

# Contents

<b>I</b>	<b>Set</b>	<b>11</b>
<b>1</b>	<b>product</b>	<b>13</b>
1.1	direct sum . . . . .	13
<b>2</b>	<b>Ring</b>	<b>15</b>
2.1	morphism . . . . .	15
<b>II</b>	<b>Sequences</b>	<b>17</b>
<b>3</b>	<b>Supremum and infimum</b>	<b>19</b>
<b>4</b>	<b>Interval</b>	<b>21</b>
<b>5</b>	<b>Enhanced real line</b>	<b>23</b>
<b>6</b>	<b>Vector space</b>	<b>25</b>
6.1	K-module . . . . .	25
6.1.1	Def . . . . .	25
6.1.2	Remark . . . . .	25
6.1.3	Notation . . . . .	26
6.1.4	K-vector space . . . . .	26
6.1.5	Association: . . . . .	26
6.1.6	Remark: . . . . .	27
6.2	sub K-module . . . . .	27
6.2.1	Def . . . . .	27
6.2.2	Example . . . . .	27
6.3	morphism of K-modules . . . . .	27
6.3.1	Def . . . . .	27
6.3.2	K-linear mapping . . . . .	27
6.3.3	Theorem . . . . .	27
6.3.4	Remark:column . . . . .	28
6.4	kernel . . . . .	28
6.4.1	Prop . . . . .	28
6.4.2	Def . . . . .	28

6.4.3	Theorem . . . . .	28
6.4.4	Def . . . . .	29
6.4.5	Remark . . . . .	29
6.4.6	Theorem . . . . .	29
6.4.7	Proof: . . . . .	29
<b>7</b>	<b>Monotone mappings</b>	<b>31</b>
7.1	Def . . . . .	31
7.2	Prop. . . . .	31
7.3	Def . . . . .	31
7.4	Prop. . . . .	31
7.5	Prop . . . . .	32
7.6	Def . . . . .	32
7.7	Prop. . . . .	32
7.8	Proof . . . . .	32
7.8.1	bijection . . . . .	32
7.8.2	uniqueness . . . . .	33
<b>8</b>	<b>sequence and series</b>	<b>35</b>
8.1	Def . . . . .	35
8.2	Remark . . . . .	35
8.3	Prop . . . . .	35
8.4	proof . . . . .	35
8.5	Prop . . . . .	35
8.6	limit . . . . .	36
8.6.1	Def . . . . .	36
8.6.2	Remark . . . . .	36
8.6.3	Prop . . . . .	36
8.6.4	Prop . . . . .	37
8.6.5	Prop . . . . .	37
8.6.6	Theorem . . . . .	37
8.6.7	Def . . . . .	37
8.6.8	Prop . . . . .	37
8.6.9	Prop . . . . .	38
8.6.10	Theorem . . . . .	38
8.6.11	Notation . . . . .	38
8.6.12	Corollary . . . . .	38
8.6.13	Notation . . . . .	38
8.6.14	Theorem: Bolzano-Weierstrass . . . . .	38
<b>9</b>	<b>Cauchy sequence</b>	<b>41</b>
9.1	Def . . . . .	41
9.2	Prop . . . . .	41
9.3	Theorem: Completeness of real number . . . . .	41
9.4	Absolutely converge . . . . .	42
9.4.1	Prop . . . . .	42

<b>10 Comparison and Technics of Computation</b>	<b>43</b>
10.1 Def . . . . .	43
10.2 Prop. . . . .	43
10.3 Theorem . . . . .	43
10.4 Prop. . . . .	44
10.5 Prop. . . . .	45
10.6 Theorem . . . . .	45
10.7 Prop. . . . .	45
10.8 Theorem . . . . .	46
10.9 Remark . . . . .	46
10.10 Calculates on $O(), o()$ . . . . .	46
10.10.1 Plus . . . . .	46
10.10.2 Transform . . . . .	47
10.10.3 Transition . . . . .	47
10.10.4 Times . . . . .	47
10.11 On the limit . . . . .	47
10.12 Prop . . . . .	47
10.13 Prop . . . . .	48
10.14 Prop . . . . .	48
10.15 Theorem: d'Alembert ratio test . . . . .	48
10.15.1 Lemma . . . . .	49
10.15.2 (2) . . . . .	49
10.16 Prop . . . . .	49
10.16.1 Corollary . . . . .	50
10.16.2 Corollary . . . . .	50
10.17 Theorem: Cauchy root test . . . . .	50

### III Axiom of choice 51

<b>11 Preparation</b>	<b>53</b>
11.1 Statement of axiom of choice . . . . .	53
11.2 Def . . . . .	53
11.3 Theorem . . . . .	53
11.4 Zorn's lemma . . . . .	53
11.5 Prop. . . . .	53
11.6 Proof . . . . .	54
11.7 Def: Initial Segment . . . . .	54
11.8 Example . . . . .	54
11.9 Prop. . . . .	54
11.10 Proof . . . . .	54
11.11 Prop. . . . .	54
11.12 Proof . . . . .	54
11.13 Lemma . . . . .	55
11.14 Prop . . . . .	55
11.15 Def . . . . .	55

11.16Def . . . . .	55
11.17Prop. . . . .	56
11.18Lemma . . . . .	56
<b>12 Zorn's lemma</b>	<b>59</b>
12.1 Proof . . . . .	59
 <b>IV Topology</b>	 <b>61</b>
<b>13 Absolute value and norms</b>	<b>63</b>
13.1 Def . . . . .	63
13.2 Notation . . . . .	63
13.3 Prop . . . . .	63
13.4 Def . . . . .	64
<b>14 Quotient Structure</b>	<b>65</b>
14.1 Def . . . . .	65
14.2 equivalence class . . . . .	65
14.3 Prop. . . . .	65
14.4 Def . . . . .	66
14.5 Remark . . . . .	66
14.6 Prop . . . . .	66
14.7 Notation on Equivalence Class . . . . .	66
14.8 Proof . . . . .	67
14.9 Quotient set . . . . .	67
14.9.1 Example . . . . .	67
14.10Def . . . . .	67
14.11Remark . . . . .	67
14.12Prop . . . . .	67
14.13Theorem . . . . .	68
14.14Def . . . . .	68
14.15Prop . . . . .	68
14.16Def . . . . .	69
14.17Theorem . . . . .	69
14.17.1 Reside Class . . . . .	70
14.18Theorem . . . . .	70
14.19Theorem . . . . .	71
<b>15 Topology</b>	<b>73</b>
15.1 Def . . . . .	73
15.2 Remark . . . . .	73
15.2.1 Example . . . . .	73
15.3 Def . . . . .	73
15.3.1 Example . . . . .	74
15.4 Def . . . . .	74

15.4.1 Example . . . . .	74
15.5 Prop. . . . .	74
15.6 Def . . . . .	74
15.7 Def . . . . .	75
15.7.1 Example . . . . .	75
<b>16 Filter</b>	<b>77</b>
16.1 Def . . . . .	77
16.1.1 Example . . . . .	77
16.2 Def: Filter Basis . . . . .	77
16.2.1 Remark . . . . .	77
16.2.2 Example . . . . .	78
16.3 Remark . . . . .	78
16.3.1 Example . . . . .	78
16.4 Def . . . . .	78
16.5 Remark . . . . .	79
16.6 Extra Episode . . . . .	79
16.7 Prop. . . . .	79
<b>17 Limit point and accumulation point</b>	<b>81</b>
17.1 Def . . . . .	81
17.2 Prop . . . . .	81
17.3 Def . . . . .	82
17.4 Def . . . . .	82
17.5 Prop . . . . .	82
17.6 Def: dense . . . . .	82
<b>18 Limit of mappings</b>	<b>83</b>
18.1 Def . . . . .	83
18.2 Remark . . . . .	83
18.2.1 Example . . . . .	83
18.3 Remark . . . . .	83
18.4 Remark . . . . .	84
18.5 Prop . . . . .	84
18.6 Theorem . . . . .	84
18.7 Prop . . . . .	85
18.8 Def . . . . .	85
18.9 Remark . . . . .	85
18.10 Prop . . . . .	85
18.11 Theorem . . . . .	86
18.12 Prop. . . . .	87
18.12.1 Proof . . . . .	87

<b>19 Continuity</b>	<b>89</b>
19.1 Def . . . . .	89
19.2 Remark . . . . .	89
19.3 Theorem . . . . .	89
19.4 Proof . . . . .	89
19.5 Prop . . . . .	90
19.6 Def . . . . .	90
19.7 Prop . . . . .	90
19.8 Proof . . . . .	90
19.9 Prop . . . . .	91
19.10 Def . . . . .	92
19.11 Remark . . . . .	92
19.12 Prop . . . . .	92
19.13 Theorem . . . . .	94
19.13.1 Proof . . . . .	94
19.14 Remark . . . . .	94
19.14.1 Example . . . . .	95
<b>20 Uniform continuity and convergency</b>	<b>97</b>
20.1 Def . . . . .	97
20.2 Remark . . . . .	97
20.3 Prop . . . . .	97
20.4 Def . . . . .	98
20.5 Prop . . . . .	98
20.5.1 Proof . . . . .	98
20.6 Def . . . . .	99
20.7 Prop . . . . .	99
20.7.1 Proof . . . . .	99
20.8 Def . . . . .	100
20.9 Theorem . . . . .	100
20.9.1 Proof . . . . .	100
20.10 Theorem . . . . .	101
20.10.1 Proof . . . . .	101
20.10.2 Def . . . . .	101
20.11 Remark . . . . .	101
20.12 Example . . . . .	101
<b>V Normed Vector Space</b>	<b>103</b>
<b>21 Linear Algebra</b>	<b>105</b>
21.1 Def . . . . .	105
21.1.1 Notation . . . . .	105
21.2 Def . . . . .	105
21.3 Def . . . . .	106
21.4 Remark . . . . .	106

21.5 Theorem . . . . .	106
21.6 Theorem . . . . .	107
21.7 Corollary . . . . .	109
21.8 Def . . . . .	110
21.9 Theorem . . . . .	110
21.10 Proof . . . . .	110
21.11 Prop . . . . .	114
21.11.1 Proof . . . . .	114
<b>22 Matrices</b>	<b>115</b>
22.1 Def . . . . .	115
22.1.1 Example . . . . .	116
22.2 Def . . . . .	116
22.2.1 Example . . . . .	116
22.3 Def . . . . .	116
22.4 Calculate Matrices . . . . .	117
22.4.1 Remind . . . . .	117
<b>23 Transpose</b>	<b>119</b>
23.1 Def . . . . .	119
23.2 Def . . . . .	120
23.2.1 Example . . . . .	120
23.3 Prop . . . . .	121
23.4 Corollary . . . . .	121
23.5 Remark . . . . .	122
<b>24 Linear Equation</b>	<b>123</b>
24.1 Def . . . . .	123
24.2 Prop . . . . .	124
24.3 Linear Equation . . . . .	124
24.4 Prop . . . . .	124
24.5 Prop . . . . .	125
24.6 Def . . . . .	125
24.7 Theorem . . . . .	125
<b>25 Normed Vector Space</b>	<b>127</b>
25.1 Def . . . . .	127
25.2 Prop . . . . .	127
25.2.1 Proof . . . . .	127
25.3 Def . . . . .	128
25.4 Def: The completion . . . . .	128
25.5 Theorem . . . . .	128
25.6 Remark . . . . .	129
25.7 Prop . . . . .	129
25.8 Theorem . . . . .	130

<b>26 Norms</b>	<b>133</b>
26.1 Def . . . . .	133
26.2 Remark . . . . .	133
26.3 Def . . . . .	134
26.4 Prop . . . . .	134
26.5 Def . . . . .	135
26.6 Remark . . . . .	135
26.7 Def . . . . .	135
26.8 Prop . . . . .	135
26.9 Def: Operator Seminorm . . . . .	137
26.10 Prop . . . . .	137
26.11 Remark . . . . .	138
26.12 Def . . . . .	138
26.13 Theorem . . . . .	138
<b>27 Differentiability</b>	<b>141</b>
27.1 Def . . . . .	141
27.2 Def . . . . .	142
27.3 Prop . . . . .	142
27.4 Example . . . . .	143
27.4.1 . . . . .	143
27.4.2 . . . . .	143
27.4.3 . . . . .	143
27.4.4 . . . . .	143
27.5 Theorem: Chain rule . . . . .	144
27.6 Prop . . . . .	144
27.7 Def . . . . .	145
27.8 Corollary . . . . .	145
27.9 Corollary . . . . .	146
27.10 Corollary . . . . .	147
27.11 Prop . . . . .	147
27.12 Corollary . . . . .	148
27.13 Def: Equivalence of Norms . . . . .	148
27.14 Prop . . . . .	148
27.15 Remark . . . . .	149
27.16 Prop . . . . .	149
27.17 Theorem . . . . .	149
27.18 Prop . . . . .	152
27.19 Theorem . . . . .	152
<b>28 Compactness</b>	<b>153</b>
28.1 Def: cover . . . . .	153
28.2 Def: compact . . . . .	153
28.3 Def . . . . .	153
28.4 Prop . . . . .	154
28.5 Theorem . . . . .	154



28.6 Theorem . . . . .	155
28.7 Lemma . . . . .	156
28.8 Prop . . . . .	157
28.9 Prop . . . . .	157
28.10Prop . . . . .	158
28.11Prop . . . . .	158
28.12Theorem . . . . .	159
28.13Def . . . . .	159
28.14Theorem . . . . .	160
28.15Def . . . . .	161
28.16Prop . . . . .	161
28.17Theorem . . . . .	162
<b>29 Mean Value Theorems</b>	<b>165</b>
29.1 Rolle Theorem . . . . .	165
29.2 Mean value theorem(Lagrange) . . . . .	165
29.3 Mean value inequality . . . . .	166
29.4 Theorem . . . . .	167
29.5 Theorem(Heine) . . . . .	168
<b>30 Fixed Point Theorem</b>	<b>169</b>
30.1 Def . . . . .	169
30.2 Def . . . . .	169
30.3 Fixed Point Theorem . . . . .	169



# Part I

## Set



# Chapter 1

## product

### 1.1 direct sum

$\oplus$  is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0 \}$$



## Chapter 2

# Ring

### 2.1 morphism

#### Def

Let  $A$  and  $B$  be unitary rings .We call morphism of unitary rings from  $A$  to  $B$  .only mapping  $A \rightarrow B$  is a morphism of group from  $(A,+)$  to  $(B,+)$ ,and a morphism of monoid from  $(A,\cdot)$  to  $(B,\cdot)$

#### Properties

- Let  $R$  be a unitary ring. There is a unique morphism from  $\mathbb{Z}$  to  $R$
- 

#### algebra

we call  $k$ -algebra any pair  $(R,f)$ ,when  $R$  is a unitary ring ,and  $f : k \rightarrow R$  is a morphism of unitary rings such that  $\forall (b,x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring  $R$ ,the unique morphism of unitary rings  $\mathbb{Z} \rightarrow R$  define a structure of  $\mathbb{Z}$ -algebra on  $R$  (extra:  $\mathbb{Z}$  is commutative despite  $R$  isn't guaranteed)

Notation: Let  $k$  be a commutative unitary ring , $(A,f)$  be a  $k$ -algebra. If there is no ambiguity on  $f$ ,for any  $(\lambda,a) \in k \times A$ ,we denote  $f(\lambda)a$  as  $\lambda a$

#### Formal power series

reminder:  $n \in \mathbb{N}$  is possible infinite ,so  $\sum_{n \in \mathbb{N}}$  couldn't be executed directly.

Def:

(extended polynomial actually) Let  $k$  be a commutative unitary ring. Def : Let  $T$  be a formal symbol.We denote  $k^{\mathbb{N}}$  as  $k[T]$  If  $(a_n)_{n \in \mathbb{N}}$  is an element of  $k^{\mathbb{N}}$ ,when we denote  $k^{\mathbb{N}}$  as  $k[T]$  this element is denote as  $\sum_{n \in \mathbb{N}} a_n T^n$  Such

element is called a formal power series over  $k$  and  $a_n$  is called the Coefficient of  $T^n$  of this formal power series Notation:

- omit terms with coefficient 0
- write  $T'$  as  $T$
- omit Coefficient those are 1;
- omit  $T^0$

Example  $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$  is written as  $1 + 2T + T^2$

Def Remind that  $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$ , define two composition laws on  $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$  form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$  is a morphism
- $(FG)H = \left( \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left( \sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left( \sum_{p,q,l=n} a_p b_q c_l \right) T^n$   
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by  $F'(T)$  or  $\mathcal{D}(F(T))$  the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$  is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote  $\exp(T) \in k[T]$  as  $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$ , which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence:  $(F_i(T))_{i \in \mathbb{N}}$  be a sequence of elements in  $k[T]$ , and  $F(T) \in k[T]$  We say that  $(F_i(T))_{i \in \mathbb{N}}$  is a Cauchy sequence if  $\forall l \in \mathbb{N}$ , there exists  $N(l) \in \mathbb{N}$  such that  $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$ ,  $\text{ord}(F_i(T) - F_j(T)) \geq l$



# Part II

## Sequences



## Chapter 3

# Supremum and infimum

Def:

Let  $(X, \leq)$  be a partially ordered set  $A$  and  $Y$  be subsets of  $X$ , such that  $A \subseteq Y$

- If the set  $\{y \in Y \mid \forall a \in A, a \leq y\}$  has a least element then we say that  $A$  has a Supremum in  $Y$  with respect to  $\leq$  denoted by  $\sup_{(Y, \leq)} A$  this least element and called it the Supremum of  $A$  in  $Y$  (this respect to  $\leq$ )
- If the set  $\{y \in Y \mid \forall a \in A, y \leq a\}$  has a greatest element, we say that  $A$  has an infimum in  $Y$  with respect to  $\leq$ . We denote by  $\inf_{(Y, \leq)} A$  this greatest element and call it the infimum of  $A$  in  $Y$
- Observation:  $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set.

- If  $f$  is a function from  $I$  to  $X$   $\sup f$  denotes the supremum of  $f(I)$  is  $X$ .  $\inf f$  takes the same
- If  $(x_i)_{i \in I}$  is a family of element in  $X$ , then  $\sup x_i$  denotes  $\sup\{x_i \mid i \in I\}$  (in  $X$ )

If moreover  $\mathbb{P}(\cdot)$  denotes a statement depending on a parameter in  $I$  then  $\sup_{i \in I, \mathbb{P}(i)} x_i$  denotes  $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let  $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$  We equip  $\mathbb{R}$  with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So  $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence  $\inf A = 0$

Example: For  $n \in \mathbb{N}$ , let  $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let  $(X, \leq)$  be a partially ordered set,  $A, Y, Z$  be subset of  $X$ , such that  $A \subseteq Z \subseteq Y$

- If  $\max A$  exists, then it is also equal to  $\sup_{(y, \leq)} A$
- If  $\sup_{(y, \leq)} A$  exists and belongs to  $Z$ , then it is equal to  $\sup A$

$\inf$  takes the same Prop.

Let  $X, \leq$  be a partially ordered set,  $A, B, Y$  be subsets of  $X$  such that  $A \subseteq B \subseteq Y$

- If  $\sup_{(y, \leq)} A$  and  $\sup_{(y, \leq)} B$  exists, then  $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If  $\inf_{(y, \leq)} A$  and  $\inf_{(y, \leq)} B$  exists, then  $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set and  $f, g : I \rightarrow X$  be mappings such that  $\forall t \in I, f(t) \leq g(t)$

- If  $\inf f$  and  $\inf g$  exists, then  $\inf f \leq \inf g$
- If  $\sup f$  and  $\sup g$  exists, then  $\sup f \leq \sup g$

## Chapter 4

# Interval

We fix a totally ordered set  $(X, \leq)$

Notation:

If  $(a, b) \in X \times X$  such that  $a \leq b$ ,  $[a, b]$  denotes  $\{x \in X \mid a \leq x \leq b\}$

Def:

Let  $I \subseteq X$ . If  $\forall (x, y) \in I \times I$  with  $x \leq y$ , one has  $[x, y] \subseteq I$  then we say that  $I$  is an interval in  $X$

Example:

Let  $(a, b) \in X \times X$ , such that  $a \leq b$ . Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let  $\Lambda$  be a non-empty set and  $(I_\lambda)_{\lambda \in \Lambda}$  be a family of intervals in  $X$ .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$
- If  $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ ,  $\bigcup_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$

We check that  $[a, b] \subseteq I_\lambda \cup I_\mu$

- If  $b \leq x$   $[a, b] \subseteq [a, x] \subseteq I_\lambda$  because  $\{a, x\} \subseteq I_\lambda$
- If  $x \leq a$   $[a, b] \subseteq [x, b] \subseteq I_\mu$  because  $\{b, x\} \subseteq I_\mu$
- If  $a < x < b$  then  $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let  $(X, \leq)$  be a totally ordered set.  $I$  be a non-empty interval of  $X$ . If  $\sup I$  exists in  $X$ , we call  $\sup I$  the right endpoint;  $\inf$  takes the similar way.

Prop.

Let  $I$  be an interval in  $X$ .

- Suppose that  $b = \sup I$  exists.  $\forall x \in I, [x, b] \subseteq I$
- Suppose that  $a = \inf I$  exists.  $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let  $I$  be an interval in  $X$ . Suppose that  $I$  has supremum  $b$  and an infimum  $a$  in  $X$ . Then  $I$  is equal to one of the following sets  $[a, b]$   $[a, b[$   $]a, b]$   $]a, b[$

Def

let  $(X, \leq)$  be a totally ordered set. If  $\forall (x, z) \in X \times X$ , such that  $x < z \quad \exists y \in X$  such that  $x < y < z$ , then we say that  $(X, \leq)$  is thick

Prop.

Let  $(X, \leq)$  be a thick totally ordered set.  $(a, b) \in X \times X, a < b$  If  $I$  is one of the following intervals  $[a, b]; [a, b[; ]a, b]; ]a, b[$  Then  $\inf I = a \quad \sup I = b$  (for it's thick empty set is impossible)

Proof:

Since  $X$  is thick, there exists  $x_0 \in ]a, b[$  By definition,  $b$  is an upper bound of  $I$ . If  $b$  is not the supremum of  $I$ , there exists an upper bound  $M$  of  $I$  such that  $M \neq b$ . Since  $X$  is thick, there is  $M' \in X$  such that  $x_0 \leq M, M' < b$  Since  $[x, b] \subseteq I, a, b \in I$  Hence  $M$  and  $M'$  belong to  $I$ , which conflicts with the uniqueness of supremum.

## Chapter 5

# Enhanced real line

Def:

Let  $+\infty$  and  $-\infty$  be two symbols that are different and don't belong to  $\mathbb{R}$ . We extend the usual total order  $\leq$  on  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus  $\mathbb{R} \cup \{-\infty, +\infty\}$  becomes a totally ordered set, and  $\mathbb{R} = ]-\infty, +\infty[$ . Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in ]-\infty, +\infty[ \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in ]0, +\infty[ \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$   
**ARE NOT DEFINED**

Def

Let  $(X, \leq)$  be a partially ordered set. If for any subset  $A$  of  $X$ ,  $A$  has a supremum and an infimum in  $X$ , then we say that  $X$  is order complete.

Example

Let  $\Omega$  be a set.  $(\mathcal{P}(\Omega), \subseteq)$  is order complete. If  $\mathcal{F}$  is a subset of  $\mathcal{P}(\Omega)$ ,  $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$ .

Interesting tip:  $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

**Axiom:**

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$  is order complete.

In  $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any  $A \subseteq \mathbb{R} \cup -\infty, +\infty$  and  $c \in \mathbb{R}$  We denote by  $A + c$  the set  $\{a + c \mid a \in A\}$
- If  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\lambda A$  denotes  $\{\lambda a \mid a \in A\}$
- $-A$  denotes  $(-1)A$

Prop.

For any  $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\sup(-A) = -\inf A$ ,  $\inf(-A) = -\sup A$  Def

We denote by  $(\mathbb{R}, \leq)$  a field  $\mathbb{R}$  equipped with a total order  $\leq$ , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$  such that  $a < b$ , one has  $\forall c \in \mathbb{R}$ ,  $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ ,  $ab > 0$
- $\forall A \subseteq \mathbb{R}$ , if  $A$  has an upper bound in  $\mathbb{R}$ , then it has a supremum in  $\mathbb{R}$

Prop.

Let  $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

$\inf$  takes the same

Theorem:

Let  $I$  and  $J$  be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If  $\{a, b\} \neq \{+\infty, -\infty\}$  then  $c = a + b$

$\inf$  takes the same if  $(-\infty) + (+\infty)$  doesn't happen

Corollary:

Let  $I$  be a non-empty set,  $f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

Then  $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

$\inf$  takes the similar ( $\leq \rightarrow \geq$ ) (provided when the sum are defined)



## Chapter 6

# Vector space

In this section:

$K$  denotes a unitary ring.

Let  $0$  be zero element of  $K$

$1$  be the unity of  $K$

### 6.1 $K$ -module

#### 6.1.1 Def

Let  $(V, +)$  be a commutative group. We call left/right  $K$ -module structure: any mapping  $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group  $(V, +)$  equipped with a left/right  $K$ -module structure is called a left/right  $K$ -module.

#### 6.1.2 Remark

Let  $K^{op}$  be the set  $K$  equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then  $K^{op}$  forms a unitary ring  
 Any left  $K^{op}$  - module is a right  $K$ -module  
 Any right  $K^{op}$  - module is a left  $K$ -module  
 $(K^{op})^{op} = K$

### 6.1.3 Notation

When we talk about a left/right  $K$ -module  $(V, +)$ , we often write its left  $K$ -module structure as  $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

### 6.1.4 $K$ -vector space

If  $K$  is commutative, then  $K^{op} = K$ , so left  $K$ -module and right  $K$ -module structure are the same. We simply call them  $K$ -module structure. A commutative group equipped with a  $K$ -module structure is called a  $K$ -module. If  $K$  is a field, a  $K$ -module is also called a  $K$ -vector space

Let  $\Phi : K \times V \rightarrow V$  be a left or right  $K$ -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence  $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$   
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$  is a morphism of groups. Hence  $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$  (*is a var*)

### 6.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping  $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$  is a left  $K$ -module structure

The mapping  $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$  is a right  $K$ -module structure

**6.1.6 Remark:**

We can also write an element  $\mu$  of  $K^I$  in the form of a family  $(\mu_i)_{i \in I}$  of elements in  $K$  ( $\mu_i$  is the image of  $i \in I$  by  $\mu$ )  
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

**6.2 sub K-module****6.2.1 Def**

Let  $V$  be a left/right  $K$ -module. If  $W$  is a subgroup of  $V$ . Such that  $\forall a \in K, \forall x \in W \quad ax/xa \in W$ , then we say that  $W$  is left/right sub- $K$ -module of  $V$ .

**6.2.2 Example**

Let  $I$  be a set. Let  $K^{\oplus I}$  be the subset of  $K^I$  composed of mappings  $f : I \rightarrow K$  such that  $I_f = \{x \in I \mid f(x) \neq 0\}$  is finite. It is a left and right sub- $K$ -module of  $K^I$

In fact,  $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$   
Hence  $f - g \in K^{\oplus I}$  So  $K^{\oplus I}$  is a subgroup of  $K^I$   
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

**6.3 morphism of K-modules****6.3.1 Def**

Let  $V$  and  $W$  be left  $K$ -module, A morphism of groups  $\phi : V \rightarrow W$  is called a morphism of left  $K$ -modules if  $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

**6.3.2 K-linear mapping**

If  $K$  is commutative, a morphism of  $K$ -modules is also called a  $K$ -linear mapping. We denote by  $\text{hom}_{K\text{-Mod}}(V, W)$  the set of all morphism of left- $K$ -module from  $V$  to  $W$ . This is a subgroup of  $W^V$

**6.3.3 Theorem**

Let  $V$  be a left  $K$ -module. Let  $I$  be a set.  
The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$  is a bijection where  
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.3.4 Remark:column

In the case where  $I = 1, 2, 3, \dots, n$   $V^I$  is denoted as  $V^n$ ,  $K^I$  is denoted as  $K^n$ . For any  $(x_1, \dots, x_n) \in V^n$ , by the theorem, there exists a unique morphism of left  $K$ -modules  $\phi : K^n \rightarrow V$  such that  $\forall i \in 1, \dots, n, \phi(e_i) = x_i$ .

We write this  $\phi$  as a column  $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ . It sends  $(a_1, \dots, a_n) \in K^n$  to  $a_1x_1 + \dots + a_nx_n$ .

## 6.4 kernel

### 6.4.1 Prop

Let  $G$  and  $H$  be groups and  $f : G \rightarrow H$  be a morphism of groups

- $Im(f) \subseteq H$  is a subgroup of  $H$
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- $f$  is injection iff  $\ker(f) = \{e_G\}$

### 6.4.2 Def

$\ker(f)$  is called the kernel of  $f$

### 6.4.3 Theorem

$f$  is injection iff  $\ker(f) = \{e_G\}$

### Proof

Let  $e_G$  and  $e_H$  be neutral element of  $G$  and  $H$  respectively

- (1) Let  $x$  and  $y$  be element of  $G$   
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ . So  $Im(f)$  is a subgroup of  $H$
- (2) Let  $x$  and  $y$  be element of  $\ker(f)$ . One has  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$ . So  $xy^{-1} \in \ker(f)$ . So  $\ker(f)$  is a subgroup of  $G$ .
- (3) Suppose that  $f$  is injection.  
 Since  $f(e_G) = e_H$  one has  $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ . Suppose that  $\ker(f) = \{e_G\}$ . If  $f(x) = f(y)$  then  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$ .  
 Hence  $xy^{-1} = e_G \Rightarrow x = y$

### 6.4.4 Def

Let  $(V, +)$  be a commutative group,  $I$  be a set. We define a composition law  $+$  on  $V^I$  as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then  $V^I$  forms a commutative group

### 6.4.5 Remark

Let  $E$  and  $F$  be left  $K$ -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$  is a subgroup of  $F^E$

In fact  $f$  and  $g$  are elements of  $\text{hom}_{K\text{-Mod}}(E, F)$ , then  $f - g$  is also a morphism of left  $K$ -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

### 6.4.6 Theorem

Let  $V$  be a left  $K$ -module,  $I$  be a set The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$  is an isomorphism of groups, where  $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.4.7 Proof:

One has  $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So  $\Psi$  is a morphism of groups

injectivity Let  $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$  Such that  $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence  $\phi$  is the neutral element.

surjectivity Let  $x = (x_i)_{i \in I} \in V^I$  We define  $\phi_x : K^{\oplus I} \rightarrow V$  such that  $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left  $K$ -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that  $K'$  is a unitary ring, and  $V$  is also equipped with a right  $K'$ -module structure, Then  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$  is a right sub- $k'$ -module, and  $\Psi$  in the theorem is a right  $K'$ -module isomorphism



## Chapter 7

# Monotone mappings

### 7.1 Def

Let  $I$  and  $X$  be partially ordered sets,  $f : I \rightarrow X$  be a mapping.

- If  $\forall (a, b) \in I \times I$  such that  $a < b$ . One has  $f(a) \leq f(b)$ , then we say that  $f$  is increasing. decreasing takes similar way.
- If  $f$  is (strictly) increasing or decreasing, we say that  $f$  is (strictly) monotone.

### 7.2 Prop.

Let  $X, Y, Z$  be partially ordered sets.  $f : X \rightarrow Y, g : Y \rightarrow Z$  be mappings

- If  $f$  and  $g$  have the same monotonicity, then  $g \circ f$  is increasing
- If  $f$  and  $g$  have different monotonicities, then  $g \circ f$  is decreasing

strict monotonicities takes the same

### 7.3 Def

Let  $f$  be a function from a partially ordered set  $I$  to another partially ordered set  $X$ . If  $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$  is (strictly) increasing/decreasing then we say that  $f$  is (strictly) increasing/decreasing

### 7.4 Prop.

Let  $I$  and  $X$  be partially ordered sets.  $f$  be function from  $I$  to  $X$ .

- If  $f$  is increasing/decreasing and  $f$  is injection, then  $f$  is strictly increasing/decreasing
- Assume that  $I$  is totally ordered and  $f$  is strictly monotone, then  $f$  is injection

## 7.5 Prop

Let  $A$  be totally ordered set,  $B$  be a partially ordered set,  $f$  be an injective function from  $A$  to  $B$

If  $f$  is increasing/decreasing, then so is  $f^{-1}$

## 7.6 Def

Let  $X$  and  $Y$  be partially ordered sets.  $f : X \rightarrow Y$  be a bijection. If both  $f$  and  $f^{-1}$  are increasing, then we say that  $f$  is an isomorphism of partially ordered sets.

(If  $X$  is totally, then a mapping  $f : X \rightarrow Y$  is an isomorphism of partially ordered sets iff  $f$  is a bijection and  $f$  is increasing)

## 7.7 Prop.

Let  $I$  be a subset of  $\mathbb{N}$  which is infinite. Then there is a unique increasing bijection  $\lambda_I : \mathbb{N} \rightarrow I$

## 7.8 Proof

### 7.8.1 bijection

We construct  $f : \mathbb{N} \rightarrow I$  by induction as follows.

Let  $f(0) = \min I$  Suppose that  $f(0), \dots, f(n)$  are constructed

then we take  $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since  $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$ . Therefore  $f(n) \leq f(n+1)$

Since  $f(n+1) \notin \{f(0), \dots, f(n)\}$ , we have  $f(n) < f(n+1)$

Hence  $f$  is strictly increasing and this is injective

If  $f$  is not surjective, then  $I \setminus \text{Im}(f)$  has a element  $N$ .

Let  $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$ .

Since  $N \notin \text{Im}(f)$ ,  $N < f(m)$ .

So  $m \neq 0$ . Hence  $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition,  $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$ ,

Hence  $f(m) \leq N$ , causing contradiction.



**7.8.2 uniqueness**

exercise: Prove that  $Id_{\mathbb{N}}$  is the only isomorphism of partially ordered sets from  $\mathbb{N}$  to  $\mathbb{N}$



# Chapter 8

## sequence and series

Let  $I \subseteq \mathbb{N}$  be a infinite subset

### 8.1 Def

Let  $X$  be a set. We call sequence in  $X$  parametrized by  $I$  a mapping from  $I$  to  $X$ .

### 8.2 Remark

If  $K$  is a unitary ring and  $E$  is a left  $K$ -module then the set of sequence  $E^I$  admits a left- $K$ -module structure. If  $x = (x_n)_{n \in I}$  is a sequence in  $E$ , we define a sequence  $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ , called the series associated with the sequence  $x$ .

### 8.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$  is a morphism of left- $K$ -module

### 8.4 proof

Let  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$  be elements of  $E^I$

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

### 8.5 Prop

Let  $I$  be a totally ordered set .  $X$  be a partially ordered set,  $f : I \rightarrow X$  be a mapping ,  $J \in I$  Assume that  $J$  does not have any upper bound in  $I$

- If  $f$  is increasing ,then  $f(I)$  and  $f(J)$  have the same upper bounds in  $X$
- If  $f$  is decreasing ,then  $f(I)$  and  $f(J)$  have the same lower bounds in  $X$

## 8.6 limit

### 8.6.1 Def

Let  $i \subseteq \mathbb{N}$  be a infinite subset.  $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$  where  $[-\infty, +\infty]$  denotes  $\mathbb{R} \cup \{-\infty, +\infty\}$ , we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left( \inf_{i \in I, i \geq n} x_i \right)$$

If  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$ , we then say that  $(x_n)_{n \in I}$  tends to  $l$  and that  $l$  is the limit of  $(x_n)_{n \in I}$ . If in addition  $(x_n)_{n \in I} \in \mathbb{R}^I$  and  $l \in \mathbb{R}$ , we say that  $(x_n)_{n \in I}$  converges to  $l$

### 8.6.2 Remark

If  $J \subseteq \mathbb{N}$  is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} = \inf_{n \in J} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left( \inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in  $(x_i)_{i \in I}$  the limit superior and the limit inferior do not change.

In fact, if we take  $J = \mathbb{N} \setminus \{0, \dots, m\}$ , then  $\inf_{n \in J}(\dots)$  and  $\sup_{n \in J}(\dots)$  only depends on the values of  $x_i, i \in I, i \geq m$

### 8.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

**8.6.4 Prop**

Let  $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

**8.6.5 Prop**

Let  $(x_n)_{n \in I}$  be elements in  $[-\infty, +\infty]^I$ . Suppose that there exists  $N_0 \in \mathbb{N}$  such that  $\forall n \in I, n \geq N_0$ , one has  $x_n \leq y_n$ . Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

**8.6.6 Theorem**

Let  $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$  one has  $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$  and  $(z_n)_{n \in I}$  tend to the same limit  $l$

Then  $(y_n)_{n \in I}$  tends to  $l$

**8.6.7 Def**

Let  $I$  be an infinite subset of  $\mathbb{N}$ , and  $(x_n)_{n \in I}$  be a sequence in some set  $X$ . We call subsequence of  $(x_n)_{n \in I}$  a sequence of the form  $(x_n)_{n \in J}$ , where  $J$  is an infinite subset of  $I$

**8.6.8 Prop**

Let  $I$  and  $J$  be infinite subset of  $\mathbb{N}$  such that  $J \subseteq I$ .  $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$ , one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if  $(x_n)_{n \in I}$  tends to  $l \in [-\infty, +\infty]$ , then  $(x_n)_{n \in J}$  tends to  $l$

### 8.6.9 Prop

$\forall n \in \mathbb{N}$ , one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

### 8.6.10 Theorem

Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$

- If the mapping  $(n \in I) \mapsto x_n$  is increasing, then  $(x_n)_{n \in I}$  tends to  $\sup_{n \in I} x_n$
- If the mapping  $(n \in I) \mapsto x_n$  is decreasing, then  $(x_n)_{n \in I}$  tends to  $\inf_{n \in I} x_n$

### 8.6.11 Notation

If a sequence  $(x_n)_{n \in I} \in [-\infty, +\infty]$  tends to some  $l \in [-\infty, +\infty]$  the expression  $\lim_{n \in I, n \rightarrow} x_n$  denotes this limit  $l$

### 8.6.12 Corollary

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{N}_{\geq 0}$ . Then the series  $\sum_{n \in I} x_n$  (the sequence  $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ ) tends to an element in  $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ . It converges in  $\mathbb{R}$  iff it is bounded from above (namely has an upper bound in  $\mathbb{R}$ )

### 8.6.13 Notation

If a series  $\sum_{n \in I} x_n$  in  $[-\infty, +\infty]$  tends to some limit, we use the expression  $\sum_{n \in I} x_n$  to denote the limit

### 8.6.14 Theorem: Bolzano-Weierstrass

Let  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\limsup_{n \in I, n \rightarrow +\infty} x_n$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\liminf_{n \in I, n \rightarrow +\infty} x_n$ .

**Proof**

Let  $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If  $J$  is infinite, the sequence  $(x_n)_{n \in J}$  is decreasing so it tends to  $\inf_{n \in J} x_n$

$\forall n \in J$  by definition  $x_n = \sup_{i \in I, i \geq n} x_i$  so  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that  $J$  is finite. Let  $n_0 \in I$  such that  $\forall n \in J, n < n_0$ . Denote by

$$l = \sup_{n \in I, n \geq n_0} x_n$$

Let  $N \in \mathbb{N}$  such that  $N \geq n_0$ . By definition  $\sup_{i \in I, i \geq n_0} x_i \leq l$ . If the strict inequality  $\sup_{i \in I, i \geq N} x_i < l$  holds, then  $\sup_{i \in I, i \geq N} x_i$  is NOT an upper bound of  $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists  $n \in I$  such that  $n_0 \leq n < N$  such that  $x_n > \sup_{i \in I, i \geq N} x_i$ . We may also assume that  $n$  is largest among elements of  $I \cap [n_0, N[$  that satisfies this inequality.

Then  $\forall m \in I$  if  $m \geq n$  then  $x_m \leq x_n$ . Thus  $n \in J$  that contradicts the maximality of  $n_0$ .

Therefore

$$l = \sup_{i \in I, i \geq N} x_i$$

, which leads to

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = l$$

Moreover, if  $m \in I, m \geq n_0$  then  $m \notin J$ , so  $x_m < l$  (since otherwise  $x_m = \sup_{i \in I, i \geq m} x_i$  and hence  $m \in J$ ). Hence,  $\forall$  finite subset  $I'$  of  $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$  and hence  $\exists n \in I$ , such that  $n > \max I'$ , and  $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence  $(n_j)_{j \in \mathbb{N}}$  in  $I$

Let  $n_0$  be as above. Let  $f : \mathbb{N} \rightarrow I_{\geq n_0}$  be a surjective mapping.

If  $n_j$  is chosen, we choose  $n_{j+1} \in I$  such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence  $(x_{n_j})_{j \in \mathbb{N}}$  is increasing

And

$$\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n = l$$

$$l = \sup_{n \in I, n \geq n_0} x_n$$

So  $(x_{n_j})_{j \in \mathbb{N}}$  tends to  $l$





## Chapter 9

# Cauchy sequence

### 9.1 Def

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{R}$   
If  $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$  then  
we say that  $(x_n)_{n \in I}$  is a Cauchy sequence

### 9.2 Prop

- If  $(x_n)_{i \in I} \in \mathbb{R}^I$  converges to some  $l \in \mathbb{R}$ , then it is a Cauchy sequence
- If  $(x_n)_{i \in I}$  is a Cauchy sequence, there exists  $M > 0$  such that  $\forall n \in I \quad |x_n| \leq M$
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite,  $(x_n)_{n \in I}$  is a Cauchy sequence.
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite and  $l \in \mathbb{R}$  such that  $(x_n)_{n \in I}$  converges to  $l$ , then  $(x_n)_{n \in J}$  converges to  $l$  too.

### 9.3 Theorem: Completeness of real number

If  $(x_n)_{n \in I} \in \mathbb{R}^I$  is a Cauchy sequence, then it converges in  $\mathbb{R}$

#### Proof

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists M \in \mathbb{R}_{>0}$  such that  $-M \leq x_n \leq M \quad \forall x \in I$ . So  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . By Bolzano-Weierstrass theorem.  $\exists J \subseteq I$  infinite such that  $(x_n)_{n \in I}$  converges to  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . Therefore  $(x_n)_{n \in I}$  converges to the same limit.

## 9.4 Absolutely converge

We say that a series  $\sum_{n \in I} x_n \in \mathbb{R}$  converges absolutely if  $\sum_{n \in I} |x_n| < +\infty$

### 9.4.1 Prop

If a series  $\sum_{n \in I} x_n$  converges absolutely, then it converges in  $\mathbb{R}$

## Chapter 10

# Comparison and Technics of Computation

### 10.1 Def

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be sequence in  $\mathbb{R}$

- If there exists  $M \in \mathbb{R}_{>0}$  and  $N \in \mathbb{N}$  such that  $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$  then we write  $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists  $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$  and  $N \in \mathbb{N}$  such that  $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$  and  $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$ , then we write  $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

### 10.2 Prop.

Let  $I$  and  $X$  be partially ordered sets and  $f : I \rightarrow X$  be an increasing/decreasing mapping. Let  $J$  be a subset of  $I$ . Assume that any elements of  $I$  has an upper bound in  $J$ . Then  $f(I)$  and  $f(J)$  have the same upper/lower bounds in  $X$

### 10.3 Theorem

Let  $I$  be a totally ordered set,  $f : I \rightarrow [-\infty, +\infty]$  and  $g : I \rightarrow [-\infty, +\infty]$  be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

### Proof

We can assume  $f$  and  $g$  increasing. Let  $a = \sup f(I), b = \sup g(I)$

Let  $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip  $A$  with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let  $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$ .

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

$h$  is increasing.

Let  $(x, y) \in A$ . Assume that  $x \leq y$

If  $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$  then  $(y, y) \in B$  and  $(x, y) \leq (y, y)$

If  $\{f(y), g(y)\} = \{-\infty, +\infty\}$  and for  $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$ . So  $a = +\infty$ , Hence  $b > -\infty$

So  $\exists z \in I$  such that  $g(z) > -\infty$ . We should have  $y \leq z$  Hence  $f(z) + g(z)$  is well defined,  $(z, z) \in B$  and  $(x, y) \leq (z, z)$  Similarly, if  $x \geq y$ ,  $(x, y)$  has also an upper bound in  $B$ . Therefore:  $\sup h(A) = \sup h(B)$

## 10.4 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

### Proof

$\forall n \in \mathbb{N}$ , let  $A_N = \sup_{n \in I, n \geq N} x_n$   $B_N = \sup_{n \in I, n \geq N} y_n$ .  $(A_N)_{N \in \mathbb{N}}$  and  $(B_N)_{N \in \mathbb{N}}$  are decreasing, and  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$   $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let  $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$  if  $A_N + B_N$  is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

## 10.5 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + \left( \limsup_{n \in I, n \rightarrow +\infty} y_n \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + \left( \liminf_{n \in I, n \rightarrow +\infty} y_n \right)$$

### Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with  $\infty$

## 10.6 Theorem

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Assume that  $\forall n \in I, y_n \in \mathbb{R}$  and  $(y_n)_{n \in I}$  converges to some  $l \in \mathbb{R}$ .  
Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

## 10.7 Prop.

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ .  
Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

### Proof

About the first inequality. Since  $\max\{x_n, y_n\} \geq x_n$  and  $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset  $J$  of  $I$  such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let  $J_1 = \{n \in J \mid x_n \geq y_n\}$   $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$  So either  $J_1$  or  $J_2$  is infinite

Suppose that  $J_1$  is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If  $J_2$  is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

## 10.8 Theorem

Let  $(a_n)_{n \in I} \in \mathbb{R}^I$   $l \in \mathbb{R}$ . The following statements are equivalent

- $(a_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

### Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\left\{\left(\limsup_{n \in I, n \rightarrow +\infty} a_n\right) - l, l - \left(\liminf_{n \in I, n \rightarrow +\infty} a_n\right)\right\}$$

(1)  $\Rightarrow$  (2):

If  $(a_n)_{n \in I}$  converges to  $l$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2)  $\Rightarrow$  (1):

If  $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore:  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

## 10.9 Remark

Let  $(a_n)_{n \in I}$  be a sequence in  $\mathbb{R}$ ,  $l \in \mathbb{R}$

The sequence  $(a_n)_{n \in I}$  converges to  $l$  iff  $a_n - l = o(1), n \in I, n \rightarrow +\infty$

## 10.10 Calculates on $O()$ , $o()$

### 10.10.1 Plus

Let  $(a_n)_{n \in I}$   $(a'_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$

### 10.10.2 Transform

Let  $(a_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be two sequence in  $\mathbb{R}$  If  $a_n = o(b_n), n \in I, n \rightarrow +\infty$ , then  $a_n = O(b_n), n \in I, n \rightarrow +\infty$

### 10.10.3 Transition

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  and  $(c_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = O(b_n)$  and  $b_n = o(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$

### 10.10.4 Times

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$  be sequences in  $\mathbb{R}$

- If  $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If  $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

## 10.11 On the limit

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be elements of  $\mathbb{R}^I$  that converges to  $l \in \mathbb{R}$  and  $l' \in \mathbb{R}$  respectively. Then:

- $(a_n + b_n)_{n \in I}$  converges to  $l + l'$
- $(a_n b_n)_{n \in I}$  converges to  $ll'$

## 10.12 Prop

Let  $a \in \mathbb{R}$  then  $a^n = o(n!)$   $n \rightarrow +\infty$

### Proof

Let  $N \in \mathbb{N}$  such that  $|a| < N$   
For  $n \in \mathbb{N}$  such that  $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^n - N$$

And  $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$ . Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

### 10.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

**Proof**

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

### 10.14 Prop

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be the elements of  $\mathbb{R}^I$ . If the series  $\sum_{n \in I} b_n$  converges absolutely and if  $a_n = O(b_n) \quad n \rightarrow +\infty$  Then  $\sum_{n \in I} a_n$  converges absolutely

**Proof**

By definition  $\sum_{n \in I} |b_n| < +\infty$ . If  $|a_n| \leq M|b_n|$  for  $n \in I, n \geq N$  where  $N \in \mathbb{N}$ . Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

### 10.15 Theorem: d'Alembert ratio test

Let  $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely
- If  $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  does not converge (diverges)



**Proof****(1)**

Let  $\alpha \in \mathbb{R}$  such that  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$ ,  $\alpha$  isn't a lower bound of  $\left( \sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$   
 So  $\exists N \in \mathbb{N}$  such that  $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$  Hence for  $n \geq N$   $|a_n| \leq \alpha^{n-N} |a_N|$  since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore  $a_n = O(\alpha^n)$  since  $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$ ,  $\sum_{n \in \mathbb{N}} a_n$  converge absolutely.

**10.15.1 Lemma**

If a series  $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$  converges, then  $\lim_{n \rightarrow +\infty} a_n = 0$

**Proof**

If  $\left( \sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$  converges to some  $l \in \mathbb{R}$ , then  $\left( \sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$  converges to  $l$ ,  
 too. Hence  $\left( a_n = \left( \sum_{i=0}^n a_i \right) - \left( \sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$  converges to  $l - l = 0$

**10.15.2 (2)**

Let  $\beta \in \mathbb{R}$  such that  $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 So there exists  $N \in \mathbb{N}$  such that  $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$   
 Hence  $(|a_n|)_{n \in \mathbb{N}}$  is not bounded since  $|a_n| \geq \beta^{n-N} |a_N|$   
 By the lemma:  $\sum_{n \in \mathbb{N}} a_n$  diverges.

**10.16 Prop**

Let  $a \in \mathbb{R}, a > 1$  Then  $n = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $\epsilon > 0$  such that  $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

**10.16.1 Corollary**

Let  $a > 1, t \in \mathbb{R}_{\geq 0}$  Then  $n^t = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $d \in \mathbb{N}_{\geq 1}$  such that  $t \leq d$  Then  $n^{t-d} \leq 1$  So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let  $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence  $n^t = o(a^n)$

**10.16.2 Corollary**

There exists  $M \geq 1$  such that  $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

**Proof**

Let  $a \in \mathbb{R}$  such that  $1 < a < e$

**10.17 Theorem: Cauchy root test**

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Let  $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If  $\alpha < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.
- If  $\alpha > 1$  then  $\sum_{n \in \mathbb{N}} a_n$  diverges

**Proof**

(1)

Let  $\beta \in \mathbb{R}, \alpha < \beta < 1$ . There exists  $N \in \mathbb{N}$  such that  $|a_n|^{\frac{1}{n}} \leq \beta$  for  $n \geq N$ . That means  $|a_n| = O(\beta^n)$  since  $0 < \beta < 1$ ,  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.

(2)

If  $\alpha > 1$  then  $\forall N \in \mathbb{N} \exists n \geq N$  such that  $|a_n|^{\frac{1}{n}} \geq 1$ , since otherwise  $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$  contradiction  
Hence  $(|a_n|)_{n \in \mathbb{N}}$  cannot converge to 0.

## Part III

# Axiom of choice



# Chapter 11

## Preparation

### 11.1 Statement of axiom of choice

For any set  $I$  and any family  $(A_i)_{i \in I}$  of non-empty sets, there exists a mapping  $f : I \rightarrow \bigcup_{i \in I} A_i$  such that  $\forall i \in I, f(i) \in A_i$

### 11.2 Def

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$   $A$  is non-empty, there exists a least element of  $A$  then we say that  $(X, \leq)$  is a well ordered set.

### 11.3 Theorem

For any set  $X$ , there exists an order relation  $\leq$  on  $X$  such that  $(X, \leq)$  forms a well ordered set.

### 11.4 Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$  that is totally ordered with respect to  $\leq$ , there exists an upper bound of  $A$  inside  $X$ . Then, there exists a maximal element  $x_0$  of  $X$  ( $\forall y \in X, y > x_0$  does not hold)

### 11.5 Prop.

Let  $(X, \leq)$  be a well ordered set,  $y \notin X$ . We extend  $\leq$  to  $X \cup \{y\}$ , such that  $\forall x \in X, x < y$ . Then  $(X \cup \{y\}, \leq)$  is well ordered.

## 11.6 Proof

Let  $A \subseteq X \cup \{y\}$ ,  $A \neq \emptyset$ . If  $A = \{y\}$  then  $Y$  is the least element of  $A$ . If  $A \neq \{y\}$  then  $B = A \setminus \{y\}$  is non-empty. Let  $b$  be the least element of  $B$ . Since  $b < y$  it's also the least element of  $A$

## 11.7 Def: Initial Segment

Let  $(X, \leq)$  be a well ordered set.  $S \subseteq X$ , If  $\forall s \in S, x \in X \quad x < s$  initial  $x \in S$  ( $X_{<s} \subseteq S$ ), then we say that  $S$  is an initial segment of  $X$

If  $S$  is a initial segment such that  $S = X$  then we sat that  $S$  is a proper initial segment.

## 11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$  Then  $X_{<x}$  is a proper initial segment of  $X$ .

## 11.9 Prop.

Let  $(X, \leq)$  be a well ordered set , If  $(S_i)_{i \in I}$  is a family of initial segment of  $X$ , then  $\bigcup_{i \in I} S_i$  is an initial segment of  $X$

## 11.10 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$  such that  $s \in S_i, i \in I$  Therefore  $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

## 11.11 Prop.

Let  $(X, \leq)$  be a well ordered set.

- (1) Let  $S$  be a proper initial segment of  $X$ ,  $x = \min(X \setminus S)$  Then  $S = X_{<x}$
- (2)  $X \rightarrow \wp(X)$   
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of  $X$  forms a well ordered subset of  $(\wp(X), \subseteq)$

## 11.12 Proof

- (1)  $\forall s \in S$  if  $x \leq s$  then  $x \in S$  contradiction. Hence  $s < x$ , This shows  $S \subseteq X_{<x}$  Conversely , if  $t \in X, t \notin X \setminus S$  Hence  $t \in S$ . Hence  $X_{<x} \subseteq S$

- (2) Let  $x, y \in X, x < y$  By definition  $X_{<x} \subseteq X_{<y}$  Moreover  $x \in X_{<y} \setminus X_{<x}$  So  $X_{<x} \subsetneq X_{<y}$
- (3) Let  $\mathcal{F} \subseteq \wp(X)$  be a set of initial segments.  $\mathcal{F} \neq \emptyset$ . Then there exists  $A \subseteq X$  such that  $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$  If  $A = \emptyset$  then  $\mathcal{F} = \{X\}$ , and  $\{X\}$  is the least element of  $\mathcal{F}$ . Otherwise  $A \neq \emptyset$  and  $A$  has a least element  $a$ . Then by (2)  $X_{<a}$  is the least element of  $\mathcal{F}$

### 11.13 Lemma

Let  $(X, \leq)$  be a well ordered set,  $f : X \rightarrow X$  be a strictly increasing mapping. Then  $\forall x \in X, x \leq f(x)$

#### Proof

Let  $A = \{x \in X \mid f(x) < x\}$  If  $A \neq \emptyset$ , let  $a$  be the least element of  $A$ . By definition  $f(a) < a$ . Hence  $f(f(a)) < f(a)$  since  $f$  is strictly increasing. This shows  $f(a) \in A$ . But  $a$  is the least element of  $A$ ,  $f(a) < a$  cannot hold: contradiction.

### 11.14 Prop

Let  $(X, \leq)$  be a well ordered set,  $S$  and  $T$  be two initial segment of  $X$ . If  $f : S \rightarrow T$  is a bijection that's strictly increasing, then  $S = T, f = Id_S$

#### Proof

We may assume  $T \subseteq S$ . Let  $l : T \rightarrow S$  be the inclusion mapping and  $g = l \circ f : S \rightarrow S$ . Since  $g$  is strictly increasing, by the lemma,  $\forall s \in S, s \leq g(s) = f(s) \in T$ . Since  $T$  is an initial segment,  $s \in T$ . Hence  $S = T$ . Apply the lemma to  $f^{-1}$  we get  $\forall s \in S, s \leq f^{-1}(s)$  Hence  $f(s) \leq s$  Therefore  $f(s) = s$

### 11.15 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be partially ordered sets. If  $\exists f : X \rightarrow Y$  that's increasing and bijective, we say that  $(X, \leq)$  and  $(Y, \leq)$  are isomorphic

### 11.16 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be well ordered sets. If  $(X, \leq)$  is isomorphic to an initial segment of  $Y$ . We note  $X \preceq Y$  or  $Y \succeq X$ . If  $X$  is isomorphic to  $Y$ , we note  $X \sim Y$ . If  $X \preceq Y$  but  $X \not\sim Y$ , we note  $X \prec Y$  or  $Y \succ X$

### 11.17 Prop.

Let  $X$  and  $Y$  be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

#### Proof

We construct a correspondence  $f$  from  $X$  to  $Y$ , such that  $(x, y) \in \Gamma_f$ , iff  $X_{<x} \sim Y_{<y}$   
By the last proposition of Oct. 11,  $f$  is a function.

- If  $a, b \in \text{Dom}(f)$ ,  $a < b$ , then  $X_{<a} \subsetneq X_{<b}$   
By definition,  $Y_{<f(b)} \sim X_{<b}$   $Y_{<f(a)} \sim X_{<a}$   
Hence  $Y_{<f(a)}$  is isomorphic to a proper initial segment of  $Y_{<f(b)}$ . Therefore  $Y_{f(a)}$  is a proper initial segment of  $Y_{<f(b)}$ . We then get  $f(a) < f(b)$ . Thus  $f$  is strictly increasing.
  - Let  $a \in \text{Dom}(f)$  Let  $x \in X, x < a$  Then  $X_{<x}$  is a initial segment of  $X_{<a} \sim Y_{<f(a)}$  Hence  $\exists y \in Y$   $X_{<x} \sim Y_{<y}$  This shows that  $x \in \text{Dom}(f)$ . Hence  $\text{Dom}(f)$  is an initial segment of  $X$ . Applying this to  $f^{-1}$ , we get :  $\text{Im}(f) = \text{Dom}(f)$  is an initial segment of  $Y$
  - Either  $\text{Dom}(f) = X$  or  $\text{Im}(f) = Y$ .  
Assume that  $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$  are respectively the least elements of  $X \setminus \text{Dom}(f)$  and  $Y \setminus \text{Im}(f)$ .  
Then we get  $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$ .  
We obtain  $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$ . Contradiction
  -
- Case 1  $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$   $X \prec Y$   
Case 2  $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$   $X \succ Y$   
Case 3  $\text{Dom}(f) = X, \text{Im}(f) = Y$   $X \sim Y$

### 11.18 Lemma

Let  $(X, \leq)$  be a partially ordered set .  $\mathfrak{S} \subseteq \wp(X)$ . Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$  is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$ , either  $A$  is an initial segment of  $B$ , or  $B$  is an initial segment of  $A$ .

Let  $Y = \bigcup_{A \in \mathfrak{S}} A$ . Then  $(Y, \leq)$  is a well ordered set, and  $\forall A \in \mathfrak{S}, A$  is an initial segment of  $Y$ .



**Proof**

- Let  $A \in \mathfrak{S}, x \in A, y \in Y, y < x$ . Since  $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$ , such that  $y \in B$ . If  $y \notin A$  then  $B \not\subseteq A$ . Hence  $A$  is an initial segment of  $B$ . Hence  $y \in A$ . Contradiction
- Let  $Z \subseteq Y, Z \neq \emptyset$ . Then  $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$ . Let  $m$  be the least element of  $A \cap Z$ . Let  $z \in Z, B \in \mathfrak{S}$ , such that  $z \in B$ . If  $z \in A$ , then  $m \leq z$ . If  $z \notin A$ , then  $A$  is an initial segment of  $B$ .

Since  $B$  is well ordered, if  $m \not\leq z$  then  $z < m$ . Since  $m \in A$ , we get  $z \in A$ . Contradiction.

Therefore,  $m$  is the least element of  $Z$ .



## Chapter 12

# Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. Suppose that any well-ordered subset of  $X$  has an upper bound on  $X$ , then  $X$  has a maximal element (a maximal element  $m$  of  $\{x \mid x > m\} = \emptyset$ )

### 12.1 Proof

Suppose that  $X$  doesn't have any maximal element.  $\forall A \in \omega. \exists f(A)$  such that  $\forall a \in A, a < f(A)$

Let

$$\omega = \{\text{well ordered subset of } X\}$$

. (guaranteed by axiom of choice)

Let  $f : \omega \rightarrow X$  such that  $f(A)$  is an upper bound of  $A \in \omega$ .

If  $A \in \omega$  satisfies

$$\forall a \in A, a = f(A_{<a})$$

, we say that  $A$  is a  $f$ -set

Let

$$\mathfrak{S} = \{f\text{-sets}\}$$

Note that

$$\emptyset \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$$

In fact, if  $a \in A$ , then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If  $a = f(A) \notin A$  then

$$(A \cup \{f(A)\})_{<a} = A$$

Let  $A$  and  $B$  be elements of  $\mathfrak{S}$ . Let  $I$  be the union of all common initial segments of  $A$  and  $B$ . This is also a common initial segment of  $A$  and  $B$ .

If  $I \neq A$  and  $I \neq B$ , then

$$\exists(a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then  $I \cup \{a\}$  is also a common initial segment of  $A$  and  $B$ , contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and  $\forall A \in \mathfrak{S}$  is an initial segment of  $Y$ .

Since  $A$  is an initial segment of  $Y$

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A \quad A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus  $Y$  is the greatest element of  $(\mathfrak{S}, \subseteq)$ . However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If  $f(y)$  is not a maximal element of  $X$

$$\exists x \in X, f(y) < x$$

**Part IV**

**Topology**



## Chapter 13

# Absolute value and norms

### 13.1 Def

Let  $K$  be a field. By absolute value on  $K$ , we mean a mapping  $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- (1)  $\forall a \in K \quad |a| = 0$  iff  $a = 0$
- (2)  $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3)  $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$  (triangle inequality)

### 13.2 Notation

$\mathbb{Q}$  Take a prime num  $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$  there exists a integer  $ord_p(\alpha) \frac{a}{b}$ , where  
 $a \in \mathbb{Z} \setminus \{0\}$   
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

### 13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on  $\mathbb{Q}$

### Proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd$$

$$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d}$$

Assume  $ord_p(\alpha) \geq ord_p(\beta)$

$$\alpha + \beta$$

$$= p^{ord_p(\beta)} \left( p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right)$$

$$= p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd$$

So

$$ord_p(\alpha + \beta) \geq ord_p(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$$

$$\text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha + \beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$$

$$\max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

### 13.4 Def

Let  $K$  be a field and  $|\cdot|$  be an absolute value. We call  $(K, |\cdot|)$  a valued field.



## Chapter 14

# Quotient Structure

### 14.1 Def

Let  $X$  be a set and  $\sim$  be a binary relation on  $X$   
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$ , if  $x \sim y$  then  $y \sim x$
- $\forall (x, y, z) \in X^3$ , if  $x \sim y, y \sim z$  then  $x \sim z$

then we say that  $\sim$  is an equivalence relation

### 14.2 equivalence class

$\forall x \in X$  we denote by  $[x]$  the set  $\{y \in X \mid y \sim x\}$  and call it the equivalence class of  $x$  on  $X$ . Let  $X/\sim$  be the set  $\{[x] \mid x \in X\}$

### 14.3 Prop.

Let  $X$  be a set and  $\sim$  be an equivalence relation on  $X$

- (1)  $\forall x \in X, y \in [x]$  on has  $[x] = [y]$
- (2) If  $\alpha$  and  $\beta$  are elements of  $X/\sim$  such that  $\alpha \neq \beta$  then  $\alpha \cap \beta = \emptyset$
- (3)  $X = \bigcup_{\alpha \in X/\sim} \alpha$

**Proof**

- (1) Let  $z \in [y]$ . Then  $y \sim z$ . Since  $y \in [x]$  one has  $x \sim y$ . Therefore,  $x \sim z$  namely  $z \in [x]$ . This proves  $[y] \subseteq [x]$ . Moreover, since  $x \sim y$ , one has  $x \in [y]$ . Hence  $[x] \subseteq [y]$ . Thus we obtain  $[x] = [y]$ .
- (2) Suppose that  $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$ .  
By (1),  $\alpha = [y], \beta = [y]$ . Thus leads to a contradiction.
- (3)  $\forall x \in X \quad x \in [x]$  Hence  $x \in \bigcup_{\alpha \in X/\sim} \alpha$ . Hence  $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$ . Conversely,  
 $\forall \alpha \in X/\sim, \alpha$  is a subset of  $X$ . Hence  $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$ . Then  $X = \bigcup_{\alpha \in X/\sim} \alpha$ .

**14.4 Def**

Let  $G$  be a group and  $X$  be a set.  
We call left/right action of  $G$  on  $X$  an mapping  $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$  that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

**14.5 Remark**

If we denote by  $G^{op}$  the set  $G$  equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of  $G$  on  $X$  is just a left action of  $G^{op}$  on  $X$ .

**14.6 Prop**

Let  $G$  be a group and  $X$  be a set. Assume given a left action of  $G$  on  $X$ . Then the binary relation  $\sim$  on  $X$  defined as  $x \sim y$  iff  $\exists g \in G \quad y = gx$  is an equivalence relation

**14.7 Notation on Equivalence Class**

We denote by  $G/X$  the set  $X/\sim \forall x \in X$  the equivalence class of  $x$  is denoted as  $Gx/xG$  or  $orb_G(x)$  call the orbit of  $x$  under the action of  $G$

## 14.8 Proof

- $\forall x \in X \quad x = 1x$  so  $x \sim x$
- $\forall (x, y) \in X^2$  if  $y = gx$  for same  $g \in G$  then  $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$ . ( $y \sim x$ )
- $\forall (x, y, z) \in X^3$ , if  $\exists (g, h) \in G^2$ , such that  $y = gx$  and then  $z = h(gx) = (hg)x$  So  $x \sim z$

## 14.9 Quotient set

Let  $X$  be a set and  $\sim$  be an equivalence relation, the mapping  $X \rightarrow X/\sim$ :  
 $(x \in X) \mapsto [x]$  is called the projection mapping.

$X/\sim$  is called the quotient set of  $X$  by equivalence relation  $\sim$

### 14.9.1 Example

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of  $H$  on  $G$ . Thus we obtain two quotient sets  $H/G$  and  $G/H$

## 14.10 Def

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . If  $\forall g \in G, h \in H \quad ghg^{-1} \in H$ ,  
 Then we say that  $H$  is a normal subgroup of  $G$

## 14.11 Remark

$\forall g \in G, gH = Hg$ , provided that  $H$  is a normal subgroup of  $G$ . In fact  $\forall h \in$ ,

- $\exists h' \in H$  such that  $ghg^{-1} = h'$  Hence  $gh = h'g$ . This shows  $gH \subseteq Hg$
- $\exists h'' \in H$  such that  $g^{-1}hg = h''$  Hence  $hg = gh''$ . This shows  $Hg \subseteq gH$

Thus  $gH = Hg$

## 14.12 Prop

If  $G$  is commutative, any subgroup of  $G$  is normal

### 14.13 Theorem

Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set  $G/H$ . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

#### Proof

- If  $xH = x'H, yH = y'H$  then  $\exists h_1 \in H, h_2 \in H$  such that  $x' = xh_1, y' = yh_2$ . Hence  $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$ . For  $y^{-1}h_1y, h_2 \in H$  then  $(x'y')H = (xy)H$ . So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

### 14.14 Def

Let  $K$  be a unitary ring and  $E$  be a left  $K$ -module. We say that a subgroup  $F$  of  $(E, +)$  is a left sub- $K$ -module of  $E$  if  $\forall (a, x) \in K \times F, ax \in F$

### 14.15 Prop

Let  $K$  be a unitary ring,  $E$  be a left  $K$ -module and  $F$  be a sub- $K$ -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- $K$ -module structure on  $E/F$ . Moreover, the projection mapping  $\pi : E \rightarrow E/F$  is a morphism of left- $K$ -modules

**Proof**

Let  $x$  and  $x'$  be elements of  $E$  such that  $[x] = [x']$ , that means:  $x' - x \in F$   
Hence  $a(x' - x) = ax' - ax \in F$  So  $[ax] = [ax']$   
Let us check that  $E/F$  forms a left  $K$ -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition,  $\pi$  is a morphism of groups. Moreover  $\forall x \in E, a \in K$   $\pi(ax) = [ax] = a[x] = a\pi(x)$

**14.16 Def**

Let  $A$  be a unitary ring . We call two-sided ideal any subgroup  $I$  of  $(A, +)$  that satisfies :  $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$  ( $I$  is a left and right sub- $K$ -module of  $A$ )

**14.17 Theorem**

Let  $A$  be a unitary ring and  $I$  be a two sided ideal of  $A$  . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover ,  $A/I$  becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely  $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$ )

**Proof**

If  $a' \sim a, b' \sim b$  that means  $a' - a \in I, b' - b \in I$  then  $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$ . For  $(a' - a), (b' - b) \in I$ , then  $a'b' - ab \in I$   
Therefore  $a'b' \sim ab$

### 14.17.1 Reside Class

Let  $d \in \mathbb{Z}$  and  $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$   $d\mathbb{Z}$  is a two sided ideal of  $\mathbb{Z}$   
 If  $m \in \mathbb{Z}$ , for any  $a \in \mathbb{Z}$   $adm = dma \in d\mathbb{Z}$

Denote by  $\mathbb{Z}/d\mathbb{Z}$  the quotient ring. The class of  $n \in \mathbb{Z}$  in  $\mathbb{Z}/d\mathbb{Z}$  is called the residue class of  $n$  modulo  $d$

If  $A$  is a commutative unitary ring, a two sided ideal of  $A$  is simply called an ideal of  $A$

### 14.18 Theorem

Let  $f : G \rightarrow H$  be a morphism of groups

- (1)  $Im(f)$  is a subgroup of  $H$
- (2)  $\ker(f) := \{x \in G \mid f(x) = 1_H\}$  is a normal subgroup of  $G$
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4)  $f$  is injective iff  $\ker(f) = \{1_G\}$

### Proof

- (1) Let  $\alpha$  and  $\beta$  be elements of  $Im(f)$ . Let  $(x, y) \in G^2$  such that  $\alpha = f(x), \beta = f(y)$  Then  $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$  So  $Im(f)$  is a subgroup
- (2) Let  $x$  and  $y$  be elements of  $\ker(f)$ .  
 One has  $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$   
 So  $xy^{-1} \in \ker f$ . Hence  $\ker f$  is a subgroup of  $G$   
 Let  $x \in \ker f, y \in G$ .  
 One has  $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$  Hence  $yxy^{-1} \in \ker f$ . So  $\ker f$  is a normal subgroup
- (3) If  $x \sim y$  then  $\exists z \in \ker f$  such that  $y = xz$  Hence  $f(y) = f(x)f(z) = f(x)1_H = f(x)$  So  $f$  is well defined.  
 Moreover  $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$  Hence  $\tilde{f}$  is a morphism of groups.  
 By definition  $Im(\tilde{f}) = Im(f)$  If  $x$  and  $y$  are elements of  $G$  such that  $f(x) = f(y)$  then  $f(xy^{-1}) = 1_H$   
 Hence  $xy^{-1} \in \ker f$  Since  $x = (xy^{-1})y$ ,  $x \sim y$  that means  $[x] = [y]$   
 Therefore  $\tilde{f}$  is injective.

- (4) If  $f$  is injective,  $\forall x \in \ker f$   $f(x) = 1_H = f(1_G)$ , so  $x = 1_G$ . Therefore  $\ker f = \{1_G\}$   
 Conversely, suppose that  $\ker f = \{1_G\}$   $\forall (x, y) \in G^2$  if  $f(x) = f(y)$  then  $f(x)f(y)^{-1} = 1_H$ . Hence  $xy^{-1} = 1_G, x = y$

### 14.19 Theorem

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Then

- (1)  $\text{Im}(f)$  is a left-sub- $K$ -module of  $F$
- (2)  $\ker(f)$  is a left-sub- $K$ -module of  $E$
- (3)  $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$  is a isomorphism of left  $K$ -modules  
 $[x] \mapsto f(x)$

#### Proof

- (1)  $\forall x \in E, f(ax) = af(x)$  So  $af(x) \in \text{Im}(f)$
- (2)
- (3)





# Chapter 15

## Topology

### 15.1 Def

Let  $X$  be a set. We call topology on  $X$  any subset  $\mathcal{J}$  of  $\wp(X)$  that satisfies:

- $\emptyset \in \mathcal{J}$  and  $X \in \mathcal{J}$
- If  $(u_i)_{i \in I}$  is an arbitrary family of elements in  $\mathcal{J}$ , then  $\bigcup_{i \in I} u_i \in \mathcal{J}$
- If  $u$  and  $v$  are elements of  $\mathcal{J}$ , then  $u \cap v \in \mathcal{J}$

### 15.2 Remark

If  $(u_i)_{i=1}^n$  is a finite family of elements of  $\mathcal{J}$ , then  $\bigcap_{i=1}^n u_i \in \mathcal{J}$  (by induction, this follows from (3))

#### 15.2.1 Example

$\{\emptyset, X\}$  is a topology. call the trivial topology on  $X$  is a topology called the discrete topology.

### 15.3 Def

Let  $X$  be a set. We call metric on  $X$  any mapping  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , that satisfies  
 $(x, y) \mapsto d(x, y)$

- $d(x, y) = 0$  iff  $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called a metric space

### 15.3.1 Example

Let  $X$  be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

## 15.4 Def

Let  $(X, d)$  be a metric space. For any  $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$ , let  $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ . We call the open ball of radius  $\epsilon$  centered at  $x$

### 15.4.1 Example

Consider  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$ , then  $B(x, \epsilon) = ]x - \epsilon, x + \epsilon[$

## 15.5 Prop.

Let  $(X, d)$  be a metric space. Let  $\mathcal{J}_d$  be the set of  $U \subseteq X$  such that  $\forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subseteq U$ . Then  $\mathcal{J}_d$  is a topology on  $X$

### Proof

- $\emptyset \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let  $(u_i)_{i \in I}$  be a family of elements of  $\mathcal{J}_d$ . Let  $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$  such that  $x \in u_i$ . Since  $u_i \in \mathcal{J}_d, \exists \epsilon > 0$  such that  $B(x, \epsilon) \subseteq u_i \subseteq U$ . Hence  $U \in \mathcal{J}_d$
- Let  $U$  and  $V$  be elements of  $\mathcal{J}_d$ . Let  $x \in U \cap V, \exists a, b \in \mathbb{R}_{\geq 0}$  such that  $B(x, a) \subseteq U, B(x, b) \subseteq V$ . Taking  $\epsilon = \min\{a, b\}$ , Then  $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ . Therefore  $U \cap V \in \mathcal{J}_d$

## 15.6 Def

$\mathcal{J}_d$  is called the topology induced by the metric  $d$

## 15.7 Def

We call topology space any pair  $(X, \mathcal{J})$  where  $X$  is a set and  $\mathcal{J}$  is a topology on  $X$

Given a topological space  $(X, \mathcal{J})$  If  $U \in \mathcal{J}$  then we say that  $U$  is an open subset of  $X$ . If  $F \in \wp(X)$  such that  $X \setminus F \in \mathcal{J}$ , then we say that  $F$  is closed subset of  $X$

If there exists  $d$  a metric on  $X$  such that  $\mathcal{J} = \mathcal{J}_d$  then we say that  $\mathcal{J}$  is metrizable

### 15.7.1 Example

Let  $X$  be a set . The discrete topology on  $X$  is metrizable. In fact, if  $d$  denote the metric defined as  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$   
 $\forall x \in X \quad B(x, 1) = \{x\}$  So  $\{x\} \in \mathcal{J}_d$  Hence  $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{J}_d$



# Chapter 16

## Filter

### 16.1 Def

Let  $X$  be a set. We call filter if  $\mathcal{F} \subseteq \wp(X)$  that satisfies:

- (1)  $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2)  $\forall A \in \mathcal{F}, \forall B \in \wp(X), \text{ if } A \subseteq B, \text{ then } B \in \mathcal{F}$
- (3)  $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

#### 16.1.1 Example

- (1) Let  $Y \subseteq X, Y \neq \emptyset$ .  $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$  is a filter, called the principal filter of  $Y$ .
- (2) Let  $X$  be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is infinite}\}$$

is a filter called the Fréchet filter of  $X$ .

- (3) Let  $(X, \mathcal{J})$  be a topological space,  $x \in X$ . We call neighborhood of  $x$  any  $V \in \wp(X)$  such that  $\exists u \in \mathcal{J}$ , satisfying  $x \in U \subseteq V$ . Then  $\mathcal{V} = \{\text{neighborhoods of } x\}$  is a filter.

### 16.2 Def: Filter Basis

Let  $X$  be a set.  $\mathcal{B} \subseteq \wp(X)$ . If  $\emptyset \notin \mathcal{B}$  and  $\forall (B_1, B_2) \in \mathcal{B}^2, \exists B \in \mathcal{B}$ , such that  $B \subseteq B_1 \cap B_2$ . We say that  $\mathcal{B}$  is a filter basis.

#### 16.2.1 Remark

If  $\mathcal{B}$  is a filter basis, then  $\mathcal{F}(\mathcal{B}) = \{A \subseteq X \mid \exists B \in \mathcal{B} \quad B \subseteq A\}$  is a filter

**Proof**

$\emptyset \notin \mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{B}) \neq \emptyset$  since  $0 \neq B \subseteq \mathcal{F}(\mathcal{B})$ . If  $A \in \mathcal{F}(\mathcal{B}), A' \in \wp(X)$  such that  $A \subseteq A'$ , then  $\exists B \in \mathcal{B}$  such that  $B \subseteq A \subseteq A'$ . Hence  $A' \in \mathcal{F}(\mathcal{B})$ . If  $A_1, A_2 \in \mathcal{F}(\mathcal{B})$ , then  $\exists(B_1, B_2) \in \mathcal{B}^2$  such that  $B_1 \subseteq A_1, B_2 \subseteq A_2$ . Since  $\mathcal{B}$  is a filter basis,  $\exists B \in \mathcal{B}$  such that  $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ . Hence  $A_1 \cap A_2 \in \mathcal{F}(\mathcal{B})$ .

**16.2.2 Example**

- Let  $Y \subseteq X, Y \neq \emptyset$   
 $\mathcal{B} = \{Y\}$  is a filter basis.  $\mathcal{F}(\mathcal{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let  $(X, \mathcal{J})$  be a topological space  $x \in X$ . If  $\mathcal{B}_x$  is a filter basis such that  $\mathcal{F}(\mathcal{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$ , then we say that  $\mathcal{B}_x$  is a neighborhood basis of  $x$ .

**16.3 Remark**

Let  $\mathcal{B}_x$  is a neighborhood basis of  $x$  iff

- $\mathcal{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathcal{B}_x$  such that  $U \subseteq V$
- Let  $(X, d)$  be a metric space,  $x \in X \forall \epsilon > 0$ , Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Then

- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$
- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$

**16.3.1 Example**

$\mathcal{V}_x \cap \mathcal{J}$  is a neighborhood basis of  $x$

**16.4 Def**

$V \in \wp(X)$  is called a neighborhood of  $x$  if  $\exists U \in \mathcal{J}$  such that  $x \in U \subseteq V$

## 16.5 Remark

Let  $(X, \mathcal{J})$  be a topological space,  $x \in X$  and  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Suppose that  $\mathcal{B}$  is countable. We choose a surjective mapping  $(B_n)_{n \in \mathbb{N}}$  from  $\mathbb{N}$  to  $\mathcal{B}_x$ . For any  $n \in \mathbb{N}$ , let  $A_n = B_0 \cap B_1 \cap \dots \cap B_n \in \mathcal{V}_x$ . The sequence  $(A_n)_{n \in \mathbb{N}}$  is decreasing and  $\{A_n \mid n \in \mathbb{N}\}$  is a neighborhood basis of  $x$ .

## 16.6 Extra Episode

$\wp(\mathbb{N})$  is NOT countable

Suppose that  $f : \wp(\mathbb{N}) \rightarrow \mathbb{N}$  is injective. Then  $\exists g : \mathbb{N} \rightarrow \wp(\mathbb{N})$  surjective. Taking  $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$ . Since  $g$  is surjective,  $\exists a \in \mathbb{N}$  such that  $A = g(a)$ .

If  $a \in A$ , then  $a \in g(a)$ , hence  $a \notin A$

If  $a \notin A$ , then  $a \in g(a) = A$

Contradiction

## 16.7 Prop.

Let  $Y$  and  $E$  be sets,  $g : Y \rightarrow E$  be a mapping,

- If  $\mathcal{F}$  is a filter of  $Y$ , then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on  $E$

- If  $\mathcal{B}$  is a filter basis of  $Y$ , then

$$g(\mathcal{B}) := \{g(B) \mid B \in \mathcal{B}\}$$

is a filter of  $E$ , and  $\mathcal{F}(g(\mathcal{B})) = g_*(\mathcal{F}(\mathcal{B}))$

### Proof

- (1)  $E \in g_*(\mathcal{F})$  since  $g^{-1}(E) = Y$   
 $\emptyset \notin g_*(\mathcal{F})$  since  $g^{-1}(\emptyset) = \emptyset$

If  $A \in g_*(\mathcal{F})$  and  $A' \supseteq A$ , then  $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{F}$ , so  $g^{-1}(A') \in \mathcal{F}$ ,  
Hence  $A' \in g_*(\mathcal{F})$

If  $A_1, A_2 \in g_*(\mathcal{F})$ . Then  $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ . Hence  $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$ . So  $A_1 \cap A_2 \in g_*(\mathcal{F})$ .

- (2) Since  $g$  is a mapping, and  $\emptyset \notin \mathcal{B}$ , we get  $\emptyset \notin g(\mathcal{B})$ , since  $\mathcal{B} \neq \emptyset, g(\mathcal{B}) \neq \emptyset$ .

Let  $B_1, B_2 \in \mathcal{B}$ , there exists  $C \in \mathcal{B}$  such that  $C \subseteq B_1 \cap B_2$ . Hence  $g(C) \subseteq g(B_1) \cap g(B_2)$ , namely  $g(\mathcal{B})$  is a filter basis.





## Chapter 17

# Limit point and accumulation point

We fix a topological space  $(X, \mathcal{T})$

### 17.1 Def

Let  $\mathcal{F}$  be a filter of  $X$  and  $x \in X$

- If  $\mathcal{V}_x \subseteq \mathcal{F}$  then we say that  $x$  is a limit point of  $\mathcal{F}$
- If  $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \emptyset$ , we say that  $x$  is an accumulation point of  $\mathcal{F}$

So any limit point of  $\mathcal{F}$  is necessarily a accumulation point of  $\mathcal{F}$

### 17.2 Prop

Let  $\mathcal{B}$  be a filter basis of  $X$ ,  $x \in X$ ,  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Then  $x$  is an accumulation point of  $\mathcal{F}(\mathcal{B})$  iff  $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \emptyset$

#### Proof

##### Necessity

Since  $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B})$ ,  $\mathcal{B} \subseteq \mathcal{V}_x$ , the necessity is true.

##### Sufficiency

Let  $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$ . There exist  $B \in \mathcal{B}, U \in \mathcal{B}_x$ , such that  $B \subseteq A, U \subseteq V$ . Hence  $\emptyset \neq B \cap U \subseteq A \cap V$

### 17.3 Def

Let  $Y \subseteq X, Y \neq \emptyset$ . We call accumulation point of  $Y$  any accumulation point of the principal filter  $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$ .

### 17.4 Def

We denote by  $\overline{Y} = \{\text{accumulation points of } Y\}$ , called the closure of  $Y$ . Note that  $x \in \overline{Y}$  iff  $\forall U \in \mathcal{B}_x, Y \cap U \neq \emptyset$

By convention  $\overline{\emptyset} = \emptyset$

### 17.5 Prop

Let  $Y \subseteq X$ . Then  $\overline{Y}$  is the smallest closed subset of  $X$  containing  $Y$ .

#### Proof

$\forall x \in X \setminus \overline{Y}$ , then there exists  $U_x = \mathcal{V} \cap \mathcal{J}$ , such that  $Y \cap U_x = \emptyset$ . Moreover,  $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{J}$ . This shows that  $\forall y \in U_x, y \notin \overline{Y}$ . Therefore  $X \setminus \overline{Y} = \bigcap_{x \in X \setminus \overline{Y}} U_x \in \mathcal{J}$

Let  $Z \subseteq X$  be a closed subset that contain  $Y$ . Suppose that  $\exists y \in \overline{Y} \setminus Z$ . Then  $U = X \setminus Z \in \mathcal{V}_y \cap \mathcal{J}$  and  $U \cap Y \subseteq U \cap Z = \emptyset$ . So  $y \notin \overline{Y}$  contradiction. Hence  $\overline{Y} \subseteq Z$ .

### 17.6 Def: dense

Let  $(X, \mathcal{J})$  be a topological space,  $Y$  a subset of  $X$ . We call  $Y$  is dense in  $X$  if

$$\overline{Y} = X$$

## Chapter 18

# Limit of mappings

### 18.1 Def

Let  $(E, \mathcal{J}_E)$  be a topological space.  $f : Y \rightarrow E$  a mapping, and  $\mathcal{F}$  be a filter of  $Y$ . If  $a \in E$  is a limit point of  $F_*(\mathcal{F})$  namely,  $\forall$  neighborhood  $V$  of  $a$ ,  $f^{-1}(V) \in \mathcal{F}$ , then we say that  $a$  is a limit of the filter  $\mathcal{F}$  by  $f$

### 18.2 Remark

Let  $\mathcal{B}_a$  be a neighborhood basis of  $a$ . Then  $\mathcal{V}_a \subseteq f_*(\mathcal{F})$ , iff  $\mathcal{B} \subseteq f_*(\mathcal{F})$ . Therefore,  $a$  is a limit of  $\mathcal{F}$  by  $f$  iff  $\forall V \in \mathcal{B}_a, f^{-1}(V) \in \mathcal{F}$

#### 18.2.1 Example

Let  $(E, \mathcal{J}_E)$  be a topological space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $x = (x_n)_{n \in I} \in E^I$ . If the Fréchet filter  $\mathcal{F}_{Fr}(I)$  has a limit  $a \in E$  by the mapping  $x : I \rightarrow E$ , we say that  $(x_n)_{n \in I}$  converges to  $a$ , denote as

$$a = \lim_{n \in I, n \rightarrow +\infty} x_n$$

### 18.3 Remark

$a = \lim_{n \in I, n \rightarrow +\infty} x_n$  iff,  $\forall U \in \mathcal{B}_a$  (where  $\mathcal{B}_a$  is a neighborhood basis of  $a$ ),  $\exists N \in \mathbb{N}$  such that  $x_n \in U$  for any  $n \in I_{\geq N}$

Suppose that  $\mathcal{J}_E$  is induced by a metric  $d$ .  $\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}, \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}, \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  are all neighborhood basis of  $a$ . Therefore, the following are equivalent

- $a = \lim_{n \in I, n \rightarrow +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$
- $(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\})$ ? unknown position )

## 18.4 Remark

We consider the metric  $d$  on  $\mathbb{R}$  defined as

$$\forall (x, y) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of  $\mathbb{R}$  defined by this metric is called the usual topology on  $\mathbb{R}$

## 18.5 Prop

Let  $(x_n)_{n \in I} \in \mathbb{R}^I$ , where  $I \subseteq \mathbb{N}$  is an infinite subset. Let  $l \in \mathbb{R}$ . The following statements are equivalent:

- The sequence  $(x_n)_{n \in I}$  converges to  $l$  in the topological space  $\mathbb{R}$
- $\liminf_{n \in I, n \rightarrow +\infty} x_n = \limsup_{n \in I, n \rightarrow +\infty} x_n = l$
- $\limsup_{n \in I, n \rightarrow +\infty} |x_n - l| = 0$

## 18.6 Theorem

Let  $(X, d)$  be a metric space. Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be an element of  $X^I$ . Let  $l \in X$ . The following statements are equivalent:

- $(x_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$  (equivalent to  $\lim_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ )

### Proof

- (1)  $\Rightarrow$  (2) The condition (1) is equivalent to  $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$ .  
 We then get  $\sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ . Therefore  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) < \epsilon$ . We obtain that  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ .
- (2)  $\Rightarrow$  (1) Let  $\epsilon \in \mathbb{R}_{>0}$ . If  $\inf_{N \in \mathbb{N}} \sup_{n \in I_{\geq N}} d(x_n, l) = 0$ . Then  $\exists N \in \mathbb{N} \quad \sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ .  
 Hence,  $\forall n \in I_{\geq N} d(x_n, l) < \epsilon$ . Since  $\epsilon$  is arbitrary, (\*) is true, Hence (1) is also true.

## 18.7 Prop

Let  $(X, \mathcal{J})$  be a topological space .  $Y \subseteq X, p \in \overline{Y} \setminus Y$ . Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of  $Y$ .

### Proof

$Y$  is not empty otherwise  $\overline{Y} = \emptyset$ .

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$   
 $\emptyset \notin \mathcal{V}_{p,Y}$  since  $p \in \overline{Y}$
- Let  $V \in \mathcal{V}_p$  and  $A \subseteq Y$  such that  $V \cap Y \subseteq A$ . Let  $U = V \cup (A \setminus (V \cap Y)) \in \mathcal{V}_p$   
and  $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let  $U$  and  $V$  be elements of  $\mathcal{V}_p$  Let  $W = U \cap V \in \mathcal{V}_p$  Then  $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

## 18.8 Def

Let  $(X, \mathcal{J}_x)$  and  $(E, \mathcal{J}_E)$  be topological spaces,  $Y \subseteq X, p \in \overline{Y} \setminus Y$ , and  $f : Y \rightarrow E$  be a mapping . If  $a$  is a limit point of  $(F_*(\mathcal{V}_{p,Y}))$ , then we say that  $a$  is a limit of  $f$  when the variable  $y \in Y$  tends to  $p$ , denoted as  $a = \lim_{y \in Y, y \rightarrow p} f(y)$

## 18.9 Remark

If  $\mathcal{B}_a$  is a neighborhood basis of  $a$ . Then  $a = \lim_{y \in Y, y \rightarrow p} f(y)$  is equivalent to  
 $\forall U \in \mathcal{B}_a \quad \exists V \in \mathcal{V}_p$  such that  $Y \cap V \subseteq f^{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

## 18.10 Prop

Let  $X$  be a set,  $\mathcal{B}$  be a filter basis,  $\mathcal{G}$  be a filter. If  $\mathcal{B} \subseteq \mathcal{G}$ , then  $\mathcal{F} \subseteq \mathcal{G}$ .

### Proof

Let  $V \in \mathcal{F}(\mathcal{B})$  By definition  $\exists U \in \mathcal{B}$  such that  $U \subseteq V$ , since  $U \in \mathcal{G}$  ( for  $\mathcal{B} \subseteq \mathcal{G}$ ) and since  $\mathcal{G}$  is a filter,  $V \in \mathcal{G}$

### 18.11 Theorem

Let  $(X, \mathcal{J}_x)$  and  $(E, \mathcal{J}_E)$  be topological spaces.  $Y \subseteq X$ ,  $p \in \overline{T} \setminus Y$ ,  $a \in E$ . We consider the following conditions.

- (i)  $a = \lim_{y \in Y, y \rightarrow p} f(y)$
- (ii)  $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  if  $\lim_{n \rightarrow +\infty} y_n = p$  then  $\lim_{n \rightarrow \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that  $p$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

#### Proof

- (1) Let  $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  such that  $p = \lim_{n \rightarrow +\infty} y_n$ . For any  $U \in \mathcal{V}_p$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}$   $y_n \in U \cap Y$ . Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \rightarrow +\infty} f(y_n) = a$$

- (2) Assume that  $p$  has a countable neighborhood basis. There exists a decreasing sequence  $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_p^{\mathbb{N}}$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $p$ .

Assume that (i) does not hold. Then there exists  $U \in \mathcal{V}_a$  such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \rightarrow +\infty} y_n = \emptyset$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However  $f(y_n) \notin U$  for any  $n \in \mathbb{N}$ , so  $(f(y_n))_{n \in \mathbb{N}}$  cannot converges to  $a$ .

## 18.12 Prop.

Let  $X$  be a set. If  $(\mathcal{J}_i)_{i \in I}$  is a family of topologies on  $X$ , then  $\mathcal{J} = \bigcap_{i \in I} \mathcal{J}_i$  is a topology. In particular, for any  $\mathcal{A} \subseteq \wp(X)$ , there is a smallest topology on  $X$  that contains  $\mathcal{A}$ .

### 18.12.1 Proof

- $\forall i \in I \quad \{\emptyset, X\} \subseteq \mathcal{J}_i$  So  $\{\emptyset, X\} \subseteq \mathcal{J}$
- Let  $(u_j)_{j \in J}$  be a family of elements of  $\mathcal{J} \quad \forall j \in J, i \in I \quad u_j \in \mathcal{J}_i$  So  $\bigcup_{j \in J} u_j \in \mathcal{J}_i$  We then get  $\bigcup_{j \in J} u_j \in \mathcal{J}$
- Let  $U$  and  $V$  be elements of  $\mathcal{J} \quad \forall i \in I, \{u, v\} \subseteq \mathcal{J}_i$  So  $U \cap V \in \mathcal{J}_i$ . Therefore we get  $U \cap V \in \mathcal{J}$  Let  $\mathcal{A} \subseteq \wp(X)$  Let  $\mathcal{J}(\mathcal{A}) = \bigcap_{\substack{\mathcal{J} \subseteq \wp(X) \text{ a topology} \\ \mathcal{A} \subseteq \mathcal{J}}} \mathcal{J}$  Then  $\mathcal{J}(\mathcal{A})$  is a topology. By definition, if  $\mathcal{J}$  is a topology containing  $\mathcal{A}$ , then  $\mathcal{J}(\mathcal{A}) \subseteq \mathcal{J}$  Hence  $\mathcal{J}(\mathcal{A})$  is the smallest topology containing  $\mathcal{A}$ .





## Chapter 19

# Continuity

### 19.1 Def

Let  $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$  be topological spaces  $f$  be a function from  $X$  to  $Y$ ,  $x \in \text{Dom}(f)$ . If for any neighborhood  $U$  of  $f(x)$ , there exists a neighborhood  $V$  of  $x$  such that  $f(V) \subseteq U$ . Then we say that  $f$  is continuous at  $x$ . If  $f$  is continuous at any  $x \in \text{Dom}(f)$  then we say  $f$  is continuous.

### 19.2 Remark

Let  $\mathcal{B}_{f(x)}$  be a neighborhood basis of  $f(x)$  If  $\forall U \in \mathcal{B}_{f(x)}$  there exist  $V \in \mathcal{B}_{f(x)}$  such that  $f(V) \subseteq U$ , then  $f$  is continuous at  $x$  Suppose that  $X$  and  $Y$  are metric space. Then  $f$  is continuous at  $x$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in \text{Dom}(f) \quad d(y, x) < \delta \text{ implies } d(f(y), f(x)) < \epsilon$$

### 19.3 Theorem

Let  $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$  be topological spaces,  $f$  be a function from  $X$  to  $Y$   $x \in \text{Dom}(f)$  Consider the following condition

- $f$  is continuous at  $x$
- $\forall (x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ , if  $\lim_{n \rightarrow +\infty} x_n = x$ , then  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$  THEN  
(i) implies (ii) Moreover, if  $x$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

### 19.4 Proof

(i)  $\Rightarrow$  (ii) Let  $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$  that converges to  $x$   $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$  Since  $\lim_{n \rightarrow +\infty} x_n = x$ , there exists  $N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}, x_n \in V$ .

Hence  $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$ . Thus  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$

(ii)  $\Rightarrow$  (i) under the hypothesis that  $x$  has countable neighborhood basis. actually we will prove  $NOT(i) \Rightarrow NOT(ii)$

Let  $(V_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $\mathcal{V}_x$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $x$

If (i) does not hold, then  $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$  Pick  $x_n \in V_n$  such that  $f(x_n) \notin U \quad \forall n \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$ . Hence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ . However,  $f(x_n) \notin U$  for any  $n$  So  $(f(x_n))_{n \in \mathbb{N}}$  does not converges to  $f(x)$ . Therefore (ii) does not hold.

## 19.5 Prop

Let  $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$  be topological spaces.  $f$  be a function from  $X$  to  $Y$ ,  $g$  be a function from  $Y$  to  $Z$ . Let  $x \in \text{Dom}(g \circ f)$  If  $f$  and  $g$  are continuous at  $x$ . then  $g \circ f$  is continuous at  $x$  sectionProof Let  $U \in \mathcal{V}_{g(f(x))}$  Since  $g$  is continuous at  $f(x)$ :

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since  $f$  is continuous at  $x$ :

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore,  $g(f(V)) \subseteq g(W) \subseteq U$  Hence  $g \circ f$  is continuous at  $x$

## 19.6 Def

Let  $(X, \mathcal{J})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{J}$ , If any element of  $\mathcal{J}$  can be written as the union of a family of sets in  $\mathcal{B}$  we say that  $\mathcal{B}$  is a topological basis of  $\mathcal{J}$

## 19.7 Prop

Let  $(X, \mathcal{J})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{J}$   $\mathcal{B}$  is a topological basis iff

$$\forall x \in X, \mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$$

is a neighborhood basis of  $x$

## 19.8 Proof

$\Rightarrow$ :

$$\forall x \in X \mathcal{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since  $\mathcal{B}$  is a topological basis of  $\mathcal{J}$ ,

$$\exists W \in \mathcal{B}, x \in W \subseteq V \subseteq U$$

Hence  $\mathcal{V}_x$  is generated by  $\mathcal{B}_x$

$\Leftarrow$  Let  $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathcal{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

## 19.9 Prop

Let  $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$  be topological spaces.  $\mathcal{B}_Y$  be a topological basis of  $\mathcal{J}_Y$   
 $f : X \rightarrow Y$  be a mapping. The following conditions are equivalent:

- (1)  $f$  is continuous
- (2)  $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$
- (3)  $\forall U \in \mathcal{B}_Y, f^{-1}(U) \in \mathcal{J}_X$

### Proof

(1)  $\Rightarrow$  (2)

Lemma Let  $(X, \mathcal{J})$  be a topological space,  $V \in \wp(X)$ , Then  $V \in \mathcal{J}$  iff  
 $\forall x \in V, V$  is a neighborhood of  $x$

Proof of lemma  $\Rightarrow$  is by definition

*Leftarrow:*

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W_x - x \in \mathcal{J}$$

Let  $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood  $W$  of  $x$  such that  $f(W) \subseteq U$   
 Since  $f$  is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since  $x$  is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2)  $\Rightarrow$  (3) For (3) is a special situation of (2), it's natural.

(3)  $\Rightarrow$  (1) Let  $x \in X$

$$\forall U \in \mathcal{B}_Y \text{ s.t. } f(x) \in U, f^{-1}(U)$$

is an open neighborhood of  $x$ , and

$$f(f^{-1}(U)) \subseteq U$$

Hence  $f$  is continuous at  $x$

### 19.10 Def

Let  $X$  be a set ,  $((Y_i, \mathcal{J}_i))_{i \in I}$  be a family of topological spaces.  $\forall i \in I$  let  $f_i : X \rightarrow Y_i$  be a mapping. We call initial topology of  $(f_i)_{i \in I}$  on  $X$  the smallest topology on  $X$  making all  $f_i$  continue

### 19.11 Remark

If  $\mathcal{J}$  is the initial topology of  $(f_i)_{i \in I}$ ,  $\forall i \in I, U_i \in \mathcal{J}_i$   $f_i^{-1}(U_i) \in \mathcal{J}$  If  $J \subseteq I$  is a finite subset,  $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j$  then  $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{J}$

### 19.12 Prop

$$\mathcal{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ is finite } (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology  $\mathcal{J}$

**Proof**

First

$$\mathcal{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{\text{subset } V \text{ of } X \text{ that can be written as the union of a family of sets in } \mathcal{B}\}$

- $\emptyset \in \mathcal{J}' \quad X \in \mathcal{B} \subseteq \mathcal{J}'$
- $\mathcal{J}'$  is stable by taking the union of any family of elements in  $\mathcal{J}'$
- If  $V_1, V_2$  are elements of  $\mathcal{J}'$ , then

$$V_1 \cap V_2 \in \mathcal{J}'$$

In fact,  $V_1, V_2$  are of the form of the union of some sets of  $\mathcal{B}$

The intersection of two elements of  $\mathcal{B}$  is still a element of  $\mathcal{B}$

$$\begin{aligned} & \left( \bigcap_{j \in J} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J'} f_j^{-1}(U'_j) \right) \\ &= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \setminus J' \\ U'_j & j \in J' \setminus J \\ U_j \cap U'_j & j \in J \cap J' \end{cases} \\ & \left( \bigcap_{j \in J \setminus J'} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U'_j) \right) \cap \left( \bigcap_{j \in J' \setminus J} f_j^{-1}(U'_j) \right) \end{aligned}$$

So  $\mathcal{J}'$  is a topology making all  $f_i$  continuous. Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

**Example**

Let  $((Y_i, \mathcal{J}_i))_{i \in I}$  be topological spaces.  $Y = \prod_{i \in I} Y_i$  and

$$\begin{aligned} \pi_i : Y &\rightarrow Y_i \\ (y_j)_{j \in I} &\mapsto y_i \end{aligned}$$

The product topology on  $Y$  is by definition the initial topology of  $(\pi_i)_{i \in I}$

### 19.13 Theorem

Let  $X$  be a set,  $((Y_i, \mathcal{J}_i))_{i \in I}$  be a family of topological spaces,

$$((f_i : X \rightarrow Y_i))_{i \in I}$$

be a family of mappings and we equip  $X$  with the initial topology  $\mathcal{J}_X$  of  $(f_i)_{i \in I}$ .  
Let  $(Z, \mathcal{J}_Z)$  be a topological space and

$$h : Z \rightarrow X$$

be a mapping. Then  $h$  is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{ is continuous}$$

#### 19.13.1 Proof

$\Rightarrow$  If  $h$  is continuous, since each  $f_i$  is continuous,  $f_i \circ h$  is also continuous.

$\Leftarrow$  Suppose that  $\forall i \in I, f_i \circ h$  is continuous. Hence

$$\forall U_i \in \mathcal{J}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Let

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

$\forall U \in \mathcal{B}$

$$h^{-1}(U) = \bigcap_{j \in J} h^{-1}(f_j^{-1}(U_j)) \in \mathcal{J}_Z$$

Therefore,  $h$  is continuous.

### 19.14 Remark

We keep the notation of the definition of initial topology. If  $\forall i \in I, \mathcal{B}_i$  is a topological basis of  $\mathcal{J}_i$ , then

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{B}_j \right\}$$

is also a topological basis of the initial topology,

**19.14.1 Example**

Let  $((X_i, d_i))_{i \in \{1, \dots, n\}}$  be a family of metric spaces.

$$X = \prod_{i \in \{1, \dots, n\}} X_i$$

We define a mapping

$$d: (x, y) \in X \times X \rightarrow \mathbb{R}_{\geq 0}$$

$$d: ((x_i)_{i \in \{1, \dots, n\}}, (y_i)_{i \in \{1, \dots, n\}}) \mapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i)$$

$d$  is a metric on  $X$ . If  $x = (x_i)_{i \in \{1, \dots, n\}}$ ,  $y = (y_i)_{i \in \{1, \dots, n\}}$ ,  $z = (z_i)_{i \in \{1, \dots, n\}}$  are elements of  $X$ , then

$$d(x, z) = \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \leq \max_{i \in \{1, \dots, n\}} (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i: X \rightarrow X_i$$

$$(x_i)_{i \in \{1, \dots, n\}} \mapsto x_i$$

is continuous. Hence the product topology  $\mathcal{J}$  is contained in  $\mathcal{J}_d$

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in X$ ,  $\epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i \in \{1, \dots, n\}} \mid \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i \in \{1, \dots, n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i \in \{1, \dots, n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$





## Chapter 20

# Uniform continuity and convergency

### 20.1 Def

Let  $(X, d)$  be a metric space.  $\forall A \subseteq X$ , we define

$$\text{diam}(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A. By convention

$$\text{diam}(\emptyset) := 0$$

If  $\text{diam}(A) < +\infty$ , we say that A is bounded

### 20.2 Remark

- If A is finite, then it's bounded
- If  $A \subseteq B$  then  $\text{diam}(A) \leq \text{diam}(B)$

### 20.3 Prop

Let  $(X, d)$  be a metric space.  $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$ . Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

In particular, if A, B are bounded, then  $A \cup B$  is bounded.

**Proof**

Let  $(x, y) \in (A \cup B)^2$ . If  $\{x, y\} \subseteq A$ , then  $d(x, y) \leq \text{diam}(A)$   
 If  $\{x, y\} \subseteq B$  then  $\text{diam}(B) \geq d(x, y)$   
 If  $x \in A, y \in B$ ,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

Similarly if  $x \in B, y \in A$

$$d(x, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

**20.4 Def**

Let  $(X, d)$  be a metric space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $(x_n)_{n \in I} \in X^I$ . If

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \quad \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \epsilon$$

then we say that  $(x_n)_{n \in I}$  is a Cauchy sequence.

**20.5 Prop**

- (1) If  $(x_n)_{n \in I}$  converges, then it's a Cauchy sequence.
- (2) If  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\{x_n \mid n \in I\}$  is bounded
- (3) Suppose that  $(x_n)_{n \in I}$  is a Cauchy sequence. If there exists an infinite subset  $J$  of  $I$  such that  $(x_n)_{n \in J}$  converges to some  $x \in X$ , then  $(x_n)_{n \in I}$  converges to  $x$

**20.5.1 Proof**

- (1) trivial
- (2) trivial
- (3) Let  $\epsilon > 0, \exists N \in \mathbb{N}$

$$\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take  $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence  $(x_n)_{n \in I}$  converges to  $x$

## 20.6 Def

Let  $(X, d_X), (Y, d_Y)$  be metric space.  $f$  be a function from  $X$  to  $Y$ . If  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

we say that  $f$  is uniformly continuous.

## 20.7 Prop

Let  $(X, d_X), (Y, d_Y)$  be metric spaces  $f$  be a function from  $X$  to  $Y$  which is uniformly continuous.

- (1) If  $I \subseteq \mathbb{N}$  is finite, and  $(x_n)_{n \in I}$  is a Cauchy sequence in  $\text{Dom}(f)^I$  then  $(f(x_n))_{n \in I}$  is Cauchy sequence
- (2)  $f$  is continuous

### 20.7.1 Proof

- (1)  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists N \in \mathbb{N}$  such that

$$\forall (n, m) \in I_{\geq N}^2, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

Therefore  $(f(x_n))_{n \in I}$  is a Cauchy sequence.

- (2) Let  $(x_n)_{n \in I}$  be a sequence in  $\text{Dom}(f)^{\mathbb{N}}$  that converges to  $x \in \text{Dom}(f)$  We define  $(y_n)_{n \in \mathbb{N}}$  as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then  $(y_n)_{n \in \mathbb{N}}$  converges to  $x$ . Hence  $(y_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Since  $f$  is uniformly continuous,  $(f(y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $Y$ .

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to  $f(x)$ . Hence  $(f(y_n))_{n \in \mathbb{N}}$  converges to  $f(x)$

## 20.8 Def

Let  $X$  be a set,  $Z \subseteq X$ ,  $(Y, d)$  be a metric space,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as their common domain of definition.

- If  $\forall x \in Z, (f_n(x))_{n \in I}$  converges to  $f(x)$ , we say that  $(f_n)_{n \in I}$  converges pointwisely to  $f$
- If

$$\lim_{n \in I, n \rightarrow +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$

we say that  $(f_n)_{n \in I}$  converges uniformly to  $f$

## 20.9 Theorem

Let  $X$  and  $Y$  be metric space,  $Z \subseteq X$ ,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$ ,  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is uniformly continuous

Then  $f$  is uniformly continuous.

### 20.9.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\lim_{n \in I, n \rightarrow +\infty} A_n = 0$$

$\forall (x, y) \in Z^2, n \in I$

$$\begin{aligned} & d(f(x), f(y)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & \leq 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f_n(x), f_n(y)) = 0$$

Hence

$$0 \leq \inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$

Take  $\lim_{n \rightarrow +\infty}$ , by squeeze theorem, we get

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

## 20.10 Theorem

Let  $X$  be a topological space,  $Y$  be a metric space,  $Z \subseteq X, p \in Z, I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  function from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that:

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is continuous at  $p$

Then  $f$  is continuous at  $p$

### 20.10.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since  $f_n$  is continuous  $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{B}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x), f(p)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$$f(U) \subseteq \overline{B}(f(p), \epsilon)$$

### 20.10.2 Def

Let  $X, Y$  be metric spaces,  $f$  be a function from  $X$  to  $Y$ ,  $\epsilon > 0$ . If

$$\forall (x, y) \in \text{Dom}(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that  $f$  is  $\epsilon$ -Lipschitzian

If  $\exists \epsilon > 0$  such that  $f$  is  $\epsilon$ -Lipschitzian, then it's uniformly continuous.

## 20.11 Remark

If  $f$  is Lipschitzian, then it's uniformly continuous.

## 20.12 Example

- Let  $((X_i, d_i))_{i \in I}$  be metric space.  $X = \prod_{i \in I} X_i$  where  $I$  is finite

$$\begin{aligned} X \times X &\rightarrow \mathbb{R}_{\geq 0} \\ d : d((x_i), (y_i)_{i \in I}) &= \max_{i \in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \rightarrow X_i$$

is Lipschitzian. ( $\forall x = (x_i)_{i \in I}, \forall y = (y_i)_{i \in I}$ )

- Let  $(X, d)$  be a metric space

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

Part V

**Normed Vector Space**





## Chapter 21

# Linear Algebra

We fix a unitary ring  $K$

### 21.1 Def

Let  $M$  be a left  $K$ -module, and let  $x = (x_i)_{i \in I}$  be a family of elements of  $M$ . We define a morphism of left  $K$ -module as following:

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \quad (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

#### 21.1.1 Notation

$$\begin{aligned} K^{\oplus I} &:= \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{ finite, such that } a_i = 0 \text{ for } i \in I \setminus J\} \\ \varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) &= \varphi_x((a_i)_{i \in I}) + \varphi_x((b_i)_{i \in I}) \end{aligned}$$

### 21.2 Def

Let  $M$  be a left  $K$ -module,  $I$  be a set,  $x = (x_i)_{i \in I} \in M^I$ . If

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say  $(x_i)_{i \in I}$  is  $K$ -linearly independent

surjective then we say  $(x_i)_{i \in I}$  is system of generator

a bijection then we say  $(x_i)_{i \in I}$  is a basis of  $M$

**Example**

Let  $e_i$  be the element  $(\delta_{ij})_{j \in I}$  with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i \in I} \in (K^{\oplus I})^I$$

is a basis of  $K^{\oplus I}$

**21.3 Def**

Let  $M$  be a left  $K$ -module

- If  $M$  has a basis, we say that  $M$  is a free  $K$ -module
- If  $M$  has finite system of generated  
( $\exists$  a finite set  $I$  and a family  $(x_i)_{i \in I} \in M^I$  that forms a system of generator),  
then we say that  $M$  is of finite type.

**21.4 Remark**

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in M^n$ , where  $n \in \mathbb{N}$

- $x$  is linearly independent iff

$$\forall a \in K^n \quad \sum a_i x_i = 0$$

implies

$$a = 0$$

- $x$  is a system of generator iff for any element of  $M$  can be written in the form

$$\sum b_i x_i \quad b \in K^n$$

Such expression is called a  $K$ -linear combination of  $x_1, \dots, x_n$

**21.5 Theorem**

Let  $K$  be a division ring ( $0 \neq 1$  and  $\forall k \in K \setminus \{0\}$   $k$  is invertible)

Let  $V$  be a left  $K$ -module of finite type and  $(x_i)_{i \in I}$  be a system of generators of  $V$ . Then, there exists a subset  $I$  of  $\{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ . (In particular,  $V$  is a free  $K$ -module)

**Proof**

(By induction on  $n$ )

If  $n = 0$ , then  $V = \{0\}$

In this case  $\emptyset$  is a basis of  $V$

**Induction hypothesis**

True for a system of generators of  $n - 1$  elements. Let  $(x_i)_{i \in \{1, \dots, n\}}$  be a system of generators of  $V$ . If  $(x_i)_{i \in \{1, \dots, n\}}$  is linearly independent, it's a basis. Otherwise,  $\exists (a_i)_{i \in I} \in K^n$  such that

$$(a_i, \dots, a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose  $a_n \neq 0$ . Then

$$x_n = -a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right)$$

Since  $(x_i)_{i \in \{1, \dots, n\}}$  is a system of generators, any elements of  $V$  can be written as

$$\begin{aligned} \sum b_i x_i &= \left( \sum_{i=1}^{n-1} b_i x_i \right) - b_n a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right) \\ &= \sum_{i=1}^{n-1} (b_i - b_n a_n^{-1} a_i) x_i \end{aligned}$$

Thus  $(x_i)_{i \in \{1, \dots, n\}}$  forms a system of generators. By the induction hypothesis, there exists  $I \subseteq \{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ .

**21.6 Theorem**

Let  $K$  be a unitary ring and  $B$  be a left  $K$ -module.  $W$  be a left  $K$ -submodule of  $V$ . Let  $(x_i)_{i=1}^n$  be an element of  $W^n$

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where  $(n, l) \in \mathbb{N}^2 \forall j \in \{1, \dots, l\}$ , let  $x_{n+j}$  be an element in the equivalence class  $\alpha_j$

- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are linearly independent, then  $(x_i)_{i=1}^{n+l}$  is also linearly independent
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are system of generators of  $W$  and  $V/W$  respectively, then  $(x_i)_{i=1}^{n+l}$  is also a system of generators
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are basis, then  $(x_i)_{i=1}^{n+l}$  is also a basis

**Proof**

(1) Suppose that  $(b_i)_{i=1}^{n+l}$  such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \rightarrow V/W$$

be the projection morphism ( $\pi(x) = [x]$ )

$$0 = \pi\left(\sum_{i=1}^{n+l} b_i x_i\right) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^l b_{n+j} \pi(x_{n+j}) = \sum_{j=1}^l b_{n+j} \alpha_j$$

$$\{x_1, \dots, x_n\} \subseteq W \text{ So } \forall i \in \{1, \dots, n\}$$

$$\pi(x_i) = 0$$

Since  $(\alpha_j)_{j=1}^l$  is linearly independent,

$$b_{n+1} = \dots = b_{n+l} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since  $(x_i)_{i=1}^n$  is linearly independent,

$$b_1 = \dots = b_n = 0$$

(2) Let  $y \in V$ . Then  $\pi(y) \in V/W$ . So there exists

$$(c_{n+1}, \dots, c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^l c_{n+j} \alpha_j \\ &= \sum_{j=1}^l c_{n+j} \pi(x_{n+j}) = \pi\left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \end{aligned}$$

So

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \in W$$

$\exists c \in K^n$  such that

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) = \left(\sum_{i=1}^n c_i x_i\right)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

## 21.7 Corollary

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. If  $(x_i)_{i=1}^n$  is a linearly independent family of elements of  $V$  ( $n \in \mathbb{N}$ ), then

$$\exists l \in \mathbb{N} \quad \exists (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of  $V$

### Proof

Let  $W$  be the image of

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow V \\ (a_i)_{i=1}^n &\mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left  $K$ -submodule of  $V$ .

Note that  $(x_i)_{i=1}^n$  forms a basis of  $W$ .

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow W \\ \varphi(x_i)_{i=1}^n(e_j) &= x_j \in W \end{aligned}$$

Moreover, since  $V$  is of finite type there exists  $d \in \mathbb{N}$  and a surjective morphism of left  $K$ -modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \rightarrow V/W$$

is surjective.

Hence the composite morphism

$$K^d \begin{array}{c} \xrightarrow{\psi} \\ \xrightarrow{\pi \circ \psi} \end{array} V \xrightarrow{\pi} V/W$$

is surjective. Thus  $V/W$  is of finite type. There exist then a basis

$$(a_j)_{j=1}^l$$

of  $V/W$ .

Taking  $x_{n+j} \in \alpha_j$  for  $j \in \{1, \dots, l\}$ , we get a basis of  $V$ :

$$(x_i)_{i=1}^{n+l}$$

## 21.8 Def

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. We call rank of  $V$  the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If  $K$  is a field  $rk(V)$  is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of  $V$ .

## 21.9 Theorem

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. Let  $W$  be a left  $K$ -submodule of  $V$ .

(1)  $W$  and  $V/W$  are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of  $V$  has exactly  $rk(V)$  elements

## 21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

In  $(\pi(x_i))_{i=1}^n$  we extract a basis of  $V/W$ , say

$$(\pi(x_i))_{i=1}^l$$

For  $j \in \{l+1, \dots, n\}$ ,

$$\exists(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any  $x \in W, \exists(a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^n a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^l a_i x_i + \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \\ &= \sum_{j=l+1}^n a_j y_j + \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^n a_j y_j$$

Hence  $W$  is of finite type, and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of  $V/W$ .

We extract a subfamily, say  $(x_i)_{i=1}^l$  such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of  $V/W$ .

For  $j \in \{1, \dots, l\}$ , there exists:

$$(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Let  $x \in W, \exists (a_i)_{i=1}^n \in K^n$  let  $x = \sum a_i x_i$ , then

$$\begin{aligned} x &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \right) \\ &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i \right) + \left( \sum_{j=l+1}^n a_j y_j \right) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i)$$

Therefore  $(y_j)_{j=l+1}^n$  is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + rk(V/W)$$



(2) All basis of  $V$  have  $rk(V)$  elements.

We reason by induction on  $rk(V)$

(1)

$$rk(V) = 0$$

In this case  $V = \{0\}$  The only basis of  $V$  is  $\emptyset$ . So the statement holds.

(2) Assume that there exists  $e \in V \setminus \{0\}$  such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of  $V$  is of the form

$$ae$$

where  $a \in K \setminus \{0\}$

Let  $(e_i)_{i=1}^m$  be a basis of  $V$ . We reason by induction on  $m$  to prove that

$$m = rk(V)$$

The cases where  $m = 0$  or  $1$  are proved in (1)(2) respectively. Induction hypothesis: true for a basis of  $< m$  elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of  $V/W$ .

If  $(a_i)_{i=2}^m \in K^{m-1}$  such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for  $(e_i)_{i=1}^m$  a basis of  $V$ ,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of  $V/W$ . We then obtain that

$$rk(V/W) \leq m - 1 \leq n - 1$$

By the induction hypothesis,

$$m - 1 = rk(V/W)$$

By (2),  $rk(W) = 1$ . Hence

$$m = (m - 1) + 1 = rk(V/W) + rk(W) = rk(V)$$

## 21.11 Prop

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Let  $I$  be a set and  $(x_i)_{i \in I} \in E^I$

- If  $(x_i)_{i \in I}$  is linearly independent and  $f$  is injective, then  $(f(x_i))_{i \in I}$  is linearly independent.
- If  $(x_i)_{i \in I}$  is a system of generators and  $f$  is surjective, then  $(f(x_i))_{i \in I}$  is a system of generators.
- If  $(x_i)_{i \in I}$  is a basis and  $f$  is an isomorphism, then  $(f(x_i))_{i \in I}$  is a basis.

### 21.11.1 Proof

$$\varphi_{(f(x_i))_{i \in I}} = f \circ \varphi_{(x_i)_{i \in I}}$$

## Chapter 22

# Matrices

We fix unitary ring  $K$

### 22.1 Def

Let  $n \in \mathbb{N}$  and  $V$  be a left  $K$ -module.

For any  $(x_i)_{i=1}^n \in V^n$ , we denote by  $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$  the morphism

$$\begin{aligned} & \phi_{(x_i)_{i=1}^n} : K^n \rightarrow V \\ (a_i)_{i=1}^n & \mapsto \sum_{i=1}^n a_i n_i \end{aligned}$$

### 22.1.1 Example

Suppose that  $V = K^p$  ( $p \in \mathbb{N}$ ) Then each  $x_i \in K^p$  is of the form  $(x_{i,1}, \dots, x_{i,p})$

Hence  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  can be written:

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,p} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,p} \end{pmatrix}$$

## 22.2 Def

Let  $(n, p) \in \mathbb{N}^2$ . We call  $n$  by  $p$  matrix of coefficient in  $K$  any morphism of left  $K$ -modules from  $K^n$  to  $K^p$

### 22.2.1 Example

- Denote by  $I_n$  then identity mapping. Then  $(e_i)_{i=1}^n$  is a basis of  $K^n$  called the canonical basis of  $K^n$

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, \dots, a_n)) = \sum_{i=1}^n a_i e_i = (a_1, \dots, a_n)$$

- Let  $(x_1, \dots, x_n) \in K^n$ , Denote by

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) (= \varphi_{(x_i e_i)_{i=1}^n}) : K^n &\rightarrow K^n \\ (a_1, \dots, a_n) &\mapsto (a_1 x_1, \dots, a_n x_n) \end{aligned}$$

## 22.3 Def

We denote by  $M_{n,p}(K)$  the set of all  $n$  by  $p$  matrices of coefficients in  $K$ . For  $(n, p, r) \in \mathbb{N}^3$ , we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) &\rightarrow M_{n,r}(K) \\ (A, B) &\mapsto AB := B \circ A \end{aligned}$$

## 22.4 Calculate Matrices

Let  $K$  be a unitary ring, and  $V$  be a left  $K$ -module. Let  $n \in \mathbb{N}$  and

$$x = (x_1, \dots, x_n) \in V^n$$

### 22.4.1 Remind

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \varphi : (a_1, \dots, a_n) \mapsto a_1 x_1, \dots, a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1, \dots, p\} \times \{1, \dots, n\}} \in M_{p,n}(K)$$

$A$  is a morphism of left  $K$ -modules from  $K^p$  to  $K^n$ . Recall that

$$A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let  $(b_1, \dots, b_n) \in K^p$

$$\begin{aligned} A((b_1, \dots, b_n)) &= \sum_{i=1}^p b_i(a_{i,1}, \dots, a_{i,n}) \\ \varphi(A((b_1, \dots, b_n))) &= \sum_{i=1}^p b_i \varphi_x((a_{i,1}, \dots, a_{i,n})) \\ &= \sum_{i=1}^p b_i(a_{i,1}x_1, \dots, a_{i,n}x_n) \end{aligned}$$

Let  $B = \{b_{ij}\}_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\}} : K^n \rightarrow K^r$

$$AB = \left\{ \sum_{j=1}^n a_{lj} b_{jm} \right\}_{(l,m) \in \{1, \dots, p\} \times \{1, \dots, r\}}$$



## Chapter 23

# Transpose

We fix a unitary ring  $K$

### 23.1 Def

Let  $E$  be a left- $K$ -module. Denote by

$$E^\vee := \{\text{morphisms of left } K\text{-modules } E \rightarrow K\}$$

$\forall (f, g) \in E^\vee$  let

$$\begin{aligned} f + g : E &\rightarrow K \\ x &\mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$  forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : E &\rightarrow K \\ x &\mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} K \times E^\vee &\rightarrow E^\vee \\ (a, f) &\mapsto fa : x \in E \rightarrow f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right  $K$ -module on  $E^\vee$

## 23.2 Def

Let  $E$  and  $F$  be two left  $K$ -modules.  $\varphi : E \rightarrow F$  be a morphism of left  $K$ -modules. We denote by

$$\varphi^\vee : F^\vee \rightarrow E^\vee$$

the morphism of right  $K$ -modules sending  $g \in F^\vee$  to  $g \circ \varphi \in E^\vee$ .  
Actually  $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

### 23.2.1 Example

Suppose that  $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,p} \end{pmatrix}$$

$\varphi$  sends  $(a_1, \dots, a_n)$  to  $\{\sum_{i=1}^n a_i b_{ij}\}_{j \in \{1, \dots, p\}}$ . Let  $g \in F^\vee$   $g : K^p \rightarrow K$ , then  $g$  is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$  sends  $(a_1, \dots, a_n)$  to  $\sum_{i=1}^p (\sum_{j=1}^n a_j b_{ij} y_i)$

Assume that  $K$  is commutative. We denote by

$$\iota_p : (K^p)^\vee \rightarrow K^p$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, \dots, x_p)$$

$$\iota_n : (K^n)^\vee \rightarrow K^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, \dots, x_n)$$

are isomorphisms of  $K$ -modules



For any morphism of K-modules  $\varphi : K^n \rightarrow K^p$ , we denote by  $\varphi^\tau$  the morphism of K-modules  $K^p \rightarrow K^n$  given by  $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{array}{ccc} (K^p)^\vee & \xrightarrow{\varphi^\vee} & (K^n)^\vee \\ \cong \downarrow \iota_p & \circlearrowleft & \cong \downarrow \iota_n \\ K^p & \xrightarrow{\varphi^\tau} & K^n \end{array}$$

$\varphi^\tau$  is called the transpose of  $\varphi$

### 23.3 Prop

Let E,F,G be left K-modules.  $\varphi : E \rightarrow F, \psi : F \rightarrow G$  be morphisms of left K-modules. Then  $(\psi \circ \varphi)^\vee$  is equal to  $\varphi^\vee \circ \psi^\vee$

#### Proof

$$\forall f \in G^\vee$$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

### 23.4 Corollary

Assume that K is commutative. Let  $n, p, q$  be neutral numbers.  $A \in M_{n,p}(K), B \in M_{p,q}(K)$ . Then

$$(AB)^\tau = B^\tau A^\tau$$

#### Proof

$$A^t au = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^t au = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned} B^\tau A^\tau &= A^\tau \circ B^\tau \\ &= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\ &= (AB)^t au \end{aligned}$$

### 23.5 Remark

(1) For  $A \in M_{n,p}(K)$ , one has  $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} E &\rightarrow (E^\vee)^\vee \\ x &\mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a K-linear mapping.

If K is a field and E is of finite dimension, this is a isomorphism of K-modules.

In fact, if  $e = (e_i)_{i=1}^n$  is a basis of E over K. For  $i \in \{1, \dots, n\}$ , let

$$\begin{aligned} e_i^\vee : E &\rightarrow K \\ \lambda_1 e_1, \dots, \lambda_n e_n &\mapsto \lambda_i \end{aligned}$$

is called the dual basis of  $e$

$$\begin{array}{ccc} K^n & \xleftarrow[\iota_n]{\cong} & (K^n)^\vee \\ \varphi_e \downarrow \cong & \searrow \varphi_{e^\vee} & \downarrow \varphi_e^\vee \\ E & \xrightarrow[\cong]{} & E^\vee \end{array}$$

$(e^\vee)^\vee$  gives a basis of  $(E^\vee)^\vee$  Hence  $E \rightarrow (E^\vee)^\vee$  is an isomorphism.

## Chapter 24

# Linear Equation

We fix a unitary ring  $K$ .

### 24.1 Def

For  $a = (a_1, \dots, a_n) \in K^n \setminus \{(0, \dots, 0)\}$ . Denote by  $j(a)$  the first index  $j \in \{1, \dots, n\}$  such that  $a_j \neq 0$ . Let  $(n, p) \in \mathbb{N}^2$ ,  $A \in M_{n,p}(K)$ . We write  $A$  as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \in K^p$$

We say that  $A$  is of row echelon form if,  $\forall i \in \{1, \dots, n-1\}$  one of following conditions is satisfied.

- $a^{(i+1)} = (0, \dots, 0)$
- $a^{(i)}, a^{(i+1)}$  are non-zero, and  $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, \dots, n\}$  such that  $a^{(i)} \neq (0, \dots, 0)$ , one has

$$a_{j(a^{(i)})}^{(i)} = 1$$

and

$$\forall k \in \{1, \dots, n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that  $A$  is of reduced row echelon form.

## 24.2 Prop

Suppose that  $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$  is of row echelon form. Then  $\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$  is of cardinal  $\leq p$

### Proof

Let  $k = \text{card}\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$   $a^{(k+1)} = \dots = a^{(n)} = (0, \dots, 0)$  and  $j(a^{(1)}) < j(a^{(2)}) < \dots < j(a^{(k)})$  Hence

$$\{1, \dots, k\} \rightarrow \{1, \dots, p\}, i \mapsto j(a^{(i)})$$

is injection. So  $k \leq p$

## 24.3 Linear Equation

Let  $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$ . Let  $V$  be a left  $K$ -module and  $(b_1, \dots, b_n) \in V^n$ . We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

The set of  $(x_1, \dots, x_p) \in V^p$  that satisfies  $(*)$  is called the solution set of  $(*)$

## 24.4 Prop

Suppose that  $A$  is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, \dots, n\} \mid (a_{i,1}, \dots, a_{i,p}) \neq (0, \dots, 0)\}$$

$$J_0(A) = \{1, \dots, p\} \setminus \{j((a_{i,1}, \dots, a_{i,p})) \mid i \in I(A)\}$$

- If  $\exists i \in \{1, \dots, n\} \setminus I(A)$  such that  $b_i \neq 0$  then  $(*)$  does not have any solution in  $K^n$
- Suppose that  $\forall i \in \{1, \dots, n\} \setminus I(A), b_i = 0$ . Then  $(*)$  has at least one solution. Moreover

$$V^{J_0(A)} \rightarrow V^p$$

$$(z_k)_{k \in J_0(A)} \mapsto (x_1, \dots, x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, \dots, a_{i,p})) \end{cases}$$

is an injective mapping, whose image is equal to the set of solution of (\*)

## 24.5 Prop

Let  $m \in \mathbb{N}, S \in M_{m,n}(K)$ . If  $(x_1, \dots, x_p) \in V^p$  is a solution of (\*), then  $(x_1, \dots, x_p)$  is a solution of  $(*)_S$ :

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

In the case where S is left invertible, namely there exist  $R \in M_{n,m}(K)$  such that  $RS = I_n \in M_{n,n}(K)$ . Then (\*) and  $(*)_S$  have the same solution set.

## 24.6 Def

Let  $G_n(K)$  be the set of  $S \in M_{n,n}(K)$  that can be written as  $U_1 \dots U_N$  (by convention  $S = I_n$  where  $N = 0$ ) where each  $U_i$  is of one of the following forms.

- $P_\sigma$  where  $\sigma \in \mathfrak{S}_n$
- $\text{diag}(r_1, \dots, r_n)$  where each  $r_i \in K$  is left invertible
- $S_{i,c}$  with  $i \in \{1, \dots, n\}$   $c = (c_1, \dots, c_n) \in K^n, c_i = 0$

Let  $p \in \mathbb{N}$ , we say that  $A \in M_{n,p}(K)$  is reducible by Gauss elimination if  $\exists S \in G_n(K)$  such that  $SA$  is of reduced row echelon form

## 24.7 Theorem

Assume that K is a division ring  $\forall (n, p) \in \mathbb{N}$  any  $A \in M_{n,p}(K)$  is reducible by Gauss elimination

### Proof

The case where  $n = 0$  or  $p = 0$  is trivial. We assume  $n \geq 1, p \geq 1$  We write A as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} B \quad \text{where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If  $\lambda_1 = \dots = \lambda_n = 0$

Applying the induction hypothesis to B, for  $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} SB$$

- Suppose that  $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

By permuting the rows we may assume  $\lambda_1 \neq 0$ . As K is division ring, by multiplying the first row by  $\lambda_1^{-1}$ , we may assume  $\lambda_1 = 1$ . We add  $(-\lambda_i)$  times the first row to the  $i^{th}$  row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 & \dots & \mu_p \\ 0 & & & \\ \vdots & C & & \\ 0 & & & \end{pmatrix} \quad \begin{array}{l} C \in M_{n-1, p-1}(K) \\ (\mu_2, \dots, \mu_p) \in K^{p-1} \end{array}$$

Applying the induction hypothesis to C, we may assume that C is of reduced row echelon form. For  $i \in \{2, \dots, k\}$  we add  $-\mu_{j(c_i)}$  times the  $i^{th}$  row of A to the first line to obtain a matrix of reduced row echelon form

## Chapter 25

# Normed Vector Space

### 25.1 Def

Let  $(X, d)$  be a metric space. If  $(x_n)_{n \in \mathbb{N}}$  is an element of  $X^{\mathbb{N}}$  such that

$$\lim_{N \rightarrow +\infty} \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

we say that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. If any Cauchy sequence in  $X$  converges, then we say that  $(X, d)$  is complete.

Let  $Cau(X, d)$  be the set of all Cauchy sequences in  $X$ . We define a binary relation  $\sim$  on  $Cau(X, d)$  as

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$$

### 25.2 Prop

$\sim$  is an equivalence relation.

#### 25.2.1 Proof

$$\lim_{n \rightarrow +\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$  be elements of  $Cau(X, d)$ . For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n \rightarrow +\infty} d(x_n, z_n) = 0$$

### 25.3 Def

$$\hat{X} := \text{Cau}(X, d) \setminus \sim$$

### 25.4 Def: The completion

The completion of  $(X, d)$  is defined as

$$\text{Cau}(X) / \sim$$

and is denoted as

$$\hat{X}$$

### 25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} : \hat{X} \times \hat{X} &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n \rightarrow +\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on  $\hat{X}$

#### Proof

TO check that  $\hat{d}$  is well defined, it suffices to prove that  $\forall ([x], [y]) \in \hat{X} \times \hat{X}$ ,  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is Cauchy sequence and its limit doesn't depend on the choice of the representation  $x$  and  $y$

For  $N \in \mathbb{N}$  and  $(n, m) \in \mathbb{N}_{\geq N}$  for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n, m) \in \mathbb{N}_{\geq N}} |d(x_n, y_n) - d(x_m, y_m)| &\leq \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(x_n, x_m) \right) \\ &\quad + \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(y_n, y_m) \right) \end{aligned}$$



Taking  $\lim_{n \rightarrow +\infty}$  we obtain that  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence.

Hence it converges in  $\mathbb{R}$ . If  $x' = (x'_n)_{n \in \mathbb{N}} \in [x], y' = (y'_n)_{n \in \mathbb{N}} \in [y]$ , thus

$$\lim_{n \rightarrow +\infty} d(x_n, x'_n) = \lim_{n \rightarrow +\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\lim_{n \rightarrow +\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(x'_n, y'_n)$$

In the following, we check that  $\hat{d}$  is a metric

- $\hat{d}([x], [y]) = 0$  iff  $[x] = [y]$ : trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$ : trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$ :

$$\begin{aligned} d([x], [y]) &= \lim_{n \rightarrow +\infty} \\ &\leq \lim_{n \rightarrow +\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

## 25.6 Remark

Let

$$\begin{aligned} i_X : X &\rightarrow \hat{X} \\ a &\mapsto [(a, a, \dots)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular,  $i_x$  is injective (if  $i_X(a) = i_X(b)$  then  $d(a, b) = 0$  hence  $a = b$ )

## 25.7 Prop

$i_X(X)$  is dense in  $\hat{X}$  (the closure of  $i_X(X)$  in  $\hat{X}$  is equal to  $i_X(X)$  (or to say  $\hat{X}$ ))

**Proof**

Let  $[x]$  be an equivalence class in  $\hat{X}$ . We claim that  $\forall (x_n)_{n \in \mathbb{N}} \in [x]$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} i_X(x_n)$$

For any  $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n \rightarrow +\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \end{aligned}$$

Taking  $\lim_{N \rightarrow +\infty}$  we get

$$\lim_{N \rightarrow +\infty} \hat{d}(i_X(x_N), [x]) = 0$$

**25.8 Theorem**

$(\hat{X}, \hat{d})$  is a complete metric space

**Proof**

Let  $([x^{(N)}])_{N \in \mathbb{N}}$  be a Cauchy sequence in  $\hat{X}$ , where  $\forall N \in \mathbb{N}$ ,  $x^{(N)} = (x_n^{(N)})_{n \in \mathbb{N}}$  is a Cauchy sequence  
 $\forall \epsilon > 0$ ,  $\exists N_0 \in \mathbb{N}$  such that  $\forall (k, l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n \rightarrow +\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any  $(\mu, \nu) \in \mathbb{N}_{\geq \alpha(N)}$

Let  $y_N = x_{\alpha(N)}^{(N)}$ . Without loss of generality, we assume that

$$\alpha(0) \leq \alpha(1) \leq \dots$$

Let  $\epsilon > 0$  Take  $N_0 \in \mathbb{N}$  such that

$$(1) \quad \forall (k, l) \in \mathbb{N}, \quad k, l \geq N_0$$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

$$(2)$$

$$\frac{1}{N_0 + 1} \leq \frac{\epsilon}{3}$$

Let  $(k, l) \in \mathbb{N}_{N_0}^2$ ,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since  $\alpha(k) \geq N_0, \forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking  $\lim_{n \rightarrow +\infty}$  get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So  $y = (y_N)_{N \in \mathbb{N}}$  is a Cauchy sequence. We check that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \hat{d}([x^{(N)}], [y]) &= 0 \\ 0 &\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \rightarrow +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$\begin{aligned} d(x_n^{(N)}, y_n) &\leq d(x_n^{(N)}, y_N) + d(y_n, y_N) \\ \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, y_n) &\leq \limsup_{N \rightarrow +\infty} \left( \frac{1}{N+1} + \lim_{n \rightarrow +\infty} d(y_n, y_N) \right) \end{aligned}$$

Since  $y$  is Cauchy sequence

$$\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(y_n, y_N) = 0$$

### Example

Let  $(K, |\cdot|)$  be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$  iff  $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$\text{Cau}(K)$  forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in \text{Cau}_0(K)$$

where

$$\text{Cau}_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of  $\text{Cau}(K)$

Hence

$$\hat{K} = \text{Cau}(K) \setminus \text{Cau}_0(K)$$

is a quotient ring of  $\text{Cau}(K)$

$|\cdot|$  extend to  $\hat{K}$ :

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \rightarrow +\infty} |a_n|$$

that forms an absolute value.

# Chapter 26

## Norms

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 26.1 Def

Let  $V$  be a vector space over  $K$  ( $K$ -module). We call seminorm on  $V$  any mapping

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}_{\geq 0} \\ s &\mapsto \|s\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V, \|s\| = 0$  iff  $s = 0$

We say that  $\|\cdot\|$  is a norm and  $(V, \|\cdot\|)$  is normed space over  $K$ .

### 26.2 Remark

If  $\|\cdot\|$  is a norm then

$$\begin{aligned} d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let  $(V, \|\cdot\|)$  be a vector space over  $K$  equipped with a seminorm, and  $W$  be a vector space subspace of  $V$  (sub- $K$ -module)

- The restriction of  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  to  $W$  forms a seminorm on  $W$ . It is a norm if  $\|\cdot\|$  is a norm.

$$\begin{aligned}\|\cdot\|_W : W &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto \|x\|\end{aligned}$$

- The mapping

$$\begin{aligned}\|\cdot\|_{V/W} : V/W &\rightarrow \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\|\end{aligned}$$

is a seminorm on  $V/W$

**Attention:** Even if  $\|\cdot\|$  is a norm,  $\|\cdot\|_{V/W}$  **might only be a seminorm**

### 26.3 Def

$\|\cdot\|_{V/W}$  is called the quotient seminorm of  $\|\cdot\|$

### 26.4 Prop

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of  $V$ . Moreover,  $\|\cdot\|_{V/N}$  is a norm

#### Proof

If  $(a, s) \in K \times N$  then  $\|as\| = |a| \cdot \|s\| = 0$  so  $as \in N$

If  $(s_1, s_2) \in N \times N$  then  $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$  so  $s_1 + s_2 \in N$

#### Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let  $\alpha \in V/N$  such that  $\|\alpha\|_{V/N} = 0$  Let  $s \in \alpha, \forall t \in N$

$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$

$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$

Hence  $\|\alpha\|_{V/N} = \|s\| = 0$  We obtain that  $\alpha = N = [0]$

## 26.5 Def

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. For any  $x \in V$  and  $r \geq 0$ , we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$

$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

## 26.6 Remark

If  $N = \{s \in V, \|s\| = 0\}$  then when  $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

## 26.7 Def

We equip the topology such that  $\forall U \subseteq V, U$  is open iff  $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

## 26.8 Prop

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorms. Let  $f : V_1 \rightarrow V_2$  be a  $K$ -linear mapping

- If  $f$  is continuous,  $\forall s \in V_1$  if  $\|s\|_1 = 0$  then  $\|f(s)\|_2 = 0$
- If there exists  $C > 0$  such that  $\forall x \in V_1, \|f(x)\|_2 \leq C\|x\|_1$  then  $f$  is continuous.

The converse is true

when  $|\cdot|$  is non-trivial

or  $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$  is of finite type

### Proof

- (1) Lemma If  $(V, \|\cdot\|)$  is a vector space over  $K$ , equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let  $s \in V \setminus N_{\|\cdot\|}$  Then  $\|s\| > 0$ . Let  $\epsilon = \frac{\|s\|}{2}$ ,  $\forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq \|s\| - \|s - x\| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteq V \setminus N_{\|\cdot\|}$$

– Then  $f^{-1}(N_{\|\cdot\|_2})$  is closed.

Note that

$$0 \in f^{-1}(N_{\|\cdot\|_2})$$

hence

$$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$

$$\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$$

$$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$

and

$$0 \in \mathcal{B}(x, \epsilon)$$

Therefore  $x \in \overline{\{0\}}$

(2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $V_1$  that converges to some  $x \in V_1$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n \rightarrow +\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n \rightarrow +\infty} C \|x_n - x\|_1 \\ &= C \limsup_{n \rightarrow +\infty} \|x_n - x\| \\ &= 0 \end{aligned}$$

So  $(f(x_n))_{n \in \mathbb{N}}$  converges to  $f(x)$ . Hence  $f$  is continuous at  $x$

Assume that  $|\cdot|$  is non-trivial and  $f$  is continuous. Then

$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

is an open subset of  $V_1$  containing  $0 \in V_1$

So there exists  $\epsilon > 0$  such that

$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

namely  $\forall x \in V_1$  if  $\|x\|_1 < \epsilon$  then  $\|f(x)\|_2 < 1$

Since  $|\cdot|$  is nontrivial,  $\exists a \in K, 0 < |a| < 1$  We prove that  $\forall x \in V_1$

$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|} \|x\|_1$$

If  $\|x\|_1 = 0$  by (1) we obtain

$$\|f(x)\|_2 = 0$$

Suppose that  $\|x\|_1 > 0$  then  $\exists n \in \mathbb{Z}$  such that

$$\begin{aligned} \|a^n x\|_1 &= |a|^n \|x\|_1 \\ &< \epsilon \leq \\ &\|a^{n-1} x\|_1 = |a|^{n-1} \|x\|_1 \end{aligned}$$



Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned} \|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}} \frac{1}{|a|} \\ &\leq \frac{1}{\epsilon} \|x\|_1 \frac{1}{|a|} = \frac{\|x\|_1}{\epsilon |a|} \end{aligned}$$

## 26.9 Def: Operator Seminorm

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorm. We say that a  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  is bounded if there exists  $C > 0$  that

$$\forall x \in V_1 \quad \|f(x)\|_2 \leq C \|x\|_1$$

For a general  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  we denote

$$\|f\| := \begin{cases} \sup_{x \in V_1, \|x\|_1 > 0} \left( \frac{\|f(x)\|_2}{\|x\|_1} \right) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

$f$  is bounded iff

$$\|f\| < +\infty$$

$\|f\|$  is called the operator seminorm of  $f$

We denote by  $\mathcal{L}(V_1, V_2)$  the set of all bounded  $K$ -linear mappings from  $V_1$  to  $V_2$

## 26.10 Prop

$\mathcal{L}(V_1, V_2)$  is a vector subspace of  $\text{Hom}_K(V_1, V_2)$ . Moreover  $\|\cdot\|$  is a seminorm on  $\mathcal{L}(V_1, V_2)$

### Proof

Let  $f, g$  be elements of  $\mathcal{L}(V_1, V_2)$

$$\begin{aligned} \|f + g\| &= \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x) + g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1} \right) + \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1} \right) \\ &\leq +\infty \end{aligned}$$

Hence  $f + g \in \mathcal{L}(V_1, V_2)$

Let  $\lambda \in K$ ,  $\lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

### 26.11 Remark

Let  $f \in \mathcal{L}(V_1, V_2)$ . Suppose that  $\exists x \in V_1$  such that  $f(x) \neq 0$ . Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore  $\|\cdot\|$  is a norm

### 26.12 Def

Let  $(V, \|\cdot\|)$  be a normed vector space. If  $V$  is complete with respect to the metric

$$\begin{aligned}d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|x - y\|\end{aligned}$$

then we say that  $(V, \|\cdot\|)$  is a Banach space.

### 26.13 Theorem

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with semi-norm. If  $(V_2, \|\cdot\|_2)$  is a Banach space, then

$$(\mathcal{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

**Proof**

Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{L}(V_1, V_2)$ .  
 $\forall x \in V_1$ , the mapping

$$(f \in \mathcal{L}(V_1, V_2)) \mapsto f(x)$$

is  $\|x\|_1$ -Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\| \|x\|_1$$

So  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence, for  $V_2$  is complete, that converges to some  $g(x) \in V_2$ . Then we obtain a mapping  $g : V_1 \rightarrow V_2$ . We prove that  $g$  is an element of  $\mathcal{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n \rightarrow +\infty} f_n(x + y) = \lim_{n \rightarrow +\infty} f_n(x) + f_n(y)$$

$$\begin{aligned} \|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \rightarrow +\infty) \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n \rightarrow +\infty} f_n(\lambda x) = \lim_{n \rightarrow +\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n \rightarrow +\infty} \|f_n(x)\| \leq (\lim_{n \rightarrow +\infty} \|f_n\|) \cdot \|x\|$$

(because  $\forall (a, b) \in V_2^2 \quad \|a\| - \|b\| \leq \|a - b\|$ ) Then

$$\|f_n(x)\| - \|g_n(x)\| \leq \|f_n(x) - g_n(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g \in \mathcal{L}(V_1, V_2)$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall (n, m) \in \mathbb{N}_{\geq N}, \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

So  $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$



## Chapter 27

# Differentiability

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 27.1 Def

Let  $X$  be a topological space and  $p \in X$ . Let  $K$  be a complete valued field and  $(E, \|\cdot\|)$  be a normed vector space over  $K$ .

Let  $f : X \rightarrow E$  be a mapping and  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative mapping.

- We say that

$$f(x) = O(g(x)) \text{ } x \rightarrow p$$

if there is a neighborhood  $V$  of  $p$  in  $X$  and a constant  $C > 0$  such that  $\forall x \in V$

$$\|f(x)\| \leq Cg(x)$$

- We say that

$$f(x) = o(g(x)) \text{ } x \rightarrow p$$

if there exists a neighborhood  $V$  of  $p$  in  $X$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\lim_{x \in V, x \rightarrow p} \epsilon(x) = 0$$

which is equivalent to

$$\forall \delta > 0, \exists \text{ neighborhood } U \text{ of } p \text{ } U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$

and  $\forall x \in V$

$$\|f(x)\| \leq \epsilon(x)g(x)$$

## 27.2 Def

Let  $E$  and  $F$  be normed vector space over  $K$ .  $U \subseteq E$  be an open subset,  $f : U \rightarrow F$  be a mapping and  $p \in U$ . If there exists  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

we say that  $f$  is differentiable at  $p$ , and  $\varphi$  is the differential of  $f$  at  $p$ . Suppose that  $|\cdot|$  is not trivial.  $\varphi(x - p)$  also written as

$$d_p f$$

### Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

means there exists an open neighborhood  $V$  of  $p$  with  $V \subseteq U$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that  $\lim_{x \rightarrow p} \epsilon(x) = 0$  and that  $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

## 27.3 Prop

If  $f$  is differentiable at  $p$ , then its differential at  $p$  is unique

### Proof

Suppose that there exists  $\varphi$  and  $\psi$  in  $\mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|(\varphi - \psi)y\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)y\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\| (y - p)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \rightarrow p} \epsilon(y) = 0 \end{aligned}$$

## 27.4 Example

### 27.4.1

$$f : U \rightarrow F : f(x) = y_0 \quad \forall x \in U$$

$$\forall p \in U$$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence  $\forall x \in E$

$$d_p(f(x)) = 0$$

### 27.4.2

Let  $f \in \mathcal{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence  $d_p f = f$

### 27.4.3

$$A : E \times E \rightarrow E$$

$$(x, y) \mapsto x + y$$

Let  $E$  be a normed space. Then  $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

### 27.4.4

$$m : K \times E \rightarrow E$$

$$(\lambda, x) \mapsto \lambda x$$

Let  $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when  $(\lambda, x) \rightarrow (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

is a  $K$ -linear mapping.

$$\begin{aligned}
- & (\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2) \\
- & b\mu p + a(by) = b(\mu p + ay) \\
- & \|\mu p + ay\| \leq |\mu| \|p\| + |a| \|y\| \\
& \leq \max\{|\mu|, \|y\|\}(|a| + \|p\|)
\end{aligned}$$

Hence  $m$  is differentiable and  $\forall(\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

## 27.5 Theorem:Chain rule

Let  $E, F, G$  be normed vector spaces,  $U \subseteq E, V \subseteq F$  be open subsets.

Let  $f : U \rightarrow F, g : V \rightarrow G$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . Assume that  $f$  is differentiable at  $p$  and  $g$  differentiable at  $f(p)$ . Then  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

### Proof

Let  $x \in U$ . By definition

$$\begin{aligned}
f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\
f(x) - f(p) &= O(\|x - p\|)
\end{aligned}$$

and

$$\begin{aligned}
(g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|)
\end{aligned}$$

So  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

## 27.6 Prop

Let  $n$  be a positive integer. Let  $(F_i)_{i \in \{1, \dots, n\}}$  be normed vector spaces over  $K$ . Let  $U \subseteq E$  be an open subset,  $p \in U$ .

$\forall i \in \{1, \dots, n\}$  let  $f_i : U \rightarrow F_i$  be a mapping. Let

$$f : U \rightarrow F = \prod F_i$$

be the mapping that sends  $x \in U$  to  $(f_i(x))_{i \in \{1, \dots, n\}}$ . We equip  $F$  with the norm  $\|\cdot\|$  defined as :

$$\|(y_i)_{i \in \{1, \dots, n\}}\| = \max_{i \in \{1, \dots, n\}} \|y_i\|$$



Then  $f$  is differentiable at  $p$  iff each  $f_i$  is differentiable at  $p$ . Moreover, when this happens, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i \in \{1, \dots, n\}}$$

### Proof

$\Leftarrow$  Suppose that  $(f_i)_{i \in \{1, \dots, n\}}$  are differentiable at  $p$

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i \in \{1, \dots, n\}} \\ &= (d_p f_i(x - p))_{i \in \{1, \dots, n\}} + o(\|x - p\|) \end{aligned}$$

Therefore  $f$  is differentiable at  $p$  and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i \in \{1, \dots, n\}}$$

$\Rightarrow$  Let

$$\begin{aligned} \pi_i : F &\rightarrow F_i \\ (x_i)_{i \in \{1, \dots, n\}} &\mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has  $\|\pi_i\| \leq 1$  because

$$\|x_i\| \leq \max_{i \in \{1, \dots, n\}} \|x_i\| = \|(x_i)_{i \in \{1, \dots, n\}}\|$$

$\pi_i$  is differentiable at  $p$  then  $\pi_i \circ f = f_i$  is differentiable at  $p$

## 27.7 Def

Let  $U$  be an open subset of  $K$  and  $(F, \|\cdot\|)$  be a normed vector space. If  $f : U \rightarrow F$  is a mapping that is differentiable at some  $p \in U$ . We denote by  $f'(p)$  the element

$$d_p f(1) \in F$$

called the derivative of  $f$  at  $p$

## 27.8 Corollary

Let  $U$  and  $V$  be open subsets of  $K$ ,  $(F, \|\cdot\|)$  be a normed vector space over  $K$ .  $f : U \rightarrow K$ ,  $g : V \rightarrow F$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$  then

$$(g \circ f)'(p) = f'(p)g'(f(p))$$

By definition

$$\begin{aligned} d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p) \cdot 1) \\ &= f'(p) \cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

Let  $E$  and  $F$  be normed vector spaces,  $U \subseteq E$  an open subset.  $f : U \rightarrow L$  and  $g : U \rightarrow F$  be mappings and  $p \in U$ . If both  $f, g$  differentiable at  $p$  then

is also differentiable at  $p$  and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_p f(l) + g(p)d_p f(l)$$

Consider

$$\begin{aligned} m : K \times F &\rightarrow F \\ (a, y) &\rightarrow ay \end{aligned}$$

We have shown  $m$  is differentiable and

$$d_{a,y}m(b,z) = by = az$$

$fg$  is the following composite:

$$U \xrightarrow{h} K \times F \xrightarrow{m} F$$

$fg$

$$x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m \circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p), g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

## 27.10 Corollary

Let  $U$  be an open subset of  $K$ ,  $f, g$  be mappings from  $U$  to  $K$  and to a normed space  $F$  respectively. If  $f, g$  are differentiable at  $p \in U$  then

$$(fg)'(p) = d_p(fg)(1) = d_p f(1)g(p) + f(p)d_p g(1) = f'(p)g(p) + f(p)g'(p)$$

### Example

$$\begin{aligned} f_n : K &\rightarrow K \\ x &\mapsto x^n \end{aligned}$$

is differentiable at any  $x \in K$

$$f'_n(x) = nx^{n-1}$$

### Proof

$f_1 : K \rightarrow K$  is differentiable  $\forall x \in K$

$$d_x f_1 = f_1$$

If  $f'_n(x) = nx^{n-1}$  then

$$\begin{aligned} f'_{n+1}(x) &= (f_n f_1)'(x) \\ &= f_n(x)f'_1(x) + f'_n(x)f_1(x) \\ &= x^n + x'_n(x) = x^n + nx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(l) &= l d_x f_n(1) \\ &= l n x^{n-1} \end{aligned}$$

## 27.11 Prop

Let  $E, F, G$  be normed vector spaces.  $U \subseteq E$  be an open subset,  $\varphi \in \mathcal{L}(F, G)$ ,  $p \in U$  if  $f : U \rightarrow E$  is differentiable at  $p$  then so is  $\varphi \circ f$ . Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

### Proof

$\varphi$  is differentiable at  $f(p)$  nad  $d_{f(p)}\varphi = \varphi$

### 27.12 Corollary

Let  $E$  and  $F$  be normed vector spaces  $U \subseteq E$  be an open subset,  $p \in U$ . Let  $f : U \rightarrow F$  and  $g : U \rightarrow F$  be mappings that are differentiable at  $p$ ,  $(a, b) \in K \times K$ . Then  $af + bg$  is differentiable at  $p$  and

$$d_p(af + bg) = ad_p f + bd_p g$$

#### Proof

$af + bg$  is composite:

$$U \xrightarrow{h} K \times F \xrightarrow{m} F$$

$ay+bz$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

### 27.13 Def: Equivalence of Norms

Let  $E$  be a vector space over  $K$  and  $\|\cdot\|_1, \|\cdot\|_2$  be norms on  $E$ . We say that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if there exist constants  $C_1, C_2 > 0$  such that  $\forall s \in E$

$$C_1 \|s\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

### 27.14 Prop

If  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover  $\|\cdot\|_1, \|\cdot\|_2$  defines the same topology on  $E$ .

#### Proof

$$\|s\|_2 \leq C_2 \|s\|_1 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are continuous. So  $\forall$  open subset  $U$  of  $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in  $(E, \|\cdot\|_1)$ . Conversely if  $V$  is open in  $(E, \|\cdot\|_1)$  then

$$V = Id_E^{-1}(V)$$

is open in  $(E, \|\cdot\|_2)$

## 27.15 Remark

If  $\|\cdot\|_1, \|\cdot\|_2$  are two norms on  $E$  that define the same topology on  $E$ , then they are equivalent (under the assumption that  $|\cdot|$  is not trivial)

## 27.16 Prop

Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$  be norms on  $E$  and  $F$  that are equivalent to  $\|\cdot\|_E, \|\cdot\|_F$  respectively. Let  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping.

Let  $p \in U$  Then  $f$  is differentiable at  $p$  with respect to  $\|\cdot\|_E$  and  $\|\cdot\|_F$  iff it's differentiable with respect to  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$

Moreover the differentiable of  $f$  at  $p$  is not changed in the change of norms from  $(\|\cdot\|_E, \|\cdot\|_F)$  to  $(\|\cdot\|'_E, \|\cdot\|'_F)$

### Proof

$$U \xrightarrow{Id_U} U \xrightarrow{f} F \xrightarrow{Id_F} F$$

$f$

$$(E, \|\cdot\|'_E) \quad (E, \|\cdot\|_E) \quad \|\cdot\|_F \quad \|\cdot\|'_F$$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$$d'_p f : (E, \|\cdot\|'_E) \rightarrow (F, \|\cdot\|'_F)$$

## 27.17 Theorem

Let  $V$  be a finite dimensional vector space over  $K$ . Then all norms on  $V$  are equivalent. Moreover  $V$  is complete with respect to any norm on  $V$ .

**Proof**

Let  $(e_i)_{i=1}^n$  be a basis of  $V$  (linear independent system of generators) The mapping:

$$V \rightarrow \mathbb{R}_{\geq 0}$$

$$\sum_{i \in \{1, \dots, n\}} a_i e_i \mapsto \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

is a norm on  $V$

Let  $\|\cdot\|$  be another norm on  $V$ . One has

$$\left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\| \leq \sum_{i \in \{1, \dots, n\}} |a_i| \|e_i\|$$

$$\leq \left( \sum_{i \in \{1, \dots, n\}} \|e_i\| \right) \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

We reason by induction that there exists  $C > 0$  such that

$$\max_{i \in \{1, \dots, n\}} \{|a_i|\} \leq C \left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\|$$

The case where  $n = 0$  is trivial.

$n=1$

$$\|a_1 e_1\| = |a_1| \|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension  $< n$

Let

$$W = \left\{ \sum_{i \in \{1, \dots, n-1\}} a_i e_i \mid (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1} \right\}$$

equipped with  $\|\cdot\|$  restricted to  $W$

The induction hypothesis shows that  $W$  is complete. Hence it's closed in  $V$ . Let  $Q = V/W$  and  $\|\cdot\|_Q$  be the quotient norm on  $Q$  that's defined as

$$\forall \alpha \in Q \quad \|\alpha\|_Q = \inf_{s \in \alpha} \|s\|$$

– If  $s \in V \setminus W$ ,  $\exists \epsilon > 0$  such that

$$\overline{B}(s, \epsilon) \cap W = \emptyset$$

$\forall t \in W$ ,

$$s + t \notin \overline{B}(0, \epsilon)$$

since otherwise

$$-t \in W \cap \overline{B}(s, \epsilon)$$

Therefore

$$\|[s]\|_Q = \inf_{i \in W} \|s + t\| \geq \epsilon > 0$$

–  $\forall \lambda \in K$

$$\|\lambda \alpha\|_Q = \inf_{s \in \alpha} \|\lambda s\| = |\lambda|$$

$$\inf_{s \in \alpha} \|s\| = |\lambda| \cdot \|\alpha\|_Q$$

–

$$\begin{aligned} \|\alpha + \beta\|_Q &= \inf_{s \in \alpha + \beta} \|s\| \\ &= \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \inf_{x \in \alpha} \|x\| + \inf_{y \in \beta} \|y\| \end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some  $A > 0$  such that  $\forall (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1}$

$$\max_{i \in \{1, \dots, n-1\}} \{|a_i|\} \leq A \left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\|$$

Take

$$s = \sum_{i \in \{1, \dots, n\}} a_i e_i \in V$$

Let  $\alpha = [s] = a_n [e_n] \in Q$

$$\left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \leq \|s\| + |a_n| \cdot \|e_n\| \leq \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \|[e_n]\|_Q = |a_n| \inf_{t \in W} \|e_n + t\|$$

Take  $e'_n \in V$  such that  $[e'_n] = [e_n]$  and  $\|e'_n\| \leq \|[e_n]\|_Q + \epsilon$

Note that  $(e_1, \dots, e_{n-1}, e'_n)$  forms also basis of  $V$  over  $K$ . Hence by replacing  $e_n$  by  $e'_n$  we may assume that  $\|e_n\| \leq \|[e_n]\|_Q + \epsilon$

$s = a_n e_n + t \in V$  with  $t \in W$

$$\|s\| \geq \|a_n e_n\|_Q = |a_n| \|[e_n]\|_Q \geq B^{-1} |a_n| \cdot \|e_n\|$$

– If  $\|a_n e_n\| < \frac{1}{2} \|t\|$

$$\|s\| \geq \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \geq \frac{1}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

– If  $\|a_n e_n\| \geq \frac{1}{2} \|t\|$

$$\|s\| \geq B^{-1} |a_n| \cdot \|e_n\| \geq \frac{B^{-1}}{2} \|t\| \geq \frac{B^{-1}A}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

We take  $C = \max\{B^{-1} \|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$  Then

$$\|s\| \geq C \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Another proof

completeness Under the norm  $\max_{i \in \{1, \dots, n\}}$ , a sequence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  is a Cauchy sequence iff  $\forall i \in \{1, \dots, n\}$   $(a_i^{(k)})_{k \in \mathbb{N}}$  is a Cauchy sequence. Since  $K$  is complete each  $(a_i^{(k)})_{k \in \mathbb{N}}$  converges to some  $a_i \in K$  Hence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  converges.

## 27.18 Prop

Let  $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . Assume that  $E$  is finite dimensional. Then any  $K$ -linear mapping  $\varphi : E \rightarrow F$  is bounded.

### Proof

Let  $(e_i)_{i=1}^n$  be a basis of  $E$ . For any two norms on  $E$  are equivalent.  
 $\forall (a_1, \dots, a_n) \in K$

$$\left\| \sum_{i=1}^n a_i e_i \right\|_E = \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Then for any  $s = \sum_{i=1}^n a_i e_i$

$$\|\varphi(s)\|_F = \left\| \sum_{i=1}^n a_i e_i \right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq \left( \sum_{i=1}^n \|\varphi(e_i)\|_F \right) \|s\|_E$$

## 27.19 Theorem

Let  $E, F$  be normed vector spaces over a complete valued field,  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping. If  $f$  is differentiable at  $p$  then  $f$  is continuous at  $p$

### Proof

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ &= f(p) + O(\|x - p\|) \\ &= f(p) + o(1) \quad x \rightarrow p \\ &\Rightarrow \lim_{x \rightarrow p} f(x) = f(p) \end{aligned}$$



## Chapter 28

# Compactness

### 28.1 Def: cover

Let  $X$  be a topological space,  $Y \subseteq X$  we call open cover of  $Y$  any family  $(U_i)_{i \in I}$  open subset of  $X$  such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If  $I$  is finite set, we say that  $(U_i)_{i \in I}$  is a finite open cover. If  $J \subseteq I$  such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that  $(U_j)_{j \in J}$  is a sub cover of  $(U_i)_{i \in I}$

### 28.2 Def: compact

If any open cover of  $Y$  has a finite subcover, we say that  $Y$  is quasi-compact. If in addition  $X$  is Hausdorff, namely  $\forall (x, y) \in X \times X$  with  $x \neq y \exists$  open neighborhoods  $U$  and  $V$  of  $x$  and  $y$  such that  $U \cap V = \emptyset$ , we say that  $Y$  is compact

### 28.3 Def

Let  $X$  be a set and  $\mathcal{F}$  be a filter on  $X$ . If there does not exist any filter  $\mathcal{F}'$  of  $X$  such that  $\mathcal{F} \subsetneq \mathcal{F}'$ , then we say that  $\mathcal{F}$  is an ultrafilter.

**Zorn's lemma** implies that  $\forall \mathcal{F}_0$  of  $X$  there exist an ultrafilter  $\mathcal{F}$  if  $X$  containing  $\mathcal{F}_0$

## 28.4 Prop

Let  $\mathcal{F}$  be a filter on a set  $X$ . The following statements are equivalent.

- (1)  $\mathcal{F}$  is an ultrafilter
- (2)  $\forall A \subseteq X$  either  $A \in \mathcal{F}$  or  $X \setminus A \in \mathcal{F}$
- (3)  $\forall (A, B) \in \wp(X)^2$  if  $A \cap B \in \mathcal{F}$  then  $A \in \mathcal{F}$  or  $B \in \mathcal{F}$

### Proof

- (1)  $\Rightarrow$  (2) Suppose that  $A \in \wp(X)$  such that  $A \notin \mathcal{F}$  and  $X \setminus A \notin \mathcal{F} \forall B \in \mathcal{F}$  one has

$$B \cap A \neq \emptyset$$

since otherwise  $B \subseteq X \setminus A$  and hence  $X \setminus A \in \mathcal{F}$  contradiction.

- (2)  $\Rightarrow$  (3) Suppose that  $B \notin \mathcal{F}$  then  $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So  $A \in \mathcal{F}$

- (3)  $\Rightarrow$  (1) Suppose that  $\mathcal{F}'$  is a filter such that  $\mathcal{F} \subsetneq \mathcal{F}'$  Take  $A \in \mathcal{F}' \setminus \mathcal{F}$  Then by  $X = A \cup (X \setminus A) \in \mathcal{F}$  Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \emptyset = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

## 28.5 Theorem

Let  $(X, \mathcal{J})$  be a topological space . The following are equivalent

- (1)  $X$  is quasi-compact
- (2) Any filter of  $X$  has an accumulation point
- (3) Any ultrafilter of  $X$  is converges.

### Proof

- (1)  $\Rightarrow$  (2) Assume that a filter  $\mathcal{F}$  of  $X$  does not have any accumulation point.  $\forall x \in X \exists A_x \in \mathcal{F} \exists$  open neighborhood  $V_x$  of  $x$  such that  $A_x \cap V_x = \emptyset$  Since  $X = \bigcup_{x \in X} V_x$  there is

$$\{x_1, \dots, x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^n V_{x_i}$$

$$\text{Take } B = \bigcap_{i=1}^n A_{x_i} \in \mathcal{F}$$

$$B \cap X = B = \emptyset$$

Since  $\forall i \ B \cap V_x = \emptyset$  contradiction.

- (2)  $\Rightarrow$  (3) Let  $\mathcal{F}$  be an ultrafilter of  $X$ . By (2) there exist  $x \in X$  such that  $\mathcal{F} \cup \mathcal{V}_x$  generates a filter  $\mathcal{F}'$  Since  $\mathcal{F}$  is an ultrafilter  $\mathcal{F} = \mathcal{F}'$  and hence  $\mathcal{V}_x \subseteq \mathcal{F}$
- (3)  $\Rightarrow$  (1) Let  $(U_i)_{i \in I}$  be an open cover of  $X$  we suppose that this have no finite subcover.  $\forall i \in I$  let

$$F_i = X \setminus U_i$$

For any  $J \subseteq I$  finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \emptyset$$

Let  $\mathcal{F}$  be the smallest filter on  $X$  that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let  $\mathcal{F}'$  be ultrafilter containing  $\mathcal{F}$ . It has a limit point  $x$  There exist  $i \in I$  such that  $x \in U_i$ . Since  $U_i$  is a neighborhood of  $x$  and  $\mathcal{V}_x \subseteq \mathcal{F}'$  we get  $U_i \in \mathcal{F}'$  This is impossible since  $F_i \in \mathcal{F}'$

## 28.6 Theorem

Let  $(X, d)$  be a metric space. The following statements are equivalent:

- (1)  $X$  is complete and  $\forall \epsilon > 0 \ \exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

- (2)  $X$  is compact

### Proof

- (1)  $\Rightarrow$  (2) Let  $\mathcal{F}$  be an ultrafilter Let  $\epsilon > 0$  and  $\{x_1, \dots, x_n\} \subseteq X$  such that

$$X = \bigcup_{i=1}^n \mathcal{B}(x_i, \epsilon)$$

There exists some  $i \in \{1, \dots, n\}$  such that  $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$  That means  $\mathcal{F}$  is a Cauchy filter (namely  $\forall \delta > 0 \ \exists A \in \mathcal{F}$  of diameter  $\leq \delta$ ) Since  $X$  is complete  $\mathcal{F}$  has a limit point. So  $\mathcal{F}$  is compact.

(2)  $\Rightarrow$  (1) Let  $\epsilon > 0$  One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since  $X$  is compact  $\exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$\mathcal{F}$  is an ultrafilter

$$\Leftrightarrow \forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}$$

$$\Leftrightarrow \forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F}$$

$$\Leftrightarrow \forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup \dots \cup A_n \ \exists i \in \{1, \dots, n\}, A_i \in \mathcal{F}$$

Let  $\mathcal{F}$  be a Cauchy filter Let  $x \in X$  be an accumulation point of  $\mathcal{F}$   
 $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$  with diameter  $\leq \frac{\epsilon}{2}$  Note that  $A \cup \mathcal{B}(x, \frac{\epsilon}{2}) \neq \emptyset$  Take  
 $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2}) \ \forall z \in A$

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore  $A \subseteq \mathcal{B}(x, \epsilon)$  So  $\mathcal{B}(x, \epsilon) \in \mathcal{F}$  This implies  $\mathcal{V}_x \subseteq \mathcal{F}$

## 28.7 Lemma

Let  $(X, d)$  be a metric space

- (1) Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$
- (2)  $X$  is complete iff any Cauchy filter of  $X$  has a limit point

### Proof

(1)

- Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$

- (2) Suppose that  $X$  is complete. Let  $\mathcal{F}$  be a Cauchy filter.  $\forall n \in \mathbb{N}_{\geq 1}$  let  $A_n \in \mathcal{F}$  such that  $\text{diam}(A_n) \leq \frac{1}{n}$  Take  $x_n \in \bigcap_{k=1}^n A_k \in \mathcal{F}$  Then  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  is a Cauchy sequence since  $\forall \epsilon > 0$  if we take  $N \in \mathbb{N}$  with  $\frac{1}{N} \leq \epsilon$  then  $\forall (n, m) \in \mathbb{N}_{\geq N} \ d(x_n, x_m) \leq \frac{1}{N}$  Hence  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  converges to some  $x \in X$  Note that  $x$  is an accumulation point of  $\mathcal{F}$  since  $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$  with  $A_n \subseteq \mathcal{B}(x, \epsilon)$  It suffices to take  $n$  such that  $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$  Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $X$ . Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N}, \{x_N, x_{N+1}, \dots\} \subseteq A\}$$

This is a Cauchy filter on  $X$  since

$$\lim_{N \rightarrow +\infty} \text{diam}\{x_N, x_{N+1}, \dots\} = 0$$

Hence  $\mathcal{F}$  has a limit point  $x \in X$  By definition  $\forall U \in \mathcal{V}_x \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, \dots\} \subseteq U$$

$$\text{So } x = \lim_{n \rightarrow +\infty} x_n$$

## 28.8 Prop

Let  $f : X \rightarrow Y$  be a continuous mapping of topological spaces. If  $A \subseteq X$  is quasi-compact then  $f(A) \subseteq Y$  is also quasi-compact.

### Proof

Let  $(V_i)_{i \in I}$  be an open cover of  $f(A)$  Then

$$(f^{-1}(V_i))_{i \in I}$$

is an open cover of  $A$  So  $\exists J \subseteq I$  such that

$$A \subseteq \bigcup_{j \in J} f^{-1}(V_j)$$

This implies

$$f(A) \subseteq \bigcup_{j \in J} V_j$$

So  $f(A)$  is quasi-compact.

## 28.9 Prop

Let  $X$  be a topological space and  $A \subseteq X$  be a quasi-compact subset. For any closed subset  $F$  of  $X$   $A \cap F$  is quasi-compact.

### Proof

Let  $(U_i)_{i \in I}$  be an open cover of  $A \cap F$ . Then

$$A \subseteq \left( \bigcup_{i \in I} U_i \right) \cup (X \setminus F)$$

Since  $A$  is quasi-compact there exist  $J \subseteq I$  finite such that

$$A \subseteq \left( \bigcup_{j \in J} U_j \right) \cup (X \setminus F)$$

Hence  $A \cap F \subseteq \bigcup_{j \in J} U_j$

### 28.10 Prop

Let  $X$  be a Hausdorff topological space. Any compact subset  $A$  of  $X$  is closed.

#### Proof

Let  $x \in X \setminus A$   $\forall y \in A, \exists$  open subsets  $U_y$  and  $V_y$  such that  $y \in U_y, x \in V_y$  and  $U_y \cap V_y = \emptyset$  Since  $A \subseteq \bigcup_{y \in A} U_y$   $\exists \{y_1, \dots, y_n\} \subseteq A$  such that

$$A \subseteq \bigcup_{i=1}^n U_{y_i}$$

Let

$$U = \bigcup_{i=1}^n U_{y_i} \quad V = \bigcap_{i=1}^n V_{y_i}$$

These are open subset Moreover  $A \subseteq U, x \in V$  and  $U \cap V = \bigcup_{i=1}^n (U_{y_i} \cap V) = \emptyset$   
In particular  $x \in V \subseteq X \setminus A$  So  $X \setminus A$  is open

### 28.11 Prop

Let  $X$  be a Hausdorff topological space and  $A$  and  $B$  be compact subsets of  $X$  such that  $A \cap B = \emptyset$  Then there exists open subsets  $U$  and  $V$  such that

$$A \subseteq U, B \subseteq V \text{ and } U \cap V = \emptyset$$

#### proof

We have seen in the proof of the previous proposition that  $\forall x \in B, \exists U_x, V_x$  open such that  $A \subseteq U_x, x \in V_x$  and  $U_x \cap V_x = \emptyset$  Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, \dots, x_m\} \subseteq B$  such that

$$B \subseteq \bigcup_{i=1}^m V_{x_i}$$

We take

$$U = \bigcap_{i=1}^m U_{x_i} \quad V = \bigcup_{i=1}^m U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq U \quad U \cap V = \emptyset$$

## 28.12 Theorem

Let  $(X, \mathcal{J})$  be a Hausdorff topological space. If  $(A_n)_{n \in \mathbb{N}}$  is a sequence of non-empty compact subsets of  $X$  such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

Then

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$$

### Proof

Suppose that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

then

$$A_0 \subseteq \bigcup_{n \in \mathbb{N}} (X \setminus A_n)$$

Since  $A_0$  is compact,  $\exists N \in \mathbb{N}$  such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^N (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^N A_n \\ &= X \setminus A_N \end{aligned}$$

So

$$A_N = \emptyset$$

## 28.13 Def

Let  $(X, \tau)$  be a topological space. If any sequence in  $X$  has a convergent subsequence, we say that  $X$  is sequentially compact.

**Example**

By Bolzano-Weierstrass, any bounded sequence in  $\mathbb{R}$  has a convergent subsequence. So any bounded and closed subset of  $\mathbb{R}$  is sequentially compact.

**Note**

bounded and closed together implies sequentially compact.

**28.14 Theorem**

Let  $(X, d)$  be a metric space. Then the following statements are equivalent:

- (1)  $(X, d)$  is compact
- (2)  $(X, d)$  is sequentially compact

**Proof**

- (1)  $\Rightarrow$  (2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Assume that no subsequence of  $(x_n)_{n \in \mathbb{N}}$  converges in  $X$ . For any  $p \in X$  there exists  $\epsilon_p > 0$  such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For  $X$  is compact  $\exists (p_i)_{i \in \{1, \dots, n\}}$

$$X \subseteq \bigcup_{i=1}^n \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^n \{n \in \mathbb{N} : d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

- (2)  $\Rightarrow$  (1)

prove  $(X, d)$  is complete Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences  $(x_{k_n})_{n \in \mathbb{N}}$  must converges to the same limit.

So  $(X, d)$  is complete



If  $X$  is not covered by finitely many balls of radius  $\epsilon$  we can construct a sequence  $(x_{k_n})_{n \in \mathbb{N}}$  such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^n \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

## 28.15 Def

Let  $X$  be a Hausdorff topological space. If for any  $x \in X$  there exist a compact neighborhood  $\mathcal{C}_x$  we say that  $X$  is locally compact.

### Example

$\mathbb{R}$  is locally compact.

## 28.16 Prop

Assume that  $(K, |\cdot|)$  is a locally compact non-trivial valued field. Let  $(E, \|\cdot\|)$  be a finite dimensional normed  $K$ -vector space. A subset  $Y \subseteq E$  is compact iff it's closed and bounded.

### Proof

$\Rightarrow$  Let  $Y \subseteq X$  be compact. Then for  $Y$  is Hausdorff,  $Y$  is closed. Moreover

$$Y \subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq \dots \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^k \mathcal{B}(0, n_i)$$

$\Rightarrow Y$  is bounded.

$\Leftarrow$  We prove sequentially compact by a theorem proved before.

Let  $(e_i)_{i=1}^d$  be a basis of  $E$ . Again we assume

$$\left\| \sum_{i=1}^d a_i e_i \right\| = \max_{i \in \{1, \dots, d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n \in \mathbb{N}} = \left( \sum_{i=1}^d a_i^{(n)} e_i \right)_{n \in \mathbb{N}}$$

Since  $Y$  is bounded for any  $i \in \{1, \dots, d\}$  the sequence  $(a_i^{(n)})$  is bounded. In particular we find  $M > 0$  such that  $\forall i \in \{1, \dots, n\}$

$$|a_i^{(n)}| < M$$

Since  $(K, |\cdot|)$  is locally compact, there exists a compact set  $\mathcal{C} = \mathcal{C}_0 \subseteq K$  that's a neighborhood of 0. Let  $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since  $K$  is not trivially valued, then exists  $a \in K$  such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$  is compact. We have the  $K$ -linear mapping

$$\begin{aligned} K &\rightarrow K \\ y &\mapsto ay \end{aligned}$$

is bounded, then continuous. Hence  $a\mathcal{C}$  is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find  $(I_i)_{i=1}^d$  are infinite subsets of  $\mathbb{N}$  with

$$I_1 \supseteq \dots \supseteq I_d$$

such that  $(a_j)_{j \in I_i}^{(n)}$  converges to some  $a_i \in K$ . It follows that our original

sequence has a convergent subsequence converges to  $\sum_{i=1}^d a_i e_i$ .

So  $Y$  is sequentially compact.

## 28.17 Theorem

Let  $X$  be a topological space and  $f : X \rightarrow \mathbb{R}$  be a continuous mapping. If  $Y \subseteq X$  is a quasi-compact subset, then there exists  $a \in Y$  and  $b \in Y$  such that  $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of  $f$  to  $Y$  attains its maximum and minimum.

**Proof**

$f(Y) \subseteq \mathbb{R}$  is a non-empty compact subset since  $Y$  is quasi-compact and  $\mathbb{R}$  is Hausdorff. Moreover, since  $\mathbb{R}$  is locally compact. SO  $f(Y)$  is bounded and closed.

Note that there exists sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  in  $f(Y)$  that tends to  $\sup f(Y)$  and  $\inf f(Y)$  respectively. Since  $f(Y)$  is closed,  $\sup f(Y), \inf f(Y)$  belongs to  $f(Y)$ . So  $f(Y)$  has a greatest and a least element.



## Chapter 29

# Mean Value Theorems

### 29.1 Rolle Theorem

Let  $a, b$  be real numbers such that  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping that is differentiable on  $]a, b[$ . If  $f(a) = f(b)$  then  $\exists t \in ]a, b[$  such that

$$f'(t) = 0$$

#### Proof

Since  $[a, b]$  is closed and bounded then it's compact,  $f$  attains its maximum and minimum. Let  $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If  $M \neq l \exists t \in ]a, b[$  such that  $f(t) = M$

$$f(t+x) = f(t) + f'(t)x + o(|X|)$$

$$f(t-x) = f(t) - f'(t)x + o(|X|)$$

$$0 \leq (f(t+x) - f(t))(f(t-x) - f(t))$$

$$= -f'(t)^2 x^2 + o(|x|^2)$$

$$0 \leq -f'(t)^2 + o(1) \quad x \rightarrow 0$$

Taking the limit when  $x \rightarrow 0$  we get  $f'(t)^2 = 0$

If  $m \neq l$  then any  $t \in ]a, b[$  such that  $f(t) = m$  verifies  $f'(t) = 0$

If  $m = l = M$   $f$  is constant, so  $\forall t \in ]a, b[, f'(t) = 0$

### 29.2 Mean value theorem(Lagrange)

Let  $a, b$  be real numbers  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping differentiable on  $]a, b[$ , then  $\exists t \in ]a, b[$  such that

$$f(b) - f(a) = f'(t)(b - a)$$

**Proof**

Let  $g : [a, b] \rightarrow \mathbb{R}$  be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then  $g(a) = f(a)$   $g(b) = f(a)$  then apply Rolle Theorem to  $g$  we get the proof.

**29.3 Mean value inequality**

Let  $a, b$  be real numbers such that  $a < b$   $(E, \|\cdot\|)$  be a normed vector space over  $\mathbb{R}$   $f : [a, b] \rightarrow E$  be a continuous mapping such that  $f$  is differentiable on  $]a, b[$  Then

$$\|f(b) - f(a)\| \leq \left( \sup_{x \in ]a, b[} \|f'(x)\| \right) (b - a)$$

**Proof**

Suppose that

$$\sup_{x \in ]a, b[} \|f'(x)\| < +\infty$$

Let  $M \in \mathbb{R}$  such that

$$M > \sup_{x \in ]a, b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition  $J$  is an interval containing  $a$ , so  $J$  is of form  $[a, c[$  or  $[a, c]$  Since  $f$  is continuous by taking a sequence  $(c_n)_{n \in \mathbb{N}}$  in  $[a, b[$  that converges to  $c$  we obtain

$$\begin{aligned} \|f(c) - f(a)\| &= \lim_{n \rightarrow +\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n \rightarrow +\infty} M(c_n - a) \\ &= M(c - a) \end{aligned}$$

Hence  $c \in J$  namely  $J = [a, c]$

$c > a$  We will prove that  $c = b$  by contradiction

Suppose that  $c < b$   $\forall h \in ]0, b - c[$

$$\begin{aligned} \|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h) \end{aligned}$$

Since  $M > \|f'(c)\|$ ,  $\exists h_0 > 0$  such that  $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$

Hence

$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h) - f(c)\| + \|f(c) - f(a)\| \\ &\leq M(c_h - c + c - a) \\ &= M(c + h - a)\end{aligned}$$

So  $c + h_0 \in J$  Contradiction. Thus

$$\|f(b) - f(a)\| \leq M(b - a)$$

for any  $M > \sup_{x \in ]a, b[} \|f'(x)\|$  since  $M$  is arbitrary the expected inequality holds.

$c = a$  In general, we apply the particular case (fis-extendable to a differentiable mapping at  $a$ ) to  $[\frac{a+b}{2}, b]$  and  $[a, \frac{a+b}{2}]$  to get

$$\begin{aligned}\left\|f(b) - f\left(\frac{a+b}{2}\right)\right\| &\leq C \frac{b-a}{2} \\ \left\|f\left(\frac{a+b}{2}\right) - f(a)\right\| &\leq C \frac{b-a}{2}\end{aligned}$$

with  $C = \sup_{x \in ]a, b[} \|f'(x)\|$

**Remark** If  $f$  is defined on an open neighborhood of  $a$  and is differentiable at  $a$  the the same arguments hold without the assumption

## 29.4 Theorem

Let  $I$  be an interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a continuous mapping, then  $f(I)$  is an interval.

### Proof

Let  $x \neq y$  be two elements of  $f(I)$  Let  $a, b$  elements of  $I$  such that  $x = f(a)$   $y = f(b)$  without loss of generality, we assume  $a < b$   
Let  $z \in \mathbb{R}$  such

$$(z - x)(z - y) \leq 0$$

We construct by induction three sequences  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$  such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If  $a_n, b_n, c_n$  are constructed, satisfying

$$c_n = \frac{1}{2}(a_n + b_n)$$

$$(z - f(a_n))(z - f(b_n)) \leq 0$$

we let

$$\begin{aligned} (a_{n+1}, b_{n+1}) &= (a_n, c_n) & \text{if } (z - f(a_n))(z - f(c_n)) \leq 0 \\ (a_{n+1}, b_{n+1}) &= (c_n, b_n) & \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\ & & ((z - f(c_n))(z - f(b_n)) \leq 0) \end{aligned}$$

$$c_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

The sequence  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  are increasing and decreasing respectively and bounded, hence converges to some  $l, m \in [a, b]$

Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \rightarrow 0 (n \rightarrow +\infty)$$

So  $l = m$ , by  $(z - f(a_n))(z - f(b_n)) \leq 0$  we obtain by letting  $n \rightarrow +\infty$

$$(z - f(l))^2 \leq 0$$

So  $z = f(l)$

## 29.5 Theorem(Heine)

Let  $I$  be an open interval of  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a differentiable mapping. Then  $f'(I)$  is an interval.

### Proof

Let  $(a, b) \in I^2$  such that  $a < b$ . Consider the following mappings:

$$\begin{aligned} g : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(x) - f(a)}{x - a} & x \neq a \\ f'(a) & x = a \end{cases} \\ h : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(b) - f(x)}{b - x} & x \neq b \\ f'(b) & x = b \end{cases} \end{aligned}$$

$g, h$  are continuous ( $\frac{f(x) - f(a)}{x - a} = f'(a) + o(1) \ x \rightarrow a$ )

So  $g([a, b])$  and  $h([a, b])$  are intervals. Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$

$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that  $g(b) = h(a)$  so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence  $f'(I)$  is an interval.



## Chapter 30

# Fixed Point Theorem

### 30.1 Def

Let  $X$  be a set and  $T : X \rightarrow X$  be a mapping. If  $x \in X$  satisfies  $T(x) = x$  we say that  $x$  is a fixed point of  $T$ .

### 30.2 Def

Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be a mapping. If  $\exists \epsilon \in [0, 1[$  such that  $T$  is  $\epsilon$ -Lipschitzian then we say that  $T$  is a contraction.

### 30.3 Fixed Point Theorem

Let  $(X, d)$  be a COMPLETE non-empty metric space, and  $T : X \rightarrow X$  be a contraction. Then  $T$  has a unique fixed point. Moreover,  $\forall x_n \in X$  if we let

$$x_{n+1} = T(x_n), x_0 \in X$$

then  $(x_n)_{n \in \mathbb{N}}$  converges to the fixed point.

#### Proof

If  $p$  and  $q$  are two fixed point of  $T$ , then

$$d(p, q) = d(T(p), T(q)) \leq \epsilon d(p, q)$$

So  $d(p, q) = 0$ .

Let

$$x_{n+1} = T(x_n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any  $N \in \mathbb{N}$ ,  $\forall (n, m) \in \mathbb{N}_{\geq N}^2$   $n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \end{aligned}$$

So

$$\lim_{N \rightarrow +\infty} \sup_{(n, m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, hence converges to some  $p \in X$

$$d(T(p), p) = \lim_{n \rightarrow +\infty} d(T(x_n), x_n) = 0$$

since  $d : X^2 \rightarrow \mathbb{R}_{\geq 0}$  is continuous.