Contents

Part I

Set

Chapter 1

product

1.1 direct sum

$\bigoplus$ is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{(x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0\}$$

Chapter 2

Ring

2.1 morphism

Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \to B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
-

algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \to R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \to R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n \in \mathbb{N}} a_n T^n$ Such element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_n T^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]

$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$

let $F + G = (a_0 + b_0) + ...$

$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n$

Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_i b_j) T^n\right) (\sum\limits_{n\in\mathbb{N}} c_n T^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_p b_q c_l\right) T^n$ is a trick applied on integral

Derivative:

let $F(T) \in k[T]$

We denote by $F'(T)$ *or* $\mathscr{D}(F(T))$ the formal power series

$\mathscr{D}(F) = \sum\limits_{n\in\mathbb{N}} (n+1) a_{n+1} T^n$

Properties:

- $\mathscr{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathscr{D}(FG) = F'G + FG'$

exp

We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!} T^n$,which fulfil the differential equation $\mathscr{D}(exp(T)) = exp(T)$(interesting)

Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall (i,j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 3

Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i \in I} is\ a\ family\ of\ element\ in\ X, then\ \sup\limits_{i \in I} x_i\ denotes\ \sup\{x_i \mid i \in I\}(in X)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup\limits_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

Chapter 4

Interval

We fix a totally ordered set $(X, \leq)$
Notation:
If $(a, b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$
Def:
Let $I \subseteq X$. If $\forall(x, y) \in I \times I$ with $x \leq y$,one has $[x, y] \subseteq I$ then we say that I is a interval in X
Example:
Let $(a, b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.
Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X
- If $\bigcap\limits_{\lambda \in \Lambda} I_\lambda \neq \varnothing$, $\bigcup\limits_{\lambda \in \Lambda} I_\lambda$ is a interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_] \mu$

- If $b \leq x \quad [a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a \quad [a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If a < x < b then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:
Let $(X, \leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.
Prop.
Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x, b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let $(X, \leq)$ be a totally ordered set .If $\forall (x, z) \in X \times X$,such that $x < z$ $\exists y \in X$ such that $x < y < z$,than we say that $(X, \leq)$ is thick

Prop.

Let $(X, \leq)$ be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a$ $\sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in]a, b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b[\subseteq]a, b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

Chapter 5

Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq$ *on* $\mathbb{R}$ *to* $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ become a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$ Obviously,this is a thick totally ordered set.
We define:

- $\forall x \in]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete

Example

Let Ω be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega)$, $\sup \mathscr{F} = \bigcup\limits_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$

$\mathscr{AXION}$:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ *and* $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \backslash \{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup\limits_{x \in I} f(x) \quad b = \sup\limits_{y \in J} g(y) \quad c = \sup\limits_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup\limits_{x \in I, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup\limits_{x \in I} f(x))(\sup\limits_{x \in I} g(x))$

inf takes the similar($\leq \to \geq$) (provided when the sum are defined)

Chapter 6

Vector space

In this section:
K denotes a unitary ring.
Let 0 be zero element of K
1 be the unity of K

6.1 K-module

6.1.1 Def

Let $(V, +)$ be a commutative group.We call left/right K-module structure:
any mapping Φ:$K \times V \to V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x))/\Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

6.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a, b) \mapsto a + b$
- $K \times K \to K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
Any left $K^{op}-module$ is a right K-module
Any right $K^{op}-module$ is a left K-module
$(K^{op})^{op} = K$

6.1.3 Notation

When we talk about a left/right K-module $(V,+)$,we often write its left K-module structure as $K \times V \to V \quad (a,x) \mapsto ax$

The axioms become:

$$\begin{aligned}
&\forall (a,b) \in K \times K, \forall x \in V \quad (ab)x = a(bx)/b(ax) \\
&\forall (a,b) \in K \times K, \forall x \in V \quad (a+b)x = ax + bx \\
&\forall a \in K, \forall (x,y) \in V \times V \quad a(x+y) = ax + ay \\
&\forall x \in V \quad 1x = x
\end{aligned}$$

6.1.4 K-vector space

If K is commutative,then $K^{op} = K$,so left K-module and right K-module structure are the same .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space

Let $\Phi : K \times V \to V$ be a left or right K-module structure

$$\forall x \in V, \Phi(\cdot, x) : K \to V \quad (a \in K) \mapsto \Phi(a,x)$$

is a morphism of addition groups.Hence $\Phi(0,x) = 0, \Phi(-a,x) = -\Phi(a,x)$
$\forall a \in K, \Phi(a,\cdot) : V \to V$ is a morphism of groups.Hence $\Phi(a,0) = 0, \Phi(a,-x) = -\Phi(a,x)(\cdot \ is \ a \ var)$

6.1.5 Association:

$\forall x \in K$

$$\begin{aligned}
(f(f+g)+h)(x) &= (f+g)(x) + h(x) = f(x) + g(x) + h(x) \\
(f+(g+h))(x) &= f(x) + ((g+h)(x)) = f(x) + g(x) + h(x)
\end{aligned}$$

Let $0 : I \to K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$
Let $-f : f + (-f) = 0$
The mapping $K \times K^I \to K^I : (a,f) \mapsto af \quad (af)(x) = af(x)$ is a left K-module structure
The mapping $K \times K^I \to K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K-module structure

6.1.6 Remark:

We can also write an element $\mu\ of\ K^I$ is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I\ by\ \mu$)
Then

$$\begin{aligned}
&(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I} \\
&a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I} \\
&(\mu_i)_{i\in I} a = (\mu_i a)_{i\in I}
\end{aligned}$$

6.2 sub K-module

6.2.1 Def

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub-K-module of V.

6.2.2 Example

Let I be a set .Let $K^{\bigoplus I}$ be the subset of K^I composed of mappings $f : I \to K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub-K-module of K^I

In fact,$\forall (f, g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$
Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of K^I
$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x\mapsto f(x)a)} \subseteq I_f$

6.3 morphism of K-modules

6.3.1 Def

Let V and W be left K-module, A morphism of groups $\phi : V \to W$ is called a morphism of left K-modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

6.3.2 K-linear mapping

If K is commutative, a morphism of K-modules is also called a K-linear mapping. We denote by $\hom_{K-Mod}(V, W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of W^V

6.3.3 Theorem

Let V be a left K-module. Let I be a set.
The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I :\ \phi \to (\phi(e_i))_{i\in I}$ is a bijection where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

6.3.4 Remark:column

In the case where $I = 1, 2, 3, ..., n$ V^I is denoted as V^n, K^I is denoted as K^n For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n \phi(e_i) = x_i$

We write this ϕ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$

6.4 kernel

6.4.1 Prop

Let G and H be groups and $f : G \to H$ be a morphism of groups

- $I_m(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

6.4.2 Def

$\ker(f)$ is called the kernel of f

6.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

Proof

Let e_G and e_H be neutral element of G and H respectively

(1) Let x and y be element of G
$f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H

(2) Let x and y be element of $\ker(f)$ One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H \quad e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$ So $\ker(f)$ is a subgroup of G

(3) Suppose that f is injection.
Since $f(E_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ Suppose that $\ker(f) = \{e_G\}$ If $f(x) = f(y)$then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$
Hence $xy^{-1} = e_G \Rightarrow x = y$

6.4.4 Def

Let (V,+) be a commutative group, I be a set. We define a composition law + on V^I as follows

$$(x_i)_{i\in I} + (y_i)_{i\in I} := (x_i + y_i)_{i\in I}$$

Then V^I forms a commutative group

6.4.5 Remark

Let E and F be left K-modules
$\hom_{K=Mod}(E,F) := \{$morphisms of left K-modules from E to F$\} \subseteq F^E$ is a subgroup of F^E
In fact f and g are elements of $\hom_{K-Mod}(E,F)$, then $f-g$ is also a morphism of left K-module
$(f-g)(x+y) = f(x+y) - g(x+y) = (f(x)+f(y)) - (g(x)+g(y)) = (f(x) - g(x)) + (f(y)-g(y)) = (f-g)(x) + (f-y)(x)$

6.4.6 Theorem

Let V be a left K-module,I be a set The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I \ : \ \phi \mapsto (\phi(e_i))_i \in I$ is an isomorphism of groups, where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$

6.4.7 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_I) + \psi(e_i)$
$\forall(\phi, psi) \in \hom_{K-Mod}(K^{\bigoplus I}, V)^2$
Hence $\Psi(\phi,\psi) = (\phi(e_i) + \psi(e_i))_{i\in I} = \Psi(\phi) + \Psi(\psi)$
So Ψ is a morphism of groups

injectivity Let $\phi \in \hom_{K-Mod}(K^{\bigoplus I}, V)$ Such that $\forall i \in I(\forall\phi \in \ker(\Psi)) \quad \phi(e_i) = 0$
Let $a = (a_i)_{i\in I} \in K^{\bigoplus I}$ One has $a = \sum\limits_{i\in I} a_i e_i$
If fact,$\forall j \in I, a_j = \sum\limits_{i\in I, a_i \neq 0} a_i e_i(j)$
Thus $\phi(a) = \sum\limits_{i\in I, a_i\neq 0} a - I\phi(e_i) = 0$
Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i\in I} \in V^I$ We define $\phi_x : K^{\bigoplus I} \to V$ such that $\forall a = (a_i)_{i\in I} \in K^{\bigoplus I}, \phi_x(a) = \sum\limits_{i\in I, a_i\neq 0} a_i x_i$
This is a morphism of left K-modules
$foralli \in I, \phi_x(e_i) = 1x_i = x_i$ So $\Psi(\phi_x) = x$

Suppose that K' is a unitary ring,and V is also equipped with a right K'-module structure, Then $\hom_{K-Mod}(K^{\bigoplus I}, V) \subseteq V^{K^{\bigoplus I}}$ is a right sub-k'-module ,and Ψ in the theorem is a right K'-module isomorphism

Chapter 7

Monotone mappings

7.1 Def

Let I and X be partially ordered sets,$f : I \to X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)/f(a) < f(b)$,then we say that f is increasing/strictly increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

7.2 Prop.

Let X,Y,Z be partially ordered sets.$f : X \to Y, g : Y \to Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

7.3 Def

Let f be a function from a partially ordered set I to another partially ordered set .If $f \mid_{Dom(f)} \to X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

7.4 Prop.

Let I and X be partially ordered sets. f be function from I to X.

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

7.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B
If f is increasing/decreasing ,then so is f^{-1}

7.6 Def

Let X and Y be partially ordered sets. $f : X \to Y$ be a bijection. If both f and f^{-1} are increasing ,then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \to Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

7.7 Prop.

Let I be a subset of $\mathbb{N}$ which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \to I$

7.8 Proof

7.8.1 bijection

We construct $f : \mathbb{N} \to I$ by induction as follows.
Let $f(0) = \min I$ Suppose that $f(0), ..., f(n)$ are constructed
then we take $f(n+1) := \min(I\backslash\{f(0), ..., f(n)\})$
Since $I\backslash\{f(0), ..., f(n-1)\} \supseteq I\backslash\{f(0), ..., f(n)\}$.Therefore $f(n) \leq f(n+1)$
$Since f(n+1) \notin \{f(0), ..., f(n)\}$,we have $f(n) < f(n+1)$
Hence f is strictly increasing and ths is injective
If f is not surjective,then $I\backslash Im(f)$ has a element N.
Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.
Since $N \notin Im(f), N < f(m)$.
So $m \neq 0$.Hence $f(m-1) < N < f(m) = \min(I\backslash\{f(0), ..., f(m-1)\})$
By definition,$N \in I\backslash Im(f) \subseteq I\backslash\{f(0), ..., f(m-1)\}$,
Hence $f(m) \leq N$,causing contradiction.

7.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from $\mathbb{N}$ to $\mathbb{N}$

Chapter 8

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

8.1 Def

Let X be a set.We call sequence in X parametrized by I a mapping from I to X.

8.2 Remark

If K is a unitary ring and E is a left K-module then the set of sequence E^I admits a left-K-module structure. If $x = (x_n)_{n \in I}$ is a sequence in E, we define a sequence $\sum(x) := (\sum\limits_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$,called the series associated with the sequence x.

8.3 Prop

$\sum : E^I \to E^{\mathbb{N}}$ is a morphism of left-K-module

8.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I
$\sum\limits_{i \in I, i \leq n} (x_i + y_i) = (\sum\limits_{i \in I, i \leq n} x_i) + (\sum\limits_{i \in I, i \leq n} y_i), \lambda \sum\limits_{i \in I, i \leq n} x_i = \sum\limits_{i \in I, i \leq n} \lambda x_i$

8.5 Prop

Let I be a totally ordered set . X be a partially ordered set,$f : I \to X$ be a mapping ,$J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

8.6 limit

8.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset.$\forall (x_i)_{n\in I} \in [-\infty,+\infty]^I$ where $[-\infty,+\infty]$ denotes $\mathbb{R}\cup\{-\infty,+\infty\}$,we define:

$$\limsup_{n\in I,n\to+\infty} x_n := \inf_{n\in I}(\sup_{i\in I,i\geq n} x_i)$$

$$\liminf_{n\in I,n\to+\infty} x_n := \sup_{n\in I}(\inf_{i\in I,i\geq n} x_i)$$

If $\limsup\limits_{n\in I,n\to+\infty} x_n = \liminf\limits_{n\in I,n\to+\infty} x_n = l$, we then say that $(x_n)_{n\in I}$ tends to l and that l is the limit of $(x_n)_{n\in I}$. If in addition $(x_n)_{n\in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$,we say that $(x_n)_{n\in I}$ converges to l

8.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n\in I,n\to+\infty} = \inf_{n\in J}(\sup_{i\in I,i\geq n} x_i)$$

$$\liminf_{n\in I,n\to+\infty} x_n = \sup_{n\in J}(\inf_{i\in I,i\geq n} x_i)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i\in I}$ the limit superior and the limit inferior do not change.
In fact, if we take $J = \mathbb{N}\backslash\{0,...,m\}$, then $\inf\limits_{n\in J}(...)$ and $\sup\limits_{n\in J}(...)$ only depends on the values of $x_i, i \in I, i \geq m$

8.6.3 Prop

$\forall (x_n)_{n\in I} \in [-\infty,+\infty]^I$, $\liminf\limits_{n\in I,n\to+\infty} x_n \leq \limsup\limits_{n\in I,n\to+\infty} x_n$

8.6.4 Prop

Let $(x_n)_{n\in I} \in [-\infty,+\infty]^I$

$$\forall c \in \mathbb{R} \qquad \limsup_{n\in I, n\to+\infty}(x_n+c) = (\limsup_{n\in I, n\to+\infty} x_n)+c$$
$$\liminf_{n\in I, n\to+\infty}(x_n+c) = (\liminf_{n\in I, n\to+\infty} x_n)+c$$

$$\forall c \in \mathbb{R}_{>0} \qquad \limsup_{n\in I, n\to+\infty}(\lambda x_n) = \lambda \limsup_{n\in I, n\to+\infty} x_n$$
$$\liminf_{n\in I, n\to+\infty}(\lambda x_n) = \lambda \liminf_{n\in I, n\to+\infty} x_n$$

$$\forall c \in \mathbb{R}_{<0} \qquad \limsup_{n\in I, n\to+\infty}(\lambda x_n) = \lambda \liminf_{n\in I, n\to+\infty} x_n$$
$$\liminf_{n\in I, n\to+\infty}(\lambda x_n) = \lambda \limsup_{n\in I, n\to+\infty} x_n$$

8.6.5 Prop

Let $(x_n)_{n\in I}$ be elements in $[-\infty,+\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$,one has $x_n \leq y_n$ Then

$$\limsup_{n\in I, n\to+\infty}(x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

,

$$\liminf_{n\in I, n\to+\infty}(x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

8.6.6 Theorem

Let $(x_n)_{n\in I}, (y_n)_{n\in I}, (z_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n\in I}$ and $(z_n)_{n\in I}$ tend to the same limit l

Then $(y_n)_{n\in I}$ tends to l

8.6.7 Def

Let I be an infinite subset of $\mathbb{N}$, and $(x_n)_{n\in I}$ be a sequence in some set X. We call subsequence of $(x_n)_{n\in I}$ a sequence of the form $(x_n)_{n\in J}$,where J is an infinite subset of I

8.6.8 Prop

Let I and J be infinite subset of $\mathbb{N}$ such that $J \subseteq I$
$\forall (x_n)_{n\in I} \in [-\infty,+\infty]^I$,one has

$$\liminf_{n\in I, n\to+\infty}(x_n) \leq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in I, n\to+\infty}(x_n) \geq \limsup_{n\in I, n\to+\infty} y_n$$

In particular, if $(x_n)_{n\in I}$ tends to $l \in [-\infty, +\infty]$,then $(x_n)_{n\in J}$ tends to l

8.6.9 Prop

$\forall n \in \mathbb{N}$,one has

$$\liminf_{n\in J, n\to+\infty}(x_n) \geq \liminf_{n\in I, n\to+\infty} y_n$$

$$\limsup_{n\in J, n\to+\infty}(x_n) \leq \limsup_{n\in I, n\to+\infty} y_n$$

8.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_N)_{n\in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing,then $(x_N)_{i\in I}$ tends to $\sup\limits_{n\in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing,then $(x_N)_{i\in I}$ tends to $\inf\limits_{n\in I} x_n$

8.6.11 Notation

If a sequence $(x_N)_{n\in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim\limits_{n\in I, n\to} x_n$ denotes this limit l

8.6.12 Corollary

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{N}_{\geq 0}$ Then the series $\sum\limits_{n\in I} x_n$(the sequence $(\sum\limits_{i\in I, i\leq n})_{n\in\mathbb{N}})$ tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ It converges in $\mathbb{R}$ iff it is bounded from above (namely has an upper bound in $\mathbb{R}$)

8.6.13 Notation

If a series $\sum\limits_{n\in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum\limits_{n\in I} x_n$ to denote the limit

8.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n\in I}$ be a sequence in $[-\infty, +\infty]$ There exists a subsequence of $(x_n)_{n\in I}$ that tends to $\limsup\limits_{n\in I, n\to+\infty} x_n$ There exists a subsequence of $(x_n)_{n\in I}$ that rends to $\liminf\limits_{n\in I, n\to+\infty} x_n$

Proof

Let $J = \{n \in I \mid \forall m \in I, \text{if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_N)_{n\in J}$ is decreasing so it tends to $\inf\limits_{n\in J} x_n$

$\forall n \in J$ by definition $x_n = \sup\limits_{i\in I, i\geq n} x_i$ so $\limsup\limits_{n\in I, n\to+\infty} x_n = \inf\limits_{n\in J} \sup\limits_{i\in I, i\geq n} x_i = \inf\limits_{n\in J} x_n = \lim\limits_{n\in J, n\to+\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$.Denote by $l = \sup\limits_{n\in I, n\geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup\limits_{i\in I, i\geq n_0} x_i \leq l$. If the strict inequality $\sup\limits_{i\in I, i\geq N} x_i < l$ holds, then $\sup\limits_{i\in I, i\geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup\limits_{i\in I, i\geq N} x_i$ We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality.

Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$ Thus $n \in J$ that contradicts the maximality of n_0

Therefore

$$l = \sup_{i\in I, i\geq N} x_i$$

, which leads to

$$\limsup_{n\in I, n\to+\infty} x_n = l$$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$,so $x_m < l$(since otherwise $x_m = \sup\limits_{i\in I, i\geq m} x_i$ and hence $m \in J$)Hence,$\forall\ finite\ subset\ I'\ of \{m \in I \mid m \geq n_0\}$

$\max\limits_{i\in I} x_i < l$ and hence $\exists n \in I$,such that $n > \max I'$,and $\max\limits_{i\in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j\in\mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \to I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is increasing

And

$$\sup_{j\in\mathbb{N}} x_{n_j} \leq \sup_{j\in\mathbb{N}} x_{f(j)} = \sup_{n\in I, n\geq n_0} x_n = l$$

$$l = \sup_{n\in I, n\geq n_0}$$

So $(x_{n_j})_{j\in\mathbb{N}}$ tends to l

Chapter 9

Cauchy sequence

9.1 Def

Let $(x_n)_{n\in I}$ be a sequence in $\mathbb{R}$
If $\inf\limits_{N\in\mathbb{N}} \sup\limits_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = \lim\limits_{N\to+\infty} \sup\limits_{(n,m)\in I\times I,\ n,m\geq N} |x_n - x_m| = 0$ then we say that $(x_n)_{n\in I}$ is a Cauchy sequence

9.2 Prop

- If $(x_n)_{i\in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_N)_{i\in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite,$(x_n)_{n\in I}$ is a Cauchy sequence.
- If $(x_n)_{n\in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n\in I}$ converges to l, then $(x_n)_{n\in J}$ converges to l too.

9.3 Theorem: Completeness of real number

If $(x_n)_{n\in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in $\mathbb{R}$

Proof

Since $(x_n)_{n\in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$ So $\limsup\limits_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n\in I}$ converges to $\limsup\limits_{n\in I, n\to+\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n\in I}$ converges to the same limit.

9.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

9.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in $\mathbb{R}$

Chapter 10

Comparison and Technics of Computation

10.1 Def

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be sequence in $\mathbb{R}$

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_N| \leq M|y_m|$ then we write $x_n = O(y_n), n \in I, n \to +\infty$
- If there exists $(\epsilon_n)_{n\in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim\limits_{n\in I, n\to+\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_N| \leq |\epsilon y_m|$, then we write $x_n = \circ(y_n), n \in I, n \to +\infty$
  Example:
  $$\lim_{n\to+\infty} \frac{1}{n} = 0$$

10.2 Prop.

Let I and X be partially ordered sets and $f : I \to X$ be an increasing/decreasing mapping. Let J ba a subset of I. Assume that any elements of I has an upper bound in J. Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

10.3 Theorem

Let I be a totally ordered set, $f : I \to [-\infty, +\infty]$ and $g : I \to [-\infty, +\infty]$ be two mappings that are both increasing/decreasing.Then the following equalities holds,provided that the sum on the right hand side of the equality is wel defined.

$$\sup_{x\in I, \{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\sup_{x\in I} f(x)) + (\sup_{y\in I} g(y))$$

$$\inf_{x\in I,\{f(x),g(x)\}\neq\{-\infty,+\infty\}} = (\inf_{x\in I} f(x)) + (\inf_{y\in I} g(y))$$

Proof

We can assume f and g increasing. Let $a = \sup f(I)$,$b = \sup g(I)$
Let $A = \{(x,y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$
We equip A with the following order relation.

$$(x,y) \leq (x',y') \ iff \ x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x,y) \in A \mid x = y\}$.
Consider

$$h : A \to [-\infty, +\infty] \quad h(x,y) = f(x) + g(y)$$

h is increasing.
Let $(x,y) \in A$. Assume that $x \leq y$
If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y,y) \in B$ and $(x,y) \leq (y,y)$
If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x,y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$
So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined,$(z,z) \in B$ and $(x,y) \leq (z,z)$ Similarly, if$x \geq y$, (x,y) has also an upper bound in B. Therefore: $\sup h(A) = \sup h(B)$

10.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty, +\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty} (x_n + y_n) \leq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty} (x_n + y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n\in I, n\geq N} x_n \quad B_N = \sup_{n\in I, n\geq N} y_n$. $(A_N)_{N\in\mathbb{N}}$ and $(B_N)_{N\in\mathbb{N}}$ are decreasing, and $\limsup_{n\in I, n\to+\infty} x_n = \inf_{N\in\mathbb{N}} A_N \quad \limsup_{n\in I, n\to+\infty} y_n = \inf_{N\in\mathbb{N}} B_N$
By theorem:

$$\inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N = \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n\in I, n\geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.
Therefore

$$\inf_{N\in\mathbb{N}} C_N \leq \inf_{N\in\mathbb{N},\{A_N,B_N\}\neq\{-\infty,+\infty\}} (A_N + B_N) = \inf_{N\in\mathbb{N}} A_N + \inf_{N\in\mathbb{N}} B_N$$

10.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$ such that ,$\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty,+\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) \geq (\limsup_{n\in I, n\to+\infty} x_n) + (\limsup_{n\in I, n\to+\infty} y_n)$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) \geq (\liminf_{n\in I, n\to+\infty} x_n) + (\liminf_{n\in I, n\to+\infty} y_n)$$

Proof

a tricky proof ?:

$$\limsup_{n\in I, n\to} x_n = \limsup_{n\in I, n\to}(x_n+y_n-y_n) \leq \limsup_{n\in I, n\to}(x_n+y_n) - \liminf_{n\in I, n\to} y_n$$

to have a true proof,only need to discuss conditions with ∞

10.6 Theorem

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n\in I}$ converges to some $i \in \mathbb{R}$.
Then:

$$\limsup_{n\in I, n\to+\infty}(x_n+y_n) = (\limsup_{n\in I, n\to+\infty} x_n) + l$$

$$\liminf_{n\in I, n\to+\infty}(x_n+y_n) = (\liminf_{n\in I, n\to+\infty} x_n) + l$$

10.7 Prop.

Let $(x_n)_{n\in I}$ and $(y_n)_{n\in I}$ be elements of $[-\infty,+\infty]^I$
Then:

$$\liminf_{n\in I, n\to+\infty} \max\{x_n, y_n\} = \max\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

$$\liminf_{n\in I, n\to+\infty} \min\{x_n, y_n\} = \min\{\liminf_{n\in I, n\to+\infty} x_n, \liminf_{n\in I, n\to+\infty} y_n\}$$

Proof

About the first inequality.Since $\max\{x_n, y_n\} \geq x_n \quad \max\{x_n, y_N\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n\in J, n\to+\infty} = \limsup_{n\in J, n\to+\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $\quad J_1 = \{n \in J \mid x_n \leq y_n\}$
$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite
Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \to} \max\{x_n, y_n\} = \lim_{n \in J_1, n \to} \max\{x_n, y_n\} = \lim_{n \in J, n \to} x_n \leq \limsup_{n \in I, n \to +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \to +\infty} = \lim_{n \in J_2, n \to +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \to +\infty} y_n$$

10.8 Theorem

Let $(a_N)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_N)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$

Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$
$$\limsup_{n \in I, n \to +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \to +\infty} a_n) - l, l - (\liminf_{n \in I, n \to +\infty} a_n)\}$$

(1) $\Rightarrow$ (2):
If $(a_n)_{n \in I}$ converges to l, then $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$
(2) $\Rightarrow$ (1):
If $\limsup_{n \in I, n \to +\infty} |a_n - l| = 0$,then $\limsup_{n \in I, n \to +\infty} a_n \leq l \leq \liminf_{n \in I, n \to +\infty} a_n$
Therefore: $\limsup_{n \in I, n \to +\infty} a_n = \liminf_{n \in I, n \to +\infty} a_n = l$

10.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in $\mathbb{R}$, $l \in \mathbb{R}$
The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \to +\infty$

10.10 Calculates on O(),o()

10.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2$ $\quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \to +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \to +\infty$
  then $\forall (\lambda, \mu) \in \mathbb{R}^2$ $\quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \to +\infty$

10.10.2 Transform

Let $(a_n)_{n\in I}$ and $(b_n)_{n\in I}$ be two sequence in $\mathbb{R}$ If $a_n = o(b_n), n \in I, n \to +\infty$,then $a_n = O(b_n), n \in I, n \to +\infty$

10.10.3 Transition

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ and $(c_n)_{n\in I}$ be elements in $\mathbb{R}^I$

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = O(c_n), n \in I, n \to +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \to +\infty$
  then$a_n = o(c_n), n \in I, n \to +\infty$

10.10.4 Times

Let $(a_n)_{n\in I}, (b_n)_{n\in I}, (c_n)_{n\in I}, (d_n)_{n\in I}$ be sequences in $\mathbb{R}$

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = O(b_n d_n), n \in I, n \to +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \to +\infty$
  then $a_n c_n = o(b_n d_n), n \in I, n \to +\infty$

10.11 On the limit

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ be elements of $\mathbb{R}^I$ that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n\in I}$ converges to $l + l'$
- $(a_n b_n)_{n\in I}$ converges to ll'

10.12 Prop

Let $a \in \mathbb{R}$ then $a^n = o(n!) \quad n \to +\infty$

Proof

Let $N \in \mathbb{N}$ such that $|a| < N$
For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} (\frac{|a|}{N})^n - N$$

And $0 < \frac{|a|}{<} 1 \Rightarrow \lim_{n\to+\infty} (\frac{|a|}{N})^n = 0$. Therefore:

$$\lim_{n\to+\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

10.13 Prop

$n! = o(n^n) \quad n \to +\infty$

Proof

Let $N \in \mathbb{N}_{\geq 1}$
$0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n\to+\infty} \frac{n!}{n^n} = 0$

10.14 Prop

Let $(a_n)_{n\in I}, (b_n)_{n\in I}$ be the elements of $\mathbb{R}^I$ If the series $\sum_{n\in I} b_n$ converges absolutely and if $on = O(b_n) \quad n \to +\infty$Then $\sum_{n\in I} a_n$ converges absolutely

Proof

By definition $\sum_{n\in I} |b_N| < +\infty$ If $|a_N| \leq M|b_N|$ fro $n \in I, n \geq N$ where $N \in \mathbb{N}$ Then

$$\sum_{n\in I} |a_n| = \sum_{n\in I, n<N} |a_n| + \sum_{n\in I, n\geq N} |a_n| \leq \sum_{n\in I, n<N} |a_n| + \sum_{n\in I, n\geq N} |b_n| < +\infty$$

10.15 Theorem: d'Alembert ratio test

Let $(a_N)_{n\in\mathbb{N}} \in (\mathbb{R}\backslash\{0\})^{\mathbb{N}}$

- If $\limsup_{n\to+\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum_{n\in\mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n\to+\infty} |\frac{a_{n+1}}{a_n}| > 1$, then $\sum_{n\in\mathbb{N}} a_n$ does not converge (diverges)

Proof

(1)

Let $\alpha \in \mathbb{R}$ such that $\limsup\limits_{n\to+\infty}|\frac{a_{n+1}}{a_n}| < \alpha < 1$, $alpha$ isn't a lower bound of $(\sup\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|)_{N\in\mathbb{N}}$
So $\exists N \in \mathbb{N}$ such that $\sup\limits_{n\geq N}|\frac{a_{n+1}}{a_n}| < \alpha$Hence for $n \geq N \quad |a_n| \leq \alpha^{n-N}|a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N}\frac{a_{N+2}}{a_{N+1}}...\frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum\limits_{n\in\mathbb{N}} = \frac{1}{1-\alpha} < +\infty$, $\sum\limits_{n\in\mathbb{N}} a_n$ converge absolutely.

10.15.1 Lemma

If a series $\sum\limits_{n\in\mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim\limits_{n\to+\infty} a_n = 0$

Proof

If $(\sum\limits_{i=0}^{n} a_i)_{n\in\mathbb{N}}$ converges to some $l \in \mathbb{R}$,then $(\sum\limits_{i=0}^{n-1} a_i)_{n\in\mathbb{N},n\geq1}$ converges to l, too. Hence $\left(a_n = \left(\sum\limits_{i=0}^{n} a_i\right) - \left(\sum\limits_{i=0}^{n-1} a_i\right)\right)_{n\in\mathbb{N}}$ converges to $l - l = 0$

10.15.2 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf\limits_{n\to+\infty}|\frac{a_{n+1}}{a_n}| = \sup\limits_{N\in\mathbb{N}} \inf\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|$
So there exists $N \in \mathbb{N}$ such that $\beta < \inf\limits_{n\geq N}|\frac{a_{n+1}}{a_n}|$
$\forall n \in \mathbb{N}, n \geq N \quad |\frac{a_{n+1}}{a_n}| \geq \beta$
Hence $(|a_n|)_{n\in\mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N}|a_n|$
By the lemma: $\sum\limits_{n\in\mathbb{N}} a_n$ diverges.

10.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \to +\infty$

Proof

Let $\epsilon > 0$ such that $a = (1+\epsilon)^2$

$$a^n = (1+\epsilon)^{2n} = (1+\epsilon)^n(1+\epsilon)^n \geq (1+n\epsilon)(1+n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

10.16.1 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \to +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

10.16.2 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

10.17 Theorem: Cauchy root test

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Let $\alpha = \limsup\limits_{n\to+\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum\limits_{n\in\mathbb{N}} a_n$ converges absolutely.
- If $a > 1$ then $\sum\limits_{n\in\mathbb{N}} a_n$ diverges

Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_N|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum\limits_{n\in\mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \quad \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N}\ \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n\in\mathbb{N}}$ cannot converge to 0.

Part III

Axiom of choice

Chapter 11

Preparation

11.1 Statement of axiom of choice

For any set I and any family $(A_i)_{i\in I}$ of non-empty sets , there exists a mapping $f : I \to \bigcup\limits_{i\in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

11.2 Def

Let $(X, \leq)$ be a partially ordered set If $\forall A \subseteq X$ A is non-empty , there exists a least element of A then we say that $(X, \leq)$ is a well ordered set.

11.3 Theorem

For any set X, there exists an order relation $\leq$ on such that $(X, \leq)$ forms a well ordered set.

11.4 Zorn's lemma

Let $(X, \leq)$ be a partially ordered set . If $\forall A \subseteq X$ that is totally ordered with respect to $\leq$, there exists an upper bound of A inside X. Then , there exists a maximal element x_0 of $X(\forall y \in X, y > x_0$ does not hold)

11.5 Prop.

Let $(X, \leq)$ be a well ordered set , $y \notin X$. We extends $\leq$ to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

11.6 Proof

Let $A \subseteq X \cup \{y\}, A \neq \varnothing$. If $A = \{y\}$ then Y is the least element of A. If $A \neq \{y\}$ then $B = A \backslash \{y\}$ is non-empty. Let b be the least element of B. Since $b < y$ it's also the least element of A

11.7 Def: Initial Segment

Let $(X, \leq)$ be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S (X_{<s} \subseteq S)$, then we say that S is an initial segment of X

If S is a initial segment such that $S \neq X$ then we sat that S is a proper initial segment.

11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X.

11.9 Prop.

Let $(X, \leq)$ be a well ordered set , If $(S_i)_{i \in I}$ is a family of initial segment of X, then $\bigcup\limits_{i \in I} S_i$ is an initial segment of X

11.10 Proof

$\forall s \in \bigcup\limits_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup\limits_{i \in I} S_i$

11.11 Prop

Let $(X, \leq)$ be a well ordered set.

(1) Let S be a proper initial segment of X, $x = \min(X \backslash S)$ Then $S = X_{<x}$

(2) $\begin{array}{l} X \to \wp(X) \\ x \mapsto X_{<x} \end{array}$

(3) The set of all initial segments of X forms a well ordered subset of $(\wp(x), \subseteq)$

11.12 Proof

(1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction.Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely , if $t \in X, t \notin X \backslash S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

(2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \backslash X_{<x}$ So $X_{<x} \subsetneqq X_{<y}$

(3) Let $\mathscr{F} \subseteq \wp(X)$ be a set of initial segments. $\mathscr{F} \neq \varnothing$. Then there exists $A \subseteq X$ such that $\mathscr{F} \backslash \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \varnothing$ then $\mathscr{F} = \{X\}$, and $\{X\}$is the least element of $\mathscr{F}$. Otherwise $A \neq \varnothing$ and A has a least element a. Then by(2) $X_{<}a$ is the least element of $\mathscr{F}$

11.13 Lemma

Let $(X, \leq)$ be a well ordered set, $f : X \to X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \varnothing$, let a be the least element of A. By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing . This shows $f(a) \in A$. But a is the least element of A, $f(a) < a$ cannot hold: contradiction.

11.14 Prop

Let $(X, \leq)$ be a well ordered set, S and T be two initial segment of X . If $f : S \to T$ is a bijection that's strictly increasing , then $S = T, f = Id_S$

Proof

We may assume $T \subseteq S$.Let $l : T \to S$ be the induction mapping and $g = l \circ f : S \to S$. Since g is strictly increasing , by the lemma ,$\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$
Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

11.15 Def

Let $(X, \leq)$ and $(Y, \leq)$ be partially ordered sets. If $\exists f : X \to Y$ that's increasing and bijective, we say that $(X, \leq)$ and $(Y, \leq)$ are isomorphic

11.16 Def

Let $(X, \leq)$ and $(Y, \leq)$ be well ordered sets. If $(X, \leq)$ is isomorphic to an initial segment of Y. We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y, we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \prec X$

11.17 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

Proof

We construct a correspondence f from X to Y, such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$
By the last proposition of Oct. 11, f is a function.

- If $a, b \in Dom(f)^2, a < b$,then $X_{<a} \subsetneqq X_{<b}$
  By definition, $Y_{<f(b)} \sim X_{<b} \quad Y_{<f(a)} \sim X_{<a}$
  Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y < f(b)$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_{<} f(B)$. We then get $f(a) < f(b)$. Thus f is strictly increasing.

- Let $a \in Dom(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y \quad X_{<x} \sim Y_{<y}$ This shows that $x \in Dom(f)$. Hence $Dom(f)$ is an initial segment of X. Applying this to f^{-1}, we get : $Im(f) = Dom(f)$ is an initial segment of Y

- Either $Dom(f) = X$ or $Im(f) = Y$.
  Assume that $x \in X \backslash Dom(f), y \in Y \backslash Im(f)$ are respectively the least elements of $X \backslash Dom(f)$ and $Y \backslash Im(f)$.
  Then we get $Dom(f) = X_{<x}, Im(f) = Y_{<y}$.
  We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction

-

Case 1 $Dom(f) = X, Im(f) \subsetneqq Y \quad X \prec Y$

Case 2 $Dom(f) \subsetneqq X, Im(f) = Y \quad X \succ Y$

Case 3 $Dom(f) = X, Im(f) = Y \quad X \sim Y$

11.18 Lemma

Let $(X, \leq)$ be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .

- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B, or B is a initial segment of A.

Let $Y = \bigcup\limits_{A \in \mathfrak{S}} A$. Then $(Y, \leq)$ is a well ordered set, and ,$\forall A \in \mathfrak{S}, A$ is an initial segment of Y.

Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$.Since $Y = \bigcup_{B \in \mathfrak{S}} B$,$\exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B. Hence $y \in A$. Contradiction

- Let $Z \subseteq Y, Z \neq \varnothing$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \mathfrak{S}$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$, then A is an initial segment of B.

Since B is well ordered , if $m \not\leq z$ then $z < m$. Since $m \in A$, we het $z \in A$. Contradiction.

Therefore, m is the least element of Z.

Chapter 12

Zorn's lemma

Let $(X, \leq)$ be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X, the X has a maximal element(a maximal element m of $\{x \mid x > m\} = \varnothing$)

12.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega.\exists f(A)$ such that $\forall a \in A, a < f(A)$
Let

$$\omega = \{\text{well ordered subset of X}\}$$

. (guaranteed by axiom of choice)
Let $f : \omega \to X$ such that $f(A)$ is an upper bound of $A \in \omega$.
If $A \in \omega$ satisfies

$$\forall a \in Aa = f(A_{<a})$$

, we say that A is a f-set
Let

$$\mathfrak{S} = \{f - sets\}$$

Note that

$$\varnothing \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cap \{f(A)\} \in \mathfrak{S}$$

In fact, if $a \in A$, then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If $a = f(A) \notin A$ then

$$(A \cup \{f(A)\})_{<a} = A$$

Let A and B be elements of $\mathfrak{S}$. Let I be the union of all common initial segments of A and B. This is also a common initial segment of A and B.
If $I \neq A$ and $I \neq B$, then

$$\exists (a,b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then $I \cup \{a\}$ is also a common initial segment of A and B, contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and $\forall A \in \mathfrak{S}$ is an initial segment of Y.

Since A is an initial segment of Y

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If $f(y)$ is not a maximal element of X

$$\exists x \in X, f(y) < x$$

Part IV

Topology

Chapter 13

Absolute value and norms

13.1 Def

Let K be a field . By absolute value on K, we mean a mapping $|\cdot| : K \to \mathbb{R}_{\geq 0}$ that satisfies:

(1) $\forall a \in K \quad |a| = 0$ iff $a = 0$

(2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$

(3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$(triangle inequality)

13.2 Notation

$\mathbb{Q}$ Take a prime num p $\forall \alpha \in \mathbb{Q}\backslash\{0\}$ there exists a integer $ord_p(\alpha)\frac{a}{b}$, where $\begin{matrix} a \in \mathbb{Z}\backslash\{0\} \\ b \in \mathbb{N}\backslash\{0\} \end{matrix}, p \nmid a, p \nmid b$

13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \to \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & if\ \alpha \neq 0 \\ 0 & if\ \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on $\mathbb{Q}$

Proof

(1) Obviously

(2) If $\alpha = p^{ord_p(\alpha)}\frac{a}{b}, \beta = p^{ord_p(\beta)}\frac{c}{d} \quad p \nmid abcd$
$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)}\frac{ac}{bd} \quad p \nmid ac, p \nmid bd$

(3) $\alpha + \beta = p^{ord_p(\alpha)}\frac{a}{b} + p^{ord_p(\beta)}\frac{c}{d}$
Assume $ord_p(\alpha) \geq ord_p(\beta)$
$\alpha + \beta$
$= p^{ord_p(\beta)}\left(p^{ord_p(\alpha)-ord_p(\beta)}\frac{a}{b} + \frac{c}{d}\right)$
$= p^{ord_p(\beta)}\frac{p^{ord_p(\alpha)-ord_p(\beta)}ad+bc}{bd} \quad p \nmid bd$
So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

Hence $ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$
So $|\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$
$\max\{|\alpha|_p, |\alpha|_p\} \leq |\alpha|_p, |\alpha|_p$

13.4 Def

Let K be a filed and $|\cdot|$ be an absolute value. We call $(K, |\cdot|)$ a valued field.

Chapter 14

Quotient Structure

14.1 Def

Let X be a set and $\sim$ be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that $\sim$ is an equivalence relation

14.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X.Let $X/\sim$ be the set $\{[x] \mid x \in X\}$

14.3 Prop.

Let X be a set and $\sim$ be an equivalence relation on X

(1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$

(2) If α and β are elements of $X/\sim$ such that $\alpha \neq \beta$ then $\alpha \cap \beta = \varnothing$

(3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

Proof

(1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ on has $x \sim y$
Therefore ,$x \sim z$ namely $z \in [x]$. This proves $y[] \subseteq [x]$. Moreover ,since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$

(2) Suppose that $\alpha \cap \beta \neq \varnothing, y \in \alpha \cap \beta$
By (1),$\alpha = [y], \beta = [y]$, Thus leads to a contradiction.

(3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup\limits_{\alpha \in X/\sim} \alpha$Hence $X \subseteq \bigcup\limits_{\alpha \in X/\sim} \alpha$.Conversely, $\forall \alpha \in X/\sim, \alpha$ is a subset of X. Hence $\bigcup\limits_{\alpha \in X/\sim} \alpha \subseteq X$.Then $X = \bigcup\limits_{\alpha \in X/\sim} \alpha$

14.4 Def

Let G be a group and X be a set
We call left/right action of G on X ant mapping $G \times X \to X : (g,x) \mapsto gx/(g,x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x \; / \; x1 = x$
- $\forall (g,h) \in G^2, x \in X \quad g(hx) = (gh)x \; / \; (xg)h = x(gh)$

14.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \to G$$

$$(g,h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X.

14.6 Prop

Let G be a group and X be a set . Assume given a left action of G on X. Then the binary relation $\sim$ on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

14.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim \forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

14.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for same $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x.(y \sim x)$
- $\forall (x, y, z) \in X^3$,if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

14.9 Quotient set

Let X be a set and $\sim$ be an equivalence relation, the mapping $X \to X/\sim$: $(x \in X) \mapsto [x]$ is called the projection mapping.
$X/\sim$ is called the quotient set of X by equivalence relation $\sim$

14.9.1 Example

Let G be a group and H be a subgroup of G.Then the mapping

$$H \times G \to G$$

$$(h, g) \mapsto hg/(h, g) \mapsto gh$$

is a left/right action of H on G. Thus we obtain two quotient sets H/G and G/H

14.10 Def

Let G be a group and H be a subgroup of G. Ig $\forall g \in G, h \in H \quad ghg^{-1} \in H$, Then we say that H is a normal subgroup of G

14.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G. In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

14.12 Prop

If G is commutative, any subgroup of G is normal

14.13 Theorem

Let G be a group and H be a normal subgroup of G. Then the mapping

$$G/H \times H/G \to G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H Moreover the projection mapping

$$\pi : G \to G/H$$

$$x \mapsto xH$$

is a morphism of groups.

Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$ Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.

- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH)$

- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$

- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

14.14 Def

Let K be a unitary ring and E be a left K-module. We say that a subgroup F og $(E, +)$ is a left sub-K-module of E if $\forall (a, x) \in K \times F, ax \in F$

14.15 Prop

Let K be a unitary ring , E be a left K-module and F be a sub-K-module. Then the mapping

$$K \times (E/F) \to E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined , and defines a left-K-module structure on E/F. Moreover, the projection mapping $pi : E \to E/F$ is a morphism of left-K-modules

Proof

Let x and x' be elements of E such that $[x] = [x']$, that meas: $x' - x \in F$ Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K-module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K \quad \pi(ax) = [ax] = a[x] = a\pi(x)$

14.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I()$ (I is a left and right sub-K-module of A)

14.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \to A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$ Therefore $a'b' \sim ab$

14.17.1 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of $\mathbb{Z}$
If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z} \quad adm = dma \in d\mathbb{Z}$
Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the reside class of n modulo d
If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

14.18 Theorem

Let $f : G \to H$ be a morphism of groups

(1) $Im(f)$ is a subgroup of H

(2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G

(3) The mapping

$$\begin{aligned} \widetilde{f} :& G/Ker(f) \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

(4) f is injective iff $\ker(f) = \{1_G\}$

Proof

(1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup

(2) Let x and y be elements of $\ker(f)$.
One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
Let $x \in \ker f, y \in G$.
One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup

(3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
Moreover $\widetilde{f}([x][y]) = \widetilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence $\widetilde{f}$ is a morphism of groups.
By definition $Im(\widetilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y, x \sin y$ that means $[x] = [y]$
Therefore $\widetilde{f}$ is injective.

(4) If f is injective ,$\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f\{1_G\}$
Conversely , suppose that $\ker f = \{1_G\} \quad \forall (x,y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$

14.19 Theorem

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Then

(1) $Im(f)$ is a left-sub-K-module of F

(2) $\ker(f)$ is a left-sub-K-module of E

(3) $\begin{aligned} \widetilde{f} :& E/\ker f \to Im(f) \\ & [x] \mapsto f(x) \end{aligned}$ is a isomorphism of left K-modules

Proof

(1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in Im(f)$

(2)

(3)

Chapter 15

Topology

15.1 Def

Let X be a set. We call topology on X any subset $\mathcal{J}$ of $\wp(x)$ that satisfies:

- $\varnothing \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i\in I}$ is an arbitrary family of elements in $\mathcal{J}$, then $\bigcup\limits_{i\in I} u_i \in \mathcal{J}$
- If u and v are elements of $\mathcal{J}$, then $u \cap v \in \mathcal{J}$

15.2 Remark

If $(u_i)_i^n = 1$ is a finite family of elements of $\mathcal{J}$, then $\bigcap\limits_{i=1}^{n} u_i \in \mathcal{J}$(by induction, this follows from (3))

15.2.1 Example

$\{\phi, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

15.3 Def

Let X be a set. We call metric on X any mapping $d : \begin{array}{c} X \times X \to \mathbb{R}_{\geq 0} \\ (x, y) \mapsto d(x, y) \end{array}$,that satisfies

- $d(x, y) = 0$ iff x=y
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$(triangle inequality)

(X, d) is called a metric space

15.3.1 Example

Let X be a set

$$d : X^2 \to \mathbb{R}_{\geq 0}$$
$$d(x,y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$$

is a metric

15.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x,y) < \epsilon\}$ We call the open ball of radius ϵ centered at x

15.4.1 Example

Consider $(\mathbb{R}, d)$ with $d(x,y) = |x - y|$,then $B(x,\epsilon) =]x - \epsilon, x + \epsilon[$

15.5 Prop.

Let (X,d) be a metric space . let $\mathcal{J}_d$ be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x,\epsilon) \subseteq U$ THen $\mathcal{J}_d$ is a topology on X

Proof

- $\varnothing \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of $\mathcal{J}_d$ Let $U = \bigcup\limits_{i \in I} u_i$, $\forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x,y) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of $\mathcal{J}_d$ Let $x \in U \cap V$ $\exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x,a) \subseteq U, B(x,b) \subseteq V$ Taking $\epsilon = \min\{a,b\}$, Then $B(x,\epsilon) = B(x,a) \cap B(x,b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

15.6 Def

$\mathcal{J}_d$ is called the topology induced by the metric d

15.7 Def

We call topology space any pair $(X, \mathcal{J})$ where X is a set and $\mathcal{J}$ is a topology on X

Given a topological space $(X, \mathcal{J})$ If $U \in \mathcal{J}$ then we say that U is an open subset of X. If $F \in \wp(X)$ such that $X \backslash F \in \mathcal{J}$, then we say that F is closed subset of X
If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that $\mathcal{J}$ is metrizable

15.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact,m if d denote the metric defined as $d(x,y) = \begin{cases} 1 & if\ x \neq y \\ 0 & if\ x = y \end{cases}$
$\forall x \in X \quad B(x,1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup\limits_{x \in A} \{x\} \in \mathcal{J}_d$

Chapter 16

Filter

16.1 Def

Let Xbe a set. We call filter if X any $\mathscr{F} \subseteq \wp(x)$ that satisfies:

(1) $\mathscr{F} \neq \varnothing, \varnothing \notin \mathscr{F}$

(2) $\forall A \in \mathscr{F}, \forall B \in \wp(X)$, if $A \subseteq B$, then $B \in \mathscr{F}$

(3) $\forall (A, B) \in \mathscr{F} \times \mathscr{F}, A \cap B \in \mathscr{F}$

16.1.1 Example

(1) Let $Y \subseteq X, Y \neq \varnothing$. $\mathscr{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y.

(2) Let X be an infinite set.

$$\mathscr{F}_{Fr}(X) := \{A \in \wp(X) \mid X \backslash A \text{ is infinite}\}$$

is a filter called the Fréchet filter of X.

(3) Let $(X, \mathscr{G})$ be a topological space, $x \in X$ We call neighborhood of x any $V \in \wp(X)$ such that $\exists u \in \mathscr{G}$, satisfying $x \in U \subseteq V$. Then $\mathscr{V} = \{\text{neighborhoods of x}\}$ is a filter.

16.2 Def: Filter Basis

Let X ba a set. $\mathscr{B} \subseteq \wp(X)$. If $\varnothing \notin \mathscr{B}$ and $\forall (B_1, b_2) \in \mathscr{B}^2, \exists B \in \mathscr{B}$, such that $B \subseteq B_1 \cap B_2$. We say that $\mathscr{B}$ is a filter basis.

16.2.1 Remark

If $\mathscr{B}$ is a filter basis, then $\mathscr{F}(\mathscr{B}) = \{A \subseteq X \mid \exists B \in \mathscr{B} \quad B \subseteq A\}$ is a filter

Proof

$\varnothing \notin \mathcal{F}(\mathscr{B}), \mathcal{F}(\mathscr{B}) \neq \varnothing$ since $0 \neq B \subseteq \mathcal{F}(\mathscr{B})$. If $A \in \mathcal{F}(\mathscr{B}), A' \in \wp(X)$such that $A \subseteq A'$, then $\exists B \in \mathscr{B}$ such that $B \subseteq A \subseteq A'$. Hence $A' \in \mathcal{F}(\mathscr{B})$ If $A_1, A_2 \in \mathcal{F}(\mathscr{B})$, then $\exists (B_1, B_2) \in \mathscr{B}^2$ such that $B_1 \subseteq A_1, B_2 \subseteq A_2$. Since $\mathscr{B}$ is a filter basis, $\exists B \in \mathscr{B}$ such that $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ Hence $A_1 \cap A_2 \in A_1 \cap A_2 \in \mathcal{F}(\mathscr{B})$

16.2.2 Example

- Let $Y \subseteq X, Y \neq \varnothing$
  $\mathscr{B} = \{Y\}$ is a filter basis. $\mathcal{F}(\mathscr{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let $(X, \mathcal{J})$ be a topological space $x \in X$. If $\mathscr{B}_x$ is a filter basis such that $\mathcal{F}(\mathscr{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$, then we say that $\mathscr{B}_x$ is a neighborhood basis of x

16.3 Remark

Let $\mathscr{B}_x$ is a neighborhood basis of x iff

- $\mathscr{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathscr{B}_x$ such that $U \subseteq V$
- Let (X, d) be a metric space , $x \in X \forall \epsilon > 0$, Let

  $$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

  $$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

  Then

  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x
  - $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
  - $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x

16.3.1 Example

$\mathcal{V}_x \cap \mathcal{J}$ is a neighborhood basis of x

16.4 Def

$V \in \wp(X)$ is called a neighborhood of x if $\exists U \in \mathcal{J}$ such that $x \in U \subseteq V$

16.5 Remark

Let $(X, \mathcal{G})$ be a topological space, $x \in X$ and $\mathscr{B}_x$ a neighborhood basis os x. Suppose that $\mathscr{B}$ is countable . We choose a surjective mapping $(B_n)_{n\in\mathbb{N}}$ from $\mathbb{N}$ to$\mathscr{B}_x$. For any $n \in \mathbb{N}$, let $A_n = B_0 \cap B_1 \cap ... \cap B_n \in \mathcal{V}_x$ The sequence $(A_n)_{n\in\mathbb{N}}$ is decreasing and $\{A_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of x.

16.6 Extra Episode

$\wp(\mathbb{N})$is NOT countable

Suppose that $f : \wp(\mathbb{N}) \to \mathbb{N}$ injective. Then $\exists g : \mathbb{N} \to \wp(\mathbb{N})$ surjective. Taking $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$. Since g is surjective, $\exists a \in \mathbb{N}$ such that $A = g(a)$.

If $a \in A$, then $a \in g(a)$, hence $a \notin A$

If $a \notin A$, then $a \in g(a) = A$

Contradiction

16.7 Prop.

Let Y and R be sets, $g : Y \to E$ be a mapping ,

- If $\mathcal{F}$ is a filter of Y, then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

  is a filter on E

- If $\mathscr{B}$ is a filter basis of Y, then

$$g(\mathscr{B}) := \{g(B) \mid B \in \mathscr{B}\}$$

  is a filter of E, and $\mathcal{F}(g(\mathscr{B})) = g_*(\mathcal{F}(\mathscr{B}))$

Proof

(1) $E \in g_x(\mathcal{F})$ since $g^{-1}(E) = Y$
$\varnothing \notin g_x(\mathcal{F})$ since $g^{-1}(\varnothing) = \varnothing$

If $A \in g_x(\mathcal{F})$ and $A' \supseteq A$, then $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{G}$, so $g^{-1}(A') \in \mathcal{G}$, Hence $A' \in g_x(\mathcal{F})$

If $A_1, A_2 \in g_x(\mathcal{F})$. Then $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ Hence $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$. So $A_1 \cap A_2 \in g_x(\mathcal{F})$.

(2) Since g is a mapping , and $\varnothing \notin \mathscr{B}$, we get $\varnothing \notin g(\mathscr{B})$,since $\mathscr{B} \neq \varnothing, g(\mathscr{B}) \neq \varnothing$.

Let $B_1, B_2 \in \mathscr{B}$, there exists $C \in \mathscr{B}$ such that $C \subseteq B_1 \cap B_2$. Hence $g(C) \subseteq g(B_1) \cap g(B_2)$, namely $g(\mathscr{B})$ is a filter basis.

Chapter 17

Limit point and accumulation point

We fix a topological space $(X, \mathcal{J})$

17.1 Def

Let $\mathcal{F}$ be a filter of X and $x \in X$

- If $\mathcal{V}_x \subseteq \mathcal{F}$ then we say that x is an limit point of $\mathcal{F}$
- If $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \varnothing$, we say that x is an accumulation point of $\mathcal{F}$

So any limit point of $\mathcal{F}$ is necessarily a accumulation point of $mathcalF$

17.2 Prop

Let $\mathcal{B}$ be a filter basis of X, $x \in X, \mathcal{B}_x$ a neighborhood basis of x. Then x is an accumulation point of $\mathcal{F}(\mathcal{B})$ iff $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \varnothing$

Proof

Necessity

Since $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B}), \mathcal{B} \subseteq \mathcal{V}_x$, the necessity is true.

Sufficiency

Let $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$. There exist $B \in \mathcal{B}, U \in \mathcal{B}_x$, such that $B \subseteq A, U \subseteq V$. Hence $\varnothing \neq B \cap U \subseteq A \cap V$

17.3 Def

Let $Y \subseteq X, Y \neq \varnothing$. W call accumulation point of Y any accumulation point of the principal filter $\mathscr{F} = \{A \subseteq X \mid Y \subseteq A\}$.

17.4 Def

We denote by $\overline{Y} = \{\text{accumulation points of } Y\}$.,called the closure of Y Note that $x \in \overline{Y}$ iff $\forall U \in \mathscr{B}_x, Y \cap U \neq \varnothing$

By convention $\overline{\varnothing} = \varnothing$

17.5 Prop

Let $Y \subseteq X$. Then $\overline{Y}$ is the smallest closed subset of X containing Y.

Proof

$\forall x \in X \backslash \overline{Y}$, then there exists $U_x = \mathcal{V} \cap \mathcal{G}$, such that $Y \cap U_x = \varnothing$. Moreover , $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{G}$. This shows that $\forall y \in U_x, y \notin \overline{Y}$. Therefore $X \backslash \overline{Y} = \bigcap_{x \in X \backslash \overline{Y}} U_x \in \mathcal{G}$

Let $Z \subseteq X$ be a closed subset that contain Y. Suppose that $\exists y \in \overline{Y} \backslash Z$. Then $U = X \backslash Z \in \mathcal{V}_y \cap \mathcal{G}$ and $U \cap Y \subseteq U \cap Z = \varnothing$. So $y \notin \overline{Y}$ contradiction. Hence $\overline{Y} \subseteq Z$.

17.6 Def: dense

Let $(X, \mathcal{G})$ be a topological space,Y a subset of X. We call Y is dense in X if

$$\overline{Y} = Y$$

Chapter 18

Limit of mappings

18.1 Def

Let $(E, \mathcal{J}_E)$ be a topological space . $f : Y \to E$ a mapping , and $\mathcal{F}$ eb a filter of Y. If $a \in E$ is a limit point of $F_*(\mathcal{F})$ namely , $\forall$neighborhood$V$of$a, f^{-1}(V) \in \mathcal{F}$, then we say that a is a limit of the filter $\mathcal{F}$ by f

18.2 Remark

Let $\mathscr{B}_a$ be a neighborhood basis of a. Then $\mathcal{V}_a \subseteq f_x(\mathcal{F})$, iff $\mathscr{B} \subseteq f_*(\mathcal{F})$ Therefore , a is a limit of $\mathcal{F}$ by f iff $\forall V \in \mathscr{B}_a, f^{-1}(V) \in \mathcal{F}$

18.2.1 Example

Let $(E, \mathcal{J}_E)$ be a topological space. $I \subseteq \mathbb{N}$ be an infinite subset, $x = (x_n)_{n \in I} \in E^I$. If the Fréchet filter $\mathcal{F}_{Fr}(I)$ has a limit $a \in E$ by the mapping $x : I \to E$, we say that $(x_n)_{n \in I}$ converges to a ,denote as

$$a = \lim_{n \in I, n \to +\infty} x_n$$

18.3 Remark

$a = \lim_{n \in I, n \to +\infty} x_n$ iff, $\forall U \in \mathscr{B}_a$(where $\mathscr{B}_a$ is a neighborhood basis of a), $\exists N \in \mathbb{N}$ such that $x_n \in U$ for any $n \in I_{\geq N}$

Suppose that $\mathcal{J}_E$ is induced by a metric $d.\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}\{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}\{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ are all neighborhood basis of a .There fore , the following are equivalent

- $a = \lim_{n \in i, n \to +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
- $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$

$(x^{-1}(B(a,\epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\}$? unknown position)

18.4 Remark

We consider the metric d on $\mathbb{R}$ defined as

$$\forall (x,x) \in \mathbb{R}^2 \quad d(x,y) := |x-y|$$

The topology of $\mathbb{R}$ defined by this metric is called the usual topology on $\mathbb{R}$

18.5 Prop

Let $(x_n)_{n\in I} \in \mathbb{R}^I$, where $I \subseteq \mathbb{N}$ is an infinite subset. Let $l \in \mathbb{R}$. The following statements are equivalent:

- The sequence $(x_n)_{n\in I}$ converges to l in the topological space $\mathbb{R}$
- $\liminf\limits_{n\in I, n\to+\infty} x_n = \limsup\limits_{n\in I, n\to+\infty} x_n = l$
- $\limsup\limits_{n\in I, n\to} |x_n - l| = 0$

18.6 Theorem

Let (X,d) be a metric space .Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n\in I}$ be an element of X^I. Let $l \in X$. The following statements are equivalent:

- $(x_n)_{n\in I}$ converges to l
- $\limsup\limits_{n\in I, n\to+\infty} d(x_n, l) = 0$(equivalent to $\lim\limits_{n\in I, n\to+\infty} d(x,l) = 0$)

Proof

(1) $\Rightarrow$ (2) The condition (1) is equivalent to $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$. We then get $\sup\limits_{n \in I_{geqN}} d(x,l) \leq \epsilon$. Therefore $\limsup\limits_{n\in I, n\to+\infty} d(x_n, l) \leq \epsilon$ We obtain that $\limsup\limits_{n\in I, n\to+\infty} = 0$

(2) $\Rightarrow$ (1) Let $\epsilon \in \mathbb{R}_{>0}$ If $\inf\limits_{N\in\mathbb{N}} \sup\limits_{n\in I_{\geq N}} d(x_n, l) = 0$. Then $\exists N \in \mathbb{N} \quad \sup\limits_{n \in I_{\leq N}} d(x_n, l) < \epsilon$. Hence ,$\forall n \in I_{\geq N} d(x_n, l) < \epsilon$. Since ϵ is arbitrary, $(*)$ is true, Hence (1) is also true .

18.7 Prop

Let $(X, \mathcal{J})$ be a topological space . $Y \subseteq X, p \in \overline{Y} \backslash Y$.Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of Y.

Proof

Y is not empty otherwise $\overline{Y} = \varnothing$.

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$
  $\varnothing \notin \mathcal{V}_{p,Y}$ since $p \in \overline{Y}$
- Let $V \in \mathcal{V}_p$ and $A \subseteq Y$ such that $V \cap Y \subseteq A$. Let $U = V \cup (A \backslash (V \cap Y)) \in \mathcal{V}_p$ and $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let U and V be elements of $\mathcal{V}_p$ Let $W = U \cap V \in \mathcal{V}_p$ Then $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

18.8 Def

Let $(X, \mathcal{J}_x)$ and $(E, \mathcal{J}_E)$ be topological spaces, $Y \subseteq X, p \in \overline{Y} \backslash Y$, and $f : Y \to E$ be a mapping . If a is a limit point of $(F_*(\mathcal{V}_{p,Y}))$, then we say that a is a limit of f when the variable $y \in Y$ tends to p, denoted as $a = \lim\limits_{y \in Y, y \to p} f(y)$

18.9 Remark

If $\mathscr{B}_a$ is a neighborhood basis of a. Then $a = \lim\limits_{y \in Y, y \to p} f(y)$ is equivalent to $\forall U \in \mathscr{B}_a \quad \exists V \in \mathcal{V}_p$ such that $Y \cap V \subseteq f_{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

18.10 Prop

Let X be a set, $\mathscr{B}$ be a filter basis, $\mathscr{G}$ be a filter. If $\mathscr{B} \subseteq \mathscr{G}$, then $\mathcal{F} \subseteq \mathscr{G}$.

Proof

Let $V \in \mathcal{F}(\mathscr{B})$ By definition $\exists U \in \mathscr{B}$ such that $U \subseteq V$, since $U \in \mathscr{G}($ for $\mathscr{B} \subseteq \mathscr{G})$ and since $\mathscr{G}$ is a filter, $V \in G$

18.11 Theorem

Let $(X, \mathcal{J}_x)$ and $(E < \mathcal{J}_E)$ be topological spaces. $Y \subseteq X,\ p \in \overline{T} \backslash Y, a \in E$. We consider the following conditions.

(i) $a = \lim\limits_{y \in Y, y \to p} f(y)$

(ii) $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ if $\lim\limits_{n \to +\infty} y_n = p$ then $\lim\limits_{n \to \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that p has a countable neighborhood basis, then (i) and (ii) are equivalent.

Proof

(1) Let $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ such that $p = \lim\limits_{n \to +\infty} y_n$. For any $U \in \mathcal{V}_p, \exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N} \quad y \in U \cap Y . y_n \in U \cap Y$ Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \to +\infty} f(y_n) = a$$

(2) Assume that p has a countable neighborhood basis . There exists a decreasing sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_P^{\mathbb{N}}$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of p.
Assume that (i) does not hold. Then there exists $U \in \mathcal{V}_a$ such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \backslash f^{-1}(U)$$

Therefore ,

$$\lim_{n \to +\infty} y_n = \varnothing$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However $f(y_n) \notin U$ for any $n \in \mathbb{N}$,so $(f(y_n))_{n \in \mathbb{N}}$ cannot converges to a.

18.12 Prop.

Let X be a set. If $(\mathcal{G}_i)_{i\in I}$ is a family of topologies on X, then $\mathcal{G} = \bigcap\limits_{i\in I} \mathcal{G}_i$ is a topology. In particular. for any $\mathcal{A} \subseteq \wp(X)$, there is a smallest topology on X that contain $\mathcal{A}$

18.12.1 Proof

- $\forall i \in I \quad \{\varnothing, X\} \subseteq \mathcal{G}_i$ So $\{\varnothing, X\} \subseteq \mathcal{G}$
- Let $(u_j)_{j\in J}$ be a family of elements of $\mathcal{G}$ $\forall j \in J, i \in I \quad u_i \in \mathcal{G}_i$ So $\bigcup\limits_{j\in J} u_j \in \mathcal{G}_i$ We then get $\bigcup\limits_{j\in J} u_j \in \mathcal{G}$
- Let U and V be elements of $\mathcal{G}$ $\forall i \in I, \{u, v\} \subseteq \mathcal{G}_i$ So $U \cap V \in \mathcal{G}_i$. THerefore we get $U \cap V \in \mathcal{G}$ Let $\mathcal{A} \subseteq \wp(X)$ Let $\mathcal{G}(\mathcal{A}) = \bigcap\limits_{\mathcal{G}\subseteq\wp(X)\text{a topology}\mathcal{A}\subseteq\mathcal{G}} \mathcal{G}$ Then $\mathcal{G}(\mathcal{A})$ is a topology. By definition, if $\mathcal{G}$ is a topology containing $\mathcal{A}$, then $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}$ Hence $\mathcal{G}(\mathcal{A})$ is the smallest topology containing $\mathcal{A}$

Chapter 19

Continuity

19.1 Def

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces f be a function from X to Y, $x \in Dom(f)$. If for any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$. Then we say that f is continuous at x. If f is continuous at any $x \in Dom(f)$ then we say f is continuous.

19.2 Remark

Let $\mathscr{B}_{f(x)}$ be a neighborhood basis of $f(x)$ If $\forall U \in \mathscr{B}_{f(x)}$ there exist $V \in \mathscr{B}_{f(x)} V_x$ such that $f(V) \subseteq U$, then f is continuous at x Suppose that X and Y are metric space. Then f is continuous at x iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in Dom(f) \quad d(y,x) < \delta \text{ implies } \ d(f(y), f(x)) < \epsilon$$

19.3 Theorem

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces, f be a function from X to Y $x \in Dom(f)$ Consider the following condition

- f is continuous at x
- $\forall (x_n)_{n\in\mathbb{N}} \in Dom(f)^{\mathbb{N}}$, if $\lim\limits_{n\to+\infty} x_n = x$, then $\lim\limits_{n\to+\infty} f(x_n) = f(x)$ THen (i) implies (ii) Moreover, if x has a countable neighborhood basis, then (i) and (ii) are equivalent.

19.4 Proof

$(i) \Rightarrow (ii)$ Let $(x_n)_{n\in\mathbb{N}} \in Dom(f)^{\mathbb{N}}$ that converges to x $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$ Since $\lim\limits_{n\to+\infty} x_n = x$,there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}$, $x_n \in V$.

Hence $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$. Thus $\lim_{n \to +\infty} f(x_n) = f(x)$

$(ii) \Rightarrow (i)$ under the hypothesis that x has countable neighborhood basis. actually we will prove $NOT(i) \Rightarrow NOT(ii)$
Let $(V_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{V}_x$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of x
If (i) does not hold, then $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$ Pick $x_n \in V_n$ such that $f(x_n) \notin U \quad \forall N \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$. Hence $(x_n)_{n\in\mathbb{N}}$ converges to x. However, $f(x_n) \notin U$ for any n So $(f(x_n))_{n\in\mathbb{N}}$ does not converges to $f(x)$. Therefore (ii) does not hold.

19.5 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$ be topological spaces. f be a function from X to Y, g be a function from Y to Z. Let $x \in Dom(g \circ f)$ If f and g are continuous at x. then $g \circ f$ is continuous at x sectionProof Let $U \in \mathcal{V}_{g(f(x))}$ Since g is continuous at $f(x)$:

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since f is continuous at x:

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore, $g(f(V)) \subseteq g(W) \subseteq U$ Hence $g \circ f$ is continuous at x

19.6 Def

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$, If any element of $\mathcal{J}$ can be written as the union of a family of sets in $\mathscr{B}$ we say that $\mathscr{B}$ is a topological basis of $\mathcal{J}$

19.7 Prop

Let $(X, \mathcal{J})$ be a topological space, $\mathscr{B} \subseteq \mathcal{J}$ $\mathscr{B}$ is a topological basis iff

$$\forall x \in X, \mathscr{B}_x := \{V \in \mathscr{B} \mid x \in V\}$$

is a neighborhood basis of x

19.8 Proof

$\Rightarrow$:

$$\forall x \in X \mathscr{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since $\mathscr{B}$ is a topological basis of $\mathcal{J}$,

$$\exists W \in \mathscr{B}, x \in W \subseteq V \subseteq U$$

Hence $\mathcal{V}_x$ is generated by $\mathscr{B}_x$

$\Leftarrow$ Let $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathscr{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

19.9 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces. $\mathscr{B}_Y$ be a topological basis of $\mathcal{J}_Y$ $f : X \to Y$ be a mapping. The following conditions are equivalent:

(1) f is continuous

(2) $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$

(3) $\forall U \in \mathscr{B}, f^{-1}(U) \in \mathcal{J}_X$

Proof

$(1) \Rightarrow (2)$

Lemma Let $(X, \mathcal{J})$ be a topological space, $V \in \wp(X)$, THen $V \in \mathcal{J}$ iff $\forall x \in V, V$ is a neighborhood of x

Proof of lemma $\Rightarrow$ is by definition
$Leftarrow$:

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W - x \in \mathcal{J}$$

Let $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood W of x such that $f(W) \subseteq U$ Since f is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since x is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2) $\Rightarrow$ (3) For (3) is a special situation of (2), it's natural.

(3) $\Rightarrow$ (1) Let $x \in X$

$$\forall U \in \mathscr{B}_Y \ s.t. \ f(x) \in U, f^{-1}(U)$$

is an open neighborhood of x, and

$$f(f^{-1}(U)) \subseteq U$$

Hence f is continuous at x

19.10 Def

LEt X be a set ,$((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces. $\forall i \in I$ let $f_i : X \to Y_i$ be a mapping. We call initial topology of $(f_i)_{i \in I}$ on X the smallest topology on X making all f_i continue

19.11 Remark

If $\mathcal{J}$ is the initial topology of $(f_i)_{i \in I}$, $\forall i \in I, U_i \in \mathcal{J}_i \quad f_i^{-1}(U_i) \in \mathcal{J}$ If $J \subseteq I$ is a finite subset, $(U_j)_{j \in J} \in \prod\limits_{j \in J} \mathcal{J}_j$ then $\bigcap\limits_{j \in J} f_j^{-1}(U_j) \in \mathcal{J}$

19.12 Prop

$$\mathscr{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{is finite} (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology $\mathcal{J}$

Proof

First

$$\mathscr{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{$subset V of X that can be written as the union of a family of sets in$\mathscr{B}\}$

- $\varnothing \in \mathcal{J}' \quad X \in \mathscr{B} \subseteq \mathcal{J}'$
- $\mathcal{J}'$ is stable by taking the union of any family of elements in $\mathcal{J}'$
- If V_1, V_2 are elements of $\mathcal{J}'$, then

$$V_1 \cap V_2 \in \mathcal{J}'$$

  In fact, V_1, V_2 are of the form of the union of some sets of $\mathscr{B}$

The intersection of two elements of $\mathscr{B}$ is still a element of $\mathscr{B}$

$$\left(\bigcap_{j \in J} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j \in J'} f_j^{-1}(U_j')\right)$$

$$= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \backslash J' \\ U_j' & j \in J' \backslash J \\ U_j \cap U_j' & j \in J \cap J' \end{cases}$$

$$\left(\bigcap_{j \in J \backslash J'} f_j^{-1}(U_j)\right) \cap \left(\bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U_j')\right) \cap \left(\bigcap_{j \in J' \backslash J} f_j^{-1}(U_j')\right)$$

So $\mathcal{J}'$ is a topology making all f_i continuous.Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

Example

Let $((Y_i, \mathcal{J}_i))_{i \in I}$ be topological spaces. $Y = \prod_{i \in I} Y_i$ and

$$\pi_i : \begin{array}{l} Y \to Y_i \\ (y_j)_{j \in I} \mapsto y_i \end{array}$$

The product topology on Y is by definition the initial topology of $(\pi_i)_{i \in I}$

19.13 Theorem

Let X be a set , $((Y_i, \mathscr{G}_i))_{i\in I}$ be a family of topological spaces,

$$((f_i : X \to Y_i))_{i\in I}$$

be a family of mappings and we equip X with the initial topology $\mathscr{G}_X$ of $(f_i)_{i\in I}$
Let $(Z, \mathscr{G}_Z)$ be a topological space and

$$h : Z \to X$$

be a mapping. Then h is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{is continuous}$$

19.13.1 Proof

$\Rightarrow$ If h is continuous, since each f_i is continuous, $f_i \circ h$ is also continuous.

$\Leftarrow$ Suppose that $\forall i \in I, f_i \circ h$ is continuous .Hence

$$\forall U_i \in \mathscr{G}_i, (f_i \circ h)_{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathscr{G}_Z$$

Let

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_j^{-1}(U_j) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathscr{G}_j \right\}$$

$\forall U \in \mathscr{B}$

$$h^{-1}(U) = \bigcap_{j\in J} h^{-1}(f_i^{-1}(U_i)) \in \mathscr{G}_Z$$

Therefore, h is continuous.

19.14 Remark

We keep the notation of the definition of initial topology If $\forall i \in I, \mathscr{B}_i$ is a topological basis of $\mathscr{G}_i$, then

$$\mathscr{B} = \left\{ \bigcap_{j\in J} f_i^{-1}(U_i) \mid J \subseteq I \text{finite} (U_j)_{j\in J} \in \prod_{j\in J} \mathscr{B}_j \right\}$$

is also a topological basis of the initial topology,

19.14.1 Example

Let $((X_i, d_i))_{i\in\{1,\dots,n\}}$ be a family of metric spaces.

$$X = \prod_{i\in\{1,\dots,n\}} X_i$$

We define a mapping

$$d : \begin{array}{l} X \times X \to \mathbb{R}_{\geq 0} \\ ((x_i)_i \in \{1,...,n\}(y_i)_{i\in\{1,\dots,n\}}) \mapsto \max_{i\in\{1,\dots,n\}} d_i(x_i, y_i) \end{array}$$

d is a metric on X. If $x = (x_i)_{i\in\{1,\dots,n\}}$ $y = (y_i)_{i\in\{1,\dots,n\}}$ $z = (z_i)_{i\in\{1,\dots,n\}}$ are elements of X, then

$$d(x, z) = \max_{i\in\{1,\dots,n\}} d_i(x_i, z_i) \leq \max_{i\in\{1,\dots,n\}} (d_i(x_i, y_i) + d(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i : \begin{array}{l} X \to X_i \\ (x_i)_{i\in\{1,\dots,n\}} \mapsto x_i \end{array}$$

is continuous. Hence the product topology $\mathcal{J}$ is contained in $\mathcal{J}_d$
Let $x = (x_i)_{i\in\{1,\dots,n\}} \in X, \epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i\in\{1,\dots,n\}} \mid \max_{i\in\{1,\dots,n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i\in\{1,\dots,n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i\in\{1,\dots,n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$

Chapter 20

Uniform continuity and convergency

20.1 Def

Let (X,d) be a metric space. $\forall A \subseteq X$, we define

$$diam(A) := \sup_{(x,y)\in A\times A} d(x,y)$$

called the diameter of A.By convention

$$diam(\varnothing) := 0$$

If $diam(A) < +\infty$, we say that A is bounded

20.2 Remark

- If A is finite, then it's bounded
- If $A \subseteq B$ then $diam(A) \leq diam(B)$

20.3 Prop

Let (X,d) be a metric space. $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$. Then

$$diam(A \cup B) \leq diam(A) + d(x_0, y_0) + diam(B)$$

In particular, if A,B are bounded, then $A \cup B$ is bounded.

Proof

Let $(x,y) \in (A \cup B)^2$. If $\{x,y\} \subseteq A$, then $d(x,y) \leq diam(A)$
If $\{x,y\} \subseteq B$ then $diam(B) \geq d(x,y)$
If $x \in A, y \in B$,

$$d(x,y) \leq d(x,x_0) + d(x_0,y_0) + d(y_0,y) \leq diam(A) + d(x_0,y_0) + diam(B)$$

Similarly if $x \in B, y \in A$

$$d(x,y) \leq diam(A) + d(x_0,y_0) + diam(B)$$

20.4 Def

Let (X,d) be a metric space. $I \subseteq \mathbb{N}$ be an infinite subset, $(x_n)_{n\in I} \in X^I$. If

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \quad diam(\{x_n \mid n \in I_{\geq \mathbb{N}}\}) \leq \epsilon$$

then we say that $(x_n)_{n\in I}$ is a Cauchy sequence.

20.5 Prop

(1) If $(x_n)_{n\in I}$ converges, then it's a Cauchy sequence.

(2) If $(x_n)_{n\in I}$ is a Cauchy sequence, $\{x_n \mid n \in I\}$ is bounded

(3) Suppose that $(x_n)_{n\in I}$ is a Cauchy sequenceIf there exists an infinite subset J of I such that $(x_n)_{n\in J}$ converges to some $x \in X$,then $(x_n)_{n\in I}$ converges to x

20.5.1 Proof

(1) trivial

(2) trivial

(3) Let $\epsilon > 0, \exists N \in \mathbb{N}$

$$diam(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n,x) \leq \frac{\epsilon}{2}$$

- Take $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n,x) \leq d(x_n,x_{n_0}) + d(x_{n_0},x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence $(x_n)_{n\in I}$ converges to x

20.6 Def

Let $(X, d_X), (Y, d_Y)$ be metric space. f be a function from X to Y. If $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in Dom(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x,y) \in Dom(f)^2, d(x,y) \leq \delta} d(f(x), f(y)) = 0$$

we say that f is uniformly continuous.

20.7 Prop

Let $(X, d_X), (Y, d_Y)$ be metric spaces f be a function from X to Y which is uniformly continuous.

(1) If $I \subseteq \mathbb{N}$ is finite, and $(x_n)_{n \in I}$ is a Cauchy sequence in $Dom(f)^I$ then $(f(x_n))_{n \in I}$ is Cauchy sequence

(2) f is continuous

20.7.1 Proof

(1) $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in Dom(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that

$$\forall (n, m) \in I^2_{\geq N}, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

THerefore $(f(x_n))_{n \in I}$ is a Cauchy sequence.

(2) Let $(x_n)_{n \in I}$ be a sequence in $Dom(f)^{\mathbb{N}}$ that converges to $x \in Dom(f)$ We define$(y_n)_{n \in \mathbb{N}}$ as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then $(y_n)_{n \in \mathbb{N}}$ converges to x. Hence $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since f is uniformly continuous, $(f(y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y.

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to $f(x)$. Hence $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x)$

20.8 Def

Let X be a set , $Z \subseteq X, (Y, d)$ be a metric space, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}$ and f be functions from X to Y, having Z as their common domain of definition.

- If $\forall x \in Z, (f_n(x))_{n\in I}$ converges to $f(x)$, we say that $(f_n)_{n\in I}$ converges pointwisely to f
- If
$$\lim_{n\in I, n\to+\infty} \sup_{x\in Z} d(f_n(x), f(x)) = 0$$
we say that $(f_n)_{n\in I}$ converges uniformly to f

20.9 Theorem

Let X and Y be metric space, $Z \subseteq X, I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}, f$ be functions from X to Y, having Z as domain of definition. Suppose that

- $(f_n)_{n\in I}$ converges uniformly to f
- each f_n is uniformly continuous

Then f is uniformly continuous.

20.9.1 Proof

$\forall n \in I$ let
$$A_n = \sup_{x\in Z} d(f_n(x), f(x))$$
$$\lim_{n\in I, n\to+\infty} A_n = 0$$
$\forall (x, y) \in Z^2, n \in I$
$$\begin{aligned}&d(f(x), f(y))\\ \le&d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y))\\ \le&2A_n + d(f_n(x), f_n(y))\end{aligned}$$
$$\inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) \le 2A_n + \inf_{(x,y)\in Z^2, d(x,y)\le\delta} d(f_n(x), f_n(y)) = 0$$
Hence
$$0 \le \inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) \le 2A_n$$
Take $\lim_{n\to+\infty}$, by squeeze theorem, we get
$$\inf_{\delta>0} \sup_{(x,y)\in Z^2, d(x,y)\le\delta} d(f(x), f(y)) = 0$$

20.10 Theorem

Let X be a topological space, Y be a metric space, $Z \subseteq X, p \in Z$, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n\in I}$ and f function from X to Y, having Z as domain of definition. Suppose that:

- $(f_n)_{n\in I}$ converges uniformly to f
- each f_n is continuous at p

Then f is continuous at p

20.10.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x\in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \; \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since f_n is continuous $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{\mathcal{B}}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned}\forall x \in U \cap Z \quad & d(f(x)f(p)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ & \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3}\end{aligned}$$

$f(U) \subseteq \overline{\mathcal{B}}(f(p), \epsilon)$

20.10.2 Def

Let X Y be metric spaces , f be a function from X to Y, $\epsilon > 0$. If

$$\forall (x, y) \in Dom(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that f is ϵ-Lipschitzian
If $\exists \epsilon > 0$ such that f is ϵ-Lipschitzian, then it's uniformly continuous.

20.11 Remark

If f is Lipschitzian, then it's uniformly continuous.

20.12 Example

- Let $((X_i, d_i))_{i\in I}$ be metric space. $X = \prod_{i\in I} X_i$ where I is finite

$$d : \begin{array}{l} X \times X \to \mathbb{R}_{\geq 0} \\ d((x_i), (y_i)_{i\in I}) = \max_{i\in I} d_i(x_i, y_i) \end{array}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \to X_i$$

is Lipschitzian.($\forall x = (x_i)_{i \in I}, \forall x = (x_i)_{i \in I}$)

- Let (X, d) be a metric space

  $$d : X \times X \to \mathbb{R}_{\geq 0}$$

  is Lipschitzian.

  $$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

Part V

Normed Vector Space

Chapter 21

Linear Algebra

We fix a unitary ring K

21.1 Def

Let M be a left K-module ,and let $x = (x_i)_{i \in I}$ be a family of elements of M. We define a morphism of left K-module as following:

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} & \to M \\ & (a_i)_{i \in I} & \mapsto \sum_{i \in I} a_i x_i (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

21.1.1 Notation

$$K^{\bigoplus I} := \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{finite,such that} a_i = 0 \text{ for } i \in I \setminus J\}$$
$$\varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) = \varphi_x((a_i)_{i \in I}) \varphi_x((b_i)_{i \in I})$$

21.2 Def

Ler M be a left K-module, I be a set, $x = (x_i)_{i \in I} \in M^I$ If

$$\begin{aligned} \varphi_x : & K^{\bigoplus I} \to M \\ & (a_i)_{i \in I} \mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say $(x_i)_{i \in I}$ is K-linearly independent

surjective then we say $(x_i)_{i \in I}$ is system of generator

a bijection then we say $(x_i)_{i \in I}$ is a basis of M

Example

Let e_i be the element $(\delta_{ij})_{j\in I}$ with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i\in I} \in (K^{\bigoplus I})^I$$

is a basis of $K^{\bigoplus I}$

21.3 Def

Let M be a left K-module

- If M bas a basis, we say that M is a free K-module
- If M has finite system of generated
  ($\exists a$ finite set I and a family $(x_i)_{i\in I} \in M^I$ that forms a system of generator),
  then we say that M is of finite type.

21.4 Remark

Let $x = (x_i)_{i\in\{1,\dots,n\}} \in M^n$, where $n \in \mathbb{N}$

- x is linearly independent iff

  $$\forall a \in K^n \quad \sum a_i x_i = 0$$

  implies

  $$a = 0$$

- x is a system of generator iff for any element of M can be written in the form

  $$\sum b_i x_i \quad b \in K^n$$

  Such expression is called a K-linear combination of $x_1, ...x_n$

21.5 Theorem

Let K be a division ring ($0 \neq 1$ and $\forall k \in K \setminus \{0\}$ k is invertible)
Let V be a left K-module of finite type and $(x_i)_{i\in I}$ be a system of generators of V. Then ,there exists a subset I of $\{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V. (In particular, V is a free K-module)

Proof

(By induction on n)
If $n = 0$, then $V = \{0\}$
In this case $\varnothing$ is a basis of V

Induction hypothesis

True for a system of generators of $n-1$ elements. Let $(x_i)_{i\in\{1,\ldots,n\}}$ be a system of generators of V. If $(x_i)_{i\in\{1,\ldots,n\}}$ is linearly independent, it's a basis. Otherwise, $\exists(a_i)_{i\in I} \in K^n$ such that

$$(a_i, ...a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose $a_n \neq 0$. Then

$$x_n = -a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i)$$

Since $(x_i)_{i\in\{1,\ldots,n\}}$ is a system of generators, any elements of V can be written as

$$\begin{aligned}\sum b_i x_i &= (\sum_{i=1}^{n-1} b_i x_i) - b_n a_n^{-1}(\sum_{i=1}^{n-1} a_i x_i) \\ &= \sum_{i=1}^{n-1}(b_i - b_n a_n^{-1} a_i) x_i\end{aligned}$$

Thus $(x_i)_{i\in\{1,\ldots n\}}$ forms a system of generators . By the induction hypothesis, there exists $I \subseteq \{1, ..., n\}$ such that $(x_i)_{i\in I}$ forms a basis of V.

21.6 Theorem

Let K be a unitary ring and B be a left K-module. W be a left K-submodule of V. Let $(x_i)_{i=1}^n$ be an element of W^n

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where $(n, l) \in \mathbb{N}^2$ $\forall j \in \{1, ...l\}$,let x_{n+j} be an element in the equivalence class α_j

- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are linearly independent, then $(x_i)_{i=1}^{n+l}$ is also linearly independent
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are system of generators of W and V/W respectively, then $(x_i)_{i=1}^{n+l}$ is also a system of generators
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are basis, then $(x_i)_{i=1}^{n+l}$ is also a basis

Proof

(1) Suppose that $(b_i)_{i=1}^{n+l}$ such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \to V/W$$

be the projection morphism($\pi(x) = [x]$)

$$0 = \pi(\sum_{i=1}^{n+l} b_i x_i) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^{l} b_{n+j}\pi(x_{n+j}) = \sum_{j=1}^{l} b_{n+j}\alpha_j$$

$\{x_1, ...x_n\} \subseteq W$ So$\forall i \in \{1, ..., n\}$

$$\pi(x_i) = 0$$

Since $(\alpha_j)_{j=1}^{l}$ is linearly independent,

$$b_{n+1} = ... = b_{n+j} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since $(x_i)_{i=1}^{n}$ is linearly independent,

$$b_1 = ...b_n = 0$$

(2) Let $y \in V$. Then $\pi(y) \in V/W$. So there exists

$$(c_{n+1}, ..., c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^{l} c_{n+j}\alpha_j \\ &= \sum_{j=1}^{l} c_{n+j}\pi(x_{n+j}) \quad = \pi(\sum_{j=1}^{l} c_{n+j}x_{n+j}) \end{aligned}$$

So

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) \in W$$

$\exists c \in K^n$ such that

$$y - (\sum_{j=1}^{l} c_{n+j}x_{n+j}) = (\sum_{i=1}^{n} c_i x_i)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

21.7 Corollary

Let K be a division ring and V be a left K-module of finite type. If $(x_i)_{i=1}^n$ is a linearly independent family of elements of V($n \in \mathbb{N}$), then

$$\exists\ l \in \mathbb{N} \quad \exists\ (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of V

Proof

Let W be the image of

$$\varphi_(x_i)_{i=1}^n : K^n \to V$$
$$(a_i)_{i=1}^n \mapsto \sum_{i=1}^n a_i x_i$$

It's a left K-submodule of V.

Note that $(x_i)_{i=1}^n$ forms a basis of W.

$$\varphi_(x_i)_{i=1}^n : K^n \to W$$
$$\varphi_(x_i)_{i=1}^n(e_j) = x_j \in W$$

Moreover ,since V is of finite type there exists $d \in \mathbb{N}$ and a surjective morphism of left K-modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \to V/W$$

is surjective.

Hence the composite morphism

$$\begin{tikzcd} K^d \arrow[r, "\psi"] \arrow[rr, bend right, "\pi\circ\psi"'] & V \arrow[r, "\pi"] & V/W \end{tikzcd}$$

is surjective. Thus V/W is of finite type. There exist then a basis

$$(a_j)_{j=1}^{l}$$

of V/W.

Taking $x_{n+j} \in \alpha_j$ for $j \in \{1, ..., l\}$, we get a basis of V:

$$(x_i)_{i=1}^{n+l}$$

21.8 Def

Let K be a division ring and V be a left K-module of finite type. We call rank of V the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If K is a field $rk(V)$ is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of V.

21.9 Theorem

Let K be a division ring and V be a left K-module of finite type. Let W be a left K-submodule of V.

(1) W and V/W are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of V has exactly $rk(V)$ elements

21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let $(x_i)_{i=1}^{n}$ be a basis of V. Let

$$\begin{aligned}\pi : & V \to V/W \\ & x \mapsto [x]\end{aligned}$$

In $(\pi(x_i))_{i=1}^n$ we extract a basis of V/W, say

$$(\pi(x_i))_{i=1}^l$$

For $j \in \{l+1, ..., n\}$,

$$\exists (b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^{l} b_{j,i}\pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^{l} b_{j,i}x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any $x \in W$,$\exists (a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^{n} a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^{l} a_i x_i + \sum_{j=l+1}^{n} a_j (y_j + \sum_{i=1}^{l} b_{j,i}x_i) \\ &= \sum_{j=l+1}^{n} a_j y_j + \sum_{i=1}^{l} (a_i + \sum_{j=l+1} a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^{l} (a_i + \sum_{j=l+1}^{n} a_j b_{j,i})\pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^{n} a_i y_i$$

Hence W is of finite type , and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let $(x_i)_{i=1}^n$ be a basis of V. Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of V/W.
We extract a subfamily , say $(x_i)_{i=1}^l$ such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of V/W.
For $j \in \{1, ..., l\}$, there exists:

$$(b_{j,1}, ..., b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^{l} b_{j,i}\pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^{l} b_{j,i}x_i \in W$$

Let $x \in W$,$\exists (a_i)_{i=1}^n \in K^n$ let $x = \sum a_i x_i$,then

$$\begin{aligned}
x &= (\sum_{i=1}^{l} a_i x_i) + (\sum_{j=l+1}^{n} a_j(y_j + \sum_{i=1}^{l} b_{j,i}x + i)) \\
&= (\sum_{i=1}^{l} a_i x_i) + (\sum_{i=1}^{l}\sum_{j=l+1}^{n} a_j b_{j,i}x_i) + (\sum_{j=l+1}^{n} a_j y_j) \\
&= \sum_{i=1}^{l}(a_i + \sum_{j=l+1}^{n} a_j b_{j,i})x_i + \sum_{j=l+1}^{n} a_j y_j
\end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^{l}(a_i + \sum_{j=l+1}^{n} a_j b_{j.i})\pi(x_i)$$

Therefore $(y_j)_{j=l+1}^n$ is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + tk(V/W)$$

(2) All basis of V have $rk(V)$ elements.

We reason by induction on $rk(V)$

(1)
$$rk(V) = 0$$

In this case $V = \{0\}$ The only basis of V is $\varnothing$. So the statement holds.

(2) Assume that there exists $e \in V \setminus \{0\}$ such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of V is of the form

$$ae$$

where $a \in K\{0\}$

Let $(e_i)_{i=1}^m$ be a basis of V. We reason by induction on m to prove that

$$m = rk(V)$$

The cases where $m = 0$ or 1 are proved in (1)(2) respectively. Induction hypothesis: true for a basis of $< m$ elements

Let
$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let
$$\pi : V \to V/W$$
$$x \mapsto [x]$$

Then
$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of V/W.

If $(a_i)_{i=2}^m \in K^{m-1}$ such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then
$$\sum_{i=2}^m a_i e_i \in W$$

Hence
$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for $(e_i)_{i=1}^m$ a basis of V,
$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of V/W. We then obtain that

$$rk(V/W) \leq m-1 \leq n-1$$

By the induction hypothesis,

$$m-1 = rk(V/W)$$

By (2), $rk(W) = 1$. Hence

$$m = (m-1)+1 = rk(V/W) + rk(W) = rk(V)$$

21.11 Prop

Let K be a unitary ring and $f : E \to F$ be a morphism of left K-modules. Let I be a set and $(x_i)_{i\in I} \in E^I$

- If $(x_i)_{i\in I}$ is linearly independent and f is injective, then $(f(x_i))_{i\in I}$ is linearly independent.
- If $(x_i)_{i\in I}$ is a system of generators and f is surjective, then $(f(x_i))_{i\in I}$ is a system of generators.
- If $(x_i)_{i\in I}$ is a basis and f is an isomorphism, then $(f(x_i))_{i\in I}$ is a basis.

21.11.1 Proof

$$\varphi_{(f(x_i))_{i\in I}} = f \circ \varphi_{(x)_{i\in I}}$$

Chapter 22

Matrices

We fix unitary ring K

22.1 Def

Let $n \in \mathbb{N}$ and V be a left K-module.

For any $(x_i)_{i=1}^n \in V^n$, we denote by $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ the morphism

$$\begin{aligned} \phi_{(x_i)_{i=1}^n} : & K^n \to V \\ (a_i)_{i=1}^n \mapsto & \sum_{i=1}^n a_i n_i \end{aligned}$$

22.1.1 Example

Suppose that $V = K^p (p \in \mathbb{N})$ Then each $x_i \in K^p$ is of the form $(x_{i,1}, ..., x_{i,p})$

Hence $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$ can be written:

$$\begin{pmatrix} x_{1,1} & . \; . & . \; x_{1,p} \\ . & & . \\ . & & . \\ . & & . \\ x_{n,1} & . \; . & . \; x_{n,p} \end{pmatrix}$$

22.2 Def

Let $(n,p) \in \mathbb{N}^2$. We call n by p matrix of coefficient in K any morphism of left K-modules from K^n to K^p

22.2.1 Example

- Denote by I_n then identity mapping.Then $(e_i)_{i=1}^n$ is a basis of K^n called the canonical basis of K^n

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, ..., a_n)) = \sum_{i=1}^n a_i e_i = (a_1, ..., a_n)$$

- Let $(x_1, ..., x_n) \in K^n$, Denote by

$$\begin{aligned} diag(x_1, ...x_n)(= \varphi_{(x_i e_i)_{i=1}^n}) :& K^n \to K^n \\ & (a_1, ..., a_n) \mapsto (a_1 x_1, ..., a_n x_n) \end{aligned}$$

22.3 Def

We denote by $M_{n,p}(K)$ the set of all n by p matrices of coefficients in K. For $(n,p,r) \in \mathbb{N}^3$,we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) \to & \quad M_{n,r}(K) \\ (A, B) \mapsto & \quad AB := B \circ A \end{aligned}$$

22.4 Calculate Matrices

Let K be a unitary ring , and V be a left K-module. Let $n \in \mathbb{N}$ and

$$x = (x_1, ..., x_n) \in V^n$$

22.4.1 Remind

$$\begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \varphi : (a_1, ..., a_n) \mapsto a_1 x_1, ..., a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1,...,p\} \times \{1,...,n\}} \in M_{p,n}(K)$$

A is a morphism of left K-modules from K^p to K^n Recall that

$$A \begin{pmatrix} x1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let $(b_1, ..., b_n) \in K^p$

$$\begin{aligned} A((b_1, ..., b_n)) &= \sum_{i=1}^{p} b_i (a_{i,1}, ..., a_{i,n}) \\ \varphi(A((b_1, ..., b_n))) &= \sum_{i=1}^{p} b_i \varphi_x((a_{i,1}, ..., a_{i,n})) \\ &= \sum_{i=1}^{p} b_i (a_{i,1} x_1, ..., a_{i,n} x_n) \end{aligned}$$

Let $B = \{b_{ij}\}_{(i,j) \in \{1,...,n\} \times \{1,...,r\}} : K^n \to K^r$

$$AB = \left\{ \sum_{j=1}^{n} a_{lj} b_{jm} \right\}_{(l,m) \in \{1,...,p\} \times \{1,...,r\}}$$

Chapter 23

Transpose

We fix a unitary ring K

23.1 Def

Let E be a left-K-module. Denote by

$$E^\vee := \{\text{morphisms of left K-modules } E \to K\}$$

$\forall (f,g) \in E^\vee$ let

$$\begin{aligned} f+g : & E \to K \\ & x \mapsto f(x)+g(x) \end{aligned}$$

$(E^\vee, +)$ forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : & E \to K \\ & x \mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} & K \times E^\vee \to E^\vee \\ & (a,f) \mapsto fa : x \in E \to f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right K-module on $E^\vee$

23.2 Def

Let E and F be two left K-modules. $\varphi : E \to F$ be a morphism of left K-modules. We denote by

$$\varphi^\vee : F^\vee \to E^\vee$$

the morphism of right K-modules sending $g \in F^\vee$ to $g \circ \varphi \in E^\vee$
Actually $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

23.2.1 Example

Suppose that $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & . & . & . & b_{1,p} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ b_{n,1} & . & . & . & b_{n,p} \end{pmatrix}$$

φ sends $(a_1, ..., a_n)$ to $\{\sum_{i=1}^{n} a_i b_i j\}_{j\in\{1,...,p\}}$ Let $g \in F^\vee \quad g : K^p \to K$, then g is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$ sends $(a_1, ..., a_n)$ to $\sum_{i=1}^{p}(\sum_{i=1}^{n} a_i b_{ij} y_j)$

Assume that K is commutative. We denote by

$$\iota_p : (K^p)^\vee \to K^p$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, ..., x_p)$$

$$\iota_n : (K^n)^\vee \to K^n$$
$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, ..., x_n)$$

are isomorphisms of K-modules

For any morphism of K-modules $\varphi : K^n \to K^p$, we denote by φ^τ the morphism of K-modules $K^p \to K^n$ given by $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{tikzcd}
(K^p)^\vee \arrow[r, "\varphi^\vee"] \arrow[d, "\iota_p", "\cong"'] \arrow[dr, phantom, "\circlearrowleft"] & (K^n)^\vee \arrow[d, "\cong"'] \\
K^p \arrow[r, "\varphi^\tau"'] & K^n
\end{tikzcd}$$

φ^τ is called the transpose of φ

23.3 Prop

Let E,F,G be left K-modules. $\varphi : E \to F, \psi : F \to G$ be morphisms of left K-modules. Then $(\psi \circ \varphi)^\vee$ is equal to $\varphi^\vee \circ \psi^\vee$

Proof

$\forall f \in G^\vee$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

23.4 Corollary

Assume that K is commutative. Let n, p, q be neutral numbers. $A \in M_{n,p}(K), B \in M_{p,q}(K)$. Then

$$(AB)^\tau = B^\tau A^\tau$$

Proof

$$A^tau = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^tau = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned}
B^\tau A^\tau &= A^\tau \circ B^\tau \\
&= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\
&= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\
&= (AB)^tau
\end{aligned}$$

23.5 Remark

(1) For $A \in M_{n,p}(K)$, one has $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} &E \to (E^\vee)^\vee \\ &x \mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a K-linear mapping.

If K is a field and E is of finite dimension, this is a isomorphism of K-modules.

In fact, if $e = (e_i)_{i=1}^n$ is a basis of E over K.For $i \in \{1, ..., n\}$, let

$$\begin{aligned} &e_i^\vee : E \to K \\ &\lambda_1 e_1, ..., \lambda_n e_n \mapsto \lambda_i \end{aligned}$$

is called the dual basis of e

$$\begin{tikzcd} K^n \arrow[d, "\varphi_e"', "\cong"] \arrow[dr, "\varphi_{e^\vee}"] & (K^n)^\vee \arrow[l, "\cong"', "\iota_n"] \arrow[d, "\cong"', "\varphi_e^\vee"] \\ E \arrow[r, "\cong"'] & E^\vee \end{tikzcd}$$

$(e^\vee)^\vee$ gives a basis of $(E^\vee)^\vee$ Hence $E \to (E^\vee)^\vee$ is an isomorphism.

Chapter 24

Linear Equation

We fix a unitary ring K.

24.1 Def

For $a = (a_1, ..., a_n) \in K^n \setminus \{(0, ..., 0)\}$. Denote by $j(a)$ the first index $j \in \{1, ..., n\}$ such that $a_j \neq 0$.Let $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$. We write A as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, ..., a_n^{(i)}) \in K^p$$

We say that A is of row echelon form if,$\forall i \in \{1, ..., n-1\}$ one of following conditions is satisfied.

- $a^{(i+1)} = (0, ..., 0)$
- $a^{(i)}, a^{(i+1)}$ are non-zero, and $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, ..., n\}$ such that $a^{(i)} \neq (0, ..., 0)$,one has

  $$a_{j(a^{(i)})}^{(i)} = 1$$

  and

  $$\forall k \in \{1, ..., n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that A is of reduced row echelon form.

24.2 Prop

Suppose that $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$ is of row echelon form. Then $\{i \in \{1, ..., n\} \mid a^{(i)} \neq (0, ..., 0)\}$ is of cardinal $\leq p$

Proof

Let $k = card\{i \in \{1, ..., n\} \mid a^{(i)} \neq (0, ..., 0)\}$ $a^{(k+1)} = ... = a^{(n)} = (0, ..., 0)$ and $j(a^{(1)}) < j(a^{(2)}) < ... < j(a^{(k)})$ Hence

$$\{1, ..., k\} \to \{1, ..., p\}, i \mapsto j(a^{(i)})$$

is injection. So $k \leq p$

24.3 Linear Equation

Let $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$. Let V be a left K-module and $(b_1, ..., b_n) \in V^n$. We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

The set of $(x_1, .., x_p) \in V^p$ that satisfies (*) is called the solution set of (*)

24.4 Prop

Suppose that A is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, ..., n\} \mid (a_{i,1}, ..., a_{i,p}) \neq (0, ..., 0)\}$$

$$J_0(A) = \{1, ..., p\} \setminus \{j((a_{i,1}, ..., a_{i,p})) \mid i \in I(A)\}$$

- If $\exists i \in \{1, ..., n\} \setminus I(A)$ such that $b_i \neq 0$ then (*) does not have any solution in K^n
- Suppose that $\forall i \in \{1, ..., n\} \setminus I(A), b_i = 0$. Then (*) has at least one solution. Moreover
$$V^{J_0(A)} \to V^p$$
$$(z_k)_{k \in J_0(A)} \mapsto (x_1, ..., x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, ..., a_{i,p})) \end{cases}$$

is an injective mapping , whose image is equal to the set of solution of (*)

24.5 Prop

Let $m \in \mathbb{N}, S \in M_{m,n}(K)$. If $(x_1, ..., x_p) \in V^p$ is a solution of (*), then $(x_1, ..., x_p)$ is a solution of $(*)_S$:

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (*)$$

In the case where S is left invertible, namely there exist $R \in M_{n,m}(K)$ such that $RS = I_n \in M_{m,n}(K)$. Then (*) and $(*)_S$ have the same solution set.

24.6 Def

Let $G_n(K)$ be the set of $S \in M_{n,n}(K)$ that can be written as $U_1 ... U_N$(by convention $S = I_n$ where $N = 0$) where each U_i is of one of the following forms.

- P_σ where $\sigma \in \mathfrak{S}_n$
- $diag(r_1, ..., r_n)$ where each $r_i \in K$ is left invertible
- $S_{i,c}$ with $i \in \{1, ..., n\}$ $c = (c_1, ..., c_n) \in K^n, c_i = 0$

Let $p \in \mathbb{N}$,we say that $A \in M_{n,p}(K)$ is reducible by Gauss elimination if $\exists S \in G_n(K)$ such that SA is of reduced row echelon form

24.7 Theorem

Assume that K is a division ring $\forall (n, p) \in \mathbb{N}$ any $A \in M_{n,p}(K)$ is reducible by Gauss elimination

Proof

The case where $n = 0$ or $p = 0$ is trivial. We assume $n \geq 1, p \geq 1$ We write A as

$$\begin{pmatrix} \lambda_1 & \\ \vdots & B \\ \lambda_n & \end{pmatrix} \text{ where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If $\lambda_1 = ... = \lambda_n = 0$

  Applying the induction hypothesis to B, for $S \in G_n(K)$

$$SA = \begin{pmatrix} S\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 & \\ \vdots & SB \\ 0 & \end{pmatrix}$$

- Suppose that $(\lambda_1, ..., \lambda_n) \neq (0, ..., 0)$

  By permuting the rows we may assume $\lambda_1 \neq 0$. As K is division ring, by multiplying the first row by λ_1^{-1} , we amy assume $\lambda_1 = 1$. We add $(-\lambda_i)$ times the first row to the i^{th} row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 \cdots & \mu_p \\ 0 & & \\ \vdots & C & \\ 0 & & \end{pmatrix} \qquad \begin{array}{l} C \in M_{n-1,p-1}(K) \\ (\mu_2, ..., \mu_p) \in K^{p-1} \end{array}$$

  Applying the induction hypothesis to C, we say assume that C is of reduced row echelon form . For $i \in \{2, ..., k\}$ we add $-\mu_{j(c_i)}$ times the i^{th} row of A to the first line to obtain a matrix of reduced row echelon form

Chapter 25

Normed Vector Space

25.1 Def

Let (X, d) be a metric space. If $(x_n)_{n\in\mathbb{N}}$ is an element of $X^{\mathbb{N}}$ such that

$$\lim_{N\to+\infty} \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} d(x_n, x_m) = 0$$

we say that $(x_n)_{n\in N}$ is a Cauchy sequence. If any Cauchy sequence in X converges, then we say that (X, d) is complete.

Let $Cau(X, d)$ be the set of all Cauchy sequences in X. We define a binary relation $\sim$ on $Cau(X, d)$ as

$$(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$$

iff

$$\lim_{n\to+\infty} d(x_n, y_n) = 0$$

25.2 Prop

$\sim$ is an equivalence relation.

25.2.1 Proof

$$\lim_{n\to+\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ be elements of $Cau(X, d)$. For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n\to+\infty} d(x_n, z_n) = 0$$

25.3 Def

$\hat{X} := Cau(X, d)\backslash \sim$

25.4 Def: The completion

The completion of (X, d) is defined as

$$Cau(X)/\sim$$

and is denoted as

$$\hat{X}$$

25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} :& \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0} \\ &(x, y) \mapsto \lim_{n\to+\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on $\hat{X}$

Proof

TO check that $\hat{d}$ is well defined , it suffices to prove that $\forall([x], [y]) \in \hat{X} \times \hat{X}$, $(d(x_n, y_n))_{n\in\mathbb{N}}$ is Cauchy sequence and its limit doesn't depend on the choice of the representation x and y
For $N \in \mathbb{N}$ and $(n, m) \in \mathbb{N}_{\geq N}$ for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|dd(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n,m\in\mathbb{N}_{\geq N})} |d(x_n, y_n) - d(x_m, y_m)| \leq &(\sup_{(n,m\in\mathbb{N}_{\geq N})} d(x_n, x_m)) \\ &+(\sup_{(n,m\in\mathbb{N}_{\geq N})} d(y_n, y_m)) \end{aligned}$$

Taking $\lim\limits_{N\to+\infty}$ we obtain that $(d(x_n, y_n))_{n\in\mathbb{N}}$ is a Cauchy sequence.
Hence it converges in $\mathbb{R}$. If $x' = (x'_n)_{n\in\mathbb{N}} \in [x], y' = (y'_n)_{n\in\mathbb{N}} \in [y]$, thus

$$\lim_{n\to+\infty} d(x_n, x'_n) = \lim_{n\to+\infty} d(y_n, y'_n) = 0$$

$$0 \le |d(x_n, y_n) - d(x'_n, y'_n)| \le d(x_n, x'_n) + d(y_n, y'_n)$$

Taking $\lim\limits_{n\to+\infty}$ we get

$$\lim_{n\to+\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n\to+\infty} d(x_n, y_n) = \lim_{n\to+\infty} d(x'_n, y'_n)$$

In the following, we check that $\hat{d}$ is a metric

- $\hat{d}([x], [y]) = 0$ iff $[x] = [y]$: trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$: trivial
- $\hat{d}([x], [y]) \le \hat{d}([x], [z]) + \hat{d}([z], [y])$:

$$\begin{aligned} d([x], [y]) &= \lim_{n\to+\infty} \\ &\le \lim_{n\to+\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

25.6 Remark

Let

$$\begin{aligned} i_X :& X \to \hat{X} \\ & a \mapsto [(a, a, ...)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular, i_x is injective(if $i_X(a) = i_X(b)$ then $d(a, b) = 0$ hence $a = b$)

25.7 Prop

$i_X(X)$ is dense in $\hat{X}$ (the closure of $i_X(X)$ in $\hat{X}$ is equal to $i_X(X)$ (or to say $\hat{X}$))

Proof

Let $[x]$ be an equivalence class in $\hat{X}$. We claim that $\forall (x_n)_{n\in\mathbb{N}} \in [x]$

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} i_X(x_n)$$

For any $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n\to+\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m)\in\mathbb{N}^2_{\geq N}} d(x_n, x_m) \end{aligned}$$

Taking $\lim\limits_{N\to+\infty}$ we get

$$\lim_{N\to+\infty} \hat{d}(i_X(x_N), [x]) = 0$$

25.8 Theorem

$(\hat{X}, \hat{d})$ is a complete metric space

Proof

Let $([x^{(N)}])_{N\in\mathbb{N}}$ be a Cauchy sequence in $\hat{X}$, where $\forall N \in \mathbb{N}$, $x^{(N)} = (x_n^{(N)})_{n\in\mathbb{N}}$ is a Cauchy sequence
$\forall \epsilon > 0,\ \exists N_0 \in \mathbb{N}$ such that $\forall (k,l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n\to+\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any $(\mu,\nu) \in \mathbb{N}_{\geq \alpha(N)}$
Let $y_N = x_{\alpha(N)}^{(N)}$. Without loss of generality , we assume that

$$\alpha(0) \leq \alpha(1) \leq ...$$

Let $\epsilon > 0$ Take $N_0 \in \mathbb{N}$ such that

(1) $\forall (k,l) \in \mathbb{N},\ k,l \geq N_0$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

(2)

$$\frac{1}{N_0+1} \leq \frac{\epsilon}{3}$$

Let $(k,l) \in \mathbb{N}_{N_0}^2$,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since $\alpha(k) \geq N_0$, $\forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking $\lim\limits_{n \to +\infty}$ get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So $y = (y_N)_{N \in \mathbb{N}}$ is a Cauchy sequence. We check that

$$\lim_{N \to +\infty} \hat{d}([x^{(N)}], [y]) = 0$$

$$\begin{aligned} 0 &\leq \limsup_{N \to +\infty} \lim_{n \to +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \to +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$d(x_n^{(N)}, y_n) \leq d(x_n^{(N)}, y_N) + d(y_n, y_N)$$

$$\limsup_{N \to +\infty} \lim_{n \to +\infty} d(x_n^{(N)}, y_n) \leq \limsup_{N \to +\infty} (\frac{1}{N+1} + \lim_{n \to +\infty} d(y_n, y_N))$$

Since y is Cauchy sequence

$$\leq \limsup_{N \to +\infty} \lim_{n \to +\infty} d(y_n, y_N) = 0$$

Example

Let $(K, |\cdot|)$ be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$ iff $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a + b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$Cau(K)$ forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n\to+\infty}(a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n\in\mathbb{N}} \in Cau_0(K)$$

where

$$Cau_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of $Cau(K)$
Hence

$$\hat{K} = Cau(K) \setminus Cau_0(K)$$

is a quotient ring of $Cau(K)$
$|\cdot|$ extend to $\hat{K}$:

$$|[(a_n)_{n\in\mathbb{N}}]| = \lim_{n\to+\infty}|a_n|$$

that forms an absolute value.

Chapter 26

Norms

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\to \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

26.1 Def

Let V be a vector space over K (K-module). We call seminorm on V any mapping

$$\begin{aligned} \|\cdot\| : & V \to \mathbb{R}_{\geq 0} \\ & s \mapsto \|\cdot\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V,\ \|s\| = 0$ iff $s = 0$

We say that $\|\cdot\|$ is a norm and $(V, \|\cdot\|)$ is normed space over K.

26.2 Remark

If $\|\cdot\|$ is a norm then

$$\begin{aligned} d : & V \times V \to \mathbb{R}_{\geq 0} \\ & (s, t) \mapsto \|s - t\| \end{aligned}$$

sectionDef Let $(V, \|\cdot\|)$ be a vector space over K equipped with a seminorm, and W be a vector space subspace of V (sub-K-module)

- The restriction of $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ to W forms a seminorm on W. It is a norm if $\|\cdot\|$ is a norm.
$$\begin{aligned}\|\cdot\|_W : & W \to \mathbb{R}_{\geq 0} \\ & x \mapsto \|x\|\end{aligned}$$
- The mapping
$$\begin{aligned}\|\cdot\|_{V/W} : & V/W \to \mathbb{R}_{\geq 0} \\ & \alpha \mapsto \inf_{s\in\alpha} \|s\| \\ \|[s]\|_{V/W} = & \inf_{w\in W} \|s+w\|\end{aligned}$$
is a seminorm on V/W
Attention: Even if $\|\cdot\|$ is a norm, $\|\cdot\|_{V/W}$ **might only be a seminorm**

26.3 Def

$\|\cdot\|_{V/W}$ is called the quotient seminorm of $\|\cdot\|$

26.4 Prop

Let $(V, \|\cdot\|)$ be a vector space over K, equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of V. Moreover, $\|\cdot\|_{V/N}$ is a norm

Proof

If $(a, s) \in K \times N$ then $\|as\| = |a| \cdot \|s\| = 0$ so $as \in N$
If $(s_1, s_2) \in N \times N$ then $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$ so $s_1 + s_2 \in N$

Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s\in\alpha} \|\lambda s\| = \inf_{s\in\alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s\in\alpha+\beta} \|s\| = \inf_{(x,y)\in\alpha\times\beta} \|x+y\| \\ &\leq \inf_{(x,y)\in\alpha\times\beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let $\alpha \in V/N$ such that $\|\alpha\|_{W/N} = 0$ Let $s \in \alpha, \forall t \in N$

$$\|s+t\| \leq \|s\| + \|t\| = \|s\| = \|(s+t) + (-t)\| \leq \|s+t\| + \|-t\| = \|s+t\|$$

$$\|\alpha\|_{V/N} = \inf_{t\in N} \|s+t\| = \|s\|$$

Hence $\|\alpha\|_{V/N} = \|s\| = 0$ We obtain that $\alpha = N = [0]$

26.5 Def

Let $(V, \|\cdot\|)$ be a vector space over K,equipped with a seminorm. For any $x \in V$ and $r \geq 0$, we denote by

$$\mathcal{B}(x,r) = \{y \in V \mid \|y - x\| < r\}$$
$$\overline{\mathcal{B}}(x,r) = \{y \in V \mid \|y - x\| \leq r\}$$

26.6 Remark

If $N = \{s \in V, \|s\| = 0\}$ then when $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x,r)$$

$$x + N \subseteq \mathcal{B}(x,r)$$

26.7 Def

We equip the topology such that $\forall U \subseteq V, U$ is open iff $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

26.8 Prop

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorms. Let $f : V_1 \to V_2$ be a K-linear mapping

- If f is continuous, $\forall s \in V_1$ if $\|s\|_1 = 0$ then $\|f(s)\|_2 = 0$
- If there exists $C > 0$ such that $\forall x \in V_1,\ \|f(x)\|_2 \leq C\|x\|_1$ then f is continuous.

  The converse is true

when $|\cdot|$ is non-trivial

or $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$ is of finite type

Proof

(1) Lemma If $(V, \|\cdot\|)$ is a vector space over K, equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let $s \in V \setminus N_{\|\cdot\|}$ Then $\|s\| > 0$.Let $\epsilon = \frac{\|s\|}{2},\ \forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq |\|s\| - \|s - x\|| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteqq V \setminus N_{\|\cdot\|}$$

- Then $f^{-1}(N_{\|\cdot\|_2})$ is closed.
  Note that
  $$0 \in f^{-1}(N_{\|\cdot\|_2})$$
  hence
  $$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$
  $\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$
  $$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x,\epsilon)$$
  and
  $$0 \in \mathcal{B}(x,\epsilon)$$
  Therefore $x \in \overline{\{0\}}$

(2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of V_1 that converges to some $x \in V_1$
Hence
$$\begin{aligned}\limsup_{n\to+\infty}\|f(x_n)-f(x)\|_2 &= \limsup_{n\to+\infty}\|f(x_n-x)\|\\ &\leq \limsup_{n\to+\infty} C\|x_n-x\|_1\\ &= C\limsup_{n\to+\infty}\|x_n-x\|\\ &= 0\end{aligned}$$
So $(f(x_n))_{n\in\mathbb{N}}$ converges to $f(x)$. Hence f is continuous at x
Assume that $|\cdot|$ is non-trivial and f is continuous.Then
$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
is an open subset of V_1 containing $0 \in V_1$
So there exists $\epsilon > 0$ such that
$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$
namely $\forall x \in V_1$ if $\|x\|_1 < \epsilon$ then $\|f(x)\|_2 < 1$
Since $|\cdot|$ si nontrivial, $\exists a \in K,\ 0 < |a| < 1$ We prove that $\forall x \in V_1$
$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|}\|x\|_1$$
If $\|x\|_1 = 0$ by (1) we obtain
$$\|f(x)\|_2 = 0$$
Suppose that $\|x\|_1 > 0$ then $\exists n \in \mathbb{Z}$ such that
$$\|a^n x\|_1 = |a|^n\|x\|_1 < \epsilon \leq \|a^{n-1}x\|_1 = |a|^{n-1}\|x\|_1$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned}\|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}}\frac{1}{|a|}\\ &\leq \frac{1}{\epsilon}\|x\|_1\frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|}\end{aligned}$$

26.9 Def: Operator Seminorm

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with seminorm. We say that a K-linear mapping $f : V_1 \to V_2$ is bounded if there exists $C > 0$ that

$$\forall x \in V_1 \quad \|f(x)\| \leq C\|x\|_1$$

For a general K-linear mapping $f : V_1 \to V_2$ we denote

$$\|f\| := \begin{cases} \sup\limits_{x\in V_1, \|x\|_1>0}(\dfrac{\|f(x)\|_2}{\|x\|_1}) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \nsubseteq N_{\|\cdot\|_2}) \end{cases}$$

f is bounded iff

$$\|f\| < +\infty$$

$\|f\|$ is called the operator seminorm of f

We denote by $\mathscr{L}(V_1, V_2)$ the set of all bounded K-linear mappings from V_1 to V_2

26.10 Prop

$\mathscr{L}(V_1, V_2)$ is a vector subspace of $Hom_K(V_1, V_2)$. Moreover $\|\cdot\|$ is a seminorm on $\mathscr{L}(V_1, V_2)$

Proof

Let f, g be elements of $\mathscr{L}(V_1, V_2)$

$$\begin{aligned}\|f+g\| &= \sup_{x\in V_1, \|x\|_1\neq 0}\frac{\|f(x)+g(x)\|_2}{\|x\|_1}\\ &\leq \sup_{x\in V_1, \|x\|_1\neq 0}\frac{\|f(x)\|_2+\|g(x)\|_2}{\|x\|_1}\\ &\leq (\sup_{x\in V_1, \|x\|_1\neq 0}\frac{\|f(x)\|_2}{\|x\|_1}) + (\sup_{x\in V_1, \|x\|_1\neq 0}\frac{\|g(x)\|_2}{\|x\|_1})\\ &\leq +\infty\end{aligned}$$

Hence $f+g \in \mathscr{L}(V_1, V_2)$

Let $\lambda \in K,\ \lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x\in V_1, \|x\|_1>0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x\in V_1, \|x\|_1>0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda|\|f\| < +\infty\end{aligned}$$

26.11 Remark

Let $f \in \mathscr{L}(V_1, V_2)$. Suppose that $\exists x \in V_1$ such that $f(x) \neq 0$. Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore $\|\cdot\|$ is a norm

26.12 Def

Let $(V, \|\cdot\|)$ be a normed vector space. If V is complete with respect to the metric

$$\begin{aligned} d : &V \times V \to \mathbb{R}_{\geq 0} \\ &(x,y) \mapsto \|x-y\|\end{aligned}$$

then we say that $(V, \|\cdot\|)$ is a Banach space.

26.13 Theorem

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K, equipped with semi-norm. If $(V_2, \|\cdot\|_2)$ is a Banach space, then

$$(\mathscr{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

Proof

Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{L}(V_1, V_2)$.
$\forall x \in V_1$, the mapping

$$(f \in \mathscr{L}(V_1, V_2)) \mapsto f(x)$$

is $\|x\|_1$-Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f-g)(x)\|_2 \leq \|f-g\|\|x\|_1$$

So $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, for V_2 is complete, that converges to some $g(x) \in V_2$ Then we obtain a mapping $g : V_1 \to V_2$.We prove that g is an element of $\mathscr{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n\to+\infty} f_n(x+y) = \lim_{n\to+\infty} f_n(x) + f_n(y)$$

$$\begin{aligned}\|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \to +\infty)\end{aligned}$$

  So

$$\lim_{n\to+\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n\to+\infty} f_n(\lambda x) = \lim_{n\to+\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1)(n \to +\infty)$$

  So $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n\to+\infty} \|f_n(x)\| \leq (\lim_{n\to+\infty} \|f_n\|) \cdot \|x\|$$

  (because $\forall (a, b) \in V_2^2 \quad |\|a\| - \|b\|| \leq \|a - b\|$)Then

$$|\|f_n(x)\| - \|g_n(x)\|| \leq \|f_n(x) - g_n(x)\| = o(1) \ (n \to +\infty)$$

  So $g \in \mathscr{L}(V_1, V_2)$

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \ \forall (n, m) \in \mathbb{N}_{\geq N}, \ \|f_n - f_m\| \leq \epsilon$$

  $\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

  Taking $\lim_{n\to+\infty}$ we get

$$\|(f_n - g)(x)\| \leq \epsilon \, \|x\|$$

  So $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$

Chapter 27

Differentiability

In this chapter we fix a field K and an absolute value $|\cdot|$ on K. We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\to \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

27.1 Def

Let X be a topological space and $p \in X$. Let K be a complete valued field and $(E, \|\cdot\|)$ be a normed vector space over K.

Let $f : X \to E$ be a mapping and $g : X \to \mathbb{R}_{\geq 0}$ be a non-negative mapping.

- We say that
$$f(x) = O(g(x)) \ x \to p$$
if there is a neighborhood V of p in X and a constant $C > 0$ such that $\forall x \in V$
$$\|f(x)\| \leq Cg(x)$$

- We say that
$$f(x) = o(g(x)) \ x \to p$$
if there exists a neighborhood V of p in X and a mapping $\epsilon : V \to \mathbb{R}_{\geq 0}$ such that
$$\lim_{x \in V, x \to p} \epsilon(x) = 0$$
which is equivalent to
$$\forall \delta > 0, \exists \text{ neighborhood U of } p \ U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$
and $\forall x \in V$
$$\|f(x)\| \leq \epsilon(x)g(x)$$

27.2 Def

Let E and F be normed vector space over K $U \subseteq E$ be an open subset, $f : U \to F$ be a mapping and $p \in U$ If there exists $\varphi \in \mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|) \quad x \to p$$

we say that f is differentiable at p, and φ is the differential of φ at p Suppose that $|\cdot|$ is not trivial.$\varphi(x-p)$ also written as

$$d_p f$$

Reminder

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|) \quad x \to p$$

means there exists an open neighborhood V of p with $V \subseteq U$ and a mapping $\epsilon V \to \mathbb{R}_{\geq 0}$ such that $\lim\limits_{x \to p} \epsilon(x) = 0$ and that $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x-p)\| \leq \epsilon(x) \cdot \|x-p\|$$

27.3 Prop

If f is differentiable at p ,then its differential at p is unique

Proof

Suppose that there exists φ and ψ in $\mathscr{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x-p) + o(\|x-p\|)$$

$$f(x) = f(p) + \psi(x-p) + o(\|x-p\|)$$

then

$$(\varphi - \psi)(x-p) = o(\|x-p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|\varphi - \psi\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y-p\| \leq \delta} \frac{\|\varphi - \psi\| (y-p)}{\|y-p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y-p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \to p} \epsilon(y) = 0 \end{aligned}$$

27.4 Example

27.4.1

$$f : U \to F : \ f(x) = y_0 \ \forall x \in U$$

$\forall p \in U$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence $\forall x \in E$

$$d_p(f(x)) = 0$$

27.4.2

Let $f \in \mathscr{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence $d_p f = f$

27.4.3

$$\begin{aligned} A :& E \times E \to E \\ &(x, y) \mapsto x + y \end{aligned}$$

Let E be a normed space. Then $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

27.4.4

$$\begin{aligned} m :& K \times E \to E \\ &(\lambda, x) \mapsto \lambda x \end{aligned}$$

Let $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when $(\lambda, x) \to (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

  $$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

  is a K-linear mapping.

- $(\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2)$
- $b\mu p + a(by) = b(\mu p + ay)$
- $$\begin{aligned} \|\mu p + ay\| &\leq |\mu| \, \|p\| + |a| \, \|y\| \\ &\leq \max\{|\mu| \, , \|y\|\}(|a| + \|p\|) \end{aligned}$$

Hence m is differentiable and $\forall(\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

27.5 Theorem:Chain rule

Let E, F, G be normed vector spaces, $U \subseteq E, V \subseteq F$ be open subsets.

Let $f : U \to F, \ g : V \to G$ be mappings such that $f(U) \subseteq V$ Let $p \in U$ Assume that f is differentiable at p and g differentiable at $f(p)$ Then $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

Proof

Let $x \in U$ By definition

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ f(x) - f(p) &= O(\|x - p\|) \end{aligned}$$

and

$$\begin{aligned} (g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\ &= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|) \end{aligned}$$

So $g \circ f$ is differentiable at p and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

27.6 Prop

Let n be a positive integer $E, (F_i)_{i \in \{1,\ldots,n\}}$ be normed vector spaces over K. $U \subseteq E$ an open subset, $p \in U$

$\forall i \in \{1, ..., n\}$ let $f_i : U \to F_i$ be a mapping. Let

$$f : U \to F = \prod F_i$$

be the mapping that sends $x \in U$ ot $(f_i(x))_{i \in \{1,\ldots,n\}}$ We equip F with the norm $\|\cdot\|$ defined as :

$$\left\|(y_i)_{i \in \{1,\ldots,n\}}\right\| = \max_{i \in \{1,\ldots,n\}} \|y_i\|$$

Then f is differentiable at p iff each f_i is differentiable at p. Moreover, when this happen, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i\in\{1,\ldots,n\}}$$

Proof

$\Leftarrow$ Suppose that $(f_i)_{i\in\{1,\ldots,n\}}$ are differentiable at p

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i\in\{1,\ldots,n\}} \\ &= (d_p f_i(x-p))_{i\in\{1,\ldots,n\}} + o(\|x-p\|) \end{aligned}$$

Therefore f is differentiable at p and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i\in\{1,\ldots,n\}}$$

$\Rightarrow$ Let

$$\begin{aligned} \pi_i : & F \to F_i \\ & (x_i)_{i\in\{1,\ldots,n\}} \mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has $\|\pi_i\| \leq 1$ because

$$\|x_i\| \leq \max_{i\in\{1,\ldots,n\}} \|x_i\| = \left\|(x_i)_{i\in\{1,\ldots,n\}}\right\|$$

π_i is differentiable at p then $\pi_i \circ f = f_i$ is differentiable at p

27.7 Def

Let U be an open subset of K and $(F, \|\cdot\|)$ be a normed vector space. If $f : U \to F$ is a mapping that is differentiable at some $p \in U$. We denote by $f'(p)$ the element

$$d_p f(1) \in F$$

called the derivative of f at p

27.8 Corollary

Let U and V be open subsets of K, $(F, \|\cdot\|)$ be a normed vector space over K. $f : U \to K,\ g : V \to F$ be mappings such that $f(U) \subseteq V$. Let $p \in U$. If f is differentiable at p and g is differentiable at $f(p)$ then

$$(g \circ f)'(p) = f'(p)g'(f(p))$$

Proof

By definition

$$\begin{aligned} d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p) \cdot 1) \\ &= f'(p) \cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

27.9 Corollary

Let E and F be normed vector spaces, $U \subseteq E$ an open subset. $f : U \to L$ and $g : U \to F$ be mappings and $p \in U$ If both f, g differentiable at p then

$$\begin{aligned} fg :&U \to F \\ &x \mapsto f(x)g(x) \end{aligned}$$

is also differentiable at p and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_pf(l) + g(p)d_pf(l)$$

Proof

Consider

$$\begin{aligned} m :&K \times F \to F \\ &(a, y) \to ay \end{aligned}$$

We have shown m is differentiable and

$$d_{a,y}m(b, z) = by = az$$

fg is the following composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "fg"'] & K \times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m \circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p),g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

27.10 Corollary

Let U be an open subset of K, f, g be mappings from U to K and to a normed space F respectively.If f, g are differentiable at $p \in U$ then

$$(fg)'(p) = d_p(fg)(1) = d_p f(1)g(p) + f(p)d_p g(1) = f'(p)g(p) + f(p)g'(p)$$

Example

$$\begin{aligned} f_n :& K \to K \\ & x \mapsto x^n \end{aligned}$$

is differentiable at any $x \in K$

$$f_n'(x) = nx^{n-1}$$

Proof

$f_1 : K \to K$ is differentiable $\forall x \in K$

$$d_x f_1 = f_1$$

If $f_n'(x) = nx^{n-1}$ then

$$\begin{aligned} f_{n+1}'(x) &= (f_n f_1)'(x) \\ &= f_n(x)f_1'(x) + f_n'(x)f_1(x) \\ &= x^n + x_n'(x) = x^n + xnx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(l) &= l d_x f_n(1) \\ &= lnx^{n-1} \end{aligned}$$

27.11 Prop

Let E, F, G be normed vector spaces. $U \subseteq E$ be an open subset, $\varphi \in \mathscr{L}(F, G), p \in U$ if $f : U \to E$ is differentiable at p then so is $\varphi \circ f$. Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

Proof

φ is differentiable at $f(p)$ nad $d_{f(p)}\varphi = \varphi$

27.12 Corollary

Let E and F be normed vector spaces $U \in E$ be an open subset, $p \in U$. Let $f : U \to F$ and $g : U \to F$ be mappings that are differentiable at $p, (a, b) \in K \times K$. Then $af + bg$ is differentiable at p and

$$d_p(af_bg) = ad_pf + bd_pg$$

Proof

$af + bg$ is composite:

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[rr, bend right, "ay+bz"'] & K \times F \arrow[r, "m"] & F \end{tikzcd}$$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

27.13 Def: Equivalence of Norms

Let E be a vector space over K and $\|\cdot\|_1, \|\cdot\|_2$ be norms on E. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if there exist constants $C_1, C_2 > 0$ such that $\forall s \in E$

$$C_1 \|S\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

27.14 Prop

If $\|\cdot\|_1, \|\cdot\|_2$ are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover $\|\cdot\|_1, \|\cdot\|_2$ defines the same topology on E.

Proof

$$\|s\|_2 \leq C_2 \quad \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \to (E, \|\cdot\|_1)$$

are continuous. So $\forall$ open subset U of $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in $(E, \|\cdot\|_1)$. Conversely if V is open in $(E, \|\cdot\|_1)$ then

$$V = Id_E^{-1}(V)$$

is open in $(E, \|\cdot\|_2)$

27.15 Remark

If $\|\cdot\|_1, \|\cdot\|_2$ are two norms on E that define the same topology on E, then they are equivalent (under the assumption that $|\cdot|$ is not trivial)

27.16 Prop

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be normed vector spaces $\|\cdot\|'_E$ and $\|\cdot\|'_F$ be norms on E and F that are equivalent to $\|\cdot\|_E, \|\cdot\|_F$ respectively. Let $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping.

Let $p \in U$ Then f is differentiable at p with respect to $\|\cdot\|_E$ and $\|\cdot\|_F$ iff it's differentiable with respect to $\|\cdot\|'_E$ and $\|\cdot\|'_F$

Moreover the differentiable of f at p is not changed in the change of norms from $(\|\cdot\|_E, \|\cdot\|_F)$ to $(\|\cdot\|'_E, \|\cdot\|'_F)$

Proof

$$\begin{tikzcd} U \arrow[r, "Id_U"] \arrow[rrr, bend right, "f"'] & U \arrow[r, "f"] & F \arrow[r, "Id_F"] & F \end{tikzcd}$$

$(E, \|\cdot\|'_E)$ $(E, \|\cdot\|_E)$ $\|\cdot\|_F$ $\|\cdot\|'_F$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$d'_p f : (E, \|\cdot\|'_E) \to (F, \|\cdot\|'_F)$

27.17 Theorem

Let V be a finite dimensional vector space over K. Then all norms on V are equivalent. Moreover V is complete with respect to any norm on V.

Proof

Let $(e_i)_{i=1}^n$ be a basis of V(linear independent system of generators)The the mapping:

$$V \to \mathbb{R}_{\geq 0}$$
$$\sum_{i\in\{1,\ldots,n\}} a_i e_i \mapsto \max_{i\in\{1,\ldots,n\}}\{|a_i|\}$$

is a norm on V

Let $\|\cdot\|$ be another norm on V. One has

$$\left\|\sum_{i\in\{1,\ldots,n\}} a_i e_i\right\| \leq \sum_{1\in\{1,\ldots,n\}} |a_i|\,\|e_i\|$$
$$\leq (\sum_{i\in\{1,\ldots,n\}} \|e_i\|) \max_{i\in\{1,\ldots,n\}}\{|a_i|\}$$

We reason by induction that there exists $C > 0$ such that

$$\max_{i\in\{1,\ldots,n\}}\{|a_1|\} \leq C\left\|\sum_{i\in\{1,\ldots,n\}} a_i e_i\right\|$$

The case where $n = 0$ is trivial.

n=1

$$\|a_1 e_1\| = |a_1|\,\|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension $< n$

Let

$$W = \{\sum_{i\in\{1,\ldots,n-1\}} a_i e_i \mid (a_i)_{i\in\{1,\ldots,n-1\}} \in K^{n-1}\}$$

equipped with $\|\cdot\|$ restricted to W

The induction hypothesis shows that W is complete. Hence it's closed in V. Let $Q = V/W$ and $\|\cdot\|_Q$ be the quotient norm on Q that's defined as

$$\forall\alpha \in Q \quad \|\alpha\|_Q = \inf_{s\in\alpha}\|s\|$$

– If $s \in V \setminus W, \exists\epsilon > 0$ such that

$$\overline{B}(s,\epsilon) \cap W = \phi$$

$\forall t \in W,$

$$s + t \notin \overline{B}(0,\epsilon)$$

since otherwise

$$-t \in W \cap \overline{B}(s,\epsilon)$$

Therefore

$$\|[s]\|_Q = \inf_{i\in W}\|s + t\| \geq \epsilon > 0$$

- $\forall \lambda \in K$
$$\|\lambda\alpha\|_Q = \inf_{s\in\alpha} \|\lambda s\| \quad = |\lambda|$$
$$\inf_{s\in\alpha} \|s\| = |\lambda| \cdot \|s\|_Q$$

-
$$\begin{aligned}\|\alpha+\beta\|_Q &= \inf_{s\in\alpha+\beta} \|s\| \\ &= \inf_{(x,y)\in\alpha\times\beta} \|x+y\| \\ &\le \inf_{(x,y)\in\alpha\times\beta} (\|x\|+\|y\|) \\ &= \inf_{x\in\alpha} \|x\| + \inf_{y\in\beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some $A > 0$ such that $\forall (a_i)_{i\in\{1,\dots,n-1\}} \in K^{n-1}$

$$\max_{i\in\{1,\dots,n-1\}} \{|a_i|\} \le A \left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\|$$

Take
$$s = \sum_{i\in\{1,\dots,n\}} a_i e_i \in V$$

Let $\alpha = [s] = a_n[e_n] \in Q$

$$\left\| \sum_{i\in\{1,\dots,n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \le \|s\| + |a_n| \cdot \|e_n\| \le \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \, \|[e_n]\|_Q = |a_n| \inf_{t\in W} \|e_n + t\|$$

Take $e'_n \in V$ such that $[e'_n] = [e_n]$ and $\|e'_n\| \le \|[e_n]\|_Q + \epsilon$
Note that $(e_1, \dots, e_{n-1}, e'_n)$ forms also basis of V over K. Hence by replacing e_n by e'_n we may assume that $\|e_n\| \le \|[e_n]\|_Q + \epsilon$
$s = a_n e_n + t \in V$ with $t \in W$

$$\|s\| \ge \|[a_n e_n]\|_Q = |a_n| \, \|[e_n]\|_Q \ge B^{-1} |a_n| \cdot \|e_n\|$$

- If $\|a_n e_n\| < \frac{1}{2} \|t\|$
$$\|s\| \ge \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \ge \frac{1}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

- If $\|a_n e_n\| \ge \frac{1}{2} \|t\|$
$$\|s\| \ge B^{-1} |a_n| \cdot \|e_n\| \ge \frac{B^{-1}}{2} \|t\| \ge \frac{B^{-1} A}{2} \max_{i\in\{1,\dots,n-1\}} \{|a_i|\}$$

We take $C = \max\{B^{-1}\|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$ Then

$$\|s\| \geq C \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Another proof

completeness Under the norm $\max\limits_{i\in\{1,...,n\}}$, a sequence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ is a Cauchy sequence iff $\forall i \in \{1,...,n\} \quad (a_i^{(k)})_{k\in\mathbb{N}}$ is a Cauchy sequence. Since K is complete each $(a_i^{(k)})_{k\in\mathbb{N}}$ converges to some $a_i \in K$ Hence $(a_i^{(k)}e_i)_{k\in\mathbb{N},i\in\{1,...,n\}}$ converges.

27.18 Prop

Let $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$ be normed vector spaces over K. Assume that E is finite dimensional. Then any K-linear mapping $\varphi : E \to F$ is bounded.

Proof

Let $(e_i)_{i=1}^n$ be a basis of E. For any two norms on E are equivalent.
$\forall(a_1, ..., a_n) \in K$

$$\left\|\sum_{i=1}^n a_ie_i\right\|_E = \max_{i\in\{1,...,n\}}\{|a_i|\}$$

Then for any $s = \sum\limits_{i=1}^n a_ie_i$

$$\|\varphi(s)\|_F = \left\|\sum_{i=1}^n a_ie_i\right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq (\sum_{i=1}^n \|\varphi(e_i)\|_F) \|s\|_E$$

27.19 Theorem

Let E, F be normed vector spaces over a complete valued field, $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping . If f is differentiable at p then f is continuous at p

Proof

$$\begin{aligned} f(x) &= f(p) + d_pf(x-p) + o(\|x-p\|) \\ &= f(p) + O(\|x-p\|) \\ &= f(p) + o(1) \quad x \to p \\ \Rightarrow &\lim_{x\to p} f(x) = f(p) \end{aligned}$$

Chapter 28

Compactness

28.1 Def: cover

Let X be a topological space, $Y \subseteq X$ we call open cover of Y any family $(U_i)_{i \in I}$ open subset of X such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If I is finite set, we say that $(U_i)_{i \in I}$ iss a finite open cover. If $J \subseteq I$ such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that $(U_j)_{j \in J}$ is a sub cover of $(U_i)_{i \in I}$

28.2 Def: compact

If any open cover of Y has a finite subcover ,we say that Y is quasi-compact. If in addition X is Hausdorff, namely $\forall (x, y) \in X \times X$ with $x \neq y$ $\exists$ open neighborhoods U and V of x and y such that $U \cap V = \varnothing$, we say that Y is compact

28.3 Def

Let X be a set and $\mathscr{F}$ be a filter on X. If there does not exists any filter $\mathscr{F}'$ of X such that $\mathscr{F} \subsetneq \mathscr{F}'$, then we say that $\mathscr{F}$ is an ultrafilter.

Zorn's lemma implies that $\forall \mathscr{F}_0$ of X there exist an ultrafilter $\mathscr{F}$ if X containing $\mathscr{F}_0$

28.4 Prop

Let $\mathcal{F}$ be a filter on a set X. The following statements are equivalent.

(1) $\mathcal{F}$ is an ultrafilter

(2) $\forall A \subseteq X$ either $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$

(3) $\forall (A, B) \in \wp(X)^2$ if $A \cap B \in \mathcal{F}$ then $A \in \mathcal{F}$ or $B \in \mathcal{F}$

Proof

(1) $\Rightarrow$ (2) Suppose that $A \in \wp(X)$ such that $A \notin \mathcal{F}$ and $X \setminus A \notin \mathcal{F}$ $\forall B \in \mathcal{F}$ one has

$$B \cap A \neq \varnothing$$

since otherwise $B \subseteq X \setminus A$ and hence $X \setminus A \in \mathcal{F}$ contradiction.

(2) $\Rightarrow$ (3) Suppose that $B \notin \mathcal{F}$ then $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So $A \in \mathcal{F}$

(3) $\Rightarrow$ (1) Suppose that $\mathcal{F}'$ is a filter such that $\mathcal{F} \subsetneqq \mathcal{F}'$ Take $A \in \mathcal{F}' \setminus \mathcal{F}$ Then by $X = A \cup (X \setminus A) \in \mathcal{F}$ Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \varnothing = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

28.5 Theorem

Let $(X, \mathcal{J})$ be a topological space . The following are equivalent

(1) X is quasi-compact

(2) Any filter of X has an accumulation point

(3) Any ultrafilter of X is converges.

Proof

(1) $\Rightarrow$ (2) Assume that a filter $\mathcal{F}$ of X does not have any accumulation point. $\forall x \in X$ $\exists A_x \in \mathcal{F}$ $\exists$ open neighborhood V_x of x such that $A_x \cap V_x = \varnothing$ Since $X = \bigcup_{x \in X} V_x$ there is

$$\{x_1, ..., x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^{n} V_{x_i}$$

Take $B = \bigcap_{i=1}^{n} A_{x_i} \in \mathcal{F}$

$$B \cap X = B = \varnothing$$

Since $\forall i\ B \cap V_x = \varnothing$ contradiction.

(2) $\Rightarrow$ (3) Let $\mathcal{F}$ be an ultrafilter of X. By (2) there exist $x \in X$ such that $\mathcal{F} \cup \mathcal{V}_x$ generates a filter $\mathcal{F}'$ Since $\mathcal{F}$ is an ultrafilter $\mathcal{F} = \mathcal{F}'$ and hence $\mathcal{V}_x \subseteq \mathcal{F}$

(3) $\Rightarrow$ (1) Let $(U_i)_{i \in I}$ be an open cover of X we suppose that this have no finite subcover. $\forall i \in I$ let

$$F_i = X \setminus U_i$$

For any $J \subseteq I$ finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \varnothing$$

Let $\mathcal{F}$ be the smallest filter on X that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let $\mathcal{F}'$ be ultrafilter containing $\mathcal{F}$. It has a limit point x There exist $i \in I$ such that $x \in U_i$.Since U_i is a neighborhood of x and $\mathcal{V}_x \subseteq \mathcal{F}'$ we get $U_i \in \mathcal{F}'$ This is impossible since $F_i \in \mathcal{F}'$

28.6 Theorem

Let (X, d) be a metric space. The following statements are equivalent:

(1) X is complete and $\forall \epsilon > 0\ \exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

(2) X is compact

Proof

(1) $\Rightarrow$ (2) Let $\mathcal{F}$ be an ultrafilter Let $\epsilon > 0$ and $\{x_1, ..., x_n\} \subseteq X$ such that

$$X = \bigcup_{i=1}^{n} \mathcal{B}(x, \epsilon)$$

There exists some $i \in \{1, ..., n\}$ such that $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$ That means $\mathcal{F}$ is a Cauchy filter (namely $\forall \delta > 0\ \exists A \in \mathcal{F}$ of diameter $\leq \delta$) Since X is complete $\mathcal{F}$ has a limit point. So $\mathcal{F}$ is compact.

(2) $\Rightarrow$ (1) Let $\epsilon > 0$ One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since X is compact $\exists X_\epsilon \subseteq X$ finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$$\begin{aligned}
&\mathcal{F} \text{ is an ultrafilter}\\
\Leftrightarrow &\forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}\\
\Leftrightarrow &\forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F}\\
\Leftrightarrow &\forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup ... \cup A_n \ \exists i \in \{1, ..., n\}, A_i \in \mathcal{F}
\end{aligned}$$

Let $\mathcal{F}$ be a Cauchy filter Let $x \in X$ be an accumulation point of $\mathcal{F}$ $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$ with diameter $\leq \frac{\epsilon}{2}$ Note that $A \cup \mathcal{B}x, \frac{\epsilon}{2} \neq \varnothing$ Take $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2})$ $\forall z \in A$

$$\begin{aligned}
d(x,z) &\leq d(x,y) + d(y,z)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Therefore $A \subseteq \mathcal{B}(x, \epsilon)$ So $\mathcal{B}(x, \epsilon) \in \mathcal{F}$ This implies $\mathcal{V}_x \subseteq \mathcal{F}$

28.7 Lemma

Let (X, d) be a metric space

(1) Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point $\mathcal{F}$ a limit point of $\mathcal{F}$

(2) X is complete iff any Cauchy filter of X has a limit point

Proof

(1)

- Let $\mathcal{F}$ be a Cauchy filter on X. Any accumulation point of $\mathcal{F}$ is a limit point of $\mathcal{F}$

(2) Suppose that X is complete.Let $\mathcal{F}$ be a Cauchy filter.$\forall n \in \mathbb{N}_{\geq 1}$ let $A_n \in \mathcal{F}$ such that $diam(A_n) \leq \frac{1}{n}$ Take $x_n \in \bigcap_{k=1}^{n} A_k \in \mathcal{F}$ Then $(x_n)_{n \in \mathbb{N}_{\geq 1}}$ is a Cauchy sequence since $\forall \epsilon > 0$ if we take $N \in \mathbb{N}$ with $\frac{1}{N} \leq \epsilon$ then $\forall (n,m) \in \mathbb{N}_{\geq N}$ $d(x_n, x_m \leq \frac{1}{N})$ Hence $(x_n)_{N \in \mathbb{N}_{\geq 1}}$ converges to some $x \in X$ Note that x is an limit point of $\mathcal{F}$ since $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$ with $A_n \subseteq \mathcal{B}(x, \epsilon)$ It suffices to take n such that $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$ Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N},\ \{x_N, x_{N+1}, ...\} \subseteq A\}$$

This is a Cauchy filter on X since

$$\lim_{N\to+\infty} diam\{x_N, x_{N+1}, ...\} = 0$$

Hence $\mathcal{F}$ has a limit point $x \in X$ By definition $\forall U \in \mathcal{V}_x\ \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, ...\} \subseteq U$$

So $x = \lim_{n\to+\infty} x_n$

28.8 Prop

Let $f : X \to Y$ be a continuous mapping of topological spaces. If $A \subseteq X$ is quasi-compact then $f(A) \subseteq Y$ is also quasi-compact.

Proof

Let $(V_i)_{i\in I}$ be an open cover of $f(A)$ Then

$$(f^{-1}(V_i))_{i\in I}$$

is an open cover of A So $\exists J \subseteq I$ such that

$$A \subseteq \bigcup_{j\in J} f^{-1}(V_i)$$

This implies

$$f(A) \subseteq \bigcup_{j\in J} V_j$$

So $f(A)$ is quasi-compact.

28.9 Prop

Let X be a topological space and $A \subseteq X$ be a quasi-compact subset. For any closed subset F of X $A \cap F$ is quasi-compact.

Proof

Let $(U_i)_{i\in I}$ be an open cover of $A \cap F$.Then

$$A \subseteq (\bigcup_{i\in I} U_i) \cup (X \setminus F)$$

Since A is quasi-compact there exist $J \subseteq I$ finite such that

$$A \subseteq (\bigcup_{j \in J} U_j) \cup (X \setminus F)$$

Hence $A \cap F \subseteq \bigcup_{j \in J} U_j$

28.10 Prop

Let X be a Hausdorff topological space. Any compact subset A of X is closed.

Proof

Let $x \in X \setminus A$ $\forall y \in A, \exists$ open subsets U_y nad V_y such that $y \in U_y, x \in V_y$ and $U_y \cap V_y = \varnothing$ Since $A \subseteq \bigcup_{y \in A} U_y$ $\exists \{y_1, ..., y_n\} \subseteq A$ such that

$$A \subseteq \bigcup_{i=1}^{n} U_{y_i}$$

Let

$$U = \bigcup_{i=1}^{n} U_{y_i} \quad V = \bigcap_{i=1}^{n} V_{y_i}$$

These are open subset Moreover $A \subseteq U, x \in V$ and $U \cap V = \bigcup_{i=1}^{n} (U_{y_i} \cap V) = \varnothing$ In particular $x \in V \subseteq X \setminus A$ So $X \setminus A$ is open

28.11 Prop

Let X be a Hausdorff topological space and A and B be compact subsets of X such that $A \cap B = \varnothing$ Then there exists open subsets U and V such that

$$A \subseteq U, B \subseteq B and U \cap V = \varnothing$$

proof

We have seen in the proof of the previous proposition that $\forall x \in B, \exists U_x, V_x$ open such that $A \subseteq U_x, x \in V_x$ and $U_x \cap V_x = \varnothing$ Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, ..., x_m\} \subseteq B$ such that

$$B \subseteq \bigcup_{i=1}^{n} V_{x_i}$$

We take

$$U = \bigcap_{i=1}^{m} U_{x_i} \quad V = \bigcup_{i=1}^{m} U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq U \quad U \cap V = \varnothing$$

28.12 Theorem

Let $(X, \mathcal{J})$ be a Hausdorff topological space If $(A_n)_{n\in\mathbb{N}}$ is a sequence of non-empty compact subsets of X such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq ...$$

Then

$$\bigcap_{n\in\mathbb{N}} A_n \neq \varnothing$$

Proof

Suppose that

$$\bigcap_{n\in\mathbb{N}} A_n = \varnothing$$

then

$$A_0 \subseteq \bigcup_{n\in\mathbb{N}} (X \setminus A_n)$$

Since A_0 is compact, $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^{N} (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^{N} A_n \\ &= X \setminus A_n \end{aligned}$$

So

$$A_n = \varnothing$$

28.13 Def

Let (X, τ) be a topological space if any sequence in X has a convergent subsequence, we say that X is sequentially compact.

Example

By Bolzano-Weierstrass, any bounded sequence in $\mathbb{R}$ has a convergent subsequence. So any bounded and closed subset of $\mathbb{R}$ is sequentially compact.

Note

bounded and closed together implies sequentially compact.

28.14 Theorem

Let (X, d) be a metric space. Then the following statements are equivalent:

(1) (X, d) is compact

(2) (X, d) is sequentially compact

Proof

(1) $\Rightarrow$ (2) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Assume that no subsequence of $(x_n)_{n\in\mathbb{N}}$ converges in X. For any $p \in X$ there exists $\epsilon_p > 0$ such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For X is compact $\exists (p_i)_{i\in\{1,\ldots,n\}}$

$$X \subseteq \bigcup_{i=1}^{n} \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^{n} \{n \in \mathbb{N}\ d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

(2) $\Rightarrow$ (1)

prove (X, d) is complete Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences $(x_{k_n})_{n\in\mathbb{N}}$ must converges to the same limit.

So (X, d) is complete

If X is not covered by finitely many balls of radius ϵ we can construct a sequence $(x_{k_n})_{n\in\mathbb{N}}$ such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^{n} \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

28.15 Def

Let X be a Hausdorff topological space. If for any $x \in X$ there exist a compact neighborhood C_x we say that X is locally compact.

Example

$\mathbb{R}$ is locally compact.

28.16 Prop

Assume that $(K, |\cdot|)$ is a locally compact non-trivial valued field. field. Let $(E, \|\cdot\|)$ be a finite dimensional normed K-vector space. A subset $Y \subseteq E$ is compact iff it's closed and bounded.

Proof

$\Rightarrow$ Let $Y \subseteq X$ be compact. Then for Y is Hausdorff, Y is closed. Moreover

$$Y \subseteq \bigcup_{n\in\mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq ... \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^{n} \mathcal{B}(0, n_i)$$

$\Rightarrow$ Y is bounded.

$\Leftarrow$ We prove sequentially compact by a theorem proved before.
Let $(e_i)_{i=1}^d$ be a basis of E. Again we assume

$$\left\|\sum_{i=1}^{d} a_i e_i\right\| = \max_{i\in\{1,\ldots,d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n\in\mathbb{N}} = (\sum_{i=1}^{d} a_i^{(n)} e_i)_{n\in\mathbb{N}}$$

Since Y is bounded for any $i \in \{1, ..., d\}$ the sequence $(a_i^{(n)})$ is bounded. In particular we find $M > 0$ such that $\forall i \in \{1, ..., n\}$

$$\left|a_i^{(n)}\right| < M$$

Since $(K, |\cdot|)$ is locally compact, there exists a compact set $\mathcal{C} = \mathcal{C}_0 \subseteq K$ that's a neighborhood of 0. Let $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since K is not trivially valued, then exists $a \in K$ such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$ is compact. We have the K-linear mapping

$$\begin{aligned} K \to & \quad K \\ y \mapsto & \quad ay \end{aligned}$$

is bounded, then continuous. Hence $a\mathcal{C}$ is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find $(I_i)_{i=1}^{d}$ are infinite subsets of $\mathbb{N}$ with

$$I_1 \supseteq ... \supseteq I_d$$

such that $(a_j)_{j\in I_i}^{(n)}$ converges to some $a_i \in K$. It follows that our original sequence has a convergent subsequence converges to $\sum\limits_{i=1}^{d} a_i e_i$.

So Y is sequentially compact.

28.17 Theorem

Let X be a topological space and $f : X \to \mathbb{R}$ be a continuous mapping. If $Y \subseteq X$ is a quasi-compact subset, then there exists $a \in Y$ and $b \in Y$ such that $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of f to y attains its maximum and minimum.

Proof

$f(Y) \subseteq \mathbb{R}$ is a non-empty compact subset since Y is quasi-compact and $\mathbb{R}$ is Hausdorff. Moreover, since $\mathbb{R}$ is locally compact. SO $f(Y)$ is bounded and closed.

Note that there exists sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ is $f(Y)$ that tends to $\sup f(Y)$ and $\inf f(Y)$ respectively. Since $f(Y)$ is closed, $\sup f(Y), \inf f(Y)$ belongs to $f(Y)$. So $f(Y)$ has a greatest and a least element.

Chapter 29

Mean Value Theorems

29.1 Rolle Theorem

Let a, b be real numbers such that $a < b$ Let $f : [a, b] \to \mathbb{R}$ be a continuous mapping that is differentiable on $]a, b[$ If $f(a) = f(b)$ then $\exists t \in]a, b[$ such that

$$f'(t) = 0$$

Proof

Since $[a, b]$ is closed and bounded then it's compact, f attains its maximum and minimum. Let $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If $M \neq l$ $\exists t \in]a, b[$ such that $f(t) = M$

$$f(t + x) = f(t) + f'(t)x + o(|X|)$$

$$f(t - x) = f(t) - f'(t)x + o(|X|)$$

$$\begin{aligned} 0 &\leq (f(t + x) - f(t))(f(t - x) - f(t)) \\ &= -f'(t)^2 x^2 + o(|x|^2) \\ 0 &\leq -f'(t)^2 + o(1) \quad x \to 0 \end{aligned}$$

Taking the limit when $x \to 0$ we get $f'(t)^2 = 0$

If $m \neq l$ then any $t \in]a, b[$ such that $f(t) = m$ verifies $f'(t) = 0$

If $m = l = M$ f is constant, so $\forall t \in]a, b[, f'(t) = 0$

29.2 Mean value theorem(Lagrange)

Let a, b be real numbers $a < b$, $f : [a, b] \to \mathbb{R}$ be a continuous mapping differentiable on $]a, b[$, then $\exists t \in]a, b[$ such that

$$f(b) - f(a) = f'(t)(b - a)$$

Proof

Let $g : [a, b] \to \mathbb{R}$ be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then $g(a) = f(a) \quad g(b) = f(a)$ then apply Rolle Theorem to g we get the proof.

29.3 Mean value inequality

Let a, b be real numbers such that $a < b$ $(E, \|\cdot\|)$ be a normed vector space over $\mathbb{R}$ $f : [a, b] \to E$ be a continuous mapping such that f is differentiable on $]a, b[$ Then

$$\|f(b) - f(a)\| \leq (\sup_{x\in]a,b[} \|f'(x)\|)(b - a)$$

Proof

Suppose that

$$\sup_{x\in]a,b[} \|f'(x)\| < +\infty$$

Let $M \in \mathbb{R}$ such that

$$M > \sup_{x\in]a,b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition J is an interval containing a, so J is of form $[a, c[$ or $[a, c]$ Since f is continuous by taking a sequence $(c_n)_{n\in\mathbb{N}}$ in $[a, b[$ that converges to c we obtain

$$\begin{aligned}\|f(c) - f(a)\| &= \lim_{n\to+\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n\to+\infty} M(c_n - a) \\ &= M(c - a)\end{aligned}$$

Hence $c \in J$ namely $J = [a, c]$

$c > a$ We will prove that $c = b$ by contradiction

Suppose that $c < b$ $\forall h \in]0, b - c[$

$$\begin{aligned}\|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h)\end{aligned}$$

Since $M > \|f'(c)\|$, $\exists h_0 > 0$ such that $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$

Hence
$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h)-f(c)\|+\|f(c)-f(a)\|\\ &\leq M(c_h-c+c-a)\\ &= M(c+h-a)\end{aligned}$$
So $c+h_0 \in J$ Contradiction. Thus
$$\|f(b)-f(a)\| \leq M(b-a)$$
for any $M > \sup\limits_{x\in]a,b[} \|f'(x)\|$ since M is arbitrary the expected inequality holds.

$c=a$ In general, we apply the particular case (fis-extendable to a differentiable mapping at a) to $[\frac{a+b}{2}, b]$ and $[a, \frac{a+b}{2}]$ to get
$$\left\|f(b)-f(\frac{a+b}{2})\right\| \leq C\frac{b-a}{2}$$
$$\left\|f(\frac{a+b}{2})-f(a)\right\| \leq C\frac{b-a}{2}$$
with $C = \sup\limits_{x\in]a,b[} \|f'(x)\|$

Remark If f is defined on an open neighborhood of a and is differentiable at a the the same arguments hold without the assumption

29.4 Theorem

Let I be an interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ be a continuous mapping, then $f(I)$ is an interval.

Proof

Let $x \neq y$ be two elements of $f(I)$ Let a, b elements of I such that $x = f(a)$ $y = f(b)$ without loss of generality, we assume $a < b$
Let $z \in \mathbb{R}$ such
$$(z-x)(z-y) \leq 0$$
We construct by induction three sequences $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If a_n, b_n, c_n are constructed, satisfying
$$c_n = \frac{1}{2}(a_n+b_n)$$
$$(z-f(a_n))(z-f(b_n)) \leq 0$$

we let

$$\begin{aligned}
(a_{n+1}, b_{n+1}) &= (a_n, c_n) && \text{if } (z - f(a_n))(z - f(c_n)) \le 0 \\
(a_{n+1}, b_{n+1}) &= (c_n, b_n) && \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\
& && \quad ((z - f(c_n))(z - f(b_n)) \le 0) \\
c_{n+1} &= \frac{a_{n+1} + b_{n+1}}{2}
\end{aligned}$$

The sequence $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ are increasing and decreasing respectively and bounded, hence converges to some $l, m \in [a, b]$

Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \to 0 (n \to +\infty)$$

So $l = m$,by $(z - f(a_n))(z - f(b_n)) \le 0$ we obtain by letting $n \to +\infty$

$$(z - f(l))^2 \le 0$$

So $z = f(l)$

29.5 Theorem(Heine)

Let I be an open interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a differentiable mapping. Then $f'(I)$ is an interval.

Proof

Let $(a, b) \in I^2$ such that $a < b$. Consider the following mappings:

$$\begin{aligned}
g : [a, b] &\to \mathbb{R} \\
x &\mapsto \begin{cases} \dfrac{f(x) - f(a)}{x - a} & x \ne a \\ f'(a) & x = a \end{cases} \\
h : [a, b] &\to \mathbb{R} \\
x &\mapsto \begin{cases} \dfrac{f(b) - f(x)}{b - x} & x \ne b \\ f'(b) & x = b \end{cases}
\end{aligned}$$

g, h are continuous ($\frac{f(x)-f(a)}{x-a} = f'(a) + o(1)\ x \to a$)

So $g([a, b])$ and $h([a, b])$ are intervals . Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$

$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that $g(b) = h(a)$ so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence $f'(I)$ is an interval.

Chapter 30

Fixed Point Theorem

30.1 Def

Let X be a set and $T : X \to X$ be a mapping. If $x \in X$ satisfies $T(x) = x$ we say that x is a fixed point of T.

30.2 Def

Let (X, d) be a metric space and $T : X \to X$ be a mapping. If $\exists \epsilon \in [0, 1[$ such that T is ϵ-Lipschitzian then we say that T is a contraction.

30.3 Fixed Point Theorem

Let (X, d) be a COMPLETE non-empty metric space, and $T : X \to X$ eb a contraction. Then T has a unique fixed point. Moreover, $\forall x_n \in X$ if we let

$$x_{n+1} = T(x + n), x_0 \in X$$

then $(x_n)_{n \in \mathbb{N}}$ converges to the fixed point.

Proof

If p and q are two fixed point of T, then

$$d(p, q) = d(T(p), T(q)) \leq \epsilon d(p, q)$$

So $d(p, q) = 0$.
Let

$$x_{n+1} = T(x + n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any $N \in \mathbb{N}$, $\forall (n,m) \in \mathbb{N}^2_{\geq N} \quad n < m$

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\
&\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\
&\leq \frac{\epsilon^n}{1-\epsilon} d(x_0, x_1) \\
&\leq \frac{\epsilon^n}{1-\epsilon} d(x_0, x_1)
\end{aligned}$$

So

$$\lim_{N \to +\infty} \sup_{(n,m) \in \mathbb{N}^2_{\geq N}} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence, hence converges to some $p \in X$

$$d(T(p), p) = \lim_{n \to +\infty} d(T(x_n), x_n) = 0$$

since $d : X^2 \to \mathbb{R}_{\geq 0}$ is continuous.

Part VI

Higher differentials

Chapter 31

Multilinear mapping

Let K be a commutative cenitary ring.

31.1 Def

Let $n \in \mathbb{N}$, $V_1, ..., V_n, W$ be K-modules. We call n-linear mapping from $V_1 \times ... \times V_n$ to W any mapping $f : V_1 \times ... \times V_n \to W$ such that $\forall i \in \{1, ..., n\}$ $\forall (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) \in V_1 \times ... \times V_{i-1} \times V_{i+1} \times ... \times V_n$ the mapping

$$\begin{aligned} f(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n) : & V_i \to W \\ & x_i \mapsto f(x_i) \end{aligned}$$

is a morphism of K-modules

We denote by $Hom^{(n)}(V_1 \times ... \times V_n, W)$ the set of all n-linear mappings from $V_1 \times ... \times V_n$ to W.

31.2 Example

$$\begin{aligned} & K \times K \to K \\ & (a, b) \mapsto ab \end{aligned}$$

is a 2-linear mapping (bilinear mapping)

31.3 Remark

$$Hom^{(0)}(\{0\}, W) := W (\text{by convention})$$

$$Hom^{(1)}(V_1, W) = Home(V_1, W) = \{\text{morphism of K-module from } V_1 \text{ to } W\}$$

31.4 Prop

Suppose that $n \geq 2$ For any $i \in \{1, ..., n-1\}$

$$Hom^{(n)}(V_1 \times ... \times V_n, W) \xrightarrow{\Phi} Hom^{(i)}(V_1 \times ... \times V_i, Hom^{(n-i)}(V_{i+1} \times ... \times V_n))$$
$$f \mapsto ((x_1, ..., x_i) \mapsto ((x_{i+1}, ..., x_n) \mapsto f(x_1, ..., x_n)))$$

is a bijection

Proof

The inverse of Φ is given by

$$g \in Hom^{(i)}(V_1 \times ... \times V_i, Home^{(n-i)}(V_{i+1} \times ... \times V_n), W) \mapsto (((x_1, ..., x_n) \in V_1 \times ... \times V_n) \mapsto g(x_1, ..., x_i)(x_{i+1},$$

31.5 Remark

$Hom^{(n)}(V_1 \times ... \times V_n, W)$ is a sub-K-module of $W^{V_1 \times ... \times V_n}$ and Φ is an isomorphism of K-modules.

Chapter 32

Operator norm of Multilinear field

Let $(K, |\cdot|)$ be a complete valued field

32.1 Def

Let $V_1 \times ... \times V_n$ and W be normed vector spaces over K. We define

$$\|\cdot\| : Hom^{(n)}(V_1 \times ... \times V_n, W) \to [0, +\infty]$$

as

$$\|f\| := \sup_{(x_1,...,x_n) \in V_1 \times ... \times V_n, x_1 ... x_n \neq 0} \frac{\|f(x_1, ..., x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

If $\|f\| < \infty$ we say that f is bounded. We denote by $\mathscr{L}^{(n)}(V_1 \times ... \times V_n, W)$ the set of bounded n-linear mappings from $V_1 \times ... \times V_n$ to W.

32.2 Theorem

For any $i \in \{1, ..., n-1\}$, $\forall f \in \mathscr{L}^{(n)}(V_1 \times ... \times V_n, W)$ $\forall (x_1, ..., x_i) \in V_1 \times ... \times V_i$ the $(n-i)$-linear mapping

$$\begin{aligned} f(x_1, ..., x_i, \cdot) : & V_{i+1} \times ... \times V_n \to W \\ & (x_{i+1}, ..., x_n) \mapsto f(x_1, ..., x_n) \end{aligned}$$

belongs to $\mathscr{L}^{(n-i)}(V_{i+1} \times ... \times V_n, W)$. Moreover

$$\|f\| = \sup_{(x_1,...,x_n) \in V_1 \times ... \times V_n, x_1 ... x_n \neq 0} \frac{\|f(x_1, ..., x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

Proof

$\forall(x_{i+1},...,x_n) \in V_{i+1} \times ... \times V_n$

$$\begin{aligned}\|f(x_1,...,x_n)\| &\le \|f\| \, \|x_1\| ... \|x_n\| \\ &= (\|f\| \, \|x_1\| ... \|x_i\|) \, \|x_{i+1}\| ... \|x_n\|\end{aligned}$$

So

$$\|f(x_1,...,x_i,\cdot)\| \le \|f\| \, \|x_1\|, ..., \|x_i\|$$

If we define

$$\|f\|' := \sup_{(x_1,...,x_i)\in V_1\times...\times V_i, x_1...x_i\neq 0} \frac{\|f(x_1,...,x_i,\cdot)\|}{\|x_1\|\cdots\|x_i\|}$$

then

$$\|f\|' \le \|f\|$$

32.3 Corollary

(1) $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$ is a vector subspace of $Hom^{(n)}(V_1 \times \cdots \times V_n, W)$

(2) $\|\cdot\|$ is a norm on $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$

(3) $\forall i \in \{1,...,n\}$

$$\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W) \xrightarrow{\Phi} \mathscr{L}^{(n)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))$$

is a K-linear isomorphism that preserves operator norms.

$$\|f\| = \|\Phi(f)\|$$

32.3.1 Proof

Conversely $\forall(x_1,\cdot,x_n) \in V_1 \times \cdots \times V_n$ such that $x_1 \cdots x_n \neq 0$

$$\|f(x_1,...,x_n)\| \le \|f(x_1,...,x_i,\cdot)\| \, \|x_{i+1}\| \cdots \|x_n\|$$

Hence

$$\frac{f(x_1,...,x_n)}{\|x_1\|\cdots\|x_n\|} \le \frac{\|f(x_1,...,x_i,\cdot)\|}{\|x_1\|\cdots\|x_i\|} \le \|f\|'$$

Taking sup, we get

$$\|f\| \le \|f\|'$$

We reason by induction on n

$n = 1$

$$\mathscr{L}^{(1)}(V_1, W) = \mathscr{L}(V_1, W)$$

$i \in \{1, ..., n-1\}$ Suppose that the corollary is true for m-linear mappings with $m < n$ We consider the following diagram of mapping

To show that $\mathscr{L}^{(n)}(V_1 \times \cdots \times V_n, W)$ is a vector subspace, it suffices to check that $\forall g \in \mathscr{L}^{(i)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))$ one has $\left\|\Phi^{-1}(g)\right\| = \|g\| < +\infty$

$$\begin{aligned}\mathscr{L}^{(i)}(V_{i+1} \times \cdots \times V_n, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W)) &\subseteq Hom^{(i)}(V_1 \times \cdots \times V_i, \mathscr{L}^{(n-i)}(V_{i+1} \times \cdots \times V_n, W)) \\ &\subseteq Hom^{(i)}(V_1 \times \cdots \times V_i, Hom^{(n-i)}(V_{i+1} \times \cdots \times V_n, W))\end{aligned}$$

For any $(x_1, ..., x_n) \in V_1 \times \vdots \times V_n$

$$\begin{aligned}\left\|\Phi^{-1}(g)(x_1, ..., x_n)\right\| &= \|g(x_1, ..., x_i)(x_{i+1}, ...x_n)\| \\ &\leq \|g(x_1, ..., x_i)\| \|x_{i+1}\| \cdots \|x_n\| \\ &\leq \|g\| \|x_1\| \cdots \|x_i\| \|x_{i+1}\| \cdots \|x_n\|\end{aligned}$$

Therefore

$$\left\|\Phi^{-1}(g)\right\| \leq \|g\| = \left\|\Phi^{-1}(g)\right\|$$

Chapter 33

Higher differentials

We fix a complete non-trivial valued field $(K, |\cdot|)$ and normed K-vector space E and F.

33.1 Def

Let $U \subseteq E$ be an open subset and $f : U \to F$ be a mapping

(1) If f is continuous, we say that f is of class C^0 and f is 0-times differentiable

(2) If f is differentiable on an open neighborhood $V \subseteq U$ of some point $p \in U$ and

$$\begin{aligned} df : & V \to \mathscr{L}(E, F) \\ & x \mapsto d_x f \end{aligned}$$

is n-times differentiable at p, then we say that f is $(n+1)$-times differentiable at p. If f is $(n+1)$-times differentiable at any point $p \in U$, we denote by

$$D^{n+1} f : U \to \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

the mapping that sends $x \in U$ to the image of $D^n(df)(x)$ by the K-linear bijection

$$\mathscr{L}^{(n)}(E^n, \mathscr{L}(E, F)) \to \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

$$df : U \to \mathscr{L}(E, F)$$

$$D^n(df) : U \to \mathscr{L}^{(n)}(E^n, \mathscr{L}(E, F)) \xrightarrow{\Phi} \mathscr{L}^{(n+1)}(E^{n+1}, F)$$

If $D^{n+1}f$ is continuous, we say that f is of class $C^{n+1}(n \geq 0)$ (Any mapping $f : U \to F$ is considered as o-times differential $D^0 f := f$)

33.2 Remark

If f is n-times differentiable $\forall i \in \{1, ..., n-1\}$
$\forall p \in U, (h_1, ..., h_n) \in E^n$ one has

$$D^i(D^{n-i}f)(p)(h_1, ..., h_i)(h_{i+1}, ..., h_n) = D^n f(p)(h_1, ..., h_n)$$

$$D^{n-i}f : U \to \mathscr{L}^{(n-i)}(E^{n-i}, F)$$

$$D^i(D^{n-i}f): \quad \begin{tikzcd} U \arrow[r] \arrow[dr, "D^n f"'] & \mathscr{L}^{(i)}(E^i, \mathscr{L}^(n-i)(E^{n-i}, F)) \; U \to \arrow[d, leftrightarrow, "\cong"] \\ & \mathscr{L}^{(n)}(E^n, F) \end{tikzcd}$$

33.3 Theorem

Assume that $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$
Let $f : U \to F$ be a mapping that is $(n+1)$-times differentiable on U. Let $p \in U$ and $h \in E$ such that $p + th \in U$ $\forall t \in [0,1]$ Then

$$\left\| f(p+h) - f(p) - \sum_{k=1}^{n} \frac{1}{k!} D^k f(p)(h, ..., h) \right\| \leq$$
$$(\sup_{t \in]0,1[} \frac{(1-y)^n}{n!} \left\| D^{n+1} f(p+th) \right\|) \cdot \|h\|^{n+1}$$

(Taylor-Lagrange formula)

33.4 Prop(Gronwall inequality)

Let F be a normed vector space over $\mathbb{R}$ $(a,b) \in \mathbb{R}^2, a < b$ Let $f : [a,b] \to F$ and $g : [a,b] \to \mathbb{R}$ be continuous mappings that are differentiable on $]a,b[$

Suppose that $\forall t \in]a,b[$

$$\|f'(t)\| \leq g'(t)$$

then

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

Proof

Let $c \in]a,b[$ Let $\epsilon > 0$ Let

$$J = \{t \in [c,b] \mid \forall s \in [c,t] \ , \|f(s) - f(c)\| \leq g(s) - g(c)\}$$

By definition J is an interval.

Since f, g are continuous, J is a closed interval, hence J is of the form $[c, t]$. If $t < b$ then for $h > 0$ Sufficiently small.

$$f(t+h) - f(t) = hf'(t) + o(h)$$

$$g(t+h) - g(t) = hg'(t) + o(h)$$

$\exists \delta > 0 \ \forall h \in [0, \delta]$

$$\|f(t+h)\| \leq \|f'(t)\| \cdot h + \frac{\epsilon}{2}h$$

$$g(t+h) - g(t) \geq g'(t)h - \frac{\epsilon}{2}h$$

So

$$\|f(t+h) - f(t)\| \leq g(t+h) - g(t) + \epsilon h$$

Moreover

$$\|f(t) - f(c)\| \leq g(t) - g(c) + \epsilon(t-c)$$

$\Rightarrow$

$$\|f(t+h) - f(c)\| \leq g(t+h) - g(c) + \epsilon(t+h-c)$$

$\Rightarrow$

$$J \supseteq [c, t+\delta]$$

Contradiction, hence

$$\|f(b) - f(c)\| \leq g(b) - g(c) + \epsilon(b-c)$$

For the same reason

$$\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c-a)$$

Hence

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b-a)$$

Since $\epsilon > 0$ is arbitrary

$$\|f(b) - f(c)\| \leq g(b) - g(c)$$

Mean value theorem:

$$g(t) = (\sup(\|f'(\cdots)\|))$$

33.5 Theorem

Let $n \in \mathbb{N}$, E, F be normed vector spaces over $\mathbb{R}$ $U \subseteq E$ open and $f : U \to F$ be a mapping that is $(n+1)$-times differentiable. Let $p \in U$ and $h \in E$. Assume that $\forall \epsilon \in [0, 1], p + th \in U$

Let

$$M = \sup_{t \in]0,1[} \left\| D^{n+1} f(p+th) \right\|$$

Then

$$\left\|f(p+h)-\sum_{k=0}^{n}\frac{1}{k!}D^k f(p)(h,\cdots,h)\right\| \leq \frac{M}{(n+1)!}\|h\|^{n+1}$$

If $E=\mathbb{R}$ Then the formula become

$$\left\|f(p+h)-\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(p)h^k\right\| \leq \frac{M}{(n+1)!}|h|^{h+1}$$

Proof

Consider $\phi:[0,1]\to F$

$$\phi(t)=\sum_{k=0}^{n}\frac{(1-t)^k}{k!}D^k f(p+th)(h,\cdots,h)$$

$$\phi(1)=f(p+h)$$

$$\phi(0)=\sum_{k=0}^{n}\frac{1}{k!}D^k f(p)(h,\cdots,h)$$

$$\phi'(t)=\sum_{k=0}^{n}\frac{(1-t)^k}{k!}D^{k+1}f(p+th)(\underbrace{h,\cdots,h}_{\text{k+1 copies}})-\sum_{k=1}^{n}\frac{(1-t)^{k-1}}{(k-1)!}D^k f(p+th)(h,\cdots,h)$$

$$=\frac{(1-t)^n}{k!}D^{k+1}f(p+th)(h,\cdots,h)$$

then

$$\|\phi'(t)\|\leq M\frac{(1-t)^n}{n!}=(-M\frac{(1-t)^{n+1}\|h\|^{n+1}}{(n+1)!})'$$

By Gronwall inequality,

$$\|\phi(1)-\phi(0)\|\leq\frac{M}{(n+1)!}\|h\|^{n+1}$$

33.6 Def

Let $n\in\mathbb{N}$ $E_1,\cdots,E_n$ and F be normed vector spaces over a complete non-trivial valued field $(K,|\cdot|)$ Let $U\in E_1\times\cdots\times E_n$ be an open subset. $p=(p_1,...,p_n)\in U$ $i\in\{1,\cdots,n\}$, $f:U\to F$ If there exists an open neighborhood U_i of p_i in E_i such that

$$\begin{aligned}&U_i\to F\\&x_i\mapsto f(p_1,\cdots,p_{i-1},x_i,p_{i+1},\cdots,p_n)\end{aligned}$$

is well defined and is differentiable at p_i

We denote by $\frac{\partial f}{\partial x_i}(p)$ the differential of this mapping $U_i\to F$ and say that f admits the i^{th} partial differentials at p

33.7 Prop

Suppose that $(K, |\cdot|)$ and f has all partial differentials on U and

$$\frac{\partial f}{\partial x_i} : U \to \mathscr{L}(E_i, F)$$

is continuous for any $i \in \{1, \cdots, n\}$ Then f is of class C^1 and $\forall h = (h_1, \cdots, h_n) \in E_1 \times \cdots \times E_n$

$$\forall p \in U \quad d_p(h) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(h_i)$$

Proof

By induction, it suffices to treat thr case where $n = 2$
$\forall \epsilon > 0 \ \exists \delta > 0$

$$\forall (h,k) \in E_1 \times E_2 \quad \max\{|h|, |k|\} \leq \delta$$

one has

$$\left\| \frac{\partial f}{\partial x_i}(a+h, b+k) - \frac{\partial f}{\partial x_2}(a,b) \right\| \leq \epsilon (\text{by continuity of } \frac{\partial f}{\partial x_2})$$

Consider the mapping $\phi : [0,1] \to F$

$$\begin{aligned}
\phi(t) &= f(a+h, b+tk) - f(a+b, b) - t \underbrace{\frac{\partial f}{\partial x_2}(\overbrace{a+h, b}^{\in U})}_{\in \mathscr{L}(E_2, F)}(k) \\
\|\phi'(t)\| &= \left\| \frac{\partial f}{\partial x_2}(a+h, b+tk)(k) - \frac{\partial f}{\partial x_2}(a+h, b)(k) \right\| \\
&\leq 2\epsilon \|k\| \\
\|\phi(1) - \phi(0)\| &\leq 2\epsilon \|k\|
\end{aligned}$$

then

$$\left\| f(a+h, b+k) - f(a+h, b) - \frac{\partial f}{\partial x_2}(a+h, b)(k) \right\| \leq 2\epsilon \|k\|$$

So

$$\left\| f(a+h, b+k) - f(a+h, b) - \frac{\partial f}{\partial x_2}(a+h, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

f has 1^{st} partial differential

$$\left\|f(a+h,b)-f(a,b)-\frac{\partial f}{\partial x_1}(a,b)(k)\right\| = o(\max\{\|h\|,\|k\|\})$$

by continuity of $\frac{\partial f}{\partial x_i}$

$$\left\|\frac{\partial f}{\partial x_2}(a+h,b)(k)-\frac{\partial f}{\partial x_2}(a,b)(k)\right\| = o(\max\{\|h\|,\|k\|\})$$

take the sum of above three statements, we get:

$$\left\|f(a+h,b+k)-f(a,b)-\frac{\partial f}{\partial x_1}(a,b)(h)-\frac{\partial f}{\partial x_2}(a,b)(k)\right\| = o(\max\{\|h\|,\|k\|\})$$

33.8 Theorem

Let E, F be normed vector spaces over $\mathbb{R}$ $U \subseteq E$ open $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable mapping form U to F Let $g : U \to \mathscr{L}(E, F)$ Suppose that

(1) $(df_n)_{n\in\mathbb{N}}$ converges uniformly to g

(2) $(f_n)_{n\in\mathbb{N}}$ converges pointwisely to some mapping $f : U \to F$

Then f is differentiable and $df = g$

Proof

Let $p \in U, \forall (m,n) \in \mathbb{N}^2, \forall x \in \mathscr{B}(p,r) \in U (r > 0)$

$\|f_n(x) - f(_m(x) - (f_n(p) - f_m(p)))\| \leq (\sup_{\xi\in U} \|d_\xi f_m - d_\xi f_n\|) \cdot \|x-p\|$ (mean value inequality)

Take $\lim_{m\to+\infty}$ we get:

$$\|(f_n(x)-f(x))-(f_n(p)-f(p))\| \leq \epsilon_n \|x-p\|$$

where $\epsilon_n = \sup_{\xi\in U} \|d_\xi f_m - g\|$.

So

$$\begin{aligned}\|f(x)-f(p)-g(p)(x-p)\| &\leq \|(f(x)-f_n(x))-(f(p)-f_n(p))\| \\ &+ \|f_n(x)-f_n(p)-d_pf_n(x-p)\| \\ &+ \|d_pf_n(x-p)-g(p)(x-p)\| \\ &\leq \epsilon_n\|x-p\| + \|f_n(x)-f_n(p)-d_pf_n(x-p)\| + \epsilon_n\|x-p\|\end{aligned}$$

$$\limsup_{x\to p} \frac{\|f(x)-f(p)-g(p)(x-p)\|}{\|x-p\|} \leq 2\epsilon_n$$

Take $\lim_{n\to+\infty}$ we get:

$$\limsup_{x\to p} \frac{\|f(x)-f(p)-g(p)(x-p)\|}{\|x-p\|} = 0$$

Chapter 34

Permutations

34.1 Def

Let X be a set. We denote with $\mathfrak{S}_X$ the set of all bijections from X to itself. The elements of $\mathfrak{S}_X$ are called permutations if the set X is finite. If $x_1, \cdots, x_n \in X$ are distinct elements then

$$(x_1, \cdots, x_n) \in \mathfrak{S}_X$$

such that

$$x_i \mapsto x_{i+1}$$
$$x_n \to x_1$$

this is called an n-cycle. A 2-cycle is called a transposition.

34.1.1 Example

$X = \{1, \cdots, 7\}$

$$\begin{aligned} & 1 \mapsto 4 \\ & 2 \mapsto 1 \\ & 3 \mapsto 2 \\ (2\ 3)(4\ 2\ 1) = {} & 4 \mapsto 3 \\ & 5 \mapsto 5 \\ & 6 \mapsto 6 \\ & 7 \mapsto 7 \\ = {} & (1\ 4\ 3\ 2) \end{aligned}$$

34.2 Def

We denote with

$$orb_\sigma(x) = \{\underbrace{\sigma \circ \cdots \circ}_{\text{n-times}} \quad n \in \mathbb{N}\}$$

$x \in X, \sigma \in \mathfrak{S}_X$

34.3 Prop

If $orb_\sigma(x)$ is a finite set of d elements, then one has

$$\sigma^d(x) = x \quad orb_\sigma(x) = \{x, \sigma(x), \cdots, \sigma^{d-1}(x)\}$$

moreover

$$\sigma^{-1}(x) \in orb_\sigma(x)$$

34.3.1 Proof

The set

$$\{(n,m) \in \mathbb{N}^2, n < m, \sigma^n(X) = \sigma^m(x)\}$$

is not empty. Let

$$d' := \min\{m - n \quad (n,m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

therefore $x, \sigma(x), \cdots, \sigma^{d'-1}(x)$ are all distinct.

Now use the each deass division

$$h = qd' + r \quad r < d'$$

$$\sigma^h(x) = \sigma^r(x) \quad 0 \leq r < d'$$

then

$$d' \geq d$$

and for

$$\{x, \sigma(x), \cdots, \sigma^{d'-1}(x)\} \subseteq orb_\sigma(x)$$

$\Rightarrow$

$$d' \leq d$$

then

$$d' = d$$

34.4 Remark

Let $Y \subseteq X$, then we have a homomorphism of groups:

$$\begin{aligned} &\mathfrak{S}_Y \to \mathfrak{S}_X \\ &\sigma \mapsto \left(x \to \begin{cases} \sigma(x) & \text{if } x \in Y \\ x \text{ if } x \in X \setminus Y \end{cases} \right) \end{aligned}$$

If Y and Z are subset of X

$$Y \cap Z = \varnothing, \sigma \in \mathfrak{S}_Y, \tau \in \mathfrak{S}_Z$$

then

$$\sigma \circ \tau = \tau \circ \sigma$$

If X is finite with n elements $\mathfrak{S}_X = S_n$ permutation group of n elements.

34.5 Theorem

Let X be a finite set and let $\sigma \in \mathfrak{S}_X$ then exist $d \in \mathbb{N}$ and $(n_1, \cdots, n_d) \in \mathbb{N}^d_{\geq 2}$ and pairwise disjoint subsets $X_1, \cdots X_d$ of X of cardinalities $n_1, \cdots, n_d$, together with n_i-cycle τ_i of X_i such that

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

In other words. Any permutation can be decomposed in composition of finitely many cycles on disjoint subsets.

Proof

By induction on the cardinality of X.

The case $\sigma = id_X$ is trivial.($d = 0$) So the case when $N = 0, 1$ is clear.

Assume $N \geq 2$.Take $x \in X$ such that $\sigma(x) \neq x$ and let $X_1 = orb_\sigma(x)$ $Y = X \setminus X_1$ $\forall y \in Y$ we have $\sigma(y) \in Y$(because if $\sigma(y) \in X$ by the previous proposition $\sigma(y) \in X_1$)

Let $\tau = \sigma \mid_Y \in \mathfrak{S}_Y$ Use the induction hypothesis, we get $X_2, \cdots, X_d$ of cardinalities $n_2, \cdots, n_d$ and n_i-cycle τ_1 such that

$$\tau = \tau_2 \circ \cdots \circ \tau_d$$

Consider $\tau_1 = \sigma \mid_{X_1}$ then τ_1 is a n_1-cycle of X_1
$\Rightarrow$

$$\tau = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_d$$

34.5.1 Remark

This theorem say that the groups of permutation si generated by cycles.

34.6 Corollary

Let X be a finite set. Then $\mathfrak{S}_X$ is generated by transpositions.

Proof

Note that

$$(x_1, \cdots, x_n) = (x_1, x_2) \circ (x_2, \cdots, x_n)$$

By induction

$$(x_1, \cdots, x_n) = (x_1, x_2) \circ \cdots \circ (x_{n-1}, x_n)$$

34.6.1 Remark

The decomposition of transposition in unique.

34.7 Def

Let $\tau \in \mathfrak{S}_n := G_{\{1,\cdots,n\}}$ is called adjacent if τ is of the form $(j, j+1)$ for $j = 1, \cdots, n-1$

34.8 Corollary

$\mathfrak{S}_n$ si generated by adjacent transposition.

34.8.1 Proof

Note that

$$(i, j) = (i, i+1) \circ (i+1, i+2) \circ \cdots \circ (j-1, j) \circ (j-2, j-1) \circ \cdots \circ (i+2, i+1)$$

Some other information on $\mathfrak{S}_n$

34.9 Caybey Theorem

Any finite group can be embedded (injective morphism) in a $\mathfrak{S}_n$ for some $n \in \mathbb{N}$

Proof

Let G be a finite group and $n = card(G)$. Let

$$\begin{aligned} \varphi : G &\to \mathfrak{S} \\ g &\mapsto l_g \end{aligned}$$

be the mapping sends $g \in G$ to $l_g(x) = gx, \forall x \in G$

34.10 Theorem

Let X be a finite set. Assume that $\sigma \in \mathfrak{S}_X$ can be written as

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

where τ_1 is transposition.
We put

$$sgn(\sigma) := (-1)^{\sigma}$$

This is a well-define function. Moreover sgn is a morphism from $\mathfrak{S}_X$ to $(\{-1,1\}, \times)$

Proof

Let's define the mapping:

$$\begin{aligned}\phi : \mathfrak{S}_n &\to \mathbb{Q}^{\times}\\ \sigma &\mapsto \prod_{(i,j)\in\{1,\cdots,n\}^2, i<j} \frac{\sigma(i)-\sigma(j)}{i-j}\end{aligned}$$

To show that ϕ is a morphism of groups. Let

$$\theta = \{U \in \wp(\{1,\cdots,n\}) \mid \ card(U) = 2\}$$

$$\begin{aligned}\phi(\sigma\circ\tau) &= \prod_{(i,j)\in\theta} \frac{\sigma(\tau(i))-\sigma(\tau(j))}{i-j}\\ &= (\frac{\sigma(\tau(i))-\sigma(\tau(j))}{\tau(i)-\tau(j)}) \times (\prod_{(i,j)\in\theta} \frac{\tau(i)-\tau(j)}{i-j})\\ &= \phi(\sigma)\phi(\tau)\end{aligned}$$

When τ is a transition, $\phi(\tau) = -1$. Therefore

$$\phi(\sigma) = \prod_{i=1}^{d} \phi(\tau_1)$$

since

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

34.11 Remark

Let $A_n \subsetneqq \mathfrak{S}_n$ such that

$$A_n = \{\sigma \in \mathfrak{S}_n \mid sgn(\sigma) = 1\}$$

is an alternating symmetric group.

34.12 Exercise

Let X be a set of cardinality n. Let $\sigma : X \to \{1, \cdots, n\}$ be a bijection. Prove that

$$\begin{aligned}\phi : \mathfrak{S}_X &\to \mathfrak{S}_n \\ \tau &\mapsto \sigma^{-1} \circ \tau \circ \sigma\end{aligned}$$

is an isomorphism.

34.13 Symmetric of multilinear mapping

We fix a commutative unitary ring K and K-modules E, F

34.14 Def: Symmetric and Alternating

symmetric Let $n \in \mathbb{N}$ and $f \in Hom^{(n)}(E^n, F)$. If for any $\sigma \in \mathfrak{S}_n$ one has $\forall x \in E^n$

$$f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Then we say f is symmetric

alternating If for any $(i, j) \in \{1, \cdots, n\}^2$ such that $i \neq j$ and any $(x_1, \cdots, x_n) \in E^n$ such that $x_i = x_j$

$$f(x_1, \cdots, x_n) = 0$$

then we say that f is alternating.

34.15 Prop

Suppose that $f \in Hom^{(n)}(E^n, F)$ is alternating, then $\forall (x_1, \cdots, x_n) \in E^n$, $\sigma \in \mathfrak{S}_n$

$$f(x_1, \cdots, x_n) = sgn(\sigma) f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Proof

By corollary 34.8, it's enough to prove the proposition for adjacent transitions. Let $i \in \{1, , \cdots, n-1\}$ then

$$\begin{aligned}0 = &f(x_1, \cdots, x_{i-1, x_i + x_{i+1}, x_{i+2}, \cdots, x_n}) \\ &= f(x_1, \cdots, x_{i-1}, x_i, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_{i+1}, x_{i+1}, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \cdots, x_n) \\ &+ f(x_1, \cdots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \cdots, x_n)\end{aligned}$$

The adjacent transition σ is $(i, i+1)$

34.16 Def:

Hom_s and Hom_a

We denote with $Hom_s^{(n)}(E^n, F)$ and $Hom_a^{(n)}(E^n, F)$ the set of symmetric and alternating n-linear mappings from E to F.

$Hom_s^{(n)}(E^n, F)$ and $Hom_a^{(n)}(E^n, F)$ are sub-K-modules of $Hom^{(n)}(E^n, F)$ and when $n = 1$, by convention

$$Hom_s^{(1)}(E, F) = Hom_a^{(1)}(E, F) = Hom(E, F)$$

34.17 Reminder

Let E, F be two normed vector spaces over $\mathbb{R}$ $f : E \to F$ is differentiable(twice)

$$df : E \to \mathscr{L}(E, F)$$
$$D^2 d : E \to \mathscr{L}(E, \mathscr{L}(E, F))$$
$$A \mapsto ((x, y) \to A(x)(y))$$

34.18 Theorem(Schwarz)

$U \subseteq E$ is an open set, $f : U \to F$ is a function of class C^n. Then for any $p \in U$

$$D^n f(p) \in \mathscr{L}^n(E^n, F)$$

is symmetric

Proof

By induction and by the fact that permutation are decomposed in transpositions, we can reduce to prove only the case $n = 2$

$$d_{p+u}f - d_p f = D^2 f(p)(u, \cdot) + o(u)$$

$\forall \epsilon > 0, \exists \delta > 0$ such that $0 < \|u\| < \delta$,then

$$\left\| d_{p+u}f - d_p f - D^2 f(p)(u, \cdot) + o(u) \right\| \le \epsilon \|u\|$$

For any $x \in \mathscr{B}(p, \frac{\epsilon}{2})$ let's introduce the following function

$$\varphi(x) = f(x + k) - f(x) - D^2 f(p)(k, x)$$

We use the mean value inequality on φ

$$\begin{aligned}
&\|\varphi(p + h) - \varphi(p)\| \\
=&\left\| f(p + h + k) + f(p) - f(p + h) - D^2 f(p)(k, p + h) - f(p + k) - f(p) - D^2 f(p)(k, p) \right\| \\
=&\left\| f(p + h + k) + f(p) - f(p + h) - f(p + k) - D^2 f(p)(k, h) \right\| \\
\le&\left(\sup_{t \in [0,1]} \|d_{p+th}\varphi\| \right) \|h\|
\end{aligned}$$

$$\|d_{p+th}(\varphi)\| = \left\|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k,\cdot)\right\|$$

add and subtract $d_pf, D^2f(p)(th,\cdot)$ then by triangle inequality

$$\begin{aligned}
&\left\|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k,\cdot)\right\| \\
\leq &\left\|d_{p+th+k}(f) - d_pf - D^2f(p)(k+th,\cdot)\right\| \\
+ &\left\|d_{p+th}f - d_pf - D^2f(p)(th,\cdot)\right\| \\
\leq &\epsilon\left\|th+k\right\| + \epsilon(th) \\
\leq &2\epsilon(\|h\| + \|k\|)
\end{aligned}$$

then

$$\begin{aligned}
&\left\|f(p+h+k) + f(p) - f(p+k) - f(p+h) - D^2f(p)(k,h)\right\| \\
=&o(\max\{\|h\|, \|k\|\}^2)
\end{aligned}$$

exchange the role of h, k then we get

$$\begin{aligned}
&\left\|f(p+h+k) + f(p) - f(p+k) - f(p+h) - f(p+k) - D^2f(p)(h,k)\right\| \\
\leq &o(\max\{\|h\|, \|k\|\}^2)
\end{aligned}$$

then

$$\underbrace{\left\|D^2f(p)(k,h) - D^2f(p)(h,k)\right\|}_{\text{bilinear function}} = o(\underbrace{\max\{\|h\|, \|k\|\}^2}_{\text{quachetic}})$$

this implies that the LHS is 0

34.19 Def

Let E¡F be normed vector spaces over a complete value field $(K, |\cdot|)$ let $U \subseteq E, V \subseteq F$ be open subsets and $f : U \to V$ is a bijection.

(1) If f and f^{-1} are both continuous we say that f is a homeomorphism

(2) If a f and f^{-1} are both of class C^n we say that f is a e^n-diffeomorphism

If (2) is true for any $n \in \mathbb{N}$ we say that f is a C^∞-diffeomorphism

34.20 Prop

Let E,F be two normed Banach spaces. Let $I(E,F) \in \mathscr{L}(E,F)$ be the set of linear continuous and invertible mappings such that $norm\varphi^{-1} \leq +\infty$. Then $I(E,F)$ is open in $\mathscr{L}(E,F)^\vee$ Moreover the mapping

$$\begin{aligned}
I(E,F) &\to I(F,E) \\
\phi &\mapsto \varphi^{-1}
\end{aligned}$$

is a e^1-diffeomorphism

Proof

Let $\varphi \in I(E,F)$ we want to show that

$$\varphi - \psi \in I(E,F)$$

for $\psi \in \mathscr{E},\mathscr{F}$ such that $\|\psi\| < \frac{1}{\|\varphi^{-1}\|}$ Notice that

$$\varphi - \psi = \varphi \circ (Id_E - \varphi^{-1} \circ \psi)$$

Since

$$\left\|\varphi^{-1}\psi\right\| \leq \left\|\varphi^{-1}\right\| \|\psi\| < 1$$

This means that the series

$$\sum_{n\in\mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n}$$

is absolutely convergent in $\mathscr{L}(E,E)$ This series is the inverse of $(Id_E - \varphi^{-1}\psi)$

$$(Id_e - \varphi^{-1}\psi) \circ \sum_{n=0}^{N-1} (\varphi^{-1} \circ \psi) \overbrace{\circ n}^{\text{composite n times}} = Id_E - (\varphi^{-1} \circ \psi)^{\circ N}$$

take $\lim\limits_{N\to+\infty}$, then

$$(\varphi - \psi)^{-1} = \sum_{n\in\mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n} \circ \varphi^{-1}$$

and

$$(\varphi - \psi)^{-1} = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

replace the inverse with i

$$i(\varphi - \psi) - i(\varphi) = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

then

$$d_\varphi i(\psi) = i(\varphi) \circ (-\psi) \circ i(\varphi)$$

so i is differentiable.Moreover i and i^{-1} are continuous.

Remark

By induction we can show that i is a $C^{+\infty}$-diffeomorphism

34.21 Prop

Let $n \in \mathbb{N} \cup \{\infty\}$ Let E F G be normed vector spaces over a complete valued field $(K, |\cdot|)$ $U \subseteq E, V \subseteq F$ be open sets. $f : U \to V$ $g : V \to G$ be mappings of class C^n, then $g \circ f$ also of class C^n

34.21.1 Proof

The case where $n = 0$ is known
Denote by

$$\begin{aligned}\Phi : \mathscr{L}(E, F) \times E &\to F \\ (\beta, \alpha) &\mapsto \beta \circ \alpha\end{aligned}$$

Φ is a bounded bilinear mapping

$$\|\Phi(\beta, \alpha)\| \leq \|\beta\| \cdot \|\alpha\|$$

Suppose that $n \geq 1$ and the statement is true for mappings of class C^{n-1} $g \circ f$ is differentiable.

$$\forall p \in U \quad d_p(g \circ f) = d_{f(p)} g \circ d_p f$$

$$D^1(g \circ f) : U \to \mathscr{L}(E, G)$$

$$D^1 = \Phi \circ (D^1 g \circ f, D^1 f)$$

$$\begin{aligned}(D^1 g \circ f, D^1 f) : U &\to \mathscr{L}(F, G) \times \mathscr{L}(E, F) \\ p &\mapsto (d_{f(p)} g, d_p f)\end{aligned}$$

$$d_{\beta_0, \alpha_0} \Phi(\beta, \alpha) = \beta_0 \circ \alpha + \beta_0 \circ \alpha$$

$$\begin{aligned}D^1 \Phi : \mathscr{L}(F, G) \times \mathscr{L}(E, F) &\to \mathscr{L}(\mathscr{L}(F, G) \times \mathscr{L}(E, F), \mathscr{L}(E, G)) \\ (\alpha_0, \beta_0) &\mapsto ((\alpha, \beta) \mapsto \beta_0 = \alpha + \beta_0 \alpha_0)\end{aligned}$$

Since g, f are of class C^n $D^1 f, D^1 g$ are of class C^{n-1} Thus, by induction hypothesis,

$$(D^1 g \circ f, D^1 f)$$

is of class C^{n-1} Since Φ is of class C^{∞}, we obtain that

$$D^1(g \circ f)$$

is of class C^{n-1} then

$$g \circ f$$

is of class C^n

34.22 Prop

Let E and F be Banach space over a complete valued field $(K, |\cdot|)$. U and V be open subsets of E and F respectively.$n \in \mathbb{N} \cup \{\infty\}$ and $f : U \to V$ be a bijection. If f is of class C^n, then f^{-1} is differentiable, then f^{-1} is of class C^n

Proof

$$f \circ f^{-1} = Id_V$$

$\forall y \in V$

$$d_y(f \circ f^{-1}) = d_{f^{-1}(p)} f \circ d_y f^{-1} = Id_F$$

For $x \in U, y = f(x)$

$$d_y(f \circ f^{-1}) = d_x f \circ d_y f^{-1} = Id_F$$

$$d_x(f^{-1} \circ f) = d_y f \circ d_x f^{-1} = Id_E$$

So

$$d_y f^{-1} - (d_x f)^{-1}$$

that is

$$D^1 f^{-1} = \iota \circ (D^1 f \circ f^{-1})$$

where

$$\iota : I(E,F) \to I(F,E)$$
$$\phi \mapsto \phi^{-1}$$

Suppose that f^{-1} is of class C^{n-1} then

$$D^1 f^{-1} = \iota D^1 f \circ f^{-1}$$

is of class C^{n-1}

34.23 Local Inversion Theorem

Let E and F be Banach space over $\mathbb{R}$ $U \in E$ open, $f : U \to F$ be a mapping of class C^n and $a \in U$. Suppose that $d_a f \in I(E,F)(d_a f$ is invertible and of bounded inverse) Then there exists open neighborhoods V and W of a and $f(a)$ respectively, such that

- $V \subseteq U$ and $f(V) \subseteq W$
- The restriction of f to V defines a bijection from V to W
- $$(f \mid_V)^{-1} W \to V$$

  is of class C^n

34.23.1 Proof

For $y \in F$ consider the mapping:

$$\begin{aligned}\phi_y : U &\to F\\ x &\mapsto x - (d_a f)^{-1}(f(x) - y)\end{aligned}$$

$f(x) = y$ iff $\phi_y(x) = x$ i.e. x is a fix point of ϕ_y ϕ_y is of class C^1 and

$$d_x\phi_y(v) = v - d_a f^{-1}(d_x f(v))$$

$\forall v$

$$d_a\phi y^{(v)} = 0$$

By the continuity of $D^1 f$ there exists $r > 0$ such that

$$\overline{\mathcal{B}}(a, r) \subseteq U$$

and $\forall y \in F, \forall x \in \overline{\mathcal{B}}(a, r)$

$$\|d_x\phi_y\| \le \frac{1}{2}$$

By the mean value inequality. $\forall (x_1, x_2) \in \overline{\mathcal{B}}(a, r)$

$$\left\|\phi_y(x_1) - \phi_{=y(x_2)}\right\| \le \frac{1}{2}\|x_1 - x_2\|$$

Hence ϕ_y is contraction.

By the boundedness of $(d_a f)^{-1}$ $\exists \delta > 0$ such that

$$\forall y \in \overline{\mathcal{B}}(f(a), \delta) \quad \left\|(d_a f)^{-1}(f(a) - y)\right\| \le \frac{r}{2}$$

Then $\forall x \in \overline{\mathcal{B}}(a, r)$ $y \in \overline{\mathcal{B}}(f(a), \delta)$

$$\begin{aligned}\|\phi_y(x) - a\| &\le \|\phi_y(x) - \phi_y(a)\| + \|\phi_y(a) - a\|\\ &\le \frac{1}{2}\|x - a\| + \frac{r}{2}\\ &\le \frac{r}{2} + \frac{r}{2} = r\end{aligned}$$

$\phi_y(\overline{a, \imath}) \in \overline{\mathcal{B}}(a, r)$. By the fixed point theorem

$$\exists g : \overline{\mathcal{B}}(f(a), \delta) \to \overline{\mathcal{B}}(a, r)$$

sending y to the fixed point of ϕ_y Let $W = \mathcal{B}(f(a), g)$,then

$$g\mid_W : W \to V$$

is the inverse of $f\mid_V : V \to W$ Hence $f^{-1}(W) = V$ is open.

In the following, we prove that g is of class C^n on an open neighborhood of $f(a)$. By reducing V and W, we may assume that $\forall x \in V$

$$d_x f \in I(E,F)$$

Let $x_0 \in V$ $y_0 = f(x_0)$ $\quad x_0 = g(y_0)$

$$y - y_0 = f(g(y)) - f(g(y_0)) = d_{x_0} f(g(y) - g(y_0)) + o(\|g(y) - g(y_0)\|)$$

So

$$g(y) - g(y_0) = (d_x f)^{-1}(y - y_0) + o(\|g(y) - g(y_0)\|)$$

Thus leads to

$$g(y) - g(y_0) = O(\|y - y_0\|)$$

($\exists \epsilon > 0 \quad (1-\epsilon)\|g(y) - g(y_0)\| \leq \|d_{x_0} f\|^{-1}$ when $\|y - y_0\|$ is sufficiently small)
So

$$d_{y_0} g = (d_x f)^{-1}$$

By the previous proposition, g is of class C^n

Part VII

Integration

Chapter 35

Integral operators

We fix a set Ω and a vector subspace S of $\mathbb{R}^\Omega$ over $\mathbb{R}$ We suppose that $\forall (f,g) \in S^2$

$$\begin{aligned} f \wedge g : \Omega &\to \mathbb{R} \\ \omega &\mapsto \min\{f(\omega), g(\omega)\} \end{aligned}$$

belongs to S

35.1 Prop

(1) $\forall (f,g) \in S^2$

$$\begin{aligned} f \vee g : \Omega &\to \mathbb{R} \\ \omega &\mapsto \max\{f(\omega), g(\omega)\} \end{aligned}$$

$f \vee g \in S$

(2) $\forall f \in S$

$$\begin{aligned} |f| : \Omega &\to \mathbb{R} \\ \omega &\mapsto |f(\omega)| \end{aligned}$$

$|f| \in S$

Proof

(1)

$$f \vee g = f + g - f \wedge g$$

(2)

$$|f| = f \vee (-f)$$

35.2 Def

We call integral operator on S any $\mathbb{R}$-linear mapping $I : S \to \mathbb{R}$ that satisfies the following conditions:

(1) If $f \in S$ is such that $\forall \omega \in \Omega, f(\omega) \geq 0$ then $I(f) \geq 0$

(2) If $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence of elements in S such that $\forall \omega \in \Omega \lim_{n \to +\infty} f_n(\omega) = 0$ then

$$\lim_{n \to +\infty} I(f_n) = 0$$

$(\forall \omega \in \Omega, n \in \mathbb{N}, f_n(\omega) \geq f_{n+1}(\omega))$

35.3 Example

(1) $\Omega = \mathbb{R}$ S=vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by mappings of the form $\mathbb{1}_{]a,b]}$ $(a, b) \in \mathbb{R}^2, a < b$

$$\mathbb{1}_{]a,b]} = \begin{cases} 1, x \in]a, b] \\ \qquad 0, else \end{cases}$$

Any element of S is of the form

$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i,b_i]}$$

$I : S \to R$ is defined as

$$I(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{]a_i,b_i]}) = \sum_{i=1}^{n} \lambda_i (b_i - a_i)$$

More generally if $\varphi : \mathbb{R} \to R$ is increasing and right continuous ($\forall x \in \mathbb{R}, \lim_{\epsilon > 0, \epsilon \to 0} \varphi(x + \epsilon) = \varphi(x)$) We define

$$I_\varphi : S \to \mathbb{R}$$

$$I(\sum_{i=1}^{n} \lambda \mathbb{1}_{]a_i,b_I]}) = \sum_{i=1}^{n} \lambda_i (\varphi(b_i) - \varphi(a_i))$$

(2) (Radon measure)

Let Ω be a quasi-compact topological space

$$S = C^0(\Omega) := \{f \Omega \to \mathbb{R} \text{ continuous}\}$$

Let $I : S \to \mathbb{R}$ $\mathbb{R}$-linear, such that $\forall f \in S, f \geq 0$ one has $I(f) \geq 0$

35.4 Dini's theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $C^0(\Omega)$, that converges pointwisely to some $f \in C^0(\Omega)$ Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f

Proof

Let $g_n = f_n - f > 0$ Fix $\epsilon > 0$ $\forall n \in \mathbb{N}$ let

$$U_n = \{\omega \in \Omega \mid g_n(\omega) < \epsilon\}$$

is open

Moreover

$$\bigcup_{n\in\mathbb{N}} U_n = \Omega \quad (U_0 \subseteq U_1 \subseteq \cdots)$$

Since Ω is quasi-compact, $\exists N \in \mathbb{N}, \Omega = U_N$ Therefore $\forall n \in \mathbb{N}, n \geq N, \forall \omega \in \Omega$

$$g_n(\omega) < \epsilon$$

Consequence. If $(f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is decreasing and converges pointwisely to 0, then

$$\|f_n\|_{\sup} := \sup_{\omega\in\Omega} |f_n(\omega)|$$

converges to 0 when $n \to +\infty$ $\forall n \in \mathbb{N}$

$$f_n \leq \|f_n\|_{\sup} \cdot \mathbb{1}_\Omega$$

So

$$0 \leq I(f_n) \leq \|f_n\|_{\sup} I(\mathbb{1}_\Omega) \to 0 \quad (n \to +\infty)$$

(If $f \leq g$ then $g - f \geq 0$ so $I(g-f) = I(g) - I(f) \geq 0$ $I(g) \geq I(f)$)

35.5 Def

We call σ-algebra any subset $\mathscr{A}$ of $\wp(\Omega)$ that satisfies the following conditions:

- $\varnothing \in \mathscr{A}$
- If $A \in \mathscr{A}$ then $\Omega \setminus A \in \mathscr{A}$
- If $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$

Given a σ-algebra $\mathscr{A}$ on Ω , we mean by measure on $(\Omega, \mathscr{A})$ any mapping $\mu : \mathscr{A} \to [0, +\infty]$ such that :

- $\mu(\varnothing) = 0$
- If $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ such that A_i are pairwisely disjoint, then

$$\mu(\bigcup_{n\in\mathbb{N}} A_n) = \sum_{n\in\mathbb{N}} \mu(A_n)$$

Chapter 36

Riemann integral

36.1 Def

Let Ω be a non-empty set and S be a vector subspace of $\mathbb{R}^\Omega$ If $\forall(f,g) \in S^2, f \wedge g \in S$, we say that S is a Riesz space.

In this section, we fix a Riesz space and an integral operator $I : S \to \mathbb{R}$

36.2 Def

For any $f : \Omega \to \mathbb{R}$ let

$$I^*(f) := \inf_{\mu \in S, \mu \geq f} I(\mu)$$

$$I_*(f) := \sup_{l \in S, l \leq f} I(f)$$

If $I^*(f) = I_*(f)$ then we say that f is I-Riemann integral, and denote by $I(f)$ the value $I^*(f)$ (or $I_*(f)$)

36.3 Theorem

The set $\mathscr{R}$ of all I-Riemann integral mappings form a vector space of $\mathbb{R}^\Omega$ that contains S. Moreover,$I : \mathscr{R} \to \mathbb{R}$ is an $\mathbb{R}$-linear mapping extending $I : S \to \mathbb{R}$

Proof

$\forall h \in S$

$$I^*(h) = I_*(h) = I(h)$$

So $h \in \mathscr{R}$

Let $(f_1, f_2) \in \mathscr{R}$ If $(\mu_1, \mu_2) \in S^2, \mu_1 \geq f_1, \mu_2 \geq f_2$ then

$$\mu_1 + \mu_2 \in S, \mu_1 + \mu_2 \geq f_1 + f_2$$

Hence
$$I(\mu_1) + I(\mu_2) \geq I^*(f_1 + f_2)$$
Take the infimum with respect to (μ_1, μ_2) we get
$$I^*(f_1) + I^*(f_2) \geq I^*(f_1 + f_2)$$
Similarly
$$I_*(f_1) + I_*(f_2) \leq I_*(f_1 + f_2)$$
Hence
$$I^*(f_1 + f_2) = I_*(f_1 + f_2) = I(f_1) + I(f_2)$$
Let $f : \Omega \to \mathbb{R}$ be a mapping ,$\lambda \in \mathbb{R}_{>0}$
$$I^*(\lambda f) = \inf_{\mu \in S, \mu \geq \lambda f} I(\mu) = \inf_{\nu \in S, \nu \geq f} I(\lambda \nu) = \lambda I^*(f)$$
Similarly
$$I_*(\lambda f) = \lambda I_*(f)$$
Hence if $f \in \mathscr{R}$ then $\lambda f \in \mathscr{R}$ and $I(\lambda f) - \lambda I(f)$
$$I^*(-f) = \inf_{\mu \in S, \mu \geq -f} I(\mu) = \inf_{l \in S, l \leq f} I(-l) = -\sup_{l \in S, l \leq f} I(l) = -I_*(f)$$
Similarly
$$I_*(-f) = -I^*(f)$$
Hence if $f \in \mathscr{R}$ then $-f \in \mathscr{R}$ and $I(-f) = -I(f)$

Chapter 37

Daniell integral

We fix an integral operator $I : S \to \mathbb{R}$

37.1 Prop

37.1.1

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in S that converges pointwisely to some $f \in S$. Then
$$\lim_{n\to+\infty} I(f_n) = I(f)$$

Proof

Let $g_n = f - f_n \in S$ $(g_n)_{n\in\mathbb{N}}$ is decreasing and converges pointwisely to 0. Then
$$\lim_{n\to+\infty} I(g_n) = 0$$
Hence
$$\lim_{n\to+\infty} I(f_n) = I(f)$$

37.1.2

Let $(f_n)_{n\in\mathbb{N}}$ be an increasing sequence in S, $f \in S$ If $f \leq \lim\limits_{n\to+\infty} f_n$, then
$$I(f) \leq \lim_{n\to+\infty} I(f_n)$$

Proof

$$f = \lim_{n\to+\infty} f \wedge f_n$$
So
$$I(f) = \lim_{n\to+\infty} I(f \wedge f_n) \leq \lim_{n\to+\infty} I(f_n)$$

37.2 Def

Let

$$S^{\uparrow} = \left\{ f : \Omega \to \mathbb{R} \cup \{+\infty\} \mid \begin{array}{r} \exists (f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}} \text{ increasing such that} \\ f = \lim\limits_{n\to+\infty} f_n \text{ pointwisely} \end{array} \right\}$$

37.3 Prop

Let f, g be elements of $S^{\uparrow}$ such that $f \leq g$ Let $(f_n)_{n\in\mathbb{N}}$ and $(g_m)_{n\in\mathbb{N}}$ be increasing sequences in S such that $f = \lim\limits_{n\to+\infty} f_n, g = \lim\limits_{n\to+\infty} g_n$ THen

$$\lim_{n\to+\infty} I(f_n) \leq \lim_{n\to+\infty} I(g_n)$$

Proof

For any $m \in \mathbb{N}$

$$f_m \leq f \leq g$$

Hence

$$I(f_m) \leq \lim_{n\to+\infty} I(g_n)$$

Taking $\lim\limits_{m\to+\infty}$ we get

$$\lim_{m\to+\infty} I(f_m) \leq \lim_{n\to+\infty} I(g_n)$$

37.4 Corollary

Let $f \in S^{\uparrow}$ If $(f_n)_{n\in\mathbb{N}}$ and $(\widetilde{f}_n)_{n\in\mathbb{N}}$ be increasing sequence in S such that

$$f = \lim_{n\to+\infty} f_n = \lim_{n\to+\infty} \widetilde{f}_n$$

then

$$\lim_{n\to+\infty} I(f_n) = \lim_{n\to+\infty} I(\widetilde{f}_n)$$

We denote by $I(f)$ the limit $\lim\limits_{n\to+\infty} I(f_n)$

Thus we obtain a mappign $I : S^{\uparrow} \to \mathbb{R} \cup \{+\infty\}$ such that

- If $(f_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$ is increasing then

$$I(\lim_{n\to+\infty} f_n) = \lim_{n\to+\infty} I(f_n)$$

- If $(f, g) \in {S^{\uparrow}}^2$ $f \leq g$ then $I(f) \leq I(g)$
- If $(f, g) \in {S^{\uparrow}}^2$ then $f + g \in S^{\uparrow}$ and

$$I(f + g) = I(f) + I(g)$$

- If $f \in S^{\uparrow}, \lambda \geq 0$then $\lambda f \in S^{\uparrow}$ and $I(\lambda f) = \lambda I(f)$

37.5 Prop

Let $(f_n)_{n\in\mathbb{N}} \in (S^\uparrow)^\mathbb{N}$ be an increasing sequence and $f = \lim_{n\to+\infty} f_n$. Then

$$f \in S^\uparrow$$

and

$$I(f) = \lim_{n\to+\infty} I(f_n)$$

Proof

For $k \in \mathbb{N}$ let $(g_{k,m})_{m\in\mathbb{N}} \in S^\mathbb{N}$ be an increasing sequence such that

$$f_k = \lim_{m\to+\infty} g_{k,m}$$

For $n \in \mathbb{N}$ let $h_n = g_{0,n} \vee \cdots \vee g_{n,n} \in S$ The sequence $(h_n)_{n\in\mathbb{N}}$ is increasing. Moreover

$$f_n \geq k_n \geq g_{k,n} \quad (k \leq n)$$

Hence

$$f_n \geq h_n$$

Taking $\lim_{n\to+\infty}$ we get $\forall k \in \mathbb{N}$

$$f = \lim_{n\to+\infty} f_n \geq \lim_{n\to+\infty} h_n \geq \lim_{n\to+\infty} g_{k,n} = f_k$$

Taking $\lim_{k\to+\infty}$ we get

$$f = \lim_{n\to+\infty} h_n$$

Hence $f \in S^\uparrow$ and

$$I(f) = \lim_{n\to+\infty} I(h_n) \leq \lim_{n\to+\infty} I(f_n)$$

Conversely, $\forall n \in \mathbb{N}, f \geq f_n$ Hence

$$I(f) \geq \lim_{n\to+\infty} I(f_n)$$

37.6 Def

Let $S^\downarrow = \{-f \mid f \in S^\uparrow\}$ We extend I to $I : S^\downarrow \to \mathbb{R}U{-\infty}$ by letting $I(-f) := -I(f)$ for $f \in S^\uparrow$

37.7 Prop

Let $(f, g) \in (S^\uparrow \cup S^\downarrow)^2$ If $f \leq g$ then

$$I(f) \leq I(g)$$

Proof

It suffices to treat the cases where $(f,g) \in S^{\uparrow} \times S^{\downarrow}$ and $(f,g) \in S^{\uparrow} \times S^{\downarrow}$

If $(f,g) \in S^{\uparrow} \times S^{\downarrow}$ then $-f \in S^{\downarrow}$ and hence $g-f \in S^{\uparrow}, g-f \geq 0$ In both cases,

$$0 \leq I(g-f) = I(g) + I(-f) = I(g) - I(f)$$

37.8 Def

Let $f : \Omega \to \mathbb{R}$ be a mapping. We define

$$\overline{I}(f) := \inf_{\mu \in S^{\uparrow}, \mu \geq f} I(\mu) \leq \inf_{\mu \in S, \mu \geq f} I(\mu) = I^{*}(f)$$

$$\underline{I}(f) := \sup_{\mu \in S^{\downarrow}, \mu \leq f} I(\mu) \geq \sup_{\mu \in S, \mu \leq f} I(\mu) = I_{*}(f)$$

If $\overline{I}(f) = \underline{I}(f)$ then we say that f is I-integrable(in the sense of Daniell)

37.9 Remark

If f is I-integrable in the sense of Riemann, then it is I-integrable in sense of Daniell

37.10 Daniell Theorem

The set $L^1(I)$ of all I-integrable mappings forms a vector subspace of $\mathbb{R}$. Moreover

- $\forall (f,g) \in L^1(I)$ $f \wedge g \in L^1(I)$
- $I : L^1(I) \to \mathbb{R}$ is an integral operator extending $I : S \to \mathbb{R}$

Proof

Let $(f_1, f_2) \in L^1(I)^2$ let $(l_1, l_2) \in {S^{\downarrow}}^2, l_1 \leq f_1, l_2 \leq f_1$ Let $(\mu_1, \mu_2) \in {S^{\uparrow}}^2, f_1 \leq \mu_1, f_2 \leq \mu_2$

We have

$$l_1 + l_2 \leq f_1 + f_2 \leq \mu_1 + \mu_2$$

Taking the supremum with respect to (l_1, l_2), we get

$$I(f_1) + I(f_2)(= \underline{I}(f_1) + \underline{I}(f_2)) \leq \underline{I}(f_1 + f_2)$$

Taking the infimum with respect to (l_1, l_2), we get

$$\overline{I}(f_1 + f_2) \leq I(f_1) + I(f_2)$$

Then

$$\overline{I}(f_1+f_2)=\underline{I}(f_1+f_2)$$

So $f_1+f_2\in L^1(I)$ and $I(f_1+f_2)=I(f_1)+I(f_2)$

Similarly, if $f\in L^1(I),\lambda\geq 0$ then

$$\begin{aligned}\underline{I}(\lambda f)&=\sup_{l\leq\lambda f,l\in S^{\downarrow}} I(l)\\&=\sup_{l\leq f,l\in S^{\downarrow}} I(\lambda l)\\&=\lambda\underline{I}(f)=\lambda I(f)\end{aligned}$$

$$\overline{I}(\lambda f)=\lambda\overline{I}(f)=\lambda I(f)$$

So $\lambda f\in L^1(I)$ and $I(\lambda f)=\lambda I(f)$
Moreover, if $f\in L^1(I),\mu\in S^{\uparrow},l\in S^{\downarrow},l\leq f\leq\mu$ then

$$-\mu\in S^{\downarrow},-l\in S^{\uparrow},-\mu\leq -f\leq -l$$

Hence

$$\overline{I}(-f)=-\underline{I}(f)=-I(f)\quad \underline{I}(-f)=-\overline{I}(f)=-I(f)$$

So $-f\in L^1(I)$ and $I(-f)=-I(f)$

We proved that $\forall(f_1,f_2)\in L^1(I)^2$

$$f_1\wedge f_2\in L^1(I)$$

Let $(f_1,f_2)\in L^1(I)^2$, for any $\epsilon>0$ $\exists(l_1,l_2)\in {S^{\downarrow}}^2,(\mu_1,\mu_2)\in {S^{\uparrow}}^2$ such that

$$l_1\leq f_1\leq\mu_1\quad l_2\leq f_2\leq\mu_2$$

such that

$$I(\mu_1-l_1)\leq\frac{\epsilon}{2}\quad I(\mu_2,l_2)\leq\frac{\epsilon}{2}$$

One has $l_1\wedge l_2\leq f_1\wedge f_2\leq\mu_1\wedge\mu_2$

$$\mu_1\wedge\mu_2-l_1\wedge l_2\leq(\mu_1-l_1)+(\mu_2-l_2)$$

$$\begin{pmatrix}\text{If}\mu_1(\omega)\leq\mu_2(\omega),l_1\leq l_1(\omega)\\ \quad LHS=\mu_1(\omega)-l_1(\omega)\\ \quad RHS=\mu_1(\omega)-l_2(\omega)+\mu_2(\omega)-l_1(\omega)\geq\mu_1(\omega)-l_2(\omega)\end{pmatrix}$$

37.11 Beppo Levi Theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a monotone sequence of elements of $L_1(I)$, which converges pointwisely to some $f:\Omega\to\mathbb{R}$ If $(I(f_n))_{n\in\mathbb{N}}$ converges to a real number α Then $f\in L^1(I)$ and $I(f)=\alpha$

Proof

Assume that $(f_n)_{n\in\mathbb{N}}$ is increasing. Moreover, by replacing f_n by $f_n - f_0$ we may assume that $f_0 = 0$
Let $\epsilon > 0$ $\forall n \in \mathbb{N}$ let $\mu_n \in S^{\uparrow}$ such that $f_n - f_{n-1} \le \mu_n$ and

$$I(f_n - f_{n-1}) \ge I(\mu_n) - \frac{\epsilon}{2}$$

the existence

$$I(f_n - f_{n-1}) = \inf_{\mu\in S^{\uparrow}, \mu\ge f_n - f_{n-1}} I(\mu)$$

If $\forall \mu \in S^{\uparrow}, \mu \ge f_n - f_{n-1}$ one has

$$I(\mu) > I(f_n - f_{n-1}) + \frac{\epsilon}{2}$$

then

$$I(f_n - f_{n-1}) + \frac{\epsilon}{2} \le I(f_n - f_{n-1})$$

contraction.

Thus

$$f_n = \sum_{k=1}^{n}(f_k - f_{k-1}) \le \mu_1 + \cdots + \mu_n$$

and

$$I(f_n) \ge \sum_{k=1}^{n}(I(\mu_k) - \frac{\epsilon}{2^k}) \ge I(\mu_1) + \cdots + I(\mu_n) - \epsilon$$

Let $\mu = \mu_1 + \cdots + \mu_n + \cdots \in S^{\uparrow}$

$$I(\mu) = \sum_{n\in\mathbb{N}} I(\mu)$$

One has $\mu \ge f$

$$\lim_{n\to+\infty} \ge I(\mu) - \epsilon \ge \overline{I}(f) - \epsilon$$

Similarly, one can choose $l_n \in S^{\downarrow}, l_n \le f_n, I(l_n) \ge I(f_n) - \epsilon$

$$\liminf_{n\to+\infty} I(l_n) \ge \alpha - \epsilon$$

Note that $l_n \le f_n \le f$, so

$$\alpha - \epsilon \le \liminf_{n\to+\infty} I(l_n) \le \underline{I}(f)$$

Thus

$$\alpha - \epsilon \le \underline{I}(f) \le \overline{I}(f) \le \alpha + \epsilon$$

Let $\epsilon \to 0$ we get

$$\overline{I}(f) = \underline{I}(f) = \alpha$$

37.12 Fatou's Lemma

Let $(f_n)_{n\in\mathbb{N}} \in L^1(I)^{\mathbb{N}}$ Assume tha there is $g \in L^1(I)$ such that

$$\forall n \in \mathbb{N} \quad f_n \geq g$$

If $\liminf\limits_{n\to+\infty} f_n$ is a mapping from Ω to $\mathbb{R}$ and $\liminf\limits_{n\to+\infty} I(f_n) < +\infty$, then $\liminf\limits_{n\to+\infty} f_n \in L^1(I)$ and

$$I(\liminf_{n\to+\infty} f_n) \leq \liminf_{n\to+\infty} I(f_n)$$

Proof

For any $n \in \mathbb{N}$, let

$$g_n = \lim_{k\to+\infty}(f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k})$$

Then

$$\liminf_{n\to+\infty} f_n = \lim_{n\to+\infty} g_n$$

For any k one has

$$f_n \wedge \cdots \wedge f_{n+k} \geq g$$

Hence

$$I(f_n) \geq \lim_{n\to+\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \geq I(g)$$

By the theorem pf Beppo Levi,

$$g_n \in L^1(I) \text{ and } I(g_n) = \lim_{n\to+\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \leq I(f_n)$$

Note that $(g_n)_{n\in\mathbb{N}}$ is increasing and $\liminf\limits_{n\to+\infty} I(f_n) < +\infty$ Hence

$$\lim_{n\to+\infty} I(g_n) = \liminf_{n\to+\infty} I(g_n) \leq \liminf_{n\to+\infty} I(f_n) < +\infty$$

By the theorem of Beppo Levi,

$$\lim_{n\to+\infty} g_n \in L^1(I)$$

and

$$I(\liminf_{n\to+\infty} f_n) = I(\lim_{n\to+\infty} g_n) = \lim_{n\to+\infty} I(g_n) \leq \liminf_{n\to+\infty} I(f_n)$$

37.13 Lebesgue dominated convergence theorem

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(I)$ that converges pointwisely to some $f : \Omega \to \mathbb{R}$ Assume that there exists $g \in L^1(I)$ such that $\forall n \in \mathbb{N}, |f_n| \leq g$ Then $f \in L^1(I)$ and $I(f) = \lim\limits_{n\to+\infty} I(f_n)$

Proof

Apply Fatou's lemma to $(f_n)_{n\in\mathbb{N}}$ adn $(-f_n)_{n\in\mathbb{N}}$ to get

$$I(f) \leq \liminf_{n\to+\infty} I(f_n)$$

and

$$\begin{aligned} I(-f) &\leq \liminf_{n\to+\infty} I(-f_n) \\ &= \limsup_{n\to+\infty} I(f_n) \\ &\leq \limsup_{n\to+\infty} I(f_n) \leq I(f) \end{aligned}$$

37.14 Notation

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing and right continuous mapping. Let S be the vector subspace of $\mathbb{R}^{\mathbb{R}}$ generated by $\mathbb{1}_{]a,b]}$ with $(a,b) \in \mathbb{R}^2, a < b$ For any $f \in L^1(I_\varphi)$ $I_\varphi(f)$ is denoted as

$$\int_{\mathbb{R}} f(x)d\varphi(x)$$

For any subset A of $\mathbb{R}$ if $\mathbb{1}_A f \in L^1(I)$ then

$$\int_A f(x)d\varphi(x) \text{ denotes } \int_{\mathbb{R}} \mathbb{1}_A(x)f(x)d\varphi(x) = I(\mathbb{1}_A f)$$

If $(a,b) \in \mathbb{R}^2, aa < b$

$$\int_a^b f(x)d\varphi(x) \text{ denotes } \int_{]a,b]} f(x)d\varphi(x)$$

$$\int_b^a f(x)d\varphi(x) \text{ denotes } -\int_{]a,b]} f(x)d\varphi(x)$$

If $\varphi(x) = x$ for any $x \in \mathbb{R}$ we replace $d\varphi(x)$ by dx.

Chapter 38

Semialgebra

38.1 Notation

Let $A, (A_i)_{i\in I}$ be sets the notation.

$$A = \bigsqcup_{i\in I} A_i$$

denotes:

- $(A_i)_{i\in I}$ is a pairwisely disjoint
- $A = \bigcup\limits_{i\in I} A_i$

38.2 Def

Let Ω be a set. We call semialgebra on Ω any $\mathcal{C} \subsetneqq \wp(\Omega)$ that verifies:

- $\varnothing \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, A \cap B \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, \exists (C_i)_{i=1}^n$ a finite family of elements in $\mathcal{C}$ such that $B \setminus A = \bigsqcup\limits_{i=1}^{n} C_i$

38.2.1 Example

$$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$$

38.3 Def

Let $\mathcal{C}$ be a semialgebra on Ω. The set

$$\{A \in \wp(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigsqcup_{i=1}^n A_i\}$$

is called the algebra generated by $\mathcal{C}$

38.4 Prop

Let $\mathcal{C}$ be a semialgebra on Ω. $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Then:

- $\varnothing \in \mathcal{A}$
- $\forall (A, B) \in \mathcal{A}^2, A \cap B \in \mathcal{A}, B \setminus A \in \mathcal{A}, A \cup B \in \mathcal{A}$

Proof

By definition, $\varnothing \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}$. Moreover, if A and B are elements of $\mathcal{A}$ such that $A \cap B = \varnothing$ then $A \cup B \in \mathcal{A}$. Let $A = \bigsqcup_{i=1}^n A_i$ and $B = \bigsqcup_{i=1}^n B_i$ be elements of $\mathcal{A}$ then

$$A \cap B = \bigsqcup_{(i,j) \in \{1, \cdots, n\}^2} (A_i \cap B_i)$$

Hence $A \cap B \in \mathcal{A}$ Finally

$$A \cup B = (A \cap B) \sqcup (A \setminus B) \sqcup (B \setminus A) \in \mathcal{A}$$

38.5 Prop

Let $\mathcal{C}$ be a semialgebra on Ω. $\mathcal{A}$ be the algebra generated by $\mathcal{C}$. Let S be the $\mathbb{R}$-vector subspace of $\mathbb{R}^\Omega$, $I : S \to \mathbb{R}$ be an $\mathbb{R}$-linear mapping generated by mappings of the form $\mathbb{1}_A, A \in \mathcal{C}(f \in S, f = \sum \lambda_i \mathbb{1}_{A_i})$
Assume that

$$\forall (f, g) \in S^2, f \leq g \text{ one has } I(f) \leq I(g)$$

Then I is an integral operator iff, for any decreasing sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{A}^\mathbb{N}$ such that $\bigcap_{n \in \mathbb{N}} A_n = \varnothing$, one has

$$\lim_{n \to +\infty} I(\mathbb{1}_{A_n}) = 0$$

38.5.1 Proof

$$\forall A \in \mathscr{A}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigsqcup_{i=1}^n A_i \text{ so } \mathbb{1}_A = \sum_{i=1}^n \mathbb{1}_{A_i} \in S$$

Lemma $\forall (f,g) \in S^2, f \wedge g \in S$

$\Rightarrow$ Suppose that I is an integral operator $(\mathbb{1}_{A_n})_{n\in\mathbb{N}}$ is a decreasing sequence in S and

$$\lim_{n\to+\infty} \mathbb{1}_{A_n} = 0$$

Hence

$$\lim_{n\to+\infty} I(\mathbb{1}_{A_n}) = 0$$

$\Leftarrow$ Let $(f_n)_{n\in\mathbb{N}}$ be a decreasing sequence in S that converges pointwisely to 0. Let

$$B = \{\omega \in \Omega \mid f_0(\omega) > 0\} \in \mathscr{A} \quad M = \max\{f_0(\omega) \mid \omega \in \Omega\}$$

- For any $\epsilon > 0$ let

$$A_n^\epsilon = \{\omega \in \Omega \mid f_n(\omega) \geq \epsilon\} \in \mathscr{A}$$

Moreover, since $\lim\limits_{n\to+\infty} f_n(\omega) = 0$, $\bigcap\limits_{n\in\mathbb{N}} A_n^\epsilon = \varnothing$

$$f(\omega) \begin{matrix} \lambda_i & \text{if} \omega \in A_i \\ 0 & \text{if} \omega \in \Omega \setminus \bigcap_{i=1}^n A_i \end{matrix}$$

($\forall f \in S, \exists (A_i)_{i=1}^n$ pairwisely disjoint and $(\lambda_i)_{i=1}^n \in \mathbb{R}$ $f = \sum\limits_{i=1}^n \lambda_i \mathbb{1}_{A_i}$)
Note that

$$0 \leq f_n \leq \epsilon \mathbb{1}_B + M \mathbb{1}_{A_n^\epsilon}$$

So

$$0 \leq I(f_n) \leq \epsilon I(\mathbb{1}_B) + MI(\mathbb{1}_{A_n^\epsilon})$$

which leads to

$$\limsup_{n\to+\infty} I(f_n) \leq \epsilon I(\mathbb{1}_B) \quad \forall \epsilon > 0$$

So

$$\lim_{n\to+\infty} I(f_n) = 0$$

38.5.2 Example

Let $\Omega = \mathbb{R}$ and $\mathcal{C} = \{]a,b] \mid (a,b) \in \mathbb{R}^2, a \leq b\}$
$\mathcal{A}$ be algebra generated by $\mathcal{C}$. $\varphi : \mathbb{R} \to \mathbb{R}$ increasing, right continuous. S be $\mathbb{R}$-vector subspace generated by $\mathbb{1}_{]a,b],(a,b)\in\mathbb{R},a\leq b}$

$$\begin{aligned} I_\varphi &: S \to \mathbb{R} \\ \mathbb{1}_{]a,b]} &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

Lemma $\forall \epsilon > 0, A \in \mathcal{A}, A \neq \varnothing, \exists B \in \mathcal{A},$

$$\varnothing \neq \overline{B} \subseteq A \text{ and } I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) \leq \epsilon$$

Proof

We first consider the the case where $A \in \mathcal{C}, A =]a,b]a,b], a < b$ By the right continuous of $\varphi, \exists]a,b[$ such that $\varphi(a') - \varphi(a) \leq \epsilon$. Let $B =]a',b]$, $\overline{B} = [a',b] \subseteq]a,b]$.

$$I_\varphi(\mathbb{1}_B) = \varphi(b) - \varphi(a')$$

$$I_\varphi(\mathbb{1}_A) = \varphi(b) - \varphi(a)$$

$$I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) = \varphi(a') - \varphi(a) \leq \epsilon$$

In general

$$A = \bigsqcup_{i=1}^{n} A_i$$

with $A_i \in \mathcal{C} \forall i \in \{1, \cdots, n\}, \exists B_i \in \mathcal{C}$

$$\varnothing \neq \overline{B_i} \subseteq A_i \quad I(\mathbb{1}_A) - I(\mathbb{1}_B) \leq \frac{\epsilon}{n}$$

Let $B = \bigsqcup_{i=1}^{n} B_i$ then

$$I(\mathbb{1}_A) - I(\mathbb{1}_B) = \sum_{i=1}^{n} I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \epsilon$$

38.6 Theorem

I_φ is an integral operator

Proof

Let $(A_n)_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathcal{A}$ such that

$$\bigcap_{n\in\mathbb{N}} A_n = \varnothing$$

Let $\epsilon > 0$ For any $n \in \mathbb{N}$ let $B_n \in \mathscr{A}$ such that

$$\overline{B}_n \subseteq A_n \text{ and } I_\varphi(A_n) - I_\varphi(B_n) \leq \frac{\epsilon}{2^n}$$

Note that $\overline{B}_n$ is compact. For any $n \in \mathbb{N}$ let

$$\begin{aligned} C_n &= \bigcap_{i=1}^{n} B_i \\ &\subseteq \bigcap_{i=1}^{n} \overline{B}_i \end{aligned}$$

Since $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$, $\bigcap_{n\in\mathbb{N}} \overline{B}_n = \varnothing$
So

$$I_\varphi(\mathbb{1}_{A_n}) \leq \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}} \cdots \frac{\epsilon}{2} \leq \epsilon$$

Thus

$$\lim_{n\to+\infty} I_\varphi(\mathbb{1}_{A_n}) = 0$$

Let Ω be a set $\mathscr{C}$ be a semialgebra on Ω and $\mathscr{A}$ be the algebra generated by $\mathscr{C}$. Let S be the $\mathbb{R}$-vector subspace of $\mathbb{R}^\Omega$ generated by mappings of the form $\mathbb{1}_A, A \in \mathscr{C}$

38.7 Prop

For any $f \in S$,$\exists (A_i)_{i=1}^n \in \mathscr{A}^n$ pairwisely disjoint and $(\lambda_i)_{i=1}^n$ such that

$$f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i}$$

Proof

f is of the form

$$f = \sum_{j=1}^{m} a_j \mathbb{1}_{B_j} \quad B_j \in \mathscr{C}$$

We reason by induction on m. For any $I \subseteq \{1, \cdots, m\}$ let

$$B_I = \bigcap_{j\in I} B_j \cap \bigcap_{j\in\{1,\cdots,m\}\setminus I} (\Omega \setminus B_j)$$

Then $(B_I)_{I\subseteq\{1,\cdots,m\}}$ are pairwisely disjoint.
Moreover. if $I = \varnothing, B_I \in \mathscr{A}$

$$B_i = \bigsqcup_{I\subseteq\{1,\cdots,m\}, i\in I} B_I$$

Hence

$$f = \sum_{U\subseteq\{1,\cdots,m\}} (\sum_{j\in I} a_j \mathbb{1}_{B_I})$$

38.8 Corollary

(1) If $f \in S$ then

$$f \wedge 0 \in S$$

(2) If $(f, g) \in S^2$ then

$$f \wedge g = (f - g) \wedge 0 + g \in S$$

Proof

We intend to define

$$I_\mu(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i})$$

as

$$\sum_{i=1}^{n} \lambda_i I_\mu(\mathbb{1}_{A_i})$$

for $A_i \in \mathcal{C}$ We need to check that if $f \in S$ is written as

$$f = \sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i} = \sum_{j=1}^{m} \xi_j \mathbb{1}_{B_j}$$

then

$$\sum_{i=1}^{n} \xi_i I_\mu(\mathbb{1}_{A_i}) = \sum_{j=1}^{m} \xi_j I_\mu(\mathbb{1}_{B_j})$$

so

$$0 = \sum_{i=1}^{n} \xi_i \mathbb{1}_{A_i} - \sum_{j=1}^{m} \xi_j \mathbb{1}_{B_j}$$

I t suffices to prove that if

$$\sum_{i=1}^{n} \xi_i \mathbb{1}_{A_i} = 0$$

then

$$\sum_{j=1}^{m} \xi_j \mathbb{1}_{B_j} = 0$$

For $I \subseteq \{1, \cdots, n\}$ let

$$A_I = \{\omega \in \Omega \mid \forall i \in I, \omega \in A_i, \forall i \in \{1, \cdots, n\} \setminus I, \omega \in \Omega \setminus A_i\}$$

$A_I \in \mathcal{A}$ when $I \neq \varnothing$

38.9 Lemma

Let $B \in \mathcal{A}$ If
$$B = \bigsqcup_{i=1}^{n} B_i = \bigsqcup_{j=1}^{m} C_j$$
with $B_i, C_i \in \mathcal{C}$, then
$$\sum_{i=1}^{n} \mu(B_i) = \sum_{j=1}^{m} \mu(C_j)$$
In particular, we can extend $\mu : \mathcal{C} \to \mathbb{R}_{\geq 0}$ to $\mu : \mathcal{A} \to \mathbb{R}_{\geq 0}$ such that $\forall D_1, \cdots, D_n$ in $\mathcal{A}$ disjoint
$$\mu(D_1 \cup \cdots \cup D_n) = \sum_{i=1}^{n} \mu(D_i)$$

38.9.1 Proof

$$B_i = \bigsqcup_{j=1}^{m} (B_i \cap C_j) \qquad \mu(B_i) = \sum_{j=1}^{m} \mu(B_i \cap C_j)$$
$$\begin{aligned}\sum_{i=1}^{n} \mu(B_i) &= \sum_{i=1}^{n} \sum_{j=1}^{m} \mu(B_i \cap C_j) \\ &= \sum_{j=1}^{m} \sum_{i=1}^{n} \mu(B_i \cap C_j) \\ &= \sum_{j=1}^{m} \mu(C_j)\end{aligned}$$
Back to the proof
$$0 = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i} = \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} (\sum_{i \in I} a_i) \mathbb{1}_{A_i}$$
hence
$$\sum_{i \in I} a_i = 0$$
$$\begin{aligned}0 &= \sum_{i=1}^{n} a_i \mu(A_i) \\ &= \sum_{i=1}^{n} a_i \sum_{i \in I \subseteq \{1, \cdots, n\}} \mu(A_i) \\ &= \sum_{\varnothing \neq I \subseteq \{1, \cdots, n\}} \mu(A_i) \sum_{i \in I} a_i\end{aligned}$$

Chapter 39

Integral function

39.1 Setting

Let Ω be a set. $S \subseteq \mathbb{R}^{\Omega}$ be $\mathbb{R}$-vector subspace, $\forall (f, g) \in S^2, f \wedge g \in S$
$I : S \to \mathbb{R}$ integral operator.

39.2 Prop

Suppose that $\mathbb{1}_{\Omega} \in L^1(I)^{\uparrow}$ The set

$$\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I)^{\uparrow}\}$$

is a σ-algebra on Ω

Moreover, if we denote by $\mu : \mathcal{G} \to \mathbb{R}_{\geq 0}$ the mapping define as

$$\mu(A) := I(\mathbb{1}_A)$$

then μ satisfies :

$\forall (A_n)_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ that's is pairwisely disjoint, then

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

39.2.1 Proof

(1)

$$\varnothing \in \mathcal{G}$$

since

$$0 = \mathbb{1} \in L^1(I)^{\uparrow}$$

(2) If A and B are elements of $\mathcal{G}, A \subseteq B$, then

$$\mu(A) \leq \mu(B)$$

so

$$\mathbb{1}_B - \mathbb{1}_{B\setminus A} \in L^1(I)^{\uparrow} \Rightarrow B \setminus A \in \mathcal{G}$$

(3) If $(A, B) \in \mathcal{G}^2$

$$\mathbb{1}_{A\cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A\cap B} \in L^1(I)^{\uparrow}$$

So $A \cup B \in \mathcal{G}$

If $(A_n)_{n\in\mathbb{N}} \in \mathcal{G}, A = \bigcup\limits_{n\in\mathbb{N}} A_n$ then

$$\mathbb{1}_A = \lim_{n\to+\infty} \mathbb{1}_{A_1\cup\cdots\cup A_n} \in L^1(I)^{\uparrow} \quad \Rightarrow A \in \mathcal{G}$$

$$\begin{aligned}
\underbrace{I(\mathbb{1}_{\sum_{n\in\mathbb{N}} A_n})}_{\mu(\bigcup_{n\in\mathbb{N}} A_n)} &= \lim_{n\to+\infty} \mathbb{1}_{A_0\cup\cdots\cup A_n} \\
&= \lim_{n\to+\infty} \sum_{i=0}^{n} \mathbb{1}_{A_n} \in L^1(I)^{\uparrow} \\
&= -\sum_{n\in\mathbb{N}} \mu(A_n)
\end{aligned}$$

Chapter 40

Limit and Differential of Integrals with Parameters

Let Ω be a set. $S \subseteq \mathbb{R}^{\Omega}$ be $\mathbb{R}$-vector subspace such that $\forall (f,g) \in S^2, f \wedge g \in S$. Let $I : S \to \mathbb{R}$ be an integral operator.

40.1 Theorem

Let X be a topological space, $p \in X$, $f : \Omega \times X \to \mathbb{R}$ be a mapping, $g \in L^1(I)$ Suppose that

(1) $\forall \omega \in \Omega$

$$\begin{aligned} f(\omega, \cdot) : \Omega &\to \mathbb{R} \\ x &\mapsto f(\omega, x) \end{aligned}$$

is continuous at p

(2) $\forall x \in X$

$$\begin{aligned} f(\cdot, x) : \Omega &\to \mathbb{R} \\ \omega &\mapsto f(\omega, x) \end{aligned}$$

belongs to $L^1(I)$ and $\forall \omega \in \Omega \ |f(\omega, x)| \leq g(\omega)$

(3) p has a countable neighborhood basis in X

Then

$$(x \in X) \mapsto I(f(\cdot, x))$$

is continuous at p

Proof

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X that converges to p. For any $n \in \mathbb{N}$ let $f_n : \Omega \to \mathbb{R}, f_n(\omega) := f(\omega, x_n)$. One has $|f_n| \le g$. Moreover $\forall \omega \in \Omega$

$$\lim_{n\to+\infty} f_n(\omega) = \lim_{n\to+\infty} f(\omega, x_n) = f(\omega, p)$$

Hence, by dominate convergence theorem

$$\lim_{n\to+\infty} I(f_n) = I(f(\cdot, p))$$

40.2 Theorem

Let J be an open interval in $\mathbb{R}$. $f : \Omega \times J \to \mathbb{R}$ be a mapping. $g \in L^1(I)$. Assume that

(1) $\forall \omega \in \Omega$

$$f(\omega, \cdot) : J \to \mathbb{R}$$

is differentiable(we denote by $\frac{\partial f}{\partial t}(\omega, t)$ its derivative at t) and $\forall t \in J$

$$\left| \frac{\partial f}{\partial t}(\omega, t) \right| \le g(\omega)$$

(2) $\forall t \in J$

$$\begin{aligned} f(\cdot, t) : \Omega &\to \mathbb{R} \\ \omega &\mapsto f(\omega, t) \end{aligned}$$

belongs to $L^1(I)$

Then

$$\begin{aligned} \varphi : J &\to \mathbb{R} \\ t &\mapsto I(f(\cdot, t)) \end{aligned}$$

is differentiable and

$$\varphi'(t) = I(\frac{\partial f}{\partial t}(\cdot, t))(= \frac{d}{dt} I(f(\cdot, t)))$$

Proof

Let $a \in J$ and $(t_n)_{n\in\mathbb{N}}$ be a sequence in $J \setminus \{a\}$ such that

$$\lim_{n\to+\infty} t_n = a$$

. Then

$$\frac{\varphi(t_n) - \varphi(a)}{t_n - a} = I(\frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a})$$

$\forall \omega \in \Omega$

$$\left| \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} \right| \leq g(\omega) \text{(by mean value theorem)}$$

and

$$\lim_{n \to +\infty} \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} = \frac{\partial f}{\partial t}(\omega, a)$$

Hence

$$\lim_{n \to +\infty} \frac{\varphi(t_n) - \varphi(a)}{t_n - a} = \frac{d}{dt} I(f(\cdot, t))$$

Chapter 41

Measure theory

41.1 Def

We call measure space any pair $(E, \mathcal{E})$, where E is a set and $\mathcal{E}$ is a σ-algebra on E.

41.2 Prop

Let Ω be a set. And $(\mathcal{G}_i)_{i\in I}$ be a family of σ-algebras on Ω. Then $\bigcap\limits_{i\in I} \mathcal{G}_i$ is a σ-algebra

Proof

- $\forall i \in I$
$$\varnothing \in \mathcal{G}_i$$
Hence
$$\varnothing \in \bigcap_{i\in I} \mathcal{G}_i$$
- If $A \in \bigcap\limits_{i\in I} \mathcal{G}_i$ then $\forall i \in I$ $A \in \mathcal{G}_i$ Hence $\forall i \in I$
$$\Omega \setminus A \in \mathcal{G}_i$$
So
$$\Omega \setminus A \in \bigcap_{i\in I} \mathcal{G}_i$$
- Let $(A_n)_{n\in\mathbb{N}} \in (\bigcap\limits_{i\in I} \mathcal{G}_i)^{\mathbb{N}}$. For any $i \in I$
$$(A_n)_{n\in\mathbb{N}} \in \mathcal{G}_i^{\mathbb{N}}$$

So

$$\bigcap_{n\in\mathbb{N}}(A_n) \in \mathcal{G}_i$$

so

$$\bigcap_{n\in\mathbb{N}}(A_n) \in \bigcap_{i\in I}\mathcal{G}_i$$

41.3 Def

Let $\mathcal{C} \subseteq \wp(\Omega)$. We denote by $\sigma(\mathcal{C})$ the intersection of all σ-algebras on Ω containing $\mathcal{C}$. It's the smallest σ-algebra containing $\mathcal{C}$

41.4 Example

- Let $(X, \mathcal{J})$ be a topological space. $\sigma(\mathcal{J})$ is called the Borel σ-algebra of X
- On $[-\infty, +\infty]$ the following σ-algebras are the same:

$$\begin{aligned}
g_1 &= \sigma(\{[a, +\infty] \mid a \in \mathbb{R}\}) \\
g_2 &= \sigma(\{]a, +\infty] \mid a \in \mathbb{R}\}) \\
g_3 &= \sigma(\{[-\infty, a] \mid a \in \mathbb{R}\}) \\
g_4 &= \sigma(\{[-\infty, a] \mid a \in \mathbb{R}\})
\end{aligned}$$

Moreover

$$\mathscr{B} = \{A \subseteq \mathbb{R} \mid A \in g_1\}$$

is equal to the Borel σ-algebra of $\mathbb{R}$

proof $\forall a \in \mathbb{R}$

$$\begin{aligned}
[a, +\infty] &= \bigcap_{n\in\mathbb{N}_{\geq 1}}]a - \frac{1}{n}, +\infty] \in g_2 && \Rightarrow g_1 \in g_2 \\
]a, +\infty] &= [-\infty, +\infty] \setminus [-\infty, a] \in g_3 && \Rightarrow g_2 \in g_3 \\
[-\infty, a] &= \bigcap_{n\in\mathbb{N}_{\geq 1}} [-\infty, a + \frac{1}{n}[\in g_4 && \Rightarrow g_3 \in g_4 \\
[-\infty, a[&= [-\infty, +\infty] \setminus [a, +\infty] \in g_1 && \Rightarrow g_4 \in g_1
\end{aligned}$$

$$\sigma(\{]a, b[| \; a < b, (a, b) \in \mathbb{R}^2\}) = \text{Borel } \sigma - \text{algebra of } \mathbb{R}$$

$J \subseteq \mathbb{R}$ open We define $\sim$ a binary relation on J such that $x \sim y \Leftrightarrow$ there exists an interval I such that $\{x, y\} \subseteq I \subseteq J$
Any equivalence class of $\sim$ is a non-empty open interval.

$$]a, b[= [a, +\infty] \cup [-\infty, b[$$

Hence Borel σ-algebra of $\mathbb{R} \subseteq \{A \subseteq \mathbb{R} \mid A \in g_1\}$

41.5 Def

Let $f : X \to Y$ be a mapping of sets.

- For any $\mathcal{C}_Y \subseteq \wp(Y)$ we denote by
$$f^{-1}(\mathcal{C}_Y) := \{f^{-1}(B) \mid B \in \mathcal{C}_Y\}$$
- For any $\mathcal{C}_X \subseteq \wp(X)$ we denote by
$$f_*(\mathcal{C}_X) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{C}_X\}$$

41.6 Prop

Let $f :\to Y$ be a mapping.

(1) If g_Y is a σ-algebra on Y then $f^{-1}(g_Y)$ is a σ-algebra on X

(2) If $\mathcal{G}_X$ is a σ-algebra on X then $f_*(\mathcal{G}_X)$ is a σ-algebra on Y

Proof

(1)
$$\varnothing = f^{-1}(\varnothing) \in f^{-1}(g_Y)$$
$\forall B \in g_Y$
$$X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$$
If $(A_n)_{n\in\mathbb{N}} \in g_Y^{\mathbb{N}}, A = \bigcup_{n\in\mathbb{N}} A_n, A_n \in g_Y$,then
$$\bigcup_{n\in\mathbb{N}} f^{-1}(A_n) = f^{-1}(A) \in f^{-1}(g_Y)$$

(2)
$$f^{-1}(\varnothing) = \varnothing = \mathcal{G}_X$$
so
$$\varnothing \in f_8(\mathcal{G}_X)$$
$\forall B \in f_*(\mathcal{G}_X)$
$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(Y) \in \mathcal{G}_X$$
so
$$Y \setminus B \in f_*(\mathcal{G}_X)$$
$\forall (B_n)_{n\in\mathbb{N}} \in f_*(\mathcal{G}_X)^{\mathbb{N}}, B = \bigcup_{n\in\mathbb{N}} B_n$
$$f^{-1}(B) = \bigcap_{n\in\mathbb{N}} f^{-1}(B_n)$$

So $B \in f_*(\mathcal{G}_X)$

41.7 Def

Let $(X, \mathcal{G}_X)$ and (Y, g_Y) be measurable spaces, $f : X \to Y$ be a mapping. If $f^{-1}(g_Y) \subseteq \mathcal{G}_X$ or equivalently $g_Y \subseteq f_8(\mathcal{G}_X)$ (or $\forall B \in g_Y, f^{-1}(B) \in \mathcal{G}_X$) then we say that f is measurable.

41.8 Prop

Let $(X, \mathcal{G}_X), (Y, g_Y), (Z, g_Z)$ be measurable spaces. $f : X \to Y$ and $g : Y \to Z$ be measurable mappings. Then $g \circ f$ is measurable.

Proof

$\forall B \in g_Z$

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

and

$$g^{-1}(B) \in g_Y$$

so

$$f^{-1}(g^{-1}(B)) \in \mathcal{G}_X$$

41.9 Def

Let Ω be a set $((E_i, \mathcal{E}))_{i\in I}$ be a family of measurable spaces. $f = (f_i)_{i\in I}$ where $f_i : \Omega \to E_i$ is a mapping. We denote by $\sigma(f)$ the σ-algebra $\sigma(\bigcup\limits_{i\in I} f_i^{-1}(\mathcal{E}_i))$ It's the smallest σ-algebra on Ω making all f_i measurable.

41.10 Prop

We keep the notation of the above definition. For any $i \in I$, let $\mathcal{C} \subseteq \wp(E_i)$ such that $\sigma(\mathcal{C}_i) = \mathcal{E}_i$Then

$$\sigma(f) = \sigma(\bigcup_{i\in I} f_i^{-1}(\mathcal{C}_i))$$

Proof

Let $g = \sigma(\bigcup\limits_{i\in I} f_i^{-1}(\mathcal{C}_i))$. By definition

$$g \subseteq \sigma(f)$$

For any $i \in I$, $f_{i,*}(\bigcup\limits_{i\in I} f_i^{-1}(\mathcal{C}_i))$ is a σ-algebra on Ω containing $\mathcal{C}_i$. So

$$\mathcal{E} \subseteq f_{i,*}(\sigma(f_i^{-1}(\mathcal{C}_i)))$$

which leads to

$$f_i^{-1}(\mathcal{C}_i) \subseteq \sigma(f_i^{-1}(\mathcal{C}_i)) \subseteq g$$

Hence

$$\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i) \subseteq g$$

$\Rightarrow$

$$\sigma(f) \subseteq g$$

$(f_{i,*}(\mathcal{A}) = \{B \subseteq E_i \mid f_i(B) \in \mathcal{A}\})$

41.11 Corollary

Let $(X, \mathcal{G}_X), (Y, g_Y)$ be measurable spaces. $f : X \to Y$ be a mapping. $\mathcal{C}_Y \subseteq g_Y$ such that $g_Y = \sigma(\mathcal{Y})$ Then f is measurable iff

$$\forall B \in \mathcal{C}_Y \quad f^{-1}(B) \in \mathcal{G}_X$$

Proof

$$\sigma(f) = \sigma(f^{-1}(\mathcal{C}_Y))$$

f is measurable iff $\sigma(f) \subseteq \mathcal{G}_X$

41.12 Example

Let $((E_i, \mathcal{E}_i))_{i \in I}$ be a family of measurable spaces.

$$E = \prod_{i \in I} E_i$$

$\forall i \in I$

$$\pi_i : E \to E_i$$
$$(x_j)_{j \in I} \mapsto x_i$$

We denote by $\bigotimes_{i \in I} \mathcal{E}_i$ the σ-algebra $\sigma((\pi_i)_{i \in I})$

41.13 Prop

Let X be a set $((E_i, \mathcal{E}_i))_{i \in I}$ be a family of measurable spaces. (Ω, g) be a measurable space. $f = (f_i : X \to E_i)_{i \in I}$ be a mappings, $\varphi : \Omega \to X$ be a mapping. Then

$$\varphi : (\Omega, g) \to (X, \sigma(f))$$

is measurable iff

$$\forall i \in I \quad f_i \circ \varphi : (\Omega, g) \to (E_i, \mathcal{E}_i) \text{ is measurable.}$$

Proof

$\Rightarrow$ If φ is measurable, since each f_i is measurable, one has $f_i \circ \varphi$ is measurable.

$\Leftarrow$ If $f_i \circ \varphi$ is measurable,$\forall B \in \mathcal{E}_i$

$$(f_i \circ \varphi)^{-1}(B) = \varphi^{-1}(f_i^{-1}(B)) \in g$$

Hence

$$\varphi^{-1}(\bigcup_{i \in I} f_i^{-1}(B)) \subseteq g$$

Since

$$\sigma(f) = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i))$$

φ is measurable.

41.14 Example

Let $(\Omega, \mathcal{G})$ be a measurable space

- $\forall A \in \mathcal{G}$ $\mathbb{1}_A : \Omega \to \mathbb{R}$ is measurable. For any $U \subseteq \mathbb{R}$

$$\mathbb{1}_A^{-1}(U) = A \text{ or } \Omega \setminus A \text{ or } \Omega \text{ or } \varnothing$$

- If X and Y be topological spaces. $f : X \to Y$ be a continuous mapping, then f is measurable with respect to Borel σ-algebra. In fact, $\forall V \subseteq Y$ open $f^{-1}(V) \subseteq X$ open.
- Let $(\Omega, \mathcal{G})$ be a measurable space If $f : \Omega \to \mathbb{R}, g : \Omega \to \mathbb{R}$ are measurable then $f + g, fg, f \wedge g, f \vee g, |f|$ are measurable.
- Let $(f_n)_{n \in \mathbb{N}}$ be a family of measurable mappings from Ω to $[-\infty, +\infty]$

$$f = \sup_{n \in \mathbb{N}} f_n \quad (f(\omega) = \sup f_n(\omega))$$

Then f is measurable.
Similarly $\inf\limits_{n \in \mathbb{N}} f_n$ is measurable.

In fact, for any $a \in \mathbb{R}$

$$\{\omega \in \Omega \mid f(\omega) > a\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid f_n(\omega) > a\}$$

Chapter 42

Measure

42.1 Def

Let Ω be a set.$\mathcal{C}$ be a semi-algebra on Ω. $\mu : \mathcal{C} \to \mathbb{R}_{\geq 0}$ be a mapping. If $\forall n \in \mathbb{N}, \forall (A_i)_{i=1}^n \in \mathcal{C}^n$ pairwisely disjoint, with $A = \bigcup\limits_{i=1}^{n} A_i$ one has

$$\mu(A) = \sum_{i=1}^{n} \mu(A_i)$$

we say that μ is additive.

Let

$$S = \text{ vector subspace of } \mathbb{R}^{\Omega} \text{ generated by } (\mathbb{1}_A)_{A \in \mathcal{C}}$$

Then

$$I_\mu : S \to \mathbb{R}$$
$$\sum_{i=1}^{n} \lambda_i \mathbb{1}_{A_i} \mapsto \sum_{i=1}^{n} \lambda_i \mu(A_i)$$

is well defined. If I_μ is an integral operator, we say that μ is σ-additive.

42.2 Def

Let $(\Omega, \mathcal{G})$ be a measurable space. $\mu : \mathcal{G} \to [0, +\infty]$ be a mapping. If $\mu(\varnothing) = 0$ and if for any $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$ pairwisely disjoint.

$$\mu(\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

we say that μ is a measure.

$(\Omega, \mathcal{G}, \mu)$ is called a measure space.

42.3 Def

If $\exists (A_n)_{n\in\mathbb{N}}$ such that $\Omega = \bigcup\limits_{n\in\mathbb{N}} A_n$ and $\mu(A_n) < +\infty$ then μ is said to be σ-finite.

42.4 Carathéodory Theorem

Let Ω be a set, $\mathcal{C}$ be a semi-algebra on Ω, $\mu : \mathcal{C} \to \mathbb{R}_{\geq 0}$ be a σ-additive mapping. Assume that there is a sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ such that $\Omega = \bigcup\limits_{n\in\mathbb{N}} A_n$. Then μ extends to a σ-finite measure on $\sigma(\mathcal{C})$

Proof

Let $S \subseteq \mathbb{R}^{\Omega}$ be the vector subspace generated by $(\mathbb{1}_A), A \in \mathcal{C}$ Let $\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I_\mu)^{\uparrow}\}$ then $\mathcal{G}$ is a σ-algebra containing $\mathcal{C}$. Hence $\sigma(\mathcal{C}) = \mathcal{G}$ Moreover, $(A \in \mathcal{G}) \mapsto I_\mu(\mathbb{1}_A)$ is a measure on $\mathcal{G}$ which is σ-finite.

42.5 Example

$\Omega = \mathbb{R}, \mathcal{C} = \{]a,b] \mid (a,b) \in \mathbb{R}^2, a < b\}$ $\sigma(\mathcal{C}) =$ Borel σ-algebra $\varphi : \mathbb{R} \to \mathbb{R}$ increasing and right continuous.

$$\begin{aligned} \mu_\varphi : \mathcal{C} &\to \mathbb{R}_{\geq 0} \\ ]a,b] &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

is σ-additive.

Hence μ_φ extends to a measure:

$$\sigma(\mathcal{C}) \to [0, +\infty]$$

called the Stieltjes measure. In the particular case where $\varphi(x) = x$ $(\forall x \in \mathbb{R})$ μ_φ is called a Lebesgue measure.

42.6 Def

Let $(\Omega, \mathcal{G}, \mu)$ be a measure space. Then

$$\{A \in \mathcal{G} \mid \mu(A) < +\infty\}$$

is a semialgebra. $\sigma(\mathcal{C}) = \mathcal{G}$ and $\mu\mid_{\mathcal{C}}$ is σ-additive.

- $$\mu(A_0) = \sum_{n\in\mathbb{N}} \mu(B_n) < +\infty$$
  $$\sum_{k\geq n} \mu(B_k) = \mu(A_n) \to 0 \ (n \to +\infty)$$

We denote by $L^1(\Omega, \mathcal{G}, \mu)$ the set of measurable mappings $f : \Omega \to \mathbb{R}$ that belongs to $L^1(I_\mu)$. For $f \in L^1(\Omega, \mathcal{G}, \mu)$

$$I_\mu(f)$$

is denoted as

$$\int_\Omega f(\omega)\mu(d\omega)$$

42.6.1 Particular case

If $\Omega = \mathbb{R}$ $\mu = \mu_\varphi$ Stieltjes measure.

$$\int_\mathbb{R} f(x)\mu_\varphi(dx)$$

is denoted as

$$\int_\mathbb{R} f(x)d\varphi(x)$$

42.7 Prop

Let $(\Omega, \mathcal{G}, \mu)$ be a σ-finite measure space. $f : \Omega \to \mathbb{R}$ is measurable. If

$$\exists g \in L^1(\Omega, \mathcal{G}, \mu), g \leq f$$

then

$$f \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

Proof

By replacing f by $f - g$, we may assume that $g = 0$ Consider first the case where

$$f = \mathbb{1}_B, B \in \mathcal{G}$$

Let $(A_n)_{n\in\mathbb{N}}$ be a increasing sequence in $\mathcal{G}$, $\mu(A_n) < +\infty$, $\bigcup\limits_{n\in\mathbb{N}} A_n = \Omega$ Then

$$\mathbb{1}_B = \lim_{n\in\mathbb{N}} \mathbb{1}_{B\cap A_n} \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

For general $f \geq 0$

$$f = \lim_{n\in\to+\infty} f_n \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

where

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega\in\Omega | \frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}\}} + n\mathbb{1}_{\{\omega\in\Omega | f(\omega)\geq n\}}$$

42.8 Corollary

Let $f : \Omega \to \mathbb{R}$ be a measurable mapping. Then

$$f \in L^1(\Omega, \mathcal{G}, \mu)$$

iff

$$\int_\Omega |f(\omega)| \, \mu(d\omega) < +\infty$$

Proof

$\Rightarrow$ One has $f \in L^1(I_\mu)$. Hence $|f| \in L^1(I_\mu)$ So $I_\mu(|f|) < +\infty$

$\Leftarrow$ Suppose that

$$\int_\Omega |f(\omega)| \, \mu(d\omega) < +\infty$$

Since $f \vee 0$ adn $-(f \wedge 0)$ belongs to $L^1(\Omega, \mathcal{G}, \mu)^\uparrow$ and $f \vee 0 \leq |f|, -(f \wedge 0) \leq |f|$ so

$$f \vee 0 \text{ and } -(f \wedge 0) \in L^1(\Omega, \mathcal{G}, \mu)$$

Hence

$$f = f \vee 0 + f \wedge 0 \in L^1(\Omega, \mathcal{G}, \mu)$$

Chapter 43

Fundamental theorem of calculus

43.1 Theorem

Let J be an open interval in $\mathbb{R}$ $x_0 \in J$ $f : J \to \mathbb{R}$ be a continuous mapping.

(1) $\forall (a,b) \in J^2, a < b$

$$\begin{aligned} \mathbb{1}_{]a,b]} : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto f(x) \quad \text{if } x \in]a,b] \\ &\mapsto 0 \qquad \text{if } x \notin]a,b] \end{aligned}$$

belongs to $L^1(\mathbb{R}, \mathscr{B}, \mu)$($\mathscr{B}$ is Borel σ-algebra,μ is Lebesgue measure)

(2) Let $F : J \to \mathbb{R}$ $F(x) : +\int_{x_0}^{x} f(t)dt$. Then F is differentiable on J with $F'(x) = f(x), \forall x \in J$

43.2 Corollary

If $G : J \to \mathbb{R}$ is a mapping such that $G' = f$ then $\forall (a,b) \in J^2, a < b$

$$G(b) - G(a) = \int_a^b f(t)dt$$

43.2.1 Proof

(1) f is bounded on $[a,b]$ Hence

$$\int_{\mathbb{R}} \mathbb{1}_{]a,b]}^{(x)} \left| f(t)^{(x)} \right| dx < +\infty$$

(2) Let $x \in J, h > 0$ such that $[x, x+h] \subseteq J$, f is uniformly continuous on $[x, x+h]$
For $0 < t \leq h$

$$\inf f \mid_{[x,x+t]} \leq \frac{F(x+t) - F(x)}{t} = \frac{1}{t} \int_x^{x+t} f(s)ds \leq \sup f \mid_{[x,x+t]}$$

Since f is continuous

$$\lim_{t \to 0} \inf f \mid_{[x,x+t]} = \lim_{t \to 0} \sup f \mid_{[x,x+t]} = f(x)$$

So

$$\lim_{t>0, t \to 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Similarly

$$\lim_{t>0, t \to 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Hence

$$F'(x) = f(x)$$

Application

- Let F and G be two mapping of class C^1 from J to $\mathbb{R}$ Then FG is of class C^1 and
$$FG'(x) = F'(x)G(x) + F(x)G'(x)$$
Let $f = F', g = G'$, then $\forall (a, b) \in J^2, a, b$
$$\int_a^b f(t)G(t)dt = F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t)dt$$

- Let $\varphi : I \to J$ be a mapping of class C^1 , where I is open interval. Let $F : J \to \mathbb{R}$ be a mapping of class C^1.
$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x)$$

- Hence $\forall (\alpha, \beta) \in I^2, \alpha < \beta$
$$\int_\alpha^\beta F(\varphi(t))\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha))$$

Chapter 44

L^p space

44.1 Def

We fix a measure space $(\Omega, \mathcal{G}, \mu)$ the set of measurable mappings $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^p} := (\int_\Omega |f(\omega)|^p \,\mu(dx))^{\frac{1}{p}} < +\infty$$

Lemma Let $(p,q) \in \mathbb{R}^2_{\geq 1}$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

for any $(a,b) \in \mathbb{R}^2_{\geq 0}$

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

Proof We may assume that $(a,b) \in \mathbb{R}_{\geq 0}$

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{1}{p}\exp(p\ln a) + \frac{1}{q}\exp(q\ln b) \geq \exp(\ln a + \ln b) = ab$$

$$\begin{aligned}\int_\Omega |\varphi(x)\psi(x)| \,\mu(dx) &\leq \frac{\int_\Omega |\varphi(x)|^p \,\mu(dx)}{p\,\|f\|^p_{L^p}} + \frac{\int_\Omega |\psi(x)|^q \,\mu(dx)}{q\,\|g\|^q_{L^q}} \\ &= \frac{1}{p} + \frac{1}{q} = 1\end{aligned}$$

44.2 Hölder inequality

Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be measurable mappings. One has

$$\|fg\|_{L^1} \leq \|f\|_{L^p}\,\|g\|_{L^q}$$

Proof

Take

$$\varphi = \frac{f}{\|f\|_{L^p}}, \psi = \frac{g}{\|g\|_{L^q}}$$

then

$$|\varphi(x)\psi(x)| \leq \frac{|\varphi(x)|^p}{p} + \frac{|\psi(x)|^q}{q}$$

44.3 Corollary

Let $p \geq 1$ $\forall (f,g) \in L^p(\Omega, \mathcal{G}, \mu)$

$$\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

Proof

Apply Hölder inequality to $f(f+g)^{p-1}$ and $g(f+g)^{p-1}$

Part VIII

tensor

Chapter 45

tensor product

Let R be a commutative ring with unity

45.1 Theorem

Let M and N be two R-modules. Then exists an R-module denoted by $M \otimes_R N$ and a bilinear mapping

$$t : M \times N \to M \otimes_R N$$

having the following properties:

(1) For any R-module P and any bilinear mapping $s : M \times N \to P$. There exists a unique linear mapping $f_s : M \otimes_R N \to P$ such that $s = f_s \circ t$

$$\begin{tikzcd} M \times N \arrow[r, "s"] \arrow[d, "t"] & P \\ M \otimes_R N \arrow[ur, "f_s"'] & \end{tikzcd}$$

(2) If T, t' is another couple that satisfies (1) with $s \mapsto g_s$ then there exists a unique isomorphism

$$T \cong M \otimes_R N$$

Proof

(2) note that the the morphisms on R-module category are just linear mapping.

$$\begin{tikzcd} M \times N \arrow[r, "t'"] \arrow[d, "t"] & M \otimes_R N \arrow[dl, bend right, "f_{t'}"'] \\ T \arrow[ur, bend right, "g_t"'] & \end{tikzcd}$$

$$(f_{t'} \circ g_t) \circ t' = f_{t'} \circ t' = t$$

It means that we have the following structure

$$f_{t'} \circ g_t = id_{M \otimes_R N}$$

$$g_t \circ f_{t'} = id_T$$

Then isomorphic.

(1) let $\mathcal{F}$ be the free R-module generated by $M \times N$

$$\mathcal{F} = \{ \sum_{finite} a_{ij}(m_i, n_i) : a_{ij} \in R, m_i \in M, n_i \in N\}$$

let $\mathcal{G}$ be the R-submodule generated by the elements of the following shape $m, m' \in M \quad n, n' \in N \quad \mathfrak{r} \in R$

$$\begin{aligned}
&(m + m', n) - (m, n) - (m', n) \\
&(m, n + n') - (m, n) - (m, n') \\
&(\mathfrak{r}m, n) - \mathfrak{r}(m, n) \\
&(m, \mathfrak{r}n) - \mathfrak{r}(m, n)
\end{aligned}$$

$$M \otimes_R N := \mathcal{F}/\mathcal{G}$$

45.2 Def

$$f_s(\mathcal{G} + (m, n)) := s(m, n)$$

Extend this mapping to linearity. This makes the diagram commutative. It's clearly the unique mapping

45.3 Def

The R-module $M \otimes_R N$ constructed above is called the tensor product of M and N. An element of $M \otimes_R N$ is called tensor. We denote

$$t(m, n) := m \otimes n$$

and any elements of this form is called pure tensor.

45.4 Remark

Pure tensors generate $M \otimes_R N$. In particular any tensor can be written as sum of pure tensors.

Example

$$0 = (m + m') \otimes n - m \otimes n - m \otimes n'$$

45.5 Corollary

The mapping $s \mapsto f_s$ defined above gives an isomorphism

$$\mathscr{L}(M, N; P) \cong \mathscr{L}(M \otimes_R N, P)$$

for any R-module P

Proof

surjection Take $\varphi \in \mathscr{L}(M \otimes_R N, P)$, the $\varphi \circ t$ is clearly bilinear ($\in \mathscr{L}(M, N; P)$),so $\varphi = f_{\varphi \circ t}$. This shows surjectivity.

injection if $0 \neq s = f_s \circ t \Rightarrow f_s \neq 0$, hence

45.6 exercise

45.6.1

show that

$$M \otimes_R N \cong N \otimes_R M$$

45.6.2

show that

$$(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$$

so we can remove parenthesis and write

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$$

(call it the n-fold tensor product of $M_1, \cdots, M_n$)

45.6.3

show that $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$ factorizes the multi-linear mappings, and

$$\mathscr{L}(M_1, \cdots, M_n; P) \cong \mathscr{L}(M_1 \otimes_R \cdots \otimes_R M_n, P)$$

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "s"] \arrow[d, "t"'] & P \\
M_1 \otimes_R M_2 \arrow[ur, "f_s"'] \arrow[d, squiggly] & \\
M_1 \otimes_R \cdots \otimes_R M_n &
\end{tikzcd}$$

We have the general definition of tensor products for R-modules. But we're interested in the case $R = K$ when K is a field.I denotes: $V_1 \otimes \cdots \otimes V_n$

$$\mathscr{L}(V_1, \cdots, V_n; K) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

This is the pervious corollary $f \sim P = K$

45.7 Lemma

Let $V_1, \cdots, V_n$ be K-vector spaces of finite dimension $d_i > 0$ let

$$e_{i1}, \cdots, e_{id_i}$$

be a basis for V_i. Let's define the following functions.

$$\begin{aligned} \varphi_{i_1,\cdots,i_n} :& V_1 \times \cdots \times V_n \to K \\ & (v_1, \cdots, v_n) \mapsto \prod_{j=1}^{n} e_{ji_j}^\vee(V_j) \end{aligned}$$

Then the set $\{\varphi_{i_1,\cdots,i_n}\}$ is a basis for $\mathscr{L}(V_1, \cdots, V_n; K)$

45.7.1 Proof

We do the proof for $n = 2$. Then the general case follows by induction.

$$\begin{aligned} V_1 &= \langle e_1, \cdots, e_m \rangle \qquad m = d_1 \\ V_2 &= \langle \omega_1, \cdots, \omega_n \rangle \qquad n = d_2 \end{aligned}$$

This special our $\varphi_{i_1,\cdots,i_n}$ are denoted by

$$\xi_{ij}(x, y) = e_i^\vee(x)\omega_j^\vee(y)$$

Let's show that $\_\xi_{ij\,i,j}$ is a generating set
$\varphi \in \mathscr{L}(V_1, V_2; K)$ such that $\varphi(e_i, \omega_i) := A_{ij} \in K$

$$\begin{aligned} \varphi(x, y) &= \varphi(\sum \alpha_i e_i, \sum \beta_j \omega_j) \\ &= \sum \alpha_i \beta_j \varphi(e_i, \omega_j) \\ &= \sum \alpha_i \beta_j A_{ij} \\ &= \sum A_{ij} e_i^\vee(x) \omega_j^\vee(y) \\ &= \sum A_{ij} \xi_{ij(x,y)} \end{aligned}$$

we prove that ξ_{ij} are linearly independent

$$\sum A_{ij}\xi_{ij}(x, y) = 0 \qquad \forall (x, y) \in V_1 \times V_2$$

Evaluate in

$$(x, y) = (e_i, \omega_i) \Rightarrow A_{ij} = 0 \ \forall ij$$

45.8 Prop

Assume that $V_1, \cdots, V_n$ are vector spaces and V_i has basis: $\{e_{i1}, \cdots, e_{id_i}\}$ then

$$B = \{e_{1i_1}, \otimes \cdots \otimes e_{ni_n}, 1 \leq i_j \leq d_j\}$$

is a basis for $V_1 \otimes \cdots \otimes V_n$. In particular, $V_1 \otimes \cdots \otimes V_n$ has dimension $\prod_{i=1}^{n} d_i$

Proof

Again we assume $n = 2$,$m = d_1, n = d_2$

$$V_1 = \langle e_1, \cdots, e_m \rangle \quad V_2 = \langle \omega_1, \cdots, \omega_n \rangle$$

We know that

$$\mathscr{L}(V_1, V_2; P) \cong (V_1 \otimes V_2)^\vee$$
$$s \mapsto f_s$$

Recall that

$$\xi_{ij}(x, y) = e_i(x) w_j(y)$$
$$f_{\xi_{ij}}(x \otimes y) = \xi_{ij}(x, y) = e_i^\vee(x) w_j^\vee(y)$$
$$f_{\xi_{ij}}(e_k \otimes w_l) = \begin{cases} 1 & if (i, j) = (k, l) \\ 0 & otherwise \end{cases}$$

It follows that $\{e_k \otimes w_l\}_{k,l}$ is a basis of $V_1 \otimes V_2$

45.9 tensor product and duality

45.9.1 product

Let $V_1, \cdots, V_n$ be vector spaces as above. Then

$$(V_1^\vee \otimes \cdots \otimes V_n^\vee) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

Proof

Define

$$V_1^\vee \times \cdots \times V_n^\vee \to \mathscr{L}(V_1, \cdots, V_n; K)(\cong (V_1 \otimes \cdots \otimes V_n)^\vee)$$
$$(\varphi_1, \cdots, \varphi_n) \mapsto [(v_1, \cdots, v_n) \mapsto \prod \varphi_i(v_i)]$$

This mapping is multi-linear. It describes by the property of tensor product to a map

$$V_1^\vee \otimes \cdots \otimes V_n^\vee \to \mathscr{L}(V_1, \cdots, V_n; K)(\cong (V_1 \otimes \cdots \otimes V_n)^\vee)$$
$$\varphi_1 \otimes \cdots \otimes \varphi_n \mapsto [(v_1, \cdots, v_n) \mapsto \prod \varphi_i(v_i)]$$

By Prop 45.8 these two space have the same dim $\prod d_i$. It enough to show that the mapping is surjective. Let's do it for $n = 2$ (keep the same notation as above). Take ξ_{ij}

$$\xi_{ij}(x, y) = e_i^\vee(x) w_j^\vee(y) = F(e_i^\vee \otimes w_j^\vee)(x, y)$$

45.9.2 duality

Let V and W be vector spaces of finite dimension. Then

$$\mathscr{L}(V, W) \cong V^\vee \otimes W^\vee$$

Proof

$$\begin{aligned} s : V^\vee \times W &\to \mathscr{L}(V, W) \\ (\varphi, \omega) &\mapsto [\sigma \mapsto \varphi(\sigma) w] \end{aligned}$$

Let's check that s is bilinear. (note that $\varphi(\sigma) \in K$)

$$((\varphi + \psi)(\sigma))w = (\varphi(v) + \psi(v))w = \varphi(v)w + \psi(v)w$$

$$\varphi(v)(w + w') = w\varphi(v) + w'\varphi(v)$$

Thus map s is then bilinear. So it induces $f_s : V^\vee \otimes W \to \mathscr{L}(V, W)$. We have to show that this is the required isomorphism.

Let $\{v_1^\vee, \cdots, v_m^\vee\}$ be a basis for $V^\vee$, and let $\{w_1, \cdots, w_n\}$ be a basis for W. Let's see what happens to

$$f_s(v_i^\vee \otimes w_j) = [v_k \mapsto v_i^\vee(v_k) w_j = \delta_{ik} w_j]$$

Consider the matrix associated to f_s with respect to the basis.

$$\begin{tikzcd} (e_1, e_n)E \arrow[r, "F"] \arrow[d, dashed] & P(p_1, \cdots, p_m) \arrow[d, dashed] \\ K^n \arrow[r, "M_F"] \arrow[u, "b_1"'] & K^m \arrow[u, "b_2"'] \end{tikzcd}$$

Call this matrix M_{ab}

$$M_{ab} = \begin{cases} 1 & if\ (a, b) = (j, i) \\ 0 & otherwise \end{cases}$$

The matrices of this form are a basis of $\mathscr{L}(K^n, K^m) \cong \mathscr{L}(V, W)$

And important case of this prop is when $V = W$:

$$\mathscr{L}(V; V) \cong V^\vee \otimes V$$

More in general

$$\begin{aligned}\mathscr{L}(V, W) &\xrightarrow{\cong} V^\vee \otimes W \\ f &\mapsto \sum a_{ij}\sigma_i^\vee \otimes w_j\end{aligned}$$

note that $\sigma_i^\vee \otimes w_j$ is a basis.

For instance $V = W$

$$id_V = \mathscr{L}(V, V) \mapsto \sum_i \sigma_i^\vee \otimes \sigma_i$$

45.9.3 Exercise

Let M, N, P R-modules. Show that

$$\mathscr{L}(M \otimes_R N; P) \cong \mathscr{L}(M; \mathscr{L}(N; P))$$

45.10 Def

We went to define the tensor product of linear mappings. letM_1, M_2, N_1, N_2 be R-modules and let $f_i : M_i \to N_i$ be linear mappings. Then we define

$$\begin{aligned}f_1 \otimes f_2 : M_1 \otimes M_2 &\to N_1 \otimes N_2 \\ m_1 \otimes m_2 &\mapsto f_1(m_1) \otimes f_2(m_2)\end{aligned}$$

This is a linear mapping

$$\begin{tikzcd}
M_1 \times M_2 \arrow[r, "f_1 \times f_2"] \arrow[d] & N_1 \times N_2 \arrow[d] \\
M_1 \otimes M_2 \arrow[r, dashed, "f_1 \otimes f_2"] & N_1 \otimes N_2
\end{tikzcd}$$

45.11 Extension of scalars

Let $\varphi : R \to S$ be a commutative unitary ring homomorphism. Let M be a R-module. Goal is to give to M also a structure of S-module "conveyed by φ"

Note that S has a structure of R-module $s \in S, \mathfrak{r} \in R$

$$\mathfrak{r}s := \varphi(\mathfrak{r})s$$

Now take thw tensor product $M \otimes_R S$. Now we give a structure of S-module to $M \otimes_R S$.

Take $s \in S$

$$s(\underbrace{m \otimes s'}_{\in M \otimes_R S}) := m \otimes ss'$$

note that ss' is a multi in S and we cannot product sm.

Notice we've a mapping

$$i : M \to M \otimes_R S$$
$$m \mapsto m \otimes s$$

Be careful, in general the mapping i is NOT injective.

Example $R = \mathbb{Z} \quad S = \mathbb{Z}/2\mathbb{Z} \quad \alpha : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \quad M = \mathbb{Z}[X]$

$$i(2X) = 2X \otimes 1 = 2(X \otimes 1) = X \otimes \alpha(2) \cdot 1 = X \otimes 0 = 0$$

45.12 Prop

Let $K \subseteq L$ be a field extension and let V be a K-vector space. Moreover let's denote $V_L = V \otimes_K L$. If $\{e_i\}_{i=1}^n$ is a basis of V then $\{e_i \otimes 1\}_{i=1}^n$ is a L-basis of V_L.(V_L has the same dim of V)

Proof

The set $\{e_i \otimes 1\}_{i=1}^n$ generates V_L if fact

$$\sigma \otimes l = (\sum \underbrace{\alpha_i}_{K} e_i \otimes \underbrace{l}_{L}) = \sum l\alpha_i(e_i \otimes 1)$$

We have to show that the elements are linearly independent.

$$0 = \sum \alpha_i(e_i \otimes 1) = \sum e_i \otimes \alpha_i \quad \alpha_i \in L$$

(L is a K-vec space)

Define the mapping with $\lambda_i \in K$

$$b_i : V \times L \to L$$
$$(\sum \lambda_i e_i, \beta) \mapsto \lambda_i \beta$$

This mapping is bilinear. It induces a mapping

$$f_i = f_{b_i}(\sum \lambda_i e_i) \otimes \beta \mapsto \lambda_i \beta$$

Note that $f_i(e_j \otimes \beta) = \delta_{ij}\beta$

- $$f_i(\sum_j e_j \otimes \alpha_j) = \alpha_i$$

But

$$0 = f_i(0) = f_i(\sum_j e_i \otimes \alpha_j) = \alpha_i \quad \forall i$$

45.13 Remark

As a consequence we have that the mapping $i : V \to V_L$ (mapping of K-vet spaces) is injective.

45.14 Exercise

Show that

$$\begin{aligned} V \otimes_K K \cong & \quad V \\ \sigma \otimes a \mapsto & \quad as \end{aligned}$$

45.15 Exactness of the tensor product

fix a R-module N and consider:

$$\_ \otimes N : M \mapsto M \otimes_R N$$

for any R-module M.

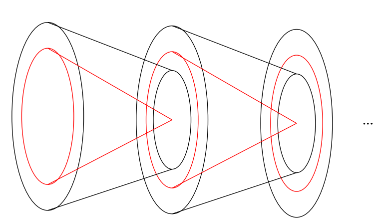
Moreover for any linear mapping $(f : M \to P) \rightsquigarrow f \otimes id_N . M \otimes_R N \to P \otimes_R N$ This association sends id_M to $id_{M \otimes_R N}$ and moreover is well defined with respect to the composition

$$f \circ g \mapsto (f \circ g) \otimes id_N = (f \otimes id_N) \circ (g \otimes id_N)$$

45.16 Def

A sequence of R-modules is a diagram of the following form (also called complex of R-modules)

$$M_1 \xrightarrow{d^1} M_2 \xrightarrow{d^2} \cdots$$



M_i is an R-module, d^i is linear mapping and

$$Ker(d^{i+1}) \supseteq Im(d^i)$$

Thus we also have:

$$d^{i+1} \circ d^i = 0$$

The diagram is called exact if

$$Ker(d^{i+1}) = Im(d^i)$$

take a morphism $f : M \to N$ then

- f is injective iff
$$0 \to M \xrightarrow{f} N$$
is exact
- f is surjective iff
$$M \xrightarrow{f} N \to 0$$
is exact

The first theorem of homomorphism

$$\overline{f} : M/Ker(f) \overset{\cong}{\to} Im(f)$$

can be written as an exact sequence

$$0 \to Ker(f) \xrightarrow{i} M \xrightarrow{f} Im(f) \to 0$$

More in general sequence like

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

are called short exact sequences.

45.17 Prop

N is a R-module

$$\_ \otimes_R N : \forall M \quad M \mapsto M \otimes_R N$$
$$f : M \to P$$
$$f \otimes id_N : M \otimes_R N \to P \otimes_R N$$

Assume that we have a short exact (also complex)sequence of R-modules

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

Then we apply $\_ \otimes_R N$ to whole sequence

$$0 \to M_1 \otimes_R N \xrightarrow{f \otimes id_N} M_2 \otimes_R N \xrightarrow{g \otimes id_N} M_3 \otimes_R N \to 0$$

is a complex sequence if it's exact then we call N a flat R-module. One significant example's that the free module is flat.

45.17.1 Example

$$
\begin{aligned}
0 \to \mathbb{Z} \xrightarrow{\mu} \mathbb{Z} \quad & \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0 \\
x \mapsto 2x & \\
& y \mapsto 2\mathbb{Z} + y
\end{aligned}
$$

Now apply $\_ \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$$
\begin{aligned}
0 \to \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mu \otimes id} \quad & \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \\
x \otimes (2\mathbb{Z} + y) \mapsto \quad & (2x) \otimes (2\mathbb{Z} + y) \\
& z \otimes (2\mathbb{Z} + y) \mapsto (2\mathbb{Z} + z)(2\mathbb{Z} + y)
\end{aligned}
$$

and

$$
\begin{aligned}
2x \otimes (2\mathbb{Z} + y) &= 2(x \otimes 2\mathbb{Z} + y) \\
&= x \otimes (2\mathbb{Z} + 2y) \\
&= x \otimes 2\mathbb{Z} \\
&= 0
\end{aligned}
$$

which is not injective, thus above isn't exact.

45.18 Exercise(important)

If $R = N$ then $\_ \otimes_R N$ (where N is a finite dim vec space) is exact. Hint: use the basis.

Chapter 46

Tensor algebra

Fix a vec space V (over K) of finite dimension

46.1 Def

We denote

$$\begin{aligned} T_p^q :=& (V^\vee)^{\otimes p} \otimes V^{\otimes q} \qquad p, q \in \mathbb{N} \\ =& \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{p \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \end{aligned}$$

An element of $T_p^q(V)$ is called a tensor of type (p, q) (or a mixed tensor which is p-covariant and q-contravariant)

Let's denote:

$$T(V) := \bigoplus_{q \in \mathbb{N}} T_0^q(V)$$

itemize some item in it:

$$\begin{aligned} &T_0^0(V) = K \\ &T_1^0(V) = V^\vee \\ &T_0^1(V) = V \\ &T_1^1(V) = V^\vee \otimes V \cong \mathscr{L}(V; V) \\ &T_2^0(V) = V^\vee \otimes V^\vee \cong (V \otimes V)^\vee \cong \mathscr{L}(V, V; K) \end{aligned}$$

If you have a R-module M, then

$$\bigoplus_{n=0}^{\infty} M = \{(m_1, \cdots, m_n, \cdots) : m_i \in M \text{ all but finite many } m_i = 0\}$$

On $T(V)$ we have following operation:

$$\begin{aligned} T_0^l(V) \times T_0^q(V) &\to T_0^{l+q}(V) \\ ((x_1 \otimes \cdots \otimes x_l), (y_1 \otimes \cdots \otimes y_q)) &\mapsto x_1 \otimes \cdots \otimes x_l \otimes y_1 \otimes \cdots \otimes y_q \end{aligned}$$

With this operation $T(V)$ becomes a K-algebra. It called the tensor algebra associated to V

46.2 exterior product

Let W be the two sided ideal of $T(V)$ generated by the element of the type $x \otimes x$

$$W = \left\{ \sum_{i(finite)} (y_1 \otimes \cdots \otimes y_{m_i}) \otimes (x_i \otimes x_i) \otimes (z_1 \otimes \cdots \otimes z_{n_i}) \right\}$$

With $x_j, y_j, z_j \in V$ and $n_i, m_j \in \mathbb{N}$

46.3 Def

The quotient algebra

$$\bigwedge(V) := T(V)/W$$

is a K-algebra, which called the exterior algebra of V

$$\begin{aligned} \pi : & T(V) && \to \bigwedge(V) \\ & x_1 \otimes \cdots \otimes x_n && \mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

This def is try to transform $\bigotimes$ to $\bigwedge$

46.4 Notation

$$\bigwedge(V) = \bigoplus_{n \in \mathbb{N}} \bigwedge^n(V)$$

$$\bigwedge^n(V) := T_0^n(V)/\left(W \cap T_0^n(V)\right)$$

this is called n-fold exterior product

46.5 Prop

Let $\sigma \in \mathfrak{S}_n$ then

$$x_1 \wedge \cdots \wedge x_n = sgn(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$$

Proof

Since any permutation can be written as the product of adjacent transpositions, it's enough to do the proof for $\sigma = (i, i+1)$

$$\begin{aligned} 0 &= (x_i + x_{i+1}) \wedge (x_i + x_{i+1}) \\ &= (x_i \wedge x_i) + (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) + (x_{i+1} \wedge x_{i+1}) \\ &= (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) \end{aligned}$$

46.6 Def

Let E be an R-module and $f : E^n \to M$ a mapping. We say that the pair $(M, f : E^n \to M)$ satisfies the universal property for the n^{th}-exterior power if

- M is an R-module, $f : E^n \to M$ is an n-linear mapping s.t.

  $\forall i \in \{1, \cdots, n-1\}$

  if

  $$x_i = x_{i+1}$$

  then

  $$f(x_1, \cdots, x_n) = 0$$

  (alternating n-linear mapping)

- If P is an R-module and $\varphi : E^n \to P$ is an alternating mapping, then

  $$\exists!\ \Phi : M \to P \text{ s.t. } \Phi \circ f = \varphi$$

46.7 Def

V is a K-vct space. A multi-linear mapping:

$$\varphi : V \times \cdots \times V \to W$$

is called skew-symmetric(alternating) if

$$\varphi(x_1, \cdots, x_n) = 0 \text{ when } \exists i \neq j : x_i = x_j$$

46.8 Prop

FIx a vct space V. For any alternating multi-linear mapping

$$s : \underbrace{V \times \cdots \times V}_{n \text{ times}} \to M$$

when M is another vct space, there exists a unique linear mapping

$$g_s : \overset{n}{\bigwedge}(V) \to M$$

such that the following diagram commutes

$$\begin{tikzcd}
V^n \arrow[r, "s"] \arrow[d, "t"'] & M \\
T_0^n(V) \arrow[ur, dashed, "f_s"] \arrow[d] & \\
\overset{n}{\bigwedge}(V) \arrow[uur, "g_s"'] &
\end{tikzcd}$$

Proof

$$g_s(\sigma_1 \wedge \cdots \wedge \sigma_n) := s(\sigma_1, \cdots, \sigma_n)$$

check the diagram is commutative

$$\mathcal{F}(V^n) \xrightarrow{\quad t \quad} \!\!\!\!\to T_0^n(V) \longrightarrow\!\!\!\!\to \overset{n}{\bigwedge}(V)$$

$$\{(\sigma_1, \cdots, \sigma_n)\} \longmapsto \{\sigma \otimes \cdots \otimes \sigma_n\} \longmapsto \{\sigma_1 \wedge \cdots \wedge \sigma_n\}$$

46.9 Remark/exercise

The couple $\overset{n}{\bigwedge} V$ with

$$V^n \to \overset{n}{\bigwedge}(V)$$

that satisfies Prop 46.8 is unique to unique isomorphism

46.10 Prop

Let V be a vct space of dimension n with a basis $\{e_1, \cdots, e_n\}$. Then $\overset{k}{\bigwedge}(V)$ is a vct space with a basis given by

$$\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \big| \ 1 \leq i_1 < \cdots < i_k \leq n\}$$

In particular, $\overset{k}{\bigwedge}(V)$ has dimension $\binom{n}{k}$

46.10.1 Proof

$\mathcal{B}$ is clearly a generating set.The different part is to show that $\mathcal{B}$ is made of linearly independent elements.

$$I = \{i_1, \cdots, i_k\}$$

with $1 \leq i_1 < \cdots < i_k \leq n$, define

$$\begin{aligned} \varphi_I : V^n &\to K \\ (e_{j_1}, \cdots, e_{j_n}) &\mapsto \begin{cases} sgn(t) \text{ if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ \quad 0 \quad \text{otherwise} \end{cases} \end{aligned}$$

φ_I is multilinear and alternating(skew-symm), hence it induce a linear mapping

$$\begin{aligned} g_{\varphi_I} = \overline{\varphi_I} : \bigwedge^k(V) &\to K \\ (e_{j_1} \wedge \cdots \wedge e_{j_k}) &\mapsto \begin{cases} sgn(t) \text{ if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ \quad 0 \quad \text{otherwise} \end{cases} \end{aligned}$$

With $\sigma \in \bigwedge^n(V)$,assume that

$$0 = \sigma = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \lambda_{j_1, \cdots, j_k} e_{j_1} \wedge \cdots \wedge e_{j_k}$$

By linearity

$$0 = \overline{\varphi}_I(\sigma) = \pm \lambda_I$$

Do it for any positive I this shows that any $\lambda_{j_1, \cdots, j_k}$ is zero.

Chapter 47

Determinant

47.1 Def

Let V be a vct space of dimension n, then

$$\det(V) = \bigwedge^n (V)$$

is called the determinant of V. It is a vct space of dimension $1 = \binom{n}{n}$ and a basis is given by

$$e_1 \wedge \cdots \wedge e_n$$

when $\{e_1, \cdots, e_n\}$ is a basis of V.

47.1.1 Proof

Let $f \in \mathscr{L}(V; V)$ then consider

$$\begin{aligned} \widetilde{f} : V^k &\to \bigwedge^k V \\ (v_1, \cdots, v_n) &\mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

This is multilinear and alternating. Therefore it induces a mapping

$$\begin{aligned} g_{\widetilde{f}} = \bigwedge^k f : \bigwedge^k (V) \quad &\to \bigwedge^k (V) \\ v_1 \wedge \cdots \wedge v_k \quad &\mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

Since $\det(V)$ has dim 1

$$\det(f) : \sigma_1 \wedge \cdots \wedge \sigma_n \mapsto \underbrace{det_f}_{\in K}(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$$

By abuse of notation we identity

$$\det(f) = det_f$$

47.2 Prop

$f \in \mathscr{L}(V;V)$ is invertible iff $\det(f) \neq 0$

47.2.1 Proof

f is not invertible iff $\{f(e_1), \cdots, f(e_n)\}$ is not a basis.
iff there's a non-trivial linear combination

$$\sum_n \lambda_i f(e_i) = 0$$

After relabelling thee e_i we can assume

$$f(e_i) = \sum_{i \geq 2} \mu_i f(e_i)$$

$$\begin{aligned}
\det(f)(e_1 \wedge \cdots \wedge e_n) &= det_f \cdot (e_1 \wedge \cdots \wedge e_n) \\
&= (\sum_{i \geq 2} \mu_i f(e_i)) \wedge f(e_1) \wedge \cdots \wedge f(e_n) \\
&= \sum_{i \geq 2} \mu_i (f(e_1) \wedge f(e_2) \wedge \cdots \wedge f(e_n)) \\
&= 0
\end{aligned}$$

47.3 Prop

$$\det(f \circ g) = \det(f) \cdot \det(g)$$

Proof

$$\begin{aligned}
\det(f \circ g) &= (f \circ g)(e_1) \wedge \cdots \wedge (f \circ g)(e_n) \\
&= f(g(e_1)) \wedge \cdots f(g(e_n)) \\
&= (\det f)(g(e_1) \wedge \cdots g(e_n)) \\
&= \det f \cdot \det g(e_1 \wedge \cdots g(e_n))
\end{aligned}$$

47.4 Prop

The determinant of f is equal to the determinant of any matrix that represents f with respect to a fixed basis. This doesn't depends on the choice of the basis.

Proof

Fix a basis $\{e_1, \cdots, e_n\}$ of V. Then

$$\begin{tikzcd} v_i & V \arrow[r, "f"] & V & \Longrightarrow & \det(V) \arrow[r, "\det(f)"] & \det(V) \\ e_i \arrow[u, mapsto] & K^n \arrow[u, "\cong"', "\mathcal{B}"] \arrow[r, "A_f"] & K^n \arrow[u, "\cong"', "\mathcal{B}"] & & \det(K^n) \arrow[u, "\bigwedge^n b"'] \arrow[r, "\det(A_f)"] & \det(K^n) \arrow[u, "\bigwedge^n b"'] \end{tikzcd}$$

$A_f^{(v_1, \cdots, v_n)}$ is the matrix associated to f with respect to the basis $\{v_1, \cdots, v_n\}$ suppose that $f(v_i) = \xi_{ij} v_i$. One we can see

$$A_f = \mathcal{B}^{-1} \circ f \circ \mathcal{B}(e_i)$$

$$\begin{aligned} \det(A_f)((a_{11}, a_{12}, \cdots, a_{1n}) \wedge (0, a_{22}, a_{23}, \cdots, a_{2n}) \wedge \cdots \wedge (0, 0, \cdots, 1)) \\ = \mathcal{B}^{-1}(f(\mathcal{B}(a_{11}, a_{12}, \cdots, a_{1n}))) \wedge \cdots \wedge \mathcal{B}^{-1(f(\mathcal{B}(0,0,\cdots,1)))} \\ = \xi_{1j}(0, \cdots, a_{1j}, \cdots, a_{1n}) \wedge \cdots \wedge \xi_{nj}(0, \cdots, a_{nj}, \cdots, a_{nn}) \end{aligned}$$

(Einstein notation used for j) We actually done the thing like

$$\begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ \vdots & \vdots \ddots & \vdots \\ 0 & 0 \cdots & a_{nn} \end{vmatrix}$$

compare the result with

$$\det(f)(v_1 \wedge \cdots v_n) = \xi_{1j}(v_1) \wedge \cdots \wedge \xi_{nj}(v_n)$$

We could find that

$$\det(A_f) = \det(f)$$

Then we got

$$\det(A) = \prod_{i=1}^{n} a_i i$$

47.5 Prop

If one column of A can be expressed as a linear combination of other columns of A, then

$$\det(A) = 0$$

The columns are images of $\{e_1, \cdots, e_n\}$, means that $A(e_1), \cdots, A(e_n)$ are linearly dependent. Then A is not an isomorphism, thus $\det(A) = 0$. If we exchange two columns of A, then $\det(A)$ changes sign.

47.6 Prop

Let (a_{ij}) be a matrix of dimension $n \times n$. Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$$

Proof

Let $\{v_1, \cdots, v_n\}$ be the columns of A, $v_i = A(e_i)$

$$\begin{aligned}
\det(A)(e_1 \wedge \cdots \wedge e_n) &= \sigma_1 \wedge \cdots \wedge \sigma_n \\
&= (\sum_i a_{i1} e_i) \wedge \cdots \wedge (\sum_i a_{in} e_i) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \prod_i a_{i\sigma(i)} \cdot e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)} \\
&= (\sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}) e_1 \wedge \cdots \wedge e_n
\end{aligned}$$

47.7 Corollary

$$\det A = \det A^T$$

Proof

$A^T = (\alpha_{ij}), A = (a_{ij}) \quad \forall ij \ a_i j = \alpha_{ji}$

$$\begin{aligned}
\det A^T &= \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \prod_{i=1}^{n} \alpha_{i\sigma(i)} \\
&= \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma) \prod_{i=1}^{n} a_{\sigma(i)i} \\
&\overset{j=\sigma(i)}{=} \sum_{\sigma \in \mathfrak{S}_n} sgn(\sigma^{-1}) \prod_{i=1}^{n} a_{j\sigma^{-1}(j)} \\
&\overset{sgn(\sigma)=sgn(\sigma^{-1})}{=} \det A
\end{aligned}$$

47.8 Prop

If you fix some basis on V and W, then A_f is the matrix associated to f^T is A_f^T

47.9 ?

Fix A of dimension of $n \times n$. Apply Gauss reduction and we get A' a upper-triangle.

By the properties listed above

$$|\det A| = |\det A'|$$

But on A' the det is just the product of elements on then diagonal

Second method to compare the determinant is to use Gauss reduction and keep track of the row/column exchanges.

47.10 Def

Fix $A = (a_{ij})(i,j) \in \{1, \cdots, n\}^2$. Denote with $A_{[i,j]}$ the matrix obtained removing the i^{th} row and j^{th} column of A.

47.11 Laplace expansion of the determinant

Let $A = (a_{ij})$ then

$$\begin{aligned}\det A =& \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{[i,j]} \\ =& \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{[i,j]}\end{aligned}$$

Proof

TEDIOUS

$$\begin{tikzcd} K^n \arrow[r, "A"] & K^n \arrow[d, "p_i"] \\ K^{n-1} \arrow[u, "t_j"] \arrow[r, "A_{[i,j]}"] & K^{n-1} \end{tikzcd}$$

$\{e'_1, \cdots, e'_n\}$ is a standard basis of K^n
$\{e_1, \cdots, e_n\}$ is a standard basis of K^{n-1} p_i is the mapping that forgets about the i-th row.

$$p_i = (x_1, \cdots, x_i, \cdots, x_n) \mapsto (x_1, \cdots, \widehat{x_i}, \cdots, x_n) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

$$\tau_j(e_i) = \begin{cases} e'_i \text{ if } i < j \\ e_i \text{ if } i \geq j \end{cases}$$

You can check that the above diagram is commutative. Now take $\bigwedge^{n-1}$ of the diagram

$$\begin{array}{ccc} \bigwedge^{n-1} K^n & \xrightarrow{\bigwedge^{n-1} A} & \bigwedge^{n-1} K^n \\ {\scriptstyle \bigwedge^{n-1} t_j}\uparrow & & \downarrow{\scriptstyle \bigwedge^{n-1} p_i} \\ \det(K^{n-1}) & \xrightarrow{\det(A_{[i,j]})} & \det(K^{n-1}) \end{array}$$

$$\begin{aligned} \det(A)(e'_1, \cdots, e'_n) &= (-1)^{i-1} \det(A)(e'_i \wedge e'_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge A(e'_1) \wedge \cdots \wedge A(\widehat{e_i}) \wedge \cdots \wedge A(e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \bigwedge^{n-1} A(e'_1, \cdots, \widehat{e_i}, \cdots, e'_n) = (*) \end{aligned}$$

Let

$$\begin{aligned} \pi_j : K^n &\to K^n \\ (x_1, \cdots, x_n) &\mapsto (0, \cdots, x_j, \cdots, 0) \end{aligned}$$

Then

$$A = \sum_i (\pi_j \circ A)$$

It means that

$$\begin{aligned} (*) &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A)(e'_1, \cdots, \widehat{e_i}, \cdots, e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \cdots, e_{n-1}) \\ &= \sum_{k,j} \left((-1)^{i-1} a_{kj} e'_k \wedge \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \cdots, e_{n-1}) \right) = (**) \end{aligned}$$

But $\pi_j(\cdot)$ is always collinear of e_j, so when $k = j$, the element in the sum is zero. We can remove the items that $k = j$

$$\begin{aligned} \rho_k := \tau_k \circ p_k : K^n &\to K^n \\ (x_1, \cdots, x_n) &\mapsto (x_1, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n) \end{aligned}$$

$\pi_k = id_{K^n} - \rho_k$ and $\sum_{j \neq k} \pi_j = \rho_k$

Then

$$\begin{aligned} (**) &= \sum_k (-1)^{i-1} a_k i e'_k \wedge \bigwedge^{n-1} \tau_k \circ \bigwedge^{n-1} (p_k \circ A \circ \tau_i)(e_1 \wedge \cdots \wedge e_{n-1}) \\ &= (***) \end{aligned}$$

But by the diagram

$$\bigwedge^{n-1}(p_k \circ A \circ \tau_i) = \det A_{[i,k]}$$

$$\bigwedge^{n-1}\tau_k(e_1 \wedge \cdots \wedge e_{n-1}) = e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n$$

Thus

$$\begin{aligned}(***) &= \sum_k (-1)^{i-1} a_{ki} \det(A_{[k,i]})(e'_k \wedge e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n) \\ &= \sum_k (-1)^{i+k} a_{ki} \det(A_[k,i])e'_1 \wedge \cdots \wedge e'_n\end{aligned}$$

Chapter 48

The Structure of Linear Mappings

48.1 Theorem

Let $f : V \to W$ be a linear mapping between vct spaces of finite and same dim. Then:

1 there exists decomposition $V = V_0 \oplus V_1$ and $W = W_1 \oplus W_2$ such that $V_0 = \ker f$ and f includes an isomorphism between V_1 and W_1(namely $f \mid_{V_1}$)

2 There exists basis in V nad W s.t. the associated matrix $A_f = a_{ij}$ satisfies $\forall 1 \leq i \leq r, \exists r \leq n$ have $a_{ii} = 1$ and have $a_{ij} = 0$ elsewhere

3 Let A be a $m \times n$ matrix Then there exists two square matrices (with det $\neq 0$) B and C of dim $m \times m$ and $n \times n$ and a num $r \leq min(m, n)$ s.t. BAC has the form in (2) Moreover the number r is unique $r = rank(A)$

48.2 Def

Let $F : V \to V$ be a linear mapping. A subspace $V_0 \subseteq V$ is said to be an invariant subspace of F is $F(V_0) \subseteq V_0$

48.3 Def

A linear mapping $f : V \to V$ (finite dim) is diagonalizable if the following equivalent conditions are satisfied

1 V decomposes as a direct sum of one-dimensional invariant subspace of f

2 There exists a basis of V, in which the matrix A_f is diagonal.

Proof of equivalence

$2 \Rightarrow 1$ Assume that in the base $\{v_1, \cdots, v_n\}$, we have $A_f = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ by the familiar diagram

$$\begin{tikzcd} V \arrow[r, "f"] & V \\ K^n \arrow[u, "b"] \arrow[r, "A_f"] & K^n \arrow[u, "b"] \end{tikzcd}$$

$$f(v_i) = b \circ A_f(e_i) = b(\lambda_i e_i) = \lambda_i v_i \in \langle v_i \rangle$$

So

$$V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle$$

$1 \Rightarrow 2$ Assume that $V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle$, where $f(\langle v_i \rangle) \subseteq \langle v_i \rangle$, then $\{v_1, \cdots, v_n\}$ forms a basis of V
Consider the previous diagram

$$A(e_1) = b^1 \circ f \circ b(e_i) = b^{-1}(f(v_i)) = b^{-1}(\lambda_i v_i) = \lambda_i e_i$$

48.3.1 Example

Take

$$A : \mathbb{R}^2 \to \mathbb{R}^2 \quad A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

A is not diagonalizable.

48.4 Def

Let L be a one-dimensional invariant subspace of $f : V \to V$. Then $f \mid_L = x \mapsto \lambda x$ with a scalar $\lambda \in K$. Such λ is called eigenvalue of f. A non-zero vector $v \in V$ is called an eigenvector of V if $\langle v \rangle$ is an invariant subspace of f

48.5 Remark

$$\{eigenvectors\} \longrightarrow \{\text{Set of invariant subspaces of dim 1}\} \longrightarrow K$$

$$v\ \langle v \rangle \longmapsto eigenvalue$$

This mapping is generally NOT injective. If V is an eigenvector, then μv is also an eigenvector, $\forall \mu \in K$

48.6 Remark/exercise

Assume that f is diagonalizable and A_f is a diagonal matrix that represents f Then A_f is unique up to permutation of the columns in the diagonal.

$$V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle = \left\langle v_{\sigma(1)} \right\rangle \oplus \cdots \oplus \left\langle v_{\sigma(n)} \right\rangle \quad \sigma \in \mathfrak{S}_n$$

48.7 Def

V a vector space over K $dim(V) = n, f \in \mathscr{L}(V;V)$ let A_f be an associated matrix (in any basis) the mapping

$$\begin{aligned} P : K &\to K \\ t &\mapsto \det(tI_n - A_f) \end{aligned}$$

This is a polynomial in $K[t]$ (with degree n)

48.8 Lemma

$P(t)$ is a monic polynomial of degree n

Proof

$$P(t) = \det(tI_n - A_f) = \sum_{\sigma} sgn(\sigma) \prod_{i=1}^{n} (t\delta_{i\sigma(i)} - A_{i\sigma(i)})$$

The only item giving t^n is when $\sigma = id$

48.9 Theorem

Use the notations introduced before

1 $P(t)$ doesn't depends on A_f (if you change basis, $P(t)$ does not change)

2 Any eigenvalue of f is a root of $P(t)$. Conversely any K-root of $P(t)$ is an eigenvalue of f

Proof

1 Put $A = A_f$ and A' be another representation of f Then $A' = B^{-1}AB$ where B invertible $n \times n$ matrix.

$$\begin{aligned} \det(tI_n - A') &= \det(tI_n - B^{-1}AB) \\ &= \det(B^{-1}(tI_n)B - B^{-1}AB) \\ &= \det(B^{-1}(tI_n - A)B) \\ &= \det(tI_n - A) \end{aligned}$$

2 Let $\lambda \in K$ be a K-root of $P(t)$, then

$$\det(\lambda I_n - A_f) = 0 = P(\lambda)$$

$\lambda I_n - A_f$ is not invertible, $\exists v \neq 0 \in \ker(\lambda I_n - A_f)$ s.t.

$$A_f(\sigma) = \lambda\sigma$$

then σ is an eigenvector

Vice versa if $\sigma \neq 0, f(\sigma) = \lambda\sigma, \sigma \in Ker(\lambda I_n - A_f)$,$\det(\lambda I_n - A_f) = 0 = P(t)$

48.10 Def

The polynomial $P(t)$ will be denoted by $P_f(t)$. It's called the characteristic polynomial of f

48.11 Corollary

If $P_f(t)$ splits with no repeated roots, then f is diagonalizable.

Proof

Natural

$$\langle\sigma_1\rangle, \cdots \langle\sigma_n\rangle$$

are all different then

$$V = \langle\sigma_1\rangle \oplus \cdots \oplus \langle\sigma_n\rangle$$

48.12 Remark

The inverse version does not hold.

48.13 Def: Jordan block

A matrix of form

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ & & & & \lambda \end{pmatrix} \in M_{r\times r}(K) \quad r \geq 1$$

is called a Jordan block (element $\lambda \in K$ is $J_1(\lambda)$)

48.14 Def: Jordan matrix

A Jordan matrix if a matrix of form

$$J = \begin{pmatrix} J_{r_1}(\lambda_1) & \cdots & \\ & J_{r_2}(\lambda_2) & \\ & & \ddots \end{pmatrix}$$

48.15 Example

Let $V_n(\lambda)$ be the vector space of complex functions:

$$F(x) := e^{\lambda x} f(x)$$

where $\lambda \in \mathbb{C}, f \in \mathbb{C}[x] \leq n-1$

Verify that $V_n(\lambda)$ is a vector space of dim n

$$\begin{aligned} \frac{d}{dx}(e^{\lambda x} f(x)) &= \lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) \\ &= e^{\lambda x}(\lambda f(x) + f'(x)) \end{aligned}$$

$\frac{d}{dx} \in \mathscr{L}(V_n(\lambda); V_n(\lambda))$ Consider

$$v_{i+1} = \frac{x^i}{i!} e^{\lambda x}$$

Show that $\{v_0, \cdots, v_{n-1}\}$ forms a basis of $V_n(\lambda)$

$$\begin{aligned} \frac{d}{dx} v_{i+1} &= \lambda v_{i+1} + \frac{x^{i-1}}{(i-1)!} e^{\lambda x} \\ &= \lambda v_{i+1} + v_i \end{aligned}$$

Then

$$A_{\frac{d}{dx}} = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & & \\ & & 1 & \lambda \end{pmatrix} = (J_n(\lambda))^T$$

48.16 Def

Let $a_0 + a_1 t + \cdots + a_n t^n = Q(t) \in K[t]$, then for $f \in \mathscr{L}(V; V)$ we define

$$Q(f) := a_0 id_V + a_1 f + a_2 f^{\circ 2} + \cdots + a_n f^{\circ n}$$

Remark From now on we write

$$f^{\circ k} = f^k$$

these are operations in $\mathscr{L}(V; V), +, \circ$

we say that Q annihilates f if $Q(f) = 0$

48.17 Prop

Let $f \in \mathscr{L}(V;V)$. There exists a polynomial $Q \in K[t] \setminus \{0\}$ that annihilates f (i.e. $Q(f) = 0$)

Proof

$$dim(\mathscr{L}(V;V)) = n^2$$

Hence the mapping $\underbrace{id_V, f^2, \cdots, f^{n^2}}_{n^2+1 \text{ mappings}} \in \mathscr{L}(V;V)$ are linear dependent. There exists a non-trivial linear comb:

$$\lambda_0 id_V + \lambda_1 f + \cdots + \lambda_{n^2} f^{n^2} = 0$$

So, take

$$Q(t) = \lambda_0 + \lambda_1 t + \cdots + \lambda_{n^2} t^{n^2}$$

This show that $Q \neq 0$ and $Q(f) = 0$

Remark

The proof of this proposition also gives the degree of a polynomial that annihilates ($\leq n^2$)

48.18 Def

Let $m(t) \in K[t] \setminus \{0\}$ be a monic polynomial of minimal degree that annihilates $f \in \mathscr{L}(V;V)$. Then $m(t)$ is called minimal polynomial of f

And by prop above (48.17), $m(t)$ exists.

48.19 Prop

If $m(t)$ is minimal polynomial of f, then $m(t)$ is unique.

Proof

Assume that $m_1(t)$ is another minimal polynomial of f. Then $m - m_1(t) \in K[t]$

$$(m - m_1)(f) = m(f) - m_1(f) = 0 - 0 = 0$$

Now m and n are both monic, so

$$deg(m - m_1) < deg(m) = deg(m_1)$$

$m - m_1$ is a polynomial of $deg < deg(m)$ that annihilates f, thus

$$m - m_1 = 0 \in K[t]$$

Notation

Form now we denote the minimal polynomial of f by m_f

Question

$f \in \mathscr{L}(V;V)$ we have $P_f, m_f \in K[t]$.
What is the relationship between P_f and m_f?

48.20 Prop

Let $Q \in K[t] \setminus \{0\}$ be a polynomial that annihilates f. Then $m_f \mid Q$

Proof

Let

$$Q(t) = m_f(t) \cdot s(t) + \mathfrak{r}(t)$$

such that $deg(\mathfrak{r}) < deg(m_f)$. So

$$0 = Q(f) = m_f(f)s(f)_{\mathfrak{r}}(f) = 0 + \mathfrak{r}(f) \Rightarrow \mathfrak{r}(f) = 0$$

But since m_f is the minimal polynomial of f, then

$$\mathfrak{r}(t) = 0$$

48.21 Def

Let A be a matrix of dim $n \times n$ and

$$M_{ij} := (-1)^{i+j} \det(A_{[i,j]}) \quad \forall (i,j) \in \{1, \cdots, n\}^2$$

In this expression

$$\det(A_{[i,j]})$$

is called the (i,j)-monic of A.
Then we define

$$Adj(A) := (M_{ij})^T$$

called adjugate matrix of A

48.22 Prop

$$Adj(A) \cdot A = A \cdot Adj(A) = \det(A) \cdot I_n$$

Proof

use Laplace expansion.

48.23 Theorem: Cayley-Hamilton Theorem

The characteristic polynomial P_f annihilates f
Consequence: $m_f \mid P_f$

Proof

Let $A = A_f$ any matrix that represents f. COnsider

$$B := Adj(tI_n - A)$$

B is a matrix with coefficient in $K[t](B \in M_{n\times n}(K[t]))$
Then

$$(tI_n - A) \cdot B = \det(tI_n - A) \cdot B == P_f(t) \cdot I_n$$

We can decompose B in the following way

$$B = \sum_{i=0}^{n-1} t^i B_i \quad B_i \in M_{n\times n}(K)$$

We have at most $n-1$, because the coefficient of B have degree at most $n-1$
(Any entry Adj is a det ogf a matrix of dim $(n-1) \times (n-1)$)

$$\begin{aligned}
P_f(t)I_n &= (tI_n - A) \cdot \sum_{i=0}^{n-1} t^i B_i \\
&= (\sum_{i=0}^{n-1} tI_n \cdot t^i B_i) - (\sum_{i=0}^{n-1} A \cdot t^i B_i) \\
&= \sum_{i=0}^{n-1} t^{i+1} B_i - \sum_{i=0}^{n-1} A \cdot t^i B_i \\
&= t^n B_{n-1} + \sum_{i=0}^{n-1} t^i (B_{i-1} - AB_i) - AB_0
\end{aligned}$$

Recall that $P_f(t) \cdot I_n = t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_1 t I_n + c_0 I_n$

$$\begin{aligned}
&t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_0 I_n \\
=&\cdots \\
=&t^n B_{n-1} + \sum_{i=1}^{n-1} t^i (B_{i-1} - AB_i) - AB_0
\end{aligned}$$

Then we can compare the coefficients:

$$B_{n-1} = I_n$$
$$B_{i-1} - AB_i = c_i I_n \quad 1 \le i \le n-1$$
$$-AB_0 = c_0 I_n$$

Multiply by A^i $0 \le i \le n$

$$A^n B_{n-1} + \sum_{i=1}^{n-1} (A^i B_{n-1} - A^{i+1} B) - AB_0 = A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I_n$$

Now the LHS we have a telescopic sum and got

$$0 = P_f(A) \Leftrightarrow 0 = P_f(f)$$

48.24 Example

(a) m_f and P_f are in general different. let $f = id_V, (dimV = n)$

$$P_f(t) = (t-1)^n \quad m_f(t) = t - 1$$

(b) Assume $f : V \to V(dimV = \mathfrak{r})$ and $A_f = J_{\mathfrak{r}}(\lambda)$. Then

$$P_f(t) = (t-\lambda)^{\mathfrak{r}}$$

Moreover

$$J_{\mathfrak{r}}(\lambda) = \lambda I_{\mathfrak{r}} + J_{\mathfrak{r}}(0)$$

and

$$J_{\mathfrak{r}}(o)^k = \begin{pmatrix} \overbrace{0 \cdots 1}^{k+1} & & \\ & \ddots & \\ & & 1 \\ & & \vdots \\ & & 0 \end{pmatrix}$$

if $k \ge \mathfrak{r}, J_{\mathfrak{r}}(o)^k = 0$

$$(J_{\mathfrak{r}}(\lambda) - \lambda I_n)^k = (\lambda I_{\mathfrak{r}} - J_{\mathfrak{r}}(0) - \lambda I_{\mathfrak{r}})^k = J_{\mathfrak{r}}(0)^k \ne 0$$

if $0 \le k \le \mathfrak{r} - 1$
We know that $m_f \mid (t-\lambda)^{\mathfrak{r}}$(by Cayley-Hamilton), m_f must be of the type

$$m_f = (t-\lambda)^k$$

But the only possibility is $k = \mathfrak{r}$, thus

$$m_f = P_f$$

48.25 Theorem

Let $f \in \mathscr{L}(V;V)$ when V is a vector space of dim n, over an algebraically closed field.
Then

(1) f can be represented by a Jordan matrix

(2) This above matrix is unique up to permutation of the Jordan blocks

(Note that a field K is algebraically closed if any non-zero polynomial has a root in K)

48.26 Def

Let $f \in \mathscr{L}(V;V)$ and let $\lambda \in K$. A vector $w \in V \setminus \{0\}$ is called a root vector of f corresponding to λ, if there exists $\mathfrak{r} \in \mathbb{N}$ s.t.

$$(f - \lambda id_V)^{\mathfrak{r}}(w) = 0$$

Remark

Eigenvector are root vectors (corresponding to their eigenvalues) take $\mathfrak{r} = 1$

Remark

Let $J_{\mathfrak{r}}(\lambda)$ be a Jordan block. Then any $\sigma \in V$ is a root vector of f corresponding to λ. In fact:

$$(J_{\mathfrak{r}}(\lambda) - \lambda I_n)^m = 0 \quad \text{if } m \geq \mathfrak{r}$$

48.27 Prop

Let K be an algebraically closed field. Let $\lambda_1, \cdots, \lambda_k$ be all of distinct eigenvalues of $f (k \geq 1)$, then

$$V = \bigoplus_{i=1}^{k} V(\lambda_i)$$

Proof

Since K is algebraically closed, then

$$P_f(t) = \prod_{i=1}^{k} (t - \lambda_i)^{r_i} \in K[t]$$

Consider

$$F_i(t) := P_f(t) \cdot (t-\lambda_i)^{-r_i} \in K[t]$$

Then we define

$$f_i := F_i(f) \in \mathscr{L}(V;V), V_i = Imf_i$$

Setp 1

We want to prove that

$$(f-\lambda_i Id_V)^{\circ r_i}(V_i) = 0 \Leftrightarrow V_i \subseteq V(\lambda_i)$$

which got from

$$(f-\lambda_i Id_V)^{\circ r_i} \circ (f_i) = (t-\lambda_i)^{r_i}(f) \circ F_i(f) = P_f(f) = 0$$

Step 2

We want to prove that

$$V = \bigoplus_{i=1}^{k} V_i$$

Since the polynomials $F_i(t)$ are coprime, then

$$\exists G_i(t) \in K[t] \text{ s.t. } \sum_{i=1}^{k} F_i(t) G_i(t) = 1$$

Let f substitute for t

$$\sum_{i=1}^{k} F_i(f) G_i(f) = Id$$

take $v \in V$

$$\sum_{i=1}^{k} f_i \circ G_i(f)(v) = v$$

$$\begin{tikzcd}
 & & \bigoplus_{i=1}^{k} V_i & & \\
V_1 \arrow[urr, "i"] & V_2 \arrow[ur, "i"'] & \cdots & V_{k-1} \arrow[ul, "i"] & V_k \arrow[ull, "i"'] \\
V \arrow[u, "f_1\circ G_1(f)"] \arrow[r, phantom, "="] & V \arrow[u, "f_2\circ G_2(f)"'] \arrow[r, phantom, "="] & \cdots \arrow[r, phantom, "="] & V \arrow[u, "f_{k-1}\circ G_{k-1}(f)"] \arrow[r, phantom, "="] & V \arrow[u, "f_k\circ G_k(f)"]
\end{tikzcd}$$

i is the inclusion mapping.

Step 3

We want to show that

$$V_i \cap (\sum_{i \neq j} V_j) = \{0\}$$

Let v be a vector in this intersection. Then by calculation,

$$(f - \lambda_i)^{r_i}(v) = 0$$

$$F_i(f)(v) = \prod_{i \neq j}(f - \lambda_j)Id^{\circ r_i}(v) = 0$$

Now $(t - \lambda_i)^{r_i}$ and $F_i(t)$ are coprime. Then there exists $G_1(t)$ and $G_2(t)$ such that:

$$G_1(t)(t - \lambda_i)^{t_i} + G_2(t)F_i(t) = 1$$

substitute f instead of t by

$$G_1(f) \circ (f - \lambda_i Id_V)^{\circ r_i} + G_2(f) \circ F_i(f) = Id_V$$

Then apply to $v = \sum_{j \neq i} v_j, v_j \in V_j$

$$G_1(f) \circ (f - \lambda_i Id_V)^{\circ r_i}(v) + G_2(f) \circ F_i(f)(v) = v = 0$$

Step 4

We want to show that

$$V_i = V(\lambda_i)$$

By step 1 we get

$$V_i \subseteq V(\lambda_i)$$

Take $v \in V(\lambda_i)$, write it as

$$v = v'(\in V(\lambda_i)) + v''(\in \bigotimes_{j \neq i} V_j)$$

By step 3,

$$v'' = v - v' \in V(\lambda_i)$$

Use same trick,substitute f for t and calculate in v''

$$v'' = 0$$

48.28 Def

Let $f \in \mathscr{L}(V; V)$. Then f is said to be nilpotent if there exists $t \in \mathbb{N}$ that $f^t = 0$

48.29 Lemma

Let f be a nilpotent mapping, then

$$Ker(f) = \{\text{set of eigenvectors of } f\} \cup \{0\}$$

Proof

Let $v \in Ker(f)$ then v is an eigenvector with eigenvalue= 0
Let v be an eigenvector, then $\forall m \geq r$

$$0 = f^m(v) = f^{m-1}(f(v)) = f^{m-1}(\lambda v) = \lambda^m v \Rightarrow \lambda^m = 0 \Rightarrow \lambda = 0$$

48.30 Lemma

Let f be a nilpotent mapping, then $Ker(f) \neq \{0\}$

Proof

Let $\mathfrak{r}$ be the minimal integer s.t. $f^{\mathfrak{r}} = 0$ then

$$f^{\mathfrak{r}-1}(V) \subseteq Ker(f)$$

but $f^{\mathfrak{r}-1}(V) \neq \{0\}$ because of the minimality of $\mathfrak{r}$

Remark

Another way to prove is that $Q(t) = t^{\mathfrak{r}}$ annihilates f. So $m_p = t^{\mathfrak{r}'}, \mathfrak{r}' \leq \mathfrak{r}$
Note that 0 is a root of m_f, by Cayley-Hamilton theorem, 0 is an eigenvalue $f(x) = 0 \cdot x = 0$ for some $x \neq 0$

48.31 Jordan matrix of form $J_{\mathfrak{r}}(0)$

Recall that

$$J_{\mathfrak{r}}(0)^k = 0 \text{ if } k \geq \mathfrak{r}$$

Then the Jordan matrix

$$\begin{pmatrix} J_{\mathfrak{r}_1}(0) & & \\ & J_{\mathfrak{r}_2}(0) & \\ & & \ddots \end{pmatrix}$$

Are nilpotent mappings since each block is nilpotent. Take one block

$$J_{\mathfrak{r}}(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \end{pmatrix} = \begin{cases} e_1 \mapsto 0 \\ e_2 \mapsto 1 \\ \vdots \\ e_{\mathfrak{r}} \mapsto e_{\mathfrak{r}-1} \end{cases}$$

We represent the action of a Jordan block on a basis as the following diagram

$$\underbrace{e_{\mathfrak{r}} \to e_{\mathfrak{r}-1} \to e_{\mathfrak{r}-2} \to \cdots \to e_1 \to 0}_{\text{lenth of the block}(\mathfrak{r})}$$

e_1 is the one which mapped to 0 (thus an eigenvector)

Given $f \in \mathscr{L}(V;V)$ if we find a basis on which f acts as in the previous diagram. Then we have found a Jordan basis made of blocks of the type "$J_{\mathfrak{r}}(0)$"

48.32 Theorem

Let $f \in \mathscr{L}(V;V)$ be a nilpotent mapping, then there exists a Jordan basis for f that gives a Jordan matrix made of blocks of the type $J_{\mathfrak{r}}(0)$

Proof

We need to find a basis that induces a diagram of the type $\mathscr{D}$:(dots in the diagram are basis)

$$\begin{tikzcd}
\cdot \arrow[d] & & & & \cdot \arrow[d] \\
\cdot \arrow[d, phantom, "\vdots"] & \cdot \arrow[d] & & & \cdot \arrow[d] \\
\cdot \arrow[d] & \cdot \arrow[d] & \cdot \arrow[d] & \cdots & \cdot \arrow[d] \\
\cdot \arrow[d] & \cdot \arrow[d] & \cdot \arrow[d] & & \cdot \arrow[d] \\
0 & 0 & 0 & & 0
\end{tikzcd}$$

(Last line of dots naturally be eigenvectors)

We work by induction on $dim(V)$. If $dim(V) = 1$, then

$$f = \mu(\cdot), f^{\mathfrak{r}} = 0 \ \mu^{\mathfrak{r}} v = 0 \ \forall v \Rightarrow \mu = 0$$

But $0 = J_1(0)$. Assume that the theorem is true for $dim(V) < n$ Let

$$V_0 = Ketf = \{\text{the set of eigenvalues}\} \cup \{0\}$$

Since f is nilpotent

$$dim(V_0) \geq 1$$

. Therefore

$$dim(V/V_0) < n$$

So define the following mapping

$$\begin{aligned} \overline{f} : & V/V_0 && \to V/V_0 \\ \overline{\sigma} = V_0 + \sigma & && \mapsto V_0 + f(v) = \overline{f(v)} \end{aligned}$$

$$\overline{f} \cdot \overline{\sigma} \mapsto \overline{f(v)}$$

is nilpotent We use the induction hypothesis

We have a Jordan basis for $\overline{f}$, so we have elements $\overline{\sigma_1}, \cdots, \overline{\sigma_m} \in V/V_0$ that give a diagram $\overline{\mathcal{D}}$:

Now left $\overline{\sigma_i}$ to some element $\sigma_i \in V$ choose $\sigma_i \in V$ s.t $\sigma + V_0 = \overline{\sigma_i}$Now start applying f to these elements $\sigma_i \neq 0$

$$v_i \to f(v_i) \to \cdots \to f^{b_i - 1}(v_i) \to f^{b_i}(v_i)$$

When b_i is the first integer such that

$$\overline{f}^{b_i}(\overline{v_i}) = 0$$

This means that

$$f^{b_i}(v_i) \in V_0$$

hence $f^{b_i}(\sigma_i)$ is an eigenvalue for ? Consider now the vector subspace generated by $f^{b_1}(v_1), f^{b_2}(v_2), \cdots, f^{b_m}(v_m)$

$$\left\langle f^{b_1}(v_1), \cdots, f^{b_m}(v_m) \right\rangle \subseteq V_0$$

Extract a basis and complete to a basis of V_0. The new vectors are denoted by $u_1, \cdots, u_t$

We want to prove that the elements of $\mathcal{D}$ form $\cdot$ basis of V

1

The elements of $\mathcal{D}$ generate V let $\sigma \in V$

$$\overline{\sigma} = \sum_{i=1}^{m} \sum_{j=0}^{b_i - 1} a_i j \overline{f^j}(\overline{v_i})$$

Now I use the properties of $\overline{f}$

$$\begin{aligned} \overline{f}(\overline{v_i}) &= \overline{f(v_i)} \\ \overline{f}(\overline{f(v)}) &= \overline{f(f(v))} \end{aligned}$$

then

$$\overline{\sigma} = \overline{\sum_{i=1}^{m} \sum_{j=0}^{b_i - 1} a_i j f^j(v_i)}$$

which gives

$$\sigma - \sum_{i=1}^{m} \sum_{j=0}^{b_i-1} a_i j f^j(v_i) \in V_0$$

this finishes. We know that

$$V_0 = \left\langle f^{b_1}(v_1), \cdots, f^{b_m}(v_m), u_1, \cdots, u_t \right\rangle$$

2

We need to prove that the elements of $\mathscr{D}$ are linearly independent

a We show that the elements of the bottom row are linearly independent

$$\sum_{i=1}^{m} a_i f^{b_i}(v_i) + \sum_{i=1}^{t} c_i u_t = 0$$

This is a non-trivial linear comb.

The first observation is that $b_i = 0$. Because if $b_j \neq 0$

$$u_j = \frac{\sum_{i=1}^{m} a_i f^{b_i}(v_i)}{\sum_{i=1}^{t} c_i}$$

But $u_1, \cdots, u_t$ were an extension of a basis. So

$$0 = \sum_{i=1}^{m} a_i f^{b_i}(v_i) = f(\sum_{i=1}^{m} a_i f^{b_i-1}(v_i)) \Rightarrow (\sum_{i=1}^{m} a_i f^{b_i-1}(v_i)) \in V_0$$

It means that

$$\sum_{i=1}^{m} a_i \overline{f}^{b_i}(v_i) = 0 \Rightarrow a_i = 0 \forall i$$

b If there is a non-trivial linear comb that equals to 0. For elements of $\mathscr{D}$. We can write it as linear comb of elements of the last row

$$f(\sum_{i=1}^{m} \sum_{j=1}^{b_i} a_{ij} f^j(v_i) + \sum_{i=1}^{t} c_i u_t) = 0$$

By applying f many times we get a linear comb of elements of the last row.

By point a,finished.

48.33 Prop

The Jordan matrix that represents a nilpotent mapping $f \in \mathscr{L}(V)$ is unique to permutations of the blocks.

Proof

Recall that a Jordan basis of f is given be diagram of the type $\mathscr{D}$

$$
\begin{tikzcd}
\cdot \arrow[d] & & & & & \\
\cdot \arrow[d] & \cdot \arrow[d] & & & & \\
\cdot \arrow[d] & \cdot \arrow[d] & \cdot \arrow[d] & \cdots & & \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & & & \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot
\end{tikzcd}
$$

These columns are ordered in a decreasing height on them, recalling that the height of a column is the dimension of a Jordan block. In the proof of existence of Jordan basis, the diagram was constructed as a lift of $\mathscr{D}$

Focus on the last row. The elements of last row generates $V_0 = \ker f$ and moreover, they are linearly independent. Then the length of the last row is exactly $dim(V_0)$, which is independent of the choice of basis.

Viewing the penultimate row, this corresponds to the last row of the diagram $\overline{\mathscr{D}}$. So if we work by induction, we done the proof:

All the rows have length independent of the choice of basis.

Remark

$$\ker(f^{\circ 3})/\ker(f^{\circ 2}) \to \ker(f^{\circ 2})/\ker(f) \to \ker f = V_0$$

48.34 Lemma

Let $f \in \mathscr{L}(V)$, λ be an eigenvalue of f. Then there exists $r \in \mathbb{N}$ s.t.

$$\forall v \in V(\lambda) \quad (f - \lambda Id)(v) = 0$$

Proof

Take a basis $\{v_1, \cdots, v_n\}$ of $V(\lambda)$. By definition, we have $(r_1, \cdots, r_n)$ such that $\forall i$ r_i is the least integer that

$$\forall v \in V \quad (f - \lambda Id)^{\circ r}(v) = 0$$

Take $r = \max\{r_i\}$,then proved by calculation.

48.35 Theorem

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. Then f admits a Jordan basis (namely there exists a basis s.t. A_f is a Jordan matrix).

Proof

Since K is algebraically closed, by Prop 48.27

$$V = \bigoplus_{i=1}^{k} V(\lambda_i)$$

where λ_i are distinct eigenvalues of f

Recall that $V(\lambda_i)$ is the set of root vectors for λ_i and 0

Consider $f\mid_{V(\lambda_i)} = g, \lambda_i = \lambda$. Only need to prove the theorem for g

$$(g - \lambda Id) : V(\lambda) \to V(\lambda)$$

This function is nilpotent on $V(\lambda)$ by definition. By lemma 48.34, we have some $J_{g-\lambda Id}$ made of blocks of the type $J_{g-\lambda Id}(0)$

Take the matrix and restrict to $J_r(0)$

$$g - \lambda Id = BJ_r(0)B^{-1}$$

One see that

$$\lambda Id + BJ_r(0)B^{-1} = B\lambda Id B^{-1} + BJ_r(0)B^{-1} = B(\lambda Id + J_r(0))B^{-1}$$

Uniqueness follows the uniqueness of $J_r(0)$

Chapter 49

Jordan Matrix

To find relations between Jordan matrix and diagonal representations

49.1 Def

Let λ be an eigenvalue of $f \in \mathscr{L}(V)$

$$E(\lambda) := \ker(f - \lambda Id)$$

This $E(\lambda)$ is called the eigenspace of λ

$$mult(\lambda)_{geo} = dim(E(\lambda))$$

is called the geometric multiplicity of λ

Moreover

$$mult(\lambda)_{alg} = \max\left\{k \in \mathbb{N} \,\middle|\, (t-\lambda)^k \mid P_f(t)\right\}$$

is called the algebraic multiplicity of λ

49.2 Prop

Let K be algebraically closed. Then $\forall \lambda$ eigenvalues of f

$$mult(\lambda)_{geo} \leq mult(\lambda)_{alg}$$

Proof

$$V = \bigoplus_{i=1}^{k} V(\lambda_i)$$

Take $\lambda = \lambda_i$. Let J_f be the Jordan matrix of f. Then

$$\det J_f = \det f$$

so

$$P_f(t) = \prod_i (t-\lambda_i)^{\dim(V(\lambda_i))} \quad \Rightarrow \quad \dim(V(\lambda)) = mult(\lambda)_{alg}$$

49.3 Corollary

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. f is diagonalizable iff

$$\forall \lambda_i \quad mult(\lambda_i)_{geo} = mult(\lambda_i)_{alg}$$

Chapter 50

Inner Product

50.1 Def

Two matrices $G, G' \in M_{n\times n}(K)$ are said conjugate if $\exists A \in \mathcal{Q}_{n\times n}(K)$ s.t. $G = G'^T$

Exercise

Verify that this is an equivalence relation

50.2 Def

Let V n-dimensional vector space over $K (K = \mathbb{R}$ or $K = \mathbb{C})$, $g \in \mathscr{L}(V, V; K)$ is said a bilinear form. Choose a basis $\{v_1, \cdots, v_n\}$ of V The matrix

$$G = (g(v_i, v_j))_{ij} \in M_{n\times n}(K)$$

is called the Gram matrix of g with respect to $\{v_1, \cdots, v_n\}$

By bilinearity, G determinant uniquely g

$$x = \sum \alpha_i v_i \to x = \sum \alpha_i e_i \quad x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$x, y \in V$

$$g(x, y) = g(\sum x_i v_i, \sum y_j v_j) = \sum_{i,j} x_i y_j g(v_i, v_j) = x^T G y$$

On the other hand, given a basis $\{v_1, \cdots, v_n\}$ and $G \in M_{n\times n}(K)$ the mapping:

$$\begin{aligned} V \times V &\to K \\ (x, y) &\mapsto x^T G y \end{aligned}$$

this is a bilinear form and the associated Gram matrix is exactly G

Fix a couple $(V, \{v_1, \cdot, v_n\})$ we have defined a bijection.

$$\begin{aligned} \mathscr{L}V, V; K &\overset{\cong}{\leftrightarrow} K \\ g &\mapsto G \end{aligned}$$

What happens if g is fixed but we change basis. We have also $\{v'_1, \cdots, v'_n\}$

$$\begin{tikzcd} & v_i & & V & & v'_i & \\ e_i \arrow[ur, mapsto] & & K^n \arrow[ur, "b"] & & K^n \arrow[ul, "b'"'] \arrow[ll, "A"] & & e_i \arrow[ul, mapsto] \end{tikzcd}$$

$$A = b^{-1} \circ (b') \quad (b')^{-1}(x) = x'$$

then A satisfies

$$Ax' = x$$

so

$$g(x, y) = x^T G y = (Ax')^T G(Ay') = (x')^T (A^T G A)(y')$$

The new Gram matrix with respect to the basis $\{v'_1, \cdots, v'_n\}$ is $A^T G A$

50.3 Prop

There exists a surjection:

$$\mathscr{L}(V, V; K) \to M_{n\times n}(K)/\sim_{conj}$$

Proof

Recall

$$\begin{aligned} \mathscr{L}(V, V; K) &\to \mathscr{L}(T_0^2(V); K) &&\to \mathscr{L}(V; V^\vee) \\ g &\mapsto g_s &&\mapsto [x \mapsto g_s(x \otimes -)] = \tilde{g} \end{aligned}$$

50.4 Def

Given $g \in \mathscr{L}(V, V; K)$ we can define several other bilinear mappings:

$$\begin{aligned} g_p : V \times V &\to K \\ (x, y) &\mapsto g(y, x) \end{aligned}$$

$$\begin{aligned} \overline{g_p} : V \times V &\to K \\ (x, y) &\mapsto \overline{g(y, x)} = \overline{g_p(x, y)} \end{aligned}$$

If $K = \mathbb{R}$ then $g_p = \overline{g_p}$

50.5 Def

A bilinear form g is said

Symmetric if $g = g_p$

Symplectic (skew-symmetric)if $g = -g_p$

hermitian if $g = \overline{g_p}$

(if $K = \mathbb{R}$ symmetric$\neq$ hermitian)

50.5.1 Example

$$\begin{aligned} K^n \times K^n &\to K \\ (x, y) &\mapsto x^T y \end{aligned}$$

is symmetric

$$\begin{aligned} K^2 \times K^2 &\to K \\ (v_1, v_2) &\mapsto \det(v_1 \mid v_2) \end{aligned}$$

is skew-symmetric

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ (x, y) &\mapsto x^T \overline{y} \end{aligned}$$

is hermitian

50.6 Def

$g \in \mathscr{L}(V, V; K)$ is an inner product of V, if g is either symmetric, symplectic or hermitian.

And (V, g) is called an inner space.

(note that $g = -\overline{g_p}$ is complicated)

50.7 Def

Let (V, g) be an inner product space. Two vectors $v_1, v_2 \in V$ are said orthogonal(with respect to g) if $g(v_1, v_2) = 0$. Two subspace $V_1, V_2 \subseteq V$ are orthogonal if $g(v_1, v_2) = 0 \ \forall v_1 \in V_1, v_2 \in V_2 (g(V_1, V_2) = 0)$

Exercise

Show the following

- If g is symmetric

$$G = G^T$$

- If g is symplectic
$$G = -G^T$$
- If g is hermitian
$$G = \overline{G^T}$$

50.8 Def

Let (V_g) be an inner product space the kernel of g

$$\ker(g) := \{v \in V \ g(v,w) = 0 \ \forall w \in V\}$$

Moreover g is said non-degenerated if

$$\ker(g) = \{0\}$$

50.9 Remark

Note that $\ker(g) = \ker(\tilde{g})$ when

$$\tilde{G} \in \mathscr{L}(V; V^\vee)$$

$$\tilde{g}_x = 0 \ \Leftrightarrow \ g(x,y) = 0 \forall y \in V$$

This implies that $\ker(g)$ is a linear subspace of V

Chapter 51

Differential Forms in $\mathbb{R}^n$

51.0.1 Notation

$$a\mid_p := (p, a)$$

51.1 Def

Let $p \in \mathbb{R}^n$ be a fixed point

$$\mathbb{R}^n_p := \{p\} \times \mathbb{R}^n$$

$(p, a) \in \mathbb{R}^n_p, a \in \mathbb{R}^n$

$$\begin{aligned}(p, a) + (p, b) &= (p, a + b)\\ \alpha(p, a) &= (p, \alpha a)\ \alpha \in \mathbb{R}\end{aligned}$$

With these operation $\mathbb{R}^n_p$ is a vector space, which is called the tangent space of $\mathbb{R}^n$ at p.

The dual space is

$$(\mathbb{R}^n_p)^\vee = \{p\} \times (\mathbb{R}^n_p)^\vee$$

A basis of $\mathbb{R}^n_p$ is denoted by

$$(e_1\mid_p, \cdots, e_n\mid_p)$$

$\bigsqcup_p \mathbb{R}^n_p$ is called the tangent bundle of $\mathbb{R}^n$

We have a projection mapping:

$$\begin{aligned}\bigsqcup_p \mathbb{R}^n_p &\overset{\pi}{\twoheadrightarrow} \mathbb{R}^n_p\\ (p, a) &\mapsto p\end{aligned}$$

and

$$\begin{aligned}\mathbb{R}^n \times \mathbb{R}^n \cong \bigsqcup_p \mathbb{R}^n_p \\ (p,a) \leftarrow\!\shortmid (p,a)\end{aligned}$$

Take $\{e_1 \mid_p, \cdots, e_n \mid_p\}$ as a basis of $\mathbb{R}^n_p$. The dual basis is denoted by

$$\{dx_1 \mid_p, \cdots, dx_n \mid_p\} = \{(e_1 \mid_p)^\vee, \cdots, (e_n \mid_p)^\vee\} \in (\mathbb{R}^n_p)^\vee$$

$$\begin{aligned} dx_i \mid_p : \mathbb{R}^n_p &\to \mathbb{R} \\ v = (\sum \alpha_i e_i \mid_p) &\mapsto \alpha_i \end{aligned}$$

$$\frac{\partial x_i}{\partial x_j} = dx_i \mid_p (e_j \mid_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recalled the wedge algebra:

$$\bigwedge (\mathbb{R}^n_p)^\vee := T(\mathbb{R}^n_p)^\vee / I = \bigoplus_{k \in \mathbb{N}} \bigwedge^k (\mathbb{R}^n_p)^\vee$$

Consider

$$\bigwedge^k (\mathbb{R}^n_p)^\vee$$

what's a basis of this vector space?

$$\left\{ dx_1 \mid_p \wedge \cdots \wedge dx_k \mid_p \middle| 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

and

$$\dim(\bigwedge^k (\mathbb{R}^n_p)^\vee) = \binom{n}{k}$$

Proved.

51.2 Do Carmo Differential forms

51.3 Def

An exterior k-form in $\mathbb{R}^n$ is a mapping:

$$\begin{aligned} \omega : \mathbb{R}^n &\to \bigsqcup_p \bigwedge^k (\mathbb{R}^n_p)^\vee \\ p &\mapsto \omega(p) \end{aligned}$$

that's a section of the projection π

$$(\pi \circ \omega = id_{\mathbb{R}}) = (\omega(p) \in \bigwedge^k (\mathbb{R}^n_p)^\vee)$$

$$\omega(p) = \sum_{1\le i_1<\cdots<i_k\le n} a_{i_1,\cdots,i_k}(p)dx_{i_1}\mid_p \wedge\cdots\wedge dx_{i_k}\mid_p \in \bigwedge^k(\mathbb{R}^n_p)^\vee$$

Note that

$$\begin{aligned}
&\coprod_p \bigwedge^k(\mathbb{R}^n_p)^\vee && \overset{\pi}{\twoheadrightarrow}\mathbb{R}^n \\
&f\mid_p && \mapsto p \\
&\omega \leftrightarrow \{a_{i_1},\cdots,a_{i_k}\}
\end{aligned}$$

if all a_{i_j} are of class $C^m(\mathbb{R})$ the ω is called a C^m-differential k-form. If $m=+\infty$ *omega* is called a smooth k-form.

51.4 Notation

$$\omega = \sum_I a_I dx_I$$

where $I=(i_1,\cdots,i_k)$

Example

take $n=4$

1-form

$$\omega = a_1dx_1 + a_2dx_2 + a_3dx_3 + a_4dx_4$$

$$\omega(p) = a_1(p)dx_1\mid_p + a_2(p)dx_2\mid_p + a_3(p)dx_3\mid_p + a_4(p)dx_4\mid_p$$

2-form

$$\begin{aligned}
\omega &= a_{12}dx_1\wedge dx_2 + a_{13}dx_1\wedge dx_3 + a_{14}dx_1\wedge dx_4 \\
&+ a_{23}dx_2\wedge dx_3 + a_{24}dx_2\wedge dx_4 + a_{34}dx_3\wedge dx_4
\end{aligned}$$

51.5 Notation

When $k=0$ a 0-form of class C^m-differential 0-form is $f\in C^m(\mathbb{R}^n)$

$$C^m(\mathbb{R}^n) = \{f:\mathbb{R}^n\to\mathbb{R} \text{ of class } C^m\}$$

51.6 Notation

$$\Omega^k_{(m)}(\mathbb{R}^n) := \{\text{set of } C^m\text{-diff } k\text{-forms}\}$$

$$\Omega^0_{(m)}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$$

m could be omitted if no confusion.

51.7 Prop

$\Omega^k_{(m)}(\mathbb{R}^n)$ is a module over $\Omega^0_{(m)}(\mathbb{R}^n)$

Proof

$\omega, \eta \in \Omega^k(\mathbb{R}^n)$

$$(\omega+\eta)(p) = \omega(p) + \eta(p) \in \bigwedge^k (\mathbb{R}^n_p)^\vee$$

$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$

$$f\omega \in \Omega^k(\mathbb{R}^n) \quad (f\omega)(p) = f(p)\omega(p) \in \bigwedge^k (\mathbb{R}^n_p)^\vee$$

51.8 Def

$f : \mathbb{R}^n \to \mathbb{R}$ differentiable then

$$df\mid_p : \mathbb{R}^n_p \to \mathbb{R}_{f(p)} \cong \mathbb{R}$$

$$df\mid_p \in (\mathbb{R}^n_p)^\vee$$

$$df\mid_p = \sum_{i=1}^n f_i(p) dx_i \mid_p$$

because

$$\{dx_1\mid_p, \cdots, dx_n\mid_p\}$$

is a basis of $(\mathbb{R}^n_p)^\vee$

By df then f_i are the partial derivatives of f. This means that df is a differential 1-form.

Moreover,

$$F : \mathbb{R}^n \to \mathbb{R}^m$$

differential, then

$$F = (F_1, \cdots, F_m)$$

when $F_i : \mathbb{R}^n \to \mathbb{R}$ differential.

$$dF\mid_p : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$$

$$dF_i\mid_p = dx_i\mid_{f(p)} (dF\mid_p) = d(x_i \circ F)\mid_p$$

$$dF_i\mid_p : \mathbb{R}^n_p \overset{dF|_p}{\to} \mathbb{R}^m_p \overset{dx_i|_{f(p)}}{\to} \mathbb{R}$$

and

$$\begin{aligned} dx_i\mid_p : \mathbb{R}^n_p &\to \mathbb{R} \\ v = \sum \alpha_i e_i\mid_p &\mapsto \alpha_i \end{aligned}$$

where $e_i \mid_p = (p, (0, \cdots, \underbrace{1}_{i-th}, 0, \cdots))$

Recall that if V is a vector space, then

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

This is a K-module with the multiplication:

$$\begin{aligned} V^{\otimes n} \times V^{\otimes n} &\to V^{\otimes n+m} \\ (x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m) &\mapsto x_1 \otimes \cdots \otimes x_n, y_1 \otimes \cdots \otimes y_m \end{aligned}$$

From $T(V)$ we construct

$$\bigwedge(V) = T(V)/I$$

$$\begin{aligned} T(V) &\to \bigwedge(V) \\ x_1 \otimes \cdots \otimes x_n &\mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

therefore also in $\bigwedge(V)$ we have the multiplication that makes $\bigwedge(V)$ a K-algebra

$$\begin{aligned} \bigwedge^k(V) &\to \bigwedge^l(V) \\ (x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_l) &\mapsto x_1 \wedge \cdots \wedge x_k \wedge y_1 \wedge \cdots \wedge y_l \end{aligned}$$

We define now a wedge product on $\Omega(\mathbb{R}^n)$

$$\begin{aligned} \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) &\to \Omega^{k+l}(\mathbb{R}^n) \\ (\omega, \eta) &\mapsto \omega \wedge \eta \end{aligned}$$

take $\omega = \sum_I a_I dx_I$ and $\eta = \sum_J b_J dx_J$

$$\omega \wedge \eta := \sum_{IJ} a_i b_J dx_{IJ}$$

where

$$IJ := (i_1, \cdots, i_k, j_1, \cdots, j_l)$$

with $I = (i_1, \cdots, i_k)$ and $J = (j_1, \cdots, j_l)$

Example

$$\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 \in \Omega^1(\mathbb{R}^3)$$

$$\eta = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$$

$$\omega \wedge \eta = (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3$$

51.9 Prop

Take $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^l(\mathbb{R}^n), \varphi \in \Omega^s(\mathbb{R}^n)$, then

(1)
$$(\omega \wedge \eta) \wedge \varphi = \omega \wedge (\eta \wedge \varphi)$$

(2)
$$(\omega + \eta) = (-1)^{kl}(\eta \wedge \omega)$$

(3) Take $\theta \in \Omega^k(\mathbb{R}^n)$
$$\omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta$$

Proof

Exercise
Try to do this. Consequence of the properties of $\wedge$ for vector spaces.

51.10 Def

Now we have
$$\Omega(\mathbb{R}^n) = \bigoplus_{k \in \mathbb{N}} \Omega^k(\mathbb{R}^n)$$
a $\mathbb{R}$-algebra with the $\wedge$-product
And it's also a $\Omega^0(\mathbb{R}^n)$ module and $\Omega^0(\mathbb{R}^n)$-algebra

51.11 Remark

$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$
$$f \wedge \omega = f\omega$$

51.12 Def: Pullback of forms

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of $C^{\mathfrak{r}}$, then it induces a mapping
$$\begin{gathered} f^* : \Omega^k_{(\mathfrak{r})}(\mathbb{R}^m) \to \Omega^k_{(\mathfrak{r})}(\mathbb{R}^n) \\ \omega \mapsto f^*\omega \end{gathered}$$
and
$$f^*(\omega)(p)(v_1, \cdots, v_k) = \omega(f(p))(df \mid_p (v_1), \cdots, df \mid_p (v_k))$$
recalling
$$df \mid_p : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)} \;\Rightarrow\; df \mid_p (v_i) \in \mathbb{R}^n_{f(p)}$$

51.13 Prop

Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be a differentiable mapping. $\omega,\eta\in\Omega^k(\mathbb{R}^n)$ and $g:\mathbb{R}^m\to\mathbb{R}$ a differentiable mapping.($g\in\Omega^0(\mathbb{R}^m)$)Then

(1)
$$f^*(\omega+\eta)=f^*(\omega)+f^*(\eta)$$

(2)
$$f^*(g\omega)=f^*g^*f^*(\omega)$$

where $f^*g:=g\circ f$

(3) If $\omega_1,\cdots,\omega_k$ are 1-forms in $\mathbb{R}^m$, then
$$f^*(\omega_1\wedge\cdots\wedge\omega_k)=f^*(\omega_1)\wedge\cdots\wedge f^*(\omega_k)$$

Proof

(1)
$$\begin{aligned}f^*(\omega+\eta)(p)(v_1,\cdots,v_k)&=(\omega+\eta)(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&=\omega(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&+\eta(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&=(f^*\omega)(p)(v_1,\cdots,v_k)+(f^*\eta)(p)(v_1,\cdots,v_k)\end{aligned}$$

(2)
$$\begin{aligned}f^*(g\omega)&=g\omega(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&=(g\circ f)(p)(f^*\omega)(p)(v_1,\cdots,v_k)\end{aligned}$$

(3)
$$(f_1f_2)(x)=f_1(x)f_2(x)$$
$$\begin{aligned}f^*(\omega_1\wedge\cdots\wedge\omega_k)(p)(v_1,\cdots,v_k)&=(\omega_1\wedge\cdots\wedge\omega_k)(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&=\omega_1(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\wedge\\&\cdots\wedge\omega_k(f(p))(df\mid_p(v_1),\cdots,df\mid_p(v_k))\\&=(f^*(\omega_1))(p)(v_1)\wedge\cdots\wedge(f^*(\omega_k))(p)(v_k)\end{aligned}$$

General fact
$$\begin{array}{lrl}f_1,\cdots,f_k: & V\to & V\\ f_1\wedge\cdots\wedge f_k: & \bigwedge^k V\to & \bigwedge^k V\\ & (v_1,\cdots,v_k)\mapsto & f_1(v_1)\wedge\cdots\wedge f_k(v_k)\end{array}$$

$$\begin{aligned}g^{\otimes n}:V^{\otimes n}&\to V^{\otimes n}\\(v_1,\cdots,v_n)&\mapsto g(v_1)\otimes\cdots\otimes g(v_n)\end{aligned}$$

Let's see what happens in terms of coordinates:

$$
\begin{aligned}
f : \mathbb{R}^n &\to \mathbb{R}^m \\
(x_1, \cdots, x_n)^T &\mapsto (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n))^T
\end{aligned}
$$

$\Omega = \sum_I a_I dy_I \in \Omega^k(\mathbb{R}^m)$

$$f^*\omega = \sum_I f^*(a_I)(f^* dy_{i_1}) \wedge \cdots \wedge (f^* dy_{i_k})$$

Note that

$$(f^* dy_i)(v) = dy_i(df(v)) = d(y_i \circ f)(v) = (df_i)(v)$$

then

$$f^*\omega = \sum_I a_I(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)) df_{i_1} \wedge \cdots \wedge df_{i_k}$$

51.14 Remark

$U \subseteq \mathbb{R}^n$ open then consider $\Omega^k(U) \subseteq \Omega^k(\mathbb{R}^n)$

Example

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\}(= U))$$

$V = \{(r, \theta) \in \mathbb{R}^2 : r > 0, 0 \leq \theta \leq 2\pi\}$

$$
\begin{aligned}
f : V &\to U \\
(r, \theta)^T &\mapsto f(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}
\end{aligned}
$$

Let's compute $f^*\omega$

$$df_1 = \cos\theta dr - r\sin\theta d\theta$$

$$df_2 = \sin\theta dr + r\cos\theta d\theta$$

$$f^*\omega = -\frac{r\sin\theta}{r^2}(\cos\theta dr - r\sin\theta d\theta) + \frac{r\cos\theta}{r^2}(\sin\theta dr + r\cos\theta d\theta) = d\theta$$

51.15

$U \in \mathbb{R}$ an open subset

$$\Omega^k_{(m)}(U)$$

this is a module over $\Omega^0_{(m)}(U)$ Moreover, $\omega \in \Omega^k(U), \eta \in \Omega^l(U)$

$$\omega \wedge \eta \in \Omega^{k+l}(U)$$

$f : \underbrace{U}_{\subseteq \mathbb{R}^n} \to \underbrace{\mathbb{R}^m}_{\subseteq \mathbb{R}^m}$ f is of class C^{m+1}

$$f^*\omega \in \Omega^k_{(m)}U()$$

df is a one-form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

where $\frac{\partial f}{\partial x_i} = a_i : \mathbb{R}^n \to \mathbb{R}$ differentiable

51.16 Prop

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable mapping. Then

(1) for any two forms in $\mathbb{R}^m$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*(\eta))$$

(2) for $g : \mathbb{R}^p \to \mathbb{R}^n$ differentiable

$$(f \circ g)^*\omega = g^*(f^*\omega)$$

Proof

1

$(y_1, \cdots, y_m) = (f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n)) \in \mathbb{R}^m$, $(x_1, \cdots, x_n) \in \mathbb{R}^n$

$$\omega = \sum_I a_I dy_I \quad \eta = \sum_J b_J dy_J$$

$$\begin{aligned} f^*(\omega \wedge \eta) &= f^*(\sum_{IJ} a_I b_J dy_I \wedge dy_J) \\ \text{(by def of pullback)} &= \sum_{IJ} a_I(f_1, \cdots, f_m) b_J(f_1, \cdots, f_m) df_I \wedge df_J \\ &= (\sum_I a_I(f_1, \cdots, f_m) df_I) \wedge (\sum_J b_J(f_1, \cdots, f_m) df_J) \\ &= (f^*\omega) \wedge (f^*(\eta)) \end{aligned}$$

2

$$\begin{aligned}(f\circ g)^*\omega &= \sum_I a_I((f\circ g)_1,\cdots,(f\circ g)_m)d(f\circ g)_I\\ &= \sum_I a_I(f_1(g_1,\cdots,g_n),\cdots,f_m(g_1,\cdots,g_n))df_I(dg_1,\cdots,dg_n)\\ &= g^*(f^*\omega)\end{aligned}$$

51.17

The deferential of a function is a one-form

$$\underbrace{f}_{\text{0-form}} \rightsquigarrow \underbrace{df}_{\text{1-form}}$$

We went to generalize this to any (exterior) differentials

$$\begin{aligned} d: \Omega^k_{(m)}(U) &\to \Omega^{k+1}_{(m)}(U)\\ \omega &\mapsto d\omega\\ \sum_I a_I dx_I &\mapsto \sum_I da_I\wedge dx_I\end{aligned}$$

where $a_I \in C^m(U)$, $da_I = \sum \frac{\partial a_I}{\partial x_i}dx_i$

51.18 Example

$\omega = xyz\mathrm{d}x + yz\mathrm{d}y + (x+z)\mathrm{d}z$

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(xyz)\wedge\mathrm{d}x + \mathrm{d}(yz)\wedge\mathrm{d}y + \mathrm{d}(x+z)\wedge\mathrm{d}z\\ &= (yz\mathrm{d}x + xz\mathrm{d}y + xy\mathrm{d}z)\wedge\mathrm{d}x + (z\mathrm{d}y + y\mathrm{d}z)\wedge\mathrm{d}y + (x\mathrm{d}z)\wedge\mathrm{d}z\\ &= -xz\mathrm{d}x\wedge\mathrm{d}y - xy\mathrm{d}x\wedge\mathrm{d}z - y\mathrm{d}y\wedge\mathrm{d}z + \mathrm{d}x\wedge\mathrm{d}z\\ &= -xz\underline{\mathrm{d}x\wedge\mathrm{d}y} + (1-xxy)\underline{\mathrm{d}x\wedge\mathrm{d}z} - y\underline{\mathrm{d}y\wedge\mathrm{d}z}\end{aligned}$$

51.19 Prop

$\forall\omega_1,\omega_2 \in \Omega^k(U), \eta\in\Omega^l(U)$

(1)
$$\mathrm{d}(\omega_1+\omega_2) = \mathrm{d}(\omega_1) + \mathrm{d}(\omega_2)$$

(2)
$$\mathrm{d}(\omega_1\wedge\omega_2) = \mathrm{d}(\omega_1)\wedge\eta + (-1)^k\omega\wedge\mathrm{d}\eta$$

(3)
$$\mathrm{d}(\mathrm{d}\omega) = 0 \quad (\mathrm{d}^2\omega = 0)$$

(4) $f : \underbrace{U}_{\subseteq \mathbb{R}^n} \to \underbrace{V}_{\subseteq \mathbb{R}^m}$
$$\mathrm{d}(f^*\omega) = f^*(\mathrm{d}\omega)$$

Proof

(1) Exercise

(2) $\omega = \sum_I a_I dx_I, \eta = \sum_J b_J dx_J;\ \omega \wedge \eta = \sum_{IJ} a_I b_J dx_I \wedge dx_J$

$$\begin{aligned}\mathrm{d}(\omega \wedge \eta) &= \sum_{IJ} \mathrm{d}(a_I b_J) \wedge dx_I \wedge dx_J \\ &= (\sum_{IJ} b_J da_I \wedge dx_I \wedge dx_J) + (\sum_{IJ} a_I db_J \wedge dx_I \wedge dx_J) \\ &= \mathrm{d}\omega \wedge \eta + (-1)^k \sum_{IJ} a_I ddx_I \wedge b_J \wedge dx_J \\ &= \mathrm{d}\omega \wedge \eta + (-1)^k \omega \wedge \mathrm{d}\eta\end{aligned}$$

(3) First assume $\omega = f \in \Omega^0(U)$

$$\begin{aligned}\mathrm{d}(\mathrm{d}f) &= \mathrm{d}(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j) \\ &= \sum_{j=1}^n \mathrm{d}(\frac{\partial f}{\partial x_j} \wedge dx_j) \\ &= \sum_{j=1}^n (\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j) \\ &= 0\end{aligned}$$

Notice that $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

By (1) we can prove for $\omega = a_I dx_I, a_I =\neq 0$, from (2) we have

$$\mathrm{d}\omega = \mathrm{d}a_I \wedge \mathrm{d}x_I + a_I \mathrm{d}^2 x_I$$

But
$$\mathrm{d}^2 x_I = d(1 \cdot \mathrm{d}x_I) = \mathrm{d}1 \wedge \mathrm{d}x_I = 0$$

Hence
$$\begin{aligned}\mathrm{d}^2\omega &= \mathrm{d}(\mathrm{d}\omega) \\ &= \mathrm{d}(\mathrm{d}a_I \wedge \mathrm{d}x_I) \\ &= 0\end{aligned}$$

(4) As above let's prove it for $\omega = g \in \Omega^0(U)$

$$\begin{aligned} g :&\mathbb{R}^m && \to \mathbb{R} \\ &(y_1, \cdots, y_m) && \mapsto g(y_1, \cdots, y_m) \end{aligned}$$

$$\begin{aligned} f^*(\mathrm{d}g) &= f^*(\sum_{i=1}^m \frac{\partial g}{\partial y_i} dy_i) \\ &= \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial f}{\partial x_j} \mathrm{d}x_j \\ &= \sum_j \frac{\partial (g \circ f)}{\partial x_j} dx_j \\ &= \mathrm{d}(g \circ f) \\ &= \mathrm{d}(f^* g) \end{aligned}$$

Now let's do the proof for $\omega \in \Omega^k(U), \omega = \sum_I a_I \mathrm{d}x_I$

$$\begin{aligned} \mathrm{d}(f^* g) &= \mathrm{d}(f^*(\sum_I a_I \mathrm{d}x_I)) \\ (\text{ by prop of } f^*) &= \mathrm{d}(\sum_I f^* a_I \wedge f^* \mathrm{d}x_I) \\ (by (1)) &= \sum_I \mathrm{d}(f^* a_I \wedge f^* \mathrm{d}x_I) \\ (\text{use}(2)) &= \sum_I f^*(\mathrm{d}a_I) \wedge f^* \mathrm{d}x_I \\ (\text{prop of } f^*) &= f^*(\sum_I \mathrm{d}a_I \wedge \mathrm{d}x_I) \\ &= f^*(\mathrm{d}\omega) \end{aligned}$$

51.20

$$df : p \mapsto df_p$$

is a differential form.

51.21 Def?

$$D_h f(p) := \lim_{t \to 0} \frac{f(p + th) - f(p)}{h} = df_p(h)$$

Chapter 52

Line integral

52.1 Def

$\omega = \sum_i a_i \mathrm{d}x_i \in \Omega^1_(m)(U)$, $U \subseteq \mathbb{R}^n$

$$\gamma : [a,b] \to U^n$$

a parametric curve

$$f : [a,b] \to \mathbb{R}$$

of class C^1

$$\gamma : t \mapsto (t, f(t)) = \text{Graph of } f$$

this is a parametric curve piecewise of class C^1: $\exists t_0 = a < t_1 < \cdots < t_k = b$ such that

$$\gamma_j := \gamma \mid_{]t_j, t_{j+1}[}$$

is of class C^1

$$\gamma_j(]t_k, t_{k+1}[) \to \mathbb{R}^n$$

we can define $\gamma_j^* \omega$ this is one form in $\Omega^1(]t_k, t_{k+1}[)$
if $\gamma_j(t) = (x_1(t), \cdots, x_n(t))$ then

$$\gamma_j^* \omega = \sum_{i=1}^n a_i(x_1(t), \cdots, x_n(t)) \frac{\mathrm{d}x_i}{\mathrm{d}t} \mathrm{d}t \qquad x_i(t) = \frac{\mathrm{d}x_i}{\mathrm{d}t}?$$

52.2 Def: Path integral

Let γ and ω be as above.

$$\int_\gamma \omega := \sum_i \int_{t_k}^{t_{k+1}} \gamma_j^* \omega$$

this is the integral of ω along the parametric curve γ with

$$\gamma = t \mapsto (x_1(t), \cdots, x_n(t))$$

where $x_i(t) = \frac{\mathrm{d}x_i}{\mathrm{d}t}$

52.3 What's this in physics?

Fix $\gamma(t), \gamma'(t) = (\frac{\mathrm{d}x_1}{\mathrm{d}t}, \cdots, \frac{\mathrm{d}x_n}{\mathrm{d}t})$ =the tangent vector of γ in $\gamma(t)$ then

$$\int_{t_k}^{t_{k+1}} \gamma_j^* \omega = \int_{t_k}^{t_{k+1}} \left\langle a \circ \gamma_j, \gamma_j' \right\rangle \mathrm{d}t$$

where $a = (a_1, \cdots, a_n), a_i : \mathbb{R}^n \to \mathbb{R}$

Chapter 53

Complement of measure theory

53.1 Def(σ-finite)

Let (X, Σ_X, μ) be a measure space. We say that it's σ-finite if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets. (namely $E_n \in \Sigma_X$) such that

$$X = \bigcup_{n\in\mathbb{N}} E_n \text{ and } \mu(E_n) < +\infty, \forall n \in \mathbb{N}$$

53.2 Example($\mathbb{R}$, Norel σ-algebra, Lebesgue measure)

this is σ-finite

$$\lambda([-n, n]) = 2n < +\infty$$

53.3 Notation

Take sets $A \subseteq X \times Y$ For $x \in X$, we define

$$A_x := \{u \in Y \mid (x, y) \in A\}$$

called a **vertical section** of A or $x-$fiber of A

For $y \in Y$ we define

$$A_y := \{x \in X \mid (x, y) \in A\}$$

called a **horizontal section** of A, or y-fiber of A

53.4 Def

Let X be a set. then $\mathscr{D} \subseteq \wp(X)$ is a **Dynkin system** if

- $X \in \mathscr{D}$ and $\varnothing \in \mathscr{D}$
- $\forall D \in \mathscr{D} \quad X \setminus D \in \mathscr{D}$
- If $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{D}$ of pairwise disjoint sets, then

$$\bigsqcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

Remark

A σ-algebra is a Dynkin system

53.5 Def

Let $(\mathscr{G} \subseteq \wp(X))$ then $\delta(\mathscr{G}) \subseteq \wp(X)$ is called the Dynkin system generated by $\mathscr{G}$ if

- $\mathscr{G} \subseteq \delta(\mathscr{G})$
- If $\mathscr{D}$ is a Dynkin system containing $\mathscr{G}$, then $\delta(\mathscr{G}) \subseteq \mathscr{D}$

Exercise

$\delta(\mathscr{G})$ exists and it's unique.

53.6 Prop

If $\mathscr{D}$ is a Dynkin system closed under the intersection, then it's a σ-algebra, namely

$$\forall (D,E) \in \mathscr{D}^2, D \cap E \in \mathscr{D} \;\Rightarrow\; \forall \{D_n\}_{n\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}} \quad \bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

Proof

We have to show that $\mathscr{D}$ is closed under any countable union. Let $\{D_n\}_{n\in\mathbb{N}}$ be any sequence in $\mathscr{D}$, let

$$E_n = D_n \cap \bigcap_{m<n} X \setminus D_m$$

and we know that

$$\bigcup_{k\in\mathbb{N}} E_k = \bigcup_{k\in\mathbb{N}} D_k \supseteq D_n \cap D_m \;\forall n, m$$

53.7 Prop

Let X be a set and let $\mathcal{G} \subseteq \wp(X)$. Assume that $\mathcal{G}$ is closed under the finite intersection. Then

$$\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$$

Proof

Prove $\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$
trivial

Prove $\sigma(\mathcal{G})$ is a σ-algebra, which gives that $\delta(\mathcal{G}) \supseteq \sigma(\mathcal{G})$
Let

$$\delta_D = \{E \subseteq X \mid E \cap D \in \delta(\mathcal{G})\}$$

Verify that $\forall D \in \mathcal{G}, \delta_D$ is a Dynkin system:

- Since $X \cap DS \in \mathcal{G} \Rightarrow X \in \delta_D$
- Take $E \in \delta_D$

  $$(X \setminus E) \cap D = X \setminus ((E \cap D) \cup (X \setminus D))$$

  Where $E \cap D \in \delta(\mathcal{G})$(since $E \in \delta_D$), $X\setminus \in \delta(\mathcal{G})$(by def)

  Hence

  $$X \setminus E \in \delta_D$$

- Let $\{E_n\}$ be elements in δ_D which are pairwisely disjoint, then

  $$(\bigcup_{n\in\mathbb{N}} E_n) \cap D = \bigcup_{n\in\mathbb{N}} (E_n \cap D)$$

Then $\forall G \in \mathcal{G}$

$$\delta(\mathcal{G}) \subseteq \delta_G$$

since $\delta(\mathcal{G})$ is the smallest Dynkin system containing $\mathcal{G}$ and $\forall G \in \mathcal{G}\ \mathcal{G} \subseteq \delta_G$ since $\mathcal{G}$ is closed under finite intersection. By definition

$$\forall H \in \delta(\mathcal{G}), H \cup G \in \delta(\mathcal{G})$$

53.8 Lemma

Let (X, Σ_X) be a measurable space. Then the mapping 0 is measurable

Proof

By def

$$0_*(\Sigma_X) = \{B \subseteq \mathbb{R} \mid 0^{-1}(B) \in \Sigma_X\}$$

Since either $0^{-1}(B) = \varnothing$ or $0^{-1}(B) = X$, then

$$0_*(\Sigma_X) = \wp(\mathbb{R})$$

Hence

$$\mathscr{B}(\mathbb{R}) \subseteq 0_*(\Sigma_X) = \wp(\mathbb{R})$$

53.9 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. Then $\forall E \in \Sigma_X \otimes \Sigma_Y$, the functions

$$\begin{aligned} f_E :X &\to \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu(E_x) \\ g_E :Y &\to \mathbb{R} \cup \{+\infty\} \\ y &\mapsto \mu(E_y) \end{aligned}$$

are respectively Σ_X-measurable and Σ_Y-measurable

Proof

We first cope with special ones that ν is finite ($\mu(Y) < +\infty$) Let

$$F = \{E \in \Sigma_X \otimes \Sigma_Y \mid f_E \text{ is measurable}\}$$

We want to have $F = \Sigma_X \otimes \Sigma_Y$ Only to show $\Sigma_X \otimes \Sigma_Y \subseteq F$ by definition of product measure

Let $S_1 \in \Sigma_X, S_2 \in \Sigma_Y$

$$(S_1 \times S_2)_x = \begin{vmatrix} S_2 & \text{if } x \in S_1 \\ 0 & \text{if } x \notin S_1 \end{vmatrix} f_{S_1 \times S_2}(x) = \nu(S_1 \times S_2) = \nu(S_2)\chi_{S_1}(x)$$

?

$f_{S_1 \times S_2}$ is measurable.

Now show that F is a Dynkin system:

- $X \times Y \in F$
- Let $D \in F$, we want to show that

  $$(X \times Y) \setminus D \in F$$

  Note that

  $$((X \times Y) \setminus D)_x = Y \setminus D_x$$

then

$$\begin{aligned} f_{(X\times Y\setminus D)}(x) &= \nu(((X\times Y)\setminus D)_x) \\ &= \nu(Y\setminus D_x) \\ &= \nu(Y)-\nu(D_x) \\ &= \nu(Y)-f_D(x) \end{aligned}$$

Which means that $f_{(X\times Y\setminus D)}$ is measurable.

- Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of disjoint sets such that $D_n \in F$. (f_{D_n} is measurable) $D = \bigcup\limits_{n\in\mathbb{N}} D_n$

$$\begin{aligned} f_D(x) &= \nu(D_x) \\ &= \nu(\bigcup_{n\in\mathbb{N}} D_n) \\ &= \sum_{n\in\mathbb{N}} \nu(D_n) \\ &= \sum_{n\in\mathbb{N}} f_{D_n}(x) \end{aligned}$$

Hence F is a Dynkin system.

Consider

$$\mathcal{G} = \{S_1 \times S_2 \mid S_1 \in \Sigma_X, S_2 \in \Sigma_Y\} \subseteq F$$

Moreover, $\mathcal{G}$ is closed under the intersection.

$$(S_1\times T_1)\cap(S_2\times T_2) = (S_1\cap S_2)\times(T_1\cap T_2)$$

So $\delta(\mathcal{G})$ is σ-algebra. By proposition53.7

$$\delta(\mathcal{G}) = \sigma(\mathcal{G}) = \Sigma_X \otimes \Sigma_Y \subseteq F$$

Secondly, for general ν, since ν is σ-finite, there exists

$$Y = \bigcup_{n\in\mathbb{N}} Y_n \quad \nu(Y_n) < +\infty$$

As above

$$F_0 = Y_0, F_n = Y_n \setminus \bigcup_{k\in\mathbb{N}} Y_k, \nu(F_n) < +\infty$$

$\{F_n\}$ are disjoint, measurable, of finite measure and $Y = \bigcup\limits_n F_n$

For all n we define a measure $\nu^{(n)}$ on Y_n

$$\nu^{(n)}(E) := \nu(E\cap F_n)$$

Notice that

$$\nu^{(n)}(Y) = \nu(Y\cap E) = \nu(F_n) < +\infty$$

Hence we have

$$\begin{aligned} f_E^{(n)} :X &\to \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu^{(n)}(E_x) \end{aligned}$$

By step 1, $f_e^{(n)}$ is measurable.
$\forall E, n$

$$\begin{aligned} f_E^{(n)}(x) &= \nu^(E_x) \\ &= \nu(E_x \cap Y) \\ &= \nu\left(E_x \cap \bigcup_n F_n\right) \\ &= \nu\left(\bigcup_n E_x \cap F_n\right) \\ &= \sum_n \nu(E_x \cap F_n) \qquad = \sum \nu^{(n)}(E_x) \\ &= \sum_n (f_E^{(n)}(x)) \end{aligned}$$

If follows that f_E is measurable

Then we need prove that $F \supseteq \Sigma_X \times \Sigma_Y$ and F is σ-algebra, so $F \supseteq \sigma(\Sigma_X \times \Sigma_Y) = \Sigma_X \otimes \Sigma_Y$

- $E_\mu \in \Sigma_X$ and $E_\epsilon \in \Sigma_Y$

$$(E_\mu \times E_\epsilon)_x = \left| \begin{array}{ll} E_\epsilon & \text{if } x \in E_\mu \\ \varnothing & \text{otherwise} \end{array} \right. \in \Sigma_X$$

- exercise

53.10 Prop

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. $\forall E \in \Sigma_X \otimes \Sigma_Y$ the functions:

$$\begin{aligned} \rho_X(E) &:= \int_X f_E(x) \mathrm{d}\mu(x) \\ \rho_Y(E) &:= \int_Y g_E(y) \mathrm{d}\nu(y) \end{aligned}$$

Define two measure on the measurable spaces $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ such that

$$\rho_\chi(S_1 \times S_2) = \rho_Y(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

Proof

We already know that f_E and g_E are measurable. So the integral makes sense. Only needs to prove for ρ_X

- Since $f_E \geq 0$ and $g_E \geq 0$, then $\rho_X(E) \geq 0$ $\forall E \in \Sigma_X \otimes \Sigma_Y$
- $\rho_X(\varnothing) = \int_X \nu(\varnothing)\mathrm{d}\mu(x) = 0$
- Assume that $\{E_n\}_{n\in\mathbb{N}}$ is a sequence in $\Sigma_X \otimes \Sigma_Y$ of disjoint subsets,

$$\begin{aligned}
\rho_X(\bigsqcup_{n\in\mathbb{N}} E_n) &= \int_X \nu(\bigsqcup_{n\in\mathbb{N}} (E_n)_x)\mathrm{d}\mu(x) \\
&= \int_X \sum_{n\in\mathbb{N}} \nu(E_n)_x \mathrm{d}\mu(x) \\
&= \sum_{n\in\mathbb{N}} \int_X \nu(E_n)_x \mathrm{d}\mu(x) \\
&= \sum_{n\in\mathbb{N}} \rho_X(E_n)
\end{aligned}$$

- $$\begin{aligned}
\rho_X(S_1 \times S_2) &= \int_X \nu(S_1 \times S_2)_x \mathrm{d}\mu(x) \\
&= \int_X \nu(S_2)\mathbb{1}_{S_1}(x)\mathrm{d}\mu(x) \\
&= \nu(S_2)\mu(S_1)
\end{aligned}$$

53.11 Prop

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. Any measure η on $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ that satisfies

$$\eta(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

is σ-finite

Proof

$$\nu(E_n \times F_m) = \mu(E_n)\nu(F_m) < +\infty$$

53.12 Prop

Let (X, Σ_X) be a measurable space and assume that $\mathcal{G} \subseteq \wp(X)$ such that $\Sigma = \sigma(\mathcal{G})$

Moreover, assume that $\mathcal{G}$ satisfies the following conditions:

(1) It's closed under finite intersection.

(2) There exists a sequence $\{G_n\}_{n\in\mathbb{N}}$ in $\mathscr{G}$ such that $\{G_m\} \uparrow X$ (namely $G_i \subseteq G_{i+1}$ and $\bigcup_n G_n = X$)

Let μ and ν be two measure on (X, Σ) such that

(a) $\forall G \in \mathscr{G} \quad \mu(G) = \nu(G)$

(b) $\forall n \in \mathbb{N} \quad \mu(G_n) = \nu(G_n)$

Then $\mu = \nu$

Proof

Define

$$\mathscr{D}_n = \{E \in \Sigma \mid \mu(G_n \cap E)\} \subseteq \Sigma$$

We show that $\mathscr{D}_n$ is a Dynkin system $\forall n$

- $G_n \cap X = G_n$
- Assume that $D \in \mathscr{D}_n$

$$\begin{aligned}\mu(G_n \cap (X \setminus D)) &= \mu(G_n \setminus D) \\ &= \mu(G_n) - \mu(G_n \cap D) \text{ (here use the fact that } \mu(G_n) < +\infty) \\ &= \nu(G_n) - \nu(G_n \cap D) \\ &= \nu(G_n \cap (X \setminus D))\end{aligned}$$

- Take $\{D_m\}_{m\in\mathbb{N}}$ in $\mathscr{D}_n$ of pairwise choice

$$\begin{aligned}\mu(G_n \cap \bigcup_m D_m) &= \mu(\bigcup_m (G_n \cap D_m)) \\ &= \sum_m \mu(G_n \cap D_m) \\ &= \sum_m \nu(G_n \cap D_m) \\ &= \nu(G_n \cap \bigcup_m D_m)\end{aligned}$$

Combining (1) and (a) $\mathscr{G} \subseteq \mathscr{D}_n$.By prop53.7, consider

$$\delta(\mathscr{G}) = \sigma(\mathscr{G}) = \Sigma$$

Moreover, since $\mathscr{G} \subseteq \mathscr{D}_n$ and $\mathscr{D}_n$ a Dynkin system

$$\delta(\mathscr{G}) \subseteq \mathscr{D}_n$$

We get

$$\Sigma = \mathscr{D}_n$$

Since $\bigcup_n G_n \cap E = E \cap \bigcup_n G_n = E$

$$\mu(E) = \lim_{x\to+\infty} \mu(G_n \cap E) = \lim_{x\to+\infty} \nu(G_n \cap E) = \nu(E)$$

53.13 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. There exists a unique σ-finite measure $\mu \times \nu$ on $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ such that

$$\mu \times \nu(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

and moreover, we have

$$(\mu \times \nu)(E) = \int_X f_E \mathrm{d}\mu = \int_Y g_E \mathrm{d}\nu$$

53.14 Corollary

On $\mathbb{R}^n$, we can define a unique measure $\lambda^{(n)}$ as product of the Lebesgue measure on $\mathbb{R}$. This is called the Lebesgue measure on $\mathbb{R}^n$

Proof

Assume that η and η' are two measures on the product satisfies the equation. Let $\mathcal{G} = \Sigma_1 \times \Sigma_2,\ \sigma(\mathcal{G}) = \Sigma_1 \otimes \Sigma_2$ And $\mathcal{G}$ is stable under finite intersection.

Since μ and ν are σ-finite, we can find some $\{E_n\} \uparrow X$ and $\{F_m\} \uparrow Y$ such that $\mu(E_n) < +\infty, \nu(F_m) < +\infty$

$$X \times Y = \bigcup_{n,m} E_n \times F_m$$

We can find some ordering of the couple

$$X \times Y = \bigcup_{i_k} E_{i_k} \times F_{i_k}$$

$$\{E_{i_k} \times F_{i_k}\} \uparrow X \times Y$$

$$G_k := E_{i_k} \times F_{i_k}$$

By the equal in conditions $\forall k$

$$\eta(G_k) = \eta'(G_k)$$

We apply prop53.12 to get

$$\eta = \eta' = \mu \times \nu$$

By prop53.11

$$\mu \times \nu$$

is σ-finite

And $\mu \times \nu$ exists by Prop53.10

53.15 Monotone convergence theorem

Let (X, Σ_X, μ) be a measure space. $f : X \to \mathbb{R}_{\geq 0}$ be a measurable function. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions

$$f_n : X \to \mathbb{R}_{\geq 0}$$

such that $f_i < f_j \quad \forall i < j$ and

$$\lim_{n\to+\infty} f_n(x) = f(x)$$

almost everywhere in $X(\forall x \in X \setminus Z$ when $Z \in \Sigma, \mu(Z) = 0)$Then

$$\int_X f \mathrm{d}\mu = \lim_{n\to+\infty} \int_X f_n \mathrm{d}\mu$$

Proof

dominated convergence theorem $\Rightarrow$ monotone convergence theorem.

53.16 Recall

Product measure on $\mathbb{R}^n$. This is the unique measure on λ^n that exact is the naive product measure on rectangles.

$$\begin{aligned}
&\Sigma_{\mathbb{R}^n} = \mathscr{B}(\mathbb{R}) \otimes \cdots \otimes \mathscr{B}(\mathbb{R}) \\
\leftrightarrow \\
&\mathscr{B}(\mathbb{R}^n);\ \lambda^n(\prod_i [a_i, b_i]) = \prod_i \lambda[a_i, b_i]
\end{aligned}$$

53.17 Def

$$\mathscr{O}^n = \{\text{set of open sets of } \mathbb{R}^n\}$$
$$\mathscr{C} = \{\text{set of closed sets of } \mathbb{R}^n\}$$
$$\mathscr{R}^n = \{\text{set of compact sets of } \mathbb{R}^n\}$$
$$\mathscr{I}^n_{ha} = \{\text{set of all half-open rectangles in } \mathbb{R}^n\}$$
$$\mathscr{I}^n_{ha,rat} = \{\text{set of all half-open rectangles of } \mathbb{R}^n, \text{ with rational end points}\}$$

53.18 Prop

$$\mathscr{B}(\mathbb{R}^n) = \sigma(\mathscr{O}^n) = \sigma(\mathscr{C}^n) = \sigma(\mathscr{R}^n) = \sigma(\mathscr{I}^n_{ha}) = \sigma(\mathscr{I}^n_{ha,rat})$$

Proof

Exercise

53.19 Recall

Let (X, Σ_X) and (Y, Σ_Y) be measurable spaces. Moreover, assume that

$$\Sigma_Y = \sigma(\mathscr{G})$$

where $\mathscr{G} \subseteq \wp(X)$.

A function $f : X \to Y$ is measurable iff

$$\forall S \in \mathscr{G} \quad f^{-1}(S) \in \Sigma(X)$$

Hint

$$\mathscr{M} := \left\{ B \subseteq Y \mid f^{-1}(B) \in \Sigma_X \right\} \subseteq \wp(X)$$

show that this is a σ-algebra

53.20 Corollary

$f : \mathbb{R}^n \to \mathbb{R}^m$, if f is continuous, the f is measurable with respect to the Lebesgue measure.

53.21 Def: Push-forward measure

Let (X, Σ_X, μ) be a measure space, and let (Y, Σ_Y) be a measurable space. If $f : X \to Y$ is a measurable function, then define:

$$f_{*\mu}(E) = \mu(f^{-1}(E)) \quad \forall E \in \Sigma^Y$$

This is a measure on Y, called the push forward of μ through f

53.22 Prop

Let $p \in \mathbb{R}$ and let $E \in \mathscr{B}(\mathbb{R}^n)$, then

$$\lambda^n(E + p) = \lambda^n(E)$$

note that

$$E + p = \{x + p \mid x \in E\}$$

Proof

$$p = (p_1, \cdots, p_n)$$

Consider the translation

$$\begin{aligned} \tau_p : &\mathbb{R}^n &&\to \mathbb{R}^n \\ &x &&\mapsto x - p \end{aligned}$$

this is continuous, so measurable. We consider

$$\lambda_p^n := \tau_{p*}\lambda^n$$

let's show that $\lambda_p^n = \lambda^n$

$$\begin{aligned} \lambda_p^n(\prod_{i=1}^n [a_i, b_i[) &\overset{\text{by def of } f_*}{=} \lambda^n(\tau_p^{-1}(\prod_{i=1}^n [a_i, b_i[)) \\ &= \lambda^n(\prod_{i=1}^n [a_i + p_i, b_i + p_i]) \\ &= \prod_{i=1}^n (b_1 - a_i) \end{aligned}$$

By thee uniqueness of the product measure, we have

$$\lambda_p^n = \lambda^n$$

53.23 Lemma

Let $f : \Omega \to \mathbb{R}_{\geq 0}$ be a mapping. Then there exists an increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ such that converges pointwisely to f

Proof

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega\in\Omega \mid \frac{k}{2^n} \leq f(\omega) \leq \frac{k+1}{2^n}\}} + n\mathbb{1}_{\{\omega\in\Omega \mid f(\omega)\geq 0\}}$$

53.24 Fubini-Tobelli Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be two σ-finite measure spaces. Let $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$ be the product space. Let $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a measurable function. Then

$$\begin{aligned} \int_{X\times Y} |f| \, \mathrm{d}(\mu \times \nu) &= \int_X (\int_Y |f(x, y)| \, \mathrm{d}\nu(y))\mathrm{d}\mu(x) \\ &= \int_X (\int_Y |f(x, y)| \, \mathrm{d}\nu(y))\mathrm{d}\mu(x) \end{aligned}$$

Proof

We can assume that $f \geq 0$ and

$$\begin{aligned} \forall x \in X \quad f_x : \quad & Y \to \mathbb{R} \cup \{+\infty\} \\ & y \mapsto \qquad f(x,y) \\ \forall y \in Y \quad f_y : \quad & X \to \mathbb{R} \cup \{+\infty\} \end{aligned}$$

Step 1: f_x and f_y are measurable

Let's do it for f_y. We have to show that for $D \in \mathscr{B}(\mathbb{R})$ where $f_y^{-1}(D) \in \Sigma_X$

$$f_y^{-1}(D) = \{x \in X \mid f(x,y) \in D\} = \{x \in X \mid (x,y) \in f^{-1}(D)\} = (f^{-1}(D))_y$$

We have shown that if E is measurable, the E_y is measurable.

Step 2

Consider the functions

$$\begin{aligned} G: \quad & Y \to \mathbb{R} \cup \{+\infty\} \\ & y \mapsto \quad \int_X f(x,y) \mathrm{d}\mu(x) \\ F: \quad & X \to \mathbb{R} \cup \{+\infty\} \\ & x \mapsto \quad \int_Y f(x,y) \mathrm{d}\nu(y) \end{aligned}$$

we want to prove that they're both measurable

Let do this for G. Assume that $f = \chi_E$ for $E \in \Sigma_X \otimes \Sigma_Y$

$$\begin{aligned} & (\chi_E)_y(x)(x) = \chi_E(x,y) = 1 \\ \Leftrightarrow & (x,y) \in E \\ \Leftrightarrow & x \in E_y \\ \Leftrightarrow & \chi_{E_y}(x) = 1 \end{aligned}$$

This chain of implications shows that

$$(\chi_E)_y = \chi_{E_y}$$

Hence

$$G(y) = \int_X (\chi_E)_y \mathrm{d}\mu = \int_X \chi_{E_y} \mathrm{d}\mu = \mu(E_y)$$

And we have proved that such functions are measurable

Now assume

$$f = \sum_{i=1}^{n} a_k \chi_{E_k} \quad E_k \in \Sigma_X \otimes \Sigma_Y, \ a_k \in \mathbb{R}_{\geq 0}$$

and

$$f_y = \sum_{i=1}^{n} a_k \chi_{E_k \cap E_y}$$

then

$$\begin{aligned} G(y) &= \int_X f_y \mathrm{d}\mu \\ &= \sum_{k=1}^{n} a_k \int_X \chi_{(E_k)_y} \mathrm{d}\mu \\ &= \sum_{k=1}^{n} a_k \mu((E_k)_y) \\ &\Rightarrow \ G \text{ is measurable} \end{aligned}$$

Now assume f measurable

By lemma 53.23, $\exists \{f_n\}$ increasing sequence such that converges pointwisely to f

Moreover, $\{(f_n)_y\}_{n\in\mathbb{N}}$(all simple functions) converges to f_y too. Consider

$$\begin{aligned} g_n : Y &\rightsquigarrow \mathbb{R} \\ y &\mapsto \int_X (f_n)_y \mathrm{d}\mu \end{aligned}$$

Since f_n are simple. By the previous claim in step 2, we know that g_n isn measurable. And since $Im(g_n) \subseteq \mathbb{R}$

$$G(y) = \lim_{n\to+\infty} g_n(y) = \sup_{n\to\infty} g_n(y)$$

G is measurable.

Step 3

First we show that the equation in theorem holds for $f = \mathbb{1}_E$. By prop 53.10

$$\int_X (\int_y f_x \mathrm{d}\nu) \mathrm{d}\mu = \int_X \nu(E_x) \mathrm{d}\mu = (\mu \times \nu)(E)$$

while

$$\int_{X\times Y} f \mathrm{d}(\mu\times\nu) = \int_{X\times Y} \mathbb{1}_E \mathrm{d}(\mu\times\nu) = (\mu\times\nu)(E)$$

By two equations above:

$$\int_X (\int_Y f_x \mathrm{d}\nu) \mathrm{d}\mu = \int_{X\times Y} f \mathrm{d}(\mu\times\nu)$$

Second we then prove for ant measurable $f \geq 0$

There exists a increasing sequence $\{f_n\}_{n\in\mathbb{N}}$ of simple non-negative functions converges pointwisely to f. Then define

$$g_n(y) = \int_X (f_n)_y \mathrm{d}\mu$$

Note that

$$\int_{X\times Y} f\mathrm{d}\mu\times\nu = \int_Y (\int_X f_y \mathrm{d}\mu)\mathrm{d}\nu = \int_Y g_n \mathrm{d}\nu$$

take the limits

$$\begin{aligned}\int_{X\times Y} f\mathrm{d}(\mu\times\nu) &= \int_{X\times Y} \lim_{n\to+\infty} f_n \mathrm{d}(\mu\times\nu)\\ &= \int_Y \lim_{n\to+\infty} g_n \mathrm{d}\nu\\ &= \int_X \lim_{n\to+\infty} (f_n)_y \mathrm{d}\mu\\ &= \int_Y (\int_X f_y \mathrm{d}\mu)\mathrm{d}\nu\end{aligned}$$

53.25 Corollary

Fubini-Tobelli holds for $f \in L^1(X\times, \Sigma_X \otimes \Sigma_Y, \mu\times\nu)$

53.25.1 Proof

Apply the theorem to $f \vee 0$ and $-(f\wedge 0)$, since

$$f = f\vee 0 - (-(f\wedge 0)) = f\vee 0 + f\wedge 0$$

53.26 Remark

Deny of the corollary neither hold nor make sense. The integral gives either $+\infty$ or $-\infty$

53.27 Remark

If $X = Y = \mathbb{R}$ and $\Sigma = \mathscr{B}(\mathbb{R})$ For $f : U \subseteq \mathbb{R}^2 \to \mathbb{R},\ f \in L^1(I_{\lambda^2})$, you can find a rectangle $E \subseteq R = [a,b]\times[c,d]$ And define

$$\tilde{f}(x) = \begin{vmatrix} f(x) & \text{if } x\in E\\ 0 & \text{otherwise}\end{vmatrix}$$

Then apply Fubini-Tobelli theorem.

Example

- $E = \{(x, y) \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq x\}$

$$\iint_E \sim (x + y)\mathrm{d}x\mathrm{d}y$$

-

$$\begin{aligned}\iint_E \sin(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^{\frac{\pi}{2}} \int_0^x \sin(x+y)\mathrm{d}y\mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} -\cos(x+y) \mid_0^x \mathrm{d}x \\ &= \int_0^{\frac{\pi}{2}} -\cos(2x) + \cos(x)\mathrm{d}x \\ &= (-\frac{\sin(2x)}{x} + \sin x) \mid_0^{\frac{\pi}{2}} \\ &= 1\end{aligned}$$

53.28 Notation

$U \subseteq \mathbb{R}^n$ is an open set. $C_c^0(U)$ denotes the set of continuous functions $f : U \to \mathbb{R}$ that have compact support

$$Supp(f) := \{x \in U \mid f(x) \neq 0\}$$

53.29 Remark

Functions of $C_c^0(U)$ are measurable

Let $g : \underbrace{U}\subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. THen the Jacobian of g is the matrix

$$J_g(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{pmatrix}$$

where

$$g(x_1, \cdots, x_n) = \begin{pmatrix} g_1(x_1, \cdots, x_n) \\ \vdots \\ g_m(x_1, \cdots, x_n) \end{pmatrix}$$

The Jacobian is related to the differential of g. In fact

$$\begin{aligned}\mathrm{d}g_p : \mathbb{R}_p^n &\to \mathbb{R}_{g(p)}^m \\ \sigma &\mapsto J_g \mid_p (\sigma)\end{aligned}$$

53.30 Theorem(Change of variables for the Lebesgue integral)

Let $V \subseteq \mathbb{R}^n$ be an open set, and let $\varphi : V \to \mathbb{R}^n$ be a C^1-differ morphism, then

$$\int_{\varphi(V)} f \mathrm{d}\lambda^n = \int_V (f \circ g) \left|\det J_\varphi\right| \mathrm{d}\lambda^n \quad \forall f \in C_c^0(\varphi(V))$$

Proof

53.31 Remark

The theorem can be generalized to a bigger classes of functions. In fact, it possible to show:
It holds whenever one of the two integrals exists
(Zorich II)

53.32 Compute integrals in $\mathbb{R}^n$

53.32.1 Example

$f(x, y) = \frac{1}{1+x^2+y^2}$
$A = \{(x, y) \in \mathbb{R}^n \mid 0 < y < \sqrt{3}x; 1 < x^2 + y^2 < 4\}$ $\int_A f \mathrm{d}x\mathrm{d}y = ?$
Use polar coordinates

$$\begin{aligned} \varphi : [0, +\infty[\times[0, 2\pi[\to & \quad \mathbb{R}^2 \\ (\rho, \theta) & \mapsto (\rho\cos\theta, \rho\sin\theta) \end{aligned}$$

φ is C^1-differentiable.
use the theorem

$$\begin{aligned} \int_A f(x, y) \mathrm{d}x\mathrm{d}y &= \int_{\tilde{A}} (f \circ \varphi) \left|\det\right| J_\varphi \mathrm{d}\rho\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{3}} \int_1^2 \frac{\rho}{1+\rho^2} \mathrm{d}\rho\mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{3}} \left[\frac{1}{2}\ln(1+\rho^2)\right]_1^2 \mathrm{d}\theta \\ &= \frac{\pi}{6}\ln(\frac{5}{2}) \end{aligned}$$

53.33 Def

$\omega = \sum a_i x_i \in \Omega^n(U)$ $\gamma : [a, b] \to U$ is piecewise of class C^1 Then we have defined $\int_\gamma \omega$ The fact that γ is differentiable is important thus we need $\gamma^*\omega$

Let

$$\varphi : \tau = [c, d] \to t = [a, b]$$

is a C^1-differmorphism. We say that φ preserves the orientation if $\varphi' > 0$, we say that φ reveres the orientation if $\varphi' < 0$

Assume it preserves orientation

$$\begin{aligned}
\int_\gamma \omega &= \int_a^b \left(\sum_i a_i(\gamma(t)) \cdot \frac{\mathrm{d}x_i}{\mathrm{d}t} \right) \mathrm{d}t \\
&= \gamma(t) = (x_1(t), \cdots, x_n(t)) \\
&= \int_a^b \left(\sum_i a_i \gamma(\varphi(\tau)) \frac{\mathrm{d}x_i}{\mathrm{d}\tau} \underbrace{\frac{\tau}{t}} = \left| J_{\varphi^{-1}} \right| \right) \mathrm{d}t \\
\text{the change of variables } &= \int_c^d \sum_i a_i \gamma(\varphi(\tau)) \frac{\mathrm{d}x_i}{\mathrm{d}\tau} \mathrm{d}\tau \\
&= \int_{\gamma \circ \varphi}
\end{aligned}$$

We call $\gamma \circ \varphi$ a reparameterization of the curve γ, with the C^1-differ φ. If φ preserves the orientation, then

$$\int_\gamma \omega = \int_{\gamma \circ \varphi} \omega$$

if reverse

$$\int_\gamma \omega = -\int_{\gamma \circ \varphi} \omega$$

53.34 Def

$\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i \in \Omega^1(U)$

- We say that ω is **closed** if $\mathrm{d}\omega = 0$
- We say that ω is **exact** in $V \subseteq U$ if there exists a mapping $f : V \to \mathbb{R}$ s.t. $\omega = \mathrm{d}f$ in V

Goal:

to relate the notions of exact forms/closed forms/integrals along curves.

53.35 Def

Let X be topological space. $U \in X$ is connected if cannot be written as disjoint union of non-empty open sets.

Equally, $U \in X$ is connected if

$$U = A \sqcup B \Rightarrow A = \varnothing \text{ or } B = \varnothing$$

53.36 Lemma

Let $U \subseteq \mathbb{R}^n$ be a connected open set. Then any two points of U can be joined by a piecewise C^1-curve.

Proof

Take $a \in U$ let $H \subseteq U$ the set of points that can be joined to a with a piecewise C^1-curve. Let $K = U \setminus H$.

Take $x \in H$ then $\exists \mathcal{B}(x, \epsilon) \subseteq U$ since U is open

Any two points in $\mathcal{B}(x, \epsilon)$ can be jointed with a segment. Take any $y \in \mathcal{B}(x, \epsilon)$, this y can be joined to a with a piecewise C^1-curve.

This means that H is open. Similarly K is also open. Since $U = H \sqcup K$ and for U connected $H = U$

53.37 Notation

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve, we define:

$$(-\gamma)(t) = \gamma(a + b - t)$$

as reserved curve of γ

$$\int_{-\gamma} \omega = -\int_{\gamma} \omega$$

53.38 Def

If $\gamma_1 : [a, b] \to U$ and $\gamma_2 : [b, c] \to U$ are two curves and $\gamma_1(b) = \gamma_2(b)$, we define

$$\begin{aligned} \gamma_1 \sqcup \gamma_2 : [a, c] &\to U \\ t &\mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t) & \text{if } t \in [b, c] \end{cases} \end{aligned}$$

One has

$$\int_{\gamma_1 \sqcup \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

53.39 Theorem

The following statements are equivalent:

1 ω is exact in a connected open set $V \subseteq U$

2 $\int_\gamma \omega$ depends only on the end-point of γ ($\forall \gamma$ in V)

3 $\int_\gamma \omega = 0$ for all closed curves γ in V

Proof

$1 \to 2$ Since $\omega = \mathrm{d}f$

$$\int_\gamma \omega = \int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(b))$$

$2 \to 3$ Take a closed curve such that $\gamma(a) = \gamma(b) = p \in V$. Take the curve $\gamma' = \{p\}$

$$\int_\gamma \omega = \int_{\{p\}} \omega = 0$$

$3 \to 2$ consider $\gamma \sqcup -\gamma$, trivial

$2 \to 1$ FIx $p \in V$ For any $(p+t) \in V$ by the lemma 53.36, there exists a curve γ_x piecewise C^1 that connects p and $(p+t)$
Let

$$\begin{aligned} f : V &\to \mathbb{R} \\ x &\mapsto \int_{\gamma_x} \omega \end{aligned}$$

Take $\omega = \sum_{i=1}^{n} a_i \mathrm{d}x_i$, one knows that

$$\inf_{x \in [p_i, p_i + t_i]} a_i(x) \leq \int_{p_i}^{p_i + t_i} a_i \mathrm{d}x_i \leq \sup_{x \in [p_i, p_i + t_i]} a_i(x)$$

Since a_i are continuous

$$\lim_{t_i \to 0} \inf_{x \in [p_i, p_i + t_i]} a_i(x) = \lim_{t_i \to 0} \sup_{x \in [p_i, p_i + t_i]} a_i(x) = a_i(p_i)$$

This means that for any curve γ with end points and $p + t$

$$\int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

Hence

$$o(\|t\|) + \mathrm{d}_p f(t) = f(p+t) - f(p) = \int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

which ends the proof.

53.40 Poincare Lemma

Let ω be a 1-form. Then $\mathrm{d}\omega = 0$ iff

$$\forall p \in U \quad \exists V \in \mathcal{V}_p, f : V \to \mathbb{R} \in C^1(\mathbb{R})$$

such that

$$\mathrm{d}f = \omega$$

Proof

Assume that ω is locally exact, then locally

$$\omega = \mathrm{d}f$$

Assume that $U = \bigcup_\alpha V_\alpha$, ω is exact,d on V_α is $\mathrm{d}(\mathrm{d}f) = 0$

Assume that $\mathrm{d}\omega = 0, \forall p = (p_1, \cdots, p_n) \in U$, consider an open ball $\mathcal{B}_p$ centered on p and all contained in V

Fix an $x \in \mathcal{B}_p$ Consider $B(t) = p + t(x-p), t \in [0,1]$ Let

$$f(x) = \int_{B(t)} \omega = \int_0^1 a_i(B(t))(x_i - p_i)\mathrm{d}t$$

We put $V = B(p)$ and show that $\mathrm{d}f = \omega$
Note that

$$0 = \mathrm{d}\omega = \sum_i \mathrm{d} \wedge \mathrm{d}x_i = \sum_i \left(\sum_j \frac{\partial a_i}{\partial x_j} \mathrm{d}x_j \right) \mathrm{d}x_i = \sum_{ij} \frac{\partial a_i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i$$

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}$$

Consider $x_i = x_1$

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &= \int_0^1 \left(\frac{\partial a_1}{\partial x_1} t(x_1 - p_1) + a_1 + \sum_{i>1} \frac{\partial a_i}{\partial x_1} t(x_i - p_i) \right) \mathrm{d}t \\
&= \int_0^1 \left(\frac{\mathrm{d}t}{\mathrm{d}} (a_1 B(t))t + a_1 \right) \mathrm{d}t \\
&= \int_0^1 \frac{\mathrm{d}t}{\mathrm{d}} \left(a_1 \circ B(t) \cdot t \right) \mathrm{d}t \\
&= a_1 \circ B(1) \\
&= a_1(x)
\end{aligned}$$

When locally exact implies exact depends on the topological properties of U

53.41 Notation

For any mapping $\gamma : [a, b] \to U$

- γ is called a closed curve if $\gamma(a) = \gamma(b)$ and γ is a curve
- γ is called a path if γ is of class C^0
- γ is called a loop if γ is a closed path

We try to integrate forms along paths.

53.42 Def

Let ω be a closed 1-form $\gamma : [a, b] \to U$ be a path. We derive a partition of $[a, b]$:

$$0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} < b$$

such that $\gamma_i := \gamma_{t_i, t_{i+1}}$ form a finite open covering

$$\bigcup_i B_i \supseteq \gamma([a, b])$$

We may assume that $\gamma_i \subseteq B_i$. From Poincare lemma, we find a function $f_i : B_i \to \mathbb{R}$ s.t. on B_i

$$\mathrm{d}f_i = \omega$$

We define the integral along path γ as

$$\int_\gamma \omega := \sum_i f_i(t_{i+1}) - f_i(t_i)$$

53.43 Prop

The def of path integral is unique and coincides with the usual definition 52.2

Proof

Let $\mathscr{P} = \{t_1, \cdots, t_n\}$ be a partition of $[a, b]$ and $\mathscr{P}'$ be a refinement of $\mathscr{P}$, namely $\mathscr{P} \in \mathscr{P}'$. We know that $\exists t' \in \mathscr{P}' \setminus \mathscr{P}$ s.t. $t' \in]t_i, t_{i+1}[, \gamma(t') \subseteq B_i$

Compute the integral with the partition $\mathscr{P}'$, in the summation:

$$f_i(\gamma_i(t_{i+1})) - f_i(\gamma_i(t')) + f_i(\gamma_i(t')) + c - f_i(\gamma_i(t_i)) - c$$

This means that if $\mathscr{P}' \subseteq \mathscr{P}$ then

$$\int_\gamma^{(\mathscr{P})} \omega = \int_\gamma^{(\mathscr{P}')} \omega$$

For any two partition $\mathscr{P}_1$ and $\mathscr{P}_2$, find the common refinement

$$\int_\gamma^{(\mathscr{P}_1)} \omega = \int_\gamma^{(\mathscr{P}_1 \cup \mathscr{P}_2)} = \int_\gamma^{(\mathscr{P}_2)} \omega$$

53.44 Def

Let $\gamma_0, \gamma_1 : [a,b] \to U$ be two paths. A **homotopy** between γ_0 and γ_1 is a continuous mapping:

$$\begin{aligned} H :&[a,b] \times [0,1] &\to& U \\ &(s,t) &\mapsto& H(s,t) \end{aligned}$$

such that

- $H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$
- $H(a, \cdot) \equiv \gamma_0(a) = \gamma_1(a) \quad H(b, \cdot) \equiv \gamma_0(b) = \gamma_1(b)$

53.45 Def: Lebesgue number

Let (X, ρ) be a metric space and $\mathcal{U} = \{U_i\}$ be an open covering X

A **Lebesgue number** $\delta = \delta_{\mathcal{U}}$ (of the open covering $\mathcal{U}$) is a non-negative number that:

If $Z \subseteq X$ is a subset with $diam(Z) < \delta$, then $Z \subseteq U_j$ for some $U_j \in \mathcal{U}$

Remark

- $\delta' < \delta$ is also a Lebesgue number
- In principle, a Lebesgue number δ can be 0

53.46 Lemma

If X is compact, then for any open covering there exists a positive Lebesgue number.

Proof

Let $\mathcal{U}$ be am open covering. Since X is compact, then

$$\mathcal{U} \supseteq \{U_1, \cdots, U_n\}$$

If one of U_i is equal to X, then any $\delta > 0$ is a Lebesgue number. So we can assume that $\forall i \in \{1, \cdots, n\}$ the set

$$C_i := X \setminus U_i$$

is non-empty. let the mapping f be:

$$\begin{aligned} f: X &\to \mathbb{R} \\ x &\mapsto \frac{1}{n}\sum_{i=1}^{n} d(x, C_i) \end{aligned}$$

f is continuous on a compact set. Hence it attains a minimum. This minimum is not 0 because $d(x, C_i) > 0$ for some i

Let $\delta = \min_{x\in X} f(x)$, we show that δ is a Lebesgue number

Let $Y \subseteq X$ s.t. $diam(Y) < \delta$. Take $x_0 \in Y$, then

$$Y \subseteq \mathcal{B}(x_0, \delta)$$

Since $f(x_0) \geq \delta$, then there exists i such that

$$d(x_0, C_i) \geq \delta$$

(otherwise $f(x_0) = \frac{1}{n}\sum\limits_{i=1}^{n} d(x_0, C_i) < \frac{1}{n} n\delta = \delta$)

this means

$$(X \setminus U_i) \cap Y = \varnothing$$

Hence $Y \subseteq U_i$

Exercise

Being homotopic is an equivalence relation.

53.47 Theorem(homotopy invariance of the integrals)

Ler ω be a closed form on an open set U. Let γ_0, γ_1 be homotopy paths in U, then

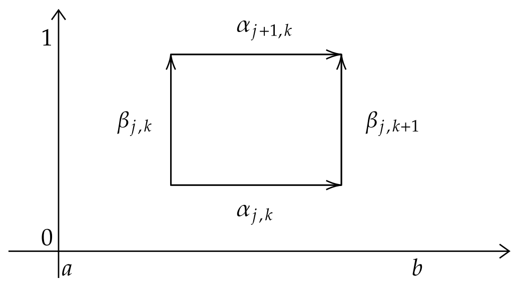
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

Proof

Since ω is closed, it's locally exact (by Poincare lemma53.40) Let $H : [a, b] \times [0, 1] \to U$ be the homotopy between γ_0 and γ_1. Let $\mathscr{B} = \{B_i\}$ be a open cover of $Im(H) \subseteq U$ made of finite many open balls (since compact) where ω is locally exact in each.

Consider $W_i = H^{-1}(B_i)$ We have covered the departure set $\mathscr{D}_H$ with $\{W_i\}$. Since $\mathscr{D}_H$ is compact, we can choose a Lebesgue number $\delta > 0$ for the covering $\{W_i\}$ (by lemma 53.36)

Divide $\mathscr{D}_H$ into rectangles $\{R_jk\}$ having diameter $< \delta$



The border of the rectangles are loops. We divide the loops in this way:

$$\partial R_{j,k} = \alpha_{j,k} \sqcup \beta_{j,k+1} \sqcap (-\alpha_{j+1,k}) \sqcup (-\beta_{j,k})$$

$$H\partial R_{j,k} = H\alpha_{j,k} \sqcup H\beta_{j,k+1} \sqcap H(-\alpha_{j+1,k}) \sqcup H(-\beta_{j,k})$$

$H\partial R_{j,k}$ is closed curve contained in some B_i, but in such balls ω is exact, then $\int_{\partial R_{j,k}} \omega = 0$. Do this $\forall j, k$

$$0 = \sum_{j,k} \int_{\partial R_{j,k}} \omega = \sum_{j,k} \left(\int_{H\alpha_{j,k}} \omega + \int_{H\beta_{j,k+1}} \omega - \int_{H\alpha_{j+1,k}} \omega - \int_{H\beta_{j,k}} \omega \right)$$

Moreover, if we do this for some particular j, k

$$0 = \int_{H(a,0)\rightsquigarrow(b,0)} \omega + \int_{H(b,0)\rightsquigarrow(b,1)} \omega - \int_{H(a,1)\rightsquigarrow(b,1)} - \int_{H(a,0)\rightsquigarrow(a,1)}$$

Since $H(a,0) \rightsquigarrow (b,0)$ and $H(a,1) \rightsquigarrow (b,1)$ are points, then

$$\int_{H(a,0)\rightsquigarrow(a,1)} \omega = \int_{H(b,0)\rightsquigarrow(b,1)} \omega$$

Proved

Chapter 54

Winding Numbers

54.1 Def: Free Homotopy

Let $\gamma_0, \gamma_1 : [a,b] \to U$ be two loops (namely $\gamma(a) = \gamma(b)$)
A **free homotopy** between γ_0 and γ_1 is a continuous mapping:

$$\begin{aligned} H : &[a,b] \times [0,1] &&\to U \\ &(s,t) &&\mapsto H(s,t) \end{aligned}$$

such that

- $$H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$$

- For any fixed t_0
$$H(\cdot, t_0)$$
is a loop

54.2 Notation

A path $\gamma : [a,b] \to I$ is said simple if $\gamma \mid_{]a,b[}$ is injective (No self-cross this is)

54.3 Jordan Theorem

Let γ be a simple loop $\gamma : [a,b] \to U$, then $\mathbb{R}^2 \setminus \gamma([a,b])$ consists exactly of two connected components. One of this is bounded (interior), the other one unbounded (exterior). Moreover $\gamma([a,b])$ is the boundary of two components.

Proof

Exercise?
Let $S^1 \subseteq \mathbb{R}$ be the unit circle. Let $c : I \to S^1$ be a closed curve. $(0 \in I)$
We want to measure the "net" turns of c around the region.

$$c(t) = (x(t, y(t)), x^2 + y^2 = 1 \quad \forall t \in I)$$

$$\Rightarrow 2(xx' + yy') = 0$$

Consider $\varphi_0 \in [0, 2\pi[$ such that

$$\cos(\varphi_0) = x(0), \sin(\varphi_0) = y(0)$$

Define the function

$$\begin{aligned} \varphi : I &\to [0, 2\pi] \\ t &\mapsto \varphi_0 + \int_0^t (xy' - yx')\mathrm{d}t \end{aligned}$$

φ is called an angle function of c, namely is of class C^0 and $c(t) = (\cos(\varphi(t)), \sin(\varphi(t)))$
And

$$F(t) = (x(t) - \cos\varphi(t))^2 + (y(t) - \sin\varphi(t))^2 \quad t \in I \cap [0, 2\pi]$$
$$F'(t) = -y \sin\varphi(yy' + xx') - x\cos\varphi(xx' + yy') = 0$$

This implies $F(t) = C$ a constant. However

$$F(0) = (x(0) - x(0))^2 + (y(0) - y(0))^2 = 0$$

So $C = 0$ It means that

$$\begin{cases} \cos(\varphi(t)) = x(t) \\ \sin(\varphi(t)) = y(t) \end{cases} \quad \forall t \in I$$

54.4 Def

Let $c : [a, b] \to S^1$ be a closed curve. Let φ be the angular function of c. We define the winding number of c as:

$$n(c) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

Since c us a closed curve, $n(c) \in \mathbb{Z}$

54.5 Def

Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\}$ be a closed curve. $(\gamma_p + \rho(t)c(t))$, when $c(t) \in S^1$

$$\gamma(t) = p + \rho(t)(\cos(\theta(t)) + \sin(\theta(t)))$$

Then we define the winding number of γ at p

$$n_p(\gamma) := n(c)$$

54.6 Prop

Let $\gamma = p + \rho(t)c(t)$ be a closed curve $\gamma : [a,b] \to \mathbb{R}^2 \setminus \{p\}$ then

$$n_p(\gamma) = \frac{1}{2\pi i}\int_C \omega_0$$

where

$$\omega_0 = -\frac{y}{x^2+y^2}\mathrm{d}x + \frac{x}{x^2+y^2}\mathrm{d}y$$

Proof

$$\begin{aligned}\frac{1}{2\pi}\int_C \omega_0 &= \frac{1}{2\pi}\int_a^b (x(t)y'(t) - y(t)x'(t))\mathrm{d}t \\ &= \frac{\varphi(b)-\varphi(a)}{2\pi} \\ &= n(c)\end{aligned}$$

54.7 Prop

Let $\gamma_0, \gamma_1 : [0,b] \to \mathbb{R}^2 \setminus \{p\}$ be two closed curves. Then they're freely homotopic iff

$$n_p(\gamma_0) = n_p(\gamma_1)$$

Proof

$\Rightarrow$ Follows fromm the invariance of the integral of ω_0 along free homotopic loops. (We use the previous prop)

$\Leftarrow$ Assume $\gamma_0 = p + \rho_0 c_0(t); \gamma_1 = p + \rho_1 c_1(t)$. The angle functions of c_0 and c_1 are φ_0 and φ_1 respectively. Consider

$$\varphi(s,t) = (1-t)\varphi_0(s) + t\varphi_1(s)$$

$(s,t) \in [0,b] \times [0,1]$

$$H(s,t) = (\cos(\varphi(s,t)), \sin(\varphi(s,t)))$$

To claim this is a free homotopy between c_0 and c_1, we have to check that any curve in the homotopy is closed.

$$\begin{aligned}\varphi(b,t) - \varphi(0,t) &= (1-t)\varphi_0(b) + t\varphi_1(b) - (1-t)\varphi_0(0) - t\varphi_1(0) \\ &= (1-t)(\varphi_0(b) - \varphi_0(0)) + t(\varphi_1(b) - \varphi_1(0)) \\ &= 2\pi(1-t)n_p(\gamma_0) + 2\pi t n_p(\gamma_1) \\ &= 2\pi n_p(\gamma_0)\end{aligned}$$

So we have proved that c_0 and c_1 are freely homotopic. What we anticipate is

$$\gamma_i \sim_{hom} \hat{\gamma_i} = p + c_i \sim_{hom} c_i$$

Now we only need to prove the first homotopic.

$$H(s,t) = p + \frac{\rho_1(s)}{(1-t) + t\,|\rho_1(s)|} c_i(s)$$

54.8 Def

Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differential mapping. We say that $p \in U$ is a zero of F if $F(p) = 0$. If then exists a neighborhood V of p such that V contains no zero of F other then p, then p is called isolated zero.

If p is a zero of F and $\mathrm{d}F\mid_p$ is non singular at p, then we say that p is a simple zero.

54.9 Remark

By the inverse function then F is one to one in a neighborhood of a simple zero. Hence a simple zero if isolated.

$$F(x,y) = (f(x,y), g(x,y))$$

$D \subseteq U$ is a closed disk, with boundary $\partial D = C$ Assume that C doesn't contain zeros of F. Consider the form

$$\theta = \frac{f\mathrm{d}g - g\mathrm{d}f}{f^2 + g^2} \in \Omega^{\vee}(U \setminus \{(x,y)\} : F(x,y) = 0)$$

54.10 Def

The index of F in D, is defined as

$$n(F,D) := \frac{1}{2\pi}\int_C \theta$$

See that $\theta = F^*\omega_0$, $\omega_0 = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}$

$$\begin{aligned} n(F,D) &= \frac{1}{2\pi}\int_C \theta \\ &= \frac{1}{2\pi}\int_C F^*\omega_0 \\ &= \frac{1}{2\pi}\int_{F\circ} \omega_0 \\ &= (\text{winding number of } F\circ C \text{ at the center of } FD) \end{aligned}$$

54.11 Remark

$$n(F, D) = \frac{1}{2\pi}\int_C \theta = \frac{1}{2\pi}\int_{F\circ C} \omega_0$$

54.12 Prop

If $n(F, D) \neq 0$ then $\exists q \in D$ s.t. $F(q) = 0$

Proof

Assume that such q doesn't exist. Let p be the center of D

$$H(s, t) = F((1 - t)C(s) + t \cdot p)$$

then H is a free homotopy between $F \circ s \mapsto p$ and $\circ C$

$$0 = \frac{1}{2\pi}\int_F \circ C\omega_0 = n(F, D)$$

Contradiction

54.13 Def

A simple zero p of F is said **positive** if $\det(\mathrm{d}_p F) > 0$, otherwise is said **negative** ?(what's =0?)

54.14 Kronecker Index Theorem

Assume that $F; U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ has only finite simple zeros in a disk $D \subseteq U$ and none of them in ∂D. Then

$$n(F, D) = P - N$$

where P is the number of positive simple zeros and N is the number of negative simple zeros.

Lemma

Assume that F has a simple zero $p \in D \subseteq U$ then $n(F, D) = \pm 1$ corresponding to $\det(\mathrm{d}_p F) > 0$ or $\det(\mathrm{d}_p F) < 0$

Proof

After translating, we assume that $p = (0,0)$ By definition of differential

$$F(q) = Tq + R(q)\,|q| \quad q \to 0 \quad \lim_{q\to 0} R(q) = 0$$

Consider the mapping

$$\begin{aligned} H :& [0,b] \times [0,1] && \to \mathbb{R}^2 \\ & (s,t) && \mapsto Tq + (1-t)R(q)\,|q| \end{aligned}$$

If we show that for D small enough, $H(q,t) \neq 0, \forall q \in D, t \in [0,1]$ $(*)$ Then $H(C(s),t)$ is a free homotopy between $F_0 \circ C$ and $T \circ C$ (Exercise) Then

$$n(F,D) = n(T,D)$$

which one should prove is $n(T,D) = 1$

For the part $(*)$, since p is non-singular, $c = \frac{1}{\|T^{-1}\|} > 0$, then

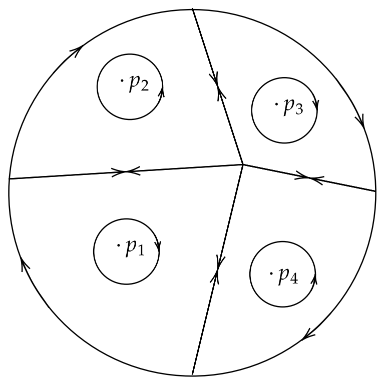
$$\|q\| = \left\|T^{-1}Tq\right\| \leq \left\|T^{-1}\right\| \|Tq\| = \frac{Tq}{c}$$

Take $\epsilon > 0$ such that $\forall p \in D$ (disk of radius $< \epsilon$)$\|R(q)\| \leq \frac{c}{2}$ Then if $q \in D_\epsilon \setminus \{0,0\}$

$$\begin{aligned} \|H(q,t)\| &= \|Tq + (1-t)R(q)\,\|q\|\| \\ &\geq \|Tq\| - (1-t)\,\|R(q)\|\,\|q\| \\ &\geq c\,\|q\| - \frac{(1-t)c}{2}\,\|q\| \\ &\geq c\,\|q\| - \frac{c}{2}\,\|q\| > 0 \end{aligned}$$

Proof

Let $p_1, \cdots, p_k$ be the zeros of F in D. Since these zeros are isolated, we choose a set of balls $\{B_i\}_{i=1}^k$ containing $\{p_i\}_{i=1}^k$ respectively. Then cut the circle into k sections correspondingly.



One can easily construct the homotopy for these balls and sections.
By lemma

$$\int_{F \circ C} \omega_0 = \sum_{i=1}^{k} \int_{FB_i} \omega_0 = \sum_{i=1}^{k} sgn(\det \mathrm{d}_{p_i} F) = P - N$$

54.15 Lemma

Let $\gamma : [a, b] \to \mathbb{R}^n$

1 If $\gamma \in C^1([a, b])$ then

$$\int_a^b \gamma'(t)\mathrm{d}t = \gamma(b) - \gamma(a)$$

2 For any $c \in \mathbb{R}^n$:

$$\left\langle c, \int_a^b \gamma(t)\mathrm{d}t \right\rangle = \int_a^b \langle c, \gamma(t) \rangle \,\mathrm{d}t$$

3

$$\left\| \int_a^b \gamma(t)\mathrm{d}t \right\| \leq \int_a^b \|\gamma(t)\| \,\mathrm{d}t$$

Proof

1 Is consequence of the fundamental theorem of calculus computation

2

$$\left\langle c, \int_a^b \gamma(t)\mathrm{d}t \right\rangle = \sum_{i=1}^n c_i \int_a^b \gamma_i(t)\mathrm{d}t$$
$$= \int_a^b \sum_i c_i\gamma_i(t)\mathrm{d}t$$
$$= \int_a^b \langle c, \gamma(t)\rangle \mathrm{d}t$$

3 We apply (2) with $c = \int_a^b \gamma(t)\mathrm{d}t \in \mathbb{R}^n$

$$\left\|\int_a^b \gamma(t)\mathrm{d}t\right\| = \left\langle \int_a^b \gamma(\tau)\mathrm{d}\tau, \int_a^b \gamma(t)\mathrm{d}t \right\rangle$$
$$\text{(by (2))} = \int_a^b \left\langle \int_a^b \gamma(\tau)\mathrm{d}\tau, \gamma(t) \right\rangle \mathrm{d}t = (*)$$

Apply Cauchy-Schwartz inequality

$$(*) \le \int_a^b \left\|\int_a^b \gamma(\tau)\mathrm{d}\tau\right\| \|\gamma(t)\| \,\mathrm{d}t = \left\|\int_a^b \gamma(\tau)\mathrm{d}\tau\right\| \int_a^b \|\gamma(t)\| \,\mathrm{d}t$$

Divide by $\left\|\int_a^b \gamma(\tau)\mathrm{d}\tau\right\|$

54.16 Def

Let $\mathscr{P} = \{t = t_a, t_1, \cdots, t_n = b\}$, $p_i = \gamma(t_i)$

$$l_{\mathscr{P}}(\gamma) = \sum_{i=0}^n \|p_{i+1} - p_i\|$$

The length of γ is

$$l(\gamma) := \sup_p \{l_p(\gamma)\}$$

If $l(\gamma) < +\infty$, then path γ is said rectifiable.

54.16.1 Example

Consider $\gamma : [0,1] \to \mathbb{R}$ $\gamma_1(t) = t$ $\gamma_2(t) = \begin{cases} t\sin\frac{\pi}{2t} & \text{if } t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\mathscr{P}_n\{0, \frac{1}{2n+1}, \frac{1}{2n-1}, \cdots, \frac{1}{3}, 1\}$$

Recall that $\sin((j-\frac{1}{2})\pi) = (-1)^{j+1}$, then for $i > 0$,

$$p_i = \gamma(t_i) = \left(\frac{1}{2j-1}, \frac{1}{2j-1}(-1)^{j+1}\right)$$

$$\begin{aligned}
\|p_i - p_j\| &= \sqrt{(\frac{2}{4j^2-1})^2 + (\frac{4j}{4j^2-1})^2} \\
&= \sqrt{\frac{4(4j^2+1)}{(4j^2-1)^2}} \\
&\geq \sqrt{\frac{4(4j^2-1)}{(4j^2-1)^2}} \\
&= \sqrt{\frac{4}{4j^2-1}} \\
&\geq \frac{1}{j}
\end{aligned}$$

then

$$l_p(\gamma) \geq \sum_{j=1}^{n} \frac{1}{j} \Rightarrow l(\gamma) = +\infty$$

54.17 Prop

Let $\gamma : [a,b] \to \mathbb{R}^n$ be of class C^1, then γ is rectifiable and

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

moreover $l(\gamma)$ doesn't depend on the parametrization of γ

Proof

Let $\mathscr{P}$ be a partition of $[a,b]$.

$$\|\gamma(t_{j+1}) - p_j\| = \left\|\int_{t_i}^{t_{j+1}} \gamma'(t)\mathrm{d}t\right\| \leq \int_{t_i}^{t_{j+1}} \|\gamma'(t)\| \,\mathrm{d}t$$

So

$$\begin{aligned}
l_{\mathscr{P}}(\gamma) &= \sum_{j=0}^{n-1} \|\gamma(t_{j+1} - \gamma(t_j))\| \\
&\leq \sum_{j=0}^{n-1} \int_{t_i}^{t_{j+1}} \|\gamma'(t)\| \,\mathrm{d}t \\
&= \int_a^b \|\gamma'(t)\| \,\mathrm{d}t
\end{aligned}$$

take sup over $\mathscr{P}$

$$l(\gamma) \leq \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

which means $l(\gamma)$ is finite.

Let's prove the other inequality:

$\gamma'(t)$ is continuous on $[a,b]$ then is uniformly continuos(Cantor-Iteiner Theorem): $\forall\epsilon > 0 \exists\delta = \delta(\epsilon)$ such that if $t,s \in [a,b]$ and $|t-s| < \delta$. Then $\|\gamma'(t) - \gamma'(s)\| < \epsilon$

Choose now a partition $\mathscr{P}$ such that $|t_{j+1} - t_j| < \delta \ \forall i$ then if $t \in [t_i, t_{i+1}]$ then$\|\gamma'(t) - \gamma'(t_j)\| < \epsilon$

$$\begin{aligned}\|\gamma'(t)\| - \|\gamma'(t_j)\| &\leq |\|\gamma'(t)\| - \|\gamma'(t_j)\|| \\ &\leq \|\gamma'(t) - \gamma'(t_j)\| \\ &\leq \epsilon\end{aligned}$$

Hence

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \|\gamma'(t)\| \,\mathrm{d}t &\leq \|\gamma'(t_j)\| (t_{j+1} - t_j) + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t) + \gamma'(t_j))\mathrm{d}t \right\| + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} \gamma'(t)\mathrm{d}t \right\| + \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t_j))\mathrm{d}t \right\| + (t_{j+1} - t_j)\epsilon \\ &\underbrace{\leq \|\gamma(t_{j+1} - \gamma(t_j))\|}_{\text{lemma 1}} + \underbrace{\int_{t_j}^{t_{j+1}} \|\gamma'(t) - \gamma'(t_j)\| \,\mathrm{d}t}_{\text{lemma 2}} + (t_{j+1} - t_j)\epsilon \\ &\leq \|\gamma(t_{j+1} - \gamma(t_j))\| + 2(t_{j+1} - t_j)\epsilon\end{aligned}$$

Hence

$$\begin{aligned}\int_a^b \|\gamma'(t)\| \,\mathrm{d}t &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| \,\mathrm{d}t \\ &\leq \sum_{j=0}^{n-1} \|\gamma(t_{j+1}) - \gamma(t_j)\| + 2(t_{j+1} - t_j)\epsilon \\ &= l_{\mathscr{P}}(\gamma) + 2\epsilon(b-a) \leq l(\gamma) + 2\epsilon(b-a)\end{aligned}$$

Take limit for $\epsilon \to 0$

Now the formula proved. We'll show that $l(\gamma)$ doesn't depends on the parametrization. Let $\varphi : [\alpha,\beta] \to [a,b]$ is of class C^1, $\varphi'(\tau) \neq 0, \forall\tau \in [\alpha,\beta]$ Let

$$\tilde{\gamma} = \gamma \circ \varphi$$

might as while assume $\varphi' > 0$

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t = \int_\alpha^\beta \|\gamma'(\varphi(\tau))\| \, \varphi'(\tau) \mathrm{d}\tau$$
$$= \int_\alpha^\beta \left\|\tilde{\gamma}'(\tau)\right\| \mathrm{d}\tau \qquad = l(\tilde{\gamma})$$

Same for $\varphi' < 0$

54.18 Corollary(exercise)

If γ is a curve (piecewise C^1), then γ is rectifiable and the length is the sum of the length of it's C^1 pieces.

54.19 Def

A C^1-curve is **regular** if $\gamma'(t) \neq 0$ for any $t \in [a, b]$ A piecewise C^1-path (curve) is regular if all its pieces are regular

54.20 Def

$$N := \frac{T}{\|T\|}$$

is **normal vector** of T

54.21 Def

Let $\gamma : [a, b] \to \mathbb{R}^n$ a C^1 curve; Let l be the length of γ (by theorem proved $l(\gamma) < +\infty$) Let's define the following function:

$$s(t) := \int_a^t \|\gamma'(t)\| \, \mathrm{d}u$$

$s(t)$ is the length of $\gamma \mid_{[a,t]}$ THe function $\|\gamma'(u)\|$ iss continuous, hence

$$s'(t) = \|\gamma'(t)\|$$

Now assume that γ is C^1 and **regucar**($\gamma'(t) \neq 0, \forall t \in [a, b]$), then $s'(t) > 0$

So $s : [a, b] \to [0, l]$ is a C^1-differmorphism, the inverse is

$$t : [0, l] \to [a, b]$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = frac1\|\gamma'(t)\|$$

We reparameterize γ with t and get

$$\tilde{\gamma}(s) = (\gamma \circ t)(s)$$

$\tilde{\gamma} : [0, l] \to \mathbb{R}^n$ we say that $\tilde{\gamma}$ is the reparameterization of γ with respect to its **curvilinear coordinate** $s(t)$

54.22 Def

$f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ f is a $C^{(k)}$-differ if

- f is of class $C^{(k)}$
- f is bijection, and the inverse is $C^{(k)}$

54.23 Def

In general

$$\gamma : [a, b] \to \mathbb{R}^n \rightsquigarrow \tilde{\gamma} : [0, l] \to \mathbb{R}^n$$

regular and C^1

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$\left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}\right\| = 1$

$$T(t) := \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

tangent: (vector) $\to$ vector of norm 1

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\|T(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle T(t), T(t)\rangle = 2\langle T(t), T'(t)\rangle \Leftrightarrow T'(t) \perp T(t)$$

use the fact that in $\mathbb{R}^n$, $u, v : \mathbb{R} \to \mathbb{R}^n$ differentiable

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v(t)\rangle = \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v(t)\right\rangle + \left\langle u(t), \frac{\mathrm{d}v}{\mathrm{d}t}\right\rangle$$

then

$$\begin{aligned}
\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}(T(t)) \\
&= \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \\
&= \frac{T'(t)}{\|\gamma'(t)\|}
\end{aligned}$$

$N(t) = \frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} / \left\| \frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} \right\|$. If $n = 2$ Along the curve we have a 'moving' canonical basis of

$$\begin{aligned}\mathbb{R}^2_{\gamma(t)} =& \{T(t), N(t)\} \\ =& \{\alpha T(t) + \beta N(t) \mid \alpha, \beta \in \mathbb{R}\}\end{aligned}$$

$\{T(t), N(t)\}$ is a orthonormal basis of $\mathbb{R}^2_{\gamma(t)}$

Chapter 55

Curvilinear integral

Let $f : U \in \mathbb{R}^n \to \mathbb{R}$ measurable and $\gamma : [a, b] \to \mathbb{R}^n$ regular

55.1 Def

The curvilinear integral of f along γ is defined as:

$$\int_\gamma f(t)\mathrm{d}s = \int_a^b f(\gamma(t))s'(t)\mathrm{d}t = \int_a^b f(\gamma(t)) \|\gamma'(t)\| \,\mathrm{d}t$$

where $s'(t)$ is curvilinear coordinate.

55.2 Prop

Let $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ be two measurable functions $\gamma, \gamma_1, \gamma_2$(piecewise) regular C^1 curves, then

1 $\alpha, \beta \in \mathbb{R}$

$$\int_\gamma (\alpha f + \beta g)\mathrm{d}s = \alpha \int_\gamma f\mathrm{d}s + \beta \int_\gamma g\mathrm{d}s$$

2 If $\gamma_a \sqcup \gamma_2$ make sense

$$\int_{\gamma_a \sqcup \gamma_2} f\mathrm{d}s = \int_{\gamma_a} f\mathrm{d}s + \int_{\gamma_2} f\mathrm{d}s$$

3

$$\int_{\gamma_1} f\mathrm{d}s = \int_{-\gamma_1} f\mathrm{d}s$$

4 $\int_\gamma f\mathrm{d}s$ doesn't depends on the reparameterization of γ

Proof

Exercise

55.3 Geometry

Let V be a finite dim vector space over $K(K = \mathbb{R}, \mathbb{C})$. Let $g : V \times V \to K$ be a bilinear function. Choose a basis $\{\sigma_1, \cdots, \sigma_n\}$ we have the Gram matrix $G = (g(\sigma_i, \sigma_j))_{i,j}$

$$g(x, y) = x^T G y$$

what happens when we change basis:

$$\begin{tikzcd}
& x \in V \arrow[dl, dotted, bend right] \arrow[dr, dotted, bend left] & \\
x \in K^n \arrow[ur, "b"] & & x' \in K^n \arrow[ul, "b'"'] \arrow[ll, "A"']
\end{tikzcd}$$

$$g(x, y) = x^T G y = (x'^T)(A^T G A) y'$$

$$\{v_1, \cdots, v_n\} \quad G \rightsquigarrow A^T G A$$

55.4 Def

Two matrices $G, G' \in M_{n\times n}(K)$ are said **congruent** if $\exists A \in GL_n(K)$ (namely A invertible) such that $G' = A^T G A$

$$\begin{array}{rcrcr}
\mathscr{L}(V, V; K) & \overset{\cong}{\to} & \mathscr{L}(T_0^2(V); K) & \overset{\cong}{\to} & \mathscr{L}(V, V^\vee) \\
g & \mapsto & g_s & \mapsto & [x \mapsto g_s(x \otimes -)]
\end{array}$$

In particular

$$\begin{array}{rcl}
\tilde{g} : V & \to & V^\vee \\
x & \mapsto & [y \mapsto g(x, y)]
\end{array}$$

- $g : V \times V \to K$
- $\begin{array}{rcr} g_p : V \times V & \to & K \\ (x, y) & \mapsto & g(y, x) \end{array}$
- $\begin{array}{rcr} \overline{g_p} : V \times V & \to & K \\ (x, y) & \mapsto & \overline{g(y, x)} \end{array}$

55.5 Def

A bilinear form g is

- **symmetric** if $g = g_p$
- **symplectic** if $g = -g_p$
- **hermitian** if $g = -\overline{g_p}$

55.6 Def

(V, g) is a vector space with an inner product. Two vector $v_1, v_2 \in V$ are said **orthogonal** (w.r.t(with respect to) g) if

$$g(v_1, v_2) = 0$$

Two subspaces $V_1, V_2 \in V$ are **orthogonal** (w.r.t g) if

$$g(v_1, v_2) = 0 \quad \forall v_1, v_2 \in V$$

55.7 Def

The kernel of an inner product g is

$$\ker g := \{\sigma \in V \mid g(\sigma, w) = 0, \forall w \in V\}$$

g is said non-degenerate if

$$\ker g = \{0\}$$

55.8 Remark

$\ker g = \ker \tilde{g}$ where $\tilde{g}$ a linear mapping $\tilde{g} : V \to V^\vee$

55.9 Prop

Let g be a bilinear form on V and let G be the Gram matrix of g w.r.t the basis $\{\sigma_1, \cdots, \sigma_n\}$ Then the matrix of $\tilde{g}$ ww.r.t. the basis $\{\sigma_1, \cdots, \sigma_n\}$ and $\{\sigma_1^\vee, \cdots, \sigma_n^\vee\}$ is G^T

$$\begin{tikzcd}
V \arrow[r, "\tilde{g}"] & V^\vee \\
K^n \arrow[u, "b"] \arrow[r, "G^T"] & K^n \arrow[u, "b^\vee"']
\end{tikzcd}$$

Proof

$$\begin{aligned}((b^\vee)^{-1} \circ \tilde{g} \circ b)(e_i) &= (b^\vee)^{-1}\tilde{g}(v_i) \\ &= (b^\vee)^{-1}g(v_i, -) \\ &= g_{v_i}(\cdot) : V \to K\end{aligned}$$

$$\begin{aligned}g_{v_i}(x) &= g(v_i, \sum_j x_j v_j) \\ &= \sum_j x_j G_{ij} \\ &= \sum_j G_{ij} v_j^\vee(x)\end{aligned}$$

so $g_{v_i} = \sum_j G_{ij} v_j^\vee$

$$\begin{aligned}(b^\vee)^{-1}(g_\vee(\cdot)) &= (b^\vee)^{-1}(\sum_j G_{ij} v_j^\vee) \\ &= \sum_j G_{ij} e_j \\ &= \text{i-th row of } G\end{aligned}$$

55.10 Prop

- g is symmetric iff $G = G^T$
- g is symplectic iff $G = -G^T$
- g is hermitian iff $G = \overline{G^T}$

Proof

Exercise

55.11 Prop

Let g be an inner product on V, THen g is non-degenerate iff

$$\det G \neq 0$$

(w.r.t. any choose of basis)

Proof

Let G and G' be two Gram matrices of g.

$$G' = A^T G A$$

for A invertible

$$\det G' = (\det A)^2 \det G$$

$$\det G' \neq 0 \Leftrightarrow \det G \neq 0$$

Fix a basis $\{v_1, \cdots, v_n\}$

$$\ker g \cong \ker G^t$$

by the fact that $\ker g = \ker \tilde{g}$. So $\ker g \neq 0$ iff

$$\det G = \det G^T \neq 0$$

55.12 Def:isometry

$$(V, g) \xrightarrow{f} (W, g')$$

a morphism f of vector space with inner product is **isometry** if

$$g(x, y) = g'(f(x), f(y))$$

55.13 Def:isometric

$V \xrightarrow{\cong} W$ up to isomorphism.

Then (V, g) and (W, g') are **isometric** if there are two isometry

$$\begin{aligned} f : (V, g) \to & \quad (W, g') \\ f' : (W, g') \to & \quad (V, g) \end{aligned}$$

such that

$$f \circ f' = f' \circ f = Id$$

Chapter 56

Complex conjugate vector space

56.1 Def:complex conjugate vector space

Let V be a vector space over $K = \mathbb{C}$. The **complex conjugate** $\overline{V}$ is the same set of V. The sum on $\overline{V}$ is the same of V, but we define

$$\alpha * v := \overline{\alpha} v \quad \alpha \in \mathbb{C}$$

56.2 Def: Semilinear

If V and W are two complex vector sapce, then a **semilinear mapping** is a mapping $f : V \to W$ such that

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha * f(v) = \overline{\alpha} f(v)$

So a semilinear mapping is a linear mapping: $f : V \to W$

For sesquilinear forms, the theory is similar to the theory of bilinear forms.

$$g \rightsquigarrow G(\text{fix a basis}) \quad g(x, y) = xG\overline{y}$$

If you change basis, then the Gram matrix changes in the following way:

$$G \rightsquigarrow A^T G \overline{A}$$

If g is bilinear

$$g \rightsquigarrow \tilde{g} : V \to V^{\vee}$$

and

$$g \rightsquigarrow \tilde{g} : V \to \overline{V^{\vee}}$$

linear if g is sesquilinear ($\tilde{g} : V \to V^{\vee}$ is semilinear)

56.3 Def

A sesquilinear form $g : V \times \overline{V} \to K$ is **hermitian** if

$$g(x, y) = \overline{g(y, x)}$$

And note that inner product is any of symmetric symplectic or hermitian.

Chapter 57

Classification (up to isometry) of vector spaces of small dim

Let (V, g) be vector space over $K(= \mathbb{R}, \mathbb{C})$ with inner product.

57.1 $\dim V = 1$ and g is symmetric

choose $v \in V \setminus \{0\}$ if $g(v, v) = 0$, then g is degenerated $\Rightarrow$ $g = 0$
If g is non-deg (non-degenerate) $\exists v$ s.t. $g(v, v) = a \neq 0$

$$\forall x \in K \quad g(xv, xv) = ax^2$$

Any v s.t. $g(v, v) = a \neq 0$ induce a set

$$\mathcal{C}(v) := \{ax^2 : x \in K^*\}$$

this is an element in $K^*/\{x^2 \mid x \in K^*\}$

57.1.1 Prop

Let $(V_1, g_1), (V_2, g_2)$ be two vector spaces of dim 1 s.t. g_1 and g_2 are symplectic. Then (V_1, g_1) and (V_2, g_2) are isometric iff

$$\exists v_1 \in V_1, v_2 \in V_2 \text{ s.t. } \mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$$

Proof

$\Rightarrow$ $f : V_1 \to V_2$ be an isometry, $v_1 \in V \setminus \{0\}$, $v_2 := f(v_1)$

$$g_2(v_2, v_2) = g_2(f(v_1), f(v_1)) = g_1(v_1, v_1)$$

$\Rightarrow$

$$\mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$$

$\Leftarrow$ Assume $\exists v_1, v_2 : \mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$ then $\forall a \in K^*$

$$g_1(v_1, v_1) = ax^2 \quad g_2(v_2, v_2) = ay^2$$

Then let

$$\begin{aligned} f : V_1 \to & \quad V_2 \\ v_1 \mapsto & \quad y^{-1} x v_2 \end{aligned}$$

for dim 1 this can def whole mapping. So

$$g_2(\frac{x}{y} v_2, \frac{xx}{y} v_2) = \frac{x^2}{y^2} g(v_2, v_2) = ax^2 = g(v_1, v_1)$$

For $K = \mathbb{R}$ $\mathbb{R}^*/\{x^2 \mid x \in \mathbb{R}^*\} \cong \{\pm 1\}$

$$\begin{aligned} f : \mathbb{R}^* \to & \quad \{\pm 1\} \\ x \mapsto & \quad sgn(x) \end{aligned}$$

For $K = \mathbb{C}$ $C^* \cong \{x^2 \mid x \in \mathbb{C}\}$, $\mathbb{C}^*/\{x^2 \mid x \in \mathbb{C}^*\} = \{[1]\}$

57.1.2 Theorem

(V, g) has dim 1, g symmetric. Then (V, g) is isometric to one of the following

- $K = \mathbb{R}$

$$(\mathbb{R}, g(x, y) = xy) \quad (\mathbb{R}, g(x, y) = -xy) \quad (\mathbb{R}, g(x, y) = 0)$$

- $K = \mathbb{C}$

$$(\mathbb{C}, g(x, y) = xy) \quad (\mathbb{C}, g(x, y) = 0)$$

Proof

Follows form last prop

57.2 $\dim V = 1$ g is hermitian

Again g degenerate $\Rightarrow$ $g = 0$We use that same reason as above. $v \in V : g(v, v) = a \neq 0$, $\forall a \in \mathbb{C}^*$

$$g(av, av) = \|a\|^2 g(v, v)$$

So any element $v \in V \setminus \{0\}$ s.t. $g(v, v) = a$ induces a coset in $\mathbb{C}^*/\mathbb{R}_{>0}$

Inside $\mathbb{C}^*$ $\mathbb{R}_{>0}$ is a (mult) subgroup

For any $z \in \mathbb{C}^*$ can be written uniquely as $z = re^{i\theta}$, hence

$$\begin{array}{rcl} C^* \cong & \mathbb{R}_{>0} \times S^1 \to & S^1 \\ z \mapsto & (r, e^{i\theta}) \mapsto & e^{i\theta} \end{array}$$

The kernel is $\mathbb{R}_{>0}$ and $S^1 \cong \mathbb{C}^*/\mathbb{R}_{>0}$.

But g is hermitian, so

$$g(v,v) \in \mathbb{R}$$

If follows that the coset

$$\{\|a\|^2 g(v,v) \mid a \in \mathbb{C}^*\} \in (\mathbb{C}^*/\mathbb{R}_{>0}) \cap (\mathbb{R}/\mathbb{R}_{>0}) \cong \{\pm 1\}$$

We repeat the proposition before with g hermitian and the following theorem

57.2.1 Theorem

(V,g) if dim 1, with g hermitian. Then (V,g) is isometry to one of the following

$$(\mathbb{C}, g(x,y) = x\overline{y}) \quad (\mathbb{C}, g(x,y) = -x\overline{y}) \quad (\mathbb{C}, g(x,y) = 0)$$

57.3 $\dim V = 1$ g symplectic

With dim 1 $\forall v_1, v_2 \in V$ can be write as $v_1 = ae, v_2 = be$ with $e \in K^*$

$$g(v_1, v_2) = ab \cdot g(v,v) = 0$$

57.3.1 Theorem

(V,g) of dim 1, g symplectic, then

$$(V,g) \cong (K, g = 0)$$

57.4 $\dim V = 2$ g symplectic

Assume that g is degenerated, then $\exists x \in V$ s.t. $g(x,y) = 0, \forall y \in V$

Extend x to a basis $\{x, x'\}$ of V

$$g(ax+a'x', bx+b'x') = ab\cdot g(x,x) + ab'\cdot g(x,x') - a'b\cdot g(x,x') + a'b'\cdot g(x',x') = 0$$

So when g degenerated $g = 0$

Take g non-degenerated $\exists v_1, v_2 \in V$ s.t. $g(v_1, v_2) = a \neq 0$.

For $g(a^{-1}v_1, v_2) = a^{-1}a = 1$, we may assume that $a = 1$

Let's show that v_1, v_2 are linearly independent. Assume by contraction: $v_1 = \lambda v_2$

$$1 = g(v_1, v_2) = g(\lambda v_2, v_2) = \lambda \cdot g(v_2, v_2) = 0$$

$\Rightarrow \{v_1, v_2\}$ is a basis of V. Then

$$\alpha_1\beta 2 - \alpha_2\beta_1 = g(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 v_1 + \beta_2 v_2) = (\alpha_1, \alpha_2)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \overset{(?)}{=} \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \end{pmatrix}$$

57.4.1 Theorem

(V, g) is dim 2, g symplectic. Then (V, g) is isometric to one of the following

$$(K^2, g(x, y) = 0) \quad (K^2, g(x, y) = x_1 y_2 - x_2 y_1)$$

Chapter 58

Compliment

58.1 Def: non-degenerate

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a subspace. We say that V_0 is non-degenerate if $g\mid_{V_0}$ is non-degenerate.

Moreover, V_0 is isotropic if $g\mid_{V_0} = 0$

Remark

Isotropic means degenerate

Example

1 $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_1 - x_2y_2$$

symmetric non-degenerate. $V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

2 $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_2 + x_2y_1$$

$V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

58.2 Def

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a subspace. The orthogonal complement of V_0 is define as

$$V_0^{\perp} := \{v \in V \mid g(v, v_0) = 0, \forall v_0 \in V_0\}$$

58.3 Prop

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a non-degenerate subspace. Then

$$V = V_0 \oplus V_0^{\perp}$$

Proof

Let $\{v_0, \cdots, v_n\}$ be a basis of V_0 ($\dim V_0 = r \leq \dim V = n$) Extend this to a basis of V

$$\{v_1, \cdots, v_r, v_{r+1}, \cdots, v_n\}$$

Let G be the Gram matrix of g with respect to such case:

$$G = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) & \cdots & g(v_1, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) & \cdots & g(v_r, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_n, v_1) & \cdots & g(v_n, v_r) & \cdots & g(v_n, v_n) \end{pmatrix}$$

while $G_0 \in M_{r,n}(K)$ is defined as a part of G

$$G_0 = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) \\ \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) \end{pmatrix}$$

Consider the dimension:

$$\begin{aligned} \dim V_0^{\perp} &= \dim \ker G_0 \\ &= n - rankG_0 \\ &= n - rankG_0^T \end{aligned}$$

Since V_0 is non-degenerated. G_0^T is non-degenerated with $rank = \dim V_0 = r$

$$\dim V_0 + \dim V_0^{\perp} = \dim V$$

We finally have to show that this sum is a direct sum:

If $x \in V_0 \cap V_0^{\perp}$, then $g(x, V_0) = 0$ since $x \in V_0^{\perp}$ and $g_0(x, V_0) = 0$ implies $x = 0$ since g_0 is non-degenerate.

58.4 Theorem

Let (V, g) be an finite dimensional inner product space. If both V_0 and $V_0^{\perp}$ are non-degenerate, then $(V^{\perp})^{\perp} = V_0$

Proof

Remark

$V_0 \subseteq (V_0^{\perp})^{\perp}$ always holds, regardless of degenerate and dimension.

58.5 Theorem

Let (V, g) be an finite dimensional inner product space. Then There exists a decomposition

$$V = V_1 \oplus \cdots \oplus V_n$$

such that $\{V_i\}_{i=1}^n$ are pairwise orthogonal and

1 They are 1-dim if g is symmetric or hermitian

2 They are 1-dim but degenerated or 2-dim non-degenerate if g is symplectic.

Proof

We work by induction on $n = \dim V$

$n = 1$

trivial

$n \geq 2$

If $g = 0$, there is nothing to prove.

g symplectic There exists x and x' such that $g(x, x') = 0$ and $\langle x, x' \rangle = V_0$. Since V_0 is non-degenerate, then apply induction on $V_0^{\perp}$

g symmetric/hermitian It's enough to show that there exists $V_0 \subseteq V$ of dim 1, which is non-degenerate.

Chapter 59

Signature

Now we discuss the uniqueness of such decomposition

59.1 Def

Let (V, g) be an inner product space with dim 1. Moreover, assume that g is symmetric or hermitian. We say that (V, g) is **positive** if (V, g) is isometry to either $(\mathbb{R}, g(x, y) = xy)$ or $(\mathbb{C}, g(x, y) = x\overline{y})$

We say that (V, g) is **negative** if (V, g) is isometry to either $(\mathbb{R}, g(x, y) = -xy)$ or $(\mathbb{C}, g(x, y) = -x\overline{y})$

59.2 Notation

By theorem 58.5, we can count the number of positive subspace of any inner product space.

- $r_0 := \dim \ker g$
- $r_+ :=$ the number of positive subspaces
- $r_- :=$ the number of negative subspaces

59.3 Def

Let (V, g) be an inner product space.

1 If g is real symmetric or, hermitian, then (r_0, r_+, r_-) is signature of V

2 If g is symplectic or complex symmetric, then $(\dim V, r_0)$ is the signature of V

59.4 Theorem

Let (V, g) and (V', g') be two inner product spaces, with g, g' that are either (both) symplectic or complex symmetric.

Then (V, g) and (V', g') are isometric iff

$$(n, r_0) = (n', r_0')$$

Proof

$\Rightarrow$

trivial

$\Leftarrow$

Use orthogonal decomposition

$$V = \bigoplus_{i=1}^{n} V_i$$

We rearrange the subspaces as $V_0, \cdots, V_r, v_{r+1}, \cdots, V_n$ such that $V_i \subseteq \ker g$, $\forall 1 \leq i \leq r$

The remaining is shown as below:

$$\begin{array}{ccccc}
(V_1, g) & \xleftrightarrow{\cong} & (\mathbb{R}, 0) & \xleftrightarrow{\cong} & (V_1', g') \\
\vdots & & & & \vdots \\
(V_r, g) & \xleftrightarrow{\cong} & (\mathbb{R}, 0) & \xleftrightarrow{\cong} & (V_r', g') \\
(V_{r+1}, g) & \xleftrightarrow{\cong} & (\mathbb{R}, 0) & \xleftrightarrow{\cong} & (V_{r+1}', g') \\
\vdots & & & & \vdots \\
(V_n, g) & \xleftrightarrow{\cong} & (\mathbb{R}, 0) & \xleftrightarrow{\cong} & (V_n', g')
\end{array}$$

59.5 Theorem

Let (V, g) and (V', g') be two inner product spaces, with g, g' that are either (both) hermitian or real symmetric.

Then (V, g) and (V', g') are isometric iff

$$(r_0, r_+, r_-) = (r'_0, r'_+, r'_-)$$

Proof

exercise

Chapter 60

Orthonormal

60.1 Def

Let (V, g) be an inner product space. The basis $\{v_1, \cdots, v_n\}$ is said **orthogonal** if $g(v_i, v_j) = 0, \forall i \neq j$

Moreover, g is said **orthonormal** if $g(v_i, v_i) \in \{0, -1, 1\}, \forall i$

Remark

If g is hermitian or symmetric. We can always find an orthonormal basis from an orthogonal basis.

60.2 Def

Let V be a vector space over $K(charK \neq 2)$. A **quadratic form** on V is a mapping $a : V \to K$ such that

- $q(\alpha v) = \alpha^2 q(v) \forall \alpha \in K, v \in V$
- $f = (u, v) \mapsto \ q(u + v) - q(u) - q(v)$ is bilinear

Remark

Any symmetric bilinear form $h : V^2 \to K$ is a quadratic form. Given a quadratic form $q : V \to K$, we can define a symmetric bilinear form

$$h_p(u, v) = \frac{1}{2} \left(q(u + v) - q(u) - q(v) \right)$$

60.3 Gram-Schmidt algorithm

Let (V, g) be an inner product space with g symmetric or hermitian. Let $\{v'_1, \cdots, v'_n\}$ be a basis of V such that $V_i = \langle v'_1, \cdots, v'_i \rangle$ $\forall i \in \{1, \cdots, n\}$ is non-degenerate.

Then there exists an orthogonal basis $\{v_1, \cdots, v_n\}$ such that $V_i = \langle v_1, \cdots, v_i \rangle$ $\forall i \in \{1, \cdots, n\}$ is non-degenerate.

Proof

We construct the v_i be induction. $v_1 = v'_1$ If $v_1, \cdots, v_{i-1}$ have been already constructed, then v_i must be of the form:

$$v_i = v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j$$

We have $\langle v_1, \cdots, v_{i-1} \rangle = \left\langle v'_1, \cdots, v'_{i-1} \right\rangle$ by induction hypothesis, then

$$\langle v_1, \cdots, v_{i-1} \rangle = \left\langle v'_1, \cdots, v'_{i-1}, v_i \right\rangle = \langle v'_1, \cdots, v'_i \rangle$$

We have to show that v_i are orthogonal

$$\begin{aligned} g\left(v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j, v_k\right) &= g(v'_i, v_k) - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} g(v_j, v_k) \\ &= g(v'_i, v_k) - \frac{g(v'_i, v_k)}{g(v_k, v_k)} g(v_k, v_k) \\ &= 0 \end{aligned}$$

Chapter 61

Euclidean and Unitary Spaces

61.1 Def:Euclidean vector space

A euclidean vector space is a finite dimensional inner product space over $\mathbb{R}$ (E, g) with g symmetric and positive definite ($g(x,x) > 0, \forall x \neq 0$)

We denote

$$\langle x, y\rangle := g(x, y)$$

Remark

Any non-zero subspace of $(E, \langle \cdot, \cdot \rangle)$ is non-degenerated:

Take $V \subseteq E$ if $\exists x \in V \setminus \{0\}$ s.t. $\langle x, V\rangle = 0$ then

$$\langle x, x\rangle = 0$$

The signature of E is of the type $(0, \mathfrak{r}_+, \mathfrak{r}_-)$ denoted by $(P, Q)(P = \mathfrak{r}_+, Q = \mathfrak{r}_-)$

An euclidean space is a normed vector space

$$\|x\| = \sqrt{\langle x, x\rangle}$$

($\langle , \rangle$ positive defined required)

Any euclidean vector admits an orthonormal basis $\{v_1, \cdots, v_n\}, \langle v_i, v_i\rangle = 1$
So orthonormal means

$$\|v_1\| = 1$$

In a euclidean space we have a distance

$$d(x, y) = \|x - y\|$$

It's a topological metric space

61.2 Prop

A euclidean space $(E, \langle,\rangle)$ of dim n is isometric to $(\mathbb{R}^n, \underbrace{\langle,\rangle}_{usualscalarproduct})$

Proof

$\{v_1, \cdots, v_n\}$ is an orthonormal basis of E. Consider the mapping

$$f : v_i \mapsto e_i = (0, \cdots, \underbrace{1}_{i\text{-th}}, \cdots, 0)$$

Let $x = \sum \alpha_i v_i, y = \sum \beta_i v_i$ then

$$\langle x, y\rangle = \sum \alpha_i \beta_i$$

and $f(x) = \sum \alpha_i e_i, f(y) = \sum \beta_i e_i$, so

$$\langle f(x), f(y)\rangle = \sum \alpha_i \beta_i = \langle x, y\rangle$$

Thus f an isometry.

61.3 Remark

Cauchy-Schwartz inequality

$$\langle x, y\rangle \leq \|x\| \, \|y\|$$

Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

61.4 Pythagona's Theorem

If $x_1, \cdots, x_k$ are pairwise orthogonal, then

$$\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$$

61.5 Def:Angles

By Cauchy-Schwartz inequality:

$$-1 \leq \frac{\langle x, y\rangle}{\|x\| \, \|y\|} \leq 1$$

Then there exists a element $\phi \in [0, \pi]$ such that

$$\cos\phi = \frac{\langle x, y\rangle}{\|x\| \, \|y\|}$$

ϕ is defined as the angle between x and y

Notice that ϕ is not an oriented angle.

61.6 Notation

Let $U, V \subseteq E$ be two subspace, then

$$d(U, V) := \inf\{\|u - v\| \mid u \in U, v \in V\}$$

61.7 Def

$V \subsetneq E$ a vector subspace, $x \in E \setminus \{0\}$. Then $E = V \oplus V^{\perp}$ (proved) Then we write (uniquely)

$$x = x_0, + x_0'$$

where $x_0 \in V, x_0' \in V^{\perp}$.

Then x_0 is called the orthogonal projection of x on V, x_0' is called the orthogonal projection of x on $V^{\perp}$

61.8 Prop

Use the above notation:

$$d(x, V) = \|x_0'\|$$

Proof

$\forall y \in V$

$$\|x - y\|^2 = \|x_0 + x_0' - y\|^2 = \|x_0 - y\|^2 + \|x_0'\|^2$$

then

$$\|x - y\| \geq \|x_0'\|^2$$

$\Rightarrow$

$$d(x, V) \geq \|x_0'\|^2$$

The equality is achieved iff $y = x_0$.

$$d(x, V) = \|x_0'\|$$

61.9 Prop

Use the previous notations. Assume that $m = \dim V$, $V \subseteq E$, $\{v_1, \cdots, v_m\}(m \leq n = \dim E)$ is an orthonormal basis of V. Then

$$x_0 = \sum_{i=1}^{m} \langle x, v_i \rangle v_i$$

Proof

Consider $y = x_0 - \sum_{i=1}^{m} \langle x, v_i \rangle v_i$, $y \in V$. Moreover

$$\begin{aligned}
\langle y, v_j \rangle &= \left\langle x_0 - \sum_{i=1}^{m} \langle x, v_i \rangle v_i, v_j \right\rangle \\
&= \left\langle x - \sum_{i=1}^{m} \langle x, v_i \rangle v_i, v_j \right\rangle \\
&= \langle x, v_j \rangle - \langle x, v_j \rangle \\
&= 0
\end{aligned}$$

y is orthonormal to any $v_j \Rightarrow \langle y, V \rangle = 0 \Rightarrow y \in V^{\perp}$ so

$$y \in V \cap V^{\perp} = \{0\}$$

61.10 Relationship with calculus

$(E, \langle , \rangle) = (\mathbb{R}^n, \langle , \rangle)$ on $\mathbb{R}^n$ we have the notion of volumes

$$vol(B) := \lambda^n(B)$$

where B is a Borel set.

A n-dimensional parallelepiped is :

$$P_n = \{t_1 v_1 + \cdots + t_n v_n \mid t_i \in [0,1] \forall i\}$$

Consider a linear mapping

$$\begin{aligned}
A_{P_n} = A : \mathbb{R}^n \to &\quad \mathbb{R}^n \\
x \mapsto &\quad Ax
\end{aligned}$$

where $A \in \mathcal{M}_{n \times n}(\mathbb{R})$, $A = (v_1 \mid \cdots \mid v_n)$

A is invertible iff $\{v_1, \cdots, v_n\}$ is a basis. Let $\prod_n = [0,1]^n$ then

$$A(\prod_n) = P_n$$

If A invertible

$$\begin{aligned} vol(P_n) &= \lambda^n(P_n) \\ &= \int_{A(\prod_n)} \chi_{P_n} \mathrm{d}\lambda^n \\ \text{change of variables } &= \int_{A(\prod_n)} |\det A| \,\mathrm{d}\lambda^n \\ &= |\det A| \\ \text{by the prop of det } &= \sqrt{\det A^T A} \end{aligned}$$

61.11 Prop

$$vol(P_n) = \sqrt{\det G}$$

Proof

The Gram matrix

$$G = (\langle v_i, v_j \rangle_{ij})$$

Notice that $\det G = 0$ iff $\{v_1, \cdots, v_n\}$ is not a basis. Let $A = (v_1 \mid \cdots \mid v_n)$ then $G = A^T A$

$$\begin{aligned} \det G &= (\det A)^2 \\ &= \det A^T A \\ &= vol(P_n) \end{aligned}$$