0.1 Def: Tensor

Let M and N be two R-modules. Then exists an R-module denoted by $M \otimes_R N$ and a bilinear mapping

$$t : M \times N \to M \otimes_R N$$

having the following properties:

(1) For any R-module P and any bilinear mapping $s : M \times N \to P$. There exists a unique linear mapping $f_s : M \otimes_R N \to P$ such that $s = f_s \circ t$

$$\begin{tikzcd} M \times N \arrow[r, "s"] \arrow[d, "t"] & P \\ M \otimes_R N \arrow[ur, "f_s"'] & \end{tikzcd}$$

(2) If T, t' is another couple that satisfies (1) with $s \mapsto g_s$ then there exists a unique isomorphism

$$T \cong M \otimes_R N$$

Let $\mathscr{F}$ be the free R-module generated by $M \times N$

$$\mathscr{F} = \{ \sum_{finite} a_{ij}(m_i, n_i) : a_{ij} \in R, m_i \in M, n_i \in N \}$$

let $\mathscr{G}$ be the R-submodule generated by the elements of the following shape $m, m' \in M \quad n, n' \in N \quad \mathfrak{r} \in R$

$$\begin{aligned} &(m + m', n) - (m, n) - (m', n) \\ &(m, n + n') - (m, n) - (m, n') \\ &(\mathfrak{r}m, n) - \mathfrak{r}(m, n) \\ &(m, \mathfrak{r}n) - \mathfrak{r}(m, n) \end{aligned}$$

$$M \otimes_R N := \mathscr{F}/\mathscr{G}$$

0.2 Def

$$f_s(\mathscr{G} + (m, n)) := s(m, n)$$

Extend this mapping to linearity. This makes the diagram commutative. It's clearly the unique mapping

0.3 Def

The R-module $M \otimes_R N$ constructed above is called the tensor product of M and N. An element of $M \otimes_R N$ is called tensor. We denote

$$t(m, n) := m \otimes n$$

and any elements of this form is called pure tensor.

0.4 Remark

Pure tensors generate $M \otimes_R N$. In particular any tensor can be written as sum of pure tensors.

0.5 tensor product and duality

0.5.1 product

Let $V_1, \cdots, V_n$ be vector spaces as above. Then

$$(V_1^\vee \otimes \cdots \otimes V_n^\vee) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

0.5.2 duality

Let V and W be vector spaces of finite dimension. Then

$$\mathscr{L}(V, W) \cong V^\vee \otimes W^\vee$$

0.6 Def

We went to define the tensor product of linear mappings. letM_1, M_2, N_1, N_2 be R-modules and let $f_i : M_i \to N_i$ be linear mappings. Then we define

$$\begin{aligned} f_1 \otimes f_2 : M_1 \otimes M_2 &\to N_1 \otimes N_2 \\ m_1 \otimes m_2 &\mapsto f_1(m_1) \otimes f_2(m_2) \end{aligned}$$

This is a linear mapping

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "f_1 \times f_2"] \arrow[d] & N_1 \times N_2 \arrow[d] \\ M_1 \otimes M_2 \arrow[r, dashed, "f_1 \otimes f_2"] & N_1 \otimes N_2 \end{tikzcd}$$

0.7 Extension of scalars

Let $\varphi : R \to S$ be a commutative unitary ring homomorphism. Let M be a R-module. Goal is to give to M also a structure of S-module "conveyed by φ"

Note that S has a structure of R-module $s \in S, \mathfrak{r} \in R$

$$\mathfrak{r}s := \varphi(\mathfrak{r})s$$

Now take thw tensor product $M \otimes_R S$. Now we give a structure of S-module to $M \otimes_R S$.

Take $s \in S$

$$s(\underbrace{m \otimes s'}_{\in M \otimes_R S}) := m \otimes ss'$$

note that ss' is a multi in S and we cannot product sm.

Notice we've a mapping

$$\begin{aligned} i : M &\to M \otimes_R S \\ m &\mapsto m \otimes s \end{aligned}$$

Be careful, in general the mapping i is NOT injective.

0.8 Prop

Let $K \subseteq L$ be a field extension and let V be a K-vector space. Moreover let's denote $V_L = V \otimes_K L$. If $\{e_i\}_{i=1}^n$ is a basis of V then $\{e_i \otimes 1\}_{i=1}^n$ is a L-basis of V_L.(V_L has the same dim of V)

0.9 Def

We denote

$$\begin{aligned} T_p^q :=& (V^\vee)^{\otimes p} \otimes V^{\otimes q} \qquad p, q \in \mathbb{N} \\ =& \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{p \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \end{aligned}$$

An element of $T_p^q(V)$ is called a tensor of type (p, q) (or a mixed tensor which is p-covariant and q-contravariant)

Let's denote:

$$T(V) := \bigoplus_{q \in \mathbb{N}} T_0^q(V)$$

On $T(V)$ we have following operation:

$$\begin{aligned} T_0^l(V) \times T_0^q(V) &\to T_0^{l+q}(V) \\ ((x_1 \otimes \cdots \otimes x_l), (y_1 \otimes \cdots \otimes y_q)) &\mapsto x_1 \otimes \cdots \otimes x_l \otimes y_1 \otimes \cdots \otimes y_q \end{aligned}$$

With this operation $T(V)$ becomes a K-algebra. It called the tensor algebra associated to V

0.10 Def

The quotient algebra

$$\bigwedge(V) := T(V)/\left\{\sum_{i(finite)} (y_1 \otimes \cdots \otimes y_{m_i}) \otimes (x_i \otimes x_i) \otimes (z_1 \otimes \cdots \otimes z_{n_i})\right\}$$

is a K-algebra, which called the exterior algebra of V

$$\begin{aligned} \pi : &T(V) && \to \bigwedge(V) \\ &x_1 \otimes \cdots \otimes x_n && \mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

0.11 Notation

$$\bigwedge(V) = \bigoplus_{n \in \mathbb{N}} \overset{n}{\bigwedge}(V)$$

$$\overset{n}{\bigwedge}(V) := T_0^n(V)/(W \cap T_0^n(V))$$

this is called n-fold exterior product

0.12 Prop

FIx a vct space V. For any alternating multi-linear mapping

$$s : \underbrace{V \times \cdots \times V}_{n \text{ times}} \to W$$

when W is another vct space, there exists a unique linear mapping

$$g_s : \overset{n}{\bigwedge}(V) \to W$$

such that the following diagram commutes

$$\begin{tikzcd} V^n \arrow[r, "s"] \arrow[d, "t"'] & W \\ T_0^n(V) \arrow[ur, dashed, "f_s"] \arrow[d] & \\ \overset{n}{\bigwedge}(V) \arrow[uur, "g_s"'] & \end{tikzcd}$$

0.13 Prop

Let V be a vct space of dimension n with a basis $\{e_1, \cdots, e_n\}$. Then $\overset{k}{\bigwedge}(V)$ is a vct space with a basis given by

$$\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \big| \, 1 \leq i_1 < \cdots < i_k \leq n\}$$

In particular, $\overset{k}{\bigwedge}(V)$ has dimension $\begin{pmatrix} n \\ k \end{pmatrix}$

0.14 Def

Let V be a vct space of dimension n, then

$$\det(V) = \overset{n}{\bigwedge}(V)$$

is called the determinant of V. It is a vct space of dimension $1 = \binom{n}{n}$ and a basis is given by

$$e_1 \wedge \cdots \wedge e_n$$

when $\{e_1, \cdots, e_n\}$ is a basis of V.

0.15 Def

$$\begin{aligned} g_{\tilde{f}} = \overset{k}{\bigwedge} f : \overset{k}{\bigwedge}(V) & \to \overset{k}{\bigwedge}(V) \\ v_1 \wedge \cdots \wedge v_k & \mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

0.16 Def

Let $F : V \to V$ be a linear mapping. A subspace $V_0 \subseteq V$ is said to be an invariant subspace of F is $F(V_0) \subseteq V_0$

0.17 Def

A linear mapping $f : V \to V$ (finite dim) is diagonalizable if the following equivalent conditions are satisfied

1 V decomposes as a direct sum of one-dimensional invariant subspace of f

2 There exists a basis of V, in which the matrix A_f is diagonal.

0.18 Def

V a vector space over K $dim(V) = n, f \in \mathscr{L}(V;V)$ let A_f be an associated matrix (in any basis) the mapping

$$\begin{aligned} P : K &\to K \\ t &\mapsto \det(tI_n - A_f) \end{aligned}$$

This is a polynomial in $K[t]$ (with degree n)

0.19 Def

Let $a_0 + a_1 t + \cdots + a_n t^n = Q(t) \in K[t]$, then for $f \in \mathscr{L}(V;V)$ we define

$$Q(f) := a_0 id_V + a_1 f + a_2 f^{\circ 2} + \cdots + a_n f^{\circ n}$$

Remark From now on we write

$$f^{\circ k} = f^k$$

these are operations in $\mathscr{L}(V;V), +, \circ$

we say that Q annihilates f if $Q(f) = 0$

0.20 Prop

Let $f \in \mathscr{L}(V;V)$. There exists a polynomial $Q \in K[t] \setminus \{0\}$ that annihilates f (i.e. $Q(f) = 0$)

Remark

The proof of this proposition also gives the degree of a polynomial that annihilates ($\leq n^2$)

0.21 Def

Let $m(t) \in K[t] \setminus \{0\}$ be a monic polynomial of minimal degree that annihilates $f \in \mathscr{L}(V;V)$. Then $m(t)$ is called minimal polynomial of f And by prop above (0.20), $m(t)$ exists.

0.22 Prop

If $m(t)$ is minimal polynomial of f, then $m(t)$ is unique.

0.23 Prop

Let $Q \in K[t] \setminus \{0\}$ be a polynomial that annihilates f. Then $m_f \mid Q$

0.24 Theorem: Cayley-Hamilton Theorem

The characteristic polynomial P_f annihilates f

0.25 Theorem

Let $f \in \mathscr{L}(V;V)$ when V is a vector space of dim n, over an algebraically closed field.
Then

(1) f can be represented by a Jordan matrix

(2) This above matrix is unique up to permutation of the Jordan blocks

0.26 Def

Let $f \in \mathscr{L}(V;V)$ and let $\lambda \in K$. A vector $w \in V \setminus \{0\}$ is called a root vector of f corresponding to λ, if there exists $\mathfrak{r} \in \mathbb{N}$ s.t.

$$(f - \lambda id_V)^{\mathfrak{r}}(w) = 0$$

Remark

Eigenvector are root vectors (corresponding to their eigenvalues) take $\mathfrak{r} = 1$

Remark

Let $J_{\mathfrak{r}}(\lambda)$ be a Jordan block. Then any $\sigma \in V$ is a root vector of f corresponding to λ. In fact:

$$(J_{\mathfrak{r}}(\lambda) - \lambda I_n)^m = 0 \quad \text{if } m \geq \mathfrak{r}$$

0.27 Prop

Let K be an algebraically closed field. Let $\lambda_1, \cdots, \lambda_k$ be all of distinct eigenvalues of $f(k \geq 1)$, then

$$V = \bigoplus_{i=1}^{k} V(\lambda_i)$$

0.28 Def

Let $f \in \mathscr{L}(V;V)$. Then f is said to be nilpotent if there exists $t \in \mathbb{N}$ that $f^t = 0$

0.29 Lemma

Let f be a nilpotent mapping, then $Ker(f) \neq \{0\}$

Proof

Let $\mathfrak{r}$ be the minimal integer s.t. $f^{\mathfrak{r}} = 0$ then

$$f^{\mathfrak{r}-1}(V) \subseteq Ker(f)$$

but $f^{\mathfrak{r}-1}(V) \neq \{0\}$ because of the minimality of $\mathfrak{r}$

0.30 Theorem

Let $f \in \mathscr{L}(V;V)$ be a nilpotent mapping, then there exists a Jordan basis for f that gives a Jordan matrix made of blocks of the type $J_{\mathfrak{r}}(0)$

0.31 Theorem

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. Then f admits a Jordan basis (namely there exists a basis s.t. A_f is a Jordan matrix).

0.32 Def

Let λ be an eigenvalue of $f \in \mathscr{L}(V)$

$$E(\lambda) := \ker(f - \lambda Id)$$

This $E(\lambda)$ is called the eigenspace of λ

$$mult(\lambda)_{geo} = dim(E(\lambda))$$

is called the geometric multiplicity of λ

Moreover

$$mult(\lambda)_{alg} = \max\left\{k \in \mathbb{N} \,\middle|\, (t-\lambda)^k \mid P_f(t)\right\}$$

is called the algebraic multiplicity of λ

0.33 Prop

Let K be algebraically closed. Then $\forall \lambda$ eigenvalues of f

$$mult(\lambda)_{geo} \leq mult(\lambda)_{alg}$$

0.34 Corollary

Let K be an algebraically closed field. Let $f \in \mathscr{L}(V)$. f is diagonalizable iff

$$\forall \lambda_i \quad mult(\lambda_i)_{geo} = mult(\lambda_i)_{alg}$$

0.35 Def

Two matrices $G, G' \in M_{n\times n}(K)$ are said conjugate if $\exists A \in \mathcal{Q}_{n\times n}(K)$ s.t. $G = G'^T$

0.36 Def

Let $p \in \mathbb{R}^n$ be a fixed point

$$\mathbb{R}^n_p := \{p\} \times \mathbb{R}^n$$

$(p, a) \in \mathbb{R}^n_p, a \in \mathbb{R}^n$

$$(p, a) + (p, b) = (p, a + b)$$
$$\alpha(p, a) = (p, \alpha a) \ \alpha \in \mathbb{R}$$

With these operation $\mathbb{R}^n_p$ is a vector space, which is called the tangent space of $\mathbb{R}^n$ at p.

The dual space is

$$(\mathbb{R}^n_p)^\vee = \{p\} \times (\mathbb{R}^n_p)^\vee$$

A basis of $\mathbb{R}^n_p$ is denoted by

$$(e_1 \mid_p, \cdots, e_n \mid_p)$$

$\bigsqcup\limits_p \mathbb{R}^n_p$ is called the tangent bundle of $\mathbb{R}^n$

0.36.1 Notation

$$a \mid_p := (p, a)$$

0.37 Def

Let $p \in \mathbb{R}^n$ be a fixed point

$$\mathbb{R}^n_p := \{p\} \times \mathbb{R}^n$$

$(p, a) \in \mathbb{R}^n_p, a \in \mathbb{R}^n$

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A basis of $\mathbb{R}^n_p$ is denoted by

$$(e_1 \mid_p, \cdots, e_n \mid_p)$$

$\bigsqcup\limits_p \mathbb{R}^n_p$ is called the tangent bundle of $\mathbb{R}^n$

We have a projection mapping:

$$\begin{aligned} \bigsqcup_p \mathbb{R}^n_p &\overset{\pi}{\twoheadrightarrow} \mathbb{R}^n_p \\ (p, a) &\mapsto p \end{aligned}$$

and

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R}^n &\cong \bigsqcup_p \mathbb{R}^n_p \\ (p, a) &\leftarrow\!\shortmid (p, a) \end{aligned}$$

Take $\{e_1 \mid_p, \cdots, e_n \mid_p\}$ as a basis of $\mathbb{R}^n_p$. The dual basis is denoted by

$$\{dx_1 \mid_p, \cdots, dx_n \mid_p\} = \{(e_1 \mid_p)^\vee, \cdots, (e_n \mid_p)^\vee\} \in (\mathbb{R}^n_p)^\vee$$

$$\begin{aligned} dx_i \mid_p : \mathbb{R}^n_p &\to \mathbb{R} \\ v = (\sum \alpha_i e_i \mid_p) &\mapsto \alpha_i \end{aligned}$$

$$\frac{\partial x_i}{\partial x_j} = dx_i \mid_p (e_j \mid_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recalled the wedge algebra:

$$\bigwedge(\mathbb{R}^n_p)^\vee := T(\mathbb{R}^n_p)^\vee / I = \bigoplus_{k \in \mathbb{N}} \bigwedge^k (\mathbb{R}^n_p)^\vee$$

Consider

$$\bigwedge^k (\mathbb{R}^n_p)^\vee$$

what's a basis of this vector space?

$$\left\{ dx_1 \mid_p \wedge \cdots \wedge dx_k \mid_p \,\middle|\, 1 \leq i_1 < \cdots < i_k \leq n \right\}$$

and

$$\dim(\bigwedge^k (\mathbb{R}^n_p)^\vee) = \binom{n}{k}$$

Proved.

0.38 Do Carmo Differential forms

0.39 Def

An exterior k-form in $\mathbb{R}^n$ is a mapping:

$$\begin{aligned} \omega : \mathbb{R}^n &\to \bigsqcup_p \bigwedge^k (\mathbb{R}^n_p)^\vee \\ p &\mapsto \omega(p) \end{aligned}$$

that's a section of the projection π

$$(\pi \circ \omega = id_{\mathbb{R}}) = (\omega(p) \in \bigwedge^k (\mathbb{R}^n_p)^\vee)$$

$$\omega(p) = \sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1, \cdots, i_k}(p) dx_{i_1} \mid_p \wedge \cdots \wedge dx_{i_k} \mid_p \in \bigwedge^k (\mathbb{R}^n_p)^\vee$$

Note that

$$\begin{aligned} \bigsqcup_p \bigwedge^k (\mathbb{R}^n_p)^\vee &\overset{\pi}{\twoheadrightarrow} \mathbb{R}^n \\ f \mid_p &\mapsto p \end{aligned}$$

$$\omega \leftrightarrow \{a_{i_1}, \cdots, a_{i_k}\}$$

if all a_{i_j} are of class $C^m(\mathbb{R})$ the ω is called a C^m-differential k-form. If $m = +\infty$ $omega$ is called a smooth k-form.

0.40 Notation

$$\omega = \sum_I a_I dx_I$$

where $I = (i_1, \cdots, i_k)$

0.41 Notation

When $k = 0$ a 0-form of class C^m-differential 0-form is $f \in C^m(\mathbb{R}^n)$

$$C^m(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \text{ of class } C^m\}$$

0.42 Notation

$$\Omega^k_{(m)}(\mathbb{R}^n) := \{\text{set of } C^m\text{-diff } k\text{-forms}\}$$
$$\Omega^0_{(m)}(\mathbb{R}^n) = C^m(\mathbb{R}^n)$$

m could be omitted if no confusion.

0.43 Def

Now we have

$$\Omega(\mathbb{R}^n) = \bigoplus_{k\in\mathbb{N}} \Omega^k(\mathbb{R}^n)$$

a $\mathbb{R}$-algebra with the $\wedge$-product

And it's also a $\Omega^0(\mathbb{R}^n)$ module and $\Omega^0(\mathbb{R}^n)$-algebra

0.44 Def: Pullback of forms

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of $C^{\mathfrak{r}}$, then it induces a mapping

$$f^* : \Omega^k_{(\mathfrak{r})}(\mathbb{R}^m) \to \Omega^k_{(\mathfrak{r})}(\mathbb{R}^n)$$
$$\omega \mapsto f^*\omega$$

and

$$f^*(\omega)(p)(v_1, \cdots, v_k) = \omega(f(p))(df\mid_p (v_1), \cdots, df\mid_p (v_k))$$

recalling

$$df\mid_p : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)} \;\Rightarrow\; df\mid_p (v_i) \in \mathbb{R}^n_{f(p)}$$

0.45 Remark

$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$

$$f \wedge \omega = f\omega$$

0.46 Prop

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable mapping. Then

(1) for any two forms in $\mathbb{R}^m$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*(\eta))$$

(2) for $g : \mathbb{R}^p \to \mathbb{R}^n$ differentiable

$$(f \circ g)^*\omega = g^*(f^*\omega)$$

0.47 Def: Path integral

Let γ and ω be as above.

$$\int_\gamma \omega := \sum_i \int_{t_k}^{t_{k+1}} \gamma_j^* \omega$$

this is the integral of ω along the parametric curve γ with

$$\gamma = t \mapsto (x_1(t), \cdots, x_n(t))$$

where $x_i(t) = \frac{\mathrm{d}x_i}{\mathrm{d}t}$

0.48 Def(σ-finite)

Let (X, Σ_X, μ) be a measure space. WE say that it's σ-finite if there exists a sequence $\{E_n\}_{n\in\mathbb{N}}$ of measurable sets. (namely $E_n \in \Sigma_X$) such that

$$X = \bigcup_{n\in\mathbb{N}} E_n \text{ and } \mu(E_n) < +\infty, \forall n \in \mathbb{N}$$

0.49 Notation

Take sets $A \subseteq X \times Y$ For $x \in X$, we define

$$A_x := \{u \in Y \mid (x, y) \in A\}$$

called a **vertical section** of A or $x-$fiber of A

For $y \in Y$ we define

$$A_y := \{x \in X \mid (x, y) \in A\}$$

called a **horizontal section** of A, or y-fiber of A

0.50 Def

Let X be a set. then $\mathscr{D} \subseteq \wp(X)$ is a **Dynkin system** if

- $X \in \mathscr{D}$ and $\varnothing \in \mathscr{D}$
- $\forall D \in \mathscr{D} \quad X \setminus D \in \mathscr{D}$
- If $\{D_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathscr{D}$ of pairwise disjoint sets, then

$$\bigsqcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

Remark

A σ-algebra is a Dynkin system

0.51 Def

Let $(\mathscr{G} \subseteq \wp(X))$ then $\delta(\mathscr{G}) \subseteq \wp(X)$ is called the Dynkin system generated by $\mathscr{G}$ if

- $\mathscr{G} \subseteq \delta(\mathscr{G})$
- If $\mathscr{D}$ is a Dynkin system containing $\mathscr{G}$, then $\delta(\mathscr{G}) \subseteq \mathscr{D}$

0.52 Prop

If $\mathscr{D}$ is a Dynkin system closed under the intersection, then it's a σ-algebra, namely

$$\forall (D, E) \in \mathscr{D}^2, D \cap E \in \mathscr{D} \ \Rightarrow \ \forall \{D_n\}_{n\in\mathbb{N}} \in \mathscr{D}^{\mathbb{N}} \quad \bigcup_{n\in\mathbb{N}} D_n \in \mathscr{D}$$

0.53 Prop

Let X be a set and let $\mathscr{G} \subseteq \wp(X)$. Assume that $\mathscr{G}$ is closed under the finite intersection. Then

$$\delta(\mathscr{G}) \subseteq \sigma(\mathscr{G})$$

0.54 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. Then $\forall E \in \Sigma_X \otimes \Sigma_Y$, the functions

$$\begin{aligned} f_E : &X \to \mathbb{R} \cup \{+\infty\} \\ &x \mapsto \nu(E_x) \\ g_E : &Y \to \mathbb{R} \cup \{+\infty\} \\ &y \mapsto \mu(E_y) \end{aligned}$$

are respectively Σ_X-measurable and Σ_Y-measurable

0.55 Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be σ-finite measure spaces. There exists a unique σ-finite measure $\mu \times \nu$ on $(X \times Y, \Sigma_X \otimes \Sigma_Y)$ such that

$$\mu \times \nu(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

and moreover, we have

$$(\mu \times \nu)(E) = \int_X f_E \mathrm{d}\mu = \int_Y g_E \mathrm{d}\nu$$

0.56 Def: Push-forward measure

Let (X, Σ_X, μ) be a measure space, and let (Y, Σ_Y) be a measurable space. If $f : X \to Y$ is a measurable function, then define:

$$f_{*\mu}(E) = \mu(f^{-1}(E)) \quad \forall E \in \Sigma^Y$$

This is a measure on Y, called the push forward of μ through f

0.57 Fubini-Tobelli Theorem

Let (X, Σ_X, μ) and (Y, Σ_Y, ν) be two σ-finite measure spaces. Let $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$ be the product space. Let $f : X \times Y \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a measurable function. Then

$$\begin{aligned} \int_{X \times Y} |f| \, \mathrm{d}(\mu \times \nu) &= \int_X (\int_Y |f(x,y)| \, \mathrm{d}\nu(y))\mathrm{d}\mu(x) \\ &= \int_X (\int_Y |f(x,y)| \, \mathrm{d}\nu(y))\mathrm{d}\mu(x) \end{aligned}$$

0.58 Notation

For any mapping $\gamma : [a, b] \to U$

- γ is called a closed curve if $\gamma(a) = \gamma(b)$ and γ is a curve
- γ is called a path if γ is of class C^0
- γ is called a loop if γ is a closed path

0.59 Def: Lebesgue number

Let (X, ρ) be a metric space and $\mathcal{U} = \{U_i\}$ be an open covering X

A **Lebesgue number** $\delta = \delta_{\mathcal{U}}$ (of the open covering $\mathcal{U}$) is a non-negative number that:

If $Z \subseteq X$ is a subset with $diam(Z) < \delta$, then $Z \subseteq U_j$ for some $U_j \in \mathcal{U}$

Remark

- $\delta' < \delta$ is also a Lebesgue number
- In principle, a Lebesgue number δ can be 0

0.60 Lemma

If X is compact, then for any open covering there exists a positive Lebesgue number.

0.61 Theorem(homotopy invariance of the integrals)

Ler ω be a closed form on an open set U. Let γ_0, γ_1 be homotopy paths in U, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

0.62 Def: Free Homotopy

Let $\gamma_0, \gamma_1 : [a, b] \to U$ be two loops (namely $\gamma(a) = \gamma(b)$)

A **free homotopy** between γ_0 and γ_1 is a continuous mapping:

$$\begin{aligned} H : &[a, b] \times [0, 1] &&\to U \\ &(s, t) &&\mapsto H(s, t) \end{aligned}$$

such that

- $$H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$$
- For any fixed t_0
$$H(\cdot, t_0)$$
is a loop

0.63 Notation

A path $\gamma : [a, b] \to I$ is said simple if $\gamma \mid_{]a,b[}$ is injective (No self-cross this is)

0.64 Jordan Theorem

Let γ be a simple loop $\gamma : [a, b] \to U$, then $\mathbb{R}^2 \setminus \gamma([a, b])$ consists exactly of two connected components. One of this is bounded (interior), the other one unbounded (exterior). Moreover $\gamma([a, b])$ is the boundary of two components.

0.65 Def

Let $c : [a, b] \to S^1$ be a closed curve. Let φ be the angular function of c. We define the winding number of c as:

$$n(c) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

Since c us a closed curve, $n(c) \in \mathbb{Z}$

0.66 Def

Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\}$ be a closed curve. $(\gamma_p + \rho(t)c(t))$, when $c(t) \in S^1$

$$\gamma(t) = p + \rho(t)(\cos(\theta(t)) + \sin(\theta(t)))$$

Then we define the winding number of γ at p

$$n_p(\gamma) := n(c)$$

0.67 Prop

Let $\gamma = p + \rho(t)c(t)$ be a closed curve $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{p\}$ then

$$n_p(\gamma) = \frac{1}{2\pi i}\int_C \omega_0$$

where

$$\omega_0 = -\frac{y}{x^2 + y^2}\mathrm{d}x + \frac{x}{x^2 + y^2}\mathrm{d}y$$

0.68 Prop

Let $\gamma_0, \gamma_1 : [0, b] \to \mathbb{R}^2 \setminus \{p\}$ be two closed curves. Then they're freely homotopic iff

$$n_p(\gamma_0) = n_p(\gamma_1)$$

0.69 Def

Let $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differential mapping. We say that $p \in U$ is a zero of F if $F(p) = 0$. If then exists a neighborhood V of p such that V contains no zero of F other then p, then p is called isolated zero.

If p is a zero of F and $\mathrm{d}F\mid_p$ is non singular at p, then we say that p is a simple zero.

0.70 Def

The index of F in D, is defined as

$$n(F,D) := \frac{1}{2\pi}\int_C \theta$$

See that $\theta = F^*\omega_0$, $\omega_0 = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2}$

$$\begin{aligned} n(F,D) &= \frac{1}{2\pi}\int_C \theta \\ &= \frac{1}{2\pi}\int_C F^*\omega_0 \\ &= \frac{1}{2\pi}\int_{F\circ} \omega_0 \\ &= (\text{winding number of } F\circ C \text{ at the center of } FD) \end{aligned}$$

0.71 Remark

$$n(F,D) = \frac{1}{2\pi}\int_C \theta = \frac{1}{2\pi}\int_{F\circ C} \omega_0$$

0.72 Prop

If $n(F,D) \neq 0$ then $\exists q \in D$ s.t. $F(q) = 0$

0.73 Def

A simple zero p of F is said **positive** if $\det(\mathrm{d}_p F) > 0$, otherwise is said **negative** ?(what's =0?)

0.74 Kronecker Index Theorem

Assume that $F; U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ has only finite simple zeros in a disk $D \subseteq U$ and none of them in ∂D. Then

$$n(F,D) = P - N$$

where P is the number of positive simple zeros and N is the number of negative simple zeros.

0.75 Def

Let $\mathscr{P} = \{t = t_a, t_1, \cdots, t_n = b\}$, $p_i = \gamma(t_i)$

$$l_{\mathscr{P}}(\gamma) = \sum_{i=0}^{n} \|p_{i+1} - p_i\|$$

The length of γ is

$$l(\gamma) := \sup_{p}\{l_p(\gamma)\}$$

If $l(\gamma) < +\infty$, then path γ is said rectifiable.

0.76 Prop

Let $\gamma : [a, b] \to \mathbb{R}^n$ be of class C^1, then γ is rectifiable and

$$l(\gamma) = \int_a^b \|\gamma'(t)\| \, \mathrm{d}t$$

moreover $l(\gamma)$ doesn't depend on the parametrization of γ

0.77 Corollary(exercise)

If γ is a curve (piecewise C^1), then γ is rectifiable and the length is the sum of the length of it's C^1 pieces.

0.78 Def

A C^1-curve is **regular** if $\gamma'(t) \neq 0$ for any $t \in [a, b]$ A piecewise C^1-path (curve) is regular if all its pieces are regular

0.79 Def

$$N := \frac{T}{\|T\|}$$

is **normal vector** of T

0.80 Def

Let $\gamma : [a, b] \to \mathbb{R}^n$ a C^1 curve; Let l be the length of γ (by theorem proved $l(\gamma) < +\infty$) Let's define the following function:

$$s(t) := \int_a^t \|\gamma'(t)\| \, \mathrm{d}u$$

$s(t)$ is the length of $\gamma \mid_{[a,t]}$ THe function $\|\gamma'(u)\|$ iss continuous, hence

$$s'(t) = \|\gamma'(t)\|$$

Now assume that γ is C^1 and **regucar**$(\gamma'(t) \neq 0, \forall t \in [a,b])$, then $s'(t) > 0$

So $s : [a,b] \to [0,l]$ is a C^1-differmorphism, the inverse is

$$t : [0,l] \to [a,b]$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = frac1\|\gamma'(t)\|$$

We reparameterize γ with t and get

$$\tilde{\gamma}(s) = (\gamma \circ t)(s)$$

$\tilde{\gamma} : [0,l] \to \mathbb{R}^n$ we say that $\tilde{\gamma}$ is the reparameterization of γ with respect to its **curvilinear coordinate** $s(t)$

0.81 Def

$f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ f is a $C^{(k)}$-differ if

- f is of class $C^{(k)}$
- f is bijection, and the inverse is $C^{(k)}$

0.82 Def

In general

$$\gamma : [a,b] \to \mathbb{R}^n \rightsquigarrow \tilde{\gamma} : [0,l] \to \mathbb{R}^n$$

regular and C^1

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\mathrm{d}\gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$\left\|\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}\right\| = 1$

$$T(t) := \frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

tangent: (vector) $\to$ vector of norm 1

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\|T(t)\|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\langle T(t), T(t)\rangle = 2\langle T(t), T'(t)\rangle \Leftrightarrow T'(t) \perp T(t)$$

use the fact that in $\mathbb{R}^n$, $u, v : \mathbb{R} \to \mathbb{R}^n$ differentiable

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), v(t)\rangle = \left\langle \frac{\mathrm{d}u}{\mathrm{d}t}, v(t)\right\rangle + \left\langle u(t), \frac{\mathrm{d}v}{\mathrm{d}t}\right\rangle$$

then

$$\begin{aligned}\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2} &= \frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}s}) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}(T(t)) \\ &= \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} \\ &= \frac{T'(t)}{\|\gamma'(t)\|}\end{aligned}$$

$N(t) = \frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2}/\left\|\frac{\mathrm{d}^2\tilde{\gamma}}{\mathrm{d}s^2}\right\|$. If $n = 2$ Along the curve we have a 'moving' canonical basis of

$$\begin{aligned}\mathbb{R}^2_{\gamma(t)} =&\{T(t), N(t)\} \\ &= \{\alpha T(t) + \beta N(t) \mid \alpha, \beta \in \mathbb{R}\}\end{aligned}$$

$\{T(t), N(t)\}$ is a orthonormal basis of $\mathbb{R}^2_{\gamma(t)}$

0.83 Def:isometry

$$(V, g) \xrightarrow{f} (W, g')$$

a morphism f of vector space with inner product is **isometry** if

$$g(x, y) = g'(f(x), f(y))$$

0.84 Def:isometric

$V \xrightarrow{\cong} W$ up to isomorphism.
Then (V, g) and (W, g') are **isometric** if there are two isometry

$$\begin{aligned}f : (V, g) \to &\quad (W, g') \\ f' : (W, g') \to &\quad (V, g)\end{aligned}$$

such that

$$f \circ f' = f' \circ f = Id$$

0.85 Def: Semilinear

If V and W are two complex vector sapce, then a **semilinear mapping** is a mapping $f : V \to W$ such that

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha * f(v) = \overline{\alpha} f(v)$

So a semilinear mapping is a linear mapping: $f : V \to W$

For sesquilinear forms, the theory is similar to the theory of bilinear forms.

$$g \rightsquigarrow G(\text{fix a basis}) \quad g(x,y) = xG\overline{y}$$

If you change basis, then the Gram matrix changes in the following way:

$$G \rightsquigarrow A^T G \overline{A}$$

If g is bilinear

$$g \rightsquigarrow \tilde{g} : V \to V^\vee$$

and

$$g \rightsquigarrow \tilde{g} : V \to \overline{V^\vee}$$

linear if g is sesquilinear ($\tilde{g} : V \to V^\vee$ is semilinear)

0.86 Def

A sesquilinear form $g : V \times \overline{V} \to K$ is **hermitian** if

$$g(x,y) = \overline{g(y,x)}$$

And note that inner product is any of symmetric symplectic or hermitian.

Chapter 1

Classification (up to isometry) of vector spaces of small dim

Let (V, g) be vector space over $K(=\mathbb{R}, \mathbb{C})$ with inner product.

1.1 $\dim V = 1$ and g is symmetric

choose $v \in V \setminus \{0\}$ if $g(v,v) = 0$, then g is degenerated $\Rightarrow\ g = 0$
If g is non-deg (non-degenerate) $\exists v$ s.t. $g(v,v) = a \neq 0$

$$\forall x \in K \quad g(xv, xv) = ax^2$$

Any v s.t. $g(v,v) = a \neq 0$ induce a set

$$\mathcal{C}(v) := \{ax^2 : x \in K^*\}$$

this is an element in $K^*/\{x^2 \mid x \in K^*\}$

1.1.1 Prop

Let $(V_1, g_1), (V_2, g_2)$ be two vector spaces of dim 1 s.t. g_1 and g_2 are symmetric. Then (V_1, g_1) and (V_2, g_2) are isometric iff

$$\exists v_1 \in V_1, v_2 \in V_2 \text{ s.t. } \mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$$

1.1.2 Theorem

(V, g) has dim 1, g symmetric. Then (V, g) is isometric to one of the following

- $K = \mathbb{R}$

$$(\mathbb{R}, g(x,y) = xy) \quad (\mathbb{R}, g(x,y) = -xy) \quad (\mathbb{R}, g(x,y) = 0)$$

- $K = \mathbb{C}$

$$(\mathbb{C}, g(x,y) = xy) \quad (\mathbb{C}, g(x,y) = 0)$$

1.2 $\dim V = 1$ g is hermitian

Again g degenerate $\Rightarrow g = 0$We use that same reason as above. $v \in V$: $g(v,v) = a \neq 0$, $\forall a \in \mathbb{C}^*$

$$g(av, av) = \|a\|^2 g(v,v)$$

So any element $v \in V \setminus \{0\}$ s.t. $g(v,v) = a$ induces a coset in $\mathbb{C}^*/\mathbb{R}_{>0}$
Inside $\mathbb{C}^*$, $\mathbb{R}_{>0}$ is a (mult) subgroup
For any $z \in \mathbb{C}^*$ can be written uniquely as $z = re^{i\theta}$, hence

$$\begin{array}{rcl} C^* \cong & \mathbb{R}_{>0} \times S^1 \to & S^1 \\ z \mapsto & (r, e^{i\theta}) \mapsto & e^{i\theta} \end{array}$$

The kernel is $\mathbb{R}_{>0}$ and $S^1 \cong \mathbb{C}^*/\mathbb{R}_{>0}$.
But g is hermitian, so

$$g(v,v) \in \mathbb{R}$$

If follows that the coset

$$\{\|a\|^2 g(v,v) \mid a \in \mathbb{C}^*\} \in (\mathbb{C}^*/\mathbb{R}_{>0}) \cap (\mathbb{R}/\mathbb{R}_{>0}) \cong \{\pm 1\}$$

We repeat the proposition before with g hermitian and the following theorem

1.2.1 Theorem

(V, g) if dim 1, with g hermitian. Then (V, g) is isometry to one of the following

$$(\mathbb{C}, g(x,y) = x\overline{y}) \quad (\mathbb{C}, g(x,y) = -x\overline{y}) \quad (\mathbb{C}, g(x,y) = 0)$$

1.3 $\dim V = 1$ g symplectic

With dim 1 $\forall v_1, v_2 \in V$ can be write as $v_1 = ae, v_2 = be$ with $e \in K^*$

$$g(v_1, v_2) = ab \cdot g(v,v) = 0$$

1.3.1 Theorem

(V, g) of dim 1, g symplectic, then

$$(V, g) \cong (K, g = 0)$$

1.4 $\dim V = 2$ g symplectic

Assume that g is degenerated, then $\exists x \in V$ s.t. $g(x, y) = 0, \forall y \in V$
Extend x to a basis $\{x, x'\}$ of V

$$g(ax+a'x', bx+b'x') = ab \cdot g(x,x) + ab' \cdot g(x,x') - a'b \cdot g(x,x') + a'b' \cdot g(x',x') = 0$$

So when g degenerated $g = 0$
Take g non-degenerated $\exists v_1, v_2 \in V$ s.t. $g(v_1, v_2) = a \neq 0$.
For $g(a^{-1}v_1, v_2) = a^{-1}a = 1$, we may assume that $a = 1$
Let's show that v_1, v_2 are linearly independent. Assume by contraction: $v_1 = \lambda v_2$

$$1 = g(v_1, v_2) = g(\lambda v_2, v_2) = \lambda \cdot g(v_2, v_2) = 0$$

$\Rightarrow \{v_1, v_2\}$ is a basis of V. Then

$$\alpha_1\beta 2 - \alpha_2\beta_1 = g(\alpha_1 v_1 + \alpha_2 v_2, \beta_1 v_1 + \beta_2 v_2) = (\alpha_1, \alpha_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \overset{(?)}{=} \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \end{pmatrix}$$

1.4.1 Theorem

(V, g) is dim 2, g symplectic. Then (V, g) is isometric to one of the following

$$(K^2, g(x,y) = 0) \quad (K^2, g(x,y) = x_1y_2 - x_2y_1)$$

Chapter 2

Compliment

2.1 Def: non-degenerate

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a subspace. We say that V_0 is non-degenerate if $g\mid_{V_0}$ is non-degenerate.

Moreover, V_0 is isotropic if $g\mid_{V_0} = 0$

Remark

Isotropic means degenerate

2.2 Def

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a subspace. The orthogonal complement of V_0 is define as

$$V_0^\perp := \{v \in V \mid g(v, v_0) = 0, \forall v_0 \in V_0\}$$

2.3 Prop

Let (V, g) be a inner product space. Let $V_0 \subseteq V$ be a non-degenerate subspace. Then

$$V = V_0 \oplus V_0^\perp$$

2.4 Theorem

Let (V, g) be an finite dimensional inner product space. If both V_0 and $V_0^\perp$ are non-degenerate, then $(V^\perp)^\perp = V_0$

2.5 Theorem

Let (V, g) be an finite dimensional inner product space. Then There exists a decomposition

$$V = V_1 \oplus \cdots \oplus V_n$$

such that $\{V_i\}_{i=1}^n$ are pairwise orthogonal and

1 They are 1-dim if g is symmetric or hermitian

2 They are 1-dim but degenerated or 2-dim non-degenerate if g is symplectic.

Chapter 3

Signature

Now we discuss the uniqueness of such decomposition

3.1 Def

Let (V, g) be an inner product space with dim 1. Moreover, assume that g is symmetric or hermitian. We say that (V, g) is **positive** if (V, g) is isometry to either $(\mathbb{R}, g(x, y) = xy)$ or $(\mathbb{C}, g(x, y) = x\overline{y})$

We say that (V, g) is **negative** if (V, g) is isometry to either $(\mathbb{R}, g(x, y) = -xy)$ or $(\mathbb{C}, g(x, y) = -x\overline{y})$

3.2 Notation

By theorem 2.5, we can count the number of positive subspace of any inner product space.

- $r_0 := \dim \ker g$
- $r_+ :=$ the number of positive subspaces
- $r_- :=$ the number of negative subspaces

3.3 Def

Let (V, g) be an inner product space.

1 If g is real symmetric or, hermitian, then (r_0, r_+, r_-) is signature of V

2 If g is symplectic or complex symmetric, then $(\dim V, r_0)$ is the signature of V

3.4 Theorem

Let (V,g) and (V',g') be two inner product spaces, with g, g' that are either (both) symplectic or complex symmetric.

Then (V,g) and (V',g') are isometric iff

$$(n, r_0) = (n', r_0')$$

3.5 Theorem

Let (V,g) and (V',g') be two inner product spaces, with g, g' that are either (both) hermitian or real symmetric.

Then (V,g) and (V',g') are isometric iff

$$(r_0, r_+, r_-) = (r_0', r_+', r_-')$$

Chapter 4

Orthonormal

4.1 Def

Let (V, g) be an inner product space. The basis $\{v_1, \cdots, v_n\}$ is said **orthogonal** if $g(v_i, v_j) = 0, \forall i \neq j$

Moreover, g is said **orthonormal** if $g(v_i, v_i) \in \{0, -1, 1\}, \forall i$

Remark

If g is hermitian or symmetric. We can always find an orthonormal basis from an orthogonal basis.

4.2 Def

Let V be a vector space over $K(charK \neq 2)$. A **quadratic form** on V is a mapping $a : V \to K$ such that

- $q(\alpha v) = \alpha^2 q(v) \forall \alpha \in K, v \in V$
- $f = (u, v) \mapsto \ q(u + v) - q(u) - q(v)$ is bilinear

Remark

Any symmetric bilinear form $h : V^2 \to K$ is a quadratic form. Given a quadratic form $q : V \to K$, we can define a symmetric bilinear form

$$h_p(u, v) = \frac{1}{2}\left(q(u + v) - q(u) - q(v)\right)$$

4.3 Gram-Schmidt algorithm

Let (V, g) be an inner product space with g symmetric or hermitian. Let $\{v'_1, \cdots, v'_n\}$ be a basis of V such that $V_i = \langle v'_1, \cdots, v'_i \rangle$ $\forall i \in \{1, \cdots, n\}$ is non-degenerate.

Then there exists an orthogonal basis $\{v_1, \cdots, v_n\}$ such that $V_i = \langle v_1, \cdots, v_i \rangle$ $\forall i \in \{1, \cdots, n\}$ is non-degenerate.

Chapter 5

Euclidean and Unitary Spaces

5.1 Def:Euclidean vector space

A euclidean vector space is a finite dimensional inner product space over $\mathbb{R}$ (E, g) with g symmetric and positive definite $(g(x, x) > 0, \forall x \neq 0)$

We denote

$$\langle x, y\rangle := g(x, y)$$

Remark

Any non-zero subspace of $(E, \langle\cdot,\cdot\rangle)$ is non-degenerated:

The signature of E is of the type $(0, \mathfrak{r}_+, \mathfrak{r}_-)$ denoted by $(P, Q)(P = \mathfrak{r}_+, Q = \mathfrak{r}_-)$

An euclidean space is a normed vector space

$$\|x\| = \sqrt{\langle x, x\rangle}$$

($\langle,\rangle$ positive defined required)

Any euclidean vector admits an orthonormal basis $\{v_1, \cdots, v_n\}, \langle v_i, v_i\rangle = 1$
So orthonormal means

$$\|v_1\| = 1$$

In a euclidean space we have a distance

$$d(x, y) = \|x - y\|$$

5.2 Prop

A euclidean space $(E, \langle,\rangle)$ of dim n is isometric to $(\mathbb{R}^n, \underbrace{\langle,\rangle}_{\text{usual scalar product}})$

5.3 Remark

Cauchy-Schwartz inequality

$$\langle x, y\rangle \leq \|x\| \, \|y\|$$

Triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

5.4 Pythagoras's Theorem

If $x_1, \cdots, x_k$ are pairwise orthogonal, then

$$\left\| \sum_{i=1}^{k} x_i \right\|^2 = \sum_{i=1}^{k} \|x_i\|^2$$

5.5 Def:Angles

By Cauchy-Schwartz inequality:

$$-1 \leq \frac{\langle x, y\rangle}{\|x\| \, \|y\|} \leq 1$$

Then there exists a element $\phi \in [0, \pi]$ such that

$$\cos\phi = \frac{\langle x, y\rangle}{\|x\| \, \|y\|}$$

ϕ is defined as the angle between x and y

Notice that ϕ is not an oriented angle.

5.6 Notation

Let $U, V \subseteq E$ be two subspace, then

$$d(U, V) := \inf\{\|u - v\| \mid u \in U, v \in V\}$$

5.7 Def

$V \subsetneq E$ a vector subspace, $x \in E \setminus \{0\}$. Then $E = V \oplus V^{\perp}$ (proved) Then we write (uniquely)

$$x = x_0, + x_0'$$

where $x_0 \in V, x_0' \in V^{\perp}$.

Then x_0 is called the orthogonal projection of x on V, x_0' is called the orthogonal projection of x on $V^{\perp}$

5.8 Prop

Use the above notation:

$$d(x, V) = \|x_0'\|$$

5.9 Prop

Use the previous notations. Assume that $m = \dim V$, $V \subseteq E$, $\{v_1, \cdots, v_m\}(m \leq n = \dim E)$ is an orthonormal basis of V. Then

$$x_0 = \sum_{i=1}^{m} \langle x, v_i \rangle \, v_i$$

5.10 Relationship with calculus

$(E, \langle , \rangle) = (\mathbb{R}^n, \langle , \rangle)$ on $\mathbb{R}^n$ we have the notion of volumes

$$vol(B) := \lambda^n(B)$$

where B is a Borel set.

A n-dimensional parallelepiped is :

$$P_n = \{t_1 v_1 + \cdots + t_n v_n \mid t_i \in [0,1] \forall i\}$$

Consider a linear mapping

$$\begin{aligned} A_{P_n} = A : \mathbb{R}^n \to & \quad \mathbb{R}^n \\ x \mapsto & \quad Ax \end{aligned}$$

where $A \in \mathcal{M}_{n \times n}(\mathbb{R})$, $A = (v_1 \mid \cdots \mid v_n)$

A is invertible iff $\{v_1, \cdots, v_n\}$ is a basis. Let $\prod_n = [0,1]^n$ then

$$A(\prod_n) = P_n$$

If A invertible

$$\begin{aligned} vol(P_n) &= \lambda^n(P_n) \\ &= \int_{A(\prod_n)} \chi_{P_n} \, \mathrm{d}\lambda^n \\ \text{change of variables } &= \int_{A(\prod_n)} |\det A| \, \mathrm{d}\lambda^n \\ &= |\det A| \\ \text{by the prop of det } &= \sqrt{\det A^T A} \end{aligned}$$

5.11 Prop

$$vol(P_n) = \sqrt{\det G}$$

Chapter 6

Unitary Space

6.1 Def

A complex inner vector space (H, h) where h is hermitian and positive define then it's called **unitary space**

As in the Euclidean space, we have orthonormal basis and define a norm,then a distance.

$$\|x\| = \sqrt{h(x, x)}$$

6.2 Decomplexification

Let V be a complex vector space of dimension n. We resist the module structure $\mathbb{C} \times V \to V$ to $\mathbb{R} \times V \to V$

The $\mathbb{R}$-vector space denoted by $V_{\mathbb{R}}$ has the same vector of V. For any $\mathbb{C}$-linear mapping $f : V \to W$, the module induces a mapping

$$f_{\mathbb{R}} \to W_{\mathbb{W}}$$

6.3 Theorem

Let V be a complex vector of dim n

- Let V be a complex vector basis of V, then $\{v_1, \cdots, v_n, iv_1, \cdots, iv_n\}$ is a real basis of $V_{\mathbb{R}}$

- $f : V \to W$ is a linear mapping. Assume that it's metric representation with respect to the basis $\{v_1, \cdots, v_n\}$ of V and $\{w_1, \cdots, w_n\}$ of W is

$$A = B + iC$$

where $B, C \in \mathbb{R}$ Then the metric representation of $f_{\mathbb{R}}$ with respect to the basis $\{v_1, \cdots, v_n, iv_1, \cdots, iv_n\}$ and $\{w_1, \cdots, w_n, iw_1, \cdots, iw_n\}$ is

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

6.4 Corollary

Let $f : V \to V$ be a $\mathbb{C}$-linear mapping. Then

$$\det f_{\mathbb{R}} = \det f \overline{\det f}$$

Chapter 7

Complexification

7.1 Def: Complex structure

Let W be a real vector space of dim n Consider $J : W \to W$ a linear mapping such that $J^{\circ 2} = -Id$. Then J is called the **complex structure** of W. Then couple (W, J) is a vector space with a complex structure.

Example

$$\begin{aligned} J : V_{\mathbb{R}} \to & \quad V_{\mathbb{R}} \\ v \mapsto & \quad iv \\ iv \mapsto & \quad -v \end{aligned}$$

7.2 Theorem

Let (W, J) be a real vector space with a complex structure. Then on W we introduces the following complex module:

$$(a + bi)w := aw + bJ(w)$$

We obtain a complex vector space W such that $(W)_{\mathbb{R}} = W$

7.3 Corollary

If (W, J) is a vector space with a complex structure,then $\dim W$ is even. Assume that if it's even, then it's possible to find a basis on W such that J is represented by

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Consider a orthonormal basis on $H\{v_1, \cdots, v_n\}$. $G_h : \mathbb{C}^n \to C^n$ is the Gram matrix of h with respect to $\{v_1, \cdots, v_n\}$

$$G_h = B_i C$$

Now

$$G_{\mathbb{R}} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

Then $G_{\mathbb{R}}$ defines an inner product and $(H_{\mathbb{R}}, \langle, \rangle)$ is Euclidean.

7.4 Notation

Now fix a inner product space. Then

$$h(x, y) = a(x, y) + ib(x, y)$$

when $a, b : V \times V \to \mathbb{R}$

7.5 Prop

In the above notation the following structure holds:

1 $a(x, y)$ and $b(x, y)$ are inner products on $V_{\mathbb{R}}$, with a symmetric and b skew-symmetric. In addition:

$$a(ix, iy) = a(x, y) \quad b(ix, iy) = b(x, y)$$

In other word a, b are invariant by the multiplication by i Invariance w.r.t. the complex structure of $H_{\mathbb{R}}$

2 The following relations hold.

$$a(x, y) = b(ix, y) \quad b(x, y) = -a(ix, y)$$

3 Any pair of J-invariant bilinear forms on $V_{\mathbb{R}}$ $a, b : V \times V \to \mathbb{R}$ that are symmetric and symplectic, respectively and s.t. (2) is satisfied. Define an hermitian inner product

$$h(x, y) := a(x, y) + ib(x, y)$$

Moreover h is positive define iff a is positive define.

7.6 Complex Cauchy-Schwartz inequality

(H, h) is a unitary space with finite dim. Then the inequality

$$|h(x, y)| \leq \|x\| \, \|y\|$$

holds iff x and y are propositional $(x = ty)$

7.7 Corollary:Complex triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

7.8 Def:Angle for unitary space

For

$$0 \leq \frac{|h(x,y)|}{\|x\| \cdot \|y\|}$$

$\exists! \phi \in [0, \frac{\pi}{2}]$

$$\cos\phi = \frac{|h(x,y)|}{\|x\| \cdot \|y\|}$$

The physical application is that ϕ can be considered as probability.

7.9 Prop

Let (V, g) be an inner product space with a non-degenerate inner product hermitian or real symm and $f : V \to V$ be a linear mapping. Then the following statements are equivalent:

0 f is isometry

1 $g(f(x), f(x)) = g(x, x)\ \forall x \in V$

2 Let $\{v_1, \cdots, v_n\}$ be a basis for V and let G be a Gram matrix of g w.r.t. such baisi. If A is the matrix of f w.r.t. $\{v_1, \cdots, v_n\}$, then

$$A^T G A = G \text{ or } A^T G \overline{A} = G$$

3 f transforms orthonormal basis into orthonormal basis

4 If the signature of g is (p, q) $(\mathfrak{r}_0 = 0)$, then the matrix of f w.r.t. any orthonormal basis $\{v_1, \cdots, v_p, v_{p+1}, \cdots, v_{p+q}\}$ where

$$(v_i, v_i) = \begin{cases} 1 & \text{if } i \leq p \\ -1 & \text{if } p < i \leq p+q \end{cases}$$

satisfies the following property:

symm case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} A = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

hermitian case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \overline{A} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

Chapter 8

Special operators

8.1 Def

$(E, \langle , \rangle)$ is a Euclidean space, then an isometry $f : E \to E$ is called an **orthogonal operator**

(H, h) a unitary space, then an isometry $f : H \to H$ is said a **unitary operator**

8.2 Corollary

Orthogonal and unitary operators have the following properties:
w.r.t. an (all) orthonormal basis, they are represented by a matrix U

$$UU^T = I_n \text{ or } UU^\dagger = I_n$$

8.3 Def: orthogonal matrices

$$O(n) := \{A \in GL_n(\mathbb{R}) \mid AA^T = I_n\}$$

8.4 Def: unitary matrices

$$U(n) := \{A \in GL_n(\mathbb{C}) \mid AA^\dagger = I_n\}$$

Remark

By the corollary 8.2, we have

- $O(n)$ is the set of orthogonal operators of $(\mathbb{R}^n, \langle , \rangle)$
- $U(n)$ is the set of unitary operators of $(\mathbb{C}^n, \langle , \rangle)$

Remark

$$\underset{\text{isometry}}{O(n)} \subseteq GL_n(\mathbb{R}) \subseteq \underset{\text{endomorphism}}{M_{n,n}(\mathbb{R})}$$

Take $T \in O(n), TT^T = I_n$

$$(\det T)^2 = 1 \;\Rightarrow\; \det T = \pm 1$$

8.5 Notation

$$SL_n(\mathbb{R}) := \{A \in GL_n(\mathbb{R}) \mid \det A = 1\}$$
$$SO(n) := \{A \in O_n \mid \det A = 1\}$$

Chapter 9

Classification of operators

$$U(1) = \{a \in \mathbb{C} \mid a\overline{a} = 1\} = \{e^{i\phi} \mid \phi \in \mathbb{R}\}$$
$$O(1) = \{1, -1\} = U(1) \cap \mathbb{R}$$

Let's study $O(2)$ and classify all its elements

$O(n)/SO(n) = \{\pm 1\}$ $SO(n)$ is a normal subgroup of $O(n)$ of index 2, namely

$$\#(O(n)/SO(n)) = 2$$

Take $T \in O(2)$, we have two cases: $\begin{cases} T \in SO(2) \\ T \in O(2) \setminus SO(2) \end{cases}$

9.0.1 $T \in SO(2)$

Assume $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $TT^T = Id_2$

$$\begin{cases} \det T = ad - bc = 1 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \\ ac + bd = 0 \end{cases}$$

This implies $\exists \alpha$ unique op to add by $2k\pi$ s.t.

$$a = \cos\alpha \quad b = \sin\alpha$$

We have shown that

$$SO(2) = \left\{ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \middle| \, \alpha \in [0, 2\pi[\right\}$$

Note that T doesn't have eigenvalues for $\alpha \neq 0, \pi$

9.0.2 $T \in O(2) \setminus SO(2)$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the reflection with respect to the line $y = 0$

$$TA \in SO(2)$$

since

$$\det(TA) = \det T \cdot \det A = 1$$

By the previous reasoning,

$$T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & -\cos\alpha \end{pmatrix}$$

The set $O(2) \setminus SO(2)$ represents reflections.

Consider the mapping

$$\begin{aligned} U(1) &\overset{\cong}{\to} & SO(2) \\ e^{i\phi} &\mapsto & \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \end{aligned}$$

$O(2) \cong U(1) \cup (O(2) \setminus SO(2))$

Remark

Reflections are diagonalizable with eigenvalues $\{\pm 1\}$ and the corresponding eigenvectors are orthogonal.

9.1 Theorem

1 Let (H, h) be a unitary space. A linear mapping $f : H \to H$ is unitary iff it's diagonalizable in an orthonormal basis and with eigenvalues in S^1

- Let $(E, \langle,\rangle)$ be a Euclidean space. A linear mapping $f : E \to E$ is orthogonal iff in some orthonormal basis f is represented by matrix:

$$\begin{pmatrix} R(\phi_1) & & & & & \\ & \ddots & & & & \\ & & R(\phi_n) & & & \\ & & & Id_1 & & \\ & & & & \ddots & \\ & & & & & Id_m \end{pmatrix}$$

where

$$R(\phi_i) = \begin{pmatrix} \cos\phi_i & -\sin\phi_i \\ \sin\phi_i & \cos\phi_i \end{pmatrix} \qquad \phi_i \in [0, 2\pi[$$

3 The eigenvectors of orthogonal/unitary operators corresponding to different eigenvalues are orthogonal.

Chapter 10

Fourier Coefficient

Goal We have an infinite dim vector space (usually a space of function) we want to express the elements as combinations of "orthogonal" vectors. (w.r.t. some nice inner product)

10.1 Def: Orthogonal and Orthonormal System

V is a vector space over $\mathbb{R}$ or $\mathbb{C}$, $\langle , \rangle$ is an inner product which either symm or hermitian.. Moreover, $\langle , \rangle$ is non-degenerate and positive define.

A set of vectors $\{l_k \mid k \in I\}$ (where I be the set of indexes) is said to be an orthogonal system if

$$\langle l_j, l_k \rangle \text{ iff } j \neq k$$

Moreover $\{l_k\}$ is an orthonormal system if

$$\langle l_j, l_k \rangle = \delta_{jk}$$

10.2 Prop

Let $\{l_k\}$ be an orthogonal system, then $\{l_k\}$ is a set of non-zero linearly independent vectors.

10.3 Prop

The inner product $\langle , \rangle$ is continuous (w.r.t. the Euclidean topology in the co-domain, and the product topology on the domain, where on V we put the topology induced by $\langle , \rangle$)

If $\{f_k\}$ is orthogonal system and $x \in V = \sum_{k=1}^{\infty} x_k l_k$, then $\forall y \in V$

$$\langle x, y\rangle = \sum_{k=1}^{\infty} \langle x_k, y\rangle$$

If $\{f_k\}$ is orthonormal system and $x = \sum_{k=1}^{\infty} x_k l_k$, $y = \sum_{k=1}^{\infty} y_k l_k$, then

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k \overline{y}_k$$

or

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k$$

10.4 Corollary: Pythagoras

1 If $\{v_k\}$ is an orthogonal system and $v = \sum_{k=1}^{\infty} v_k$, then

$$\|v\|^2 = \sum_i |v_i|^2$$

2 If $\{l_k\}$ is an orthonormal system and $x = \sum_{k=1}^{\infty} x_k l_k$, then

$$\|x\|^2 = \sum_i |x_i|^2$$

10.5 Def: Fourier coefficient

Let $\{l_k\}$ be an orthogonal system in V. Assume that $x = \sum_{k=1}^{\infty} x_k l_k$, then

$$x_k := \frac{\langle x, l_k\rangle}{\langle l_k, l_k\rangle}$$

is called a **Fourier coefficient** of x in $\{l_k\}$

Consider $x \in V$, then the Fourier series of x in $\{l_k\}$ is

$$x \sim \sum_{k=1}^{\infty} x_k l_k$$

We don't know whether it converges to x yet.

10.6 Main example

$$V = L^2(X, \mathbb{K})/\sim$$

where $X \in \mathbb{K}^n$ a measurable subset (w.r.t. λ^n), $\mathbb{K} = \mathbb{R}$ or $= \mathbb{C}$

$$L^2(X, \mathbb{K}) := \{f : X \to \mathbb{K} \mid \int_X |f|^2 \,\mathrm{d}x < +\infty\}$$

We define a equivalence relation on $L^2(X, \mathbb{K})$ by

$$f \sim g \text{ iff } \lambda^n \left(\{x \in X \mid f(x) \neq g(x)\}\right) = 0$$

we identify two functions equal if they're equal on almost everywhere(only diff on set that measures zero) From now on, we write elements in V simply by representatives.

We define an inner product on V:

$$\begin{aligned} \langle , \rangle : V \times V \to & \quad \mathbb{K} \\ (f, g) \mapsto & \int_X f\overline{g}\mathrm{d}\lambda^n \end{aligned}$$

(Well defined w.r.t. $\sim$?) Check that is $\int_X f\overline{g}\mathrm{d}\lambda^n$ well defined? Recall that

$$\left\| f(x)\overline{g(x)} \right\| = \|f(x)\| \, \|g(x)\|$$

Then inequality:

$$\begin{aligned} \|f(x)\| \, \|g(x)\| & \leq \frac{1}{2}(\|f(x)\|^2 + \|g(x)\|^2) \\ \Leftrightarrow 0 & \leq (\|f(x)\| + \|g(x)\|)^2 \end{aligned}$$

We always have $\|f(x)\| \, \|g(x)\| \leq \frac{1}{2}(\|f(x)\|^2 + \|g(x)\|^2)$ It follows that $\langle f, g\rangle$ is well-defined. It easy to show that $\langle , \rangle$ is hermitian. The only non-trivial thing is:

$$0 = \langle f, f\rangle \Leftrightarrow f(x) = 0 \text{ almost everywhere}$$

This means that if we don't use $\sim$ that we cannot say $\langle , \rangle$is positive defined.

Consider the following Integral:

$$\int_\pi^\pi e^{imx}e^{-inx}\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

So $\{x \mapsto e^{imx} \mid m \in \mathbb{Z}\}$ is an orthogonal system for $V = L^2([-\pi, \pi], \mathbb{C})/\sim$. To make it orthonormal, consider

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{imx} \middle| n \in \mathbb{Z} \right\}$$

If you want to replace $[-\pi, \pi]$ by $[-a, a]$, consider:

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{\frac{imx}{a}} \middle| n \in \mathbb{Z} \right\}$$

This is an orthonormal system for $L^2([-a, a], \mathbb{C})/\sim$ In the real case, consider the following integrals:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx)\sin(nx)\mathrm{d}x = 0$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\mathrm{d}x = \begin{cases} 0 & \text{if } m \neq n \text{ or } mn = 0 \\ \pi & \text{if } m = n \neq 0 \end{cases}$$

If follows that $\{1, \cos(nx), \sin(mx) \mid (m, n) \in \mathbb{N}^2\}$ is an orthogonal system for $L^2([-\pi, \pi], \mathbb{R})$

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k(f)\sin(kx)$$

$$\begin{cases} a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x \\ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(kx)\mathrm{d}x \end{cases}$$

For instance if $f = Id$ then $a_k = 0, b_k = (-1)^{k+1}\frac{2}{k}$

Chapter 11

Convergence

Always assume that the orthogonal system countable this chapter

11.1 Prop

Take $x \in V$ and let

$$\overline{x} = \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle}{\langle l_x, l_x \rangle} l_k$$

Then if we write $x = \overline{x} + h$ then h is orthogonal to $\overline{x}$ and h is orthogonal to the topological closure of $\langle \{l_k\} \rangle$

Remark

By Pythagoras Theorem 10.4, since $x = \overline{x} + h$

$$\|x\|^2 = \|\overline{x}\|^2 + \|h\|^2 \geq \|\overline{x}\|^2$$

If we write that inequality with respect to the Fourier coefficients, we get **Bessel's inequality**
Note that

$$\|x\|^2 = \sum_{k}^{\infty} \left| \frac{\langle x, l_m \rangle}{\langle l_k, l_k \rangle} \right| \langle l_k, l_k \rangle = \sum_{k} \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle}$$

$$\sum_{k}^{\infty} \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle} \leq \|x\|^2 \quad \text{(Bessel's inequality)}$$

So far we have assumed that the Fourier series converges to prove Bessel's inequality. But we DON'T NEED this assumption

11.2 Theorem

Assume $\{l_k\}$ is orthonormal. Let $x_k = \langle x, l_k \rangle$. If V is complete. then $\sum_k x_k l_k$ converges.

Remark

In the proof we assumed $\{l_k\}$ orthogonal. But this is not essential. **We have studied the existence of the limit x . What about the relation between $\overline{x}$ and x**

11.3 Prop

Let $\{l_k\}$ be an orthogonal system. Take $x \in V$ and assume that

$$V \ni \overline{x} = \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle \langle l_x, l_x \rangle}{l}_k$$

Then for any $y = \sum_k d_k l_k$ $(d_k \in \mathbb{F})$ it holds that:

$$\|x - \overline{x}\| \leq \|x - y\|$$

The equality is true iff $\overline{x} = y$

11.4 Def

A family of vectors $\mathcal{F} = \{x_\alpha \mid \alpha \in A\}$ in a normed vector space V is **complete** in a subset $E \subseteq V$ if every vector $x \in E$ can be approximated with arbitrary accuracy by a **finite** linear combination of elements in $\mathcal{F}$

Another statement

Let $L = \{\mathcal{F}\}$, then $\mathcal{F}$ is complete in E if $E \subseteq \overline{L}$

11.5 Weierstrass Approximation Theorem

Let $f \in \mathcal{C}([a, b])$. For any $\epsilon > 0$, there exists a polynomial $p \in \mathbb{F}[x]$ such that for any $x \in [a, b]$, we have

$$|f(x) - p(x)| < \epsilon$$

In fact

$$\|f - p\| = \sqrt{\int_a^b |f - p|^2 \, \mathrm{d}\lambda} < \epsilon\sqrt{b - a}$$

11.6 Prop

Let V be a complete vector space over $\mathbb{F}$ with inner product $\langle , \rangle$ hermitian or real symm, and positive define and non-degenerate.

Moreover, $\{l_k\}$ is an orthogonal system at most countable. Then the following conditions are equivalent:

1 $\{l_k\}$ is complete in $E \subseteq V$

2 For any $x \in E$, we have $x = \sum_k \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle} l_k$

3 Any vector $x \in E$ satisfies

$$\|x\|^2 = \sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle}$$

11.7 Def: Hamal basis

A countable family of vectors $\{b_k\}$ is a **Hamal basis** of V if any $v \in V$ there exists a unique sequence $\{\alpha_k\}$ in $\mathbb{K}$ with $\alpha_k = 0$ for all but finitely many k s.t.

$$v = \sum_k \alpha_k b_k$$

(In this def we don't need to use the topological properties of V)

11.8 Def: Schauder basis

A countable family of vectors $\{b_k\}$ is a **Schauder basis** for V if for any $v \in V$ there exists a unique sequence $\{\alpha_k\}$ such that

$$v = \sum_k a_k b_k \quad \text{(as convergent series)}$$

A Hamal basis is a Schauder basis(? to prove an element in basis can't be represented by others). In particular, a Schauder basis is a complete family of vectors (in $E = V$)

In pervious, we've proved that if $\{l_k\}$ is an orthogonal complete system (in $E = V$) with V complete, then any $x \in V$ can be written as

$$x = \sum_k \alpha_k l_k$$

when α_k are the Fourier coefficients.

In general it's FALSE that a complete family $\{b_k\}$ is a Schauder ($x \in \overline{\langle \{b_k\} \rangle}$)

11.9 Important Result

1

$L^2([-\pi,\pi[,\mathbb{K})$ is complete as topological vector space.

2

$\{1, \cos kx, \sin kx \mid k \in \mathbb{N}_{\geq 1}\}$ is a complete family.

11.10 Def

$f : X \backslash \{x_0\} \to [0,+\infty[$ we say that f is **unbounded** at x_0 if $\forall U \ni x_0,\ M > 0$

$$\exists x \in U \text{ s.t. } f(x) > M$$

11.11 Def: extend by periodicity

Let $f : [-\pi,\pi[\to \mathbb{R}$ extend this func by periodicity.

$$\tilde{f} = f(x - 2k\pi) \quad k \in \mathbb{Z}$$

11.12 Theorem

$L^p(X,\mu)$ is complete w.r.t the topology induced by $\|\cdot\|_p$

11.13 Def: Dirichlet kernel

$$D_n(x) := \sum_{k=-n}^{n} e^{iku} = \frac{\sin(n + \frac{u}{2})}{\sin \frac{u}{2}}$$

this is called **Dirichlet kernel**, which has the prop

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(u)\mathrm{d}u = \frac{1}{\pi}\int_{0}^{\pi} D_n(u)\mathrm{d}u = 1$$

Back to T_n putting $u = x - t$

$$\begin{aligned} T_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)D_n(u)\mathrm{d}u \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-u)\frac{\sin(n+\frac{u}{2})}{\sin\frac{u}{2}}\mathrm{d}u \end{aligned}$$

Now use that D_n is an even function

$$T_n(x) = \frac{1}{2\pi}\int_{0}^{\pi} \left(f(x-u) + f(x+u)\right) D_n(u)\mathrm{d}u = \frac{1}{2\pi}\int_{0}^{\pi} \left(f(x-u) + f(x+u)\right) \frac{\sin(n+\frac{u}{2})}{\sin\frac{u}{2}}\mathrm{d}u$$

11.14 Riemann-Lebesgue's Lemma

Let $f :: [a, b] \to \mathbb{R}$ be an integrable function. Then

$$\lim_{\lambda \to +\infty} \int_a^b f(x) e^{i\lambda x} \mathrm{d}x = 0$$

11.15 Corollary

$$\lim_{\lambda \to +\infty} \int_a^b f(x) \cos \lambda x \mathrm{d}x = 0$$
$$\lim_{\lambda \to +\infty} \int_a^b f(x) \sin \lambda x \mathrm{d}x = 0$$

11.16 Localization Principle

Let $f, g \in L^2([-\pi, \pi], \mathbb{K})$. If f, g coincide in a neighborhood of $x_0 \in]-\pi, \pi[$ $(f = g)$, the Fourier series

$$f \sim \sum_{-\infty}^{+\infty} c_k(f) e^{i\lambda x} \quad g \sim \sum_{-\infty}^{+\infty} c_k(g) e^{i\lambda x}$$

either both diverges *or* both converges. Moreover if they converges ar x_0, then their limits are the same (NOT to be $f(x_0) = g(x_0)$)

11.17 Def: Dini's Condition

Let $U_x^0 = [-\delta, x[\cup]x, \delta[$ and $f : U_x^0 \to \mathbb{C}$. We say that f satisfies **Dini's Condition** at x if

- $f(x_-)$ and $f(x_+)$ exists and finite
- $\exists > 0$ s.t.

$$\int_0^\epsilon \left| \frac{(f(x-t) - f(x_-)) + (f(x+t) - f(x_+))}{t} \right| \mathrm{d}t < +\infty$$

11.18 Theorem: pointwise convergence of Fourier series

Let $f : \mathbb{R} \to \mathbb{C}$ be a periodic function of period 2π, such that f is integrable in $[-\pi, \pi]$. If f satisfies the Dini's condition at $x \in \mathbb{R}$, then its Fourier series converges at x and

$$\sum_{-\infty}^{+\infty} c_k(f) e^{i\lambda x} = \frac{f(x_-) + f(x_+)}{2}$$

11.19 Lemma

$$\sum_{k=0}^{n} \sin(k+\frac{1}{2})t = \frac{\sin^2\left(\frac{n+1}{2}\right)}{\sin\frac{t}{2}}$$

$$F_n(t) = \frac{\sin^2\frac{n+1}{2}t}{(n+1)\sin^2\frac{t}{2}}$$

What happens when $t = 2k\pi$? Use Tayor's expansion:

$$F_n(t) = \frac{\left(\frac{(n+1)}{2}t + o(t)\right)^2}{(n+1)\left(\frac{t}{2}+o(t)\right)^2} \quad t \to 0$$

$$F_n(t) = n+1$$

Sso we can extend F_n at all points $2l\pi$ by putting $F_n(2k\pi) = n+1$

11.20 Def: approximated identity(delta function)

A family of functions $\{K_n\}_{n\in\mathbb{N}}$ with $K_n : \mathbb{R} \to \mathbb{R}$ is called a **approximated identify** if

- $\frac{1}{2\pi}\int_{-\infty}^{\infty} K_n(t)\mathrm{d}t = 1 \quad \forall n \geq 0$
- $K_n(t) \geq 0, \forall t \in \mathbb{R}, n \geq 0$
- For any $\delta > 0$

$$\lim_{n\to+\infty}\int_{|t|>\delta} K_n(t)\mathrm{d}t = 0$$

11.21 Prop

Consider

$$\delta_n(x) = \begin{cases} \frac{1}{2\pi}F_n(x) & \text{if } |x| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Then $\{\delta_n\}$ is an approximated identify

11.22 Fejer Theorem

Let $f : \mathbb{R} \to \mathbb{C}$ continuous and with period of 2π and integrable in $[-\pi, \pi]$. Then $\sigma_{f,n}$ converges uniformly to f

11.23 Weierstrass approximation

Let $f : [-\pi, \pi] \to \mathbb{C}$ be a continuous functions s.t. $f(-\pi) = f(\pi)$. Then such function can be approximated uniformly by σ_n arbitrarily