Contents

Part I

Set

Chapter 1

Ring

1.1 morphism

Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \rightarrow B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
-

algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n \in \mathbb{N}} a_n T^n$ Such element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_nT^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]

$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$

let $F + G = (a_0 + b_0) + ...$

$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n$

Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n\right)(\sum\limits_{n\in\mathbb{N}} c_nT^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_pb_qc_l\right) T^n$ is a trick applied on integral

Derivative:

let $F(T) \in k[T]$

We denote by $F'(T)$ *or* $\mathcal{D}(F(T))$ the formal power series

$\mathcal{D}(F) = \sum\limits_{n\in\mathbb{N}} (n + 1)a_{n+1}T^n$

Properties:

- $\mathcal{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!}T^n$,which fulfil the differential equation $\mathcal{D}(exp(T)) = exp(T)$(interesting)

Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 2

Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i \in I} is\ a\ family\ of\ element\ in\ X, then\ \sup\limits_{i \in I} x_i\ denotes\ \sup\{x_i \mid i \in I\}(inX)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup\limits_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N},k\geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

Chapter 3

Interval

We fix a totally ordered set $(X, \leq)$

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$,one has $[x, y] \subseteq I$ then we say that I is a interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X
- If $\bigcap\limits_{\lambda \in \Lambda} I_\lambda \neq \varnothing$, $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_] \mu$

- If $b \leq x \quad [a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a \quad [a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If a < x < b then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let $(X, \leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.

Prop.

Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x, b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let $(X, \leq)$ be a totally ordered set .If $\forall (x, z) \in X \times X$,such that $x < z \quad \exists y \in X$ such that $x < y < z$,than we say that $(X, \leq)$ is thick

Prop.

Let $(X, \leq)$ be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in]a, b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b[\subseteq]a, b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

Chapter 4

Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq$ *on* $\mathbb{R}$ *to* $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ become a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$ Obviously,this is a thick totally ordered set.
We define:

- $\forall x \in]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete

Example

Let Ω be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega)$, $\sup \mathscr{F} = \bigcup_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$

$\mathcal{AXION}$:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ *and* $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \backslash \{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

inf takes the similar($\leq \to \geq$) (provided when the sum are defined)

Chapter 5

Vector space

In this section:
K denotes a unitary ring.
Let 0 be zero element of K
1 be the unity of K

5.1 K-module

5.1.1 Def

Let $(V,+)$ be a commutative group.We call left/right K-module structure:
any mapping Φ:$K \times V \to V$

- $\forall (a,b) \in K \times K, \forall x \in V \quad \Phi(ab,x) = \Phi(a, \Phi(b,x))/\Phi(b, \Phi(a,x))$
- $\forall (a,b) \in K \times K, \forall x \in V, \Phi(a+b,x) = \Phi(a,x) + \Phi(b,x)$
- $\forall a \in K, \forall (x,y) \in V \times V, \Phi(a,x+y) = \Phi(a,x) + \Phi(a,y)$
- $\forall x \in V, \Phi(1,x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

5.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a,b) \mapsto a+b$
- $K \times K \to K$
- $(a,b) \mapsto ba$

Then K^{op} forms a unitary ring
Any left $K^{op}-module$ is a right K-module
Any right $K^{op}-module$ is a left K-module
$(K^{op})^{op}=K$

5.1.3 Notation

When we talk about a left/right K-module $(V,+)$,we often write its left K-module structure as $K\times V\to V \quad (a,x)\mapsto ax$

The axioms become:

$$\begin{aligned}
&\forall(a,b)\in K\times K,\forall x\in V \quad (ab)x=a(bx)/b(ax)\\
&\forall(a,b)\in K\times K,\forall x\in V \quad (a+b)x=ax+bx\\
&\forall a\in K,\forall(x,y)\in V\times V \quad a(x+y)=ax+ay\\
&\forall x\in V \quad 1x=x
\end{aligned}$$

5.1.4 K-vector space

If K is commutative,then $K^{op}=K$,so left K-module and right K-module structure are the same .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space

Let $\Phi:K\times V\to V$ be a left or right K-module structure

$$\forall x\in V,\Phi(\cdot,x):K\to V \quad (a\in K)\mapsto\Phi(a,x)$$

is a morphism of addition groups.Hence $\Phi(0,x)=0,\Phi(-a,x)=-\Phi(a,x)$
$\forall a\in K,\Phi(a,\cdot):V\to V$ is a morphism of groups.Hence $\Phi(a,0)=0,\Phi(a,-x)=-\Phi(a,x)(\cdot\ is\ a\ var)$

5.1.5 Association:

$\forall x\in K$

$$\begin{aligned}
(f(f+g)+h)(x)&=(f+g)(x)+h(x)=f(x)+g(x)+h(x)\\
(f+(g+h))(x)&=f(x)+((g+h)(x))=f(x)+g(x)+h(x)
\end{aligned}$$

Let $0:I\to K:x\mapsto 0 \quad \forall f\in K^I \quad f+0=f$
Let $-f:f+(-f)=0$
The mapping $K\times K^I\to K^I:(a,f)\mapsto af \quad (af)(x)=af(x)$ is a left K-module structure
The mapping $K\times K^I\to K^I:(a\in I)\mapsto((x\in I)\mapsto f(x)a) \quad (af)(x)=af(x)$ is a right K-module structure

5.1.6 Remark:

We can also write an element $\mu\ of\ K^I$ is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I\ by\ \mu$)
Then

$$(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I}$$
$$a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I}$$
$$(\mu_i)_{i\in I}a = (\mu_i a)_{i\in I}$$

5.2 sub K-module

5.2.1 Def

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub-K-module of V.

5.2.2 Example

Let I be a set .Let $K^{\bigoplus I}$ be the subset of K^I composed of mappings $f : I \to K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub-K-module of K^I

In fact,$\forall (f, g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$
Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of K^I
$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x\mapsto f(x)a)} \subseteq I_f$

5.3 morphism of K-modules

5.3.1 Def

Let V and W be left K-module, A morphism of groups $\phi : V \to W$ is called a morphism of left K-modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

5.3.2 K-linear mapping

If K is commutative, a morphism of K-modules is also called a K-linear mapping. We denote by $\hom_{K-Mod}(V, W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of W^V

5.3.3 Theorem

Let V be a left K-module. Let I be a set.
The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I :\ \phi \to (\phi(e_i))_{i\in I}$ is a bijection where
$e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

5.3.4 Remark:column

In the case where $I = 1, 2, 3, ..., n$ V^I is denoted as V^n, K^I is denoted as K^n For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n \phi(e_i) = x_i$

We write this ϕ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$

5.4 kernel

5.4.1 Prop

Let G and H be groups and $f : G \to H$ be a morphism of groups

- $I_m(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

5.4.2 Def

$\ker(f)$ is called the kernel of f

5.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

5.4.4 Proof

Let e_G and e_H be neutral element of G and H respectively

(1) Let x and y be element of G
$f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H

(2) Let x and y be element of $\ker(f)$ One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H \quad e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$ So $\ker(f)$ is a subgroup of G

(3) Suppose that f is injection.
Since $f(E_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ Suppose that $\ker(f) = \{e_G\}$ If $f(x) = f(y)$then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$
Hence $xy^{-1} = e_G \Rightarrow x = y$

5.4.5 Def

Let (V,+) be a commutative group, I be a set. We define a composition law + on V^I as follows

$$(x_i)_{i\in I} + (y_i)_{i\in I} \ := (x_i + y_i)_{i\in I}$$

Then V^I forms a commutative group

5.4.6 Remark

Let E and F be left K-modules
$\hom_{K=Mod}(E, F) := \{\text{morphisms of left K-modules from E to F}\} \subseteq F^E$ is a subgroup of F^E
In fact f and g are elements of $\hom_{K-Mod}(E, F)$, then $f - g$ is also a morphism of left K-module
$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - y)(x)$

5.4.7 Theorem

Let V be a left K-module,I be a set The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I \ : \ \phi \mapsto (\phi(e_i))_i \in I$ is an isomorphism of groups, where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

5.4.8 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_I) + \psi(e_i)$
$\forall(\phi, psi) \in \hom_{K-Mod}(K^{\bigoplus I}, V)^2$
Hence $\Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i\in I} = \Psi(\phi) + \Psi(\psi)$
So Ψ is a morphism of groups

injectivity Let $\phi \in \hom_{K-Mod}(K^{\bigoplus I}, V)$ Such that $\forall i \in I(\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$
Let $a = (a_i)_{i\in I} \in K^{\bigoplus I}$ One has $a = \sum\limits_{i\in I} a_i e_i$
If fact,$\forall j \in I, a_j = \sum\limits_{i\in I, a_i \neq 0} a_i e_i(j)$
Thus $\phi(a) = \sum\limits_{i\in I, a_i \neq 0} a - I\phi(e_i) = 0$
Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i\in I} \in V^I$ We define $\phi_x : K^{\bigoplus I} \to V$ such that $\forall a = (a_i)_{i\in I} \in K^{\bigoplus I}, \phi_x(a) = \sum\limits_{i\in I, a_i \neq 0} a_i x_i$
This is a morphism of left K-modules
$foralli \in I, \phi_x(e_i) = 1x_i = x_i$ So $\Psi(\phi_x) = x$

Suppose that K' is a unitary ring,and V is also equipped with a right K'-module structure, Then $\hom_{K-Mod}(K^{\bigoplus I}, V) \subseteq V^{K^{\bigoplus I}}$ is a right sub-k'-module ,and Ψ in the theorem is a right K'-module isomorphism

Chapter 6

Monotone mappings

6.1 Def

Let I and X be partially ordered sets,$f : I \to X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)/f(a) < f(b)$,then we say that f is increasing/strictly increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

6.2 Prop.

Let X,Y,Z be partially ordered sets.$f : X \to Y, g : Y \to Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

6.3 Def

Let f be a function from a partially ordered set I to another partially ordered set .If $f \mid_{Dom(f)} \to X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

6.4 Prop.

Let I and X be partially ordered sets. f be function from I to X.

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing

- Assume that I is totally ordered and f is strictly monotone, then f is injection

6.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B
If f is increasing/decreasing ,then so is f^{-1}

6.6 Def

Let X and Y be partially ordered sets. $f : X \to Y$ be a bijection. If both f and f^{-1} are increasing ,then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \to Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

6.7 Prop.

Let I be a subset of $\mathbb{N}$ which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \to I$

6.8 Proof

6.8.1 bijection

We construct $f : \mathbb{N} \to I$ by induction as follows.
Let $f(0) = \min I$ Suppose that $f(0), ..., f(n)$ are constructed
then we take $f(n+1) := \min(I\backslash\{f(0), ..., f(n)\})$
Since $I\backslash\{f(0), ..., f(n-1)\} \supseteq I\backslash\{f(0), ..., f(n)\}$.Therefore $f(n) \le f(n+1)$
$Since f(n+1) \notin \{f(0), ..., f(n)\}$,we have $f(n) < f(n+1)$
Hence f is strictly increasing and ths is injective
If f is not surjective,then $I\backslash Im(f)$ has a element N.
Let $m = \min\{n \in \mathbb{N} \mid N \le f(n)\}$.
Since $N \notin Im(f), N < f(m)$.
So $m \ne 0$.Hence $f(m-1) < N < f(m) = \min(I\backslash\{f(0), ..., f(m-1)\})$
By definition,$N \in I\backslash Im(f) \subseteq I\backslash\{f(0), ..., f(m-1)\}$,
Hence $f(m) \le N$,causing contradiction.

6.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from $\mathbb{N}$ to $\mathbb{N}$

Chapter 7

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

7.1 Def

Let X be a set.We call sequence in X parametrized by I a mapping from I to X.

7.2 Remark

If K is a unitary ring and E is a left K-module then the set of sequence E^I admits a left-K-module structure. If $x = (x_n)_{n\in I}$ is a sequence in E, we define a sequence $\sum(x) := (\sum\limits_{i\in I, i\leq n} x_i)_{n\in\mathbb{N}}$,called the series associated with the sequence x.

7.3 Prop

$\sum : E^I \to E^{\mathbb{N}}$ is a morphism of left-K-module

7.4 proof

Let $x = (x_i)_{i\in I}$ and $y = (y_i)_{i\in I}$ be elements of E^I
$\sum\limits_{i\in I, i\leq n}(x_i + y_i) = (\sum\limits_{i\in I, i\leq n} x_i) + (\sum\limits_{i\in I, i\leq n} y_i), \lambda \sum\limits_{i\in I, i\leq n} x_i = \sum\limits_{i\in I, i\leq n} \lambda x_i$

7.5 Prop

Let I be a totally ordered set . X be a partially ordered set,$f : I \to X$ be a mapping ,$J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

7.6 limit

7.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset.$\forall (x_i)_{n\in I} \in [-\infty,+\infty]^I$ where $[-\infty,+\infty]$ denotes $\mathbb{R}\cup\{-\infty,+\infty\}$,we define:

$$\limsup_{n\in I, n\to+\infty} x_n := \inf_{n\in I}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n := \sup_{n\in I}(\inf_{i\in I, i\geq n} x_i)$$

If $\limsup_{n\in I, n\to+\infty} x_n = \liminf_{n\in I, n\to+\infty} x_n = l$, we then say that $(x_n)_{n\in I}$ tends to l and that l is the limit of $(x_n)_{n\in I}$. If in addition $(x_n)_{n\in I}\in\mathbb{R}^I$ and $l\in\mathbb{R}$,we say that $(x_n)_{n\in I}$ converges to l

7.6.2 Remark

If $J\subseteq\mathbb{N}$ is an infinite subset, then:

$$\limsup_{n\in I, n\to+\infty} = \inf_{n\in J}(\sup_{i\in I, i\geq n} x_i)$$

$$\liminf_{n\in I, n\to+\infty} x_n = \sup_{n\in J}(\inf_{i\in I, i\geq n} x_i)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i\in I}$ the limit superior and the limit inferior do not change.
In fact, if we take $J=\mathbb{N}\backslash\{0,...,m\}$, then $\inf_{n\in J}(...)$ and $\sup_{n\in J}(...)$ only depends on the values of $x_i, i\in I, i\geq m$

7.6.3 Prop

$\forall (x_n)_{n\in I}\in[-\infty,+\infty]^I$, $\liminf_{n\in I, n\to+\infty} x_n \leq \limsup_{n\in I, n\to+\infty} x_n$

7.6.4 Prop

Let $(x_n)_{n\in I} \in [-\infty, +\infty]^I$

$$\forall c \in \mathbb{R} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty}(x_n + c) &= (\limsup_{n\in I, n\to+\infty} x_n) + c \\ \liminf_{n\in I, n\to+\infty}(x_n + c) &= (\liminf_{n\in I, n\to+\infty} x_n) + c \end{aligned}$$

$$\forall c \in \mathbb{R}_{>0} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \end{aligned}$$

$$\forall c \in \mathbb{R}_{<0} \quad \begin{aligned} \limsup_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \liminf_{n\in I, n\to+\infty} x_n \\ \liminf_{n\in I, n\to+\infty}(\lambda x_n) &= \lambda \limsup_{n\in I, n\to+\infty} x_n \end{aligned}$$

7.6.5 Prop

Let $(x_n)_{n\in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$,one has $x_n \leq y_n$ Then

$$\limsup_{n\in I, n\to+\infty}(x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

,

$$\liminf_{n\in I, n\to+\infty}(x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

7.6.6 Theorem

Let $(x_n)_{n\in I}, (y_n)_{n\in I}, (z_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$
Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n\in I}$ and $(z_n)_{n\in I}$ tend to the same limit l

Then $(y_n)_{n\in I}$ tends to l

7.6.7 Def

Let I be an infinite subset of $\mathbb{N}$, and $(x_n)_{n\in I}$ be a sequence in some set X. We call subsequence of $(x_n)_{n\in I}$ a sequence of the form $(x_n)_{n\in J}$,where J is an infinite subset of I

7.6.8 Prop

Let I and J be infinite subset of $\mathbb{N}$ such that $J \subseteq I$
$\forall (x_n)_{n\in I} \in [-\infty, +\infty]^I$,one has

$$\liminf_{n\in I, n\to+\infty}(x_n) \leq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in I,n\to+\infty}(x_n)\geq\limsup_{n\in I,n\to+\infty}y_n$$

In particular, if $(x_n)_{n\in I}$ tends to $l\in[-\infty,+\infty]$,then $(x_n)_{n\in J}$ tends to l

7.6.9 Prop

$\forall n\in\mathbb{N}$,one has

$$\liminf_{n\in J,n\to+\infty}(x_n)\geq\liminf_{n\in I,n\to+\infty}y_n$$

$$\limsup_{n\in J,n\to+\infty}(x_n)\leq\limsup_{n\in I,n\to+\infty}y_n$$

7.6.10 Theorem

Let $I\subseteq\mathbb{N}$ be an infinite subset and $(x_N)_{n\in I}$ be a sequence in $[-\infty,+\infty]$

- If the mapping $(n\in I)\mapsto x_n$ is increasing,then $(x_N)_{i\in I}$ tends to $\sup\limits_{n\in I}x_n$
- If the mapping $(n\in I)\mapsto x_n$ is decreasing,then $(x_N)_{i\in I}$ tends to $\inf\limits_{n\in I}x_n$

7.6.11 Notation

If a sequence $(x_N)_{n\in I}\in[-\infty,+\infty]$ tends to some $l\in[-\infty,+\infty]$ the expression $\lim\limits_{n\in I,n\to}x_n$ denotes this limit l

7.6.12 Corollary

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{N}_{\geq 0}$ Then the series $\sum\limits_{n\in I}x_n$(the sequence $(\sum\limits_{i\in I,i\leq n})_{n\in\mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0}\cup\{+\infty\}$ It converges in $\mathbb{R}$ iff it is bounded from above (namely has an upper bound in $\mathbb{R}$)

7.6.13 Notation

If a series $\sum\limits_{n\in I}x_n$ in $[-\infty,+\infty]$ tends to some limit, we use the expression $\sum\limits_{n\in I}x_n$ to denote the limit

7.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n\in I}$ be a sequence in $[-\infty,+\infty]$ There exists a subsequence of $(x_n)_{n\in I}$ that tends to $\limsup\limits_{n\in I,n\to+\infty}x_n$ There exists a subsequence of $(x_n)_{n\in I}$ that rends to $\liminf\limits_{n\in I,n\to+\infty}x_n$

7.6.15 Proof

Let $J = \{n \in I \mid \forall m \in I, \text{if } \ m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_N)_{n\in J}$ is decreasing so it tends to $\inf\limits_{n\in J} x_n$

$\forall n \in J$ by definition $x_n = \sup\limits_{i\in I, i\geq n} x_i$ so $\limsup\limits_{n\in I, n\to+\infty} x_n = \inf\limits_{n\in J} \sup\limits_{i\in I, i\geq n} x_i = \inf\limits_{n\in J} x_n = \lim\limits_{n\in J, n\to+\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$.Denote by $l = \sup\limits_{n\in I, n\geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup\limits_{i\in I, i\geq n_0} x_i \leq l$. If the strict inequality $\sup\limits_{i\in I, i\geq N} x_i < l$ holds, then $\sup\limits_{i\in I, i\geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup\limits_{i\in I, i\geq N} x_i$ We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality. Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$ Thus $n \in J$ that contradicts the maximality of n_0 Therefore $l = \sup\limits_{i\in I, i\geq N} x_i$, which leads to $\limsup\limits_{n\in I, n\to+\infty} x_n = l$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$,so $x_m < l$(since otherwise $x_m = \sup\limits_{i\in I, i\geq m} x_i$ and hence $m \in J$)Hence,$\forall$ *finite subset* I' *of*$\{m \in I \mid m \geq n_0\}$

$\max\limits_{i\in I} x_i < l$ and hence $\exists n \in I$,such that $n > \max I'$,and $\max\limits_{i\in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \to I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$ Hence the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is increasing, and $\sup\limits_{j\in\mathbb{N}} x_{n_j} \leq \sup\limits_{j\in\mathbb{N}} x_{f(j)} = \sup\limits_{n\in I, n\geq n_0} x_n = l$

$l = \sup\limits_{n\in I, n\geq n_0}$

So $(x_{n_j})_{j\in\mathbb{N}}$ tends to l

Chapter 8

Cauchy sequence

8.1 Def

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{R}$
If $\inf_{N\in\mathbb{N}} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = \lim_{N\to+\infty} \sup_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = 0$ then we say that $(x_n)_{n\in I}$ is a Cauchy sequence

8.2 Prop

- If $(x_n)_{i\in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_N)_{i\in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite,$(x_n)_{n\in I}$ is a Cauchy sequence.
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n\in I}$ converges to l, then $(x_n)_{n\in J}$ converges to l too.

8.3 Theorem: Completeness of real number

If $(x_n)_{n\in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in $\mathbb{R}$

8.3.1 Proof

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$ So $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n\in I}$ converges to the same limit.

8.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

8.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in $\mathbb{R}$

Chapter 9

Comparison and Technics of Computation

9.1 Def

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be sequence in $\mathbb{R}$

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_N| \leq M|y_m|$ then we write $x_n = O(y_n), n \in I, n \to +\infty$
- If there exists $(\epsilon_n)_{n\in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim\limits_{n\in I, n\to+\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_N| \leq |\epsilon y_m|$, then we write $x_n = \circ(y_n), n \in I, n \to +\infty$
  Example:
  $$\lim_{n\to+\infty} \frac{1}{n} = 0$$

9.2 Prop.

Let I and X be partially ordered sets and $f : I \to X$ be an increasing/decreasing mapping. Let J ba a subset of I. Assume that any elements of I has an upper bound in J. Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

9.3 Theorem

Let I be a totally ordered set, $f : I \to [-\infty, +\infty]$ and $g : I \to [-\infty, +\infty]$ be two mappings that are both increasing/decreasing.Then the following equalities holds,provided that the sum on the right hand side of the equality is wel defined.

$$\sup_{x\in I, \{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\sup_{x\in I} f(x)) + (\sup_{y\in I} g(y))$$

$$\inf_{x\in I,\{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\inf_{x\in I} f(x)) + (\inf_{y\in I} g(y))$$

9.3.1 Proof

We can assume f and g increasing. Let $a = \sup f(I)$,$b = \sup g(I)$
Let $A = \{(x,y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$
We equip A with the following order relation.

$$(x,y) \leq (x',y') \ iff \ x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x,y) \in A \mid x = y\}$.
Consider

$$h : A \to [-\infty, +\infty] \quad h(x,y) = f(x) + g(y)$$

h is increasing.
Let $(x,y) \in A$. Assume that $x \leq y$
If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y,y) \in B$ and $(x,y) \leq (y,y)$
If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x,y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$
So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined,$(z,z) \in B$ and $(x,y) \leq (z,z)$ Similarly, if$x \geq y$, (x,y) has also an upper bound in B. Therefore: $\sup h(A) = \sup h(B)$

9.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty} (x_n + y_n) \leq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty} (x_n + y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

9.4.1 Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n\in I, n\geq N} x_n \quad B_N = \sup_{n\in I, n\geq N} y_n$. $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are decreasing, and $\limsup_{n\in I, n\to+\infty} x_n = \inf_{N\in\mathbb{N}} A_N \quad \limsup_{n\in I, n\to+\infty} y_n = \inf_{N\in\mathbb{N}} B_N$
By theorem:

$$\inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N = \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n\in I, n\geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.
Therefore

$$\inf_{N\in\mathbb{N}} C_N \leq \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N) = \inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N$$

9.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty,+\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) \geq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

9.5.1 Proof

a tricky proof ?:

$$\limsup_{n\in I, n\to} x_n = \limsup_{n\in I, n\to}(x_n+y_n-y_n) \leq \limsup_{n\in I, n\to}(x_n+y_n) - \liminf_{n\in I, n\to} y_n$$

to have a true proof,only need to discuss conditions with ∞

9.6 Theorem

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n\in I}$ converges to some $i \in \mathbb{R}$.
Then:

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) = (\limsup_{n\in I, n\to+\infty} x_n) + l$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) = (\liminf_{n\in I, n\to+\infty} x_n) + l$$

9.7 Prop.

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Then:

$$\liminf_{n\in I, n\to+\infty} \max\{x_n, y_n\} = \max\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

$$\liminf_{n\in I, n\to+\infty} \min\{x_n, y_n\} = \min\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

9.7.1 Proof

About the first inequality.Since $\max\{x_n, y_n\} \geq x_n \quad \max\{x_n, y_N\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n\in J, n\to+\infty} = \limsup_{n\in J, n\to+\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $\quad J_1 = \{n \in J \mid x_n \leq y_n\}$
$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite
Suppose that J_1 is infinite, then

$$\lim_{n\in J, n\to} \max\{x_n, y_n\} = \lim_{n\in J_1, n\to} \max\{x_n, y_n\} = \lim_{n\in J, n\to} x_n \leq \limsup_{n\in I, n\to+\infty} x_n$$

If J_2 is infinite

$$\limsup_{n\in I, n\to+\infty} = \lim_{n\in J_2, n\to+\infty} \max\{x_n, y_n\} \leq \limsup_{n\in I, n\to+\infty} y_n$$

9.8 Theorem

Let $(a_N)_{n\in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_N)_{n\in I}$ converges to l
- $\limsup_{n\in I, n\to+\infty} |a_n - l| = 0$

9.8.1 Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$
$$\limsup_{n\in I, n\to+\infty} |a_n - l| = \max\{(\limsup_{n\in I, n\to+\infty} a_n) - l, l - (\liminf_{n\in I, n\to+\infty} a_n)\}$$

(1) $\Rightarrow$ (2):
If $(a_n)_{n\in I}$ converges to l, then $\limsup_{n\in I, n\to+\infty} a_n = \liminf_{n\in I, n\to+\infty} a_n = l$

(2) $\Rightarrow$ (1):
If $\limsup_{n\in I, n\to+\infty} |a_n - l| = 0$,then $\limsup_{n\in I, n\to+\infty} a_n \leq l \leq \liminf_{n\in I, n\to+\infty} a_n$
Therefore: $\limsup_{n\in I, n\to+\infty} a_n = \liminf_{n\in I, n\to+\infty} a_n = l$

9.9 Remark

Let $(a_n)_{n\in I}$ be a sequence in $\mathbb{R}$, $l \in \mathbb{R}$
The sequence $(a_n)_{n\in I}$ converges to l iff $a_n - l = o(1), n \in I, n \to +\infty$

9.10 Calculates on O(),o()

9.10.1 Plus

Let $(a_n)_{n\in I}$ $(a'_n)_{n\in I}$ and $(b_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \to +\infty$
  then $\forall(\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \to +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \to +\infty$
  then $\forall(\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \to +\infty$

9.10.2 Transform

Let $(a_n)_{n\in I}$ and $(b_n)_{n\in I}$ be two sequence in $\mathbb{R}$ If $a_n = o(b_n), n \in I, n \to +\infty$,then $a_n = O(b_n), n \in I, n \to +\infty$

9.10.3 Transition

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ and $(c_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = O(c_n), n \in I, n \to +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$

9.10.4 Times

Let $(a_n)_{n\in I}, (b_n)_{n\in I}, (c_n)_{n\in I}, (d_n)_{n\in I}$ be sequences in $\mathbb{R}$

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = O(b_n d_n), n \in I, n \to +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = o(b_n d_n), n \in I, n \to +\infty$

9.11 On the limit

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ be elements of $\mathbb{R}^I$ that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n\in I}$ converges to $l + l'$
- $(a_n b_n)_{n\in I}$ converges to ll'

9.12 Prop

Let $a \in \mathbb{R}$ THen $a^n = o(n!) \quad n \to +\infty$

9.12.1 Proof

Let $N \in \mathbb{N}$ such that $|a| < N$For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot frac|a^n - N|\frac{n!}{N!} \leq \frac{|a^N|}{N!}(\frac{|a|}{N})^n - N$$

And $0 < \frac{|a|}{<} 1 \Rightarrow \lim_{n \to +\infty} (\frac{|a|}{N})^n = 0$. Therefore:

$$\lim_{n \to +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

9.13 Prop

$n! = o(n^n) \quad n \to +\infty$

9.13.1 Proof

Let $N \in \mathbb{N}_{\geq 1}$
$0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \to +\infty} \frac{n!}{n^n} = 0$

9.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of $\mathbb{R}^I$ If the series $\sum_{n \in I} b_n$ converges absolutely and if $on = O(b_n) \quad n \to +\infty$Then $\sum_{n \in I} a_n$ converges absolutely

9.14.1 Proof

By definition $\sum_{n \in I} |b_N| < +\infty$ If $|a_N| \leq M|b_N|$ fro $n \in I, n \geq N$ where $N \in \mathbb{N}$ Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

9.15 Theorem: d'Alembert ratio test

Let $(a_N)_{n \in \mathbb{N}} \in (\mathbb{R} \backslash \{0\})^{\mathbb{N}}$

- If $\limsup_{n \to +\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \to +\infty} |\frac{a_{n+1}}{a_n}| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

9.15.1 Proof

(1)

Let $\alpha \in \mathbb{R}$ such that $\limsup\limits_{n\to+\infty}|\frac{a_{n+1}}{a_n}| < \alpha < 1$, $alpha$ isn't a lower bound of $(\sup\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|)_{N\in\mathbb{N}}$
So $\exists N \in \mathbb{N}$ such that $\sup\limits_{n\geq N}|\frac{a_{n+1}}{a_n}| < \alpha$Hence for $n \geq N \quad |a_n| \leq \alpha^{n-N}|a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N}\frac{a_{N+2}}{a_{N+1}}...\frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum\limits_{n\in\mathbb{N}} = \frac{1}{1-\alpha} < +\infty$, $\sum\limits_{n\in\mathbb{N}} a_n$ converge absolutely.

9.15.2 Lemma

If a series $\sum\limits_{n\in\mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim\limits_{n\to+\infty} a_n = 0$

Proof

If $(\sum\limits_{i=0}^{n} a_i)_{n\in\mathbb{N}}$ converges to some $l \in \mathbb{R}$,then $(\sum\limits_{i=0}^{n-1} a_i)_{n\in\mathbb{N},n\geq 1}$ converges to l, too. Hence $\left(a_n = \left(\sum\limits_{i=0}^{n} a_i\right) - \left(\sum\limits_{i=0}^{n-1} a_i\right)\right)_{n\in\mathbb{N}}$ converges to $l - l = 0$

9.15.3 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf\limits_{n\to+\infty}|\frac{a_{n+1}}{a_n}| = \sup\limits_{N\in\mathbb{N}} \inf\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|$
So there exists $N \in \mathbb{N}$ such that $\beta < \inf\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|$
$\forall n \in \mathbb{N}, n \geq N \quad |\frac{a_{n+1}}{a_n}| \geq \beta$
Hence $(|a_n|)_{n\in\mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N}|a_n|$
By the lemma: $\sum\limits_{n\in\mathbb{N}} a_n$ diverges.

9.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \to +\infty$

9.16.1 Proof

Let $\epsilon > 0$ such that $a = (1+\epsilon)^2$

$$a^n = (1+\epsilon)^{2n} = (1+\epsilon)^n(1+\epsilon)^n \geq (1+n\epsilon)(1+n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

9.16.2 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \to +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

9.16.3 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

9.17 Theorem: Cauchy root test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $\alpha = \limsup_{n\to+\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n\in\mathbb{N}} a_n$ converges absolutely.
- If $a > 1$ then $\sum_{n\in\mathbb{N}} a_n$ diverges

9.17.1 Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_N|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n\in\mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \quad \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N}\ \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n\in\mathbb{N}}$ cannot converge to 0.

Part III

Topology

Chapter 10

Absolute value and norms

10.1 Def

Let K be a field . By absolute value on K, we mean a mapping $|\cdot| : K \to \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\forall a \in K \quad |a| = 0$ iff $a = 0$

(2) $\forall (a,b) \in K^2 \quad |ab| = |a| \cdot |b|$

(3) $\forall (a,b) \in K^2 \quad |a+b| \leq |a| + |b|$(triangle inequality)

10.2 Notation

$\mathbb{Q}$ Take a prime num p $\forall \alpha \in \mathbb{Q}\backslash\{0\}$ there exists a integer $ord_p(\alpha)\frac{a}{b}$, where $\begin{matrix} a \in \mathbb{Z}\backslash\{0\} \\ b \in \mathbb{N}\backslash\{0\} \end{matrix}, p \nmid a, p \nmid b$

10.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \to \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & if\ \alpha \neq 0 \\ 0 & if\ \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on $\mathbb{Q}$

10.3.1 proof

(1) Obviously

(2) If $\alpha = p^{ord_p(\alpha)}\frac{a}{b}, \beta = p^{ord_p(\beta)}\frac{c}{d} \quad p \nmid abcd$
$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)}\frac{ac}{bd} \quad p \nmid ac, p \nmid bd$

(3) $\alpha + \beta = p^{ord_p(\alpha)}\frac{a}{b} + p^{ord_p(\beta)}\frac{c}{d}$
Assume $ord_p(\alpha) \geq ord_p(\beta)$
$\alpha + \beta$
$= p^{ord_p(\beta)}\left(p^{ord_p(\alpha)-ord_p(\beta)}\frac{a}{b} + \frac{c}{d}\right)$
$= p^{ord_p(\beta)}\frac{p^{ord_p(\alpha)-ord_p(\beta)}ad+bc}{bd} \quad p \nmid bd$
So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

Hence $ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$
So $|\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$
$\max\{|\alpha|_p, |\alpha|_p\} \leq |\alpha|_p, |\alpha|_p$

Chapter 11

Quotient Structure

11.1 Def

Let X be a set and $\sim$ be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that $\sim$ is an equivalence relation

11.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X.Let $X/\sim$ be the set $\{[x] \mid x \in X\}$

11.3 Prop.

Let X be a set and $\sim$ be an equivalence relation on X

(1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$

(2) If α and β are elements of $X/\sim$ such that $\alpha \neq \beta$ then $\alpha \cap \beta = \varnothing$

(3) $X = \bigcup\limits_{\alpha \in X/\sim} \alpha$

11.3.1 Proof

(1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ on has $x \sim y$
Therefore ,$x \sim z$ namely $z \in [x]$. This proves $y[] \subseteq [x]$. Moreover ,since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$

(2) Suppose that $\alpha \cap \beta \neq \varnothing, y \in \alpha \cap \beta$
By (1),$\alpha = [y], \beta = [y]$, Thus leads to a contradiction.

(3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup_{\alpha \in X/\sim} \alpha$Hence $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$.Conversely, $\forall \alpha \in X/\sim, \alpha$ is a subset of X. Hence $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$.Then $X = \bigcup_{\alpha \in X/\sim} \alpha$

11.4 Def

Let G be a group and X be a set
We call left/right action of G on X ant mapping $G \times X \to X : (g, x) \mapsto gx/(g, x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x \ / \ x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x \ / \ (xg)h = x(gh)$

11.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \to G$$

$$(g, h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X.

11.6 Prop

Let G be a group and X be a set . Assume given a left action of G on X. Then the binary relation $\sim$ on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

11.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim$ $\forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

11.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x.(y \sim x)$
- $\forall (x, y, z) \in X^3$,if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

11.9 Quotient set

Let X be a set and $\sim$ be an equivalence relation, the mapping $X \to X/\sim$: $(x \in X) \mapsto [x]$ is called the projection mapping.
$X/\sim$ is called the quotient set of X by equivalence relation $\sim$

11.9.1 Example

Let G be a group and H be a subgroup of G.Then the mapping

$$H \times G \to G$$

$$(h, g) \mapsto hg/(h, g) \mapsto gh$$

is a left/right action of H on G. Thus we obtain two quotient sets H/G and G/H

11.10 Def

Let G be a group and H be a subgroup of G. Ig $\forall g \in G, h \in H \quad ghg^{-1} \in H$, Then we say that H is a normal subgroup of G

11.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G. In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

11.12 Prop

If G is commutative, any subgroup of G is normal

11.13 Theorem

Let G be a group and H be a normal subgroup of G. Then the mapping

$$G/H \times H/G \to G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H Moreover the projection mapping

$$\pi : G \to G/H$$

$$x \mapsto xH$$

is a morphism of groups.

11.13.1 Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$ Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.

- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH)$

- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$

- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

11.14 Def

Let K be a unitary ring and E be a left K-module. We say that a subgroup F og $(E, +)$ is a left sub-K-module of E if $\forall (a, x) \in K \times F, ax \in F$

11.15 Prop

Let K be a unitary ring , E be a left K-module and F be a sub-K-module. Then the mapping

$$K \times (E/F) \to E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined , and defines a left-K-module structure on E/F. Moreover, the projection mapping $pi : E \to E/F$ is a morphism of left-K-modules

11.15.1 Proof

Let x and x' be elements of E such that $[x] = [x']$, that meas: $x' - x \in F$ Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K-module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K \quad \pi(ax) = [ax] = a[x] = a\pi(x)$

11.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$ (I is a left and right sub-K-module of A)

11.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \to A/I$$

$$([a], [b]) \mapsto [ab]$$

si well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

11.17.1 Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$ Therefore $a'b' \sim ab$

11.17.2 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of $\mathbb{Z}$
If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z} \quad adm = dma \in d\mathbb{Z}$
Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the reside class of n modulo d
If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

11.18 Theorem

Let $f : G \to H$ be a morphism of groups

(1) $Im(f)$ is a subgroup of H

(2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G

(3) The mapping

$$\begin{aligned} \widetilde{f} :& G/Ker(f) \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

(4) f is injective iff $\ker(f) = \{1_G\}$

11.18.1 Proof

(1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup

(2) Let x and y be elements of $\ker(f)$.
One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
Let $x \in \ker f, y \in G$.
One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup

(3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
Moreover $\widetilde{f}([x][y]) = \widetilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence $\widetilde{f}$ is a morphism of groups.
By definition $Im(\widetilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y, x \sin y$ that means $[x] = [y]$
Therefore $\widetilde{f}$ is injective.

(4) If f is injective ,$\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f\{1_G\}$
Conversely , suppose that $\ker f = \{1_G\} \quad \forall (x,y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$

11.19 Theorem

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Then

(1) $Im(f)$ is a left-sub-K-module of F

(2) $\ker(f)$ is a left-sub-K-module of E

(3) $\begin{aligned} \widetilde{f} :& E/\ker f \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$ is a isomorphism of left K-modules

11.19.1 Proof

(1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in Im(f)$

(2)

(3)

Chapter 12

Topology

12.1 Def

Let X be a set. We call topology on X any subset $\mathcal{J}$ of $\wp(x)$ that satisfies:

- $\varnothing \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitraty family of elements in $\mathcal{J}$, then $\bigcup\limits_{i \in I} u_i \in \mathcal{J}$
- If u and v are elements of $\mathcal{J}$, then $u \cap v \in \mathcal{J}$

12.2 Remark

If $(u_i)_i^n = 1$ is a finite family of elements of $\mathcal{J}$, then $\bigcap\limits_{i=1}^{n} u_i \in \mathcal{J}$(by induction, this follows from (3))

12.2.1 Example

$\{\phi, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

12.3 Def

Let X be a set. We call metric on X any mapping $d : \begin{array}{c} X \times X \to \mathbb{R}_{\geq 0} \\ (x, y) \mapsto d(x, y) \end{array}$,that satisfies

- $d(x, y) = 0$ iff x=y
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$(triangle inequality)

(X, d) is called a metric space

12.3.1 Example

Let X be a set

$$d : X^2 \to \mathbb{R}_{\geq 0}$$
$$d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a metric

12.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) \leq \epsilon\}$ We call the open ball of radius ϵ centered at x

12.4.1 Example

Consider $(\mathbb{R}, d)$ with $d(x, y) = |x - y|$,then $B(x, \epsilon) =]x - \epsilon, x + \epsilon[$

12.5 Prop.

Let (X,d) be a metric space . let $\mathcal{J}_d$ be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x, \epsilon) \subseteq U$ THen $\mathcal{J}_d$ is a topology on X

12.5.1 Proof

- $\varnothing \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of $\mathcal{J}_d$ Let $U = \bigcup_{i \in I} u_i$, $\forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, y) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of $\mathcal{J}_d$ Let $x \in U \cap V$ $\exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$ Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

12.6 Def

$\mathcal{J}_d$ is called the topology induced by the metric d

12.7 Def

We call topology space any pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a topology on X

Given a topological space $(X, \mathcal{J})$ If $U \in \mathcal{J}$ then we say that U is an open subset of X. If $F \in \wp(X)$ such that $X \backslash F \in \mathcal{J}$, then we say that F is closed subset of X
If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that $\mathcal{J}$ is metrizable

12.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact,m if d denote the metric defined as $d(x, y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$
$\forall x \in X \quad B(x, 1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup\limits_{x \in A} \{x\} \in \mathcal{J}_d$

12.8 Axiom of choice

For any set I and any family $(A_i)_{i \in I}$ of non-empty sets , there exists a mapping $f : I \to \bigcup\limits_{i \in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

12.9 Def

Let $(X, \leq)$ be a partially ordered set If $\forall A \subseteq X$ A is non-empty , there exists a least element of A then we say that $(X, \leq)$ is a well ordered set.

12.10 Theorem

For any set X, there exists an order relation $\leq$ on such that $(X, \leq)$ forms a well ordered set.

12.11 Zorn's lemma

Let $(X, \leq)$ be a partially ordered set . If $\forall A \subseteq X$ that is totally ordered with respect to $\leq$, there exists an upper bound of A inside X. Then , there exists a maximal element x_0 of $X(\forall y \in X, y > x_0$ does not hold)

12.12 Prop.

Let $(X, \leq)$ be a well ordered set , $y \notin X$. We extends $\leq$ to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

12.13 Proof

Let $A \subseteq X \cup \{y\}, A \neq \varnothing$. If $A = \{y\}$ then Y is the least element of A. If $A \neq \{y\}$ then $B = A \backslash \{y\}$ is non-empty. Let b be the least element of B. Since $b < y$ it's also the least element of A

12.14 Def

Let $(X, \leq)$ be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S(X_{<s} \subseteq S)$, then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

12.15 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X.

12.16 Prop.

Let $(X, \leq)$ be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X, then $\bigcup\limits_{i \in I} S_i$ is an initial segment of X

12.17 Proof

$\forall s \in \bigcup\limits_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup\limits_{i \in I} S_i$

12.18 Prop.

Let $(X <\leq)$ be a well erodered set.

(1) Let S be a proper initial segment of X, $x = \min(X \backslash S)$ Then $S = X_{<x}$

(2) $X \to \wp(X)$
$x \mapsto X_{<x}$

(3) The set of all initial segments of X forms a well ordered subset of $(\wp(x), \subseteq)$

12.19 Proof

(1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction.Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \backslash S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

(2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \backslash X_{<x}$ So $X_{<x} \subsetneqq X_{<y}$

(3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \varnothing$. Then there exists $A \subseteq X$ such that $\mathcal{F} \backslash \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \varnothing$ then $\mathcal{F} = \{X\}$, and $\{X\}$is the least element of $\mathcal{F}$. Otherwise $A \neq \varnothing$ and A has a least element a. Then by(2) $X_{<}a$ is the least element of $\mathcal{F}$

12.20 Lemma

Let $(X, \leq)$ be a well ordered set, $f : X \to X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

12.20.1 Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \varnothing$, let a be the least element of A. By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing . This shows $f(a) \in A$. But a is the least element of A, $f(a) < a$ cannot hold: contradiction.

12.21 Prop

Let $(X, \leq)$ be a well ordered set, S and T be two initial segment of X . If $f : S \to T$ is a bijection that's strictly increasing , then $S = T, f = Id_S$

12.21.1 Proof

We may assume $T \subseteq S$.Let $l : T \to S$ be the induction mapping and $g = l \circ f : S \to S$. Since g is strictly increasing , by the lemma ,$\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$
Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$