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Chapter 1

Countable sets

1.1 Notation

$$\mathbb{N} = \mathbb{N} \setminus \{0\}$$

1.2 Def

S is **infinitely countable** if $\exists S \rightarrow \mathbb{N}$ bijection, **countable** if S is finite or inf-countable

Remark

- for sequence $\langle S_n \rangle_{n \in \mathbb{N}}$

$$\begin{aligned} \mathbb{N} &\rightarrow S \\ n &\mapsto S_n \end{aligned}$$

- if $S \neq \emptyset$ then TFAE:

- S is countable
- \exists surjection $\mathbb{N} \rightarrow S$
- \exists injection $S \rightarrow \mathbb{N}$

- \mathbb{Q} is inf-countable

- if $m \in \mathbb{N}$ S_1, \dots, S_m are countable. Then $\prod_{j=1}^m S_j$ is countable.

1.3 Cantor Theorem

\mathbb{N} is not equinumerous with $\wp(\mathbb{N})$

Proof

$\wp(\mathbb{N}) \cong \{0, 1\}^{\mathbb{N}}$ if $A \in \wp(\mathbb{N})$ then

$$\begin{aligned} \mathbb{1}_A : \mathbb{N} &\rightarrow \{0, 1\} \\ n &\mapsto \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases} \end{aligned}$$

the identify of A :

$$\begin{aligned} \wp(\mathbb{N}) &\rightarrow \{0, 1\}^{\mathbb{N}} \\ A &\mapsto \mathbb{1}_A \end{aligned}$$

is a bijection

$$\{0, 1\}^{\mathbb{N}} = \mathcal{F}(\mathbb{N}; \{0, 1\})$$

Remark

A, B be sets. $\mathcal{F}(A; B)$ is the set of all functions from A to B .

Proof

Assume that \exists bijection

$$\begin{aligned} \mathbb{N} &\rightarrow \wp(\mathbb{N}) \\ n &\mapsto f_n \end{aligned}$$

Define

$$\begin{aligned} f : \mathbb{N} &\rightarrow \{0, 1\} \\ n &\mapsto \begin{cases} 0 & \text{if } f_n(n) = 1 \\ 1 & \text{if } f_n(n) = 0 \end{cases} \end{aligned}$$

$f \in \mathcal{F}(\mathbb{N}; \{0, 1\})$ thus $\exists m \in \mathbb{N}$ s.t. $f = f_m$. Then $f_m(m)$ broken.

Chapter 2

Number Series

2.1 Def

$\sum_{n=0}^{+\infty} a_n$ is **commutatively convergent** (CC) if for each permutation ϕ of \mathbb{N} the series $\sum_{n=0}^{+\infty} a_{\phi(n)}$ converges.

Remark

A.C. is **absolutely convergent**.

C. is **convergent**. Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a bijection.

- if $\sum_{n=0}^{+\infty} a_n$ is A.C. then $\sum_{n=0}^{+\infty} a_n$ C.
- $\sum_{n=0}^{+\infty} \frac{(-1)^n}{n}$ C. but not A.C. or C.C.

2.2 Riemann Theorem

Let $\sum_{n=0}^{+\infty} a_n$ be a convergent series in \mathbb{R} TFAE:

- $\sum_{n=0}^{+\infty} a_n$ is not A.C.
- $\forall s \in \mathbb{R} \exists$ permutation of \mathbb{N} s.t.

$$\sum_{n=0}^{+\infty} a_{\phi(n)} = s$$

- $\forall s \in \mathbb{R} \cup \{-\infty, +\infty\} \exists \text{permutation of } \mathbb{N} \text{ s.t.}$

$$\sum_{n=0}^{+\infty} a_{\phi(n)} = s$$

Chapter 3

Kurzneil-Henstock integral

3.1 Def

Cell is a non-degenerated interval

3.2 Nested cell theorem

If $\langle I_n \rangle_{n \in \mathbb{N}}$ is a decreasing sequence ($I_{n+1} \subseteq I_n$) of compact cells s.t.

$$\lim_{N \rightarrow +\infty} \text{diam} I_n = 0$$

then $\exists x \in \mathbb{R}$

$$\bigcap_{n \in \mathbb{N}} I_n = \{x\}$$

3.3 Exercises

Every cell is uncountable.

3.4 Def

Two cells are **non-overlapping** if their intersection either empty or a singleton.

3.5 Exercises

If I_1, I_2, I_3 are pairwise non-overlapping, then

$$I_1 \cap I_2 \cup I_3 = \emptyset$$

3.6 Lemma

If I is a compact cell and $N \in \mathbb{N}_0$ are pairwise non-overlapping cells s.t.
 $\bigcup_{n=1}^N I_n = I$ then renumbering them if necessary, we may get:

$$\begin{aligned}\min I &= \min I_1 \\ \max I_n &= \min I_{n+1} \\ \max I_N &= \max I\end{aligned}$$

3.7 Def

A **partial division** Δ of I is a finite set consisting of non-overlapping compact sub-cells of I . If

$$\bigcup \Delta = I$$

it's called a **division** of I

3.8 Lemma

If Δ is a partial division of I , then there exists a partial Δ' of I s.t. $\Delta \cap \Delta'$ is a division of I

3.9 Def

A **gauge** on I is a function

$$\delta : I \rightarrow \mathbb{R}$$

such that $\forall x \in I \delta(x) > 0$

Remark

If $\delta_1, \dots, \delta_N$ are gauges on I then

$$\delta(x) = \min\{\delta_1(x), \dots, \delta_N(x)\}$$

is also a gauge.

3.10 Def

A **partial P-division** of a compact cell I , is a finite Π of pairs (J, x) s.t.

- $J \subseteq I$
- J is a compact cell

- $x \in J$
- $\forall (J_1, x), (J_2, x_2) \in \Pi$ if $J_1 \neq J_2$ then J_1, J_2 are non-overlapping

x is cal tag of the pair.

3.11 Def

Given a partial P-division Π of I define

$$body(\Pi) = \bigcup \{J : (J, x) \in \Pi\}$$

A **P-division** Π of I is a partial P-division s.t. $body(\Pi) = I$

3.12 Lemmas

- If Π_1, \dots, Π_N are partial P-divisions of I s.t. for each $n, m \in \{1, \dots, N\}, n \neq m$ $body\Pi_n$ and $body\Pi_m$ are either disjoint or their intersection is a singleton, then $\bigcup_{n=1}^N \Pi_n$ is a partial P-division of I .
- If Π is a partial P-division of I and $\xi \in I$ then there're at most 2 $(J, x) \in \Pi$ s.t. $x = \xi$

3.13 Def

Let δ be a gauge on I and Π a (partial) P-division of I , we say that Π is δ -finite if

$$\forall (J, x) \in \Pi \quad J \subseteq [x - \delta(x), x + \delta(x)]$$

3.14 Def

If $f : I \rightarrow \mathbb{R}$ and Π is a (partial) P-division then the **Riemann sum** is defined as

$$S(\Pi, f) := \sum_{(J, x) \in \Pi} f(x) |J|$$

3.15 Def

Let $f : I \rightarrow \mathbb{R}$ f is **KH-integrable** on I if $\exists r \in \mathbb{R}, \forall \epsilon > 0 \exists$ gauge δ on I $\forall \delta$ -finite P-division Π of I

$$|S(\Pi, f) - r| < \epsilon$$

3.16 Prop

r is unique

Proof

Assume that r_1 and r_2 . Fix $\epsilon > 0$. For $i = 1, 2$, there's a gauge δ_i on I s.t. if Π is a δ_i -finite P-division of I then

$$|S(\Pi, f) - r_i| < \epsilon$$

$$\begin{aligned} |r_1 - r_2| &= |r_1 - S(\Pi, f) + S(\Pi, f) - r_2| \\ &\leq |r_1 - S(\Pi, f)| + |S(\Pi, f) - r_2| \\ &< 2\epsilon \end{aligned}$$

Let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$ then δ is a gauge on I . If Π is δ -finite then it's δ_1 -finite and δ_2 -finite.

3.17 Cousin Theorem

I be a compact cell and δ a gauge on I . Then there exists a δ -finite then P-division of I .

Proof

assume there's no. Then divide I into I_l, I_r by middle. Then either I_l, I_r has no δ -finite division. Then we get a decreasing sequence $(I_n)_{n \in \mathbb{N}}$ by keeping dividing. According to nested theorem, get their intersection a singleton x . Notice that x is a point of I , for $N \in \mathbb{N}$ big enough

$$\text{diam} I_N = 2^{-N} \cdot \text{diam} I < \delta(x)$$

then $\Pi = \{(I_N, x)\}$ is a δ -finite P-division of I_N .

3.18 Notation

$$r = \int_I f = \int_I f(x) dx$$

if $I = [a, b]$

$$r = \int_a^b f = \int_a^b f(x) dx$$

3.19 Prop of Riemann Sum

linearity $\forall \Pi$ (partial) P-division, $\forall f_1, f_2 : I \rightarrow \mathbb{R}, \forall \alpha \in \mathbb{R}$

$$S(\Pi, \alpha f_1 + f_2) = \alpha S(\Pi, f_1) + S(\Pi, f_2)$$

monotonicity

$$f_1 \leq f_2 \Rightarrow S(\Pi, f_1) \leq S(\Pi, f_2)$$

additivity if Π_1, Π_2 are partial P-division of I and $(body \Pi_1) \cap (body \Pi_2)$ is either empty or a finite set, then $\forall f$

$$S(\Pi_1 \cup \Pi_2, f) = S(\Pi_1, f) + S(\Pi_2, f)$$

3.20 Prop of KH-integral

I a compact cell

3.21 Prop: Constant functions

If $f : I \rightarrow \mathbb{R}$ is constant then $f \in KH(I)$ and $\int_I f = y \cdot |I|$. (y is the constant value of f)

Proof

$\forall \Pi$ P-division of I

$$S(\Pi, f) = \sum_{(J,x) \in \Pi} f(x) |J| = y \sum_{(J,x) \in \Pi} |J| = y |I|$$

3.22 Theorem

$KH(I)$ is a vector space and $KH(I) \rightarrow \mathbb{R}, f \mapsto \int_I f$ is linear and monotone.

Proof

$0, \mathbb{1}_I \in KH(I)$

If $f_1, f_2 \in KH(I)$ and $\alpha \in \mathbb{R}$, we want to show that $\alpha f_1 + f_2 \in KH(I)$ and

$$\int_I (\alpha f_1 + f_2) = \alpha \int_I f_1 + \int_I f_2$$

Let $\epsilon > 0$, δ_1 be a gauge on I , $\frac{\epsilon}{2(|\alpha|+1)}$ -adapted to f_1 and δ_2 $\frac{\epsilon}{2}$ -adapted. Def

$$\delta = \min\{\delta_1, \delta_2\}$$

Let Π be a δ -finite P-division of I

$$\begin{aligned} \left| S(\Pi, \alpha S(\Pi, f_1) + S(\Pi, f_2)) - (\alpha \int_I f_1 + \int_I f_2) \right| &= \left| \alpha S(\Pi, f_1) + S(\Pi, f_2) - (\alpha \int_I f_1 + \int_I f_2) \right| \\ &\leq |\alpha| \left| S(\Pi, f_1) - \int_I f_1 \right| + \left| S(\Pi, f_2) - \int_I f_2 \right| \\ &\leq |\alpha| \frac{\epsilon}{2(|\alpha| + 1)} + \frac{\epsilon}{2} \leq \epsilon \end{aligned}$$

3.23 Cauchy criterion

Let $f : I \rightarrow \mathbb{R}$, TFAE:

- $f \in KH(I)$
- $\forall \epsilon > 0 \exists \text{gauge } \delta \text{ on } I \text{ s.t. } \forall \Pi, \Pi \text{ is } \delta\text{-finite P-division of } I$

$$\left| S(\Pi, f) - \int_I f \right| < \epsilon$$

Proof

1 \Rightarrow 2

trivial

2 \Rightarrow 1

For each $n \in \mathbb{N}_0$, we apply hypothesis (2) with $\epsilon = \frac{1}{n}$ and we obtain a gauge δ_n , define

$$\hat{\delta}_n = \min_{i=1}^n \delta_i$$

choose Π_n a $\hat{\delta}_n$ -finite

Let $r_n := S(\Pi_n, f)$. We show that $\langle r_n \rangle_{n \in \mathbb{N}_0}$ is a Cauchy sequence. Let $0 < p < q \in \mathbb{N}_0$

$$|r_p - r_q| = |S(\Pi_p, f) - S(\Pi_q, f)| < \frac{1}{p}$$

Name $r := \lim_{n \rightarrow +\infty} r_n$, now we show that f is KH-integrable with $\int_I f = r$. Let $\epsilon > 0$, choose $n_0 \in \mathbb{N}_0$ large enough for $\frac{1}{n_0} < \epsilon$. We claim that $\hat{\delta}_n$ is a gauge with integrability of f . $\forall \Pi \hat{\delta}_n$ -finite, for each $n \geq n_0$, we have:

$$\begin{aligned} |S(\Pi, f) - r| &\leq |S(\Pi, f) - S(\Pi_n, f)| + |S(\Pi_n, f) - r| \\ &\leq \frac{1}{n_0} + |r_n - r| \\ &\leq \epsilon + \epsilon \end{aligned}$$

3.24 Example: Dirichlet function

$$f : \mathbb{R} \rightarrow \mathbb{R} : \mathbb{1}_{\mathbb{Q}}$$

Let I be a compact cell, we want to show

$$f|_I \in KH(I) \quad \int_I f = 0$$

We deal with $S(\Pi, \mathbb{1}_{\mathbb{Q}}) = \sum_{(J,x) \in \Pi, x \in \mathbb{Q}} |J|$. For \mathbb{Q} countable:

$$\exists q : \mathbb{N} \xrightarrow{q} I \cap \mathbb{Q}$$

Let $\epsilon > 0$, we define δ on $I \cap \mathbb{Q}$ as follows:

- If $x \in I \cap \mathbb{Q}$, then $x = q(n)$ for some n and let $\delta(x) = \frac{\epsilon}{2^n}$
- If $x \in I \setminus \mathbb{Q}$, then define $\delta(x) = 1$

Let Π be δ -finite,

$$\begin{aligned} S(\Pi, \mathbb{1}_{\mathbb{Q}}) &= \sum_{(J,x) \in \Pi, x \in \mathbb{Q}} |J| \\ &\leq \sum_{n=0}^{\infty} \sum_{(J,x) \in \Pi, x \in \mathbb{Q}} |J| \\ &\leq \sum_{n=0}^{\infty} 2 \cdot 2 \cdot \frac{\epsilon}{2^n} = 8\epsilon \end{aligned}$$

Exercises

If $\mathbb{1}_{\mathbb{Q} \cap I}$ Riemann integrable?

3.25 Theorem

Let $f \in KH(I), g : I \rightarrow \mathbb{R}$ s.t. $\{f \neq g\}$ is countable. Then $g \in KH(I)$ and $\int_I g = \int_I f$

3.26 Theorem: subordinate P-division

Let I be a compact cell and Δ be a division of I . There exists a gauge δ on I satisfying the following properties:

$\forall \delta$ -finite P-division Π of I ,

- $\forall K \in \Delta, \exists$ P-division Π_K of K

- There exists a P-division $\tilde{\Pi}$ of I s.t.

$$\text{A } \tilde{\Pi} = \bigcup_{K \in \Delta} \Pi_K$$

$$\text{B } \forall f : I \rightarrow \mathbb{R}$$

$$S(\Pi, f) = S(\tilde{\Pi}, f) = \sum_{K \in \Delta} S(\Pi_K, f_K)$$

- For every gauge η on I, if Π is η -finite, then each Π_K is $\eta|_K$ -finite, $K \in \Delta$

Proof

$$\delta(x) = \begin{cases} \text{dist}(x, F) & \text{if } x \notin F \\ \text{dist}(x, F \setminus \{x\}) & \text{if } x \in F \end{cases}$$

3.27 Finite-additivity

Let $\{I_1, \dots, I_N\}$ be a division of a compact cell I and $f : I \rightarrow \mathbb{R}$ TFAE

- $f \in KH(I)$
- $f|_{I_n} \in KH(I_n), \forall n \in \{1, \dots, N\}$, In this case,

$$\int_I f = \sum_{I_n} \int_{I_n} f$$

Proof

1 \Rightarrow 2

Let $J \subseteq I$ be a compact cell and assume

$$I = J_1 \sqcup J \sqcup J_2 \quad (J_1 \leq J \leq J_2)$$

We want to show that $f|_J \in KH(J)$. Apply Cauchy criterion for this. Let $\epsilon > 0$
We need to find a gauge δ_0 on J s.t. Π_0 is δ_0 -finite P-division, then

$$|S(\Pi_0, f|_J) - S(\Pi_0, f|_J)| < \epsilon$$

For $\epsilon > 0$, $\exists \delta$ gauge on I ϵ -adapted to f . We define:

- $\delta_1 = \delta|_{J_1}$ then Π_1 is δ_1 -finite
- $\delta_0 = \delta|_J$ then Π_0 is δ_0 -finite
- $\delta_2 = \delta|_{J_2}$ then Π_2 is δ_2 -finite

so

$$S(\Pi, f) = S(\Pi_1, f|_{J_1}) + S(\Pi_0, f|_J) + S(\Pi_2, f|_{J_2})$$

$2 \Rightarrow 1$

trivial

3.28 Theorem

If $f \in KH(I)$ and $J \subseteq I$ is a compact cell, then $f|_J \in KH(I)$ and

$$\int_J f|_J = \int_I \mathbb{1}_J \cdot f$$

3.29 Def: step function

$f : I \rightarrow \mathbb{R}$ is a **step function** if there exists a division Δ of I s.t. $\forall J \in \Delta, f|_J$ is constant.

3.30 Theorem

Every step function on I is JH-integrable.

3.31 Theorem

If (f_n) a sequence in $KH(I)$ that converges uniformly on I to $f : I \rightarrow \mathbb{R}$, then $f \in KH(I)$

3.32 Def:regulated function

A **regulated function** $f : I \rightarrow \mathbb{R}$ is a function which is a limit of a sequence of step functions.

3.33 Corollary

Every regulated function on I is KH-integrable.

3.34 Prop

- Every continuous function $f : I \rightarrow \mathbb{R}$ is regulated
- Every monotone function $f : I \rightarrow \mathbb{R}$ is regulated.

Chapter 4

Fundamental theorem of calculus

4.1 Theorem

If $F : I \rightarrow \mathbb{R}$ is diff. (differentiable) everywhere, then $F' \in KH(I)$ and

$$\int_I F' = F(\max I) - F(\min I)$$

4.1.1 Lemma

If f is diff. at $x \in I$ then $\forall \epsilon \exists \delta > 0$ s.t. $\forall y \leq x \leq z, y, z \in I, \max\{|y - x|, |x - z|\} < \delta$, then

$$|F(z) - F(y) - F'(x)(z - y)| < \epsilon |z - y|$$

Proof of lemma

$$\begin{aligned} & |F(z) - F(x) + F(x) - F(y) - F'(x)(z - x + x - y)| \\ & \leq \epsilon |z - x| + \epsilon |y - x| \\ & = \epsilon |y - z| \end{aligned}$$

Proof

Let $\epsilon > 0$, $\forall x \in I$, there exists $\delta(x) > 0$ s.t. \forall compact cell $J \subseteq I$, with $x \in J \subseteq [x - \delta(x), x + \delta(x)]$

$$|F(\max J) - F(\min J) - F'(x)|J|| < \epsilon |J|$$

If Π is a δ -finite P-division of I . We want to show

$$|S(\Pi, F') - F(\max I) + F(\min I)| < \epsilon |I|$$

Basically

$$\begin{aligned}
 S(\Pi, F') &= \sum_{(J,x) \in \Pi} F'(x) |J| \\
 F(\max I) - F(\min I) &= \sum_{(J,x) \in \Pi} (F(\max J) - F(\min J)) \\
 |S(\Pi, F') - F(\max I) + F(\min I)| \\
 &\leq \sum_{(J,x) \in \Pi} |F'(x) |J| - F(\max J) + F(\min J)| \\
 &\leq \epsilon |I|
 \end{aligned}$$

Chapter 5

Change of variables

5.1 Theorem: change of variable

$$I \xrightarrow{\phi} \tilde{I} \xrightarrow{f} \mathbb{R}$$

I and \tilde{I} be compact cells, $\phi : I \leftrightarrow \tilde{I}$ be a (monotone) bijection which is diff. everywhere on I . If $f \in KH(\tilde{I})$ then $(f \circ \phi) |\phi'| \in KH(I)$ and

$$\int_I (f \circ \phi) |\phi'| = \int_{\tilde{I}} f |f'|$$

Proof

Let $\epsilon > 0$, exists a gauge $\tilde{\delta}$ on \tilde{I} s.t. if $\tilde{\Pi}$ is a $\tilde{\delta}$ -finite P-division, then

$$\left| S(\tilde{\Pi}, f) - \int_{\tilde{I}} f |f'| \right| < \epsilon$$

If Π is any P-division, then we can associate with it $\tilde{\Pi} = \{(\phi(J), \phi(x)) \mid (J, x) \in \Pi\}$ which is a P-division of \tilde{I}

Since ϕ is uniformly continuous on I , there exists $\eta :]0, +\infty[\rightarrow]0, +\infty[$ s.t. $\forall \delta > 0, \forall x, y \in I$ have

$$|x - y| \leq \eta(\delta) \Rightarrow |\phi(x) - \phi(y)| \leq \delta$$

Only a different interpretation of uniformly continuous. We define a gauge δ_1 on I :

$$\delta_1(x) = \eta \circ \tilde{\delta} \circ \phi(x)$$

Remark

If Π is a δ_1 -finite P-division of I then $\tilde{\Pi}$ is a $\tilde{\delta}$ -finite P-division of \tilde{I}

$$J = [y, z] \subseteq [x - \delta_1(x), x + \delta_1(x)]$$

$$\begin{aligned}\max\{|y-x|, |x-z|\} &\leq \delta_1(x) = \eta \circ \tilde{\delta} \circ \phi(x) \\ \max\{|\phi(y) - \phi(x)|, |\phi(x) - \phi(z)|\} &\leq \tilde{\delta} \circ \phi(x)\end{aligned}$$

Given $x \in I$, we define $\epsilon(x) = \frac{\epsilon}{1+|f \circ \phi(x)|}$. By lemma 4.1.1, there exists a $\delta_2(x) > 0$ s.t. if $J = [y, z] \subseteq [x - \delta_2(x), x + \delta_2(x)] \subseteq I$ contains x and then

$$\begin{aligned}||\phi(J)| - |\phi'(x)| \cdot |J|| &= ||\phi(y) - \phi(z)| - |\phi'(x)| \cdot |z - y|| \\ &= |\phi(z) - \phi(y) - \phi'(x)(z - y)| \\ &< \epsilon(x) |z - y| = \epsilon(x) |J|\end{aligned}$$

Define a gauge δ on I by $\delta = \min\{\delta_1, \delta_2\}$. If Π is a δ -finite P-division of I then

$$\begin{aligned}\left| \int_{\tilde{I}} f - S(\Pi, (f \circ \phi) \cdot |\phi'|) \right| &\leq \left| \int_{\tilde{I}} f - S(\tilde{\Pi}, f) \right| + \left| S(\tilde{\Pi}, f) - S(\Pi, (f \circ \phi) \cdot |\phi'|) \right| \\ &\leq \sum_{(J,x) \in \Pi} |f \circ \phi(x)| \cdot ||\phi(J)| - |\phi'(x)| \cdot |J|| \\ &\leq \sum_{(J,x) \in \Pi} |f \circ \phi(x)| \cdot \epsilon(x) \cdot |J| \\ &\leq \epsilon |I|\end{aligned}$$

Chapter 6

Integral on the real line

6.1 Saks-Henstock's theorem

Let I be a compact cell and $f \in KH(I)$ and $\epsilon > 0$ and δ a gauge on I which is ϵ -adapted to f . If Π is a partial δ -finite P-division of I then:

•

$$\left| \sum_{(J,x) \in \Pi} \left(\int_J f |J - f(x)|J| \right) \right| \leq \epsilon$$

•

$$\sum_{(J,x) \in \Pi} \left| \int_J f |J - f(x)|J| \right|$$

Proof

1 \Rightarrow 2

Given Π define

$$\Pi^+ = \Pi \cap \left\{ (J, x) \mid \int_J f |J - f(x)|J| \geq 0 \right\}$$

$$\Pi^- = \Pi \cap \left\{ (J, x) \mid \int_J f |J - f(x)|J| < 0 \right\}$$

let $\pi = \Pi^+ \sqcup \Pi^-$, then

$$\sum_{(J,x) \in \Pi^+} \left| \int_J f |J - f(x)|J| \right| + \left| \sum_{(J,x) \in \Pi^+} \int_J f |J - f(x)|J| \right| \leq \epsilon$$

the same for Π^-

prove (1)

$\Delta_\Pi = \{J \mid (J, x) \in \Pi\}$ is a partial division of I . There exists another partial division Δ' of I s.t. $\Delta \cup \Delta_\Pi$ is a division of I .

Let $\eta > 0$, $\forall K \in \Delta'$, there exists a gauge δ_K on K , η -adapted to $f|_K \in KH(K)$. Define $\tilde{\delta}_K(x) = \min\{\delta_K(x), \delta(x)\}$, $x \in K$, a gauge on K . Let Π_K be a $\delta\delta_K$ -finite P-division of K . Then

$$\left| \int_K -S(\Pi_K, f) \right| < \eta$$

Define $\tilde{\Pi} = \Pi \cup \left(\bigcup_{K \in \Delta'} \Pi_K \right)$ is a P-division of I and is δ -finite. Since δ is a ϵ -adpated to f and $\tilde{\Pi}$ is δ -finite on I , we have:

$$\left| \int_I f - S(\tilde{\Pi}, f) \right| < \epsilon$$

$$S(\tilde{\Pi}, f) = \sum_{(J,x) \in \Pi} f(x) |J| + \sum_{K \in \Delta'} S(\Pi_K, f)$$

$$\int_I f = \sum_{(J,x) \in \Pi} \int_I f|_J + \sum_K \int_K f|_K$$

then

$$\begin{aligned} \left| \sum_{(J,x) \in \Pi} \int_J f|_J - f(x) |J| \right| &\leq \left| \int_I f - S(\tilde{\Pi}, f) \right| + \left| \sum_{K \in \Delta'} \left(\int_K f - S(\Pi_K, f) \right) \right| \\ &< \epsilon + \sum_{K \in \Delta'} \left| \int_K f - S(\Pi_K, f) \right| \\ &\leq \epsilon + \eta \cdot (\text{card} \Delta') \end{aligned}$$

6.2 Hake Theorem

Let I be a compact cell, $f : I \rightarrow \mathbb{R}$ and for $0 < \eta < |I|$, put

$$I_\eta = [\eta + \min I, \max I]$$

TFAE

- $f \in KH(I)$
- $\forall \eta$,

$$f|_{I_\eta} \in KH(I_\eta) \text{ and } \lim_{\eta \rightarrow 0} \int_{I_\eta} f|_{I_\eta} \text{ exists}$$

In this case,

$$\int_I f = \lim_{\eta \rightarrow 0} \int_{I_\eta} f|_{I_\eta}$$

6.3 Corollary

If $f \in KH(I)$, then the **indefinite integral** of f

$$F : I \rightarrow \mathbb{R}$$

$$x \mapsto \begin{cases} \int_{[\min I, x]} f & \text{if } x > \min I \\ 0 & \text{if } x = \min I \end{cases}$$

is continuous by Hake Theorem 6.2

$$\int f := F$$

6.4 Prop

TFAE

- $f \in KH(I)$
- \exists continuous function $F : I \rightarrow \mathbb{R}$ s.t. $\forall \epsilon > 0 \exists$ a gauge δ on I , \forall partial δ -finite P-division Π of I :

$$\sum_{(J,x) \in \Pi} |f(x)|J| - F(\max J) + F(\min J)| < \epsilon$$

6.5 Def: KH-integral

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is KH-integrable if:

$\exists F : \mathbb{R} \rightarrow \mathbb{R}$ and $\lim_{x \rightarrow -\infty} F(x)$ and $\lim_{x \rightarrow +\infty} F(x)$ exists. $\forall \epsilon > 0 \exists$ a gauge δ on \mathbb{R} s.t. $\forall \Pi$ partial δ -finite P-division :

$$\sum_{(J,x) \in \Pi} |f(x)|J| - F(\max J) + F(\min J)| < \epsilon$$

and define

$$\int_{\mathbb{R}} f := \lim_{x \rightarrow +\infty} F(x) - \lim_{x \rightarrow -\infty} F(x)$$

Chapter 7

Monotone Convergence & Lebesgue's Measure

I be a closed cell.

7.1 Def: AKH-integrable

$$AKH(I) = KH(I) \cap \{f \mid |f| \in KH(I)\}$$

7.2 Prop

TFAE

- $f \in AKH(I)$

-

$$f^+ := \max\{f, 0\} \in KH(I) \quad f^- := \min\{f, 0\} \in KH(I)$$

Proof

$1 \Rightarrow 2$

$$f^+ = \frac{|f| + f}{2} \quad f^- = \frac{|f| - f}{2}$$

$2 \Rightarrow 1$

$$f = f^+ - f^- \quad |f| = f^+ + f^-$$

7.3 Prop

Let $f \in AKH(I)$, then

$$1 \quad \left| \int_I f \right| \leq \int_I |f|$$

2 $\forall J \subseteq I$, closed cell,

$$f|_J \in AKH(J)$$

7.4 Prop

Let $f \in KH(I)$, then

$$f \in AKH(I)$$

iff

$$\sup \left\{ \sum_{K \in \Delta} \left| \int_K f|_K \right| \mid \Delta \text{ is a partial division of } I \right\} < +\infty$$