x), \forall x \in J$

### 43.2 Corollary

If  $G : J \rightarrow \mathbb{R}$  is a mapping such that  $G' = f$  then  $\forall (a, b) \in J^2, a < b$

$$G(b) - G(a) = \int_a^b f(t)dt$$

#### 43.2.1 Proof

(1)  $f$  is bounded on  $[a, b]$  Hence

$$\int_{\mathbb{R}} \mathbb{1}_{]a, b]}^{(x)} |f(t)^{(x)}| dx < +\infty$$

- (2) Let  $x \in J, h > 0$  such that  $[x, x+h] \subseteq J$ ,  $f$  is uniformly continuous on  $[x, x+h]$   
 For  $0 < t \leq h$

$$\inf_{[x, x+t]} f \leq \frac{F(x+t) - F(x)}{t} = \frac{1}{t} \int_x^{x+t} f(s) ds \leq \sup_{[x, x+t]} f$$

Since  $f$  is continuous

$$\liminf_{t \rightarrow 0} \inf_{[x, x+t]} f = \limsup_{t \rightarrow 0} \sup_{[x, x+t]} f = f(x)$$

So

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Similarly

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Hence

$$F'(x) = f(x)$$

### Application

- Let  $F$  and  $G$  be two mapping of class  $C^1$  from  $J$  to  $\mathbb{R}$ . Then  $FG$  is of class  $C^1$  and

$$FG'(x) = F'(x)G(x) + F(x)G'(x)$$

Let  $f = F', g = G'$ , then  $\forall (a, b) \in J^2, a, b$

$$\int_a^b f(t)G(t)dt = F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t)dt$$

- Let  $\varphi : I \rightarrow J$  be a mapping of class  $C^1$ , where  $I$  is open interval. Let  $F : J \rightarrow \mathbb{R}$  be a mapping of class  $C^1$ .

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x)$$

- Hence  $\forall (\alpha, \beta) \in I^2, \alpha < \beta$

$$\int_{\alpha}^{\beta} F(\varphi(t))\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha))$$

## Chapter 44

### $L^p$ space

#### 44.1 Def

We fix a measure space  $(\Omega, \mathcal{G}, \mu)$  the set of measurable mappings  $f : \Omega \rightarrow \mathbb{R}$  such that

$$\|f\|_{L^p} := \left( \int_{\Omega} |f(\omega)|^p \mu(dx) \right)^{\frac{1}{p}} < +\infty$$

Lemma Let  $(p, q) \in \mathbb{R}_{\geq 1}^2$  such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

for any  $(a, b) \in \mathbb{R}_{\geq 0}^2$

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

Proof We may assume that  $(a, b) \in \mathbb{R}_{\geq 0}$

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{1}{p} \exp(p \ln a) + \frac{1}{q} \exp(q \ln b) \geq \exp(\ln a + \ln b) = ab$$

$$\begin{aligned} \int_{\Omega} |\varphi(x)\psi(x)| \mu(dx) &\leq \frac{\int_{\Omega} |\varphi(x)|^p \mu(dx)}{p \|f\|_{L^p}^p} + \frac{\int_{\Omega} |\psi(x)|^q \mu(dx)}{q \|g\|_{L^q}^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

#### 44.2 Hölder inequality

Let  $f : \Omega \rightarrow \mathbb{R}$  and  $g : \Omega \rightarrow \mathbb{R}$  be measurable mappings. One has

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

**Proof**

Take

$$\varphi = \frac{f}{\|f\|_{L^p}}, \psi = \frac{g}{\|g\|_{L^q}}$$

then

$$|\varphi(x)\psi(x)| \leq \frac{|\varphi(x)|^p}{p} + \frac{|\psi(x)|^q}{q}$$

**44.3 Corollary**

Let  $p \geq 1$   $\forall (f, g) \in L^p(\Omega, \mathcal{G}, \mu)$

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

**Proof**

Apply Hölder inequality to  $f(f + g)^{p-1}$  and  $g(f + g)^{p-1}$

# Part VIII

## tensor



# Chapter 45

## tensor product

Let  $R$  be a commutative ring with unity

### 45.1 Theorem

Let  $M$  and  $N$  be two  $R$ -modules. Then exists an  $R$ -module denoted by  $M \otimes_R N$  and a bilinear mapping

$$t : M \times N \rightarrow M \otimes_R N$$

having the following properties:

- (1) For any  $R$ -module  $P$  and any bilinear mapping  $s : M \times N \rightarrow P$ . There exists a unique linear mapping  $f_s : M \otimes_R N \rightarrow P$  such that  $s = f_s \circ t$

$$\begin{array}{ccc} M \times N & \xrightarrow{s} & P \\ \downarrow t & \nearrow f_s & \\ M \otimes_R N & & \end{array}$$

- (2) If  $T, t'$  is another couple that satisfies (1) with  $s \mapsto g_s$  then there exists a unique isomorphism

$$T \cong M \otimes_R N$$

### Proof

- (2) note that the the morphisms on  $R$ -module category are just linear mapping.

$$\begin{array}{ccc} M \times N & \xrightarrow{t'} & M \otimes_R N \\ \downarrow t & \nearrow f_{t'} & \\ T & \nearrow g_t & \end{array}$$

$$(f_{t'} \circ g_t) \circ t' = f_{t'} \circ t' = t$$

It means that we have the following structure

$$f_{t'} \circ g_t = id_{M \otimes_R N}$$

$$g_t \circ f_{t'} = id_T$$

Then isomorphic.

(1) let  $\mathcal{F}$  be the free  $R$ -module generated by  $M \times N$

$$\mathcal{F} = \left\{ \sum_{finite} a_{ij}(m_i, n_i) : a_{ij} \in R, m_i \in M, n_i \in N \right\}$$

let  $\mathcal{G}$  be the  $R$ -submodule generated by the elements of the following shape  
 $m, m' \in M \quad n, n' \in N \quad \mathfrak{z} \in R$

$$(m + m', n) - (m, n) - (m', n)$$

$$(m, n + n') - (m, n) - (m, n')$$

$$(\mathfrak{z}m, n) - \mathfrak{z}(m, n)$$

$$(m, \mathfrak{z}n) - \mathfrak{z}(m, n)$$

$$M \otimes_R N := \mathcal{F} / \mathcal{G}$$

## 45.2 Def

$$f_s(\mathcal{G} + (m, n)) := s(m, n)$$

Extend this mapping to linearity. This makes the diagram commutative. It's clearly the unique mapping

## 45.3 Def

The  $R$ -module  $M \otimes_R N$  constructed above is called the tensor product of  $M$  and  $N$ . An element of  $M \otimes_R N$  is called tensor. We denote

$$t(m, n) := m \otimes n$$

and any elements of this form is called pure tensor.

## 45.4 Remark

Pure tensors generate  $M \otimes_R N$ . In particular any tensor can be written as sum of pure tensors.

Example

$$0 = (m + m') \otimes n - m \otimes n - m \otimes n'$$



## 45.5 Corollary

The mapping  $s \mapsto f_s$  defined above gives an isomorphism

$$\mathcal{L}(M, N; P) \cong \mathcal{L}(M \otimes_R N, P)$$

for any  $R$ -module  $P$

### Proof

surjection Take  $\varphi \in \mathcal{L}(M \otimes_R N, P)$ , the  $\varphi \circ t$  is clearly bilinear ( $\in \mathcal{L}(M, N; P)$ ), so  $\varphi = f_{\varphi \circ t}$ . This shows surjectivity.

injection if  $0 \neq s = f_s \circ t \Rightarrow f_s \neq 0$ , hence

## 45.6 exercise

### 45.6.1

show that

$$M \otimes_R N \cong N \otimes_R M$$

### 45.6.2

show that

$$(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$$

so we can remove parenthesis and write

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$$

(call it the  $n$ -fold tensor product of  $M_1, \dots, M_n$ )

### 45.6.3

show that  $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$  factorizes the multi-linear mappings, and

$$\mathcal{L}(M_1, \dots, M_n; P) \cong \mathcal{L}(M_1 \otimes_R \cdots \otimes_R M_n, P)$$

$$\begin{array}{ccc}
 M_1 \times M_2 & \xrightarrow{s} & P \\
 \downarrow t & \nearrow f_s & \\
 M_1 \otimes_R M_2 & & \\
 \downarrow \wr & & \\
 M_1 \otimes_R \cdots \otimes_R M_n & & 
 \end{array}$$

We have the general definition of tensor products for  $R$ -modules. But we're interested in the case  $R = K$  when  $K$  is a field.  $\mathcal{L}$  denotes:  $V_1 \otimes \cdots \otimes V_n$

$$\mathcal{L}(V_1, \dots, V_n; K) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

This is the pervious corollary  $f \sim P = K$

## 45.7 Lemma

Let  $V_1, \dots, V_n$  be  $K$ -vector spaces of finite dimension  $d_i > 0$  let

$$e_{i1}, \dots, e_{id_i}$$

be a basis for  $V_i$ . Let's define the following functions.

$$\begin{aligned} \varphi_{i_1, \dots, i_n} : V_1 \times \cdots \times V_n &\rightarrow K \\ (v_1, \dots, v_n) &\mapsto \prod_{j=1}^n e_{ji_j}^\vee(V_j) \end{aligned}$$

Then the set  $\{\varphi_{i_1, \dots, i_n}\}$  is a basis for  $\mathcal{L}(V_1, \dots, V_n; K)$

### 45.7.1 Proof

We do the proof for  $n = 2$ . Then the general case follows by induction.

$$\begin{aligned} V_1 &= \langle e_1, \dots, e_m \rangle & m &= d_1 \\ V_2 &= \langle \omega_1, \dots, \omega_n \rangle & n &= d_2 \end{aligned}$$

This special our  $\varphi_{i_1, \dots, i_n}$  are denoted by

$$\xi_{ij}(x, y) = e_i^\vee(x) \omega_j^\vee(y)$$

Let's show that  $\xi_{ij}$  is a generating set

$\varphi \in \mathcal{L}(V_1, V_2; K)$  such that  $\varphi(e_i, \omega_i) := A_{ij} \in K$

$$\begin{aligned} \varphi(x, y) &= \varphi\left(\sum \alpha_i e_i, \sum \beta_j \omega_j\right) \\ &= \sum \alpha_i \beta_j \varphi(e_i, \omega_j) \\ &= \sum \alpha_i \beta_j A_{ij} \\ &= \sum A_{ij} e_i^\vee(x) \omega_j^\vee(y) \\ &= \sum A_{ij} \xi_{ij}(x, y) \end{aligned}$$

we prove that  $\xi_{ij}$  are linearly independent

$$\sum A_{ij} \xi_{ij}(x, y) = 0 \quad \forall (x, y) \in V_1 \times V_2$$

Evaluate in

$$(x, y) = (e_i, \omega_i) \Rightarrow A_{ij} = 0 \quad \forall i \neq j$$

## 45.8 Prop

Assume that  $V_1, \dots, V_n$  are vector spaces and  $V_i$  has basis:  $\{e_{i1}, \dots, e_{id_i}\}$  then

$$B = \{e_{1i_1} \otimes \dots \otimes e_{ni_n}, 1 \leq i_j \leq d_j\}$$

is a basis for  $V_1 \otimes \dots \otimes V_n$ . In particular,  $V_1 \otimes \dots \otimes V_n$  has dimension  $\prod_{i=1}^n d_i$

### Proof

Again we assume  $n = 2, m = d_1, n = d_2$

$$V_1 = \langle e_1, \dots, e_m \rangle \quad V_2 = \langle \omega_1, \dots, \omega_n \rangle$$

We know that

$$\begin{aligned} \mathcal{L}(V_1, V_2; P) &\cong (V_1 \otimes V_2)^\vee \\ s &\mapsto f_s \end{aligned}$$

Recall that

$$\begin{aligned} \xi_{ij}(x, y) &= e_i(x)w_j(y) \\ f_{\xi_{ij}}(x \otimes y) &= \xi_{ij}(x, y) = e_i^\vee(x)w_j^\vee(y) \\ f_{\xi_{ij}}(e_k \otimes w_l) &= \begin{cases} 1 & \text{if } (i, j) = (k, l) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

It follows that  $\{e_k \otimes w_l\}_{k,l}$  is a basis of  $V_1 \otimes V_2$

## 45.9 tensor product and duality

### 45.9.1 product

Let  $V_1, \dots, V_n$  be vector spaces as above. Then

$$(V_1^\vee \otimes \dots \otimes V_n^\vee) \cong (V_1 \otimes \dots \otimes V_n)^\vee$$

### Proof

Define

$$\begin{aligned} V_1^\vee \times \dots \times V_n^\vee &\rightarrow \mathcal{L}(V_1, \dots, V_n; K) (\cong (V_1 \otimes \dots \otimes V_n)^\vee) \\ (\varphi_1, \dots, \varphi_n) &\mapsto [(v_1, \dots, v_n) \mapsto \prod \varphi_i(v_i)] \end{aligned}$$

This mapping is multi-linear. It describes by the property of tensor product to a map

$$\begin{aligned} V_1^\vee \otimes \dots \otimes V_n^\vee &\rightarrow \mathcal{L}(V_1, \dots, V_n; K) (\cong (V_1 \otimes \dots \otimes V_n)^\vee) \\ \varphi_1 \otimes \dots \otimes \varphi_n &\mapsto [(v_1, \dots, v_n) \mapsto \prod \varphi_i(v_i)] \end{aligned}$$

By Prop 45.8 these two space have the same  $\dim \prod d_i$ . It enough to show that the mapping is surjective. Let's do it for  $n = 2$  (keep the same notation as above). Take  $\xi_{ij}$

$$\xi_{ij}(x, y) = e_i^\vee(x)w_j^\vee(y) = F(e_i^\vee \otimes w_j^\vee)(x, y)$$

### 45.9.2 duality

Let  $V$  and  $W$  be vector spaces of finite dimension. Then

$$\mathcal{L}(V, W) \cong V^\vee \otimes W^\vee$$

#### Proof

$$\begin{aligned} s : V^\vee \times W &\rightarrow \mathcal{L}(V, W) \\ (\varphi, \omega) &\mapsto [\sigma \mapsto \varphi(\sigma)\omega] \end{aligned}$$

Let's check that  $s$  is bilinear. (note that  $\varphi(\sigma) \in K$ )

$$((\varphi + \psi)(\sigma))\omega = (\varphi(\sigma) + \psi(\sigma))\omega = \varphi(\sigma)\omega + \psi(\sigma)\omega$$

$$\varphi(\sigma)(\omega + \omega') = \varphi(\sigma)\omega + \varphi(\sigma)\omega'$$

Thus map  $s$  is then bilinear. So it induces  $f_s : V^\vee \otimes W \rightarrow \mathcal{L}(V, W)$ . We have to show that this is the required isomorphism.

Let  $\{v_1^\vee, \dots, v_m^\vee\}$  be a basis for  $V^\vee$ , and let  $\{w_1, \dots, w_n\}$  be a basis for  $W$ . Let's see what happens to

$$f_s(v_i^\vee \otimes w_j) = [v_i^\vee \mapsto v_i^\vee(w_j)]w_j = \delta_{ij}w_j$$

Consider the matrix associated to  $f_s$  with respect to the basis.

$$\begin{array}{ccc} (e_1, e_n)E & \xrightarrow{F} & P(p_1, \dots, p_m) \\ \uparrow \scriptstyle b_1 & & \uparrow \scriptstyle b_2 \\ K^n & \xrightarrow{M_F} & K^m \end{array}$$

Call this matrix  $M_{ab}$

$$M_{ab} = \begin{cases} 1 & \text{if } (a, b) = (j, i) \\ 0 & \text{otherwise} \end{cases}$$

The matrices of this form are a basis of  $\mathcal{L}(K^n, K^m) \cong \mathcal{L}(V, W)$

And important case of this prop is when  $V = W$ :

$$\mathcal{L}(V; V) \cong V^\vee \otimes V$$

More in general

$$\begin{aligned}\mathcal{L}(V, W) &\xrightarrow{\cong} V^\vee \otimes W \\ f &\mapsto \sum a_{ij} \sigma_i^\vee \otimes w_j\end{aligned}$$

note that  $\sigma_i^\vee \otimes w_j$  is a basis.

For instance  $V = W$

$$id_V = \mathcal{L}(V, V) \mapsto \sum_i \sigma_i^\vee \otimes \sigma_i$$

### 45.9.3 Exercise

Let  $M, N, P$   $R$ -modules. Show that

$$\mathcal{L}(M \otimes_R N; P) \cong \mathcal{L}(M; \mathcal{L}(N; P))$$

## 45.10 Def

We want to define the tensor product of linear mappings. let  $M_1, M_2, N_1, N_2$  be  $R$ -modules and let  $f_i : M_i \rightarrow N_i$  be linear mappings. Then we define

$$\begin{aligned}f_1 \otimes f_2 : M_1 \otimes M_2 &\rightarrow N_1 \otimes N_2 \\ m_1 \otimes m_2 &\mapsto f_1(m_1) \otimes f_2(m_2)\end{aligned}$$

This is a linear mapping

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{f_1 \times f_2} & N_1 \times N_2 \\ \downarrow & & \downarrow \\ M_1 \otimes M_2 & \xrightarrow{f_1 \otimes f_2} & N_1 \otimes N_2 \end{array}$$

## 45.11 Extension of scalars

Let  $\varphi : R \rightarrow S$  be a commutative unitary ring homomorphism. Let  $M$  be a  $R$ -module. Goal is to give to  $M$  also a structure of  $S$ -module "conveyed by  $\varphi$ "

Note that  $S$  has a structure of  $R$ -module  $s \in S, r \in R$

$$rs := \varphi(r)s$$

Now take the tensor product  $M \otimes_R S$ . Now we give a structure of  $S$ -module to  $M \otimes_R S$ .

Take  $s \in S$

$$s(\underbrace{m \otimes s'}_{\in M \otimes_R S}) := m \otimes ss'$$

note that  $ss'$  is a multi in  $S$  and we cannot product  $sm$ .

Notice we've a mapping

$$\begin{aligned} i : M &\rightarrow M \otimes_R S \\ m &\mapsto m \otimes s \end{aligned}$$

Be careful, in general the mapping  $i$  is NOT injective.

Example  $R = \mathbb{Z}$   $S = \mathbb{Z}/2\mathbb{Z}$   $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$   $M = \mathbb{Z}[X]$

$$i(2X) = 2X \otimes 1 = 2(X \otimes 1) = X \otimes \alpha(2) \cdot 1 = X \otimes 0 = 0$$

## 45.12 Prop

Let  $K \subseteq L$  be a field extension and let  $V$  be a  $K$ -vector space. Moreover let's denote  $V_L = V \otimes_K L$ . If  $\{e_i\}_{i=1}^n$  is a basis of  $V$  then  $\{e_i \otimes 1\}_{i=1}^n$  is a  $L$ -basis of  $V_L$ . ( $V_L$  has the same dim of  $V$ )

### Proof

The set  $\{e_i \otimes 1\}_{i=1}^n$  generates  $V_L$  if fact

$$\sigma \otimes l = \left( \sum \underbrace{\alpha_i}_{K} e_i \otimes \underbrace{l}_L \right) = \sum l \alpha_i (e_i \otimes 1)$$

We have to show that the elements are linearly independent.

$$0 = \sum \alpha_i (e_i \otimes 1) = \sum e_i \otimes \alpha_i \quad \alpha_i \in L$$

( $L$  is a  $K$ -vec space)

Define the mapping with  $\lambda_i \in K$

$$\begin{aligned} b_i : V \times L &\rightarrow L \\ \left( \sum \lambda_i e_i, \beta \right) &\mapsto \lambda_i \beta \end{aligned}$$

This mapping is bilinear. It induces a mapping

$$f_i = f_{b_i} \left( \sum \lambda_i e_i \right) \otimes \beta \mapsto \lambda_i \beta$$

Note that  $f_i(e_j \otimes \beta) = \delta_{ij} \beta$

•

$$f_i \left( \sum_j e_j \otimes \alpha_j \right) = \alpha_i$$

But

$$0 = f_i(0) = f_i \left( \sum_j e_i \otimes \alpha_j \right) = \alpha_i \quad \forall i$$

### 45.13 Remark

As a consequence we have that the mapping  $i : V \rightarrow V_L$  (mapping of  $K$ -vet spaces) is injective.

### 45.14 Exercise

Show that

$$\begin{aligned} V \otimes_K K &\cong V \\ \sigma \otimes a &\mapsto as \end{aligned}$$

### 45.15 Exactness of the tensor product

fix a  $R$ -module  $N$  and consider:

$$_ \otimes N : M \mapsto M \otimes_R N$$

for any  $R$ -module  $M$ .

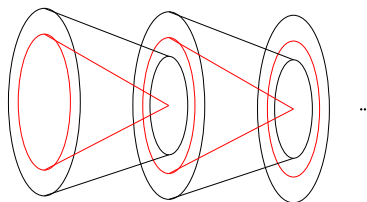
Moreover for any linear mapping  $(f : M \rightarrow P) \rightsquigarrow f \otimes id_N : M \otimes_R N \rightarrow P \otimes_R N$ . This association sends  $id_M$  to  $id_{M \otimes_R N}$  and moreover is well defined with respect to the composition

$$f \circ g \mapsto (f \circ g) \otimes id_N = (f \otimes id_N) \circ (g \otimes id_N)$$

### 45.16 Def

A sequence of  $R$ -modules is a diagram of the following form (also called complex of  $R$ -modules)

$$M_1 \xrightarrow{d^1} M_2 \xrightarrow{d^2} \dots$$



$M_i$  is an  $R$ -module,  $d^i$  is linear mapping and

$$\text{Ker}(d^{i+1}) \supseteq \text{Im}(d^i)$$

Thus we also have:

$$d^{i+1} \circ d^i = 0$$

The diagram is called exact if

$$\text{Ker}(d^{i+1}) = \text{Im}(d^i)$$

take a morphism  $f : M \rightarrow N$  then

- $f$  is injective iff

$$0 \rightarrow M \xrightarrow{f} N$$

is exact

- $f$  is surjective iff

$$M \xrightarrow{f} N \rightarrow 0$$

is exact

The first theorem of homomorphism

$$\bar{f} : M/\text{Ker}(f) \xrightarrow{\cong} \text{Im}(f)$$

can be written as an exact sequence

$$0 \rightarrow \text{Ker}(f) \xrightarrow{i} M \xrightarrow{f} \text{Im}(f) \rightarrow 0$$

More in general sequence like

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

are called short exact sequences.

## 45.17 Prop

$N$  is a  $R$ -module

$$- \otimes_R N : \forall M \quad M \mapsto M \otimes_R N$$

$$f : M \rightarrow P$$

$$f \otimes id_N : M \otimes_R N \rightarrow P \otimes_R N$$

Assume that we have a short exact (also complex) sequence of  $R$ -modules

$$0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$$

Then we apply  $- \otimes_R N$  to whole sequence

$$0 \rightarrow M_1 \otimes_R N \xrightarrow{f \otimes id_N} M_2 \otimes_R N \xrightarrow{g \otimes id_N} M_3 \otimes_R N \rightarrow 0$$

is a complex sequence if it's exact then we call  $N$  a flat  $R$ -module. One significant example's that the free module is flat.



**45.17.1 Example**

$$\begin{array}{ccc}
 0 \rightarrow \mathbb{Z} & \xrightarrow{\mu} & \mathbb{Z} \\
 & & x \mapsto 2x \\
 & & y \mapsto 2\mathbb{Z} + y
 \end{array}
 \quad
 \begin{array}{c}
 \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \rightarrow 0
 \end{array}$$

Now apply  $-\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccc}
 0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\mu \otimes id} & \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \\
 x \otimes (2\mathbb{Z} + y) \mapsto & (2x) \otimes (2\mathbb{Z} + y) & \\
 z \otimes (2\mathbb{Z} + y) \mapsto & (2\mathbb{Z} + z)(2\mathbb{Z} + y) &
 \end{array}$$

and

$$\begin{aligned}
 2x \otimes (2\mathbb{Z} + y) &= 2(x \otimes 2\mathbb{Z} + y) \\
 &= x \otimes (2\mathbb{Z} + 2y) \\
 &= x \otimes 2\mathbb{Z} \\
 &= 0
 \end{aligned}$$

which is not injective, thus above isn't exact.

**45.18 Exercise(important)**

If  $R = N$  then  $-\otimes_R N$  (where  $N$  is a finite dim vec space) is exact. Hint: use the basis.



# Chapter 46

## Tensor algebra

Fix a vec space  $V$  (over  $K$ ) of finite dimension

### 46.1 Def

We denote

$$\begin{aligned} T_p^q &:= (V^\vee)^{\otimes p} \otimes V^{\otimes q} \quad p, q \in \mathbb{N} \\ &= \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{p \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \end{aligned}$$

An element of  $T_p^q(V)$  is called a tensor of type  $(p, q)$  (or a mixed tensor which is  $p$ -covariant and  $q$ -contravariant)

Let's denote:

$$T(V) := \bigoplus_{q \in \mathbb{N}} T_0^q(V)$$

itemize some item in it:

$$\begin{aligned} T_0^0(V) &= K \\ T_1^0(V) &= V^\vee \\ T_0^1(V) &= V \\ T_1^1(V) &= V^\vee \otimes V \cong \mathcal{L}(V; V) \\ T_2^0(V) &= V^\vee \otimes V^\vee \cong (V \otimes V)^\vee \cong \mathcal{L}(V, V; K) \end{aligned}$$

If you have a  $R$ -module  $M$ , then

$$\bigoplus_{n=0}^{\infty} M = \{(m_1, \dots, m_n, \dots) : m_i \in M \text{ all but finite many } m_i = 0\}$$

On  $T(V)$  we have following operation:

$$\begin{aligned} T_0^l(V) \times T_0^q(V) &\rightarrow T_0^{l+q}(V) \\ ((x_1 \otimes \cdots \otimes x_l), (y_1 \otimes \cdots \otimes y_q)) &\mapsto x_1 \otimes \cdots \otimes x_l \otimes y_1 \otimes \cdots \otimes y_q \end{aligned}$$

With this operation  $T(V)$  becomes a  $K$ -algebra. It called the tensor algebra associated to  $V$

## 46.2 exterior product

Let  $W$  be the two sided ideal of  $T(V)$  generated by the element of the type  $x \otimes x$

$$W = \left\{ \sum_{i(finite)} (y_1 \otimes \cdots \otimes y_{m_i}) \otimes (x_i \otimes x_i) \otimes (z_1 \otimes \cdots \otimes z_{n_i}) \right\}$$

With  $x_j, y_j, z_j \in V$  and  $n_i, m_j \in \mathbb{N}$

## 46.3 Def

The quotient algebra

$$\bigwedge(V) := T(V)/W$$

is a  $K$ -algebra, which called the exterior algebra of  $V$

$$\begin{aligned} \pi : T(V) &\rightarrow \bigwedge(V) \\ x_1 \otimes \cdots \otimes x_n &\mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

This def is try to transform  $\otimes$  to  $\wedge$

## 46.4 Notation

$$\bigwedge(V) = \bigoplus_{n \in \mathbb{N}} \bigwedge^n(V)$$

$$\bigwedge^n(V) := T_0^n(V) / (W \cap T_0^n(V))$$

this is called  $n$ -fold exterior product

## 46.5 Prop

Let  $\sigma \in \mathfrak{S}_n$  then

$$x_1 \wedge \cdots \wedge x_n = \text{sgn}(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$$

**Proof**

Since any permutation can be written as the product of adjacent transpositions, it's enough to do the proof for  $\sigma = (i, i+1)$

$$\begin{aligned} 0 &= (x_i + x_{i+1}) \wedge (x_i + x_{i+1}) \\ &= (x_i \wedge x_i) + (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) + (x_{i+1} \wedge x_{i+1}) \\ &= (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) \end{aligned}$$

**46.6 Def**

Let  $E$  be an  $R$ -module and  $f : E^n \rightarrow M$  a mapping. We say that the pair  $(M, f : E^n \rightarrow M)$  satisfies the universal property for the  $n^{\text{th}}$ -exterior power if

- $M$  is an  $R$ -module,  $f : E^n \rightarrow M$  is an  $n$ -linear mapping s.t.

$$\forall i \in \{1, \dots, n-1\}$$

if

$$x_i = x_{i+1}$$

then

$$f(x_1, \dots, x_n) = 0$$

(alternating  $n$ -linear mapping)

- If  $P$  is an  $R$ -module and  $\varphi : E^n \rightarrow P$  is an alternating mapping, then

$$\exists! \Phi : M \rightarrow P \text{ s.t. } \Phi \circ f = \varphi$$

**46.7 Def**

$V$  is a  $K$ -vct space. A multi-linear mapping:

$$\varphi : V \times \dots \times V \rightarrow W$$

is called skew-symmetric(alternating) if

$$\varphi(x_1, \dots, x_n) = 0 \text{ when } \exists i \neq j : x_i = x_j$$

**46.8 Prop**

Let  $V$  be a vct space. For any alternating multi-linear mapping

$$s : \underbrace{V \times \dots \times V}_{n \text{ times}} \rightarrow M$$

when  $M$  is another vct space, there exists a unique linear mapping

$$g_s : \bigwedge^n(V) \rightarrow M$$

such that the following diagram commutes

$$\begin{array}{ccc} V^n & \xrightarrow{s} & M \\ \downarrow t & \nearrow f_s & \\ T_0^n(V) & & \\ \downarrow & \nearrow g_s & \\ \bigwedge^n(V) & & \end{array}$$

**Proof**

$$g_s(\sigma_1 \wedge \cdots \wedge \sigma_n) := s(\sigma_1, \dots, \sigma_n)$$

check the diagram is commutative

$$\mathcal{F}(V^n) \xrightarrow{t} T_0^n(V) \longrightarrow \bigwedge^n(V)$$

$$\{(\sigma_1, \dots, \sigma_n)\} \longmapsto \{\sigma \otimes \cdots \otimes \sigma_n\} \longmapsto \{\sigma_1 \wedge \cdots \wedge \sigma_n\}$$

## 46.9 Remark/exercise

The couple  $\bigwedge^n V$  with

$$V^n \rightarrow \bigwedge^n(V)$$

that satisfies Prop 46.8 is unique to unique isomorphism

## 46.10 Prop

Let  $V$  be a vct space of dimension  $n$  with a basis  $\{e_1, \dots, e_n\}$ . Then  $\bigwedge^k(V)$  is a vct space with a basis given by

$$\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

In particular,  $\bigwedge^k(V)$  has dimension  $\binom{n}{k}$

### 46.10.1 Proof

$\mathcal{B}$  is clearly a generating set. The different part is to show that  $\mathcal{B}$  is made of linearly independent elements.

$$I = \{i_1, \dots, i_k\}$$

with  $1 \leq i_1 < \dots < i_k \leq n$ , define

$$\begin{aligned} \varphi_I : V^n &\rightarrow K \\ (e_{j_1}, \dots, e_{j_n}) &\mapsto \begin{cases} \text{sgn}(t) & \text{if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$\varphi_I$  is multilinear and alternating (skew-symm), hence it induce a linear mapping

$$\begin{aligned} g_{\varphi_I} = \overline{\varphi_I} : \bigwedge^k(V) &\rightarrow K \\ (e_{j_1} \wedge \dots \wedge e_{j_k}) &\mapsto \begin{cases} \text{sgn}(t) & \text{if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

With  $\sigma \in \bigwedge^n(V)$ , assume that

$$0 = \sigma = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1, \dots, j_k} e_{j_1} \wedge \dots \wedge e_{j_k}$$

By linearity

$$0 = \overline{\varphi_I}(\sigma) = \pm \lambda_I$$

Do it for any positive  $I$  this shows that any  $\lambda_{j_1, \dots, j_k}$  is zero.





## Chapter 47

# Determinant

### 47.1 Def

Let  $V$  be a vct space of dimension  $n$ , then

$$\det(V) = \bigwedge^n(V)$$

is called the determinant of  $V$ . It is a vct space of dimension  $1 = \binom{n}{n}$  and a basis is given by

$$e_1 \wedge \cdots \wedge e_n$$

when  $\{e_1, \dots, e_n\}$  is a basis of  $V$ .

#### 47.1.1 Proof

Let  $f \in \mathcal{L}(V; V)$  then consider

$$\begin{aligned} \tilde{f} : V^k &\rightarrow \bigwedge^k V \\ (v_1, \dots, v_n) &\mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

This is multilinear and alternating. Therefore it induces a mapping

$$\begin{aligned} g_{\tilde{f}} = \bigwedge^k f : \bigwedge^k(V) &\rightarrow \bigwedge^k(V) \\ v_1 \wedge \cdots \wedge v_k &\mapsto f(v_1) \wedge \cdots \wedge f(v_k) \end{aligned}$$

Since  $\det(V)$  has  $\dim 1$

$$\det(f) : \sigma_1 \wedge \cdots \wedge \sigma_n \mapsto \underbrace{\det_f}_{\in K}(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$$

By abuse of notation we identity

$$\det(f) = \det_f$$

## 47.2 Prop

$f \in \mathcal{L}(V; V)$  is invertible iff  $\det(f) \neq 0$

### 47.2.1 Proof

$f$  is not invertible iff  $\{f(e_1), \dots, f(e_n)\}$  is not a basis.  
iff there's a non-trivial linear combination

$$\sum_n \lambda_i f(e_i) = 0$$

After relabelling the  $e_i$  we can assume

$$f(e_i) = \sum_{i \geq 2} \mu_i f(e_i)$$

$$\begin{aligned} \det(f)(e_1 \wedge \dots \wedge e_n) &= \det_f \cdot (e_1 \wedge \dots \wedge e_n) \\ &= \left( \sum_{i \geq 2} \mu_i f(e_i) \right) \wedge f(e_1) \wedge \dots \wedge f(e_n) \\ &= \sum_{i \geq 2} \mu_i (f(e_1) \wedge f(e_2) \wedge \dots \wedge f(e_n)) \\ &= 0 \end{aligned}$$

## 47.3 Prop

$$\det(f \circ g) = \det(f) \cdot \det(g)$$

### Proof

$$\begin{aligned} \det(f \circ g) &= (f \circ g)(e_1) \wedge \dots \wedge (f \circ g)(e_n) \\ &= f(g(e_1)) \wedge \dots \wedge f(g(e_n)) \\ &= (\det f)(g(e_1) \wedge \dots \wedge g(e_n)) \\ &= \det f \cdot \det g(e_1 \wedge \dots \wedge g(e_n)) \end{aligned}$$

## 47.4 Prop

The determinant of  $f$  is equal to the determinant of any matrix that represents  $f$  with respect to a fixed basis. This doesn't depend on the choice of the basis.

**Proof**

Fix a basis  $\{e_1, \dots, e_n\}$  of  $V$ . Then

$$\begin{array}{ccc}
 \begin{array}{c} v_i \\ \uparrow \\ e_i \end{array} & \begin{array}{ccc} V & \xrightarrow{f} & V \\ \cong \uparrow \mathcal{B} & & \cong \uparrow \mathcal{B} \\ K^n & \xrightarrow{A_f} & K^n \end{array} & \Longrightarrow & \begin{array}{ccc} \det(V) & \xrightarrow{\det(f)} & \det(V) \\ \uparrow \bigwedge^n b & & \uparrow \bigwedge^n b \\ \det(K^n) & \xrightarrow{\det(A_f)} & \det(K^n) \end{array}
 \end{array}$$

$A_f^{(v_1, \dots, v_n)}$  is the matrix associated to  $f$  with respect to the basis  $\{v_1, \dots, v_n\}$  suppose that  $f(v_i) = \xi_{ij}v_j$ . One we can see

$$A_f = \mathcal{B}^{-1} \circ f \circ \mathcal{B}(e_i)$$

$$\begin{aligned}
 \det(A_f) &= ((a_{11}, a_{12}, \dots, a_{1n}) \wedge (0, a_{22}, a_{23}, \dots, a_{2n}) \wedge \dots \wedge (0, 0, \dots, 1)) \\
 &= \mathcal{B}^{-1}(f(\mathcal{B}(a_{11}, a_{12}, \dots, a_{1n}))) \wedge \dots \wedge \mathcal{B}^{-1}(f(\mathcal{B}(0, 0, \dots, 1))) \\
 &= \xi_{1j}(0, \dots, a_{1j}, \dots, a_{1n}) \wedge \dots \wedge \xi_{nj}(0, \dots, a_{nj}, \dots, a_{nn})
 \end{aligned}$$

(Einstein notation used for  $j$ ) We actually done the thing like

$$\begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots & a_{nn} \end{vmatrix}$$

compare the result with

$$\det(f)(v_1 \wedge \dots \wedge v_n) = \xi_{1j}(v_1) \wedge \dots \wedge \xi_{nj}(v_n)$$

We could find that

$$\det(A_f) = \det(f)$$

Then we got

$$\det(A) = \prod_{i=1}^n a_{ii}$$

**47.5 Prop**

If one column of  $A$  can be expressed as a linear combination of other columns of  $A$ , then

$$\det(A) = 0$$

The columns are images of  $\{e_1, \dots, e_n\}$ , means that  $A(e_1), \dots, A(e_n)$  are linearly dependent. Then  $A$  is not an isomorphism, thus  $\det(A) = 0$ . If we exchange two columns of  $A$ , then  $\det(A)$  changes sign.

### 47.6 Prop

Let  $(a_{ij})$  be a matrix of dimension  $n \times n$ . Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

#### Proof

Let  $\{v_1, \dots, v_n\}$  be the columns of  $A$ ,  $v_i = A(e_i)$

$$\begin{aligned} \det(A)(e_1 \wedge \dots \wedge e_n) &= \sigma_1 \wedge \dots \wedge \sigma_n \\ &= \left( \sum_i a_{i1} e_i \right) \wedge \dots \wedge \left( \sum_i a_{in} e_i \right) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \prod_i a_{i\sigma(i)} \cdot e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(n)} \\ &= \left( \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \right) e_1 \wedge \dots \wedge e_n \end{aligned}$$

### 47.7 Corollary

$$\det A = \det A^T$$

#### Proof

$$A^T = (\alpha_{ij}), A = (a_{ij}) \quad \forall i, j \quad a_{ij} = \alpha_{ji}$$

$$\begin{aligned} \det A^T &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n \alpha_{i\sigma(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i)i} \\ &\stackrel{j=\sigma(i)}{=} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma^{-1}) \prod_{i=1}^n a_{j\sigma^{-1}(j)} \\ &\stackrel{\text{sgn}(\sigma)=\text{sgn}(\sigma^{-1})}{=} \det A \end{aligned}$$

### 47.8 Prop

If you fix some basis on  $V$  and  $W$ , then  $A_f$  is the matrix associated to  $f^T$  is  $A_f^T$

## 47.9 ?

Fix  $A$  of dimension of  $n \times n$ . Apply Gauss reduction and we get  $A'$  a upper-triangle.

By the properties listed above

$$|\det A| = |\det A'|$$

But on  $A'$  the det is just the product of elements on then diagonal

Second method to compare the determinant is to use Gauss reduction and keep track of the row/column exchanges.

## 47.10 Def

Fix  $A = (a_{ij})(i, j) \in \{1, \dots, n\}^2$ . Denote with  $A_{[i,j]}$  the matrix obtained removing the  $i^{th}$  row and  $j^{th}$  column of  $A$ .

## 47.11 Laplace expansion of the determinant

Let  $A = (a_{ij})$  then

$$\begin{aligned} \det A &= \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{[i,j]} \\ &= \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A_{[i,j]} \end{aligned}$$

## Proof

TEDIOUS

$$\begin{array}{ccc} K^n & \xrightarrow{A} & K^n \\ t_j \uparrow & & \downarrow p_i \\ K^{n-1} & \xrightarrow{A_{[i,j]}} & K^{n-1} \end{array}$$

$\{e'_1, \dots, e'_n\}$  is a standard basis of  $K^n$   
 $\{e_1, \dots, e_n\}$  is a standard basis of  $K^{n-1}$   $p_i$  is the mapping that forgets about the  $i$ -th row.

$$p_i = (x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

$$\tau_j(e_i) = \begin{cases} e'_i & \text{if } i < j \\ e_i & \text{if } i \geq j \end{cases}$$

You can check that the above diagram is commutative. Now take  $\bigwedge^{n-1}$  of the diagram

$$\begin{array}{ccc} \bigwedge^{n-1} K^n & \xrightarrow{\bigwedge^{n-1} A} & \bigwedge^{n-1} K^n \\ \bigwedge^{n-1} t_j \uparrow & & \bigwedge^{n-1} p_i \downarrow \\ \det(K^{n-1}) & \xrightarrow{\det(A_{[i,j]})} & \det(K^{n-1}) \end{array}$$

$$\begin{aligned} \det(A)(e'_1, \dots, e'_n) &= (-1)^{i-1} \det(A)(e'_i \wedge e'_1 \wedge \dots \wedge \widehat{e}_i \wedge \dots \wedge e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge A(e'_1) \wedge \dots \wedge A(\widehat{e}_i) \wedge \dots \wedge A(e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \bigwedge^{n-1} A(e'_1, \dots, \widehat{e}_i, \dots, e'_n) = (*) \end{aligned}$$

Let

$$\begin{aligned} \pi_j : K^n &\rightarrow K^n \\ (x_1, \dots, x_n) &\mapsto (0, \dots, x_j, \dots, 0) \end{aligned}$$

Then

$$A = \sum_i (\pi_j \circ A)$$

It means that

$$\begin{aligned} (*) &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A)(e'_1, \dots, \widehat{e}_i, \dots, e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \dots, e_{n-1}) \\ &= \sum_{k,j} \left( (-1)^{i-1} a_{kj} e'_k \wedge \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \dots, e_{n-1}) \right) = (**) \end{aligned}$$

But  $\pi_j(\cdot)$  is always collinear of  $e_j$ , so when  $k = j$ , the element in the sum is zero. We can remove the items that  $k = j$

$$\begin{aligned} \rho_k &:= \tau_k \circ p_k : K^n \rightarrow K^n \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n) \end{aligned}$$

$$\pi_k = id_{K^n} - \rho_k \text{ and } \sum_{j \neq k} \pi_j = \rho_k$$

Then

$$\begin{aligned} (**) &= \sum_k (-1)^{i-1} a_k i e'_k \wedge \bigwedge^{n-1} \tau_k \circ \bigwedge^{n-1} (p_k \circ A \circ \tau_i)(e_1 \wedge \dots \wedge e_{n-1}) \\ &= (***) \end{aligned}$$

But by the diagram

$$\bigwedge_{k=1}^{n-1} (p_k \circ A \circ \tau_i) = \det A_{[i,k]}$$

$$\bigwedge_{k=1}^{n-1} \tau_k(e_1 \wedge \cdots \wedge e_{n-1}) = e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n$$

Thus

$$\begin{aligned} (* * *) &= \sum_k (-1)^{i-1} a_{ki} \det(A_{[k,i]})(e'_k \wedge e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n) \\ &= \sum_k (-1)^{i+k} a_{ki} \det(A[k,i]) e'_1 \wedge \cdots \wedge e'_n \end{aligned}$$





## Chapter 48

# The Structure of Linear Mappings

### 48.1 Theorem

Let  $f : V \rightarrow W$  be a linear mapping between vct spaces of finite and same dim. Then:

- 1 there exists decomposition  $V = V_0 \oplus V_1$  and  $W = W_1 \oplus W_2$  such that  $V_0 = \ker f$  and  $f$  includes an isomorphism between  $V_1$  and  $W_1$  (namely  $f|_{V_1}$ )
- 2 There exists basis in  $V$  and  $W$  s.t. the associated matrix  $A_f = a_{ij}$  satisfies  $\forall 1 \leq i \leq r, \exists r \leq n$  have  $a_{ii} = 1$  and have  $a_{ij} = 0$  elsewhere
- 3 Let  $A$  be a  $m \times n$  matrix Then there exists two square matrices (with  $\det \neq 0$ )  $B$  and  $C$  of dim  $m \times m$  and  $n \times n$  and a num  $r \leq \min(m, n)$  s.t.  $BAC$  has the form in (2) Moreover the number  $r$  is unique  $r = \text{rank}(A)$

### 48.2 Def

Let  $F : V \rightarrow V$  be a linear mapping. A subspace  $V_0 \subseteq V$  is said to be an invariant subspace of  $F$  if  $F(V_0) \subseteq V_0$

### 48.3 Def

A linear mapping  $f : V \rightarrow V$  (finite dim) is diagonalizable if the following equivalent conditions are satisfied

- 1  $V$  decomposes as a direct sum of one-dimensional invariant subspace of  $f$
- 2 There exists a basis of  $V$ , in which the matrix  $A_f$  is diagonal.

**Proof of equivalence**

$2 \Rightarrow 1$  Assume that in the base  $\{v_1, \dots, v_n\}$ , we have  $A_f = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$  by the

familiar diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \uparrow b & & \uparrow b \\ K^n & \xrightarrow{A_f} & K^n \end{array}$$

$$f(v_i) = b \circ A_f(e_i) = b(\lambda_i e_i) = \lambda_i v_i \in \langle v_i \rangle$$

So

$$V = \langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle$$

$1 \Rightarrow 2$  Assume that  $V = \langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle$ , where  $f(\langle v_i \rangle) \subseteq \langle v_i \rangle$ , then  $\{v_1, \dots, v_n\}$  forms a basis of  $V$

Consider the previous diagram

$$A(e_1) = b^{-1} \circ f \circ b(e_i) = b^{-1}(f(v_i)) = b^{-1}(\lambda_i v_i) = \lambda_i e_i$$

**48.3.1 Example**

Take

$$A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$A$  is not diagonalizable.

**48.4 Def**

Let  $L$  be a one-dimensional invariant subspace of  $f : V \rightarrow V$ . Then  $f|_L = x \mapsto \lambda x$  with a scalar  $\lambda \in K$ . Such  $\lambda$  is called eigenvalue of  $f$ . A non-zero vector  $v \in V$  is called an eigenvector of  $V$  if  $\langle v \rangle$  is an invariant subspace of  $f$

**48.5 Remark**

$$\{\text{eigenvectors}\} \longrightarrow \{\text{Set of invariant subspaces of dim 1}\} \longrightarrow K$$

$$v \langle v \rangle \longmapsto \text{eigenvalue}$$

This mapping is generally NOT injective. If  $V$  is an eigenvector, then  $\mu v$  is also an eigenvector,  $\forall \mu \in K$

## 48.6 Remark/exercise

Assume that  $f$  is diagonalizable and  $A_f$  is a diagonal matrix that represents  $f$ . Then  $A_f$  is unique up to permutation of the columns in the diagonal.

$$V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle = \langle v_{\sigma(1)} \rangle \oplus \cdots \oplus \langle v_{\sigma(n)} \rangle \quad \sigma \in \mathfrak{S}_n$$

## 48.7 Def

V a vector space over  $K$   $\dim(V) = n$ ,  $f \in \mathcal{L}(V; V)$  let  $A_f$  be an associated matrix (in any basis) the mapping

$$\begin{aligned} P : K &\rightarrow K \\ t &\mapsto \det(tI_n - A_f) \end{aligned}$$

This is a polynomial in  $K[t]$  (with degree  $n$ )

## 48.8 Lemma

$P(t)$  is a monic polynomial of degree  $n$

### Proof

$$P(t) = \det(tI_n - A_f) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n (t\delta_{i\sigma(i)} - A_{i\sigma(i)})$$

The only item giving  $t^n$  is when  $\sigma = id$

## 48.9 Theorem

Use the notations introduced before

- 1  $P(t)$  doesn't depend on  $A_f$  (if you change basis,  $P(t)$  does not change)
- 2 Any eigenvalue of  $f$  is a root of  $P(t)$ . Conversely any  $K$ -root of  $P(t)$  is an eigenvalue of  $f$

### Proof

- 1 Put  $A = A_f$  and  $A'$  be another representation of  $f$ . Then  $A' = B^{-1}AB$  where  $B$  invertible  $n \times n$  matrix.

$$\begin{aligned} \det(tI_n - A') &= \det(tI_n - B^{-1}AB) \\ &= \det(B^{-1}(tI_n)B - B^{-1}AB) \\ &= \det(B^{-1}(tI_n - A)B) \\ &= \det(tI_n - A) \end{aligned}$$

2 Let  $\lambda \in K$  be a  $K$ -root of  $P(t)$ , then

$$\det(\lambda I_n - A_f) = 0 = P(\lambda)$$

$\lambda I_n - A_f$  is not invertible,  $\exists v \neq 0 \in \ker(\lambda I_n - A_f)$  s.t.

$$A_f(\sigma) = \lambda \sigma$$

then  $\sigma$  is an eigenvector

Vice versa if  $\sigma \neq 0, f(\sigma) = \lambda \sigma, \sigma \in \ker(\lambda I_n - A_f), \det(\lambda I_n - A_f) = 0 = P(\lambda)$

### 48.10 Def

The polynomial  $P(t)$  will be denoted by  $P_f(t)$ . It's called the characteristic polynomial of  $f$

### 48.11 Corollary

If  $P_f(t)$  splits with no repeated roots, then  $f$  is diagonalizable.

#### Proof

Natural

$$\langle \sigma_1 \rangle, \dots, \langle \sigma_n \rangle$$

are all different then

$$V = \langle \sigma_1 \rangle \oplus \dots \oplus \langle \sigma_n \rangle$$

### 48.12 Remark

The inverse version does not hold.

### 48.13 Def: Jordan block

A matrix of form

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \lambda \end{pmatrix} \in M_{r \times r}(K) \quad r \geq 1$$

is called a Jordan block (element  $\lambda \in K$  is  $J_1(\lambda)$ )

### 48.14 Def: Jordan matrix

A Jordan matrix is a matrix of form

$$J = \begin{pmatrix} J_{r_1}(\lambda_1) & & \cdots & \\ & J_{r_2}(\lambda_2) & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

### 48.15 Example

Let  $V_n(\lambda)$  be the vector space of complex functions:

$$F(x) := e^{\lambda x} f(x)$$

where  $\lambda \in \mathbb{C}$ ,  $f \in \mathbb{C}[x] \leq n-1$

Verify that  $V_n(\lambda)$  is a vector space of dim  $n$

$$\begin{aligned} \frac{d}{dx}(e^{\lambda x} f(x)) &= \lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) \\ &= e^{\lambda x} (\lambda f(x) + f'(x)) \end{aligned}$$

$\frac{d}{dx} \in \mathcal{L}(V_n(\lambda); V_n(\lambda))$  Consider

$$v_{i+1} = \frac{x^i}{i!} e^{\lambda x}$$

Show that  $\{v_0, \dots, v_{n-1}\}$  forms a basis of  $V_n(\lambda)$

$$\begin{aligned} \frac{d}{dx} v_{i+1} &= \lambda v_{i+1} + \frac{x^{i-1}}{(i-1)!} e^{\lambda x} \\ &= \lambda v_{i+1} + v_i \end{aligned}$$

Then

$$A_{\frac{d}{dx}} = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix} = (J_n(\lambda))^T$$

### 48.16 Def

Let  $a_0 + a_1 t + \dots + a_n t^n = Q(t) \in K[t]$ , then for  $f \in \mathcal{L}(V; V)$  we define

$$Q(f) := a_0 \text{id}_V + a_1 f + a_2 f^{\circ 2} + \dots + a_n f^{\circ n}$$

Remark From now on we write

$$f^{\circ k} = f^k$$

these are operations in  $\mathcal{L}(V; V)$ ,  $+$ ,  $\circ$

we say that  $Q$  annihilates  $f$  if  $Q(f) = 0$

**48.17 Prop**

Let  $f \in \mathcal{L}(V; V)$ . There exists a polynomial  $Q \in K[t] \setminus \{0\}$  that annihilates  $f$  (i.e.  $Q(f) = 0$ )

**Proof**

$$\dim(\mathcal{L}(V; V)) = n^2$$

Hence the mapping  $\underbrace{id_V, f^2, \dots, f^{n^2}}_{n^2+1 \text{ mappings}} \in \mathcal{L}(V; V)$  are linear dependent. There exists a non-trivial linear comb:

$$\lambda_0 id_V + \lambda_1 f + \dots + \lambda_{n^2} f^{n^2} = 0$$

So, take

$$Q(t) = \lambda_0 + \lambda_1 t + \dots + \lambda_{n^2} t^{n^2}$$

This show that  $Q \neq 0$  and  $Q(f) = 0$

**Remark**

The proof of this proposition also gives the degree of a polynomial that annihilates ( $\leq n^2$ )

**48.18 Def**

Let  $m(t) \in K[t] \setminus \{0\}$  be a monic polynomial of minimal degree that annihilates  $f \in \mathcal{L}(V; V)$ . Then  $m(t)$  is called minimal polynomial of  $f$

And by prop above (48.17),  $m(t)$  exists.

**48.19 Prop**

If  $m(t)$  is minimal polynomial of  $f$ , then  $m(t)$  is unique.

**Proof**

Assume that  $m_1(t)$  is another minimal polynomial of  $f$ . Then  $m - m_1(t) \in K[t]$

$$(m - m_1)(f) = m(f) - m_1(f) = 0 - 0 = 0$$

Now  $m$  and  $n$  are both monic, so

$$\deg(m - m_1) < \deg(m) = \deg(m_1)$$

$m - m_1$  is a polynomial of  $\deg < \deg(m)$  that annihilates  $f$ , thus

$$m - m_1 = 0 \in K[t]$$

## Notation

From now we denote the minimal polynomial of  $f$  by  $m_f$

## Question

$f \in \mathcal{L}(V; V)$  we have  $P_f, m_f \in K[t]$ .

What is the relationship between  $P_f$  and  $m_f$ ?

## 48.20 Prop

Let  $Q \in K[t] \setminus \{0\}$  be a polynomial that annihilates  $f$ . Then  $m_f \mid Q$

## Proof

Let

$$Q(t) = m_f(t) \cdot s(t) + \mathfrak{z}(t)$$

such that  $\deg(\mathfrak{z}) < \deg(m_f)$ . So

$$0 = Q(f) = m_f(f)s(f) + \mathfrak{z}(f) = 0 + \mathfrak{z}(f) \Rightarrow \mathfrak{z}(f) = 0$$

But since  $m_f$  is the minimal polynomial of  $f$ , then

$$\mathfrak{z}(t) = 0$$

## 48.21 Def

Let  $A$  be a matrix of dim  $n \times n$  and

$$M_{ij} := (-1)^{i+j} \det(A_{[i,j]}) \quad \forall (i, j) \in \{1, \dots, n\}^2$$

In this expression

$$\det(A_{[i,j]})$$

is called the  $(i, j)$ -monic of  $A$ .

Then we define

$$\text{Adj}(A) := (M_{ij})^T$$

called adjugate matrix of  $A$

## 48.22 Prop

$$\text{Adj}(A) \cdot A = A \cdot \text{Adj}(A) = \det(A) \cdot I_n$$

**Proof**

use Laplace expansion.

**48.23 Theorem: Cayley-Hamilton Theorem**

The characteristic polynomial  $P_f$  annihilates  $f$

Consequence:  $m_f \mid P_f$

**Proof**

Let  $A = A_f$  any matrix that represents  $f$ . Consider

$$B := \text{Adj}(tI_n - A)$$

$B$  is a matrix with coefficient in  $K[t]$  ( $B \in M_{n \times n}(K[t])$ )

Then

$$(tI_n - A) \cdot B = \det(tI_n - A) \cdot B = P_f(t) \cdot I_n$$

We can decompose  $B$  in the following way

$$B = \sum_{i=0}^{n-1} t^i B_i \quad B_i \in M_{n \times n}(K)$$

We have at most  $n-1$ , because the coefficient of  $B$  have degree at most  $n-1$   
(Any entry  $\text{Adj}$  is a det of a matrix of dim  $(n-1) \times (n-1)$ )

$$\begin{aligned} P_f(t)I_n &= (tI_n - A) \cdot \sum_{i=0}^{n-1} t^i B_i \\ &= \left( \sum_{i=0}^{n-1} tI_n \cdot t^i B_i \right) - \left( \sum_{i=0}^{n-1} A \cdot t^i B_i \right) \\ &= \sum_{i=0}^{n-1} t^{i+1} B_i - \sum_{i=0}^{n-1} A \cdot t^i B_i \\ &= t^n B_{n-1} + \sum_{i=0}^{n-1} t^i (B_{i-1} - AB_i) - AB_0 \end{aligned}$$

Recall that  $P_f(t) \cdot I_n = t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_1 t I_n + c_0 I_n$

$$\begin{aligned} &t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_0 I_n \\ &= \cdots \\ &= t^n B_{n-1} + \sum_{i=1}^{n-1} t^i (B_{i-1} - AB_i) - AB_0 \end{aligned}$$



Then we can compare the coefficients:

$$\begin{aligned} B_{n-1} &= I_n \\ B_{i-1} - AB_i &= c_i I_n \quad 1 \leq i \leq n-1 \\ -AB_0 &= c_0 I_n \end{aligned}$$

Multiply by  $A^i$   $0 \leq i \leq n$

$$A^n B_{n-1} + \sum_{i=1}^{n-1} (A^i B_{n-1} - A^{i+1} B) - AB_0 = A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I_n$$

Now the LHS we have a telescopic sum and got

$$0 = P_f(A) \Leftrightarrow 0 = P_f(f)$$

## 48.24 Example

(a)  $m_f$  and  $P_f$  are in general different. let  $f = id_V$ , ( $\dim V = n$ )

$$P_f(t) = (t-1)^n \quad m_f(t) = t-1$$

(b) Assume  $f : V \rightarrow V$  ( $\dim V = \mathfrak{z}$ ) and  $A_f = J_{\mathfrak{z}}(\lambda)$ . Then

$$P_f(t) = (t-\lambda)^{\mathfrak{z}}$$

Moreover

$$J_{\mathfrak{z}}(\lambda) = \lambda I_{\mathfrak{z}} + J_{\mathfrak{z}}(0)$$

and

$$J_{\mathfrak{z}}(0)^k = \begin{pmatrix} \overbrace{0 \cdots 0}^{k+1} & & \\ & \ddots & \\ & & 1 \\ & & \vdots \\ & & 0 \end{pmatrix}$$

if  $k \geq \mathfrak{z}$ ,  $J_{\mathfrak{z}}(0)^k = 0$

$$(J_{\mathfrak{z}}(\lambda) - \lambda I_n)^k = (\lambda I_{\mathfrak{z}} - J_{\mathfrak{z}}(0) - \lambda I_{\mathfrak{z}})^k = J_{\mathfrak{z}}(0)^k \neq 0$$

if  $0 \leq k \leq \mathfrak{z}-1$

We know that  $m_f \mid (t-\lambda)^{\mathfrak{z}}$  (by Cayley-Hamilton),  $m_f$  must be of the type

$$m_f = (t-\lambda)^k$$

But the only possibility is  $k = \mathfrak{z}$ , thus

$$m_f = P_f$$

### 48.25 Theorem

Let  $f \in \mathcal{L}(V; V)$  when  $V$  is a vector space of dim  $n$ , over an algebraically closed field.

Then

- (1)  $f$  can be represented by a Jordan matrix
- (2) This above matrix is unique up to permutation of the Jordan blocks

(Note that a field  $K$  is algebraically closed if any non-zero polynomial has a root in  $K$ )

### 48.26 Def

Let  $f \in \mathcal{L}(V; V)$  and let  $\lambda \in K$ . A vector  $w \in V \setminus \{0\}$  is called a root vector of  $f$  corresponding to  $\lambda$ , if there exists  $\tau \in \mathbb{N}$  s.t.

$$(f - \lambda id_V)^\tau(w) = 0$$

#### Remark

Eigenvector are root vectors (corresponding to their eigenvalues) take  $\tau = 1$

#### Remark

Let  $J_\tau(\lambda)$  be a Jordan block. Then any  $\sigma \in V$  is a root vector of  $f$  corresponding to  $\lambda$ . In fact:

$$(J_\tau(\lambda) - \lambda I_n)^m = 0 \quad \text{if } m \geq \tau$$

### 48.27 Prop

Let  $K$  be an algebraically closed field. Let  $\lambda_1, \dots, \lambda_k$  be all of distinct eigenvalues of  $f$  ( $k \geq 1$ ), then

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

#### Proof

Since  $K$  is algebraically closed, then

$$P_f(t) = \prod_{i=1}^k (t - \lambda_i)^{r_i} \in K[t]$$

Consider

$$F_i(t) := P_f(t) \cdot (t - \lambda_i)^{-r_i} \in K[t]$$

Then we define

$$f_i := F_i(f) \in \mathcal{L}(V; V), V_i = \text{Im} f_i$$

### Setp 1

We want to prove that

$$(f - \lambda_i \text{Id}_V)^{\circ r_i}(V_i) = 0 \Leftrightarrow V_i \subseteq V(\lambda_i)$$

which got from

$$(f - \lambda_i \text{Id}_V)^{\circ r_i} \circ (f_i) = (t - \lambda_i)^{r_i}(f) \circ F_i(f) = P_f(f) = 0$$

### Step 2

We want to prove that

$$V = \bigoplus_{i=1}^k V_i$$

Since the polynomials  $F_i(t)$  are coprime, then

$$\exists G_i(t) \in K[t] \text{ s.t. } \sum_{i=1}^k F_i(t)G_i(t) = 1$$

Let  $f$  substitute for  $t$

$$\sum_{i=1}^k F_i(f)G_i(f) = \text{Id}$$

take  $v \in V$

$$\sum_{i=1}^k f_i \circ G_i(f)(v) = v$$

$$\begin{array}{ccccccc}
 & & & \bigoplus_{i=1}^k V_i & & & \\
 & & \nearrow & & \nwarrow & & \\
 & V_1 & & & & V_{k-1} & V_k \\
 & \uparrow & \nearrow & & \nwarrow & \uparrow & \uparrow \\
 & f_1 \circ G_1(f) & f_2 \circ G_2(f) & & f_{k-1} \circ G_{k-1}(f) & f_k \circ G_k(f) & \\
 & \uparrow & \uparrow & & \uparrow & \uparrow & \\
 V & = & V & = & \dots & = & V & = & V
 \end{array}$$

$i$  is the inclusion mapping.

**Step 3**

We want to show that

$$V_i \cap \left( \sum_{j \neq i} V_j \right) = \{0\}$$

Let  $v$  be a vector in this intersection. Then by calculation,

$$(f - \lambda_i)^{r_i}(v) = 0$$

$$F_i(f)(v) = \prod_{j \neq i} (f - \lambda_j) Id^{or_i}(v) = 0$$

Now  $(t - \lambda_i)^{r_i}$  and  $F_i(t)$  are coprime. Then there exists  $G_1(t)$  and  $G_2(t)$  such that:

$$G_1(t)(t - \lambda_i)^{t_i} + G_2(t)F_i(t) = 1$$

substitute  $f$  instead of  $t$  by

$$G_1(f) \circ (f - \lambda_i Id_V)^{or_i} + G_2(f) \circ F_i(f) = Id_V$$

Then apply to  $v = \sum_{j \neq i} v_j, v_j \in V_j$

$$G_1(f) \circ (f - \lambda_i Id_V)^{or_i}(v) + G_2(f) \circ F_i(f)(v) = v = 0$$

**Step 4**

We want to show that

$$V_i = V(\lambda_i)$$

By step 1 we get

$$V_i \subseteq V(\lambda_i)$$

Take  $v \in V(\lambda_i)$ , write it as

$$v = v'(\in V(\lambda_i)) + v''(\in \bigotimes_{j \neq i} V_j)$$

By step 3,

$$v'' = v - v' \in V(\lambda_i)$$

Use same trick, substitute  $f$  for  $t$  and calculate in  $v''$

$$v'' = 0$$

**48.28 Def**

Let  $f \in \mathcal{L}(V; V)$ . Then  $f$  is said to be nilpotent if there exists  $t \in \mathbb{N}$  that  $f^t = 0$

**48.29 Lemma**

Let  $f$  be a nilpotent mapping, then

$$\text{Ker}(f) = \{\text{set of eigenvectors of } f\} \cup \{0\}$$

**Proof**

Let  $v \in \text{Ker}(f)$  then  $v$  is an eigenvector with eigenvalue  $= 0$

Let  $v$  be an eigenvector, then  $\forall m \geq r$

$$0 = f^m(v) = f^{m-1}(f(v)) = f^{m-1}(\lambda v) = \lambda^m v \Rightarrow \lambda^m = 0 \Rightarrow \lambda = 0$$

**48.30 Lemma**

Let  $f$  be a nilpotent mapping, then  $\text{Ker}(f) \neq \{0\}$

**Proof**

Let  $\tau$  be the minimal integer s.t.  $f^\tau = 0$  then

$$f^{\tau-1}(V) \subseteq \text{Ker}(f)$$

but  $f^{\tau-1}(V) \neq \{0\}$  because of the minimality of  $\tau$

**Remark**

Another way to prove is that  $Q(t) = t^\tau$  annihilates  $f$ . So  $m_p = t^{\tau'}, \tau' \leq \tau$   
Note that 0 is a root of  $m_f$ , by Cayley-Hamilton theorem, 0 is an eigenvalue  
 $f(x) = 0 \cdot x = 0$  for some  $x \neq 0$

**48.31 Jordan matrix of form  $J_\tau(0)$** 

Recall that

$$J_\tau(0)^k = 0 \text{ if } k \geq \tau$$

Then the Jordan matrix

$$\begin{pmatrix} J_{\tau_1}(0) & & \\ & J_{\tau_2}(0) & \\ & & \ddots \end{pmatrix}$$

Are nilpotent mappings since each block is nilpotent. Take one block

$$J_\tau(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} = \begin{cases} e_1 \mapsto 0 \\ e_2 \mapsto 1 \\ \vdots \\ e_\tau \mapsto e_{\tau-1} \end{cases}$$

We represent the action of a Jordan block on a basis as the following diagram

$$\underbrace{e_\tau \rightarrow e_{\tau-1} \rightarrow e_{\tau-2} \rightarrow \cdots \rightarrow e_1 \rightarrow 0}_{\text{lenth of the block}(\tau)}$$

$e_1$  is the one which mapped to 0 (thus an eigenvector)

Given  $f \in \mathcal{L}(V; V)$  if we find a basis on which  $f$  acts as in the previous diagram. Then we have found a Jordan basis made of blocks of the type " $J_\tau(0)$ "

### 48.32 Theorem

Let  $f \in \mathcal{L}(V; V)$  be a nilpotent mapping, then there exists a Jordan basis for  $f$  that gives a Jordan matrix made of blocks of the type  $J_\tau(0)$

#### Proof

We need to find a basis that induces a diagram of the type  $\mathcal{D}$  :(dots in the diagram are basis)

$$\begin{array}{ccccccc} \cdot & & & & & & \cdot \\ \downarrow & & & & & & \downarrow \\ \cdot & \cdot & & & & & \cdot \\ \vdots & \downarrow & & & & & \vdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & & \cdot \\ \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\ \cdot & \cdot & \cdot & & \cdot & & \cdot \\ \downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow \\ 0 & 0 & 0 & & 0 & & 0 \end{array}$$

(Last line of dots naturally be eigenvectors)

We work by induction on  $\dim(V)$ . If  $\dim(V) = 1$ , then

$$f = \mu(\cdot), f^\tau = 0 \quad \mu^\tau v = 0 \quad \forall v \Rightarrow \mu = 0$$

But  $0 = J_1(0)$ . Assume that the theorem is true for  $\dim(V) < n$  Let

$$V_0 = \text{Ket } f = \{\text{the set of eigenvalues}\} \cup \{0\}$$

Since  $f$  is nilpotent

$$\dim(V_0) \geq 1$$

. Therefore

$$\dim(V/V_0) < n$$

So define the following mapping

$$\begin{aligned}\bar{f}: V/V_0 &\rightarrow V/V_0 \\ \bar{\sigma} = V_0 + \sigma &\mapsto V_0 + f(v) = \overline{f(v)} \\ \bar{f} \cdot \bar{\sigma} &\mapsto \overline{f(v)}\end{aligned}$$

is nilpotent We use the induction hypothesis

We have a Jordan basis for  $\bar{f}$ , so we have elements  $\bar{\sigma}_1, \dots, \bar{\sigma}_m \in V/V_0$  that give a diagram  $\overline{\mathcal{D}}$ :

Now left  $\bar{\sigma}_i$  to some element  $\sigma_i \in V$  choose  $\sigma_i \in V$  s.t  $\sigma + V_0 = \bar{\sigma}_i$  Now start applying  $f$  to these elements  $\sigma_i \neq 0$

$$v_i \rightarrow f(v_i) \rightarrow \dots \rightarrow f^{b_i-1}(v_i) \rightarrow f^{b_i}(v_i)$$

When  $b_i$  is the first integer such that

$$\bar{f}^{b_i}(\bar{v}_i) = 0$$

This means that

$$f^{b_i}(v_i) \in V_0$$

hence  $f^{b_i}(\sigma_i)$  is an eigenvalue for ? Consider now the vector subspace generated by  $f^{b_1}(v_1), f^{b_2}(v_2), \dots, f^{b_m}(v_m)$

$$\langle f^{b_1}(v_1), \dots, f^{b_m}(v_m) \rangle \subseteq V_0$$

Extract a basis and complete to a basis of  $V_0$ . The new vectors are denoted by  $u_1, \dots, u_t$

We want to prove that the elements of  $\mathcal{D}$  form a basis of  $V$

**1**

The elements of  $\mathcal{D}$  generate  $V$  let  $\sigma \in V$

$$\bar{\sigma} = \sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} \bar{f}^j(\bar{v}_i)$$

Now I use the properties of  $\bar{f}$

$$\begin{aligned}\bar{f}(\bar{v}_i) &= \overline{f(v_i)} \\ \bar{f}(f(v)) &= \overline{f(f(v))}\end{aligned}$$

then

$$\bar{\sigma} = \overline{\sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} f^j(v_i)}$$

which gives

$$\sigma - \sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} f^j(v_i) \in V_0$$

this finishes. We know that

$$V_0 = \langle f^{b_1}(v_1), \dots, f^{b_m}(v_m), u_1, \dots, u_t \rangle$$

## 2

We need to prove that the elements of  $\mathcal{D}$  are linearly independent

a We show that the elements of the bottom row are linearly independent

$$\sum_{i=1}^m a_i f^{b_i}(v_i) + \sum_{i=1}^t c_i u_t = 0$$

This is a non-trivial linear comb.

The first observation is that  $b_i = 0$ . Because if  $b_j \neq 0$

$$u_j = \frac{\sum_{i=1}^m a_i f^{b_i}(v_i)}{\sum_{i=1}^t c_i}$$

But  $u_1, \dots, u_t$  were an extension of a basis. So

$$0 = \sum_{i=1}^m a_i f^{b_i}(v_i) = f\left(\sum_{i=1}^m a_i f^{b_i-1}(v_i)\right) \Rightarrow \left(\sum_{i=1}^m a_i f^{b_i-1}(v_i)\right) \in V_0$$

It means that

$$\sum_{i=1}^m a_i \bar{f}^{b_i}(v_i) = 0 \Rightarrow a_i = 0 \forall i$$

b If there is a non-trivial linear comb that equals to 0. For elements of  $\mathcal{D}$ .

We can write it as linear comb of elements of the last row

$$f\left(\sum_{i=1}^m \sum_{j=1}^{b_i} a_{ij} f^j(v_i) + \sum_{i=1}^t c_i u_t\right) = 0$$

By applying  $f$  many times we get a linear comb of elements of the last row.

By point a, finished.

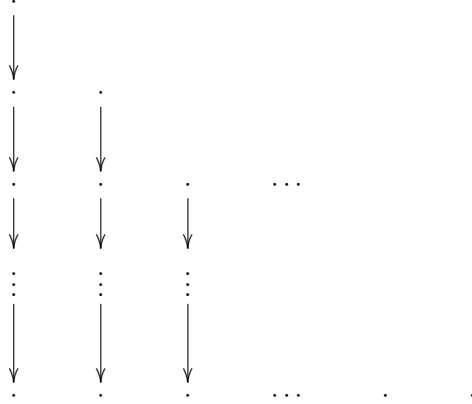
## 48.33 Prop

The Jordan matrix that represents a nilpotent mapping  $f \in \mathcal{L}(V)$  is unique to permutations of the blocks.



**Proof**

Recall that a Jordan basis of  $f$  is given by diagram of the type  $\mathcal{D}$



These columns are ordered in a decreasing height on them, recalling that the height of a column is the dimension of a Jordan block. In the proof of existence of Jordan basis, the diagram was constructed as a lift of  $\mathcal{D}$

Focus on the last row. The elements of last row generates  $V_0 = \ker f$  and moreover, they are linearly independent. Then the length of the last row is exactly  $\dim(V_0)$ , which is independent of the choice of basis.

Viewing the penultimate row, this corresponds to the last row of the diagram  $\overline{\mathcal{D}}$ . So if we work by induction, we done the proof:

All the rows have length independent of the choice of basis.

**Remark**

$$\ker(f^{\circ 3})/\ker(f^{\circ 2}) \rightarrow \ker(f^{\circ 2})/\ker(f) \rightarrow \ker f = V_0$$

**48.34 Lemma**

Let  $f \in \mathcal{L}(V)$ ,  $\lambda$  be an eigenvalue of  $f$ . Then there exists  $r \in \mathbb{N}$  s.t.

$$\forall v \in V(\lambda) \quad (f - \lambda Id)(v) = 0$$

**Proof**

Take a basis  $\{v_1, \dots, v_n\}$  of  $V(\lambda)$ . By definition, we have  $(r_1, \dots, r_n)$  such that  $\forall i$   $r_i$  is the least integer that

$$\forall v \in V \quad (f - \lambda Id)^{or}(v) = 0$$

Take  $r = \max\{r_i\}$ , then proved by calculation.

### 48.35 Theorem

Let  $K$  be an algebraically closed field. Let  $f \in \mathcal{L}(V)$ . Then  $f$  admits a Jordan basis (namely there exists a basis s.t.  $A_f$  is a Jordan matrix).

#### Proof

Since  $K$  is algebraically closed, by Prop 48.27

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

where  $\lambda_i$  are distinct eigenvalues of  $f$

Recall that  $V(\lambda_i)$  is the set of root vectors for  $\lambda_i$  and 0

Consider  $f|_{V(\lambda_i)} = g, \lambda_i = \lambda$ . Only need to prove the theorem for  $g$

$$(g - \lambda Id) : V(\lambda) \rightarrow V(\lambda)$$

This function is nilpotent on  $V(\lambda)$  by definition. By lemma 48.34, we have some  $J_{g-\lambda Id}$  made of blocks of the type  $J_{g-\lambda Id}(0)$

Take the matrix and restrict to  $J_r(0)$

$$g - \lambda Id = BJ_r(0)B^{-1}$$

One see that

$$\lambda Id + BJ_r(0)B^{-1} = B\lambda IdB^{-1} + BJ_r(0)B^{-1} = B(\lambda Id + J_r(0))B^{-1}$$

Uniqueness follows the uniqueness of  $J_r(0)$

## Chapter 49

# Jordan Matrix

To find relations between Jordan matrix and diagonal representations

### 49.1 Def

Let  $\lambda$  be an eigenvalue of  $f \in \mathcal{L}(V)$

$$E(\lambda) := \ker(f - \lambda Id)$$

This  $E(\lambda)$  is called the eigenspace of  $\lambda$

$$mult(\lambda)_{geo} = \dim(E(\lambda))$$

is called the geometric multiplicity of  $\lambda$

Moreover

$$mult(\lambda)_{alg} = \max \{k \in \mathbb{N} \mid (t - \lambda)^k \mid P_f(t)\}$$

is called the algebraic multiplicity of  $\lambda$

### 49.2 Prop

Let  $K$  be algebraically closed. Then  $\forall \lambda$  eigenvalues of  $f$

$$mult(\lambda)_{geo} \leq mult(\lambda)_{alg}$$

**Proof**

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

Take  $\lambda = \lambda_i$ . Let  $J_f$  be the Jordan matrix of  $f$ . Then

$$\det J_f = \det f$$

so

$$P_f(t) = \prod_i (t - \lambda_i)^{\dim(V(\lambda_i))} \Rightarrow \dim(V(\lambda)) = \text{mult}(\lambda)_{alg}$$

### 49.3 Corollary

Let  $K$  be an algebraically closed field. Let  $f \in \mathcal{L}(V)$ .  $f$  is diagonalizable iff

$$\forall \lambda_i \quad \text{mult}(\lambda_i)_{geo} = \text{mult}(\lambda_i)_{alg}$$

# Chapter 50

## Inner Product

### 50.1 Def

Two matrices  $G, G' \in M_{n \times n}(K)$  are said conjugate if  $\exists A \in \mathcal{Q}_{n \times n}(K)$  s.t.  
 $G = G'^T$

#### Exercise

Verify that this is an equivalence relation

### 50.2 Def

Let  $V$   $n$ -dimensional vector space over  $K$  ( $K = \mathbb{R}$  or  $K = \mathbb{C}$ ),  $g \in \mathcal{L}(V, V; K)$  is said a bilinear form. Choose a basis  $\{v_1, \dots, v_n\}$  of  $V$ . The matrix

$$G = (g(v_i, v_j))_{ij} \in M_{n \times n}(K)$$

is called the Gram matrix of  $g$  with respect to  $\{v_1, \dots, v_n\}$

By bilinearity,  $G$  determinant uniquely  $g$

$$x = \sum \alpha_i v_i \rightarrow x = \sum \alpha_i e_i \quad x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$x, y \in V$

$$g(x, y) = g\left(\sum x_i v_i, \sum y_j v_j\right) = \sum_{i,j} x_i y_j g(v_i, v_j) = x^T G y$$

On the other hand, given a basis  $\{v_1, \dots, v_n\}$  and  $G \in M_{n \times n}(K)$  the mapping:

$$\begin{aligned} V \times V &\rightarrow K \\ (x, y) &\mapsto x^T G y \end{aligned}$$

this is a bilinear form and the associated Gram matrix is exactly  $G$

Fix a couple  $(V, \{v_1, \dots, v_n\})$  we have defined a bijection.

$$\begin{array}{ccc} \mathcal{L}(V, V; K) & \xrightarrow{\cong} & K \\ g & \mapsto & G \end{array}$$

What happens if  $g$  is fixed but we change basis. We have also  $\{v'_1, \dots, v'_n\}$

$$\begin{array}{ccccc} & & v_i & & V & & v'_i & & \\ & \nearrow & & \nearrow b & & \nwarrow b' & & \nwarrow & \\ e_i & & K^n & \xleftarrow{A} & K^n & & e_i & & \end{array}$$

$$A = b^{-1} \circ (b') \quad (b')^{-1}(x) = x'$$

then  $A$  satisfies

$$Ax' = x$$

so

$$g(x, y) = x^T G y = (Ax')^T G (Ay') = (x')^T (A^T G A)(y')$$

The new Gram matrix with respect to the basis  $\{v'_1, \dots, v'_n\}$  is  $A^T G A$

### 50.3 Prop

There exists a surjection:

$$\mathcal{L}(V, V; K) \rightarrow M_{n \times n}(K) / \sim_{conj}$$

### Proof

Recall

$$\begin{array}{ccccc} \mathcal{L}(V, V; K) & \rightarrow & \mathcal{L}(T_0^2(V); K) & \rightarrow & \mathcal{L}(V; V^\vee) \\ g & \mapsto & g_s & \mapsto & [x \mapsto g_s(x \otimes -)] = \tilde{g} \end{array}$$

### 50.4 Def

Given  $g \in \mathcal{L}(V, V; K)$  we can define several other bilinear mappings:

$$\begin{array}{ccc} g_p : V \times V & \rightarrow & K \\ (x, y) & \mapsto & g(y, x) \end{array}$$

$$\begin{array}{ccc} \overline{g_p} : V \times V & \rightarrow & K \\ (x, y) & \mapsto & \overline{g(y, x)} = \overline{g_p(x, y)} \end{array}$$

If  $K = \mathbb{R}$  then  $g_p = \overline{g_p}$

## 50.5 Def

A bilinear form  $g$  is said

Symmetric if  $g = g_p$

Symplectic (skew-symmetric) if  $g = -g_p$

hermitian if  $g = \overline{g_p}$

(if  $K = \mathbb{R}$  symmetric  $\neq$  hermitian)

### 50.5.1 Example

$$\begin{array}{ccc} K^n \times K^n & \rightarrow & K \\ (x, y) & \mapsto & x^T y \end{array}$$

is symmetric

$$\begin{array}{ccc} K^2 \times K^2 & \rightarrow & K \\ (v_1, v_2) & \mapsto & \det(v_1 \mid v_2) \end{array}$$

is skew-symmetric

$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \rightarrow & \mathbb{C} \\ (x, y) & \mapsto & x^T \overline{y} \end{array}$$

is hermitian

## 50.6 Def

$g \in \mathcal{L}(V, V; K)$  is an inner product of  $V$ , if  $g$  is either symmetric, symplectic or hermitian.

And  $(V, g)$  is called an inner space.

(note that  $g = -\overline{g_p}$  is complicated)

## 50.7 Def

Let  $(V, g)$  be an inner product space. Two vectors  $v_1, v_2 \in V$  are said orthogonal (with respect to  $g$ ) if  $g(v_1, v_2) = 0$ . Two subspace  $V_1, V_2 \subseteq V$  are orthogonal if  $g(v_1, v_2) = 0 \forall v_1 \in V_1, v_2 \in V_2$  ( $g(V_1, V_2) = 0$ )

### Exercise

Show the following

- If  $g$  is symmetric

$$G = G^T$$

- If  $g$  is symplectic

$$G = -G^T$$

- If  $g$  is hermitian

$$G = \overline{G^T}$$

## 50.8 Def

Let  $(V_g)$  be an inner product space the kernel of  $g$

$$\ker(g) := \{v \in V \mid g(v, w) = 0 \ \forall w \in V\}$$

Moreover  $g$  is said non-degenerated if

$$\ker(g) = \{0\}$$

## 50.9 Remark

Note that  $\ker(g) = \ker(\tilde{g})$  when

$$\tilde{G} \in \mathcal{L}(V; V^\vee)$$

$$\tilde{g}_x = 0 \Leftrightarrow g(x, y) = 0 \ \forall y \in V$$

This implies that  $\ker(g)$  is a linear subspace of  $V$



# Chapter 51

## Differential Forms in $\mathbb{R}^n$

### 51.0.1 Notation

$$a|_p := (p, a)$$

### 51.1 Def

Let  $p \in \mathbb{R}^n$  be a fixed point

$$\mathbb{R}_p^n := \{p\} \times \mathbb{R}^n$$

$$(p, a) \in \mathbb{R}_p^n, a \in \mathbb{R}^n$$

$$(p, a) + (p, b) = (p, a + b)$$

$$\alpha(p, a) = (p, \alpha a) \quad \alpha \in \mathbb{R}$$

With these operation  $\mathbb{R}_p^n$  is a vector space, which is called the tangent space of  $\mathbb{R}^n$  at  $p$ .

The dual space is

$$(\mathbb{R}_p^n)^\vee = \{p\} \times (\mathbb{R}^n)^\vee$$

A basis of  $\mathbb{R}_p^n$  is denoted by

$$(e_1|_p, \dots, e_n|_p)$$

$\bigsqcup_p \mathbb{R}_p^n$  is called the tangent bundle of  $\mathbb{R}^n$

We have a projection mapping:

$$\begin{aligned} \bigsqcup_p \mathbb{R}_p^n &\xrightarrow{\pi} \mathbb{R}^n \\ (p, a) &\mapsto p \end{aligned}$$

and

$$\begin{aligned}\mathbb{R}^n \times \mathbb{R}^n &\cong \bigsqcup_p \mathbb{R}_p^n \\ (p, a) &\leftarrow (p, a)\end{aligned}$$

Take  $\{e_1|_p, \dots, e_n|_p\}$  as a basis of  $\mathbb{R}_p^n$ . The dual basis is denoted by

$$\{dx_1|_p, \dots, dx_n|_p\} = \{(e_1|_p)^\vee, \dots, (e_n|_p)^\vee\} \in (\mathbb{R}_p^n)^\vee$$

$$\begin{aligned}dx_i|_p : \mathbb{R}_p^n &\rightarrow \mathbb{R} \\ v = \left(\sum \alpha_i e_i|_p\right) &\mapsto \alpha_i\end{aligned}$$

$$\frac{\partial x_i}{\partial x_j} = dx_i|_p(e_j|_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recalled the wedge algebra:

$$\bigwedge (\mathbb{R}_p^n)^\vee := T(\mathbb{R}_p^n)^\vee / I = \bigoplus_{k \in \mathbb{N}} \bigwedge^k (\mathbb{R}_p^n)^\vee$$

Consider

$$\bigwedge^k (\mathbb{R}_p^n)^\vee$$

what's a basis of this vector space?

$$\{dx_{i_1}|_p \wedge \dots \wedge dx_{i_k}|_p \mid 1 \leq i_1 < \dots < i_k \leq n\}$$

and

$$\dim(\bigwedge^k (\mathbb{R}_p^n)^\vee) = \binom{n}{k}$$

Proved.

## 51.2 Do Carmo Differential forms

### 51.3 Def

An exterior  $k$ -form in  $\mathbb{R}^n$  is a mapping:

$$\begin{aligned}\omega : \mathbb{R}^n &\rightarrow \bigsqcup_p \bigwedge^k (\mathbb{R}_p^n)^\vee \\ p &\mapsto \omega(p)\end{aligned}$$

that's a section of the projection  $\pi$

$$(\pi \circ \omega = id_{\mathbb{R}}) = (\omega(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee)$$

$$\omega(p) = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(p) dx_{i_1} |_p \wedge \dots \wedge dx_{i_k} |_p \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

Note that

$$\begin{array}{ccc} \bigsqcup_p \bigwedge^k (\mathbb{R}_p^n)^\vee & \xrightarrow{\pi} & \mathbb{R}^n \\ f |_p & \mapsto & p \\ \omega \leftrightarrow & \{a_{i_1}, \dots, a_{i_k}\} & \end{array}$$

if all  $a_{i_j}$  are of class  $C^m(\mathbb{R})$  the  $\omega$  is called a  $C^m$ -differential  $k$ -form. If  $m = +\infty$  *omega* is called a smooth  $k$ -form.

## 51.4 Notation

$$\omega = \sum_I a_I dx_I$$

where  $I = (i_1, \dots, i_k)$

### Example

take  $n = 4$

1-form

$$\begin{aligned} \omega &= a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + a_4 dx_4 \\ \omega(p) &= a_1(p) dx_1 |_p + a_2(p) dx_2 |_p + a_3(p) dx_3 |_p + a_4(p) dx_4 |_p \end{aligned}$$

2-form

$$\begin{aligned} \omega &= a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{14} dx_1 \wedge dx_4 \\ &\quad + a_{23} dx_2 \wedge dx_3 + a_{24} dx_2 \wedge dx_4 + a_{34} dx_3 \wedge dx_4 \end{aligned}$$

## 51.5 Notation

When  $k = 0$  a 0-form of class  $C^m$ -differential 0-form is  $f \in C^m(\mathbb{R}^n)$

$$C^m(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ of class } C^m\}$$

## 51.6 Notation

$$\Omega_{(m)}^k(\mathbb{R}^n) := \{\text{set of } C^m\text{-diff } k\text{-forms}\}$$

$$\Omega_{(m)}^0(\mathbb{R}^n) = C^m(\mathbb{R}^n)$$

$m$  could be omitted if no confusion.

### 51.7 Prop

$\Omega_{(m)}^k(\mathbb{R}^n)$  is a module over  $\Omega_{(m)}^0(\mathbb{R}^n)$

#### Proof

$$\omega, \eta \in \Omega^k(\mathbb{R}^n)$$

$$(\omega + \eta)(p) = \omega(p) + \eta(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

$$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$$

$$f\omega \in \Omega^k(\mathbb{R}^n) \quad (f\omega)(p) = f(p)\omega(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

### 51.8 Def

$f : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable then

$$df|_p : \mathbb{R}_p^n \rightarrow \mathbb{R}_{f(p)} \cong \mathbb{R}$$

$$df|_p \in (\mathbb{R}_p^n)^\vee$$

$$df|_p = \sum_{i=1}^n f_i(p) dx_i|_p$$

because

$$\{dx_1|_p, \dots, dx_n|_p\}$$

is a basis of  $(\mathbb{R}_p^n)^\vee$

By  $df$  then  $f_i$  are the partial derivatives of  $f$ . This means that  $df$  is a differential 1-form.

Moreover,

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

differential, then

$$F = (F_1, \dots, F_m)$$

when  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  differential.

$$dF|_p : \mathbb{R}_p^n \rightarrow \mathbb{R}_{f(p)}^m$$

$$dF_i|_p = dx_i|_{f(p)} (dF|_p) = d(x_i \circ F)|_p$$

$$dF_i|_p : \mathbb{R}_p^n \xrightarrow{dF|_p} \mathbb{R}_p^m \xrightarrow{dx_i|_{f(p)}} \mathbb{R}$$

and

$$\begin{aligned} dx_i|_p : \mathbb{R}_p^n &\rightarrow \mathbb{R} \\ v = \sum \alpha_i e_i|_p &\mapsto \alpha_i \end{aligned}$$

where  $e_i|_p = (p, (0, \dots, \underbrace{1}_{i\text{-th}}, 0, \dots))$

Recall that if  $V$  is a vector space, then

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

This is a  $K$ -module with the multiplication:

$$\begin{aligned} V^{\otimes n} \times V^{\otimes m} &\rightarrow V^{\otimes n+m} \\ (x_1 \otimes \dots \otimes x_n, y_1 \otimes \dots \otimes y_m) &\mapsto x_1 \otimes \dots \otimes x_n \otimes y_1 \otimes \dots \otimes y_m \end{aligned}$$

From  $T(V)$  we construct

$$\begin{aligned} \bigwedge(V) &= T(V)/I \\ T(V) &\rightarrow \bigwedge(V) \\ x_1 \otimes \dots \otimes x_n &\mapsto x_1 \wedge \dots \wedge x_n \end{aligned}$$

therefore also in  $\bigwedge(V)$  we have the multiplication that makes  $\bigwedge(V)$  a  $K$ -algebra

$$\begin{aligned} \bigwedge^k(V) &\rightarrow \bigwedge^l(V) \\ (x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l) &\mapsto x_1 \wedge \dots \wedge x_k \wedge y_1 \wedge \dots \wedge y_l \end{aligned}$$

We define now a wedge product on  $\Omega(\mathbb{R}^n)$

$$\begin{aligned} \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) &\rightarrow \Omega^{k+l}(\mathbb{R}^n) \\ (\omega, \eta) &\mapsto \omega \wedge \eta \end{aligned}$$

take  $\omega = \sum_I a_I dx_I$  and  $\eta = \sum_J b_J dx_J$

$$\omega \wedge \eta := \sum_{IJ} a_i b_J dx_{IJ}$$

where

$$IJ := (i_1, \dots, i_k, j_1, \dots, j_l)$$

with  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$

### Example

$$\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 \in \Omega^1(\mathbb{R}^3)$$

$$\eta = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$$

$$\omega \wedge \eta = (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3$$

### 51.9 Prop

Take  $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^l(\mathbb{R}^n), \varphi \in \Omega^s(\mathbb{R}^n)$ , then

$$(1) \quad (\omega \wedge \eta) \wedge \varphi = \omega \wedge (\eta \wedge \varphi)$$

$$(2) \quad (\omega + \eta) = (-1)^{kl}(\eta \wedge \omega)$$

$$(3) \quad \text{Take } \theta \in \Omega^k(\mathbb{R}^n) \\ \omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta$$

### Proof

Exercise

Try to do this. Consequence of the properties of  $\wedge$  for vector spaces.

### 51.10 Def

Now we have

$$\Omega(\mathbb{R}^n) = \bigoplus_{k \in \mathbb{N}} \Omega^k(\mathbb{R}^n)$$

a  $\mathbb{R}$ -algebra with the  $\wedge$ -product

And it's also a  $\Omega^0(\mathbb{R}^n)$  module and  $\Omega^0(\mathbb{R}^n)$ -algebra

### 51.11 Remark

$$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n) \\ f \wedge \omega = f\omega$$

### 51.12 Def: Pullback of forms

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping of  $C^r$ , then it induces a mapping

$$f^* : \Omega_{(x)}^k(\mathbb{R}^m) \rightarrow \Omega_{(x)}^k(\mathbb{R}^n) \\ \omega \mapsto f^*\omega$$

and

$$f^*(\omega)(p)(v_1, \dots, v_k) = \omega(f(p))(df|_p(v_1), \dots, df|_p(v_k))$$

recalling

$$df|_p : \mathbb{R}^n \rightarrow \mathbb{R}_{f(p)}^m \Rightarrow df|_p(v_i) \in \mathbb{R}_{f(p)}^n$$

### 51.13 Prop

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a differentiable mapping.  $\omega, \eta \in \Omega^k(\mathbb{R}^n)$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  a differentiable mapping. ( $g \in \Omega^0(\mathbb{R}^m)$ ) Then

(1)

$$f^*(\omega + \eta) = f^*(\omega) + f^*(\eta)$$

(2)

$$f^*(g\omega) = f^*g^*f^*(\omega)$$

where  $f^*g := g \circ f$

(3) If  $\omega_1, \dots, \omega_k$  are 1-forms in  $\mathbb{R}^m$ , then

$$f^*(\omega_1 \wedge \dots \wedge \omega_k) = f^*(\omega_1) \wedge \dots \wedge f^*(\omega_k)$$

### Proof

(1)

$$\begin{aligned} f^*(\omega + \eta)(p)(v_1, \dots, v_k) &= (\omega + \eta)(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= \omega(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &\quad + \eta(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (f^*\omega)(p)(v_1, \dots, v_k) + (f^*\eta)(p)(v_1, \dots, v_k) \end{aligned}$$

(2)

$$\begin{aligned} f^*(g\omega) &= g\omega(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (g \circ f)(p)(f^*\omega)(p)(v_1, \dots, v_k) \end{aligned}$$

(3)

$$\begin{aligned} (f_1 f_2)(x) &= f_1(x) f_2(x) \\ f^*(\omega_1 \wedge \dots \wedge \omega_k)(p)(v_1, \dots, v_k) &= (\omega_1 \wedge \dots \wedge \omega_k)(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= \omega_1(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \wedge \\ &\quad \dots \wedge \omega_k(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (f^*(\omega_1))(p)(v_1) \wedge \dots \wedge (f^*(\omega_k))(p)(v_k) \end{aligned}$$

General fact

$$\begin{aligned} f_1, \dots, f_k : \quad & V \rightarrow V \\ f_1 \wedge \dots \wedge f_k : \quad & \bigwedge^k V \rightarrow \bigwedge^k V \\ & (v_1, \dots, v_k) \mapsto f_1(v_1) \wedge \dots \wedge f_k(v_k) \\ g^{\otimes n} : V^{\otimes n} & \rightarrow V^{\otimes n} \\ & (v_1, \dots, v_n) \mapsto g(v_1) \otimes \dots \otimes g(v_n) \end{aligned}$$

Let's see what happens in terms of coordinates:

$$\begin{aligned} f : \mathbb{R}^n & \rightarrow \mathbb{R}^m \\ (x_1, \dots, x_n)^T & \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))^T \end{aligned}$$

$$\Omega = \sum_I a_I dy_I \in \Omega^k(\mathbb{R}^m)$$

$$f^* \omega = \sum_I f^*(a_I) (f^* dy_{i_1}) \wedge \dots \wedge (f^* dy_{i_k})$$

Note that

$$(f^* dy_i)(v) = dy_i(df(v)) = d(y_i \circ f)(v) = (df_i)(v)$$

then

$$f^* \omega = \sum_I a_I (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) df_{i_1} \wedge \dots \wedge df_{i_k}$$

### 51.14 Remark

$U \subseteq \mathbb{R}^n$  open then consider  $\Omega^k(U) \subseteq \Omega^k(\mathbb{R}^n)$

#### Example

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0, 0)\}) (= U)$$

$$V = \{(r, \theta) \in \mathbb{R}^2 : r > 0, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned} f : V & \rightarrow U \\ (r, \theta)^T & \mapsto f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} \end{aligned}$$

Let's compute  $f^* \omega$

$$df_1 = \cos \theta dr - r \sin \theta d\theta$$

$$df_2 = \sin \theta dr + r \cos \theta d\theta$$

$$f^* \omega = -\frac{r \sin \theta}{r^2} (\cos \theta dr - r \sin \theta d\theta) + \frac{r \cos \theta}{r^2} (\sin \theta dr + r \cos \theta d\theta) = d\theta$$

### 51.15

$U \subseteq \mathbb{R}^n$  an open subset

$$\Omega_{(m)}^k(U)$$



this is a module over  $\Omega_{(m)}^0(U)$  Moreover,  $\omega \in \Omega^k(U), \eta \in \Omega^l(U)$

$$\omega \wedge \eta \in \Omega^{k+l}(U)$$

$$f : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \underbrace{\mathbb{R}^m}_{\subseteq \mathbb{R}^m} \text{ } f \text{ is of class } C^{m+1}$$

$$f^*\omega \in \Omega_{(m)}^k U()$$

$df$  is a one-form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

where  $\frac{\partial f}{\partial x_i} = a_i : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable

## 51.16 Prop

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable mapping. Then

(1) for any two forms in  $\mathbb{R}^m$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*(\eta))$$

(2) for  $g : \mathbb{R}^p \rightarrow \mathbb{R}^n$  differentiable

$$(f \circ g)^*\omega = g^*(f^*\omega)$$

### Proof

1

$$(y_1, \dots, y_m) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \in \mathbb{R}^m, (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\omega = \sum_I a_I dy_I \quad \eta = \sum_J b_J dy_J$$

$$f^*(\omega \wedge \eta) = f^*\left(\sum_{IJ} a_I b_J dy_I \wedge dy_J\right)$$

$$\begin{aligned} \text{(by def of pullback)} &= \sum_{IJ} a_I(f_1, \dots, f_m) b_J(f_1, \dots, f_m) df_I \wedge df_J \\ &= \left(\sum_I a_I(f_1, \dots, f_m) df_I\right) \wedge \left(\sum_J b_J(f_1, \dots, f_m) df_J\right) \\ &= (f^*\omega) \wedge (f^*(\eta)) \end{aligned}$$

**2**

$$\begin{aligned}
(f \circ g)^*\omega &= \sum_I a_I((f \circ g)_1, \dots, (f \circ g)_m) d(f \circ g)_I \\
&= \sum_I a_I(f_1(g_1, \dots, g_n), \dots, f_m(g_1, \dots, g_n)) df_I(dg_1, \dots, dg_n) \\
&= g^*(f^*\omega)
\end{aligned}$$

**51.17**

The differential of a function is a one-form

$$\begin{array}{ccc}
\underbrace{f} & \rightsquigarrow & \underbrace{df} \\
\text{0-form} & & \text{1-form}
\end{array}$$

We went to generalize this to any (exterior) differentials

$$\begin{array}{ccc}
d : \Omega_{(m)}^k(U) & \rightarrow & \Omega_{(m)}^{k+1}(U) \\
\omega & \mapsto & d\omega \\
\sum_I a_I dx_I & \mapsto & \sum_I da_I \wedge dx_I
\end{array}$$

where  $a_I \in C^m(U)$ ,  $da_I = \sum \frac{\partial a_I}{\partial x_i} dx_i$

**51.18 Example**

$$\omega = xyzdx + yzdy + (x+z)dz$$

$$\begin{aligned}
d\omega &= d(xyz) \wedge dx + d(yz) \wedge dy + d(x+z) \wedge dz \\
&= (yzdx + xzdy + xydz) \wedge dx + (zdy + ydz) \wedge dy + (xdz) \wedge dz \\
&= -xzdx \wedge dy - xydx \wedge dz - ydy \wedge dz + dx \wedge dz \\
&= -xz \underline{dx \wedge dy} + (1 - xxy) \underline{dx \wedge dz} - y \underline{dy \wedge dz}
\end{aligned}$$

**51.19 Prop**

$$\forall \omega_1, \omega_2 \in \Omega^k(U), \eta \in \Omega^l(U)$$

(1)

$$d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$$

(2)

$$d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \eta + (-1)^k \omega \wedge d\eta$$

$$(3) \quad d(d\omega) = 0 \quad (d^2\omega = 0)$$

$$(4) \quad f : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \underbrace{V}_{\subseteq \mathbb{R}^m} \\ d(f^*\omega) = f^*(d\omega)$$

### Proof

(1) Exercise

$$(2) \quad \omega = \sum_I a_I dx_I, \eta = \sum_J b_J dx_J; \quad \omega \wedge \eta = \sum_{IJ} a_I b_J dx_I \wedge dx_J$$

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{IJ} d(a_I b_J) \wedge dx_I \wedge dx_J \\ &= \left( \sum_{IJ} b_J da_I \wedge dx_I \wedge dx_J \right) + \left( \sum_{IJ} a_I db_J \wedge dx_I \wedge dx_J \right) \\ &= d\omega \wedge \eta + (-1)^k \sum_{IJ} a_I dd x_I \wedge b_J \wedge dx_J \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \end{aligned}$$

(3) First assume  $\omega = f \in \Omega^0(U)$

$$\begin{aligned} d(df) &= d\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j\right) \\ &= \sum_{j=1}^n d\left(\frac{\partial f}{\partial x_j} \wedge dx_j\right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j\right) \\ &= 0 \end{aligned}$$

Notice that  $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

By (1) we can prove for  $\omega = a_I dx_I, a_I \neq 0$ , from (2) we have

$$d\omega = da_I \wedge dx_I + a_I d^2 x_I$$

But

$$d^2 x_I = d(1 \cdot dx_I) = d1 \wedge dx_I = 0$$

Hence

$$\begin{aligned} d^2\omega &= d(d\omega) \\ &= d(da_I \wedge dx_I) \\ &= 0 \end{aligned}$$

(4) As above let's prove it for  $\omega = g \in \Omega^0(U)$

$$\begin{aligned} g : \mathbb{R}^m & \rightarrow \mathbb{R} \\ (y_1, \dots, y_m) & \mapsto g(y_1, \dots, y_m) \end{aligned}$$

$$\begin{aligned} f^*(dg) &= f^*\left(\sum_{i=1}^m \frac{\partial g}{\partial y_i} dy_i\right) \\ &= \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial f}{\partial x_j} dx_j \\ &= \sum_j \frac{\partial(g \circ f)}{\partial x_j} dx_j \\ &= d(g \circ f) \\ &= d(f^*g) \end{aligned}$$

Now let's do the proof for  $\omega \in \Omega^k(U), \omega = \sum_I a_I dx_I$

$$\begin{aligned} d(f^*g) &= d\left(f^*\left(\sum_I a_I dx_I\right)\right) \\ (\text{by prop of } f^*) &= d\left(\sum_I f^*a_I \wedge f^*dx_I\right) \\ (by(1)) &= \sum_I d(f^*a_I \wedge f^*dx_I) \\ (\text{use(2)}) &= \sum_I f^*(da_I) \wedge f^*dx_I \\ (\text{prop of } f^*) &= f^*\left(\sum_I da_I \wedge dx_I\right) \\ &= f^*(d\omega) \end{aligned}$$

## 51.20

$$df : p \mapsto df_p$$

is a differential form.

## 51.21 Def?

$$D_h f(p) := \lim_{t \rightarrow 0} \frac{f(p+th) - f(p)}{h} = df_p(h)$$

# Chapter 52

## Line integral

### 52.1 Def

$$\omega = \sum_i a_i dx_i \in \Omega^1_m(U), U \subseteq \mathbb{R}^n$$

$$\gamma : [a, b] \rightarrow U^n$$

a parametric curve

$$f : [a, b] \rightarrow \mathbb{R}$$

of class  $C^1$

$$\gamma : t \mapsto (t, f(t)) = \text{Graph of } f$$

this is a parametric curve piecewise of class  $C^1$ :  $\exists t_0 = a < t_1 < \dots < t_k = b$   
such that

$$\gamma_j := \gamma|_{]t_j, t_{j+1}[}$$

is of class  $C^1$

$$\gamma_j(]t_k, t_{k+1}[) \rightarrow \mathbb{R}^n$$

we can define  $\gamma_j^* \omega$  this is one form in  $\Omega^1(]t_k, t_{k+1}[)$   
if  $\gamma_j(t) = (x_1(t), \dots, x_n(t))$  then

$$\gamma_j^* \omega = \sum_{i=1}^n a_i(x_1(t), \dots, x_n(t)) \frac{dx_i}{dt} dt \quad x_i(t) = \frac{dx_i}{dt}?$$

### 52.2 Def: Path integral

Let  $\gamma$  and  $\omega$  be as above.

$$\int_{\gamma} \omega := \sum_i \int_{t_k}^{t_{k+1}} \gamma_j^* \omega$$

this is the integral of  $\omega$  along the parametric curve  $\gamma$  with

$$\gamma = t \mapsto (x_1(t), \dots, x_n(t))$$

where  $x_i(t) = \frac{dx_i}{dt}$

### 52.3 What's this in physics?

Fix  $\gamma(t), \gamma'(t) = (\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt})$  = the tangent vector of  $\gamma$  in  $\gamma(t)$  then

$$\int_{t_k}^{t_{k+1}} \gamma_j^* \omega = \int_{t_k}^{t_{k+1}} \langle a \circ \gamma_j, \gamma_j' \rangle dt$$

where  $a = (a_1, \dots, a_n), a_i : \mathbb{R}^n \rightarrow \mathbb{R}$

## Chapter 53

# Complement of measure theory

### 53.1 Def( $\sigma$ -finite)

Let  $(X, \Sigma_X, \mu)$  be a measure space. We say that it's  $\sigma$ -finite if there exists a sequence  $\{E_n\}_{n \in \mathbb{N}}$  of measurable sets. (namely  $E_n \in \Sigma_X$ ) such that

$$X = \bigcup_{n \in \mathbb{N}} E_n \text{ and } \mu(E_n) < +\infty, \forall n \in \mathbb{N}$$

### 53.2 Example( $\mathbb{R}$ , Norel $\sigma$ -algebra, Lebesgue measure)

this is  $\sigma$ -finite

$$\lambda([-n, n]) = 2n < +\infty$$

### 53.3 Notation

Take sets  $A \subseteq X \times Y$  For  $x \in X$ , we define

$$A_x := \{u \in Y \mid (x, u) \in A\}$$

called a **vertical section** of  $A$  or  $x$ -fiber of  $A$

For  $y \in Y$  we define

$$A_y := \{x \in X \mid (x, y) \in A\}$$

called a **horizontal section** of  $A$ , or  $y$ -fiber of  $A$

### 53.4 Def

Let  $X$  be a set. then  $\mathcal{D} \subseteq \wp(X)$  is a **Dynkin system** if

- $X \in \mathcal{D}$  and  $\emptyset \in \mathcal{D}$
- $\forall D \in \mathcal{D} \quad X \setminus D \in \mathcal{D}$
- If  $\{D_n\}_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{D}$  of pairwise disjoint sets, then

$$\bigsqcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$$

### Remark

A  $\sigma$ -algebra is a Dynkin system

### 53.5 Def

Let  $(\mathcal{G} \subseteq \wp(X))$  then  $\delta(\mathcal{G}) \subseteq \wp(X)$  is called the Dynkin system generated by  $\mathcal{G}$  if

- $\mathcal{G} \subseteq \delta(\mathcal{G})$
- If  $\mathcal{D}$  is a Dynkin system containing  $\mathcal{G}$ , then  $\delta(\mathcal{G}) \subseteq \mathcal{D}$

### Exercise

$\delta(\mathcal{G})$  exists and it's unique.

### 53.6 Prop

If  $\mathcal{D}$  is a Dynkin system closed under the intersection, then it's a  $\sigma$ -algebra, namely

$$\forall (D, E) \in \mathcal{D}^2, D \cap E \in \mathcal{D} \Rightarrow \forall \{D_n\}_{n \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}} \quad \bigcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$$

### Proof

We have to show that  $\mathcal{D}$  is closed under any countable union. Let  $\{D_n\}_{n \in \mathbb{N}}$  be any sequence in  $\mathcal{D}$ , let

$$E_n = D_n \cap \bigcap_{m < n} X \setminus D_m$$

and we know that

$$\bigcup_{k \in \mathbb{N}} E_k = \bigcup_{k \in \mathbb{N}} D_k \supseteq D_n \cap D_m \quad \forall n, m$$



### 53.7 Prop

Let  $X$  be a set and let  $\mathcal{G} \subseteq \wp(X)$ . Assume that  $\mathcal{G}$  is closed under the finite intersection. Then

$$\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$$

#### Proof

Prove  $\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$   
trivial

Prove  $\sigma(\mathcal{G})$  is a  $\sigma$ -algebra, which gives that  $\delta(\mathcal{G}) \supseteq \sigma(\mathcal{G})$   
Let

$$\delta_D = \{E \subseteq X \mid E \cap D \in \delta(\mathcal{G})\}$$

Verify that  $\forall D \in \mathcal{G}, \delta_D$  is a Dynkin system:

- Since  $X \cap D \in \mathcal{G} \Rightarrow X \in \delta_D$
- Take  $E \in \delta_D$

$$(X \setminus E) \cap D = X \setminus ((E \cap D) \cup (X \setminus D))$$

Where  $E \cap D \in \delta(\mathcal{G})$  (since  $E \in \delta_D$ ),  $X \setminus \in \delta(\mathcal{G})$  (by def)

Hence

$$X \setminus E \in \delta_D$$

- Let  $\{E_n\}$  be elements in  $\delta_D$  which are pairwise disjoint, then

$$\left(\bigcup_{n \in \mathbb{N}} E_n\right) \cap D = \bigcup_{n \in \mathbb{N}} (E_n \cap D)$$

Then  $\forall G \in \mathcal{G}$

$$\delta(\mathcal{G}) \subseteq \delta_G$$

since  $\delta(\mathcal{G})$  is the smallest Dynkin system containing  $\mathcal{G}$  and  $\forall G \in \mathcal{G} \mathcal{G} \subseteq \delta_G$   
since  $\mathcal{G}$  is closed under finite intersection. By definition

$$\forall H \in \delta(\mathcal{G}), H \cup G \in \delta(\mathcal{G})$$

### 53.8 Lemma

Let  $(X, \Sigma_X)$  be a measurable space. Then the mapping 0 is measurable

**Proof**

By def

$$0_*(\Sigma_X) = \{B \subseteq \mathbb{R} \mid 0^{-1}(B) \in \Sigma_X\}$$

Since either  $0^{-1}(B) = \emptyset$  or  $0^{-1}(B) = X$ , then

$$0_*(\Sigma_X) = \wp(\mathbb{R})$$

Hence

$$\mathcal{B}(\mathbb{R}) \subseteq 0_*(\Sigma_X) = \wp(\mathbb{R})$$

**53.9 Theorem**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. Then  $\forall E \in \Sigma_X \otimes \Sigma_Y$ , the functions

$$\begin{aligned} f_E : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu(E_x) \\ g_E : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto \mu(E_y) \end{aligned}$$

are respectively  $\Sigma_X$ -measurable and  $\Sigma_Y$ -measurable

**Proof**

We first cope with special ones that  $\nu$  is finite ( $\mu(Y) < +\infty$ ) Let

$$F = \{E \in \Sigma_X \otimes \Sigma_Y \mid f_E \text{ is measurable}\}$$

We want to have  $F = \Sigma_X \otimes \Sigma_Y$  Only to show  $\Sigma_X \otimes \Sigma_Y \subseteq F$  by definition of product measure

Let  $S_1 \in \Sigma_X, S_2 \in \Sigma_Y$

$$(S_1 \times S_2)_x = \begin{cases} S_2 & \text{if } x \in S_1 \\ 0 & \text{if } x \notin S_1 \end{cases} \quad f_{S_1 \times S_2}(x) = \nu(S_1 \times S_2) = \nu(S_2)_{\chi_{S_1}}(x)$$

?

$f_{S_1 \times S_2}$  is measurable.

Now show that  $F$  is a Dynkin system:

- $X \times Y \in F$
- Let  $D \in F$ , we want to show that

$$(X \times Y) \setminus D \in F$$

Note that

$$((X \times Y) \setminus D)_x = Y \setminus D_x$$

then

$$\begin{aligned}
 f_{(X \times Y \setminus D)}(x) &= \nu((X \times Y) \setminus D)_x \\
 &= \nu(Y \setminus D_x) \\
 &= \nu(Y) - \nu(D_x) \\
 &= \nu(Y) - f_D(x)
 \end{aligned}$$

Which means that  $f_{(X \times Y \setminus D)}$  is measurable.

- Let  $\{D_n\}_{n \in \mathbb{N}}$  be a sequence of disjoint sets such that  $D_n \in F$ . ( $f_{D_n}$  is measurable)  $D = \bigcup_{n \in \mathbb{N}} D_n$

$$\begin{aligned}
 f_D(x) &= \nu(D_x) \\
 &= \nu\left(\bigcup_{n \in \mathbb{N}} D_n\right) \\
 &= \sum_{n \in \mathbb{N}} \nu(D_n) \\
 &= \sum_{n \in \mathbb{N}} f_{D_n}(x)
 \end{aligned}$$

Hence  $F$  is a Dynkin system.

Consider

$$\mathcal{G} = \{S_1 \times S_2 \mid S_1 \in \Sigma_X, S_2 \in \Sigma_Y\} \subseteq F$$

Moreover,  $\mathcal{G}$  is closed under the intersection.

$$(S_1 \times T_1) \cap (S_2 \times T_2) = (S_1 \cap S_2) \times (T_1 \cap T_2)$$

So  $\delta(\mathcal{G})$  is  $\sigma$ -algebra. By proposition 53.7

$$\delta(\mathcal{G}) = \sigma(\mathcal{G}) = \Sigma_X \otimes \Sigma_Y \subseteq F$$

Secondly, for general  $\nu$ , since  $\nu$  is  $\sigma$ -finite, there exists

$$Y = \bigcup_{n \in \mathbb{N}} Y_n \quad \nu(Y_n) < +\infty$$

As above

$$F_0 = Y_0, F_n = Y_n \setminus \bigcup_{k \in \mathbb{N}} Y_k, \nu(F_n) < +\infty$$

$\{F_n\}$  are disjoint, measurable, of finite measure and  $Y = \bigcup_n F_n$

For all  $n$  we define a measure  $\nu^{(n)}$  on  $Y_n$

$$\nu^{(n)}(E) := \nu(E \cap F_n)$$

Notice that

$$\nu^{(n)}(Y) = \nu(Y \cap F_n) = \nu(F_n) < +\infty$$

Hence we have

$$\begin{aligned} f_E^{(n)} : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu^{(n)}(E_x) \end{aligned}$$

By step 1,  $f_E^{(n)}$  is measurable.  
 $\forall E, n$

$$\begin{aligned} f_E^{(n)}(x) &= \nu(E_x) \\ &= \nu(E_x \cap Y) \\ &= \nu\left(E_x \cap \bigcup_n F_n\right) \\ &= \nu\left(\bigcup_n E_x \cap F_n\right) \\ &= \sum_n \nu(E_x \cap F_n) = \sum \nu^{(n)}(E_x) \\ &= \sum_n (f_E^{(n)}(x)) \end{aligned}$$

It follows that  $f_E$  is measurable

Then we need prove that  $F \supseteq \Sigma_X \times \Sigma_Y$  and  $F$  is  $\sigma$ -algebra, so  $F \supseteq \sigma(\Sigma_X \times \Sigma_Y) = \Sigma_X \otimes \Sigma_Y$

–  $E_\mu \in \Sigma_X$  and  $E_\epsilon \in \Sigma_Y$

$$(E_\mu \times E_\epsilon)_x = \begin{cases} E_\epsilon & \text{if } x \in E_\mu \\ \emptyset & \text{otherwise} \end{cases} \in \Sigma_X$$

– exercise

### 53.10 Prop

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces.  $\forall E \in \Sigma_X \otimes \Sigma_Y$  the functions:

$$\begin{aligned} \rho_X(E) &:= \int_X f_E(x) d\mu(x) \\ \rho_Y(E) &:= \int_Y g_E(y) d\nu(y) \end{aligned}$$

Define two measure on the measurable spaces  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  such that

$$\rho_X(S_1 \times S_2) = \rho_Y(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

**Proof**

We already know that  $f_E$  and  $g_E$  are measurable. So the integral makes sense. Only needs to prove for  $\rho_X$

- Since  $f_E \geq 0$  and  $g_E \geq 0$ , then  $\rho_X(E) \geq 0 \ \forall E \in \Sigma_X \otimes \Sigma_Y$
- $\rho_X(\emptyset) = \int_X \nu(\emptyset) d\mu(x) = 0$
- Assume that  $\{E_n\}_{n \in \mathbb{N}}$  is a sequence in  $\Sigma_X \otimes \Sigma_Y$  of disjoint subsets,

$$\begin{aligned} \rho_X\left(\bigsqcup_{n \in \mathbb{N}} E_n\right) &= \int_X \nu\left(\bigsqcup_{n \in \mathbb{N}} (E_n)_x\right) d\mu(x) \\ &= \int_X \sum_{n \in \mathbb{N}} \nu(E_n)_x d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \int_X \nu(E_n)_x d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \rho_X(E_n) \end{aligned}$$

•

$$\begin{aligned} \rho_X(S_1 \times S_2) &= \int_X \nu(S_1 \times S_2)_x d\mu(x) \\ &= \int_X \nu(S_2) \mathbb{1}_{S_1}(x) d\mu(x) \\ &= \nu(S_2) \mu(S_1) \end{aligned}$$

**53.11 Prop**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. Any measure  $\eta$  on  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  that satisfies

$$\eta(S_1 \times S_2) = \mu(S_1) \nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

is  $\sigma$ -finite

**Proof**

$$\nu(E_n \times F_m) = \mu(E_n) \nu(F_m) < +\infty$$

**53.12 Prop**

Let  $(X, \Sigma_X)$  be a measurable space and assume that  $\mathcal{G} \subseteq \wp(X)$  such that  $\Sigma = \sigma(\mathcal{G})$

Moreover, assume that  $\mathcal{G}$  satisfies the following conditions:

- (1) It's closed under finite intersection.
- (2) There exists a sequence  $\{G_n\}_{n \in \mathbb{N}}$  in  $\mathcal{G}$  such that  $\{G_m\} \uparrow X$  (namely  $G_i \subseteq G_{i+1}$  and  $\bigcup_n G_n = X$ )

Let  $\mu$  and  $\nu$  be two measure on  $(X, \Sigma)$  such that

- (a)  $\forall G \in \mathcal{G} \quad \mu(G) = \nu(G)$
- (b)  $\forall n \in \mathbb{N} \quad \mu(G_n) = \nu(G_n)$

Then  $\mu = \nu$

### Proof

Define

$$\mathcal{D}_n = \{E \in \Sigma \mid \mu(G_n \cap E) = \nu(G_n \cap E)\} \subseteq \Sigma$$

We show that  $\mathcal{D}_n$  is a Dynkin system  $\forall n$

- $G_n \cap X = G_n$
- Assume that  $D \in \mathcal{D}_n$ 

$$\begin{aligned} \mu(G_n \cap (X \setminus D)) &= \mu(G_n \setminus D) \\ &= \mu(G_n) - \mu(G_n \cap D) \quad (\text{here use the fact that } \mu(G_n) < +\infty) \\ &= \nu(G_n) - \nu(G_n \cap D) \\ &= \nu(G_n \cap (X \setminus D)) \end{aligned}$$
- Take  $\{D_m\}_{m \in \mathbb{N}}$  in  $\mathcal{D}_n$  of pairwise choice

$$\begin{aligned} \mu(G_n \cap \bigcup_m D_m) &= \mu(\bigcup_m (G_n \cap D_m)) \\ &= \sum_m \mu(G_n \cap D_m) \\ &= \sum_m \nu(G_n \cap D_m) \\ &= \nu(G_n \cap \bigcup_m D_m) \end{aligned}$$

Combining (1) and (a)  $\mathcal{G} \subseteq \mathcal{D}_n$ . By prop 53.7, consider

$$\delta(\mathcal{G}) = \sigma(\mathcal{G}) = \Sigma$$

Moreover, since  $\mathcal{G} \subseteq \mathcal{D}_n$  and  $\mathcal{D}_n$  a Dynkin system

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_n$$

We get

$$\Sigma = \mathcal{D}_n$$

Since  $\bigcup_n G_n \cap E = E \cap \bigcup_n G_n = E$

$$\mu(E) = \lim_{x \rightarrow +\infty} \mu(G_n \cap E) = \lim_{x \rightarrow +\infty} \nu(G_n \cap E) = \nu(E)$$

### 53.13 Theorem

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. There exists a unique  $\sigma$ -finite measure  $\mu \times \nu$  on  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  such that

$$\mu \times \nu(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

and moreover, we have

$$(\mu \times \nu)(E) = \int_X f_E d\mu = \int_Y g_E d\nu$$

### 53.14 Corollary

On  $\mathbb{R}^n$ , we can define a unique measure  $\lambda^{(n)}$  as product of the Lebesgue measure on  $\mathbb{R}$ . This is called the Lebesgue measure on  $\mathbb{R}^n$

#### Proof

Assume that  $\eta$  and  $\eta'$  are two measures on the product satisfies the equation. Let  $\mathcal{G} = \Sigma_1 \times \Sigma_2$ ,  $\sigma(\mathcal{G}) = \Sigma_1 \otimes \Sigma_2$ . And  $\mathcal{G}$  is stable under finite intersection.

Since  $\mu$  and  $\nu$  are  $\sigma$ -finite, we can find some  $\{E_n\} \uparrow X$  and  $\{F_m\} \uparrow Y$  such that  $\mu(E_n) < +\infty, \nu(F_m) < +\infty$

$$X \times Y = \bigcup_{n,m} E_n \times F_m$$

We can find some ordering of the couple

$$X \times Y = \bigcup_{i_k} E_{i_k} \times F_{i_k}$$

$$\{E_{i_k} \times F_{i_k}\} \uparrow X \times Y$$

$$G_k := E_{i_k} \times F_{i_k}$$

By the equal in conditions  $\forall k$

$$\eta(G_k) = \eta'(G_k)$$

We apply prop53.12 to get

$$\eta = \eta' = \mu \times \nu$$

By prop53.11

$$\mu \times \nu$$

is  $\sigma$ -finite

And  $\mu \times \nu$  exists by Prop53.10

### 53.15 Monotone convergence theorem

Let  $(X, \Sigma_X, \mu)$  be a measure space.  $f : X \rightarrow \mathbb{R}_{\geq 0}$  be a measurable function. Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of measurable functions

$$f_n : X \rightarrow \mathbb{R}_{\geq 0}$$

such that  $f_i < f_j \quad \forall i < j$  and

$$\lim_{n \rightarrow +\infty} f_n(x) = f(x)$$

almost everywhere in  $X$  ( $\forall x \in X \setminus Z$  when  $Z \in \Sigma, \mu(Z) = 0$ ) Then

$$\int_X f d\mu = \lim_{n \rightarrow +\infty} \int_X f_n d\mu$$

#### Proof

dominated convergence theorem  $\Rightarrow$  monotone convergence theorem.

### 53.16 Recall

Product measure on  $\mathbb{R}^n$ . This is the unique measure on  $\lambda^n$  that exact is the naive product measure on rectangles.

$$\begin{aligned} \Sigma_{\mathbb{R}^n} &= \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}) \\ \Leftrightarrow \\ \mathcal{B}(\mathbb{R}^n); \lambda^n(\prod_i [a_i, b_i]) &= \prod_i \lambda[a_i, b_i] \end{aligned}$$

### 53.17 Def

$$\mathcal{O}^n = \{\text{set of open sets of } \mathbb{R}^n\}$$

$$\mathcal{C} = \{\text{set of closed sets of } \mathbb{R}^n\}$$

$$\mathcal{R}^n = \{\text{set of compact sets of } \mathbb{R}^n\}$$

$$\mathcal{J}_{ha}^n = \{\text{set of all half-open rectangles in } \mathbb{R}^n\}$$

$$\mathcal{J}_{ha, rat}^n = \{\text{set of all half-open rectangles of } \mathbb{R}^n, \text{ with rational end points}\}$$

### 53.18 Prop

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n) = \sigma(\mathcal{R}^n) = \sigma(\mathcal{J}_{ha}^n) = \sigma(\mathcal{J}_{ha, rat}^n)$$



**Proof**

Exercise

**53.19 Recall**

Let  $(X, \Sigma_X)$  and  $(Y, \Sigma_Y)$  be measurable spaces. Moreover, assume that

$$\Sigma_Y = \sigma(\mathcal{G})$$

where  $\mathcal{G} \subseteq \wp(X)$ .

A function  $f : X \rightarrow Y$  is measurable iff

$$\forall S \in \mathcal{G} \quad f^{-1}(S) \in \Sigma(X)$$

**Hint**

$$\mathcal{M} := \{B \subseteq Y \mid f^{-1}(B) \in \Sigma_X\} \subseteq \wp(Y)$$

show that this is a  $\sigma$ -algebra

**53.20 Corollary**

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , if  $f$  is continuous, the  $f$  is measurable with respect to the Lebesgue measure.

**53.21 Def: Push-forward measure**

Let  $(X, \Sigma_X, \mu)$  be a measure space, and let  $(Y, \Sigma_Y)$  be a measurable space. If  $f : X \rightarrow Y$  is a measurable function, then define:

$$f_*\mu(E) = \mu(f^{-1}(E)) \quad \forall E \in \Sigma_Y$$

This is a measure on  $Y$ , called the push forward of  $\mu$  through  $f$

**53.22 Prop**

Let  $p \in \mathbb{R}$  and let  $E \in \mathcal{B}(\mathbb{R}^n)$ , then

$$\lambda^n(E + p) = \lambda^n(E)$$

note that

$$E + p = \{x + p \mid x \in E\}$$

**Proof**

$$p = (p_1, \dots, p_n)$$

Consider the translation

$$\begin{aligned} \tau_p : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto x - p \end{aligned}$$

this is continuous, so measurable. We consider

$$\lambda_p^n := \tau_{p*} \lambda^n$$

let's show that  $\lambda_p^n = \lambda^n$

$$\begin{aligned} \lambda_p^n \left( \prod_{i=1}^n [a_i, b_i] \right) &\stackrel{\text{by def of } f_*}{=} \lambda^n (\tau_p^{-1} \left( \prod_{i=1}^n [a_i, b_i] \right)) \\ &= \lambda^n \left( \prod_{i=1}^n [a_i + p_i, b_i + p_i] \right) \\ &= \prod_{i=1}^n (b_i - a_i) \end{aligned}$$

By the uniqueness of the product measure, we have

$$\lambda_p^n = \lambda^n$$

**53.23 Lemma**

Let  $f : \Omega \rightarrow \mathbb{R}_{\geq 0}$  be a mapping. Then there exists an increasing sequence  $\{f_n\}_{n \in \mathbb{N}}$  such that converges pointwisely to  $f$

**Proof**

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega \in \Omega \mid \frac{k}{2^n} \leq f(\omega) \leq \frac{k+1}{2^n}\}} + n \mathbb{1}_{\{\omega \in \Omega \mid f(\omega) \geq 0\}}$$

**53.24 Fubini-Tobelli Theorem**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be two  $\sigma$ -finite measure spaces. Let  $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$  be the product space. Let  $f : X \times Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be a measurable function. Then

$$\begin{aligned} \int_{X \times Y} |f| \, d(\mu \times \nu) &= \int_X \left( \int_Y |f(x, y)| \, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left( \int_X |f(x, y)| \, d\mu(x) \right) d\nu(y) \end{aligned}$$

**Proof**

We can assume that  $f \geq 0$  and

$$\begin{aligned} \forall x \in X \quad f_x : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto f(x, y) \\ \forall y \in Y \quad f_y : X &\rightarrow \mathbb{R} \cup \{+\infty\} \end{aligned}$$

**Step 1:  $f_x$  and  $f_y$  are measurable**

Let's do it for  $f_y$ . We have to show that for  $D \in \mathcal{B}(\mathbb{R})$  where  $f_y^{-1}(D) \in \Sigma_X$

$$f_y^{-1}(D) = \{x \in X \mid f(x, y) \in D\} = \{x \in X \mid (x, y) \in f^{-1}(D)\} = (f^{-1}(D))_y$$

We have shown that if  $E$  is measurable, the  $E_y$  is measurable.

**Step 2**

Consider the functions

$$\begin{aligned} G : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto \int_X f(x, y) d\mu(x) \\ F : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \int_Y f(x, y) d\nu(y) \end{aligned}$$

we want to prove that they're both measurable

Let do this for  $G$ . Assume that  $f = \chi_E$  for  $E \in \Sigma_X \otimes \Sigma_Y$

$$\begin{aligned} (\chi_E)_y(x) &= \chi_E(x, y) = 1 \\ &\Leftrightarrow (x, y) \in E \\ &\Leftrightarrow x \in E_y \\ &\Leftrightarrow \chi_{E_y}(x) = 1 \end{aligned}$$

This chain of implications shows that

$$(\chi_E)_y = \chi_{E_y}$$

Hence

$$G(y) = \int_X (\chi_E)_y d\mu = \int_X \chi_{E_y} d\mu = \mu(E_y)$$

And we have proved that such functions are measurable

Now assume

$$f = \sum_{i=1}^n a_k \chi_{E_k} \quad E_k \in \Sigma_X \otimes \Sigma_Y, \quad a_k \in \mathbb{R}_{\geq 0}$$

and

$$f_y = \sum_{i=1}^n a_k \chi_{E_k \cap E_y}$$

then

$$\begin{aligned} G(y) &= \int_X f_y d\mu \\ &= \sum_{k=1}^n a_k \int_X \chi_{(E_k)_y} d\mu \\ &= \sum_{k=1}^n a_k \mu((E_k)_y) \\ &\Rightarrow G \text{ is measurable} \end{aligned}$$

Now assume  $f$  measurable

By lemma 53.23,  $\exists \{f_n\}$  increasing sequence such that converges pointwisely to  $f$

Moreover,  $\{(f_n)_y\}_{n \in \mathbb{N}}$  (all simple functions) converges to  $f_y$  too. Consider

$$\begin{aligned} g_n : Y &\rightsquigarrow \mathbb{R} \\ y &\mapsto \int_X (f_n)_y d\mu \end{aligned}$$

Since  $f_n$  are simple. By the previous claim in step 2, we know that  $g_n$  is measurable. And since  $Im(g_n) \subseteq \mathbb{R}$

$$G(y) = \lim_{n \rightarrow +\infty} g_n(y) = \sup_{n \rightarrow \infty} g_n(y)$$

$G$  is measurable.

### Step 3

First we show that the equation in theorem holds for  $f = \mathbb{1}_E$ . By prop 53.10

$$\int_X \left( \int_Y f_x d\nu \right) d\mu = \int_X \nu(E_x) d\mu = (\mu \times \nu)(E)$$

while

$$\int_{X \times Y} f d(\mu \times \nu) = \int_{X \times Y} \mathbb{1}_E d(\mu \times \nu) = (\mu \times \nu)(E)$$

By two equations above:

$$\int_X \left( \int_Y f_x d\nu \right) d\mu = \int_{X \times Y} f d(\mu \times \nu)$$

Second we then prove for ant measurable  $f \geq 0$

There exists a increasing sequence  $\{f_n\}_{n \in \mathbb{N}}$  of simple non-negative functions converges pointwisely to  $f$ . Then define

$$g_n(y) = \int_X (f_n)_y d\mu$$

Note that

$$\int_{X \times Y} f d\mu \times \nu = \int_Y \left( \int_X f_y d\mu \right) d\nu = \int_Y g_n d\nu$$

take the limits

$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \int_{X \times Y} \lim_{n \rightarrow +\infty} f_n d(\mu \times \nu) \\ &= \int_Y \lim_{n \rightarrow +\infty} g_n d\nu \\ &= \int_Y \lim_{n \rightarrow +\infty} \left( \int_X (f_n)_y d\mu \right) d\nu \\ &= \int_Y \left( \int_X f_y d\mu \right) d\nu \end{aligned}$$

## 53.25 Corollary

Fubini-Tobelli holds for  $f \in L^1(X \times, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$

### 53.25.1 Proof

Apply the theorem to  $f \vee 0$  and  $-(f \wedge 0)$ , since

$$f = f \vee 0 - (-(f \wedge 0)) = f \vee 0 + f \wedge 0$$

## 53.26 Remark

Deny of the corollary neither hold nor make sense. The integral gives either  $+\infty$  or  $-\infty$

## 53.27 Remark

If  $X = Y = \mathbb{R}$  and  $\Sigma = \mathcal{B}(\mathbb{R})$  For  $f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f \in L^1(I_{\lambda^2})$ , you can find a rectangle  $E \subseteq R = [a, b] \times [c, d]$  And define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Then apply Fubini-Tobelli theorem.

**Example**

- $E = \{(x, y) \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq x\}$

$$\iint_E \sim (x + y) dx dy$$

- 

$$\begin{aligned} \iint_E \sin(x + y) dx dy &= \int_0^{\frac{\pi}{2}} \int_0^x \sin(x + y) dy dx \\ &= \int_0^{\frac{\pi}{2}} -\cos(x + y) \Big|_0^x dx \\ &= \int_0^{\frac{\pi}{2}} -\cos(2x) + \cos(x) dx \\ &= \left( -\frac{\sin(2x)}{2} + \sin(x) \right) \Big|_0^{\frac{\pi}{2}} \\ &= 1 \end{aligned}$$

**53.28 Notation**

$U \subseteq \mathbb{R}^n$  is an open set.  $C_c^0(U)$  denotes the set of continuous functions  $f : U \rightarrow \mathbb{R}$  that have compact support

$$\text{Supp}(f) := \{x \in U \mid f(x) \neq 0\}$$

**53.29 Remark**

Functions of  $C_c^0(U)$  are measurable

Let  $g : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \mathbb{R}^m$  be a differentiable function. Then the Jacobian of  $g$  is the matrix

$$J_g(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{pmatrix}$$

where

$$g(x_1, \dots, x_n) = \begin{pmatrix} g_1(x_1, \dots, x_n) \\ \vdots \\ g_m(x_1, \dots, x_n) \end{pmatrix}$$

The Jacobian is related to the differential of  $g$ . In fact

$$\begin{aligned} dg_p : \mathbb{R}_p^n &\rightarrow \mathbb{R}_{g(p)}^m \\ \sigma &\mapsto J_g|_p(\sigma) \end{aligned}$$

### 53.30 Theorem(Change of variables for the Lebesgue integral)

Let  $V \subseteq \mathbb{R}^n$  be an open set, and let  $\varphi : V \rightarrow \mathbb{R}^n$  be a  $C^1$ -differ morphism, then

$$\int_{\varphi(V)} f d\lambda^n = \int_V (f \circ \varphi) |\det J_\varphi| d\lambda^n \quad \forall f \in C_c^0(\varphi(V))$$

#### Proof

### 53.31 Remark

The theorem can be generalized to a bigger classes of functions. In fact, it possible to show:

It holds whenever one of the two integrals exists  
(Zorich II)

### 53.32 Compute integrals in $\mathbb{R}^n$

#### 53.32.1 Example

$$f(x, y) = \frac{1}{1+x^2+y^2}$$

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \sqrt{3}x; 1 < x^2 + y^2 < 4\} \int_A f dx dy = ?$$

Use polar coordinates

$$\begin{aligned} \varphi : [0, +\infty[ \times [0, 2\pi[ &\rightarrow \mathbb{R}^2 \\ (\rho, \theta) &\mapsto (\rho \cos \theta, \rho \sin \theta) \end{aligned}$$

$\varphi$  is  $C^1$ -differentiable.

use the theorem

$$\begin{aligned} \int_A f(x, y) dx dy &= \int_{\tilde{A}} (f \circ \varphi) |\det J_\varphi| d\rho d\theta \\ &= \int_0^{\frac{\pi}{3}} \int_1^2 \frac{\rho}{1+\rho^2} d\rho d\theta \\ &= \int_0^{\frac{\pi}{3}} \left[ \frac{1}{2} \ln(1+\rho^2) \right]_1^2 d\theta \\ &= \frac{\pi}{6} \ln\left(\frac{5}{2}\right) \end{aligned}$$

### 53.33 Def

$\omega = \sum a_i x_i \in \Omega^n(U)$   $\gamma : [a, b] \rightarrow U$  is piecewise of class  $C^1$  Then we have defined  $\int_\gamma \omega$  The fact that  $\gamma$  is differentiable is important thus we need  $\gamma^* \omega$

Let

$$\varphi : \tau = [c, d] \rightarrow t = [a, b]$$

is a  $C^1$ -diffeomorphism. We say that  $\varphi$  preserves the orientation if  $\varphi' > 0$ , we say that  $\varphi$  reverses the orientation if  $\varphi' < 0$

Assume it preserves orientation

$$\begin{aligned} \int_{\gamma} \omega &= \int_a^b \left( \sum_i a_i(\gamma(t)) \cdot \frac{dx_i}{dt} \right) dt \\ &= \gamma(t) = (x_1(t), \dots, x_n(t)) \\ &= \int_a^b \left( \sum_i a_i(\gamma(\varphi(\tau))) \frac{dx_i}{d\tau} \underbrace{\frac{\tau}{t}}_{= |J_{\varphi^{-1}}|} \right) dt \\ \text{the change of variables} &= \int_c^d \sum_i a_i(\gamma(\varphi(\tau))) \frac{dx_i}{d\tau} d\tau \\ &= \int_{\gamma \circ \varphi} \end{aligned}$$

We call  $\gamma \circ \varphi$  a reparameterization of the curve  $\gamma$ , with the  $C^1$ -differ  $\varphi$ . If  $\varphi$  preserves the orientation, then

$$\int_{\gamma} \omega = \int_{\gamma \circ \varphi} \omega$$

if reverse

$$\int_{\gamma} \omega = - \int_{\gamma \circ \varphi} \omega$$

### 53.34 Def

$$\omega = \sum_{i=1}^n a_i dx_i \in \Omega^1(U)$$

- We say that  $\omega$  is **closed** if  $d\omega = 0$
- We say that  $\omega$  is **exact** in  $V \subseteq U$  if there exists a mapping  $f : V \rightarrow \mathbb{R}$  s.t.  $\omega = df$  in  $V$

### Goal:

to relate the notions of exact forms/closed forms/integrals along curves.



### 53.35 Def

Let  $X$  be topological space.  $U \in X$  is connected if cannot be written as disjoint union of non-empty open sets.

Equally,  $U \in X$  is connected if

$$U = A \sqcup B \Rightarrow A = \emptyset \text{ or } B = \emptyset$$

### 53.36 Lemma

Let  $U \subseteq \mathbb{R}^n$  be a connected open set. Then any two points of  $U$  can be joined by a piecewise  $C^1$ -curve.

#### Proof

Take  $a \in U$  let  $H \subseteq U$  the set of points that can be joined to  $a$  with a piecewise  $C^1$ -curve. Let  $K = U \setminus H$ .

Take  $x \in H$  then  $\exists \mathcal{B}(x, \epsilon) \subseteq U$  since  $U$  is open

Any two points in  $\mathcal{B}(x, \epsilon)$  can be jointed with a segment. Take any  $y \in \mathcal{B}(x, \epsilon)$ , this  $y$  can be joined to  $a$  with a piecewise  $C^1$ -curve.

This means that  $H$  is open. Similarly  $K$  is also open. Since  $U = H \sqcup K$  and for  $U$  connected  $H = U$

### 53.37 Notation

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a curve, we define:

$$(-\gamma)(t) = \gamma(a + b - t)$$

as reserved curve of  $\gamma$

$$\int_{-\gamma} \omega = - \int_{\gamma} \omega$$

### 53.38 Def

If  $\gamma_1 : [a, b] \rightarrow U$  and  $\gamma_2 : [b, c] \rightarrow U$  are two curves and  $\gamma_1(b) = \gamma_2(b)$ , we define

$$\begin{aligned} \gamma_1 \sqcup \gamma_2 : [a, c] &\rightarrow U \\ t &\mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t) & \text{if } t \in [b, c] \end{cases} \end{aligned}$$

One has

$$\int_{\gamma_1 \sqcup \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

### 53.39 Theorem

The following statements are equivalent:

- 1  $\omega$  is exact in a connected open set  $V \subseteq U$
- 2  $\int_\gamma \omega$  depends only on the end-point of  $\gamma$  ( $\forall \gamma$  in  $V$ )
- 3  $\int_\gamma \omega = 0$  for all closed curves  $\gamma$  in  $V$

#### Proof

1  $\rightarrow$  2 Since  $\omega = df$

$$\int_\gamma \omega = \int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

2  $\rightarrow$  3 Take a closed curve such that  $\gamma(a) = \gamma(b) = p \in V$ . Take the curve  $\gamma' = \{p\}$

$$\int_\gamma \omega = \int_{\{p\}} \omega = 0$$

3  $\rightarrow$  2 consider  $\gamma \sqcup -\gamma$ , trivial

2  $\rightarrow$  1 Fix  $p \in V$  For any  $(p+t) \in V$  by the lemma 53.36, there exists a curve  $\gamma_x$  piecewise  $C^1$  that connects  $p$  and  $(p+t)$   
Let

$$\begin{aligned} f: V &\rightarrow \mathbb{R} \\ x &\mapsto \int_{\gamma_x} \omega \end{aligned}$$

Take  $\omega = \sum_{i=1}^n a_i dx_i$ , one knows that

$$\inf_{x \in [p_i, p_i+t_i]} a_i(x) \leq \int_{p_i}^{p_i+t_i} a_i dx_i \leq \sup_{x \in [p_i, p_i+t_i]} a_i(x)$$

Since  $a_i$  are continuous

$$\lim_{t_i \rightarrow 0} \inf_{x \in [p_i, p_i+t_i]} a_i(x) = \lim_{t_i \rightarrow 0} \sup_{x \in [p_i, p_i+t_i]} a_i(x) = a_i(p_i)$$

This means that for any curve  $\gamma$  with end points  $p$  and  $p+t$

$$\int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

Hence

$$o(\|t\|) + d_p f(t) = f(p+t) - f(p) = \int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

which ends the proof.

### 53.40 Poincaré Lemma

Let  $\omega$  be a 1-form. Then  $d\omega = 0$  iff

$$\forall p \in U \quad \exists V \in \mathcal{V}_p, f : V \rightarrow \mathbb{R} \in C^1(\mathbb{R})$$

such that

$$df = \omega$$

#### Proof

Assume that  $\omega$  is locally exact, then locally

$$\omega = df$$

Assume that  $U = \bigcup_{\alpha} V_{\alpha}$ ,  $\omega$  is exact,  $d$  on  $V_{\alpha}$  is  $d(df) = 0$

Assume that  $d\omega = 0, \forall p = (p_1, \dots, p_n) \in U$ , consider an open ball  $\mathcal{B}_p$  centered on  $p$  and all contained in  $V$

Fix an  $x \in \mathcal{B}_p$  Consider  $B(t) = p + t(x - p), t \in [0, 1]$  Let

$$f(x) = \int_{B(t)} \omega = \int_0^1 a_i(B(t))(x_i - p_i) dt$$

We put  $V = B(p)$  and show that  $df = \omega$

Note that

$$\begin{aligned} 0 = d\omega &= \sum_i d \wedge dx_i = \sum_i \left( \sum_j \frac{\partial a_i}{\partial x_j} dx_j \right) dx_i = \sum_{ij} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \\ \frac{\partial a_i}{\partial x_j} &= \frac{\partial a_j}{\partial x_i} \end{aligned}$$

Consider  $x_i = x_1$

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \int_0^1 \left( \frac{\partial a_1}{\partial x_1} t(x_1 - p_1) + a_1 + \sum_{i>1} \frac{\partial a_i}{\partial x_1} t(x_i - p_i) \right) dt \\ &= \int_0^1 \left( \frac{dt}{d} (a_1 B(t)) t + a_1 \right) dt \\ &= \int_0^1 \frac{dt}{d} (a_1 \circ B(t) \cdot t) dt \\ &= a_1 \circ B(1) \\ &= a_1(x) \end{aligned}$$

When locally exact implies exact depends on the topological properties of  $U$

### 53.41 Notation

For any mapping  $\gamma : [a, b] \rightarrow U$

- $\gamma$  is called a closed curve if  $\gamma(a) = \gamma(b)$  and  $\gamma$  is a curve
- $\gamma$  is called a path if  $\gamma$  is of class  $C^0$
- $\gamma$  is called a loop if  $\gamma$  is a closed path

We try to integrate forms along paths.

### 53.42 Def

Let  $\omega$  be a closed 1-form  $\gamma : [a, b] \rightarrow U$  be a path. We derive a partition of  $[a, b]$ :

$$0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} < b$$

such that  $\gamma_i := \gamma|_{[t_i, t_{i+1}]}$  form a finite open covering

$$\bigcup_i B_i \supseteq \gamma([a, b])$$

We may assume that  $\gamma_i \subseteq B_i$ . From Poincare lemma, we find a function  $f_i : B_i \rightarrow \mathbb{R}$  s.t. on  $B_i$

$$df_i = \omega$$

We define the integral along path  $\gamma$  as

$$\int_{\gamma} \omega := \sum_i f_i(t_{i+1}) - f_i(t_i)$$

### 53.43 Prop

The def of path integral is unique and coincides with the usual definition 52.2

#### Proof

Let  $\mathcal{P} = \{t_1, \dots, t_n\}$  be a partition of  $[a, b]$  and  $\mathcal{P}'$  be a refinement of  $\mathcal{P}$ , namely  $\mathcal{P} \in \mathcal{P}'$ . We know that  $\exists t' \in \mathcal{P}' \setminus \mathcal{P}$  s.t.  $t' \in ]t_i, t_{i+1}[$ ,  $\gamma(t') \subseteq B_i$

Compute the integral with the partition  $\mathcal{P}'$ , in the summation:

$$f_i(\gamma(t_{i+1})) - f_i(\gamma(t')) + f_i(\gamma(t')) + c - f_i(\gamma(t_i)) - c$$

This means that if  $\mathcal{P}' \subseteq \mathcal{P}$  then

$$\int_{\gamma}^{(\mathcal{P})} \omega = \int_{\gamma}^{(\mathcal{P}')} \omega$$

For any two partition  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , find the common refinement

$$\int_{\gamma}^{(\mathcal{P}_1)} \omega = \int_{\gamma}^{(\mathcal{P}_1 \cup \mathcal{P}_2)} \omega = \int_{\gamma}^{(\mathcal{P}_2)} \omega$$

### 53.44 Def

Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow U$  be two paths. A **homotopy** between  $\gamma_0$  and  $\gamma_1$  is a continuous mapping:

$$\begin{aligned} H : [a, b] \times [0, 1] &\rightarrow U \\ (s, t) &\mapsto H(s, t) \end{aligned}$$

such that

- $H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$
- $H(a, \cdot) \equiv \gamma_0(a) = \gamma_1(a) \quad H(b, \cdot) \equiv \gamma_0(b) = \gamma_1(b)$

### 53.45 Def: Lebesgue number

Let  $(X, \rho)$  be a metric space and  $\mathcal{U} = \{U_i\}$  be an open covering  $X$

A **Lebesgue number**  $\delta = \delta_{\mathcal{U}}$  (of the open covering  $\mathcal{U}$ ) is a non-negative number that:

If  $Z \subseteq X$  is a subset with  $\text{diam}(Z) < \delta$ , then  $Z \subseteq U_j$  for some  $U_j \in \mathcal{U}$

#### Remark

- $\delta' < \delta$  is also a Lebesgue number
- In principle, a Lebesgue number  $\delta$  can be 0

### 53.46 Lemma

If  $X$  is compact, then for any open covering there exists a positive Lebesgue number.

#### Proof

Let  $\mathcal{U}$  be an open covering. Since  $X$  is compact, then

$$\mathcal{U} \supseteq \{U_1, \dots, U_n\}$$

If one of  $U_i$  is equal to  $X$ , then any  $\delta > 0$  is a Lebesgue number. So we can assume that  $\forall i \in \{1, \dots, n\}$  the set

$$C_i := X \setminus U_i$$

is non-empty. let the mapping  $f$  be:

$$\begin{aligned} f : X &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{n} \sum_{i=1}^n d(x, C_i) \end{aligned}$$

$f$  is continuous on a compact set. Hence it attains a minimum. This minimum is not 0 because  $d(x, C_i) > 0$  for some  $i$

Let  $\delta = \min_{x \in X} f(x)$ , we show that  $\delta$  is a Lebesgue number

Let  $Y \subseteq X$  s.t.  $\text{diam}(Y) < \delta$ . Take  $x_0 \in Y$ , then

$$Y \subseteq \mathcal{B}(x_0, \delta)$$

Since  $f(x_0) \geq \delta$ , then there exists  $i$  such that

$$d(x_0, C_i) \geq \delta$$

(otherwise  $f(x_0) = \frac{1}{n} \sum_{i=1}^n d(x_0, C_i) < \frac{1}{n} n\delta = \delta$ )

this means

$$(X \setminus U_i) \cap Y = \emptyset$$

Hence  $Y \subseteq U_i$

### Exercise

Being homotopic is an equivalence relation.

## 53.47 Theorem(homotopy invariance of the integrals)

Ler  $\omega$  be a closed form on an open set  $U$ . Let  $\gamma_0, \gamma_1$  be homotopy paths in  $U$ , then

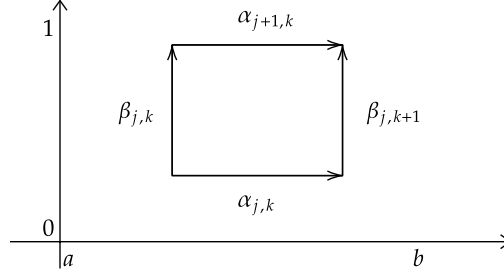
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

### Proof

Since  $\omega$  is closed, it's locally exact (by Poincare lemma 53.40) Let  $H : [a, b] \times [0, 1] \rightarrow U$  be the homotopy between  $\gamma_0$  and  $\gamma_1$ . Let  $\mathcal{B} = \{B_i\}$  be a open cover of  $\text{Im}(H) \subseteq U$  made of finite many open balls (since compact) where  $\omega$  is locally exact in each.

Consider  $W_i = H^{-1}(B_i)$  We have covered the departure set  $\mathcal{D}_H$  with  $\{W_i\}$ . Since  $\mathcal{D}_H$  is compact, we can choose a Lebesgue number  $\delta > 0$  for the covering  $\{W_i\}$  (by lemma 53.36)

Divide  $\mathcal{D}_H$  into rectangles  $\{R_j k\}$  having diameter  $< \delta$



The border of the rectangles are loops. We divide the loops in this way:

$$\partial R_{j,k} = \alpha_{j,k} \sqcup \beta_{j,k+1} \sqcup (-\alpha_{j+1,k}) \sqcup (-\beta_{j,k})$$

$$H\partial R_{j,k} = H\alpha_{j,k} \sqcup H\beta_{j,k+1} \sqcup H(-\alpha_{j+1,k}) \sqcup H(-\beta_{j,k})$$

$H\partial R_{j,k}$  is closed curve contained in some  $B_i$ , but in such balls  $\omega$  is exact, then  $\int_{\partial R_{j,k}} \omega = 0$ . Do this  $\forall j, k$

$$0 = \sum_{j,k} \int_{\partial R_{j,k}} \omega = \sum_{j,k} \left( \int_{H\alpha_{j,k}} \omega + \int_{H\beta_{j,k+1}} \omega - \int_{H\alpha_{j+1,k}} \omega - \int_{H\beta_{j,k}} \omega \right)$$

Moreover, if we do this for some particular  $j, k$

$$0 = \int_{H(a,0) \rightsquigarrow (b,0)} \omega + \int_{H(b,0) \rightsquigarrow (b,1)} \omega - \int_{H(a,1) \rightsquigarrow (b,1)} \omega - \int_{H(a,0) \rightsquigarrow (a,1)} \omega$$

Since  $H(a,0) \rightsquigarrow (b,0)$  and  $H(a,1) \rightsquigarrow (b,1)$  are points, then

$$\int_{H(a,0) \rightsquigarrow (a,1)} \omega = \int_{H(b,0) \rightsquigarrow (b,1)} \omega$$

Proved





## Chapter 54

# Winding Numbers

### 54.1 Def: Free Homotopy

Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow U$  be two loops (namely  $\gamma(a) = \gamma(b)$ )

A **free homotopy** between  $\gamma_0$  and  $\gamma_1$  is a continuous mapping:

$$\begin{array}{ccc} H : [a, b] \times [0, 1] & \rightarrow & U \\ (s, t) & \mapsto & H(s, t) \end{array}$$

such that

- 

$$H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$$

- For any fixed  $t_0$

$$H(\cdot, t_0)$$

is a loop

### 54.2 Notation

A path  $\gamma : [a, b] \rightarrow I$  is said simple if  $\gamma|_{]a, b[}$  is injective (No self-cross this is)

### 54.3 Jordan Theorem

Let  $\gamma$  be a simple loop  $\gamma : [a, b] \rightarrow U$ , then  $\mathbb{R}^2 \setminus \gamma([a, b])$  consists exactly of two connected components. One of this is bounded (interior), the other one unbounded (exterior). Moreover  $\gamma([a, b])$  is the boundary of two components.

**Proof**

Exercise?

Let  $S^1 \subseteq \mathbb{R}$  be the unit circle. Let  $c : I \rightarrow S^1$  be a closed curve. ( $0 \in I$ )

We want to measure the "net" turns of  $c$  around the region.

$$\begin{aligned} c(t) &= (x(t), y(t)), x^2 + y^2 = 1 \quad \forall t \in I \\ &\Rightarrow 2(xx' + yy') = 0 \end{aligned}$$

Consider  $\varphi_0 \in [0, 2\pi[$  such that

$$\cos(\varphi_0) = x(0), \sin(\varphi_0) = y(0)$$

Define the function

$$\begin{aligned} \varphi : I &\rightarrow [0, 2\pi] \\ t &\mapsto \varphi_0 + \int_0^t (xy' - yx') dt \end{aligned}$$

$\varphi$  is called an angle function of  $c$ , namely is of class  $C^0$  and  $c(t) = (\cos(\varphi(t)), \sin(\varphi(t)))$

And

$$\begin{aligned} F(t) &= (x(t) - \cos \varphi(t))^2 + (y(t) - \sin \varphi(t))^2 \quad t \in I \cap [0, 2\pi] \\ F'(t) &= -y \sin \varphi (yy' + xx') - x \cos \varphi (xx' + yy') = 0 \end{aligned}$$

This implies  $F(t) = C$  a constant. However

$$F(0) = (x(0) - x(0))^2 + (y(0) - y(0))^2 = 0$$

So  $C = 0$  It means that

$$\begin{cases} \cos(\varphi(t)) = x(t) \\ \sin(\varphi(t)) = y(t) \end{cases} \quad \forall t \in I$$

**54.4 Def**

Let  $c : [a, b] \rightarrow S^1$  be a closed curve. Let  $\varphi$  be the angular function of  $c$ . We define the winding number of  $c$  as:

$$n(c) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

Since  $c$  is a closed curve,  $n(c) \in \mathbb{Z}$

**54.5 Def**

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  be a closed curve. ( $\gamma_p + \rho(t)c(t)$ ), when  $c(t) \in S^1$

$$\gamma(t) = p + \rho(t)(\cos(\theta(t)) + \sin(\theta(t)))$$

Then we define the winding number of  $\gamma$  at  $p$

$$n_p(\gamma) := n(c)$$

## 54.6 Prop

Let  $\gamma = p + \rho(t)c(t)$  be a closed curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  then

$$n_p(\gamma) = \frac{1}{2\pi i} \int_C \omega_0$$

where

$$\omega_0 = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

### Proof

$$\begin{aligned} \frac{1}{2\pi} \int_C \omega_0 &= \frac{1}{2\pi} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt \\ &= \frac{\varphi(b) - \varphi(a)}{2\pi} \\ &= n(c) \end{aligned}$$

## 54.7 Prop

Let  $\gamma_0, \gamma_1 : [0, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  be two closed curves. Then they're freely homotopic iff

$$n_p(\gamma_0) = n_p(\gamma_1)$$

### Proof

$\Rightarrow$  Follows from the invariance of the integral of  $\omega_0$  along free homotopic loops. (We use the previous prop)

$\Leftarrow$  Assume  $\gamma_0 = p + \rho_0 c_0(t)$ ;  $\gamma_1 = p + \rho_1 c_1(t)$ . The angle functions of  $c_0$  and  $c_1$  are  $\varphi_0$  and  $\varphi_1$  respectively. Consider

$$\varphi(s, t) = (1 - t)\varphi_0(s) + t\varphi_1(s)$$

$$(s, t) \in [0, b] \times [0, 1]$$

$$H(s, t) = (\cos(\varphi(s, t)), \sin(\varphi(s, t)))$$

To claim this is a free homotopy between  $c_0$  and  $c_1$ , we have to check that any curve in the homotopy is closed.

$$\begin{aligned} \varphi(b, t) - \varphi(0, t) &= (1 - t)\varphi_0(b) + t\varphi_1(b) - (1 - t)\varphi_0(0) - t\varphi_1(0) \\ &= (1 - t)(\varphi_0(b) - \varphi_0(0)) + t(\varphi_1(b) - \varphi_1(0)) \\ &= 2\pi(1 - t)n_p(\gamma_0) + 2\pi t n_p(\gamma_1) \\ &= 2\pi n_p(\gamma_0) \end{aligned}$$

So we have proved that  $c_0$  and  $c_1$  are freely homotopic. What we anticipate is

$$\gamma_i \sim_{hom} \hat{\gamma}_i = p + c_i \sim_{hom} c_i$$

Now we only need to prove the first homotopic.

$$H(s, t) = p + \frac{\rho_1(s)}{(1-t) + t|\rho_1(s)|} c_i(s)$$

### 54.8 Def

Let  $F : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a differential mapping. We say that  $p \in U$  is a zero of  $F$  if  $F(p) = 0$ . If then exists a neighborhood  $V$  of  $p$  such that  $V$  contains no zero of  $F$  other than  $p$ , then  $p$  is called isolated zero.

If  $p$  is a zero of  $F$  and  $dF|_p$  is non singular at  $p$ , then we say that  $p$  is a simple zero.

### 54.9 Remark

By the inverse function then  $F$  is one to one in a neighborhood of a simple zero. Hence a simple zero if isolated.

$$F(x, y) = (f(x, y), g(x, y))$$

$D \subseteq U$  is a closed disk, with boundary  $\partial D = C$  Assume that  $C$  doesn't contain zeros of  $F$ . Consider the form

$$\theta = \frac{f dg - g df}{f^2 + g^2} \in \Omega^1(U \setminus \{(x, y) : F(x, y) = 0\})$$

### 54.10 Def

The index of  $F$  in  $D$ , is defined as

$$n(F, D) := \frac{1}{2\pi} \int_C \theta$$

See that  $\theta = F^* \omega_0$ ,  $\omega_0 = \frac{-y dx + x dy}{x^2 + y^2}$

$$\begin{aligned} n(F, D) &= \frac{1}{2\pi} \int_C \theta \\ &= \frac{1}{2\pi} \int_C F^* \omega_0 \\ &= \frac{1}{2\pi} \int_{F \circ C} \omega_0 \\ &= (\text{winding number of } F \circ C \text{ at the center of } FD) \end{aligned}$$

### 54.11 Remark

$$n(F, D) = \frac{1}{2\pi} \int_C \theta = \frac{1}{2\pi} \int_{F \circ C} \omega_0$$

### 54.12 Prop

If  $n(F, D) \neq 0$  then  $\exists q \in D$  s.t.  $F(q) = 0$

#### Proof

Assume that such  $q$  doesn't exist. Let  $p$  be the center of  $D$

$$H(s, t) = F((1 - t)C(s) + t \cdot p)$$

then  $H$  is a free homotopy between  $F \circ s \mapsto p$  and  $\circ C$

$$0 = \frac{1}{2\pi} \int_F \circ C \omega_0 = n(F, D)$$

Contradiction

### 54.13 Def

A simple zero  $p$  of  $F$  is said **positive** if  $\det(d_p F) > 0$ , otherwise is said **negative** (what's =0?)

### 54.14 Kronecker Index Theorem

Assume that  $F; U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$  has only finite simple zeros in a disk  $D \subseteq U$  and none of them in  $\partial D$ . Then

$$n(F, D) = P - N$$

where  $P$  is the number of positive simple zeros and  $N$  is the number of negative simple zeros.

#### Lemma

Assume that  $F$  has a simple zero  $p \in D \subseteq U$  then  $n(F, D) = \pm 1$  corresponding to  $\det(d_p F) > 0$  or  $\det(d_p F) < 0$

**Proof**

After translating, we assume that  $p = (0, 0)$  By definition of differential

$$F(q) = Tq + R(q) |q| \quad q \rightarrow 0 \quad \lim_{q \rightarrow 0} R(q) = 0$$

Consider the mapping

$$\begin{aligned} H : [0, b] \times [0, 1] &\rightarrow \mathbb{R}^2 \\ (s, t) &\mapsto Tq + (1 - t)R(q) |q| \end{aligned}$$

If we show that for  $D$  small enough,  $H(q, t) \neq 0, \forall q \in D, t \in [0, 1]$  (\*) Then  $H(C(s), t)$  is a free homotopy between  $F_0 \circ C$  and  $T \circ C$  (Exercise) Then

$$n(F, D) = n(T, D)$$

which one should prove is  $n(T, D) = 1$

For the part (\*), since  $p$  is non-singular,  $c = \frac{1}{\|T^{-1}\|} > 0$ , then

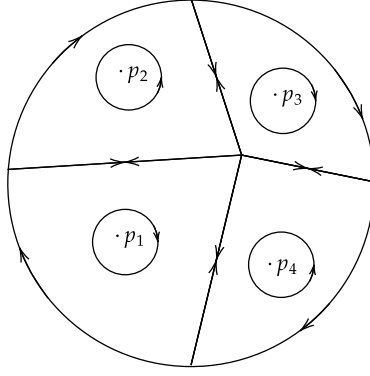
$$\|q\| = \|T^{-1}Tq\| \leq \|T^{-1}\| \|Tq\| = \frac{\|Tq\|}{c}$$

Take  $\epsilon > 0$  such that  $\forall p \in D$  (disk of radius  $< \epsilon$ )  $\|R(q)\| \leq \frac{c}{2}$  Then if  $q \in D_\epsilon \setminus \{0, 0\}$

$$\begin{aligned} \|H(q, t)\| &= \|Tq + (1 - t)R(q) |q|\| \\ &\geq \|Tq\| - (1 - t) \|R(q)\| \|q\| \\ &\geq c \|q\| - \frac{(1 - t)c}{2} \|q\| \\ &\geq c \|q\| - \frac{c}{2} \|q\| > 0 \end{aligned}$$

**Proof**

Let  $p_1, \dots, p_k$  be the zeros of  $F$  in  $D$ . Since these zeros are isolated, we choose a set of balls  $\{B_i\}_{i=1}^k$  containing  $\{p_i\}_{i=1}^k$  respectively. Then cut the circle into  $k$  sections correspondingly.



One can easily construct the homotopy for these balls and sections.  
By lemma

$$\int_{F \circ C} \omega_0 = \sum_{i=1}^k \int_{FB_i} \omega_0 = \sum_{i=1}^k \text{sgn}(\det d_{p_i} F) = P - N$$

**54.15 Lemma**

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$

1 If  $\gamma \in C^1([a, b])$  then

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a)$$

2 For any  $c \in \mathbb{R}^n$ :

$$\left\langle c, \int_a^b \gamma(t) dt \right\rangle = \int_a^b \langle c, \gamma(t) \rangle dt$$

3

$$\left\| \int_a^b \gamma(t) dt \right\| \leq \int_a^b \|\gamma(t)\| dt$$

**Proof**

1 Is consequence of the fundamental theorem of calculus computation

2

$$\begin{aligned}
\left\langle c, \int_a^b \gamma(t) dt \right\rangle &= \sum_{i=1}^n c_i \int_a^b \gamma_i(t) dt \\
&= \int_a^b \sum_i c_i \gamma_i(t) dt \\
&= \int_a^b \langle c, \gamma(t) \rangle dt
\end{aligned}$$

3 We apply (2) with  $c = \int_a^b \gamma(t) dt \in \mathbb{R}^n$

$$\begin{aligned}
\left\| \int_a^b \gamma(t) dt \right\| &= \left\langle \int_a^b \gamma(\tau) d\tau, \int_a^b \gamma(t) dt \right\rangle \\
(\text{by (2)}) &= \int_a^b \left\langle \int_a^b \gamma(\tau) d\tau, \gamma(t) \right\rangle dt = (*)
\end{aligned}$$

Apply Cauchy-Schwartz inequality

$$(*) \leq \int_a^b \left\| \int_a^b \gamma(\tau) d\tau \right\| \|\gamma(t)\| dt = \left\| \int_a^b \gamma(\tau) d\tau \right\| \int_a^b \|\gamma(t)\| dt$$

Divide by  $\left\| \int_a^b \gamma(\tau) d\tau \right\|$

## 54.16 Def

Let  $\mathcal{P} = \{t = t_a, t_1, \dots, t_n = b\}$ ,  $p_i = \gamma(t_i)$

$$l_{\mathcal{P}}(\gamma) = \sum_{i=0}^n \|p_{i+1} - p_i\|$$

The length of  $\gamma$  is

$$l(\gamma) := \sup_p \{l_p(\gamma)\}$$

If  $l(\gamma) < +\infty$ , then path  $\gamma$  is said rectifiable.

### 54.16.1 Example

Consider  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$   $\gamma_1(t) = t$   $\gamma_2(t) = \begin{cases} t \sin \frac{\pi}{2t} & \text{if } t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\mathcal{P}_n \{0, \frac{1}{2n+1}, \frac{1}{2n-1}, \dots, \frac{1}{3}, 1\}$$



Recall that  $\sin((j - \frac{1}{2})\pi) = (-1)^{j+1}$ , then for  $i > 0$ ,

$$\begin{aligned} p_i = \gamma(t_i) &= \left( \frac{1}{2j-1}, \frac{1}{2j-1}(-1)^{j+1} \right) \\ \|p_i - p_j\| &= \sqrt{\left(\frac{2}{4j^2-1}\right)^2 + \left(\frac{4j}{4j^2-1}\right)^2} \\ &= \sqrt{\frac{4(4j^2+1)}{(4j^2-1)^2}} \\ &\geq \sqrt{\frac{4(4j^2-1)}{(4j^2-1)^2}} \\ &= \sqrt{\frac{4}{4j^2-1}} \\ &\geq \frac{1}{j} \end{aligned}$$

then

$$l_p(\gamma) \geq \sum_{j=1}^n \frac{1}{j} \Rightarrow l(\gamma) = +\infty$$

## 54.17 Prop

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be of class  $C^1$ , then  $\gamma$  is rectifiable and

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt$$

moreover  $l(\gamma)$  doesn't depend on the parametrization of  $\gamma$

### Proof

Let  $\mathcal{P}$  be a partition of  $[a, b]$ .

$$\|\gamma(t_{j+1}) - p_j\| = \left\| \int_{t_i}^{t_{j+1}} \gamma'(t) dt \right\| \leq \int_{t_i}^{t_{j+1}} \|\gamma'(t)\| dt$$

So

$$\begin{aligned} l_{\mathcal{P}}(\gamma) &= \sum_{j=0}^{n-1} \|\gamma(t_{j+1}) - \gamma(t_j)\| \\ &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt \\ &= \int_a^b \|\gamma'(t)\| dt \end{aligned}$$

take sup over  $\mathcal{P}$

$$l(\gamma) \leq \int_a^b \|\gamma'(t)\| dt$$

which means  $l(\gamma)$  is finite.

Let's prove the other inequality:

$\gamma'(t)$  is continuous on  $[a, b]$  then is uniformly continuous (Cantor-Itener Theorem):  $\forall \epsilon > 0 \exists \delta = \delta(\epsilon)$  such that if  $t, s \in [a, b]$  and  $|t - s| < \delta$ . Then  $\|\gamma'(t) - \gamma'(s)\| < \epsilon$

Choose now a partition  $\mathcal{P}$  such that  $|t_{j+1} - t_j| < \delta \forall i$  then if  $t \in [t_i, t_{i+1}]$  then  $\|\gamma'(t) - \gamma'(t_j)\| < \epsilon$

$$\begin{aligned} \|\gamma'(t)\| - \|\gamma'(t_j)\| &\leq \| \|\gamma'(t)\| - \|\gamma'(t_j)\| \| \\ &\leq \|\gamma'(t) - \gamma'(t_j)\| \\ &\leq \epsilon \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt &\leq \|\gamma'(t_j)\| (t_{j+1} - t_j) + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t_j) + \gamma'(t_j)) dt \right\| + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} \gamma'(t) dt \right\| + \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t_j)) dt \right\| + (t_{j+1} - t_j)\epsilon \\ &\leq \underbrace{\|\gamma(t_{j+1}) - \gamma(t_j)\|}_{\text{lemma 1}} + \underbrace{\int_{t_j}^{t_{j+1}} \|\gamma'(t) - \gamma'(t_j)\| dt}_{\text{lemma 2}} + (t_{j+1} - t_j)\epsilon \\ &\leq \|\gamma(t_{j+1}) - \gamma(t_j)\| + 2(t_{j+1} - t_j)\epsilon \end{aligned}$$

Hence

$$\begin{aligned} \int_a^b \|\gamma'(t)\| dt &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt \\ &\leq \sum_{j=0}^{n-1} \|\gamma(t_{j+1}) - \gamma(t_j)\| + 2(t_{j+1} - t_j)\epsilon \\ &= l_{\mathcal{P}}(\gamma) + 2\epsilon(b - a) \leq l(\gamma) + 2\epsilon(b - a) \end{aligned}$$

Take limit for  $\epsilon \rightarrow 0$

Now the formula proved. We'll show that  $l(\gamma)$  doesn't depends on the parametrization. Let  $\varphi : [\alpha, \beta] \rightarrow [a, b]$  is of class  $C^1$ ,  $\varphi'(\tau) \neq 0, \forall \tau \in [\alpha, \beta]$  Let

$$\tilde{\gamma} = \gamma \circ \varphi$$

might as while assume  $\varphi' > 0$

$$\begin{aligned} l(\gamma) &= \int_a^b \|\gamma'(t)\| dt = \int_\alpha^\beta \|\gamma'(\varphi(\tau))\| \varphi'(\tau) d\tau \\ &= \int_\alpha^\beta \|\tilde{\gamma}'(\tau)\| d\tau = l(\tilde{\gamma}) \end{aligned}$$

Same for  $\varphi' < 0$

## 54.18 Corollary(exercise)

If  $\gamma$  is a curve (piecewise  $C^1$ ), then  $\gamma$  is rectifiable and the length is the sum of the length of it's  $C^1$  pieces.

## 54.19 Def

A  $C^1$ -curve is **regular** if  $\gamma'(t) \neq 0$  for any  $t \in [a, b]$  A piecewise  $C^1$ -path (curve) is regular if all its pieces are regular

## 54.20 Def

$$N := \frac{T}{\|T\|}$$

is **normal vector** of  $T$

## 54.21 Def

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  a  $C^1$  curve; Let  $l$  be the length of  $\gamma$  (by theorem proved  $l(\gamma) < +\infty$ ) Let's define the following function:

$$s(t) := \int_a^t \|\gamma'(u)\| du$$

$s(t)$  is the length of  $\gamma|_{[a,t]}$  The function  $\|\gamma'(u)\|$  is continuous, hence

$$s'(t) = \|\gamma'(t)\|$$

Now assume that  $\gamma$  is  $C^1$  and **regular** ( $\gamma'(t) \neq 0, \forall t \in [a, b]$ ), then  $s'(t) > 0$

So  $s : [a, b] \rightarrow [0, l]$  is a  $C^1$ -diffeomorphism, the inverse is

$$t : [0, l] \rightarrow [a, b]$$

$$\frac{dt}{ds} = \frac{1}{\|\gamma'(t)\|}$$

We reparameterize  $\gamma$  with  $t$  and get

$$\tilde{\gamma}(s) = (\gamma \circ t)(s)$$

$\tilde{\gamma} : [0, l] \rightarrow \mathbb{R}^n$  we say that  $\tilde{\gamma}$  is the reparameterization of  $\gamma$  with respect to its **curvilinear coordinate**  $s(t)$

### 54.22 Def

$f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$   $f$  is a  $C^{(k)}$ -differ if

- $f$  is of class  $C^{(k)}$
- $f$  is bijection, and the inverse is  $C^{(k)}$

### 54.23 Def

In general

$$\gamma : [a, b] \rightarrow \mathbb{R}^n \rightsquigarrow \tilde{\gamma} : [0, l] \rightarrow \mathbb{R}^n$$

regular and  $C^1$

$$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt} \frac{dt}{ds} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$$\left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1$$

$$T(t) := \frac{d\tilde{\gamma}}{ds} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

tangent: (vector)  $\rightarrow$  vector of norm 1

$$0 = \frac{d}{dt} \|T(t)\|^2 = \frac{d}{dt} \langle T(t), T(t) \rangle = 2 \langle T(t), T'(t) \rangle \Leftrightarrow T'(t) \perp T(t)$$

use the fact that in  $\mathbb{R}^n$ ,  $u, v : \mathbb{R} \rightarrow \mathbb{R}^n$  differentiable

$$\frac{d}{dt} \langle u(t), v(t) \rangle = \left\langle \frac{du}{dt}, v(t) \right\rangle + \left\langle u(t), \frac{dv}{dt} \right\rangle$$

then

$$\begin{aligned} \frac{d^2\tilde{\gamma}}{ds^2} &= \frac{d}{ds} \left( \frac{d\tilde{\gamma}}{ds} \right) \\ &= \frac{d}{ds} (T(t)) \\ &= \frac{dT}{dt} \frac{dt}{ds} \\ &= \frac{T'(t)}{\|\gamma'(t)\|} \end{aligned}$$

$N(t) = \frac{d^2\tilde{\gamma}}{ds^2} / \left\| \frac{d^2\tilde{\gamma}}{ds^2} \right\|$ . If  $n = 2$  Along the curve we have a 'moving' canonical basis of

$$\begin{aligned}\mathbb{R}_{\gamma(t)}^2 &= \{T(t), N(t)\} \\ &= \{\alpha T(t) + \beta N(t) \mid \alpha, \beta \in \mathbb{R}\}\end{aligned}$$

$\{T(t), N(t)\}$  is a orthonormal basis of  $\mathbb{R}_{\gamma(t)}^2$



## Chapter 55

# Curvilinear integral

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  measurable and  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  regular

### 55.1 Def

The curvilinear integral of  $f$  along  $\gamma$  is defined as:

$$\int_{\gamma} f(t) ds = \int_a^b f(\gamma(t)) s'(t) dt = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$$

where  $s'(t)$  is curvilinear coordinate.

### 55.2 Prop

Let  $f, g : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  be two measurable functions  $\gamma, \gamma_1, \gamma_2$  (piecewise) regular  $C^1$  curves, then

1  $\alpha, \beta \in \mathbb{R}$

$$\int_{\gamma} (\alpha f + \beta g) ds = \alpha \int_{\gamma} f ds + \beta \int_{\gamma} g ds$$

2 If  $\gamma_a \sqcup \gamma_2$  make sense

$$\int_{\gamma_a \sqcup \gamma_2} f ds = \int_{\gamma_a} f ds + \int_{\gamma_2} f ds$$

3

$$\int_{\gamma_1} f ds = \int_{-\gamma_1} f ds$$

4  $\int_{\gamma} f ds$  doesn't depend on the reparameterization of  $\gamma$

**Proof**

Exercise

**55.3 Geometry**

Let  $V$  be a finite dim vector space over  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ). Let  $g : V \times V \rightarrow K$  be a bilinear function. Choose a basis  $\{\sigma_1, \dots, \sigma_n\}$  we have the Gram matrix  $G = (g(\sigma_i, \sigma_j))_{i,j}$

$$g(x, y) = x^T G y$$

what happens when we change basis:

$$\begin{array}{ccc} & x \in V & \\ b \nearrow & & \nwarrow b' \\ x \in K^n & \xleftarrow{A} & x' \in K^n \end{array}$$

$$g(x, y) = x^T G y = (x'^T)(A^T G A) y'$$

$$\{v_1, \dots, v_n\} \quad G \rightsquigarrow A^T G A$$

**55.4 Def**

Two matrices  $G, G' \in M_{n \times n}(K)$  are said **congruent** if  $\exists A \in GL_n(K)$  (namely  $A$  invertible) such that  $G' = A^T G A$

$$\begin{array}{ccc} \mathcal{L}(V, V; K) & \xrightarrow{\cong} & \mathcal{L}(T_0^2(V); K) \xrightarrow{\cong} \mathcal{L}(V, V^\vee) \\ g \mapsto & & g_s \mapsto [x \mapsto g_s(x \otimes -)] \end{array}$$

In particular

$$\begin{array}{ccc} \tilde{g} : V \rightarrow V^\vee & & \\ x \mapsto & & [y \mapsto g(x, y)] \end{array}$$

- $g : V \times V \rightarrow K$
- $\begin{array}{ccc} g_p : V \times V & \rightarrow & K \\ (x, y) & \mapsto & g(y, x) \end{array}$
- $\begin{array}{ccc} \overline{g_p} : V \times V & \rightarrow & K \\ (x, y) & \mapsto & \overline{g(y, x)} \end{array}$



## 55.5 Def

A bilinear form  $g$  is

- **symmetric** if  $g = g_p$
- **symplectic** if  $g = -g_p$
- **hermitian** if  $g = -\overline{g_p}$

## 55.6 Def

$(V, g)$  is a vector space with an inner product. Two vector  $v_1, v_2 \in V$  are said **orthogonal** (w.r.t (with respect to)  $g$ ) if

$$g(v_1, v_2) = 0$$

Two subspaces  $V_1, V_2 \in V$  are **orthogonal** (w.r.t  $g$ ) if

$$g(v_1, v_2) = 0 \quad \forall v_1, v_2 \in V$$

## 55.7 Def

The kernel of an inner product  $g$  is

$$\ker g := \{\sigma \in V \mid g(\sigma, w) = 0, \forall w \in V\}$$

$g$  is said non-degenerate if

$$\ker g = \{0\}$$

## 55.8 Remark

$\ker g = \ker \tilde{g}$  where  $\tilde{g}$  a linear mapping  $\tilde{g} : V \rightarrow V^\vee$

## 55.9 Prop

Let  $g$  be a bilinear form on  $V$  and let  $G$  be the Gram matrix of  $g$  w.r.t the basis  $\{\sigma_1, \dots, \sigma_n\}$ . Then the matrix of  $\tilde{g}$  w.r.t. the basis  $\{\sigma_1, \dots, \sigma_n\}$  and  $\{\sigma_1^\vee, \dots, \sigma_n^\vee\}$  is  $G^T$

$$\begin{array}{ccc} V & \xrightarrow{\tilde{g}} & V^\vee \\ \uparrow b & & \uparrow b^\vee \\ K^n & \xrightarrow{G^T} & K^n \end{array}$$

**Proof**

$$\begin{aligned}
((b^\vee)^{-1} \circ \tilde{g} \circ b)(e_i) &= (b^\vee)^{-1} \tilde{g}(v_i) \\
&= (b^\vee)^{-1} g(v_i, -) \\
&= g_{v_i}(\cdot) : V \rightarrow K
\end{aligned}$$

$$\begin{aligned}
g_{v_i}(x) &= g(v_i, \sum_j x_j v_j) \\
&= \sum_j x_j G_{ij} \\
&= \sum_j G_{ij} v_j^\vee(x)
\end{aligned}$$

$$\text{so } g_{v_i} = \sum_j G_{ij} v_j^\vee$$

$$\begin{aligned}
(b^\vee)^{-1}(g_\vee(\cdot)) &= (b^\vee)^{-1}(\sum_j G_{ij} v_j^\vee) \\
&= \sum_j G_{ij} e_j \\
&= \text{i-th row of } G
\end{aligned}$$

**55.10 Prop**

- $g$  is symmetric iff  $G = G^T$
- $g$  is symplectic iff  $G = -G^T$
- $g$  is hermitian iff  $G = \overline{G^T}$

**Proof**

Exercise

**55.11 Prop**

Let  $g$  be an inner product on  $V$ , Then  $g$  is non-degenerate iff

$$\det G \neq 0$$

(w.r.t. any choice of basis)

**Proof**

Let  $G$  and  $G'$  be two Gram matrices of  $g$ .

$$G' = A^T G A$$

for  $A$  invertible

$$\det G' = (\det A)^2 \det G$$

$$\det G' \neq 0 \Leftrightarrow \det G \neq 0$$

Fix a basis  $\{v_1, \dots, v_n\}$

$$\ker g \cong \ker G^t$$

by the fact that  $\ker g = \ker \tilde{g}$ . So  $\ker g \neq 0$  iff

$$\det G = \det G^T \neq 0$$

**55.12 Def:isometry**

$$(V, g) \xrightarrow{f} (W, g')$$

a morphism  $f$  of vector space with inner product is **isometry** if

$$g(x, y) = g'(f(x), f(y))$$

**55.13 Def:isometric**

$V \xrightarrow{\cong} W$  up to isomorphism.

Then  $(V, g)$  and  $(W, g')$  are **isometric** if there are two isometry

$$\begin{aligned} f : (V, g) &\rightarrow (W, g') \\ f' : (W, g') &\rightarrow (V, g) \end{aligned}$$

such that

$$f \circ f' = f' \circ f = Id$$



## Chapter 56

# Complex conjugate vector space

### 56.1 Def: complex conjugate vector space

Let  $V$  be a vector space over  $K = \mathbb{C}$ . The **complex conjugate**  $\bar{V}$  is the same set of  $V$ . The sum on  $\bar{V}$  is the same of  $V$ , but we define

$$\alpha * v := \bar{\alpha}v \quad \alpha \in \mathbb{C}$$

### 56.2 Def: Semilinear

If  $V$  and  $W$  are two complex vector sapce, then a **semilinear mapping** is a mapping  $f : V \rightarrow W$  such that

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha * f(v) = \bar{\alpha}f(v)$

So a semilinear mapping is a linear mapping:  $f : V \rightarrow W$

For sesquilinear forms, the theory is similar to the theory of bilinear forms.

$$g \rightsquigarrow G(\text{fix a basis}) \quad g(x, y) = xG\bar{y}$$

If you change basis, then the Gram matrix changes in the following way:

$$G \rightsquigarrow A^T G \bar{A}$$

If  $g$  is bilinear

$$g \rightsquigarrow \tilde{g} : V \rightarrow V^\vee$$

and

$$g \rightsquigarrow \tilde{g} : V \rightarrow \overline{V}^\vee$$

linear if  $g$  is sesquilinear ( $\tilde{g} : V \rightarrow V^\vee$  is semilinear)

**56.3 Def**

A sesquilinear form  $g : V \times \overline{V} \rightarrow K$  is **hermitian** if

$$g(x, y) = \overline{g(y, x)}$$

And note that inner product is any of symmetric symplectic or hermitian.

## Chapter 57

# Classification (up to isometry) of vector spaces of small dim

Let  $(V, g)$  be vector space over  $K (= \mathbb{R}, \mathbb{C})$  with inner product.

### 57.1 $\dim V = 1$ and $g$ is symmetric

choose  $v \in V \setminus \{0\}$  if  $g(v, v) = 0$ , then  $g$  is degenerated  $\Rightarrow g = 0$   
If  $g$  is non-deg (non-degenerate)  $\exists v$  s.t.  $g(v, v) = a \neq 0$

$$\forall x \in K \quad g(xv, xv) = ax^2$$

Any  $v$  s.t.  $g(v, v) = a \neq 0$  induce a set

$$\mathcal{C}(v) := \{ax^2 : x \in K^*\}$$

this is an element in  $K^*/\{x^2 \mid x \in K^*\}$

#### 57.1.1 Prop

Let  $(V_1, g_1), (V_2, g_2)$  be two vector spaces of dim 1 s.t.  $g_1$  and  $g_2$  are symplectic. Then  $(V_1, g_1)$  and  $(V_2, g_2)$  are isometric iff

$$\exists v_1 \in V_1, v_2 \in V_2 \text{ s.t. } \mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$$

#### Proof

$\Rightarrow f : V_1 \rightarrow V_2$  be an isometry,  $v_1 \in V \setminus \{0\}$ ,  $v_2 := f(v_1)$

$$g_2(v_2, v_2) = g_2(f(v_1), f(v_1)) = g_1(v_1, v_1)$$

$\Rightarrow$

$$C_{g_1}(v_1) = C_{g_2}(v_2)$$

$\Leftarrow$  Assume  $\exists v_1, v_2 : C_{g_1}(v_1) = C_{g_2}(v_2)$  then  $\forall a \in K^*$

$$g_1(v_1, v_1) = ax^2 \quad g_2(v_2, v_2) = ay^2$$

Then let

$$\begin{aligned} f : V_1 &\rightarrow V_2 \\ v_1 &\mapsto y^{-1}xv_2 \end{aligned}$$

for dim 1 this can def whole mapping. So

$$g_2\left(\frac{x}{y}v_2, \frac{xx}{y}v_2\right) = \frac{x^2}{y^2}g(v_2, v_2) = ax^2 = g(v_1, v_1)$$

For  $K = \mathbb{R} \quad \mathbb{R}^*/\{x^2 \mid x \in \mathbb{R}^*\} \cong \{\pm 1\}$

$$\begin{aligned} f : \mathbb{R}^* &\rightarrow \{\pm 1\} \\ x &\mapsto \operatorname{sgn}(x) \end{aligned}$$

For  $K = \mathbb{C} \quad \mathbb{C}^* \cong \{x^2 \mid x \in \mathbb{C}\}, \quad \mathbb{C}^*/\{x^2 \mid x \in \mathbb{C}^*\} = \{[1]\}$

### 57.1.2 Theorem

$(V, g)$  has dim 1,  $g$  symmetric. Then  $(V, g)$  is isometric to one of the following

- $K = \mathbb{R}$

$$(\mathbb{R}, g(x, y) = xy) \quad (\mathbb{R}, g(x, y) = -xy) \quad (\mathbb{R}, g(x, y) = 0)$$

- $K = \mathbb{C}$

$$(\mathbb{C}, g(x, y) = xy) \quad (\mathbb{C}, g(x, y) = 0)$$

### Proof

Follows form last prop

## 57.2 dim $V = 1$ $g$ is hermitian

Again  $g$  degenerate  $\Rightarrow g = 0$  We use that same reason as above.  $v \in V : g(v, v) = a \neq 0, \forall a \in \mathbb{C}^*$

$$g(av, av) = \|a\|^2 g(v, v)$$

So any element  $v \in V \setminus \{0\}$  s.t.  $g(v, v) = a$  induces a coset in  $\mathbb{C}^*/\mathbb{R}_{>0}$

Inside  $\mathbb{C}^* \quad \mathbb{R}_{>0}$  is a (mult) subgroup



For any  $z \in \mathbb{C}^*$  can be written uniquely as  $z = re^{i\theta}$ , hence

$$\begin{aligned} \mathbb{C}^* &\cong \mathbb{R}_{>0} \times S^1 \rightarrow S^1 \\ z &\mapsto (r, e^{i\theta}) \mapsto e^{i\theta} \end{aligned}$$

The kernel is  $\mathbb{R}_{>0}$  and  $S^1 \cong \mathbb{C}^*/\mathbb{R}_{>0}$ .

But  $g$  is hermitian, so

$$g(v, v) \in \mathbb{R}$$

It follows that the coset

$$\{\|a\|^2 g(v, v) \mid a \in \mathbb{C}^*\} \in (\mathbb{C}^*/\mathbb{R}_{>0}) \cap (\mathbb{R}/\mathbb{R}_{>0}) \cong \{\pm 1\}$$

We repeat the proposition before with  $g$  hermitian and the following theorem

### 57.2.1 Theorem

$(V, g)$  if  $\dim 1$ , with  $g$  hermitian. Then  $(V, g)$  is isometry to one of the following

$$(\mathbb{C}, g(x, y) = x\bar{y}) \quad (\mathbb{C}, g(x, y) = -x\bar{y}) \quad (\mathbb{C}, g(x, y) = 0)$$

## 57.3 $\dim V = 1$ $g$ symplectic

With  $\dim 1 \forall v_1, v_2 \in V$  can be write as  $v_1 = ae, v_2 = be$  with  $e \in K^*$

$$g(v_1, v_2) = ab \cdot g(v, v) = 0$$

### 57.3.1 Theorem

$(V, g)$  of  $\dim 1$ ,  $g$  symplectic, then

$$(V, g) \cong (K, g = 0)$$

## 57.4 $\dim V = 2$ $g$ symplectic

Assume that  $g$  is degenerated, then  $\exists x \in V$  s.t.  $g(x, y) = 0, \forall y \in V$

Extend  $x$  to a basis  $\{x, x'\}$  of  $V$

$$g(ax + a'x', bx + b'x') = ab \cdot g(x, x) + ab' \cdot g(x, x') - a'b \cdot g(x, x') + a'b' \cdot g(x', x') = 0$$

So when  $g$  degenerated  $g = 0$

Take  $g$  non-degenerated  $\exists v_1, v_2 \in V$  s.t.  $g(v_1, v_2) = a \neq 0$ .

For  $g(a^{-1}v_1, v_2) = a^{-1}a = 1$ , we may assume that  $a = 1$

Let's show that  $v_1, v_2$  are linearly independent. Assume by contraction:

$$v_1 = \lambda v_2$$

$$1 = g(v_1, v_2) = g(\lambda v_2, v_2) = \lambda \cdot g(v_2, v_2) = 0$$

$\Rightarrow \{v_1, v_2\}$  is a basis of  $V$ . Then

$$\alpha_1\beta_2 - \alpha_2\beta_1 = g(\alpha_1v_1 + \alpha_2v_2, \beta_1v_1 + \beta_2v_2) = (\alpha_1, \alpha_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \stackrel{(?)}{=} \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \end{pmatrix}$$

**57.4.1 Theorem**

$(V, g)$  is dim 2,  $g$  symplectic. Then  $(V, g)$  is isometric to one of the following

$$(K^2, g(x, y) = 0) \quad (K^2, g(x, y) = x_1y_2 - x_2y_1)$$

## Chapter 58

# Compliment

### 58.1 Def: non-degenerate

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a subspace. We say that  $V_0$  is non-degenerate if  $g|_{V_0}$  is non-degenerate.

Moreover,  $V_0$  is isotropic if  $g|_{V_0} = 0$

#### Remark

Isotropic means degenerate

#### Example

1  $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_1 - x_2y_2$$

symmetric non-degenerate.  $V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

2  $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_2 + x_2y_1$$

$$V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

### 58.2 Def

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a subspace. The orthogonal complement of  $V_0$  is define as

$$V_0^\perp := \{v \in V \mid g(v, v_0) = 0, \forall v_0 \in V_0\}$$

### 58.3 Prop

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a non-degenerate subspace. Then

$$V = V_0 \oplus V_0^\perp$$

#### Proof

Let  $\{v_0, \dots, v_n\}$  be a basis of  $V_0$  ( $\dim V_0 = r \leq \dim V = n$ ) Extend this to a basis of  $V$

$$\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$$

Let  $G$  be the Gram matrix of  $g$  with respect to such case:

$$G = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) & \cdots & g(v_1, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) & \cdots & g(v_r, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_n, v_1) & \cdots & g(v_n, v_r) & \cdots & g(v_n, v_n) \end{pmatrix}$$

while  $G_0 \in M_{r,n}(K)$  is defined as a part of  $G$

$$G_0 = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) \\ \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) \end{pmatrix}$$

Consider the dimension:

$$\begin{aligned} \dim V_0^\perp &= \dim \ker G_0 \\ &= n - \text{rank} G_0 \\ &= n - \text{rank} G_0^T \end{aligned}$$

Since  $V_0$  is non-degenerated.  $G_0^T$  is non-degenerated with  $\text{rank} = \dim V_0 = r$

$$\dim V_0 + \dim V_0^\perp = \dim V$$

We finally have to show that this sum is a direct sum:

If  $x \in V_0 \cap V_0^\perp$ , then  $g(x, V_0) = 0$  since  $x \in V_0^\perp$  and  $g_0(x, V_0) = 0$  implies  $x = 0$  since  $g_0$  is non-degenerate.

### 58.4 Theorem

Let  $(V, g)$  be an finite dimensional inner product space. If both  $V_0$  and  $V_0^\perp$  are non-degenerate, then  $(V^\perp)^\perp = V_0$

**Proof****Remark**

$V_0 \subseteq (V_0^\perp)^\perp$  always holds, regardless of degenerate and dimension.

**58.5 Theorem**

Let  $(V, g)$  be a finite dimensional inner product space. Then There exists a decomposition

$$V = V_1 \oplus \cdots \oplus V_n$$

such that  $\{V_i\}_{i=1}^n$  are pairwise orthogonal and

- 1 They are 1-dim if  $g$  is symmetric or hermitian
- 2 They are 1-dim but degenerated or 2-dim non-degenerate if  $g$  is symplectic.

**Proof**

We work by induction on  $n = \dim V$

$n = 1$

trivial

$n \geq 2$

If  $g = 0$ , there is nothing to prove.

$g$  symplectic There exists  $x$  and  $x'$  such that  $g(x, x') = 0$  and  $\langle x, x' \rangle = V_0$ . Since  $V_0$  is non-degenerate, then apply induction on  $V_0^\perp$

$g$  symmetric/hermitian It's enough to show that there exists  $V_0 \subseteq V$  of dim 1, which is non-degenerate.



## Chapter 59

# Signature

Now we discuss the uniqueness of such decomposition

### 59.1 Def

Let  $(V, g)$  be an inner product space with  $\dim V = 1$ . Moreover, assume that  $g$  is symmetric or hermitian. We say that  $(V, g)$  is **positive** if  $(V, g)$  is isometry to either  $(\mathbb{R}, g(x, y) = xy)$  or  $(\mathbb{C}, g(x, y) = x\bar{y})$

We say that  $(V, g)$  is **negative** if  $(V, g)$  is isometry to either  $(\mathbb{R}, g(x, y) = -xy)$  or  $(\mathbb{C}, g(x, y) = -x\bar{y})$

### 59.2 Notation

By theorem 58.5, we can count the number of positive subspace of any inner product space.

- $r_0 := \dim \ker g$
- $r_+ :=$  the number of positive subspaces
- $r_- :=$  the number of negative subspaces

### 59.3 Def

Let  $(V, g)$  be an inner product space.

- 1 If  $g$  is real symmetric or, hermitian, then  $(r_0, r_+, r_-)$  is signature of  $V$
- 2 If  $g$  is symplectic or complex symmetric, then  $(\dim V, r_0)$  is the signature of  $V$

### 59.4 Theorem

Let  $(V, g)$  and  $(V', g')$  be two inner product spaces, with  $g, g'$  that are either (both) symplectic or complex symmetric.

Then  $(V, g)$  and  $(V', g')$  are isometric iff

$$(n, r_0) = (n', r'_0)$$

#### Proof

$\Rightarrow$

trivial

$\Leftarrow$

Use orthogonal decomposition

$$V = \bigoplus_{i=1}^n V_i$$

We rearrange the subspaces as  $V_0, \dots, V_r, v_{r+1}, \dots, V_n$  such that  $V_i \subseteq \ker g, \forall 1 \leq i \leq r$

The remaining is shown as below:

$$(V_1, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_1, g')$$

$\vdots$

$\vdots$

$$(V_r, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_r, g')$$

$$(V_{r+1}, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_{r+1}, g')$$

$\vdots$

$\vdots$

$$(V_n, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_n, g')$$



## 59.5 Theorem

Let  $(V, g)$  and  $(V', g')$  be two inner product spaces, with  $g, g'$  that are either (both) hermitian or real symmetric.

Then  $(V, g)$  and  $(V', g')$  are isometric iff

$$(r_0, r_+, r_-) = (r'_0, r'_+, r'_-)$$

### Proof

exercise



## Chapter 60

# Orthonormal

### 60.1 Def

Let  $(V, g)$  be an inner product space. The basis  $\{v_1, \dots, v_n\}$  is said **orthogonal** if  $g(v_i, v_j) = 0, \forall i \neq j$

Moreover,  $g$  is said **orthonormal** if  $g(v_i, v_i) \in \{0, -1, 1\}, \forall i$

### Remark

If  $g$  is hermitian or symmetric. We can always find an orthonormal basis from an orthogonal basis.

### 60.2 Def

Let  $V$  be a vector space over  $K$  ( $\text{char} K \neq 2$ ). A **quadratic form** on  $V$  is a mapping  $q : V \rightarrow K$  such that

- $q(\alpha v) = \alpha^2 q(v) \forall \alpha \in K, v \in V$
- $f = (u, v) \mapsto q(u + v) - q(u) - q(v)$  is bilinear

### Remark

Any symmetric bilinear form  $h : V^2 \rightarrow K$  is a quadratic form. Given a quadratic form  $q : V \rightarrow K$ , we can define a symmetric bilinear form

$$h_p(u, v) = \frac{1}{2} (q(u + v) - q(u) - q(v))$$

### 60.3 Gram-Schmidt algorithm

Let  $(V, g)$  be an inner product space with  $g$  symmetric or hermitian. Let  $\{v'_1, \dots, v'_n\}$  be a basis of  $V$  such that  $V_i = \langle v'_1, \dots, v'_i \rangle \quad \forall i \in \{1, \dots, n\}$  is non-degenerate.

Then there exists an orthogonal basis  $\{v_1, \dots, v_n\}$  such that  $V_i = \langle v_1, \dots, v_i \rangle \quad \forall i \in \{1, \dots, n\}$  is non-degenerate.

#### Proof

We construct the  $v_i$  by induction.  $v_1 = v'_1$  If  $v_1, \dots, v_{i-1}$  have been already constructed, then  $v_i$  must be of the form:

$$v_i = v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j$$

We have  $\langle v_1, \dots, v_{i-1} \rangle = \langle v'_1, \dots, v'_{i-1} \rangle$  by induction hypothesis, then

$$\langle v_1, \dots, v_{i-1} \rangle = \langle v'_1, \dots, v'_{i-1}, v_i \rangle = \langle v'_1, \dots, v'_i \rangle$$

We have to show that  $v_i$  are orthogonal

$$\begin{aligned} g \left( v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j, v_k \right) &= g(v'_i, v_k) - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} g(v_j, v_k) \\ &= g(v'_i, v_k) - \frac{g(v'_i, v_k)}{g(v_k, v_k)} g(v_k, v_k) \\ &= 0 \end{aligned}$$

## Chapter 61

# Euclidean and Unitary Spaces

### 61.1 Def:Euclidean vector space

A euclidean vector space is a finite dimensional inner product space over  $\mathbb{R}$   $(E, g)$  with  $g$  symmetric and positive definite ( $g(x, x) > 0, \forall x \neq 0$ )

We denote

$$\langle x, y \rangle := g(x, y)$$

#### Remark

Any non-zero subspace of  $(E, \langle \cdot, \cdot \rangle)$  is non-degenerated:

Take  $V \subseteq E$  if  $\exists x \in V \setminus \{0\}$  s.t.  $\langle x, V \rangle = 0$  then

$$\langle x, x \rangle = 0$$

The signature of  $E$  is of the type  $(0, \mathfrak{z}_+, \mathfrak{z}_-)$  denoted by  $(P, Q)$  ( $P = \mathfrak{z}_+, Q = \mathfrak{z}_-$ )

An euclidean space is a normed vector space

$$\|x\| = \sqrt{\langle x, x \rangle}$$

( $\langle \cdot, \cdot \rangle$  positive defined required)

Any euclidean vector admits an orthonormal basis  $\{v_1, \dots, v_n\}, \langle v_i, v_i \rangle = 1$

So orthonormal means

$$\|v_1\| = 1$$

In a euclidean space we have a distance

$$d(x, y) = \|x - y\|$$

It's a topological metric space

### 61.2 Prop

A euclidean space  $(E, \langle, \rangle)$  of dim  $n$  is isometric to  $(\mathbb{R}^n, \underbrace{\langle, \rangle}_{\text{usual scalar product}})$

#### Proof

$\{v_1, \dots, v_n\}$  is an orthonormal basis of  $E$ . Consider the mapping

$$f : v_i \mapsto e_i = (0, \dots, \underbrace{1}_{i\text{-th}}, \dots, 0)$$

Let  $x = \sum \alpha_i v_i, y = \sum \beta_i v_i$  then

$$\langle x, y \rangle = \sum \alpha_i \beta_i$$

and  $f(x) = \sum \alpha_i e_i, f(y) = \sum \beta_i e_i$ , so

$$\langle f(x), f(y) \rangle = \sum \alpha_i \beta_i = \langle x, y \rangle$$

Thus  $f$  an isometry.

### 61.3 Remark

Cauchy-Schwartz inequality

$$\langle x, y \rangle \leq \|x\| \|y\|$$

Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

### 61.4 Pythagona's Theorem

If  $x_1, \dots, x_k$  are pairwise orthogonal, then

$$\left\| \sum_{i=1}^k x_i \right\|^2 = \sum_{i=1}^k \|x_i\|^2$$

### 61.5 Def:Angles

By Cauchy-Schwartz inequality:

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1$$

Then there exists a element  $\phi \in [0, \pi]$  such that

$$\cos \phi = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$\phi$  is defined as the angle between  $x$  and  $y$

Notice that  $\phi$  is not an oriented angle.

## 61.6 Notation

Let  $U, V \subseteq E$  be two subspace, then

$$d(U, V) := \inf\{\|u - v\| \mid u \in U, v \in V\}$$

## 61.7 Def

$V \subsetneq E$  a vector subspace,  $x \in E \setminus \{0\}$ . Then  $E = V \oplus V^\perp$  (proved) Then we write (uniquely)

$$x = x_0 + x'_0$$

where  $x_0 \in V, x'_0 \in V^\perp$ .

Then  $x_0$  is called the orthogonal projection of  $x$  on  $V$ ,  $x'_0$  is called the orthogonal projection of  $x$  on  $V^\perp$

## 61.8 Prop

Use the above notation:

$$d(x, V) = \|x'_0\|$$

### Proof

$$\forall y \in V$$

$$\|x - y\|^2 = \|x_0 + x'_0 - y\|^2 = \|x_0 - y\|^2 + \|x'_0\|^2$$

then

$$\|x - y\| \geq \|x'_0\|$$

$\Rightarrow$

$$d(x, V) \geq \|x'_0\|$$

The equality is achieved iff  $y = x_0$ .

$$d(x, V) = \|x'_0\|$$

### 61.9 Prop

Use the previous notations. Assume that  $m = \dim V$ ,  $V \subseteq E$ ,  $\{v_1, \dots, v_m\}$  ( $m \leq n = \dim E$ ) is an orthonormal basis of  $V$ . Then

$$x_0 = \sum_{i=1}^m \langle x, v_i \rangle v_i$$

#### Proof

Consider  $y = x_0 - \sum_{i=1}^m \langle x, v_i \rangle v_i$ ,  $y \in V$ . Moreover

$$\begin{aligned} \langle y, v_j \rangle &= \left\langle x_0 - \sum_{i=1}^m \langle x, v_i \rangle v_i, v_j \right\rangle \\ &= \left\langle x - \sum_{i=1}^m \langle x, v_i \rangle v_i, v_j \right\rangle \\ &= \langle x, v_j \rangle - \langle x, v_j \rangle \\ &= 0 \end{aligned}$$

$y$  is orthonormal to any  $v_j \Rightarrow \langle y, V \rangle = 0 \Rightarrow y \in V^\perp$  so

$$y \in V \cap V^\perp = \{0\}$$

### 61.10 Relationship with calculus

$(E, \langle, \rangle) = (\mathbb{R}^n, \langle, \rangle)$  on  $\mathbb{R}^n$  we have the notion of volumes

$$\text{vol}(B) := \lambda^n(B)$$

where  $B$  is a Borel set.

A  $n$ -dimensional parallelepiped is :

$$P_n = \{t_1 v_1 + \dots + t_n v_n \mid t_i \in [0, 1] \forall i\}$$

Consider a linear mapping

$$\begin{aligned} A_{P_n} &= A : \mathbb{R}^n \rightarrow \mathbb{R}^n \\ x &\mapsto Ax \end{aligned}$$

where  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ ,  $A = (v_1 \mid \dots \mid v_n)$

$A$  is invertible iff  $\{v_1, \dots, v_n\}$  is a basis. Let  $\prod_n = [0, 1]^n$  then

$$A\left(\prod_n\right) = P_n$$



If  $A$  invertible

$$\begin{aligned}
 \text{vol}(P_n) &= \lambda^n(P_n) \\
 &= \int_{A(\Pi_n)} \chi_{P_n} d\lambda^n \\
 \text{change of variables} &= \int_{A(\Pi_n)} |\det A| d\lambda^n \\
 &= |\det A| \\
 \text{by the prop of det} &= \sqrt{\det A^T A}
 \end{aligned}$$

## 61.11 Prop

$$\text{vol}(P_n) = \sqrt{\det G}$$

### Proof

The Gram matrix

$$G = (\langle v_i, v_j \rangle_{ij})$$

Notice that  $\det G = 0$  iff  $\{v_1, \dots, v_n\}$  is not a basis. Let  $A = (v_1 \mid \dots \mid v_n)$  then  $G = A^T A$

$$\begin{aligned}
 \det G &= (\det A)^2 \\
 &= \det A^T A \\
 &= \text{vol}(P_n)
 \end{aligned}$$



## Chapter 62

# Unitary Space

### 62.1 Def

A complex inner vector space  $(H, h)$  where  $h$  is hermitian and positive define then it's called **unitary space**

As in the Euclidean space, we have orthonormal basis and define a norm, then a distance.

$$\|x\| = \sqrt{h(x, x)}$$

### 62.2 Decomplexification

Let  $V$  be a complex vector space of dimension  $n$ . We resist the module structure  $\mathbb{C} \times V \rightarrow V$  to  $\mathbb{R} \times V \rightarrow V$

The  $\mathbb{R}$ -vector space denoted by  $V_{\mathbb{R}}$  has the same vector of  $V$ . For any  $\mathbb{C}$ -linear mapping  $f : V \rightarrow W$ , the module induces a mapping

$$f_{\mathbb{R}} \rightarrow W_{\mathbb{W}}$$

### Exercise

Verify that  $(\cdot)_{\mathbb{R}}$  is a functor

### 62.3 Theorem

Let  $V$  be a complex vector of dim  $n$

- Let  $V$  be a complex vector basis of  $V$ , then  $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$  is a real basis of  $V_{\mathbb{R}}$

- $f : V \rightarrow W$  is a linear mapping. Assume that it's metric representation with respect to the basis  $\{v_1, \dots, v_n\}$  of  $V$  and  $\{w_1, \dots, w_n\}$  of  $W$  is

$$A = B + iC$$

where  $B, C \in \mathbb{R}$ . Then the metric representation of  $f_{\mathbb{R}}$  with respect to the basis  $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$  and  $\{w_1, \dots, w_n, iw_1, \dots, iw_n\}$  is

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

### Proof

1

Take  $v \in V$

$$\begin{aligned} v &= \sum_{k=1}^n a_k v_k \\ &= \sum_{k=1}^n (\Re a_k + \Im a_k) v_k \\ &= \sum_{k=1}^n (\Re a_k) v_k - \sum_{k=1}^n (\Re i a_k) i v_k \end{aligned}$$

generate. Assume  $\sum b_k v_k + \sum c_k i v_k = 0$ . But  $\{v_k\}$  is a basis, implying

$$b_k + i c_k = 0 \Rightarrow b_k = c_k = 0$$

linearly independent.

2

Decompose  $V = \bigoplus_{k=1}^n V_k$  and  $W = \bigoplus_{k=1}^n W_k$  into 1 dimensional orthogonal subspaces.

Take  $k, k'$  consider

$$\begin{aligned} f_{k,k'} : V_k &\rightarrow W_{k'} \\ v_k &\mapsto \alpha w_{k'} \end{aligned}$$

Take

$$\begin{aligned} f_{k,k',\mathbb{R}} : V_{k,\mathbb{R}} &\rightarrow W_{k',\mathbb{R}} \\ v_k &\mapsto (\Re \alpha) w_{k'} + (\Im \alpha) i w_{k'} \\ i v_k &\mapsto -(\Im \alpha) w_{k'} + (\Re \alpha) i w_{k'} \\ A_{f_{k,k',\mathbb{R}}} &= \begin{pmatrix} \Re \alpha & -\Im \alpha \\ \Im \alpha & \Re \alpha \end{pmatrix} \end{aligned}$$

Rearrange the whole mapping, we get

$$A = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

## 62.4 Corollary

Let  $f : V \rightarrow V$  be a  $\mathbb{C}$ -linear mapping. Then

$$\det f_{\mathbb{R}} = \det f \overline{\det f}$$

### Proof

Assume  $A = B + iC$  is the matrix representation of  $f$

$$\begin{aligned} \det f_{\mathbb{R}} &= \det \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \\ &= \det \begin{pmatrix} B + iC & -C + iB \\ C & B \end{pmatrix} \\ &= \det \begin{pmatrix} B + iC & 0 \\ C & B - iC \end{pmatrix} \\ &= \det(B + iC) \cdot \det(B - iC) \\ &= \det A \cdot \overline{\det A} \end{aligned}$$



## Chapter 63

# Complexification

### 63.1 Def: Complex structure

Let  $W$  be a real vector space of dim  $n$ . Consider  $J : W \rightarrow W$  a linear mapping such that  $J^{\circ 2} = -Id$ . Then  $J$  is called the **complex structure** of  $W$ . Then couple  $(W, J)$  is a vector space with a complex structure.

#### Example

$$\begin{aligned} J : V_{\mathbb{R}} &\rightarrow V_{\mathbb{R}} \\ v &\mapsto iv \\ iv &\mapsto -v \end{aligned}$$

### 63.2 Theorem

Let  $(W, J)$  be a real vector space with a complex structure. Then on  $W$  we introduces the following complex module:

$$(a + bi)w := aw + bJ(w)$$

We obtain a complex vector space  $W$  such that  $(W)_{\mathbb{R}} = W$

### 63.3 Corollary

If  $(W, J)$  is a vector space with a complex structure, then  $\dim W$  is even. Assume that if it's even, then it's possible to find a basis on  $W$  such that  $J$  is represented by

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Consider a orthonormal basis on  $H\{v_1, \dots, v_n\}$ .  $G_h : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is the Gram matrix of  $h$  with respect to  $\{v_1, \dots, v_n\}$

$$G_h = B_i C$$

Now

$$G_{\mathbb{R}} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

Then  $G_{\mathbb{R}}$  defines an inner product and  $(H_{\mathbb{R}}, \langle, \rangle)$  is Euclidean.

## 63.4 Notation

Now fix a inner product space. Then

$$h(x, y) = a(x, y) + ib(x, y)$$

when  $a, b : V \times V \rightarrow \mathbb{R}$

## 63.5 Prop

In the above notation the following structure holds:

- 1  $a(x, y)$  and  $b(x, y)$  are inner products on  $V_{\mathbb{R}}$ , with  $a$  symmetric and  $b$  skew-symmetric. In addition:

$$a(ix, iy) = a(x, y) \quad b(ix, iy) = b(x, y)$$

In other word  $a, b$  are invariant by the multiplication by  $i$  Invariance w.r.t. the complex structure of  $H_{\mathbb{R}}$

- 2 The following relations hold.

$$a(x, y) = b(ix, y) \quad b(x, y) = -a(ix, y)$$

- 3 Any pair of  $J$ -invariant bilinear forms on  $V_{\mathbb{R}}$   $a, b : V \times V \rightarrow \mathbb{R}$  that are symmetric and symplectic, respectively and s.t. (2) is satisfied. Define an hermitian inner product

$$h(x, y) := a(x, y) + ib(x, y)$$

Moreover  $h$  is positive definite iff  $a$  is positive definite.



**Proof****1**

First to check is

$$\begin{aligned}
h(x, y) &= h(y, x) \\
\Rightarrow a(x, y) + ib(x, y) &= a(y, x) - ib(y, x) \\
\Rightarrow \begin{cases} a(x, y) = a(y, x) & \text{symm} \\ b(x, y) = -b(y, x) & \text{skew-symm} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
h(ix, iy) &= i\bar{i}h(x, y) = h(x, y) \\
\Rightarrow a(ix, iy) + ib(ix, iy) &= a(x, y) + ib(x, y) \Rightarrow \begin{cases} a(ix, iy) = a(x, y) \\ b(ix, iy) = b(x, y) \end{cases}
\end{aligned}$$

**2**

$$\begin{aligned}
h(ix, y) &= ih(x, y) \\
\Rightarrow a(ix, y) + ib(ix, y) &= ia(x, y) - b(x, y) \\
\Rightarrow \begin{cases} a(ix, y) = -b(x, y) \\ b(ix, y) = a(x, y) \end{cases}
\end{aligned}$$

**3**

$$\begin{aligned}
ih(x, y) &= ia(x, y) - b(x, y) \\
(2) &= ib(ix, y) + a(ix, y) \\
&= h(ix, y)
\end{aligned}$$

We have linearity in the first variable, then positive/negative define follows from

$$b(x, x) = -b(x, x) \Rightarrow b(x, x) = 0$$

so

$$h(xx, x, ) = a(x, x)$$

**63.6 Complex Cauchy-Schwartz inequality** $(H, h)$  is a unitary space with finite dim. Then the inequality

$$|h(x, y)| \leq \|x\| \|y\|$$

holds iff  $x$  and  $y$  are propositional ( $x = ty$ )

**Proof**

When  $x = 0$ , obvious.  $x \neq 0, \forall t \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \|tx + y\|^2 \\ &= t^2 \|x\|^2 + 2t \operatorname{Re}(h(x, y)) + \|y\|^2 \end{aligned}$$

This is a quadratic inequality. The equality has at most one solution when  $y = -tx$ . In general the discriminant of the quadratic polynomial in  $t \leq 0$

$$(\operatorname{Re} h(x, y))^2 \leq \|x\|^2 \|y\|^2$$

Now take

$$h(x, y) = |h(x, y)| e^{i\theta}$$

Then

$$\operatorname{Re}(h(e^{-i\theta}, y)) = |h(x, y)|$$

Hence

$$\operatorname{Re}(h(e^{-i\theta}, y))^2 = |h(x, y)|^2 \leq \|e^{-i\theta}\|^2 \|y\|^2$$

**63.7 Corollary:Complex triangle inequality**

$$\|x + y\| \leq \|x\| + \|y\|$$

**63.8 Def:Angle for unitary space**

For

$$0 \leq \frac{|h(x, y)|}{\|x\| \cdot \|y\|}$$

$$\exists! \phi \in [0, \frac{\pi}{2}]$$

$$\cos \phi = \frac{|h(x, y)|}{\|x\| \cdot \|y\|}$$

The physical application is that  $\phi$  can be considered as probability.

**63.9 Prop**

Let  $(V, g)$  be an inner product space with a non-degenerate inner product hermitian or real symm and  $f : V \rightarrow V$  be a linear mapping. Then the following statements are equivalent:

0  $f$  is isometry

1  $g(f(x), f(x)) = g(x, x) \forall x \in V$

- 2 Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$  and let  $G$  be a Gram matrix of  $g$  w.r.t. such basis. If  $A$  is the matrix of  $f$  w.r.t.  $\{v_1, \dots, v_n\}$ , then

$$A^T G A = G \text{ or } A^T G \bar{A} = G$$

- 3  $f$  transforms orthonormal basis into orthonormal basis
- 4 If the signature of  $g$  is  $(p, q)$  ( $\mathbf{z}_0 = 0$ ), then the matrix of  $f$  w.r.t. any orthonormal basis  $\{v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q}\}$  where

$$(v_i, v_i) = \begin{cases} 1 & \text{if } i \leq p \\ -1 & \text{if } p < i \leq p + q \end{cases}$$

satisfies the following property:

symm case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} A = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

hermitian case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \bar{A} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

## Proof

$0 \Leftrightarrow 1$

$\Rightarrow$  trivial

realsymm  $q_g(x) = g(x, x)$

$$g = \frac{1}{2} (q(x+y) - q(x) - q(y)) \quad (*)$$

From (\*) it's evident that if  $f$  preserves  $q$ , then it also preserves  $g$ .

hermitian  $q(x) = g(x, x)$

$$\text{Reg}(x, y) = \frac{1}{2} (q(x, y) - q(x) - q(y))$$

and

$$\text{Img}(x, y) = -\text{Reg}(ix, y)$$

by the proposition proved. So if  $f$  preserves the quadratic form, then it also preserves  $g$ .

**0 $\Leftrightarrow$ 2**

$\mathcal{B} = \{v_1, \dots, v_n\}$  is orthonormal. let  $\mathcal{B}' = \{f(v_1), \dots, f(v_n)\}$ .

$\Rightarrow$  Since  $f$  is an isometry then the Gram matrix is the same. We also proved that if we change the Gram matrix changes in the following way

$$G \rightsquigarrow A^T G A \Rightarrow G = A^T G A$$

(real symm case)

$\Leftarrow$

$$\begin{aligned} g(x, y) &= x^T G y \\ &= x^T A^T G A y \\ &= f(x)^T G f(y) \\ &= g(f(x), f(y)) \end{aligned}$$

The same proof holds in hermitian case.(taking care of the conjugation)

**3 and 4**

they are just special cases of the others(Exercise)

### 63.10 Def

$(E, \langle, \rangle)$  is a Euclidean space, then an isometry  $f : E \rightarrow E$  is called an **orthogonal operator**

$(H, h)$  a unitary space, then an isometry  $f : H \rightarrow H$  is said a **unitary operator**

### 63.11 Corollary

Orthogonal and unitary operators have the following properties:  
w.r.t. an (all) orthonormal basis, they are represented by a matrix  $U$

$$UU^T = I_n \text{ or } UU^\dagger = I_n$$