Contents

Chapter 1

Preface

1.1 Ref

- Ahlfors: Complex analysis.
- 谭小江, 伍胜健复变函数简明教程
- Stein,? complex analysis.(extra exercises)

1.2 A brief history of complex analysis

Complex analysis refers studies on functions of complex variables, emerged in the 19th century. Cauchy proposed Cauchy 's integral theorem (1825) and the concept of residues. Riemann defined the Riemann Surface, which enlarge complex analysis to geometry field. Besides, he defined Riemann zeta function. And he gave Riemann mapping theorem. Weirstrass use power series to approach complex analysis.

Complex analysis also deeply connects to other filed in math.

- It's essential to analysis geometry and complex geometry.
- Provide powerful tool to research prime numbers.
- In dynamics, complex dynamics is active.
- Deep connected with topology of 3-manifold.
- Deep connection with harmonic analysis(Fourier analysis).

Chapter 2

Definition of complex numbers

$\mathbb{R}$ denotes the real numbers. Some polynomials equation like $x^2 + 1 = 0$ has no solutions in $\mathbb{R}$. So we formally introduce the number i (an imaginary number) s.t.

$$i^1 + 1 = 0$$

A complex number $z = a + bi$, where $a, b \in \mathbb{R}$. Let

$$\mathbb{C} = \{z = a + bi \mid a, b \in \mathbb{R}\}$$

$\mathbb{C}$ is called complex plane. The real numbers a, b are called the real and imaginary part of z respectively. Denoted by $\Re z,\ \Im z$

Similar with to $\mathbb{R}$, we can define a field structure on $\mathbb{C}$.

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

To verify $\mathbb{C}$ a field, we need to show $\forall z \neq 0,\ \exists z^{-1}$

2.1 Def: complex conjugation

Let $z \in \mathbb{C}$. The complex conjugation $\overline{z}$ of $z = a + bi$ is

$$\overline{z} = a - bi$$

Ones can verify are

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{zw} = \overline{zw}$$

As a corollary, we consider a polynomial equation

$$a_n z^n + \cdots + a_0 = 0 \quad a_i \in \mathbb{C}$$

. If z is a root, then $\overline{z}$ a root for:

$$\overline{a_n} z^n + \cdots + \overline{a_0} = 0$$

In particular, $a_i \in \mathbb{R}$, then $\overline{z}$ is also a solution to original equation.

2.2 Def:absolute value

The absolute value of complex number z is defined as:

$$|z| := \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}$$

one can verify:

$$|zw| = |z| \cdot |w|$$
$$|z + w|^2 = |z|^2 + |w|^2 + 2\Re(z\overline{w})$$
$$|z - w|^2 = |z|^2 + |w|^2 - 2\Re(z\overline{w})$$

2.3 Def: division

Let $z_1, z_2 \in \mathbb{C}$

$$\frac{z_1}{z_2} := \frac{z_1 \overline{z_2}}{|z_2|^2}$$

In particular, if $z = a + bi$

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Chapter 3

Geometry picture of complex numbers

We can identify $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector space, by using $z = a + bi$. We can also use the polar coordinates write $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$, θ is called the argument of z. Then conjugation flip z along real axis. Addition is the same with vectors' addition. Multiplication multiplicate the length of vector and rotate the vector by the other's argument.

Consider the equation $z^n = 1,\ n \geq 1$. The solution of it is called n-th root of unity.

3.1 Some inequalities

By the definition of absolute value

$$-|z| \leq \Re z \leq |z|$$
$$-|z| \leq \Im z \leq |z|$$

The equality $\Re z = |z|$ iff z is a non-negative real number. Since $Re(z\overline{w}) \leq |z|\,|w|$ recall for $z, w \in \mathbb{C}$

$$|z + w|^2 = |z|^2 + |w|^2 + 2\Re(z\overline{w})$$

Then we get triangle inequality:

$$|z + w| \leq |z| + |w|$$

3.1.1 Cauchy's inequality

Let $n \geq 1$, then

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq (\sum_{k=1}^{n} |z_k|^2)(\sum_{k=1}^{n} |w_k|^2)$$

with the equality holds iff $\exists t \in \mathbb{C}, \forall 1 \leq k \leq n, z_k + t\overline{w_k} = 0$

Proof

Let $t \in \mathbb{C}$ be any complex number

$$0 \leq \sum_{k=1}^{n} |z_k + t\overline{w_k}|^2 = \sum_{k=1}^{n} |z_k|^2 + |t|^2 \sum_{k=1}^{n} |w_k|^2 + 2\Re(\bar{t} \sum_{k=1}^{n} z_k w_k)$$

choose $t = \frac{\sum_{k=1}^{n} z_k w_k}{\sum_{k=1}^{n} |w_k|^2}$ Then we get

$$\sum_{k=1}^{n} |z_k|^2 = \frac{\left|\sum_{k=1}^{n} z_k w_k\right|^2}{\sum_{k=1}^{n} |w_k|^2} \geq 0$$

The condition of equality $\Leftarrow$ the equality $0 = \sum_{k=1}^{n} |z_k + t\overline{w_k}|$

Chapter 4

Topology and metrics on $\mathbb{C}$

4.1 Basic definitions

Recall that a topology space is a set X equipped with a collection of subsets of X as open sets, satisfying:

- X and $\varnothing$ are open.
- Arbitrary union of open sets is open
- Finite intersection of open sets is open.

A closed set is by definition the complement of an open set.

A metric space is a pair (X, d), where X be a set and $d : X^2 \to \mathbb{R}_{\geq 0}$ a mapping s.t.

- $d(x, x) = 0 \quad \forall x \in X$
- $d(x, y) > 0 \quad \forall x \neq y \in X$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

let $x \in X, r > 0 \in \mathbb{R}$ the set

$$\mathcal{B}(x, r) := \{y \in X \mid d(x, y) < r\}$$

is called an open ball. We say a subset $N \subseteq X$ is a neighborhood of x if N contains an open ball centered at x. A subset N is open if $\forall x \in N$ N is a neighborhood of x

Remark

For any subset $N \subseteq X$ (N, d) is a metric space. The diameter of X:

$$diamX := \sup_{x,y \in X} d(x, y)$$

X is bounded if $diamX < +\infty$. A sequence of points x_n in X is called converges to $x \in X$ if $\lim_n \to +\infty d(x_n, x) = 0$. A sequence (x_n) is called Cauchy sequence if $\forall \epsilon > 0, \exists N \geq 1$ s.t. $\forall n > m \geq N, d(x_n, x_m) < \epsilon$

The metric space is called complete if any Cauchy sequence converges.

4.2 Notations

$N \subseteq X$ any subset.

- $\mathring{N}$ the interior of N, is the maximal open subset contained in N, i.e.

$$\mathring{N} = \text{ union of all open subsets in N}$$

- $\overline{N}$ the closure of N, the minimal closed set contains N.
- ∂N the boundary of N,

$$\partial := \overline{N} \setminus \mathring{N}$$

  let $N \subseteq X$. A point $x \in X$ is a limit point of N if $x \in \overline{N} \Leftrightarrow$ this means $\exists$ sequence (x_n) in N s.t. $x_n \to x$ $(\lim_{n \to +\infty} d(x_n, x) = 0)$

- We say $x \in X$ is called an isolated point if $\exists$ an open ball $\mathcal{B}(x, r)$ s.t.

$$\mathcal{B}(x, r) \cap X = \{x\}$$

- We say X is connected if X is not a disjoint union of non-empty open subsets.