Chapter 1

Set

1.1 Ring

1.1.1 morphism

Def

Let A and B be unitary rings .We call morphism of unitary rings from A to B .only mapping $A \to B$is a morphism of group from (A,+) to (B,+),and a morphism of monoid from $(A, \cdot)\ to\ (B, \cdot)$

Properties

- Let R be a unitary ting. There is a unique morphism from $\mathbb{Z}\ to\ R$
-

algebra

we call k-algebra any pair(R,f),when R is a unitary ring ,and $f : k \to R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R,the unique morphism of unitary rings $\mathbb{Z} \to R$ define a structure of $\mathbb{Z} - algebra$ on R (extra: $\mathbb{Z}$ is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring ,(A,f) be a k-algebra. If there is no ambiguity on f,for any$(\lambda, a) \in k \times A$,we denote $f(\lambda)a\ as\ \lambda a$

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite ,so $\sum\limits_{n \in \mathbb{N}}$ couldn't be executed directly.
Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def : Let T be a formal symbol.We denote $k^{\mathbb{N}}$ as $k[T]$ If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$,when we denote $k^{\mathbb{N}}$ as k[T] this element is denote as $\sum_{n\in\mathbb{N}} a_nT^n$ Such element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient O
- write T' as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + ... + 0T^n + ...$ is written as $1 + 2T + T^2$

Def Remind that k[T]=$\{\sum_{n\in\mathbb{N}} a_nT^n \mid (a_n)_{n\in\mathbb{N}} \in k^{\mathbb{N}}\}$,define two composition laws on k[T]
$\forall F(T) = a_0 + a_+1T + ... \quad G(T) = b_0 + ...$
let $F + G = (a_0 + b_0) + ...$
$FG = \sum\limits_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n$
Properties:

- $(k[T], +, \cdot)$form a commutative unitary ring.
- $k \to k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n\in\mathbb{N}} \sum\limits_{i+j=n} (a_ib_j)T^n\right) (\sum\limits_{n\in\mathbb{N}} c_nT^n) = \sum\limits_{n\in\mathbb{N}} \left(\sum\limits_{p,q,l=n} a_pb_qc_l\right) T^n$
  is a trick applied on integral

Derivative:
let $F(T) \in k[T]$
We denote by $F'(T)$ *or* $\mathcal{D}(F(T))$ the formal power series
$\mathcal{D}(F) = \sum\limits_{n\in\mathbb{N}} (n + 1)a_{n+1}T^n$
Properties:

- $\mathcal{D}(k[T], +) \to (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp
We denote $exp(T) \in k[T]$ as $\sum\limits_{n\in\mathbb{N}} \frac{1}{n!}T^n$,which fulfil the differential equation $\mathcal{D}(exp(T)) = exp(T)$(interesting)
Cauchy sequence: $(F_i(T))_{i\in\mathbb{N}}$ be a sequence of elements in k[T],and $F(T) \in k[T]$We say that $(F_i(T))_{i\in\mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$,there exists $N(l) \in \mathbb{N}$ such that $\forall(i, j) \in \mathbb{N}^2_{\geq N(l)}, ord(F_i(T) - F_j(T)) \geq l$

Chapter 2

Sequences

2.1 Supremum and infimum

Def:

Let $(X, \leq)$ be a partially ordered set A and Y be subsets of X,such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq Y\}$has a least element then we say that A has a Supremum in Y with respect to $\leq$ denoted by $sup_{(y,\leq)}A$ this least element and called it the Supremum of A in Y(this respect to $\leq$)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has n infimum in Y with respect to $\leq$ .We denote by $inf_{(y,\leq)}A$ this greatest element and call it the infimum of A in Y
- Observation: $inf_{(Y,\leq)}A = sup_{(Y,\geq)}A$

Notation:

Let $(X, \leq)$ be a partially ordered set,I be a set.

- If f is a function from I to X sup f denotes the supremum of f(I) is X.inf $f$takes the same
- If $(x_i)_{i \in I} is\ a\ family\ of\ element\ in\ X, then$ $\sup\limits_{i \in I} x_i$ $denotes$ $\sup\{x_i \mid i \in I\}(in X)$

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup\limits_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i)\ holds\}$

Example:

Let $A = x \in R \mid 0 \leq x < 1 \subseteq \mathbb{R}$ We equip $\mathbb{R}$ with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$
Example: For $n \in \mathbb{N}$,let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:
Let $(X, \leq)$ be a partially ordered set,A,Y,Z be subset of X,such that $A \subseteq Z \subseteq Y$

- If max A exists,then is is also equal to $\sup_{(y,\leq)} A$
- If $\sup_{(y,\leq)} A$ exists and belongs to Z, then it is equal to $\sup A$

inf takes the same Prop.
Let $X, \leq$ be a partially ordered set ,A,B,Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y,\leq)} A$ and $\sup_{(y,\leq)} B$ exists,then $\sup_{(y,\leq)} A \leq \sup_{(y,\leq)} B$
- If $\inf_{(y,\leq)} A$ and $\inf_{(y,\leq)} B$ exists,then $\inf_{(y,\leq)} A \geq \inf_{(y,\leq)} B$

Prop.
Let $(X, \leq)$be a partially ordered set ,I be a set and $f, g : I \to X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists,then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists,then $\sup f \leq \sup g$

2.2 Interval

We fix a totally ordered set $(X, \leq)$
Notation:
If $(a, b) \in X \times X$ such that $a \leq b$,[a,b] denotes $\{x \in X \mid a \leq x \leq b\}$
Def:
Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$,one has $[x, y] \subseteq I$ then we say that I is a interval in X
Example:
Let $(a, b) \in X \times X$, such that $a \leq b$ Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.
Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X.

- $\bigcap_{\lambda\in\Lambda} I_\lambda$ is a interval in X
- If $\bigcap_{\lambda\in\Lambda} I_\lambda \neq \varnothing, \bigcup_{\lambda\in\Lambda} I_\lambda$ is a interval in X

We check that $[a,b] \subseteq I_\lambda \cup I_]\mu$

- If $b \leq x \quad [a,b] \subseteq [a,x] \subseteq I_\lambda$ because $\{a,x\} \subseteq I_\lambda$
- If $x \leq a \quad [a,b] \subseteq [x,b] \subseteq I_\mu$ because $\{b,x\} \subseteq I_\mu$
- If a < x < b then $[a,b] = [a,x] \cup [x,b] \subseteq I_\lambda \cup I_\mu$

Def:

Let $(X,\leq)$ be a totally ordered set .I be a non-empty interval of X. If $\sup I$ exists in X, we call $\sup I$ the right endpoint; inf takes the similar way.

Prop.

Let I be an interval in X.

- Suppose that $b = \sup I$exists. $\forall x \in I, [x,b[\subseteq I$
- Suppose that $a = \inf I$exists. $\forall x \in I,]a,x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X.Then I is equal to one of the following sets $[a,b] \quad [a,b[\quad]a,b] \quad]a,b[$

Def

let $(X,\leq)$ be a totally ordered set .If $\forall (x,z) \in X \times X$,such that $x < z \quad \exists y \in X$ such that $x < y < z$,than we say that $(X,\leq)$ is thick

Prop.

Let $(X,\leq)$ be a thick totally ordered set. $(a,b) \in X \times X, a < b$ If I is one of the following intervals $[a,b];[a,b[;]a,b];]a,b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick,there exists $x_0 \in]a,b[$ By definition,b is an upper bound of I. If b is not the supremum of I,there exists an upper bound M of I such that M¡b. Since X is thick ,there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x,b[\subseteq]a,b[\in I$ Hence M and M' belong to I,which conflicts with the uniqueness of supremum.

2.3 Enhanced real line

Def:

Let $+\infty$ and $-infty$ be two symbols that are different and don not belong to $\mathbb{R}$ We extend the usual total order $\leq \ on\ \mathbb{R}\ to\ \mathbb{R} \cup \{-\infty,+\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty,+\infty\}$ become a totally ordered set, and $\mathbb{R} =]-\infty,+\infty[$ Obviously,this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x = -\infty$
- $\forall x \in]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
  ARE NOT DEFINED

Def

Let $(X, \leq)$ be a partially ordered set. If for any subset A of X,A has a supremum and an infimum in X, then we say the X is order complete

Example

Let Ω be a set $(\mathscr{P}(\Omega), \subseteq)$is order complete If $\mathscr{F}$is a subset of $\mathscr{P}(\Omega), \sup \mathscr{F} = \bigcup\limits_{A \in \mathscr{F}} A$

Interesting tip: $\inf \varnothing = \Omega \quad \sup \varnothing = \varnothing$

$\mathcal{AXION}$:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \varnothing = -\infty \quad \inf \varnothing = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty \ and \ c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \backslash \{0\}, \lambda A$ denotes $\{\lambda a \mid a \in A\}$
- -A denotes (-1)A

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}, \sup(-A) = -\inf A, \inf(-A) + -\sup A$ Def

We denote by $(R, \leq)$ a field $\mathbb{R}$ equipped with a total order $\leq$, which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$,one has $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$,if A hsa an upper bound in$\mathbb{R}$,then it has a supremum in$\mathbb{R}$

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad sup(\lambda A) = \lambda \inf(A)$

inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

$a = \sup_{x\in I} f(x) \quad b = \sup_{y\in J} g(y) \quad c = \sup_{(x,y)\in I\times J, \{f(x),g(y)\}\neq\{+\infty,-\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$then $c = a + b$

inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set,$f : I \to [-\infty, +\infty], g : J \to [-\infty, +\infty]$

Then $\sup_{x\in I, \{f(x),g(y)\}\neq\{+\infty,-\infty\}} (f(x) + g(x)) \leq (\sup_{x\in I} f(x))(\sup_{x\in I} g(x))$

inf takes the similar($\leq\to\geq$) (provided when the sum are defined)

2.4 Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

2.4.1 K-module

Def

Let $(V, +)$ be a commutative group.We call left K-module structure:

any mapping Φ:$K \times V \to V$

We call right K-module structure:

any mapping Φ:$(\lambda, x) \mapsto \Phi(\lambda, x)$

- $\forall(a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x))$
- $\forall(a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall(x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x)\Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group (V,+) equipped with a left/right K-module structure is called a left/right K-module.

Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \to K$
- $(a, b) \mapsto a + b$
- $K \times K \to K$

- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
Any left $K^{op} - module$ is a right K-module
Any right $K^{op} - module$ is a left K-module
$(K^{op})^{op} = K$
Notation:

When we talk about a left K-module $(V, +)$,we often write its left K-module structure as $K \times V \to V \quad (a, x) \mapsto ax$

The axions become:

$$\begin{aligned}
&\forall (a, b) \in K \times K, \forall x \in V \quad (ab)x = a(bx)\\
&\forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x = ax + bx\\
&\forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) = ax + ay\\
&\forall x \in V \quad 1x = x
\end{aligned}$$

right K-module the similar way
Remark:

If K is commutative,then $K^{op} = K$,so left K-module and right K-module structure are the sae .We simply call them K-module structure. A commutative group equipped with a K-module structure is called a K-module.If K is a field,a K-module is also called a K-vector space

Let $\Phi : K \times V \to V$ be a left or right K-module structure

$$\forall x \in V, \Phi(\cdot, x) : K \to V, (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups.Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$ $\forall a \in K, \Phi(a, \cdot) : V \to V$ is a morphism of groups.Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)(\cdot\ is\ a\ var)$ Association: $\forall x \in K$

$$\begin{aligned}
(f(f + g) + h)(x) = (f + g)(x) + h(x) = f(x) + g(x) + h(x)\\
(f + (g + h))(x) = f(x) + ((g + h)(x)) = f(x) + g(x) + h(x)
\end{aligned}$$

Let $0 : I \to K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$
Let $-f : f + (-f) = 0$
The mapping $K \times K^I \to K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K-module structure
The mapping $K \times K^I \to K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K-module structure
Notation:

We can also write an element $\mu\ of\ K^I$ is the form of a family $(\mu_i)_{i\in I}$ of elements in K ($\mu_i$is the image of $i \in I\ by\ \mu$)
Then

$$\begin{aligned}
&(\mu_i)_{i\in I} + (\nu_i)_{i\in I} := (\mu_i + \nu_i)_{i\in I}\\
&a(\mu_i)_{i\in I} := (a\mu_i)_{i\in I}\\
&(\mu_i)_{i\in I}a = (\mu_i a)_{i\in I}
\end{aligned}$$

Def:

Let V be a left/right K-module.If W is a subgroup of V. Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W si left/right sub-K-module of V.

Example:

Let I be a set .Let $K^{\bigoplus I}$ be the sebset of K^I composed of mappings $f : I \to K$ such that $I_f = x \in I \mid f(x) \neq 0$ is finite. It is a left and right sub-K-module of K^I

In fact,$\forall (f, g) \in K^{\bigoplus I} \times K^I \quad I_{f-g} = x \in I \mid f(x) - g(x) \neq 0 \subseteq I_f \cup I_g$

Hence $f - g \in K^{\bigoplus I}$ So $K^{\bigoplus I}$ is a subgroup of K^I

$\forall a \in K, \forall f \in K^{\bigoplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$ Def:

Let V and W be left K-module, A morphism of groups$\phi : V \to W$ is called a morphism of left K-modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

If K is commutative, a morphism of K-modules is also called a K-lines mapping. We denote by $\hom_{K-Mod}(V, W)$ the set of all morphism of left-K-module from V to W.This is a subgroup of W^V

Theorem:

Let V be a left K-module. Let I be a set.

The mapping $\hom_{K-Mod}(K^{\bigoplus I}, V) \to V^I$ is a bijection where $e_i : I \to K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \phi \to (\phi(e_i))_{i \in I}$

Remark:

In the case where $I = 1, 2, 3, ..., n$ $V^I isdenotedasV^n, K^I isdenotedasK^n$

For any $(x_1, ..., x_n) \in V^n$,by the theorem, there exists a unique morphism of left K-modules $\phi : K^n \to V$ such that $\forall i \in 1, ..., n\phi(e_i) = x_i$

We write this ϕ as a column $\begin{pmatrix} x_1 \\ ... \\ x_n \end{pmatrix}$ It sends $(a_1, ..., a_n) \in K^n$ to $a_1x_1 + ... + a_nx_n$