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Part I

Set

Chapter 1

product

1.1 direct sum

\oplus is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0 \}$$

Chapter 2

Ring

2.1 morphism

Def

Let A and B be unitary rings. We call morphism of unitary rings from A to B only mapping $A \rightarrow B$ is a morphism of group from $(A, +)$ to $(B, +)$, and a morphism of monoid from (A, \cdot) to (B, \cdot)

Properties

- Let R be a unitary ring. There is a unique morphism from \mathbb{Z} to R
-

algebra

we call k -algebra any pair (R, f) , when R is a unitary ring, and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R , the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of \mathbb{Z} -algebra on R (extra: \mathbb{Z} is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring, (A, f) be a k -algebra. If there is no ambiguity on f , for any $(\lambda, a) \in k \times A$, we denote $f(\lambda)a$ as λa

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite, so $\sum_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def: Let T be a formal symbol. We denote $k^{\mathbb{N}}$ as $k[T]$. If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$, when we denote $k^{\mathbb{N}}$ as $k[T]$ this element is denoted as $\sum_{n \in \mathbb{N}} a_n T^n$. Such

element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient 0
- write T as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$ is written as $1 + 2T + T^2$

Def Remind that $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$, define two composition laws on $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$ form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left(\sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left(\sum_{p,q,l=n} a_p b_q c_l \right) T^n$
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by $F'(T)$ or $\mathcal{D}(F(T))$ the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $\exp(T) \in k[T]$ as $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$, which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence: $(F_i(T))_{i \in \mathbb{N}}$ be a sequence of elements in $k[T]$, and $F(T) \in k[T]$ We say that $(F_i(T))_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$, there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$, $\text{ord}(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 3

Supremum and infimum

Def:

Let (X, \leq) be a partially ordered set A and Y be subsets of X , such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq y\}$ has a least element then we say that A has a Supremum in Y with respect to \leq denoted by $\sup_{(Y, \leq)} A$ this least element and called it the Supremum of A in Y (this respect to \leq)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has an infimum in Y with respect to \leq . We denote by $\inf_{(Y, \leq)} A$ this greatest element and call it the infimum of A in Y
- Observation: $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let (X, \leq) be a partially ordered set, I be a set.

- If f is a function from I to X $\sup f$ denotes the supremum of $f(I)$ is X . $\inf f$ takes the same
- If $(x_i)_{i \in I}$ is a family of element in X , then $\sup x_i$ denotes $\sup\{x_i \mid i \in I\}$ (in X)

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$ We equip \mathbb{R} with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$

Example: For $n \in \mathbb{N}$, let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let (X, \leq) be a partially ordered set, A, Y, Z be subset of X , such that $A \subseteq Z \subseteq Y$

- If $\max A$ exists, then it is also equal to $\sup_{(y, \leq)} A$
- If $\sup_{(y, \leq)} A$ exists and belongs to Z , then it is equal to $\sup A$

\inf takes the same Prop.

Let X, \leq be a partially ordered set, A, B, Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y, \leq)} A$ and $\sup_{(y, \leq)} B$ exists, then $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If $\inf_{(y, \leq)} A$ and $\inf_{(y, \leq)} B$ exists, then $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let (X, \leq) be a partially ordered set, I be a set and $f, g : I \rightarrow X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists, then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists, then $\sup f \leq \sup g$

Chapter 4

Interval

We fix a totally ordered set (X, \leq)

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$, $[a, b]$ denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$, one has $[x, y] \subseteq I$ then we say that I is an interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$. Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an interval in X
- If $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$, $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_\mu$

- If $b \leq x$ $[a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a$ $[a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If $a < x < b$ then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let (X, \leq) be a totally ordered set. I be a non-empty interval of X . If $\sup I$ exists in X , we call $\sup I$ the right endpoint; \inf takes the similar way.

Prop.

Let I be an interval in X .

- Suppose that $b = \sup I$ exists. $\forall x \in I, [x, b] \subseteq I$
- Suppose that $a = \inf I$ exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X . Suppose that I has supremum b and an infimum a in X . Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let (X, \leq) be a totally ordered set. If $\forall (x, z) \in X \times X$, such that $x < z \quad \exists y \in X$ such that $x < y < z$, then we say that (X, \leq) is thick

Prop.

Let (X, \leq) be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]; [a, b[;]a, b];]a, b[$ Then $\inf I = a \quad \sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists $x_0 \in]a, b[$ By definition, b is an upper bound of I . If b is not the supremum of I , there exists an upper bound M of I such that $M \neq b$. Since X is thick, there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b] \subseteq I, a, b \in I$ Hence M and M' belong to I , which conflicts with the uniqueness of supremum.

Chapter 5

Enhanced real line

Def:

Let $+\infty$ and $-\infty$ be two symbols that are different and don't belong to \mathbb{R} . We extend the usual total order \leq on \mathbb{R} to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ becomes a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$. Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty[\quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in]0, +\infty[\quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
ARE NOT DEFINED

Def

Let (X, \leq) be a partially ordered set. If for any subset A of X , A has a supremum and an infimum in X , then we say that X is order complete.

Example

Let Ω be a set $(\mathcal{P}(\Omega), \subseteq)$ is order complete. If \mathcal{F} is a subset of $\mathcal{P}(\Omega)$, $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$

Interesting tip: $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

Axiom:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ and $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \setminus \{0\}$, λA denotes $\{\lambda a \mid a \in A\}$
- $-A$ denotes $(-1)A$

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$, $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ Def

We denote by (\mathbb{R}, \leq) a field \mathbb{R} equipped with a total order \leq , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, one has $\forall c \in \mathbb{R}$, $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, $ab > 0$
- $\forall A \subseteq \mathbb{R}$, if A has an upper bound in \mathbb{R} , then it has a supremum in \mathbb{R}

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

\inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$ then $c = a + b$

\inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set, $f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

\inf takes the similar ($\leq \rightarrow \geq$) (provided when the sum are defined)

Chapter 6

Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

6.1 K -module

6.1.1 Def

Let $(V, +)$ be a commutative group. We call left/right K -module structure: any mapping $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group $(V, +)$ equipped with a left/right K -module structure is called a left/right K -module.

6.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
 Any left K^{op} - module is a right K -module
 Any right K^{op} - module is a left K -module
 $(K^{op})^{op} = K$

6.1.3 Notation

When we talk about a left/right K -module $(V, +)$, we often write its left K -module structure as $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

6.1.4 K -vector space

If K is commutative, then $K^{op} = K$, so left K -module and right K -module structure are the same. We simply call them K -module structure. A commutative group equipped with a K -module structure is called a K -module. If K is a field, a K -module is also called a K -vector space

Let $\Phi : K \times V \rightarrow V$ be a left or right K -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$ is a morphism of groups. Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$ (*is a var*)

6.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K -module structure

The mapping $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K -module structure

6.1.6 Remark:

We can also write an element μ of K^I in the form of a family $(\mu_i)_{i \in I}$ of elements in K (μ_i is the image of $i \in I$ by μ)
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

6.2 sub K-module**6.2.1 Def**

Let V be a left/right K -module. If W is a subgroup of V . Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub- K -module of V .

6.2.2 Example

Let I be a set. Let $K^{\oplus I}$ be the subset of K^I composed of mappings $f : I \rightarrow K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub- K -module of K^I

In fact, $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$
Hence $f - g \in K^{\oplus I}$ So $K^{\oplus I}$ is a subgroup of K^I
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

6.3 morphism of K-modules**6.3.1 Def**

Let V and W be left K -module, A morphism of groups $\phi : V \rightarrow W$ is called a morphism of left K -modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

6.3.2 K-linear mapping

If K is commutative, a morphism of K -modules is also called a K -linear mapping. We denote by $\text{hom}_{K\text{-Mod}}(V, W)$ the set of all morphism of left- K -module from V to W . This is a subgroup of W^V

6.3.3 Theorem

Let V be a left K -module. Let I be a set.
The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$ is a bijection where
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

6.3.4 Remark:column

In the case where $I = 1, 2, 3, \dots, n$ V^I is denoted as V^n , K^I is denoted as K^n . For any $(x_1, \dots, x_n) \in V^n$, by the theorem, there exists a unique morphism of left K -modules $\phi : K^n \rightarrow V$ such that $\forall i \in 1, \dots, n, \phi(e_i) = x_i$.

We write this ϕ as a column $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$. It sends $(a_1, \dots, a_n) \in K^n$ to $a_1x_1 + \dots + a_nx_n$.

6.4 kernel

6.4.1 Prop

Let G and H be groups and $f : G \rightarrow H$ be a morphism of groups

- $Im(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

6.4.2 Def

$\ker(f)$ is called the kernel of f

6.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

Proof

Let e_G and e_H be neutral element of G and H respectively

- (1) Let x and y be element of G
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H
- (2) Let x and y be element of $\ker(f)$. One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$. So $\ker(f)$ is a subgroup of G .
- (3) Suppose that f is injection.
 Since $f(e_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$. Suppose that $\ker(f) = \{e_G\}$. If $f(x) = f(y)$ then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$.
 Hence $xy^{-1} = e_G \Rightarrow x = y$

6.4.4 Def

Let $(V, +)$ be a commutative group, I be a set. We define a composition law $+$ on V^I as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then V^I forms a commutative group

6.4.5 Remark

Let E and F be left K -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$ is a subgroup of F^E

In fact f and g are elements of $\text{hom}_{K\text{-Mod}}(E, F)$, then $f - g$ is also a morphism of left K -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

6.4.6 Theorem

Let V be a left K -module, I be a set The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$ is an isomorphism of groups, where $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

6.4.7 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So Ψ is a morphism of groups

injectivity Let $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i \in I} \in V^I$ We define $\phi_x : K^{\oplus I} \rightarrow V$ such that $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left K -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that K' is a unitary ring, and V is also equipped with a right K' -module structure, Then $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub- k' -module, and Ψ in the theorem is a right K' -module isomorphism

Chapter 7

Monotone mappings

7.1 Def

Let I and X be partially ordered sets, $f : I \rightarrow X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)$, then we say that f is increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

7.2 Prop.

Let X, Y, Z be partially ordered sets. $f : X \rightarrow Y, g : Y \rightarrow Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

7.3 Def

Let f be a function from a partially ordered set I to another partially ordered set X . If $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

7.4 Prop.

Let I and X be partially ordered sets. f be function from I to X .

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

7.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B

If f is increasing/decreasing, then so is f^{-1}

7.6 Def

Let X and Y be partially ordered sets. $f : X \rightarrow Y$ be a bijection. If both f and f^{-1} are increasing, then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \rightarrow Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

7.7 Prop.

Let I be a subset of \mathbb{N} which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \rightarrow I$

7.8 Proof

7.8.1 bijection

We construct $f : \mathbb{N} \rightarrow I$ by induction as follows.

Let $f(0) = \min I$ Suppose that $f(0), \dots, f(n)$ are constructed

then we take $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$. Therefore $f(n) \leq f(n+1)$

Since $f(n+1) \notin \{f(0), \dots, f(n)\}$, we have $f(n) < f(n+1)$

Hence f is strictly increasing and this is injective

If f is not surjective, then $I \setminus \text{Im}(f)$ has a element N .

Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.

Since $N \notin \text{Im}(f)$, $N < f(m)$.

So $m \neq 0$. Hence $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition, $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$,

Hence $f(m) \leq N$, causing contradiction.

7.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from \mathbb{N} to \mathbb{N}

Chapter 8

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

8.1 Def

Let X be a set. We call sequence in X parametrized by I a mapping from I to X .

8.2 Remark

If K is a unitary ring and E is a left K -module then the set of sequence E^I admits a left- K -module structure. If $x = (x_n)_{n \in I}$ is a sequence in E , we define a sequence $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$, called the series associated with the sequence x .

8.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$ is a morphism of left- K -module

8.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

8.5 Prop

Let I be a totally ordered set . X be a partially ordered set, $f : I \rightarrow X$ be a mapping , $J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

8.6 limit

8.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset. $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left(\inf_{i \in I, i \geq n} x_i \right)$$

If $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$, we then say that $(x_n)_{n \in I}$ tends to l and that l is the limit of $(x_n)_{n \in I}$. If in addition $(x_n)_{n \in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$, we say that $(x_n)_{n \in I}$ converges to l

8.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} = \inf_{n \in J} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left(\inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i \in I}$ the limit superior and the limit inferior do not change.

In fact, if we take $J = \mathbb{N} \setminus \{0, \dots, m\}$, then $\inf_{n \in J}(\dots)$ and $\sup_{n \in J}(\dots)$ only depends on the values of $x_i, i \in I, i \geq m$

8.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

8.6.4 Prop

Let $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

8.6.5 Prop

Let $(x_n)_{n \in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$, one has $x_n \leq y_n$. Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

8.6.6 Theorem

Let $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$ and $(z_n)_{n \in I}$ tend to the same limit l

Then $(y_n)_{n \in I}$ tends to l

8.6.7 Def

Let I be an infinite subset of \mathbb{N} , and $(x_n)_{n \in I}$ be a sequence in some set X . We call subsequence of $(x_n)_{n \in I}$ a sequence of the form $(x_n)_{n \in J}$, where J is an infinite subset of I

8.6.8 Prop

Let I and J be infinite subset of \mathbb{N} such that $J \subseteq I$. $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$, one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if $(x_n)_{n \in I}$ tends to $l \in [-\infty, +\infty]$, then $(x_n)_{n \in J}$ tends to l

8.6.9 Prop

$\forall n \in \mathbb{N}$, one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

8.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing, then $(x_n)_{n \in I}$ tends to $\sup_{n \in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing, then $(x_n)_{n \in I}$ tends to $\inf_{n \in I} x_n$

8.6.11 Notation

If a sequence $(x_n)_{n \in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim_{n \in I, n \rightarrow} x_n$ denotes this limit l

8.6.12 Corollary

Let $(x_n)_{n \in I}$ be a sequence in $\mathbb{N}_{\geq 0}$. Then the series $\sum_{n \in I} x_n$ (the sequence $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$. It converges in \mathbb{R} iff it is bounded from above (namely has an upper bound in \mathbb{R})

8.6.13 Notation

If a series $\sum_{n \in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum_{n \in I} x_n$ to denote the limit

8.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\limsup_{n \in I, n \rightarrow +\infty} x_n$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\liminf_{n \in I, n \rightarrow +\infty} x_n$.

Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_n)_{n \in J}$ is decreasing so it tends to $\inf_{n \in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i \in I, i \geq n} x_i$ so $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$. Denote by $l = \sup_{n \in I, n \geq n_0}$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i \in I, i \geq n_0} x_i \leq l$. If the strict inequality $\sup_{i \in I, i \geq N} x_i < l$ holds, then $\sup_{i \in I, i \geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i \in I, i \geq N} x_i$. We may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality.

Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$. Thus $n \in J$ that contradicts the maximality of n_0 .

Therefore

$$l = \sup_{i \in I, i \geq N} x_i$$

, which leads to

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = l$$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$, so $x_m < l$ (since otherwise $x_m = \sup_{i \in I, i \geq m} x_i$ and hence $m \in J$). Hence, \forall finite subset I' of $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$ and hence $\exists n \in I$, such that $n > \max I'$, and $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j \in \mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \rightarrow I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence $(x_{n_j})_{j \in \mathbb{N}}$ is increasing

And

$$\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n = l$$

$$l = \sup_{n \in I, n \geq n_0} x_n$$

So $(x_{n_j})_{j \in \mathbb{N}}$ tends to l

Chapter 9

Cauchy sequence

9.1 Def

Let $(x_n)_{n \in I}$ be a sequence in \mathbb{R}
If $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$ then
we say that $(x_n)_{n \in I}$ is a Cauchy sequence

9.2 Prop

- If $(x_n)_{i \in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_n)_{i \in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite, $(x_n)_{n \in I}$ is a Cauchy sequence.
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n \in I}$ converges to l , then $(x_n)_{n \in J}$ converges to l too.

9.3 Theorem: Completeness of real number

If $(x_n)_{n \in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in \mathbb{R}

Proof

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$. So $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n \in I}$ converges to $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n \in I}$ converges to the same limit.

9.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

9.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in \mathbb{R}

Chapter 10

Comparison and Technics of Computation

10.1 Def

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be sequence in \mathbb{R}

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$ then we write $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$, then we write $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

10.2 Prop.

Let I and X be partially ordered sets and $f : I \rightarrow X$ be an increasing/decreasing mapping. Let J be a subset of I . Assume that any elements of I has an upper bound in J . Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

10.3 Theorem

Let I be a totally ordered set, $f : I \rightarrow [-\infty, +\infty]$ and $g : I \rightarrow [-\infty, +\infty]$ be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

Proof

We can assume f and g increasing. Let $a = \sup f(I), b = \sup g(I)$

Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip A with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

h is increasing.

Let $(x, y) \in A$. Assume that $x \leq y$

If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$

If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$

So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined, $(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if $x \geq y$, (x, y) has also an upper bound in B . Therefore: $\sup h(A) = \sup h(B)$

10.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n \in I, n \geq N} x_n$ $B_N = \sup_{n \in I, n \geq N} y_n$. $(A_N)_{N \in \mathbb{N}}$ and $(B_N)_{N \in \mathbb{N}}$ are decreasing, and $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$ $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

10.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + \left(\limsup_{n \in I, n \rightarrow +\infty} y_n \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + \left(\liminf_{n \in I, n \rightarrow +\infty} y_n \right)$$

Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with ∞

10.6 Theorem

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n \in I}$ converges to some $l \in \mathbb{R}$.
Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left(\limsup_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = \left(\liminf_{n \in I, n \rightarrow +\infty} x_n \right) + l$$

10.7 Prop.

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$.
Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{ \liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n \right\}$$

Proof

About the first inequality. Since $\max\{x_n, y_n\} \geq x_n$ and $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite

Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

10.8 Theorem

Let $(a_n)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1) \Rightarrow (2):

If $(a_n)_{n \in I}$ converges to l , then $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2) \Rightarrow (1):

If $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$, then $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore: $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

10.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in \mathbb{R} , $l \in \mathbb{R}$

The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \rightarrow +\infty$

10.10 Calculates on $O()$, $o()$

10.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$

10.10.2 Transform

Let $(a_n)_{n \in I}$ and $(b_n)_{n \in I}$ be two sequence in \mathbb{R} If $a_n = o(b_n), n \in I, n \rightarrow +\infty$, then $a_n = O(b_n), n \in I, n \rightarrow +\infty$

10.10.3 Transition

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ and $(c_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$

10.10.4 Times

Let $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$ be sequences in \mathbb{R}

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

10.11 On the limit

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be elements of \mathbb{R}^I that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n \in I}$ converges to $l + l'$
- $(a_n b_n)_{n \in I}$ converges to ll'

10.12 Prop

Let $a \in \mathbb{R}$ then $a^n = o(n!)$ $n \rightarrow +\infty$

Proof

Let $N \in \mathbb{N}$ such that $|a| < N$
For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^n - N$$

And $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$. Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

10.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

10.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of \mathbb{R}^I . If the series $\sum_{n \in I} b_n$ converges absolutely and if $a_n = O(b_n) \quad n \rightarrow +\infty$ Then $\sum_{n \in I} a_n$ converges absolutely

Proof

By definition $\sum_{n \in I} |b_n| < +\infty$. If $|a_n| \leq M|b_n|$ for $n \in I, n \geq N$ where $N \in \mathbb{N}$. Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

10.15 Theorem: d'Alembert ratio test

Let $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

Proof**(1)**

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, *alpha* isn't a lower bound of $\left(\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$
 So $\exists N \in \mathbb{N}$ such that $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$ Hence for $n \geq N$ $|a_n| \leq \alpha^{n-N} |a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$, $\sum_{n \in \mathbb{N}} a_n$ converge absolutely.

10.15.1 Lemma

If a series $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n \rightarrow +\infty} a_n = 0$

Proof

If $\left(\sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$ converges to some $l \in \mathbb{R}$, then $\left(\sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$ converges to l ,
 too. Hence $\left(a_n = \left(\sum_{i=0}^n a_i \right) - \left(\sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$ converges to $l - l = 0$

10.15.2 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$
 Hence $(|a_n|)_{n \in \mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N} |a_N|$
 By the lemma: $\sum_{n \in \mathbb{N}} a_n$ diverges.

10.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \rightarrow +\infty$

Proof

Let $\epsilon > 0$ such that $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

10.16.1 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \rightarrow +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$ Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

10.16.2 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

10.17 Theorem: Cauchy root test

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} . Let $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.
- If $\alpha > 1$ then $\sum_{n \in \mathbb{N}} a_n$ diverges

Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_n|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n \in \mathbb{N}}$ cannot converge to 0.

Part III

Axiom of choice

Chapter 11

Preparation

11.1 Statement of axiom of choice

For any set I and any family $(A_i)_{i \in I}$ of non-empty sets, there exists a mapping $f : I \rightarrow \bigcup_{i \in I} A_i$ such that $\forall i \in I, f(i) \in A_i$

11.2 Def

Let (X, \leq) be a partially ordered set. If $\forall A \subseteq X$ A is non-empty, there exists a least element of A then we say that (X, \leq) is a well ordered set.

11.3 Theorem

For any set X , there exists an order relation \leq on X such that (X, \leq) forms a well ordered set.

11.4 Zorn's lemma

Let (X, \leq) be a partially ordered set. If $\forall A \subseteq X$ that is totally ordered with respect to \leq , there exists an upper bound of A inside X . Then, there exists a maximal element x_0 of X ($\forall y \in X, y > x_0$ does not hold)

11.5 Prop.

Let (X, \leq) be a well ordered set, $y \notin X$. We extend \leq to $X \cup \{y\}$, such that $\forall x \in X, x < y$. Then $(X \cup \{y\}, \leq)$ is well ordered.

11.6 Proof

Let $A \subseteq X \cup \{y\}$, $A \neq \emptyset$. If $A = \{y\}$ then Y is the least element of A . If $A \neq \{y\}$ then $B = A \setminus \{y\}$ is non-empty. Let b be the least element of B . Since $b < y$ it's also the least element of A

11.7 Def: Initial Segment

Let (X, \leq) be a well ordered set. $S \subseteq X$, If $\forall s \in S, x \in X \quad x < s$ initial $x \in S$ ($X_{<s} \subseteq S$), then we say that S is an initial segment of X

If S is a initial segment such that $S = X$ then we sat that S is a proper initial segment.

11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$ Then $X_{<x}$ is a proper initial segment of X .

11.9 Prop.

Let (X, \leq) be a well ordered set, If $(S_i)_{i \in I}$ is a family of initial segment of X , then $\bigcup_{i \in I} S_i$ is an initial segment of X

11.10 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$ such that $s \in S_i, i \in I$ Therefore $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

11.11 Prop.

Let (X, \leq) be a well ordered set.

- (1) Let S be a proper initial segment of X , $x = \min(X \setminus S)$ Then $S = X_{<x}$
- (2) $X \rightarrow \wp(X)$
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of X forms a well ordered subset of $(\wp(X), \subseteq)$

11.12 Proof

- (1) $\forall s \in S$ if $x \leq s$ then $x \in S$ contradiction. Hence $s < x$, This shows $S \subseteq X_{<x}$ Conversely, if $t \in X, t \notin X \setminus S$ Hence $t \in S$. Hence $X_{<x} \subseteq S$

- (2) Let $x, y \in X, x < y$ By definition $X_{<x} \subseteq X_{<y}$ Moreover $x \in X_{<y} \setminus X_{<x}$ So $X_{<x} \subsetneq X_{<y}$
- (3) Let $\mathcal{F} \subseteq \wp(X)$ be a set of initial segments. $\mathcal{F} \neq \emptyset$. Then there exists $A \subseteq X$ such that $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$ If $A = \emptyset$ then $\mathcal{F} = \{X\}$, and $\{X\}$ is the least element of \mathcal{F} . Otherwise $A \neq \emptyset$ and A has a least element a . Then by (2) $X_{<a}$ is the least element of \mathcal{F}

11.13 Lemma

Let (X, \leq) be a well ordered set, $f : X \rightarrow X$ be a strictly increasing mapping. Then $\forall x \in X, x \leq f(x)$

Proof

Let $A = \{x \in X \mid f(x) < x\}$ If $A \neq \emptyset$, let a be the least element of A . By definition $f(a) < a$. Hence $f(f(a)) < f(a)$ since f is strictly increasing. This shows $f(a) \in A$. But a is the least element of A , $f(a) < a$ cannot hold: contradiction.

11.14 Prop

Let (X, \leq) be a well ordered set, S and T be two initial segment of X . If $f : S \rightarrow T$ is a bijection that's strictly increasing, then $S = T, f = Id_S$

Proof

We may assume $T \subseteq S$. Let $l : T \rightarrow S$ be the inclusion mapping and $g = l \circ f : S \rightarrow S$. Since g is strictly increasing, by the lemma, $\forall s \in S, s \leq g(s) = f(s) \in T$. Since T is an initial segment, $s \in T$. Hence $S = T$. Apply the lemma to f^{-1} we get $\forall s \in S, s \leq f^{-1}(s)$ Hence $f(s) \leq s$ Therefore $f(s) = s$

11.15 Def

Let (X, \leq) and (Y, \leq) be partially ordered sets. If $\exists f : X \rightarrow Y$ that's increasing and bijective, we say that (X, \leq) and (Y, \leq) are isomorphic

11.16 Def

Let (X, \leq) and (Y, \leq) be well ordered sets. If (X, \leq) is isomorphic to an initial segment of Y . We note $X \preceq Y$ or $Y \succeq X$. If X is isomorphic to Y , we note $X \sim Y$. If $X \preceq Y$ but $X \not\sim Y$, we note $X \prec Y$ or $Y \succ X$

11.17 Prop.

Let X and Y be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

Proof

We construct a correspondence f from X to Y , such that $(x, y) \in \Gamma_f$, iff $X_{<x} \sim Y_{<y}$
By the last proposition of Oct. 11, f is a function.

- If $a, b \in \text{Dom}(f)$, $a < b$, then $X_{<a} \subsetneq X_{<b}$
By definition, $Y_{<f(b)} \sim X_{<b}$ $Y_{<f(a)} \sim X_{<a}$
Hence $Y_{<f(a)}$ is isomorphic to a proper initial segment of $Y_{<f(b)}$. Therefore $Y_{f(a)}$ is a proper initial segment of $Y_{<f(b)}$. We then get $f(a) < f(b)$. Thus f is strictly increasing.
 - Let $a \in \text{Dom}(f)$ Let $x \in X, x < a$ Then $X_{<x}$ is a initial segment of $X_{<a} \sim Y_{<f(a)}$ Hence $\exists y \in Y$ $X_{<x} \sim Y_{<y}$ This shows that $x \in \text{Dom}(f)$. Hence $\text{Dom}(f)$ is an initial segment of X . Applying this to f^{-1} , we get : $\text{Im}(f) = \text{Dom}(f)$ is an initial segment of Y
 - Either $\text{Dom}(f) = X$ or $\text{Im}(f) = Y$.
Assume that $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$ are respectively the least elements of $X \setminus \text{Dom}(f)$ and $Y \setminus \text{Im}(f)$.
Then we get $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$.
We obtain $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$. Contradiction
 -
- Case 1 $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$ $X \prec Y$
Case 2 $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$ $X \succ Y$
Case 3 $\text{Dom}(f) = X, \text{Im}(f) = Y$ $X \sim Y$

11.18 Lemma

Let (X, \leq) be a partially ordered set . $\mathfrak{S} \subseteq \wp(X)$. Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$ is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$, either A is an initial segment of B , or B is an initial segment of A .

Let $Y = \bigcup_{A \in \mathfrak{S}} A$. Then (Y, \leq) is a well ordered set, and $\forall A \in \mathfrak{S}, A$ is an initial segment of Y .

Proof

- Let $A \in \mathfrak{S}, x \in A, y \in Y, y < x$. Since $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$, such that $y \in B$. If $y \notin A$ then $B \not\subseteq A$. Hence A is an initial segment of B. Hence $y \in A$. Contradiction
- Let $Z \subseteq Y, Z \neq \emptyset$. Then $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$. Let m be the least element of $A \cap Z$. Let $z \in Z, B \in \mathfrak{S}$, such that $z \in B$. If $z \in A$, then $m \leq z$. If $z \notin A$, then A is an initial segment of B.

Since B is well ordered, if $m \not\leq z$ then $z < m$. Since $m \in A$, we get $z \in A$. Contradiction.

Therefore, m is the least element of Z .

Chapter 12

Zorn's lemma

Let (X, \leq) be a partially ordered set. Suppose that any well-ordered subset of X has an upper bound on X , the X has a maximal element (a maximal element m of $\{x \mid x > m\} = \emptyset$)

12.1 Proof

Suppose that X doesn't have any maximal element. $\forall A \in \omega. \exists f(A)$ such that $\forall a \in A, a < f(A)$

Let

$$\omega = \{\text{well ordered subset of } X\}$$

. (guaranteed by axiom of choice)

Let $f : \omega \rightarrow X$ such that $f(A)$ is an upper bound of $A \in \omega$.

If $A \in \omega$ satisfies

$$\forall a \in A, a = f(A_{<a})$$

, we say that A is a f -set

Let

$$\mathfrak{S} = \{f\text{-sets}\}$$

Note that

$$\emptyset \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$$

In fact, if $a \in A$, then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If $a = f(A) \notin A$ then

$$(A \cup \{f(A)\})_{<a} = A$$

Let A and B be elements of \mathfrak{S} . Let I be the union of all common initial segments of A and B . This is also a common initial segment of A and B .

If $I \neq A$ and $I \neq B$, then

$$\exists(a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then $I \cup \{a\}$ is also a common initial segment of A and B , contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and $\forall A \in \mathfrak{S}$ is an initial segment of Y .

Since A is an initial segment of Y

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A \quad A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus Y is the greatest element of $(\mathfrak{S}, \subseteq)$. However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If $f(y)$ is not a maximal element of X

$$\exists x \in X, f(y) < x$$

Part IV

Topology

Chapter 13

Absolute value and norms

13.1 Def

Let K be a field. By absolute value on K , we mean a mapping $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- (1) $\forall a \in K \quad |a| = 0$ iff $a = 0$
- (2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$ (triangle inequality)

13.2 Notation

\mathbb{Q} Take a prime num $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$ there exists a integer $ord_p(\alpha) \frac{a}{b}$, where
 $a \in \mathbb{Z} \setminus \{0\}$
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on \mathbb{Q}

Proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd \\ \alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d} \\ \text{Assume } ord_p(\alpha) \geq ord_p(\beta) \\ \alpha + \beta \\ = p^{ord_p(\beta)} \left(p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right) \\ = p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd \\ \text{So}$$

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\} \\ \text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} = \\ \max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

13.4 Def

Let K be a field and $|\cdot|$ be an absolute value. We call $(K, |\cdot|)$ a valued field.

Chapter 14

Quotient Structure

14.1 Def

Let X be a set and \sim be a binary relation on X
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$, if $x \sim y$ then $y \sim x$
- $\forall (x, y, z) \in X^3$, if $x \sim y, y \sim z$ then $x \sim z$

then we say that \sim is an equivalence relation

14.2 equivalence class

$\forall x \in X$ we denote by $[x]$ the set $\{y \in X \mid y \sim x\}$ and call it the equivalence class of x on X . Let X/\sim be the set $\{[x] \mid x \in X\}$

14.3 Prop.

Let X be a set and \sim be an equivalence relation on X

- (1) $\forall x \in X, y \in [x]$ on has $[x] = [y]$
- (2) If α and β are elements of X/\sim such that $\alpha \neq \beta$ then $\alpha \cap \beta = \emptyset$
- (3) $X = \bigcup_{\alpha \in X/\sim} \alpha$

Proof

- (1) Let $z \in [y]$. Then $y \sim z$. Since $y \in [x]$ one has $x \sim y$. Therefore, $x \sim z$ namely $z \in [x]$. This proves $[y] \subseteq [x]$. Moreover, since $x \sim y$, one has $x \in [y]$. Hence $[x] \subseteq [y]$. Thus we obtain $[x] = [y]$.
- (2) Suppose that $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$.
By (1), $\alpha = [y], \beta = [y]$. Thus leads to a contradiction.
- (3) $\forall x \in X \quad x \in [x]$ Hence $x \in \bigcup_{\alpha \in X/\sim} \alpha$. Hence $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$. Conversely,
 $\forall \alpha \in X/\sim, \alpha$ is a subset of X . Hence $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$. Then $X = \bigcup_{\alpha \in X/\sim} \alpha$.

14.4 Def

Let G be a group and X be a set.
We call left/right action of G on X an mapping $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$ that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

14.5 Remark

If we denote by G^{op} the set G equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of G on X is just a left action of G^{op} on X .

14.6 Prop

Let G be a group and X be a set. Assume given a left action of G on X . Then the binary relation \sim on X defined as $x \sim y$ iff $\exists g \in G \quad y = gx$ is an equivalence relation

14.7 Notation on Equivalence Class

We denote by G/X the set $X/\sim \forall x \in X$ the equivalence class of x is denoted as Gx/xG or $orb_G(x)$ call the orbit of x under the action of G

14.8 Proof

- $\forall x \in X \quad x = 1x$ so $x \sim x$
- $\forall (x, y) \in X^2$ if $y = gx$ for some $g \in G$ then $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$. ($y \sim x$)
- $\forall (x, y, z) \in X^3$, if $\exists (g, h) \in G^2$, such that $y = gx$ and then $z = h(gx) = (hg)x$ So $x \sim z$

14.9 Quotient set

Let X be a set and \sim be an equivalence relation, the mapping $X \rightarrow X/\sim$:
 $(x \in X) \mapsto [x]$ is called the projection mapping.

X/\sim is called the quotient set of X by equivalence relation \sim

14.9.1 Example

Let G be a group and H be a subgroup of G . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of H on G . Thus we obtain two quotient sets H/G and G/H

14.10 Def

Let G be a group and H be a subgroup of G . If $\forall g \in G, h \in H \quad ghg^{-1} \in H$,
 Then we say that H is a normal subgroup of G

14.11 Remark

$\forall g \in G, gH = Hg$, provided that H is a normal subgroup of G . In fact $\forall h \in$,

- $\exists h' \in H$ such that $ghg^{-1} = h'$ Hence $gh = h'g$. This shows $gH \subseteq Hg$
- $\exists h'' \in H$ such that $g^{-1}hg = h''$ Hence $hg = gh''$. This shows $Hg \subseteq gH$

Thus $gH = Hg$

14.12 Prop

If G is commutative, any subgroup of G is normal

14.13 Theorem

Let G be a group and H be a normal subgroup of G . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set G/H . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

Proof

- If $xH = x'H, yH = y'H$ then $\exists h_1 \in H, h_2 \in H$ such that $x' = xh_1, y' = yh_2$. Hence $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$. For $y^{-1}h_1y, h_2 \in H$ then $(x'y')H = (xy)H$. So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

14.14 Def

Let K be a unitary ring and E be a left K -module. We say that a subgroup F of $(E, +)$ is a left sub- K -module of E if $\forall (a, x) \in K \times F, ax \in F$

14.15 Prop

Let K be a unitary ring, E be a left K -module and F be a sub- K -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- K -module structure on E/F . Moreover, the projection mapping $\pi : E \rightarrow E/F$ is a morphism of left- K -modules

Proof

Let x and x' be elements of E such that $[x] = [x']$, that means: $x' - x \in F$
Hence $a(x' - x) = ax' - ax \in F$ So $[ax] = [ax']$
Let us check that E/F forms a left K -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition, π is a morphism of groups. Moreover $\forall x \in E, a \in K$ $\pi(ax) = [ax] = a[x] = a\pi(x)$

14.16 Def

Let A be a unitary ring . We call two-sided ideal any subgroup I of $(A, +)$ that satisfies : $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I$ (I is a left and right sub- K -module of A)

14.17 Theorem

Let A be a unitary ring and I be a two sided ideal of A . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover , A/I becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$)

Proof

If $a' \sim a, b' \sim b$ that means $a' - a \in I, b' - b \in I$ then $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$. For $(a' - a), (b' - b) \in I$, then $a'b' - ab \in I$
Therefore $a'b' \sim ab$

14.17.1 Reside Class

Let $d \in \mathbb{Z}$ and $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$ $d\mathbb{Z}$ is a two sided ideal of \mathbb{Z}
 If $m \in \mathbb{Z}$, for any $a \in \mathbb{Z}$ $adm = dma \in d\mathbb{Z}$

Denote by $\mathbb{Z}/d\mathbb{Z}$ the quotient ring. The class of $n \in \mathbb{Z}$ in $\mathbb{Z}/d\mathbb{Z}$ is called the residue class of n modulo d

If A is a commutative unitary ring, a two sided ideal of A is simply called an ideal of A

14.18 Theorem

Let $f : G \rightarrow H$ be a morphism of groups

- (1) $Im(f)$ is a subgroup of H
- (2) $\ker(f) := \{x \in G \mid f(x) = 1_H\}$ is a normal subgroup of G
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4) f is injective iff $\ker(f) = \{1_G\}$

Proof

- (1) Let α and β be elements of $Im(f)$. Let $(x, y) \in G^2$ such that $\alpha = f(x), \beta = f(y)$ Then $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ So $Im(f)$ is a subgroup
- (2) Let x and y be elements of $\ker(f)$.
 One has $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$
 So $xy^{-1} \in \ker f$. Hence $\ker f$ is a subgroup of G
 Let $x \in \ker f, y \in G$.
 One has $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$ Hence $yxy^{-1} \in \ker f$. So $\ker f$ is a normal subgroup
- (3) If $x \sim y$ then $\exists z \in \ker f$ such that $y = xz$ Hence $f(y) = f(x)f(z) = f(x)1_H = f(x)$ So f is well defined.
 Moreover $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$ Hence \tilde{f} is a morphism of groups.
 By definition $Im(\tilde{f}) = Im(f)$ If x and y are elements of G such that $f(x) = f(y)$ then $f(xy^{-1}) = 1_H$
 Hence $xy^{-1} \in \ker f$ Since $x = (xy^{-1})y$, $x \sim y$ that means $[x] = [y]$
 Therefore \tilde{f} is injective.

- (4) If f is injective, $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$, so $x = 1_G$. Therefore $\ker f = \{1_G\}$.
 Conversely, suppose that $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$ if $f(x) = f(y)$ then $f(x)f(y)^{-1} = 1_H$. Hence $xy^{-1} = 1_G, x = y$.

14.19 Theorem

Let K be a unitary ring and $f : E \rightarrow F$ be a morphism of left K -modules. Then

- (1) $\text{Im}(f)$ is a left-sub- K -module of F
- (2) $\ker(f)$ is a left-sub- K -module of E
- (3) $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$ is a isomorphism of left K -modules
 $[x] \mapsto f(x)$

Proof

- (1) $\forall x \in E, \quad f(ax) = af(x)$ So $af(x) \in \text{Im}(f)$
- (2)
- (3)

Chapter 15

Topology

15.1 Def

Let X be a set. We call topology on X any subset \mathcal{J} of $\wp(X)$ that satisfies:

- $\emptyset \in \mathcal{J}$ and $X \in \mathcal{J}$
- If $(u_i)_{i \in I}$ is an arbitrary family of elements in \mathcal{J} , then $\bigcup_{i \in I} u_i \in \mathcal{J}$
- If u and v are elements of \mathcal{J} , then $u \cap v \in \mathcal{J}$

15.2 Remark

If $(u_i)_{i=1}^n$ is a finite family of elements of \mathcal{J} , then $\bigcap_{i=1}^n u_i \in \mathcal{J}$ (by induction, this follows from (3))

15.2.1 Example

$\{\emptyset, X\}$ is a topology. call the trivial topology on $\wp(X)$ is a topology called the discrete topology.

15.3 Def

Let X be a set. We call metric on X any mapping $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$, that satisfies
 $(x, y) \mapsto d(x, y)$

- $d(x, y) = 0$ iff $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

(X, d) is called a metric space

15.3.1 Example

Let X be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

15.4 Def

Let (X, d) be a metric space. For any $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$, let $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ We call the open ball of radius ϵ centered at x

15.4.1 Example

Consider (\mathbb{R}, d) with $d(x, y) = |x - y|$, then $B(x, \epsilon) =]x - \epsilon, x + \epsilon[$

15.5 Prop.

Let (X, d) be a metric space. let \mathcal{J}_d be the set of $U \subseteq X$ such that $\forall x \in U \exists \epsilon > 0 \quad B(x, \epsilon) \subseteq U$ Then \mathcal{J}_d is a topology on X

Proof

- $\emptyset \in \mathcal{J}_d \quad X \in \mathcal{J}_d$
- Let $(u_i)_{i \in I}$ be a family of elements of \mathcal{J}_d Let $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$ such that $x \in u_i$. Since $u_i \in \mathcal{J}_d, \exists \epsilon > 0$ such that $B(x, \epsilon) \subseteq u_i \subseteq U$ Hence $U \in \mathcal{J}_d$
- Let U and V be elements of \mathcal{J}_d Let $x \in U \cap V \exists a, b \in \mathbb{R}_{\geq 0}$ such that $B(x, a) \subseteq U, B(x, b) \subseteq V$ Taking $\epsilon = \min\{a, b\}$, Then $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ Therefore $U \cap V \in \mathcal{J}_d$

15.6 Def

\mathcal{J}_d is called the topology induced by the metric d

15.7 Def

We call topology space any pair (X, \mathcal{J}) where X is a set and \mathcal{J} is a topology on X

Given a topological space (X, \mathcal{J}) If $U \in \mathcal{J}$ then we say that U is an open subset of X . If $F \in \wp(X)$ such that $X \setminus F \in \mathcal{J}$, then we say that F is closed subset of X

If there exists d a metric on X such that $\mathcal{J} = \mathcal{J}_d$ then we say that \mathcal{J} is metrizable

15.7.1 Example

Let X be a set . The discrete topology on X is metrizable. In fact, if d denote the metric defined as $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$
 $\forall x \in X \quad B(x, 1) = \{x\}$ So $\{x\} \in \mathcal{J}_d$ Hence $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{J}_d$

Chapter 16

Filter

16.1 Def

Let X be a set. We call filter on X any $\mathcal{F} \subseteq \wp(X)$ that satisfies:

- (1) $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2) $\forall A \in \mathcal{F}, \forall B \in \wp(X), \text{ if } A \subseteq B, \text{ then } B \in \mathcal{F}$
- (3) $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

16.1.1 Example

- (1) Let $Y \subseteq X, Y \neq \emptyset$. $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$ is a filter, called the principal filter of Y .
- (2) Let X be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is infinite}\}$$

is a filter called the Fréchet filter of X .

- (3) Let (X, \mathcal{J}) be a topological space, $x \in X$. We call neighborhood of x any $V \in \wp(X)$ such that $\exists u \in \mathcal{J}$, satisfying $x \in U \subseteq V$. Then $\mathcal{V} = \{\text{neighborhoods of } x\}$ is a filter.

16.2 Def: Filter Basis

Let X be a set. $\mathcal{B} \subseteq \wp(X)$. If $\emptyset \notin \mathcal{B}$ and $\forall (B_1, B_2) \in \mathcal{B}^2, \exists B \in \mathcal{B}$, such that $B \subseteq B_1 \cap B_2$. We say that \mathcal{B} is a filter basis.

16.2.1 Remark

If \mathcal{B} is a filter basis, then $\mathcal{F}(\mathcal{B}) = \{A \subseteq X \mid \exists B \in \mathcal{B} \quad B \subseteq A\}$ is a filter

Proof

$\emptyset \notin \mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{B}) \neq \emptyset$ since $0 \neq B \subseteq \mathcal{F}(\mathcal{B})$. If $A \in \mathcal{F}(\mathcal{B}), A' \in \wp(X)$ such that $A \subseteq A'$, then $\exists B \in \mathcal{B}$ such that $B \subseteq A \subseteq A'$. Hence $A' \in \mathcal{F}(\mathcal{B})$. If $A_1, A_2 \in \mathcal{F}(\mathcal{B})$, then $\exists(B_1, B_2) \in \mathcal{B}^2$ such that $B_1 \subseteq A_1, B_2 \subseteq A_2$. Since \mathcal{B} is a filter basis, $\exists B \in \mathcal{B}$ such that $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$. Hence $A_1 \cap A_2 \in \mathcal{F}(\mathcal{B})$.

16.2.2 Example

- Let $Y \subseteq X, Y \neq \emptyset$
 $\mathcal{B} = \{Y\}$ is a filter basis. $\mathcal{F}(\mathcal{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let (X, \mathcal{J}) be a topological space $x \in X$. If \mathcal{B}_x is a filter basis such that $\mathcal{F}(\mathcal{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$, then we say that \mathcal{B}_x is a neighborhood basis of x .

16.3 Remark

Let \mathcal{B}_x is a neighborhood basis of x iff

- $\mathcal{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathcal{B}_x$ such that $U \subseteq V$
- Let (X, d) be a metric space, $x \in X \forall \epsilon > 0$, Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Then

- $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x
- $\{B(x, \epsilon) \mid \epsilon > 0\}$ is a neighborhood basis of x
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ is a neighborhood basis of x

16.3.1 Example

$\mathcal{V}_x \cap \mathcal{J}$ is a neighborhood basis of x

16.4 Def

$V \in \wp(X)$ is called a neighborhood of x if $\exists U \in \mathcal{J}$ such that $x \in U \subseteq V$

16.5 Remark

Let (X, \mathcal{J}) be a topological space, $x \in X$ and \mathcal{B}_x a neighborhood basis of x . Suppose that \mathcal{B} is countable. We choose a surjective mapping $(B_n)_{n \in \mathbb{N}}$ from \mathbb{N} to \mathcal{B}_x . For any $n \in \mathbb{N}$, let $A_n = B_0 \cap B_1 \cap \dots \cap B_n \in \mathcal{V}_x$. The sequence $(A_n)_{n \in \mathbb{N}}$ is decreasing and $\{A_n \mid n \in \mathbb{N}\}$ is a neighborhood basis of x .

16.6 Extra Episode

$\wp(\mathbb{N})$ is NOT countable

Suppose that $f : \wp(\mathbb{N}) \rightarrow \mathbb{N}$ is injective. Then $\exists g : \mathbb{N} \rightarrow \wp(\mathbb{N})$ surjective. Taking $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$. Since g is surjective, $\exists a \in \mathbb{N}$ such that $A = g(a)$.

If $a \in A$, then $a \in g(a)$, hence $a \notin A$

If $a \notin A$, then $a \in g(a) = A$

Contradiction

16.7 Prop.

Let Y and E be sets, $g : Y \rightarrow E$ be a mapping,

- If \mathcal{F} is a filter of Y , then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on E

- If \mathcal{B} is a filter basis of Y , then

$$g(\mathcal{B}) := \{g(B) \mid B \in \mathcal{B}\}$$

is a filter of E , and $\mathcal{F}(g(\mathcal{B})) = g_*(\mathcal{F}(\mathcal{B}))$

Proof

- (1) $E \in g_*(\mathcal{F})$ since $g^{-1}(E) = Y$
 $\emptyset \notin g_*(\mathcal{F})$ since $g^{-1}(\emptyset) = \emptyset$

If $A \in g_*(\mathcal{F})$ and $A' \supseteq A$, then $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{F}$, so $g^{-1}(A') \in \mathcal{F}$,
Hence $A' \in g_*(\mathcal{F})$

If $A_1, A_2 \in g_*(\mathcal{F})$. Then $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$. Hence $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$. So $A_1 \cap A_2 \in g_*(\mathcal{F})$.

- (2) Since g is a mapping, and $\emptyset \notin \mathcal{B}$, we get $\emptyset \notin g(\mathcal{B})$, since $\mathcal{B} \neq \emptyset, g(\mathcal{B}) \neq \emptyset$.

Let $B_1, B_2 \in \mathcal{B}$, there exists $C \in \mathcal{B}$ such that $C \subseteq B_1 \cap B_2$. Hence $g(C) \subseteq g(B_1) \cap g(B_2)$, namely $g(\mathcal{B})$ is a filter basis.

Chapter 17

Limit point and accumulation point

We fix a topological space (X, \mathcal{T})

17.1 Def

Let \mathcal{F} be a filter of X and $x \in X$

- If $\mathcal{V}_x \subseteq \mathcal{F}$ then we say that x is a limit point of \mathcal{F}
- If $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \emptyset$, we say that x is an accumulation point of \mathcal{F}

So any limit point of \mathcal{F} is necessarily a accumulation point of \mathcal{F}

17.2 Prop

Let \mathcal{B} be a filter basis of X , $x \in X$, \mathcal{B}_x a neighborhood basis of x . Then x is an accumulation point of $\mathcal{F}(\mathcal{B})$ iff $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \emptyset$

Proof

Necessity

Since $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B})$, $\mathcal{B} \subseteq \mathcal{V}_x$, the necessity is true.

Sufficiency

Let $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$. There exist $B \in \mathcal{B}, U \in \mathcal{B}_x$, such that $B \subseteq A, U \subseteq V$. Hence $\emptyset \neq B \cap U \subseteq A \cap V$

17.3 Def

Let $Y \subseteq X, Y \neq \emptyset$. We call accumulation point of Y any accumulation point of the principal filter $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$.

17.4 Def

We denote by $\overline{Y} = \{\text{accumulation points of } Y\}$, called the closure of Y . Note that $x \in \overline{Y}$ iff $\forall U \in \mathcal{B}_x, Y \cap U \neq \emptyset$

By convention $\overline{\emptyset} = \emptyset$

17.5 Prop

Let $Y \subseteq X$. Then \overline{Y} is the smallest closed subset of X containing Y .

Proof

$\forall x \in X \setminus \overline{Y}$, then there exists $U_x = \mathcal{V} \cap \mathcal{J}$, such that $Y \cap U_x = \emptyset$. Moreover, $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{J}$. This shows that $\forall y \in U_x, y \notin \overline{Y}$. Therefore $X \setminus \overline{Y} = \bigcap_{x \in X \setminus \overline{Y}} U_x \in \mathcal{J}$

Let $Z \subseteq X$ be a closed subset that contain Y . Suppose that $\exists y \in \overline{Y} \setminus Z$. Then $U = X \setminus Z \in \mathcal{V}_y \cap \mathcal{J}$ and $U \cap Y \subseteq U \cap Z = \emptyset$. So $y \notin \overline{Y}$ contradiction. Hence $\overline{Y} \subseteq Z$.

17.6 Def: dense

Let (X, \mathcal{J}) be a topological space, Y a subset of X . We call Y is dense in X if

$$\overline{Y} = X$$

Chapter 18

Limit of mappings

18.1 Def

Let (E, \mathcal{J}_E) be a topological space. $f : Y \rightarrow E$ a mapping, and \mathcal{F} be a filter of Y . If $a \in E$ is a limit point of $F_*(\mathcal{F})$ namely, \forall neighborhood V of a , $f^{-1}(V) \in \mathcal{F}$, then we say that a is a limit of the filter \mathcal{F} by f

18.2 Remark

Let \mathcal{B}_a be a neighborhood basis of a . Then $\mathcal{V}_a \subseteq f_*(\mathcal{F})$, iff $\mathcal{B} \subseteq f_*(\mathcal{F})$. Therefore, a is a limit of \mathcal{F} by f iff $\forall V \in \mathcal{B}_a, f^{-1}(V) \in \mathcal{F}$

18.2.1 Example

Let (E, \mathcal{J}_E) be a topological space. $I \subseteq \mathbb{N}$ be an infinite subset, $x = (x_n)_{n \in I} \in E^I$. If the Fréchet filter $\mathcal{F}_{Fr}(I)$ has a limit $a \in E$ by the mapping $x : I \rightarrow E$, we say that $(x_n)_{n \in I}$ converges to a , denote as

$$a = \lim_{n \in I, n \rightarrow +\infty} x_n$$

18.3 Remark

$a = \lim_{n \in I, n \rightarrow +\infty} x_n$ iff, $\forall U \in \mathcal{B}_a$ (where \mathcal{B}_a is a neighborhood basis of a), $\exists N \in \mathbb{N}$ such that $x_n \in U$ for any $n \in I_{\geq N}$

Suppose that \mathcal{J}_E is induced by a metric d . $\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}, \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}, \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$ are all neighborhood basis of a . Therefore, the following are equivalent

- $a = \lim_{n \in I, n \rightarrow +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
 - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
 - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$
- $(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\})$? unknown position)

18.4 Remark

We consider the metric d on \mathbb{R} defined as

$$\forall (x, y) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of \mathbb{R} defined by this metric is called the usual topology on \mathbb{R}

18.5 Prop

Let $(x_n)_{n \in I} \in \mathbb{R}^I$, where $I \subseteq \mathbb{N}$ is an infinite subset. Let $l \in \mathbb{R}$. The following statements are equivalent:

- The sequence $(x_n)_{n \in I}$ converges to l in the topological space \mathbb{R}
- $\liminf_{n \in I, n \rightarrow +\infty} x_n = \limsup_{n \in I, n \rightarrow +\infty} x_n = l$
- $\limsup_{n \in I, n \rightarrow +\infty} |x_n - l| = 0$

18.6 Theorem

Let (X, d) be a metric space. Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be an element of X^I . Let $l \in X$. The following statements are equivalent:

- $(x_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ (equivalent to $\lim_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$)

Proof

- (1) \Rightarrow (2) The condition (1) is equivalent to $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$.
 We then get $\sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$. Therefore $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) < \epsilon$. We obtain that $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$.
- (2) \Rightarrow (1) Let $\epsilon \in \mathbb{R}_{>0}$. If $\inf_{N \in \mathbb{N}} \sup_{n \in I_{\geq N}} d(x_n, l) = 0$. Then $\exists N \in \mathbb{N} \quad \sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$.
 Hence, $\forall n \in I_{\geq N} d(x_n, l) < \epsilon$. Since ϵ is arbitrary, (*) is true, Hence (1) is also true.

18.7 Prop

Let (X, \mathcal{J}) be a topological space . $Y \subseteq X, p \in \overline{Y} \setminus Y$. Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of Y .

Proof

Y is not empty otherwise $\overline{Y} = \emptyset$.

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$
 $\emptyset \notin \mathcal{V}_{p,Y}$ since $p \in \overline{Y}$
- Let $V \in \mathcal{V}_p$ and $A \subseteq Y$ such that $V \cap Y \subseteq A$. Let $U = V \cup (A \setminus (V \cap Y)) \in \mathcal{V}_p$
and $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let U and V be elements of \mathcal{V}_p Let $W = U \cap V \in \mathcal{V}_p$ Then $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

18.8 Def

Let (X, \mathcal{J}_x) and (E, \mathcal{J}_E) be topological spaces, $Y \subseteq X, p \in \overline{Y} \setminus Y$, and $f : Y \rightarrow E$ be a mapping . If a is a limit point of $(F_*(\mathcal{V}_{p,Y}))$, then we say that a is a limit of f when the variable $y \in Y$ tends to p , denoted as $a = \lim_{y \in Y, y \rightarrow p} f(y)$

18.9 Remark

If \mathcal{B}_a is a neighborhood basis of a . Then $a = \lim_{y \in Y, y \rightarrow p} f(y)$ is equivalent to $\forall U \in \mathcal{B}_a \quad \exists V \in \mathcal{V}_p$ such that $Y \cap V \subseteq f^{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

18.10 Prop

Let X be a set, \mathcal{B} be a filter basis, \mathcal{G} be a filter. If $\mathcal{B} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$.

Proof

Let $V \in \mathcal{F}(\mathcal{B})$ By definition $\exists U \in \mathcal{B}$ such that $U \subseteq V$, since $U \in \mathcal{G}$ (for $\mathcal{B} \subseteq \mathcal{G}$) and since \mathcal{G} is a filter, $V \in \mathcal{G}$

18.11 Theorem

Let (X, \mathcal{J}_x) and (E, \mathcal{J}_E) be topological spaces. $Y \subseteq X$, $p \in \overline{T} \setminus Y$, $a \in E$. We consider the following conditions.

- (i) $a = \lim_{y \in Y, y \rightarrow p} f(y)$
- (ii) $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ if $\lim_{n \rightarrow +\infty} y_n = p$ then $\lim_{n \rightarrow \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that p has a countable neighborhood basis, then (i) and (ii) are equivalent.

Proof

- (1) Let $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$ such that $p = \lim_{n \rightarrow +\infty} y_n$. For any $U \in \mathcal{V}_p$, $\exists N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}$ $y_n \in U \cap Y$. Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \rightarrow +\infty} f(y_n) = a$$

- (2) Assume that p has a countable neighborhood basis. There exists a decreasing sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_p^{\mathbb{N}}$ such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of p .

Assume that (i) does not hold. Then there exists $U \in \mathcal{V}_a$ such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \rightarrow +\infty} y_n = \emptyset$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However $f(y_n) \notin U$ for any $n \in \mathbb{N}$, so $(f(y_n))_{n \in \mathbb{N}}$ cannot converges to a .

18.12 Prop.

Let X be a set. If $(\mathcal{J}_i)_{i \in I}$ is a family of topologies on X , then $\mathcal{J} = \bigcap_{i \in I} \mathcal{J}_i$ is a topology. In particular, for any $\mathcal{A} \subseteq \wp(X)$, there is a smallest topology on X that contains \mathcal{A} .

18.12.1 Proof

- $\forall i \in I \quad \{\emptyset, X\} \subseteq \mathcal{J}_i$ So $\{\emptyset, X\} \subseteq \mathcal{J}$
- Let $(u_j)_{j \in J}$ be a family of elements of $\mathcal{J} \quad \forall j \in J, i \in I \quad u_j \in \mathcal{J}_i$ So $\bigcup_{j \in J} u_j \in \mathcal{J}_i$ We then get $\bigcup_{j \in J} u_j \in \mathcal{J}$
- Let U and V be elements of $\mathcal{J} \quad \forall i \in I, \{u, v\} \subseteq \mathcal{J}_i$ So $U \cap V \in \mathcal{J}_i$. Therefore we get $U \cap V \in \mathcal{J}$ Let $\mathcal{A} \subseteq \wp(X)$ Let $\mathcal{J}(\mathcal{A}) = \bigcap_{\substack{\mathcal{J} \subseteq \wp(X) \text{ a topology} \\ \mathcal{A} \subseteq \mathcal{J}}} \mathcal{J}$ Then $\mathcal{J}(\mathcal{A})$ is a topology. By definition, if \mathcal{J} is a topology containing \mathcal{A} , then $\mathcal{J}(\mathcal{A}) \subseteq \mathcal{J}$ Hence $\mathcal{J}(\mathcal{A})$ is the smallest topology containing \mathcal{A}

Chapter 19

Continuity

19.1 Def

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces f be a function from X to Y , $x \in \text{Dom}(f)$. If for any neighborhood U of $f(x)$, there exists a neighborhood V of x such that $f(V) \subseteq U$. Then we say that f is continuous at x . If f is continuous at any $x \in \text{Dom}(f)$ then we say f is continuous.

19.2 Remark

Let $\mathcal{B}_{f(x)}$ be a neighborhood basis of $f(x)$ If $\forall U \in \mathcal{B}_{f(x)}$ there exist $V \in \mathcal{B}_{f(x)}$ such that $f(V) \subseteq U$, then f is continuous at x Suppose that X and Y are metric space. Then f is continuous at x iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in \text{Dom}(f) \quad d(y, x) < \delta \text{ implies } d(f(y), f(x)) < \epsilon$$

19.3 Theorem

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces, f be a function from X to Y $x \in \text{Dom}(f)$ Consider the following condition

- f is continuous at x
- $\forall (x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$, if $\lim_{n \rightarrow +\infty} x_n = x$, then $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$ THEN
(i) implies (ii) Moreover, if x has a countable neighborhood basis, then (i) and (ii) are equivalent.

19.4 Proof

(i) \Rightarrow (ii) Let $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ that converges to x $\forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$ Since $\lim_{n \rightarrow +\infty} x_n = x$, there exists $N \in \mathbb{N}$ such that $\forall n \in \mathbb{N}_{\geq N}, x_n \in V$.

Hence $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$. Thus $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$

(ii) \Rightarrow (i) under the hypothesis that x has countable neighborhood basis. actually we will prove $NOT(i) \Rightarrow NOT(ii)$

Let $(V_n)_{n \in \mathbb{N}}$ be a decreasing sequence in \mathcal{V}_x such that $\{V_n \mid n \in \mathbb{N}\}$ forms a neighborhood basis of x

If (i) does not hold, then $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$ Pick $x_n \in V_n$ such that $f(x_n) \notin U \quad \forall n \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$. Hence $(x_n)_{n \in \mathbb{N}}$ converges to x . However, $f(x_n) \notin U$ for any n So $(f(x_n))_{n \in \mathbb{N}}$ does not converges to $f(x)$. Therefore (ii) does not hold.

19.5 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y), (Z, \mathcal{J}_Z)$ be topological spaces. f be a function from X to Y , g be a function from Y to Z . Let $x \in \text{Dom}(g \circ f)$ If f and g are continuous at x . then $g \circ f$ is continuous at x sectionProof Let $U \in \mathcal{V}_{g(f(x))}$ Since g is continuous at $f(x)$:

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since f is continuous at x :

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore, $g(f(V)) \subseteq g(W) \subseteq U$ Hence $g \circ f$ is continuous at x

19.6 Def

Let (X, \mathcal{J}) be a topological space, $\mathcal{B} \subseteq \mathcal{J}$, If any element of \mathcal{J} can be written as the union of a family of sets in \mathcal{B} we say that \mathcal{B} is a topological basis of \mathcal{J}

19.7 Prop

Let (X, \mathcal{J}) be a topological space, $\mathcal{B} \subseteq \mathcal{J}$ \mathcal{B} is a topological basis iff

$$\forall x \in X, \mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$$

is a neighborhood basis of x

19.8 Proof

\Rightarrow :

$$\forall x \in X \mathcal{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since \mathcal{B} is a topological basis of \mathcal{J} ,

$$\exists W \in \mathcal{B}, x \in W \subseteq V \subseteq U$$

Hence \mathcal{V}_x is generated by \mathcal{B}_x

\Leftarrow Let $U \in \mathcal{J}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathcal{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{J}$$

19.9 Prop

Let $(X, \mathcal{J}_X), (Y, \mathcal{J}_Y)$ be topological spaces. \mathcal{B}_Y be a topological basis of \mathcal{J}_Y
 $f : X \rightarrow Y$ be a mapping. The following conditions are equivalent:

- (1) f is continuous
- (2) $\forall U \in \mathcal{J}_Y, f^{-1}(U) \in \mathcal{J}_X$
- (3) $\forall U \in \mathcal{B}_Y, f^{-1}(U) \in \mathcal{J}_X$

Proof

(1) \Rightarrow (2)

Lemma Let (X, \mathcal{J}) be a topological space, $V \in \wp(X)$, Then $V \in \mathcal{J}$ iff
 $\forall x \in V, V$ is a neighborhood of x

Proof of lemma \Rightarrow is by definition

Leftarrow:

$$\forall x \in V, \exists W_x \in \mathcal{J}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W_x - x \in \mathcal{J}$$

Let $U \in \mathcal{J}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood W of x such that $f(W) \subseteq U$
 Since f is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since x is arbitrary,

$$f^{-1}(U) \in \mathcal{J}_X$$

(2) \Rightarrow (3) For (3) is a special situation of (2), it's natural.

(3) \Rightarrow (1) Let $x \in X$

$$\forall U \in \mathcal{B}_Y \text{ s.t. } f(x) \in U, f^{-1}(U)$$

is an open neighborhood of x , and

$$f(f^{-1}(U)) \subseteq U$$

Hence f is continuous at x

19.10 Def

Let X be a set , $((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces. $\forall i \in I$ let $f_i : X \rightarrow Y_i$ be a mapping. We call initial topology of $(f_i)_{i \in I}$ on X the smallest topology on X making all f_i continue

19.11 Remark

If \mathcal{J} is the initial topology of $(f_i)_{i \in I}$, $\forall i \in I, U_i \in \mathcal{J}_i$ $f_i^{-1}(U_i) \in \mathcal{J}$ If $J \subseteq I$ is a finite subset, $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j$ then $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{J}$

19.12 Prop

$$\mathcal{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ is finite } (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

is a topological basis of the initial topology \mathcal{J}

Proof

First

$$\mathcal{B} \subseteq \mathcal{J}$$

Let

$\mathcal{J}' = \{\text{subset } V \text{ of } X \text{ that can be written as the union of a family of sets in } \mathcal{B}\}$

- $\emptyset \in \mathcal{J}' \quad X \in \mathcal{B} \subseteq \mathcal{J}'$
- \mathcal{J}' is stable by taking the union of any family of elements in \mathcal{J}'
- If V_1, V_2 are elements of \mathcal{J}' , then

$$V_1 \cap V_2 \in \mathcal{J}'$$

In fact, V_1, V_2 are of the form of the union of some sets of \mathcal{B}

The intersection of two elements of \mathcal{B} is still a element of \mathcal{B}

$$\begin{aligned} & \left(\bigcap_{j \in J} f_j^{-1}(U_j) \right) \cap \left(\bigcap_{j \in J'} f_j^{-1}(U'_j) \right) \\ &= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \setminus J' \\ U'_j & j \in J' \setminus J \\ U_j \cap U'_j & j \in J \cap J' \end{cases} \\ & \left(\bigcap_{j \in J \setminus J'} f_j^{-1}(U_j) \right) \cap \left(\bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U'_j) \right) \cap \left(\bigcap_{j \in J' \setminus J} f_j^{-1}(U'_j) \right) \end{aligned}$$

So \mathcal{J}' is a topology making all f_i continuous. Hence

$$\mathcal{J} \subseteq \mathcal{J}' \subseteq \mathcal{J} \Rightarrow \mathcal{J}' = \mathcal{J}$$

Example

Let $((Y_i, \mathcal{J}_i))_{i \in I}$ be topological spaces. $Y = \prod_{i \in I} Y_i$ and

$$\begin{aligned} \pi_i : Y &\rightarrow Y_i \\ (y_j)_{j \in I} &\mapsto y_i \end{aligned}$$

The product topology on Y is by definition the initial topology of $(\pi_i)_{i \in I}$

19.13 Theorem

Let X be a set, $((Y_i, \mathcal{J}_i))_{i \in I}$ be a family of topological spaces,

$$((f_i : X \rightarrow Y_i))_{i \in I}$$

be a family of mappings and we equip X with the initial topology \mathcal{J}_X of $(f_i)_{i \in I}$. Let (Z, \mathcal{J}_Z) be a topological space and

$$h : Z \rightarrow X$$

be a mapping. Then h is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{ is continuous}$$

19.13.1 Proof

\Rightarrow If h is continuous, since each f_i is continuous, $f_i \circ h$ is also continuous.

\Leftarrow Suppose that $\forall i \in I, f_i \circ h$ is continuous. Hence

$$\forall U_i \in \mathcal{J}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{J}_Z$$

Let

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{J}_j \right\}$$

$\forall U \in \mathcal{B}$

$$h^{-1}(U) = \bigcap_{j \in J} h^{-1}(f_j^{-1}(U_j)) \in \mathcal{J}_Z$$

Therefore, h is continuous.

19.14 Remark

We keep the notation of the definition of initial topology. If $\forall i \in I, \mathcal{B}_i$ is a topological basis of \mathcal{J}_i , then

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{B}_j \right\}$$

is also a topological basis of the initial topology,

19.14.1 Example

Let $((X_i, d_i))_{i \in \{1, \dots, n\}}$ be a family of metric spaces.

$$X = \prod_{i \in \{1, \dots, n\}} X_i$$

We define a mapping

$$d: (X \times X \rightarrow \mathbb{R}_{\geq 0}) \\ d: ((x_i)_{i \in \{1, \dots, n\}}, (y_i)_{i \in \{1, \dots, n\}}) \mapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i)$$

d is a metric on X . If $x = (x_i)_{i \in \{1, \dots, n\}}$, $y = (y_i)_{i \in \{1, \dots, n\}}$, $z = (z_i)_{i \in \{1, \dots, n\}}$ are elements of X , then

$$d(x, z) = \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \leq \max_{i \in \{1, \dots, n\}} (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i: X \rightarrow X_i \\ \pi_i: (x_i)_{i \in \{1, \dots, n\}} \mapsto x_i$$

is continuous. Hence the product topology \mathcal{J} is contained in \mathcal{J}_d

Let $x = (x_i)_{i \in \{1, \dots, n\}} \in X$, $\epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i \in \{1, \dots, n\}} \mid \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i \in \{1, \dots, n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i \in \{1, \dots, n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{J} \end{aligned}$$

Chapter 20

Uniform continuity and convergency

20.1 Def

Let (X, d) be a metric space. $\forall A \subseteq X$, we define

$$\text{diam}(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A. By convention

$$\text{diam}(\emptyset) := 0$$

If $\text{diam}(A) < +\infty$, we say that A is bounded

20.2 Remark

- If A is finite, then it's bounded
- If $A \subseteq B$ then $\text{diam}(A) \leq \text{diam}(B)$

20.3 Prop

Let (X, d) be a metric space. $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$. Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

In particular, if A, B are bounded, then $A \cup B$ is bounded.

Proof

Let $(x, y) \in (A \cup B)^2$. If $\{x, y\} \subseteq A$, then $d(x, y) \leq \text{diam}(A)$
 If $\{x, y\} \subseteq B$ then $\text{diam}(B) \geq d(x, y)$
 If $x \in A, y \in B$,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

Similarly if $x \in B, y \in A$

$$d(x, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

20.4 Def

Let (X, d) be a metric space. $I \subseteq \mathbb{N}$ be an infinite subset, $(x_n)_{n \in I} \in X^I$. If

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \quad \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \epsilon$$

then we say that $(x_n)_{n \in I}$ is a Cauchy sequence.

20.5 Prop

- (1) If $(x_n)_{n \in I}$ converges, then it's a Cauchy sequence.
- (2) If $(x_n)_{n \in I}$ is a Cauchy sequence, $\{x_n \mid n \in I\}$ is bounded
- (3) Suppose that $(x_n)_{n \in I}$ is a Cauchy sequence. If there exists an infinite subset J of I such that $(x_n)_{n \in J}$ converges to some $x \in X$, then $(x_n)_{n \in I}$ converges to x

20.5.1 Proof

- (1) trivial
- (2) trivial
- (3) Let $\epsilon > 0, \exists N \in \mathbb{N}$

$$\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence $(x_n)_{n \in I}$ converges to x

20.6 Def

Let $(X, d_X), (Y, d_Y)$ be metric space. f be a function from X to Y . If $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

we say that f is uniformly continuous.

20.7 Prop

Let $(X, d_X), (Y, d_Y)$ be metric spaces f be a function from X to Y which is uniformly continuous.

- (1) If $I \subseteq \mathbb{N}$ is finite, and $(x_n)_{n \in I}$ is a Cauchy sequence in $\text{Dom}(f)^I$ then $(f(x_n))_{n \in I}$ is Cauchy sequence
- (2) f is continuous

20.7.1 Proof

- (1) $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists N \in \mathbb{N}$ such that

$$\forall (n, m) \in I_{\geq N}^2, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

Therefore $(f(x_n))_{n \in I}$ is a Cauchy sequence.

- (2) Let $(x_n)_{n \in I}$ be a sequence in $\text{Dom}(f)^{\mathbb{N}}$ that converges to $x \in \text{Dom}(f)$ We define $(y_n)_{n \in \mathbb{N}}$ as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then $(y_n)_{n \in \mathbb{N}}$ converges to x . Hence $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. Since f is uniformly continuous, $(f(y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y .

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to $f(x)$. Hence $(f(y_n))_{n \in \mathbb{N}}$ converges to $f(x)$

20.8 Def

Let X be a set, $Z \subseteq X$, (Y, d) be a metric space, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$ and f be functions from X to Y , having Z as their common domain of definition.

- If $\forall x \in Z, (f_n(x))_{n \in I}$ converges to $f(x)$, we say that $(f_n)_{n \in I}$ converges pointwisely to f
- If

$$\lim_{n \in I, n \rightarrow +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$

we say that $(f_n)_{n \in I}$ converges uniformly to f

20.9 Theorem

Let X and Y be metric space, $Z \subseteq X$, $I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}, f$ be functions from X to Y , having Z as domain of definition. Suppose that

- $(f_n)_{n \in I}$ converges uniformly to f
- each f_n is uniformly continuous

Then f is uniformly continuous.

20.9.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\lim_{n \in I, n \rightarrow +\infty} A_n = 0$$

$\forall (x, y) \in Z^2, n \in I$

$$\begin{aligned} & d(f(x), f(y)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & \leq 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f_n(x), f_n(y)) = 0$$

Hence

$$0 \leq \inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$

Take $\lim_{n \rightarrow +\infty}$, by squeeze theorem, we get

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

20.10 Theorem

Let X be a topological space, Y be a metric space, $Z \subseteq X, p \in Z, I \subseteq \mathbb{N}$ infinite. $(f_n)_{n \in I}$ and f function from X to Y , having Z as domain of definition. Suppose that:

- $(f_n)_{n \in I}$ converges uniformly to f
- each f_n is continuous at p

Then f is continuous at p

20.10.1 Proof

$\forall n \in I$ let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since f_n is continuous $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{B}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x), f(p)) &\leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$$f(U) \subseteq \overline{B}(f(p), \epsilon)$$

20.10.2 Def

Let X, Y be metric spaces, f be a function from X to Y , $\epsilon > 0$. If

$$\forall (x, y) \in \text{Dom}(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that f is ϵ -Lipschitzian

If $\exists \epsilon > 0$ such that f is ϵ -Lipschitzian, then it's uniformly continuous.

20.11 Remark

If f is Lipschitzian, then it's uniformly continuous.

20.12 Example

- Let $((X_i, d_i))_{i \in I}$ be metric space. $X = \prod_{i \in I} X_i$ where I is finite

$$\begin{aligned} X \times X &\rightarrow \mathbb{R}_{\geq 0} \\ d : d((x_i), (y_i)_{i \in I}) &= \max_{i \in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \rightarrow X_i$$

is Lipschitzian. ($\forall x = (x_i)_{i \in I}, \forall y = (y_i)_{i \in I}$)

- Let (X, d) be a metric space

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$

Part V

Normed Vector Space

Chapter 21

Linear Algebra

We fix a unitary ring K

21.1 Def

Let M be a left K -module, and let $x = (x_i)_{i \in I}$ be a family of elements of M . We define a morphism of left K -module as following:

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \quad (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

21.1.1 Notation

$$\begin{aligned} K^{\oplus I} &:= \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{ finite, such that } a_i = 0 \text{ for } i \in I \setminus J\} \\ \varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) &= \varphi_x((a_i)_{i \in I}) + \varphi_x((b_i)_{i \in I}) \end{aligned}$$

21.2 Def

Let M be a left K -module, I be a set, $x = (x_i)_{i \in I} \in M^I$. If

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say $(x_i)_{i \in I}$ is K -linearly independent

surjective then we say $(x_i)_{i \in I}$ is system of generator

a bijection then we say $(x_i)_{i \in I}$ is a basis of M

Example

Let e_i be the element $(\delta_{ij})_{j \in I}$ with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i \in I} \in (K^{\oplus I})^I$$

is a basis of $K^{\oplus I}$

21.3 Def

Let M be a left K -module

- If M has a basis, we say that M is a free K -module
- If M has finite system of generated
(\exists a finite set I and a family $(x_i)_{i \in I} \in M^I$ that forms a system of generator),
then we say that M is of finite type.

21.4 Remark

Let $x = (x_i)_{i \in \{1, \dots, n\}} \in M^n$, where $n \in \mathbb{N}$

- x is linearly independent iff

$$\forall a \in K^n \quad \sum a_i x_i = 0$$

implies

$$a = 0$$

- x is a system of generator iff for any element of M can be written in the form

$$\sum b_i x_i \quad b \in K^n$$

Such expression is called a K -linear combination of x_1, \dots, x_n

21.5 Theorem

Let K be a division ring ($0 \neq 1$ and $\forall k \in K \setminus \{0\}$ k is invertible)

Let V be a left K -module of finite type and $(x_i)_{i \in I}$ be a system of generators of V . Then, there exists a subset I of $\{1, \dots, n\}$ such that $(x_i)_{i \in I}$ forms a basis of V . (In particular, V is a free K -module)

Proof

(By induction on n)

If $n = 0$, then $V = \{0\}$

In this case \emptyset is a basis of V

Induction hypothesis

True for a system of generators of $n - 1$ elements. Let $(x_i)_{i \in \{1, \dots, n\}}$ be a system of generators of V . If $(x_i)_{i \in \{1, \dots, n\}}$ is linearly independent, it's a basis. Otherwise, $\exists (a_i)_{i \in I} \in K^n$ such that

$$(a_i, \dots, a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose $a_n \neq 0$. Then

$$x_n = -a_n^{-1} \left(\sum_{i=1}^{n-1} a_i x_i \right)$$

Since $(x_i)_{i \in \{1, \dots, n\}}$ is a system of generators, any elements of V can be written as

$$\begin{aligned} \sum b_i x_i &= \left(\sum_{i=1}^{n-1} b_i x_i \right) - b_n a_n^{-1} \left(\sum_{i=1}^{n-1} a_i x_i \right) \\ &= \sum_{i=1}^{n-1} (b_i - b_n a_n^{-1} a_i) x_i \end{aligned}$$

Thus $(x_i)_{i \in \{1, \dots, n\}}$ forms a system of generators. By the induction hypothesis, there exists $I \subseteq \{1, \dots, n\}$ such that $(x_i)_{i \in I}$ forms a basis of V .

21.6 Theorem

Let K be a unitary ring and B be a left K -module. W be a left K -submodule of V . Let $(x_i)_{i=1}^n$ be an element of W^n

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where $(n, l) \in \mathbb{N}^2 \forall j \in \{1, \dots, l\}$, let x_{n+j} be an element in the equivalence class α_j

- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are linearly independent, then $(x_i)_{i=1}^{n+l}$ is also linearly independent
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are system of generators of W and V/W respectively, then $(x_i)_{i=1}^{n+l}$ is also a system of generators
- If both $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$ are basis, then $(x_i)_{i=1}^{n+l}$ is also a basis

Proof

(1) Suppose that $(b_i)_{i=1}^{n+l}$ such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \rightarrow V/W$$

be the projection morphism ($\pi(x) = [x]$)

$$0 = \pi\left(\sum_{i=1}^{n+l} b_i x_i\right) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^l b_{n+j} \pi(x_{n+j}) = \sum_{j=1}^l b_{n+j} \alpha_j$$

$$\{x_1, \dots, x_n\} \subseteq W \text{ So } \forall i \in \{1, \dots, n\}$$

$$\pi(x_i) = 0$$

Since $(\alpha_j)_{j=1}^l$ is linearly independent,

$$b_{n+1} = \dots = b_{n+l} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since $(x_i)_{i=1}^n$ is linearly independent,

$$b_1 = \dots = b_n = 0$$

(2) Let $y \in V$. Then $\pi(y) \in V/W$. So there exists

$$(c_{n+1}, \dots, c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^l c_{n+j} \alpha_j \\ &= \sum_{j=1}^l c_{n+j} \pi(x_{n+j}) = \pi\left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \end{aligned}$$

So

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \in W$$

$\exists c \in K^n$ such that

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) = \left(\sum_{i=1}^n c_i x_i\right)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

21.7 Corollary

Let K be a division ring and V be a left K -module of finite type. If $(x_i)_{i=1}^n$ is a linearly independent family of elements of V ($n \in \mathbb{N}$), then

$$\exists l \in \mathbb{N} \quad \exists (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of V

Proof

Let W be the image of

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow V \\ (a_i)_{i=1}^n &\mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left K -submodule of V .

Note that $(x_i)_{i=1}^n$ forms a basis of W .

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow W \\ \varphi(x_i)_{i=1}^n(e_j) &= x_j \in W \end{aligned}$$

Moreover, since V is of finite type there exists $d \in \mathbb{N}$ and a surjective morphism of left K -modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \rightarrow V/W$$

is surjective.

Hence the composite morphism

$$K^d \begin{array}{c} \xrightarrow{\psi} \\ \searrow \pi \circ \psi \end{array} V \xrightarrow{\pi} V/W$$

is surjective. Thus V/W is of finite type. There exist then a basis

$$(a_j)_{j=1}^l$$

of V/W .

Taking $x_{n+j} \in \alpha_j$ for $j \in \{1, \dots, l\}$, we get a basis of V :

$$(x_i)_{i=1}^{n+l}$$

21.8 Def

Let K be a division ring and V be a left K -module of finite type. We call rank of V the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If K is a field $rk(V)$ is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of V .

21.9 Theorem

Let K be a division ring and V be a left K -module of finite type. Let W be a left K -submodule of V .

(1) W and V/W are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of V has exactly $rk(V)$ elements

21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let $(x_i)_{i=1}^n$ be a basis of V . Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

In $(\pi(x_i))_{i=1}^n$ we extract a basis of V/W , say

$$(\pi(x_i))_{i=1}^l$$

For $j \in \{l+1, \dots, n\}$,

$$\exists(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any $x \in W, \exists(a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^n a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^l a_i x_i + \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \\ &= \sum_{j=l+1}^n a_j y_j + \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^n a_j y_j$$

Hence W is of finite type, and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let $(x_i)_{i=1}^n$ be a basis of V . Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of V/W .

We extract a subfamily, say $(x_i)_{i=1}^l$ such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of V/W .

For $j \in \{1, \dots, l\}$, there exists:

$$(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Let $x \in W, \exists (a_i)_{i=1}^n \in K^n$ let $x = \sum a_i x_i$, then

$$\begin{aligned} x &= \left(\sum_{i=1}^l a_i x_i \right) + \left(\sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \right) \\ &= \left(\sum_{i=1}^l a_i x_i \right) + \left(\sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i \right) + \left(\sum_{j=l+1}^n a_j y_j \right) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i)$$

Therefore $(y_j)_{j=l+1}^n$ is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + rk(V/W)$$

(2) All basis of V have $rk(V)$ elements.

We reason by induction on $rk(V)$

(1)

$$rk(V) = 0$$

In this case $V = \{0\}$ The only basis of V is \emptyset . So the statement holds.

(2) Assume that there exists $e \in V \setminus \{0\}$ such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of V is of the form

$$ae$$

where $a \in K \setminus \{0\}$

Let $(e_i)_{i=1}^m$ be a basis of V . We reason by induction on m to prove that

$$m = rk(V)$$

The cases where $m = 0$ or 1 are proved in (1)(2) respectively. Induction hypothesis: true for a basis of $< m$ elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of V/W .

If $(a_i)_{i=2}^m \in K^{m-1}$ such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for $(e_i)_{i=1}^m$ a basis of V ,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of V/W . We then obtain that

$$rk(V/W) \leq m - 1 \leq n - 1$$

By the induction hypothesis,

$$m - 1 = rk(V/W)$$

By (2), $rk(W) = 1$. Hence

$$m = (m - 1) + 1 = rk(V/W) + rk(W) = rk(V)$$

21.11 Prop

Let K be a unitary ring and $f : E \rightarrow F$ be a morphism of left K -modules. Let I be a set and $(x_i)_{i \in I} \in E^I$

- If $(x_i)_{i \in I}$ is linearly independent and f is injective, then $(f(x_i))_{i \in I}$ is linearly independent.
- If $(x_i)_{i \in I}$ is a system of generators and f is surjective, then $(f(x_i))_{i \in I}$ is a system of generators.
- If $(x_i)_{i \in I}$ is a basis and f is an isomorphism, then $(f(x_i))_{i \in I}$ is a basis.

21.11.1 Proof

$$\varphi_{(f(x_i))_{i \in I}} = f \circ \varphi_{(x_i)_{i \in I}}$$

Chapter 22

Matrices

We fix unitary ring K

22.1 Def

Let $n \in \mathbb{N}$ and V be a left K -module.

For any $(x_i)_{i=1}^n \in V^n$, we denote by $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$ the morphism

$$\begin{aligned} & \phi_{(x_i)_{i=1}^n} : K^n \rightarrow V \\ (a_i)_{i=1}^n & \mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

22.1.1 Example

Suppose that $V = K^p$ ($p \in \mathbb{N}$) Then each $x_i \in K^p$ is of the form $(x_{i,1}, \dots, x_{i,p})$

Hence $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ can be written:

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,p} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,p} \end{pmatrix}$$

22.2 Def

Let $(n, p) \in \mathbb{N}^2$. We call n by p matrix of coefficient in K any morphism of left K -modules from K^n to K^p

22.2.1 Example

- Denote by I_n then identity mapping. Then $(e_i)_{i=1}^n$ is a basis of K^n called the canonical basis of K^n

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, \dots, a_n)) = \sum_{i=1}^n a_i e_i = (a_1, \dots, a_n)$$

- Let $(x_1, \dots, x_n) \in K^n$, Denote by

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) (= \varphi_{(x_i e_i)_{i=1}^n}) : K^n &\rightarrow K^n \\ (a_1, \dots, a_n) &\mapsto (a_1 x_1, \dots, a_n x_n) \end{aligned}$$

22.3 Def

We denote by $M_{n,p}(K)$ the set of all n by p matrices of coefficients in K . For $(n, p, r) \in \mathbb{N}^3$, we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) &\rightarrow M_{n,r}(K) \\ (A, B) &\mapsto AB := B \circ A \end{aligned}$$

22.4 Calculate Matrices

Let K be a unitary ring, and V be a left K -module. Let $n \in \mathbb{N}$ and

$$x = (x_1, \dots, x_n) \in V^n$$

22.4.1 Remind

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \varphi : (a_1, \dots, a_n) \mapsto a_1 x_1, \dots, a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1, \dots, p\} \times \{1, \dots, n\}} \in M_{p,n}(K)$$

A is a morphism of left K -modules from K^p to K^n . Recall that

$$A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let $(b_1, \dots, b_n) \in K^p$

$$\begin{aligned} A((b_1, \dots, b_n)) &= \sum_{i=1}^p b_i(a_{i,1}, \dots, a_{i,n}) \\ \varphi(A((b_1, \dots, b_n))) &= \sum_{i=1}^p b_i \varphi_x((a_{i,1}, \dots, a_{i,n})) \\ &= \sum_{i=1}^p b_i(a_{i,1}x_1, \dots, a_{i,n}x_n) \end{aligned}$$

Let $B = \{b_{ij}\}_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\}} : K^n \rightarrow K^r$

$$AB = \left\{ \sum_{j=1}^n a_{lj} b_{jm} \right\}_{(l,m) \in \{1, \dots, p\} \times \{1, \dots, r\}}$$

Chapter 23

Transpose

We fix a unitary ring K

23.1 Def

Let E be a left- K -module. Denote by

$$E^\vee := \{\text{morphisms of left } K\text{-modules } E \rightarrow K\}$$

$\forall (f, g) \in E^\vee$ let

$$\begin{aligned} f + g : E &\rightarrow K \\ x &\mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$ forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : E &\rightarrow K \\ x &\mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} K \times E^\vee &\rightarrow E^\vee \\ (a, f) &\mapsto fa : x \in E \rightarrow f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right K -module on E^\vee

23.2 Def

Let E and F be two left K -modules. $\varphi : E \rightarrow F$ be a morphism of left K -modules. We denote by

$$\varphi^\vee : F^\vee \rightarrow E^\vee$$

the morphism of right K -modules sending $g \in F^\vee$ to $g \circ \varphi \in E^\vee$.
Actually $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

23.2.1 Example

Suppose that $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,p} \end{pmatrix}$$

φ sends (a_1, \dots, a_n) to $\{\sum_{i=1}^n a_i b_{ij}\}_{j \in \{1, \dots, p\}}$. Let $g \in F^\vee$ $g : K^p \rightarrow K$, then g is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$ sends (a_1, \dots, a_n) to $\sum_{i=1}^p (\sum_{j=1}^n a_j b_{ij} y_i)$

Assume that K is commutative. We denote by

$$\iota_p : (K^p)^\vee \rightarrow K^p$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, \dots, x_p)$$

$$\iota_n : (K^n)^\vee \rightarrow K^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, \dots, x_n)$$

are isomorphisms of K -modules

For any morphism of K-modules $\varphi : K^n \rightarrow K^p$, we denote by φ^τ the morphism of K-modules $K^p \rightarrow K^n$ given by $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{array}{ccc} (K^p)^\vee & \xrightarrow{\varphi^\vee} & (K^n)^\vee \\ \cong \downarrow \iota_p & \circlearrowleft & \cong \downarrow \iota_n \\ K^p & \xrightarrow{\varphi^\tau} & K^n \end{array}$$

φ^τ is called the transpose of φ

23.3 Prop

Let E,F,G be left K-modules. $\varphi : E \rightarrow F, \psi : F \rightarrow G$ be morphisms of left K-modules. Then $(\psi \circ \varphi)^\vee$ is equal to $\varphi^\vee \circ \psi^\vee$

Proof

$$\forall f \in G^\vee$$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

23.4 Corollary

Assume that K is commutative. Let n, p, q be neutral numbers. $A \in M_{n,p}(K), B \in M_{p,q}(K)$. Then

$$(AB)^\tau = B^\tau A^\tau$$

Proof

$$A^t a u = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^t a u = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned} B^\tau A^\tau &= A^\tau \circ B^\tau \\ &= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\ &= (AB)^t a u \end{aligned}$$

23.5 Remark

(1) For $A \in M_{n,p}(K)$, one has $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} E &\rightarrow (E^\vee)^\vee \\ x &\mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a K -linear mapping.

If K is a field and E is of finite dimension, this is an isomorphism of K -modules.

In fact, if $e = (e_i)_{i=1}^n$ is a basis of E over K . For $i \in \{1, \dots, n\}$, let

$$\begin{aligned} e_i^\vee : E &\rightarrow K \\ \lambda_1 e_1, \dots, \lambda_n e_n &\mapsto \lambda_i \end{aligned}$$

is called the dual basis of e

$$\begin{array}{ccc} K^n & \xleftarrow[\iota_n]{\cong} & (K^n)^\vee \\ \varphi_e \downarrow \cong & \searrow \varphi_{e^\vee} & \downarrow \varphi_e^\vee \\ E & \xrightarrow[\cong]{} & E^\vee \end{array}$$

$(e^\vee)^\vee$ gives a basis of $(E^\vee)^\vee$. Hence $E \rightarrow (E^\vee)^\vee$ is an isomorphism.

Chapter 24

Linear Equation

We fix a unitary ring K .

24.1 Def

For $a = (a_1, \dots, a_n) \in K^n \setminus \{(0, \dots, 0)\}$. Denote by $j(a)$ the first index $j \in \{1, \dots, n\}$ such that $a_j \neq 0$. Let $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$. We write A as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \in K^p$$

We say that A is of row echelon form if, $\forall i \in \{1, \dots, n-1\}$ one of following conditions is satisfied.

- $a^{(i+1)} = (0, \dots, 0)$
- $a^{(i)}, a^{(i+1)}$ are non-zero, and $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, \dots, n\}$ such that $a^{(i)} \neq (0, \dots, 0)$, one has

$$a_{j(a^{(i)})}^{(i)} = 1$$

and

$$\forall k \in \{1, \dots, n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that A is of reduced row echelon form.

24.2 Prop

Suppose that $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$ is of row echelon form. Then $\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$ is of cardinal $\leq p$

Proof

Let $k = \text{card}\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$ $a^{(k+1)} = \dots = a^{(n)} = (0, \dots, 0)$ and $j(a^{(1)}) < j(a^{(2)}) < \dots < j(a^{(k)})$ Hence

$$\{1, \dots, k\} \rightarrow \{1, \dots, p\}, i \mapsto j(a^{(i)})$$

is injection. So $k \leq p$

24.3 Linear Equation

Let $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$. Let V be a left K -module and $(b_1, \dots, b_n) \in V^n$. We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

The set of $(x_1, \dots, x_p) \in V^p$ that satisfies $(*)$ is called the solution set of $(*)$

24.4 Prop

Suppose that A is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, \dots, n\} \mid (a_{i,1}, \dots, a_{i,p}) \neq (0, \dots, 0)\}$$

$$J_0(A) = \{1, \dots, p\} \setminus \{j((a_{i,1}, \dots, a_{i,p})) \mid i \in I(A)\}$$

- If $\exists i \in \{1, \dots, n\} \setminus I(A)$ such that $b_i \neq 0$ then $(*)$ does not have any solution in K^n
- Suppose that $\forall i \in \{1, \dots, n\} \setminus I(A), b_i = 0$. Then $(*)$ has at least one solution. Moreover

$$V^{J_0(A)} \rightarrow V^p$$

$$(z_k)_{k \in J_0(A)} \mapsto (x_1, \dots, x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, \dots, a_{i,p})) \end{cases}$$

is an injective mapping, whose image is equal to the set of solution of (*)

24.5 Prop

Let $m \in \mathbb{N}, S \in M_{m,n}(K)$. If $(x_1, \dots, x_p) \in V^p$ is a solution of (*), then (x_1, \dots, x_p) is a solution of $(*)_S$:

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

In the case where S is left invertible, namely there exist $R \in M_{n,m}(K)$ such that $RS = I_n \in M_{n,n}(K)$. Then (*) and $(*)_S$ have the same solution set.

24.6 Def

Let $G_n(K)$ be the set of $S \in M_{n,n}(K)$ that can be written as $U_1 \dots U_N$ (by convention $S = I_n$ where $N = 0$) where each U_i is of one of the following forms.

- P_σ where $\sigma \in \mathfrak{S}_n$
- $\text{diag}(r_1, \dots, r_n)$ where each $r_i \in K$ is left invertible
- $S_{i,c}$ with $i \in \{1, \dots, n\}$ $c = (c_1, \dots, c_n) \in K^n, c_i = 0$

Let $p \in \mathbb{N}$, we say that $A \in M_{n,p}(K)$ is reducible by Gauss elimination if $\exists S \in G_n(K)$ such that SA is of reduced row echelon form

24.7 Theorem

Assume that K is a division ring $\forall (n, p) \in \mathbb{N}$ any $A \in M_{n,p}(K)$ is reducible by Gauss elimination

Proof

The case where $n = 0$ or $p = 0$ is trivial. We assume $n \geq 1, p \geq 1$ We write A as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} B \quad \text{where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If $\lambda_1 = \dots = \lambda_n = 0$

Applying the induction hypothesis to B, for $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} SB$$

- Suppose that $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

By permuting the rows we may assume $\lambda_1 \neq 0$. As K is division ring, by multiplying the first row by λ_1^{-1} , we may assume $\lambda_1 = 1$. We add $(-\lambda_i)$ times the first row to the i^{th} row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 & \dots & \mu_p \\ 0 & & & \\ \vdots & C & & \\ 0 & & & \end{pmatrix} \quad \begin{array}{l} C \in M_{n-1, p-1}(K) \\ (\mu_2, \dots, \mu_p) \in K^{p-1} \end{array}$$

Applying the induction hypothesis to C, we may assume that C is of reduced row echelon form. For $i \in \{2, \dots, k\}$ we add $-\mu_{j(c_i)}$ times the i^{th} row of A to the first line to obtain a matrix of reduced row echelon form

Chapter 25

Normed Vector Space

25.1 Def

Let (X, d) be a metric space. If $(x_n)_{n \in \mathbb{N}}$ is an element of $X^{\mathbb{N}}$ such that

$$\lim_{N \rightarrow +\infty} \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

we say that $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence. If any Cauchy sequence in X converges, then we say that (X, d) is complete.

Let $Cau(X, d)$ be the set of all Cauchy sequences in X . We define a binary relation \sim on $Cau(X, d)$ as

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$$

25.2 Prop

\sim is an equivalence relation.

25.2.1 Proof

$$\lim_{n \rightarrow +\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$ be elements of $Cau(X, d)$. For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n \rightarrow +\infty} d(x_n, z_n) = 0$$

25.3 Def

$$\hat{X} := \text{Cau}(X, d) \setminus \sim$$

25.4 Def: The completion

The completion of (X, d) is defined as

$$\text{Cau}(X) / \sim$$

and is denoted as

$$\hat{X}$$

25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} : \hat{X} \times \hat{X} &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n \rightarrow +\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on \hat{X}

Proof

TO check that \hat{d} is well defined, it suffices to prove that $\forall ([x], [y]) \in \hat{X} \times \hat{X}$, $(d(x_n, y_n))_{n \in \mathbb{N}}$ is Cauchy sequence and its limit doesn't depend on the choice of the representation x and y

For $N \in \mathbb{N}$ and $(n, m) \in \mathbb{N}_{\geq N}$ for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n, m) \in \mathbb{N}_{\geq N}} |d(x_n, y_n) - d(x_m, y_m)| &\leq \left(\sup_{(n, m) \in \mathbb{N}_{\geq N}} d(x_n, x_m) \right) \\ &\quad + \left(\sup_{(n, m) \in \mathbb{N}_{\geq N}} d(y_n, y_m) \right) \end{aligned}$$

Taking $\lim_{n \rightarrow +\infty}$ we obtain that $(d(x_n, y_n))_{n \in \mathbb{N}}$ is a Cauchy sequence.

Hence it converges in \mathbb{R} . If $x' = (x'_n)_{n \in \mathbb{N}} \in [x], y' = (y'_n)_{n \in \mathbb{N}} \in [y]$, thus

$$\lim_{n \rightarrow +\infty} d(x_n, x'_n) = \lim_{n \rightarrow +\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking $\lim_{n \rightarrow +\infty}$ we get

$$\lim_{n \rightarrow +\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(x'_n, y'_n)$$

In the following, we check that \hat{d} is a metric

- $\hat{d}([x], [y]) = 0$ iff $[x] = [y]$: trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$: trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$:

$$\begin{aligned} d([x], [y]) &= \lim_{n \rightarrow +\infty} \\ &\leq \lim_{n \rightarrow +\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

25.6 Remark

Let

$$\begin{aligned} i_X : X &\rightarrow \hat{X} \\ a &\mapsto [(a, a, \dots)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular, i_x is injective (if $i_X(a) = i_X(b)$ then $d(a, b) = 0$ hence $a = b$)

25.7 Prop

$i_X(X)$ is dense in \hat{X} (the closure of $i_X(X)$ in \hat{X} is equal to $i_X(X)$ (or to say \hat{X}))

Proof

Let $[x]$ be an equivalence class in \hat{X} . We claim that $\forall (x_n)_{n \in \mathbb{N}} \in [x]$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} i_X(x_n)$$

For any $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n \rightarrow +\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \end{aligned}$$

Taking $\lim_{N \rightarrow +\infty}$ we get

$$\lim_{N \rightarrow +\infty} \hat{d}(i_X(x_N), [x]) = 0$$

25.8 Theorem

(\hat{X}, \hat{d}) is a complete metric space

Proof

Let $([x^{(N)}])_{N \in \mathbb{N}}$ be a Cauchy sequence in \hat{X} , where $\forall N \in \mathbb{N}$, $x^{(N)} = (x_n^{(N)})_{n \in \mathbb{N}}$ is a Cauchy sequence
 $\forall \epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $\forall (k, l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n \rightarrow +\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any $(\mu, \nu) \in \mathbb{N}_{\geq \alpha(N)}$

Let $y_N = x_{\alpha(N)}^{(N)}$. Without loss of generality, we assume that

$$\alpha(0) \leq \alpha(1) \leq \dots$$

Let $\epsilon > 0$ Take $N_0 \in \mathbb{N}$ such that

$$(1) \quad \forall (k, l) \in \mathbb{N}, \quad k, l \geq N_0$$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

$$(2)$$

$$\frac{1}{N_0 + 1} \leq \frac{\epsilon}{3}$$

Let $(k, l) \in \mathbb{N}_{N_0}^2$,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since $\alpha(k) \geq N_0, \forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking $\lim_{n \rightarrow +\infty}$ get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So $y = (y_N)_{N \in \mathbb{N}}$ is a Cauchy sequence. We check that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \hat{d}([x^{(N)}], [y]) &= 0 \\ 0 &\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \rightarrow +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$\begin{aligned} d(x_n^{(N)}, y_n) &\leq d(x_n^{(N)}, y_N) + d(y_n, y_N) \\ \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, y_n) &\leq \limsup_{N \rightarrow +\infty} \left(\frac{1}{N+1} + \lim_{n \rightarrow +\infty} d(y_n, y_N) \right) \end{aligned}$$

Since y is Cauchy sequence

$$\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(y_n, y_N) = 0$$

Example

Let $(K, |\cdot|)$ be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$ iff $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$\text{Cau}(K)$ forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in \text{Cau}_0(K)$$

where

$$\text{Cau}_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of $\text{Cau}(K)$

Hence

$$\hat{K} = \text{Cau}(K) \setminus \text{Cau}_0(K)$$

is a quotient ring of $\text{Cau}(K)$

$|\cdot|$ extend to \hat{K} :

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \rightarrow +\infty} |a_n|$$

that forms an absolute value.

Chapter 26

Norms

In this chapter we fix a field K and an absolute value $|\cdot|$ on K . We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

26.1 Def

Let V be a vector space over K (K -module). We call seminorm on V any mapping

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}_{\geq 0} \\ s &\mapsto \|s\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V$

If additionally:

- $\forall s \in V, \|s\| = 0$ iff $s = 0$

We say that $\|\cdot\|$ is a norm and $(V, \|\cdot\|)$ is normed space over K .

26.2 Remark

If $\|\cdot\|$ is a norm then

$$\begin{aligned} d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let $(V, \|\cdot\|)$ be a vector space over K equipped with a seminorm, and W be a vector space subspace of V (sub- K -module)

- The restriction of $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$ to W forms a seminorm on W . It is a norm if $\|\cdot\|$ is a norm.

$$\begin{aligned}\|\cdot\|_W : W &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto \|x\|\end{aligned}$$

- The mapping

$$\begin{aligned}\|\cdot\|_{V/W} : V/W &\rightarrow \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\|\end{aligned}$$

is a seminorm on V/W

Attention: Even if $\|\cdot\|$ is a norm, $\|\cdot\|_{V/W}$ **might only be a seminorm**

26.3 Def

$\|\cdot\|_{V/W}$ is called the quotient seminorm of $\|\cdot\|$

26.4 Prop

Let $(V, \|\cdot\|)$ be a vector space over K , equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of V . Moreover, $\|\cdot\|_{V/N}$ is a norm

Proof

If $(a, s) \in K \times N$ then $\|as\| = |a| \cdot \|s\| = 0$ so $as \in N$

If $(s_1, s_2) \in N \times N$ then $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$ so $s_1 + s_2 \in N$

Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let $\alpha \in V/N$ such that $\|\alpha\|_{V/N} = 0$ Let $s \in \alpha, \forall t \in N$

$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$

$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$

Hence $\|\alpha\|_{V/N} = \|s\| = 0$ We obtain that $\alpha = N = [0]$

26.5 Def

Let $(V, \|\cdot\|)$ be a vector space over K , equipped with a seminorm. For any $x \in V$ and $r \geq 0$, we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$

$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

26.6 Remark

If $N = \{s \in V, \|s\| = 0\}$ then when $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

26.7 Def

We equip the topology such that $\forall U \subseteq V, U$ is open iff $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

26.8 Prop

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with seminorms. Let $f : V_1 \rightarrow V_2$ be a K -linear mapping

- If f is continuous, $\forall s \in V_1$ if $\|s\|_1 = 0$ then $\|f(s)\|_2 = 0$
- If there exists $C > 0$ such that $\forall x \in V_1, \|f(x)\|_2 \leq C\|x\|_1$ then f is continuous.

The converse is true

when $|\cdot|$ is non-trivial

or $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$ is of finite type

Proof

- (1) Lemma If $(V, \|\cdot\|)$ is a vector space over K , equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let $s \in V \setminus N_{\|\cdot\|}$ Then $\|s\| > 0$. Let $\epsilon = \frac{\|s\|}{2}$, $\forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq \|s\| - \|s - x\| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteq V \setminus N_{\|\cdot\|}$$

– Then $f^{-1}(N_{\|\cdot\|_2})$ is closed.

Note that

$$0 \in f^{-1}(N_{\|\cdot\|_2})$$

hence

$$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$

$$\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$$

$$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$

and

$$0 \in \mathcal{B}(x, \epsilon)$$

Therefore $x \in \overline{\{0\}}$

(2) Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of V_1 that converges to some $x \in V_1$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n \rightarrow +\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n \rightarrow +\infty} C \|x_n - x\|_1 \\ &= C \limsup_{n \rightarrow +\infty} \|x_n - x\| \\ &= 0 \end{aligned}$$

So $(f(x_n))_{n \in \mathbb{N}}$ converges to $f(x)$. Hence f is continuous at x

Assume that $|\cdot|$ is non-trivial and f is continuous. Then

$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

is an open subset of V_1 containing $0 \in V_1$

So there exists $\epsilon > 0$ such that

$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

namely $\forall x \in V_1$ if $\|x\|_1 < \epsilon$ then $\|f(x)\|_2 < 1$

Since $|\cdot|$ is nontrivial, $\exists a \in K, 0 < |a| < 1$ We prove that $\forall x \in V_1$

$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|} \|x\|_1$$

If $\|x\|_1 = 0$ by (1) we obtain

$$\|f(x)\|_2 = 0$$

Suppose that $\|x\|_1 > 0$ then $\exists n \in \mathbb{Z}$ such that

$$\begin{aligned} \|a^n x\|_1 &= |a|^n \|x\|_1 \\ &< \epsilon \leq \\ &\|a^{n-1} x\|_1 = |a|^{n-1} \|x\|_1 \end{aligned}$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned} \|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}} \frac{1}{|a|} \\ &\leq \frac{1}{\epsilon} \|x\|_1 \frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|} \end{aligned}$$

26.9 Def

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with seminorm. We say that a K -linear mapping $f : V_1 \rightarrow V_2$ is bounded if there exists $C > 0$ that

$$\forall x \in V_1 \quad \|f(x)\|_2 \leq C\|x\|_1$$

For a general K -linear mapping $f : V_1 \rightarrow V_2$ we denote

$$\|f\| := \begin{cases} \sup_{x \in V_1, \|x\|_1 > 0} \left(\frac{\|f(x)\|_2}{\|x\|_1} \right) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

f is bounded iff

$$\|f\| < +\infty$$

$\|f\|$ is called the operator seminorm of f

We denote by $\mathcal{L}(V_1, V_2)$ the set of all bounded K -linear mappings from V_1 to V_2

26.10 Prop

$\mathcal{L}(V_1, V_2)$ is a vector subspace of $\text{Hom}_K(V_1, V_2)$. Moreover $\|\cdot\|$ is a seminorm on $\mathcal{L}(V_1, V_2)$

Proof

Let f, g be elements of $\mathcal{L}(V_1, V_2)$

$$\begin{aligned} \|f + g\| &= \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x) + g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq \left(\sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1} \right) + \left(\sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1} \right) \\ &\leq +\infty \end{aligned}$$

Hence $f + g \in \mathcal{L}(V_1, V_2)$

Let $\lambda \in K$, $\lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

26.11 Remark

Let $f \in \mathcal{L}(V_1, V_2)$. Suppose that $\exists x \in V_1$ such that $f(x) \neq 0$. Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore $\|\cdot\|$ is a norm

26.12 Def

Let $(V, \|\cdot\|)$ be a normed vector space. If V is complete with respect to the metric

$$\begin{aligned}d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|x - y\|\end{aligned}$$

then we say that $(V, \|\cdot\|)$ is a Banach space.

26.13 Theorem

Let $(V_1, \|\cdot\|_1)$ and $(V_2, \|\cdot\|_2)$ be vector spaces over K , equipped with semi-norm. If $(V_2, \|\cdot\|_2)$ is a Banach space, then

$$(\mathcal{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

Proof

Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(V_1, V_2)$.
 $\forall x \in V_1$, the mapping

$$(f \in \mathcal{L}(V_1, V_2)) \mapsto f(x)$$

is $\|x\|_1$ -Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\| \|x\|_1$$

So $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence, for V_2 is complete, that converges to some $g(x) \in V_2$. Then we obtain a mapping $g : V_1 \rightarrow V_2$. We prove that g is an element of $\mathcal{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n \rightarrow +\infty} f_n(x + y) = \lim_{n \rightarrow +\infty} f_n(x) + f_n(y)$$

$$\begin{aligned} \|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \rightarrow +\infty) \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n \rightarrow +\infty} f_n(\lambda x) = \lim_{n \rightarrow +\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1) (n \rightarrow +\infty)$$

So $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n \rightarrow +\infty} \|f_n(x)\| \leq (\lim_{n \rightarrow +\infty} \|f_n\|) \cdot \|x\|$$

(because $\forall (a, b) \in V_2^2 \quad \| \|a\| - \|b\| \| \leq \|a - b\|$) Then

$$\| \|f_n(x)\| - \|g_n(x)\| \| \leq \|f_n(x) - g_n(x)\| = o(1) (n \rightarrow +\infty)$$

So $g \in \mathcal{L}(V_1, V_2)$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall (n, m) \in \mathbb{N}_{\geq N}, \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking $\lim_{n \rightarrow +\infty}$ we get

$$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

So $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$

Chapter 27

Differentiability

In this chapter we fix a field K and an absolute value $|\cdot|$ on K . We assume that $(K, |\cdot|)$ forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

27.1 Def

Let E and F be normed vector space over K . $U \subseteq E$ be an open subset, $f : U \rightarrow F$ be a mapping and $p \in U$. If there exists $\varphi \in \mathcal{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

we say that f is differentiable at p , and φ is the differential of f at p . Suppose that $|\cdot|$ is not trivial.

Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

means there exists an open neighborhood V of p with $V \subseteq U$ and a mapping $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$ such that $\lim_{x \rightarrow p} \epsilon(x) = 0$ and that $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

27.2 Prop

If f is differentiable at p , then its differential at p is unique

Proof

Suppose that there exists φ and ψ in $\mathcal{L}(E, F)$ such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|(\varphi - \psi)(y)\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\|(\|y - p\|)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \rightarrow p} \epsilon(y) = 0 \end{aligned}$$

27.3 Example**27.3.1**

$$f : U \rightarrow F : f(x) = y_0 \quad \forall x \in U$$

$\forall p \in U$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence $\forall x \in E$

$$d_p(f(x)) = 0$$

27.3.2

Let $f \in \mathcal{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence $d_p f = f$

27.3.3

$$A : E \times E \rightarrow E$$

$$(x, y) \mapsto x + y$$

Let E be a normed space. Then $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

27.3.4

$$\begin{aligned} m : K \times E &\rightarrow E \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

Let $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when $(\lambda, x) \rightarrow (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

is a K -linear mapping.

$$\begin{aligned} - (\mu_1 + \mu_2)p + a(y_1 + y_2) &= (\mu_1 p + ay_1) + (\mu_2 p + ay_2) \\ - b\mu p + a(by) &= b(\mu p + ay) \\ - \|\mu p + ay\| &\leq |\mu| \|p\| + |a| \|y\| \\ &\leq \max\{|\mu|, \|y\|\} (|a| + \|p\|) \end{aligned}$$

Hence m is differentiable and $\forall (\mu, y) \in K \times E$

$$d_{(a,p)} m(\mu, y) = \mu p + ay$$