Contents

Chapter 1

Preface

1.1 Ref

- Ahlfors: Complex analysis.
- 谭小江, 伍胜健复变函数简明教程
- Stein,? complex analysis.(extra exercises)

1.2 A brief history of complex analysis

Complex analysis refers studies on functions of complex variables, emerged in the 19th century. Cauchy proposed Cauchy 's integral theorem (1825) and the concept of residues. Riemann defined the Riemann Surface, which enlarge complex analysis to geometry field. Besides, he defined Riemann zeta function. And he gave Riemann mapping theorem. Weirstrass use power series to approach complex analysis.

Complex analysis also deeply connects to other filed in math.

- It's essential to analysis geometry and complex geometry.
- Provide powerful tool to research prime numbers.
- In dynamics, complex dynamics is active.
- Deep connected with topology of 3-manifold.
- Deep connection with harmonic analysis(Fourier analysis).

Part I

Review of learnt

Chapter 2

Definition of complex numbers

$\mathbb{R}$ denotes the real numbers. Some polynomials equation like $x^2 + 1 = 0$ has no solutions in $\mathbb{R}$. So we formally introduce the number i (an imaginary number) s.t.

$$i^1 + 1 = 0$$

A complex number $z = a + bi$, where $a, b \in \mathbb{R}$. Let

$$\mathbb{C} = \{z = a + bi \mid a, b \in \mathbb{R}\}$$

$\mathbb{C}$ is called complex plane. The real numbers a, b are called the real and imaginary part of z respectively. Denoted by $\Re z,\ \Im z$

Similar with to $\mathbb{R}$, we can define a field structure on $\mathbb{C}$.

Addition

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Multiplication

$$(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i$$

To verify $\mathbb{C}$ a field, we need to show $\forall z \neq 0,\ \exists z^{-1}$

2.1 Def: complex conjugation

Let $z \in \mathbb{C}$. The complex conjugation $\overline{z}$ of $z = a + bi$ is

$$\overline{z} = a - bi$$

Ones can verify are

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{zw} = \overline{z}\overline{w}$$

As a corollary, we consider a polynomial equation

$$a_n z^n + \cdots + a_0 = 0 \quad a_i \in \mathbb{C}$$

. If z is a root, then $\overline{z}$ a root for:

$$\overline{a_n} z^n + \cdots + \overline{a_0} = 0$$

In particular, $a_i \in \mathbb{R}$, then $\overline{z}$ is also a solution to original equation.

2.2 Def:absolute value

The absolute value of complex number z is defined as:

$$|z| := \sqrt{z \cdot \overline{z}} = \sqrt{a^2 + b^2}$$

one can verify:

$$|zw| = |z| \cdot |w|$$
$$|z+w|^2 = |z|^2 + |w|^2 + 2\Re(z\overline{w})$$
$$|z-w|^2 = |z|^2 + |w|^2 - 2\Re(z\overline{w})$$

2.3 Def: division

Let $z_1, z_2 \in \mathbb{C}$

$$\frac{z_1}{z_2} := \frac{z_1 \overline{z_2}}{|z_2|^2}$$

In particular, if $z = a + bi$

$$z^{-1} = \frac{\overline{z}}{|z|^2}$$

Chapter 3

Geometry picture of complex numbers

We can identify $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector space, by using $z = a + bi$. We can also use the polar coordinates write $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$, θ is called the argument of z. Then conjugation flip z along real axis. Addition is the same with vectors' addition. Multiplication multiplicate the length of vector and rotate the vector by the other's argument.

Consider the equation $z^n = 1,\ n \geq 1$. The solution of it is called n-th root of unity.

3.1 Some inequalities

By the definition of absolute value

$$-|z| \leq \Re z \leq |z|$$

$$-|z| \leq \Im z \leq |z|$$

The equality $\Re z = |z|$ iff z is a non-negative real number. Since $Re(z\overline{w}) \leq |z|\,|w|$ recall for $z, w \in \mathbb{C}$

$$|z + w|^2 = |z|^2 + |w|^2 + 2\Re(z\overline{w})$$

Then we get triangle inequality:

$$|z + w| \leq |z| + |w|$$

3.1.1 Cauchy's inequality

Let $n \geq 1$, then

$$\left|\sum_{k=1}^{n} z_k w_k\right|^2 \leq (\sum_{k=1}^{n} |z_k|^2)(\sum_{k=1}^{n} |w_k|^2)$$

with the equality holds iff $\exists t \in \mathbb{C}, \forall 1 \leq k \leq n, z_k + t\overline{w_k} = 0$

Proof

Let $t \in \mathbb{C}$ be any complex number

$$0 \leq \sum_{k=1}^{n} |z_k + t\overline{w_k}|^2 = \sum_{k=1}^{n} |z_k|^2 + |t|^2 \sum_{k=1}^{n} |w_k|^2 + 2\Re(\bar{t} \sum_{k=1}^{n} z_k w_k)$$

choose $t = \frac{\sum_{k=1}^{n} z_k w_k}{\sum_{k=1}^{n} |w_k|^2}$ Then we get

$$\sum_{k=1}^{n} |z_k|^2 = \frac{\left|\sum_{k=1}^{n} z_k w_k\right|^2}{\sum_{k=1}^{n} |w_k|^2} \geq 0$$

The condition of equality $\Leftarrow$ the equality $0 = \sum_{k=1}^{n} |z_k + t\overline{w_k}|$

Chapter 4

Topology and metrics on $\mathbb{C}$

4.1 Basic definitions

Recall that a topology space is a set X equipped with a collection of subsets of X as open sets, satisfying:

- X and $\varnothing$ are open.
- Arbitrary union of open sets is open
- Finite intersection of open sets is open.

A closed set is by definition the complement of an open set.

A metric space is a pair (X, d), where X be a set and $d : X^2 \to \mathbb{R}_{\geq 0}$ a mapping s.t.

- $d(x, x) = 0 \quad \forall x \in X$
- $d(x, y) > 0 \quad \forall x \neq y \in X$
- $d(x, y) = d(y, x)$
- $d(x, y) \leq d(x, z) + d(z, y)$

let $x \in X, r > 0 \in \mathbb{R}$ the set

$$\mathscr{B}(x, r) := \{y \in X \mid d(x, y) < r\}$$

is called an open ball. We say a subset $N \subseteq X$ is a neighborhood of x if N contains an open ball centered at x. A subset N is open if $\forall x \in N$ N is a neighborhood of x

Remark

For any subset $N \subseteq X$ (N, d) is a metric space. The diameter of X:

$$diamX := \sup_{x,y \in X} d(x, y)$$

X is bounded if $diamX < +\infty$. A sequence of points x_n in X is called converges to $x \in X$ if $\lim_n \to +\infty d(x_n, x) = 0$. A sequence (x_n) is called Cauchy sequence if $\forall \epsilon > 0, \exists N \geq 1$ s.t. $\forall n > m \geq N, d(x_n, x_m) < \epsilon$

The metric space is called complete if any Cauchy sequence converges.

4.2 Notations

$N \subseteq X$ any subset.

- $\mathring{N}$ the interior of N, is the maximal open subset contained in N, i.e.

$$\mathring{N} = \text{ union of all open subsets in N}$$

- $\overline{N}$ the **closure** of N, the minimal closed set contains N.

- ∂N the **boundary** of N,

$$\partial := \overline{N} \setminus \mathring{N}$$

  let $N \subseteq X$. A point $x \in X$ is a limit point of N if $x \in \overline{N} \Leftrightarrow$ this means $\exists$ sequence (x_n) in N s.t. $x_n \to x$ $(\lim_{n \to +\infty} d(x_n, x) = 0)$

- We say $x \in X$ is called an **isolated** point if $\exists$ an open ball $\mathcal{B}(x, r)$ s.t.

$$\mathcal{B}(x, r) \cap X = \{x\}$$

- We say X is **connected** if X is not a disjoint union of non-empty open subsets.

- a point $x \in X$ is called a **limit point** of N if $x \in \overline{N} \Leftrightarrow$ this means $\exists$ sequence (x_n) in N s.t. $x_n \to x$ $(\lim_{n \to +\infty} d(x_n, x) = 0)$

Chapter 5

Compactness

An open cover of X is a collection of open sets $\{U_\alpha\}$, $X = \bigcup\limits_{\alpha} U_\alpha$

X is called totally bounded if $\forall \epsilon > 0, \exists$ finite open cover using $\epsilon-$radius balls. It's clear that totally bounded set is bounded.

The metric space X is called compact if every open cover of X has a finite sub-cover.

5.1 Theorem

A metric space X is compact $\Leftrightarrow$ X is complete and totally bounded.

Proof

$\Rightarrow$

For completeness, assuming X isn't. Then exists a Cauchy sequence (x_n) doesn't converges. Then $\forall y \in X, x_n \not\to y$. Then $\exists r > 0$ s.t. $\cup_y := \mathscr{B}(y, r)$ then $\cup_y$ contains finite many elements. Then we get an open cover $\{\cup_y\}_{y \in X}$. For X compact, we can get a finite subcover.

$$X = \bigcup_{y \in F} \cup_y$$

where F finite.In particular $x_n \in \bigcup\limits_{y \in F} \cup_y$ so $X_n = \{x_n \mid n \in \mathbb{N}\}$ contains finite many elements. But finite Cauchy sequence converges. Contradiction.

For total boundence. For every $\epsilon > 0, y \in X$ let $\cup_y := \mathscr{B}(y, \epsilon)$ Then $\{\cup_y\}_{y \in X}$ is an open cover. For compactness, we get a finite subcover $X \subseteq \bigcap\limits_{y \in F} \cup_y$ so X totally bounded.

$\Leftarrow$

Assume X is complete and totally bounded Assume X is not compact. Then $\exists$ open cover $\{U_\alpha\}$ s.t. $\nexists$ finite subcover. For totally bounded, $X = \bigcap_{x \in F} \mathcal{B}(x, 1)$ F finite. Then consider the index in F s.t. $\mathcal{B}(x, 1) \neq \bigcup_{\alpha \in E} \cup_\alpha \cap \mathcal{B}(x, 1)$ Then exist x_0 s.t. $\mathcal{B}(x_0, 1)$ can not be covered by finite many $\cup_\alpha$ So $\exists x_1 \in \mathcal{B}(x_0, 1)$ s.t. $\mathcal{B}(x_1, 2^{-1})$ cannot be covered by finite many $\cup_\alpha$. Inductively, we get a sequence $(x_n)_{n \in \mathbb{N}}$.

$$d(x_n, x_{n+1}) \leq 2^{-n}$$

which means (x_n) is Cauchy sequence. Moreover, $\mathcal{B}(x_n, 2^{-n})$ can't be covered by finite many $\cup_\alpha$. By completeness, $(x_n) \to y \in X$ Let U be an open set s.t. $U \in \{U_\alpha\}$ and $y \in U$. Then for n large

$$\mathcal{B}(x_n, 2^{-n}) \subseteq U$$

contradiction.

5.2 Theorem

A metric space X. Compact is equiv with Cauchy compact.

Proof

$\Leftarrow$

Assume X Cauchy compact. We prove it by prove X complete and totally bounded.

For a Cauchy sequence $(x_n)_{n \in \mathbb{N}}$ converges iff $\exists$ subsequence (x_{n_i}) converges. This means every sequence in X converges, meaning X complete.

Assume X isn't totally bounded. Then $\exists \epsilon > 0$ s.t. X isn't covered by finite ϵ-balls. We inductively construct a sequence (x_n) as following: We choose x_{n+1} s.t. $x_{n+1} \notin \bigcup_{k=1}^{n} \mathcal{B}(x_k, \epsilon)$ It doesn't have a subsequence convergent. Contradiction.

$\Rightarrow$

Assume that $\exists (x_n)_{n \in \mathbb{N}}$ divergent. Then $\forall y \in X, \exists r > 0$ s.t.

$$\cup_y := \mathcal{B}(y, r)$$

$\cup_y$ contains finite points in $\{x_n\}$. Then consider the open cover $\{\cap_y\}_{y \in X}$. According to compactness, extract a finite sub-cover: $X \subseteq \bigcup_{y \in F} \cup_y$. Then $\{x_n\}$ a finite set, which means (x_n) has a convergent subsequence. For Cauchy sequence this implies convergence. Contradiction.

5.3 Def

Consider $X = \mathbb{C}$ $\forall z, w \in \mathbb{C}$

$$d(z, w) := |z - w|$$

open balls in $\mathbb{C}$ is called open disks $\mathscr{D}(z, r)$

$$\mathbb{D} := D(0, 1)$$

is called unit disk.

5.4 Lemma

A sequence $z_n \to z$ in $\mathbb{C} \Leftrightarrow$

- $\Re z_n \to \Re z$
- $\Im z_n \to = \Im z$

5.5 Theorem

$\mathbb{C}$ is complete.

Proof

This follows $\mathbb{R}$ is complete and the Lemma above.

5.6 Lemma

A bounded subset in $\mathbb{C}$ is totally bounded.

Proof

Let $K \subseteq \mathbb{C}$ bounded. $\exists R > 0$ s.t. $K \subseteq \mathscr{D}(0, R)$. It suffice to show $\mathscr{D}(0, R)$ is totally bounded. It's clear, since $\mathscr{D}(0, R)$ can be covered by finitely many $\epsilon-$balls.

5.7 Corollary

A subset $K \subseteq \mathbb{C}$ is compact $\Leftrightarrow$ K is bounded and K is closed.

Proof

K is compact $\Leftrightarrow$ K is totally bounded and complete. Since $\mathbb{C}$ complete, K is complete iff K is closed. Then K compact $\Leftrightarrow$ K closed and bounded.

5.8 Continuous mapping

$f : X \to Y$ between metric space is continuous if $\forall$ open set $U \subseteq Y$ $f^{-1}(U)$ is open

A homomorphism $f : X \to Y$ continuous and f^{-1} is also continuous.

5.9 Lemma

Let $f : X \to Y$ between metric space f is continuous $\Leftrightarrow$ $\forall$ sequence (x_n) in X $x_n \to x \Rightarrow f(x_n) \to f(x)$

5.10 Theorem

$f : X \to Y$ continuous mapping between metric spaces. Let $K \subseteq X$ compact. Then $f(K)$ is compact in Y.

Proof

Any open cover $\{V_\alpha\}$ of $f(K)$ induces an open cover $U_\alpha := f^{-1}V_\alpha$ of K. Since K is compact, $\exists$ finite set F s.t.

$$K = \bigcup_{\alpha \in F} U_\alpha$$

then

$$f(K) = \bigcup_{\alpha \in F} V_\alpha$$

so $f(K)$ is compact.

5.11 Corollary

Let X compact metric space. Let $f : X \to \mathbb{R}$ continuous function. Then $f(X)$ can take maximal and minimal values.

Proof

$f(x)$ is compact in $\mathbb{R}$

5.12 Theorem

$f : X \to Y$ continuous. If X is connected, then $f(X)$ is connected.

5.12.1 Proof

Assume that $f(X) = A \cup B$, with A, B non-empty and disjoint. Then $X = f^{-1}(A) \cup f^{-1}(B)$ is a union of non-empty open sets, meaning X not connected.

5.13 Def

Let $f : X \to Y$ continuous mapping f is called uniformly continuous if $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

5.14 Theorem

Let $f : X \to Y$ continuous. X compact. Then f uniformly continuous.

Chapter 6

Path connected and homotopy

A curve in $\mathbb{C}$ is a continuous mapping $\gamma : [a, b] \to \mathbb{C}$

6.1 Def

A subseteq $S \subseteq \mathbb{C}$ is called path-connected if $\forall z, w \in S, \exists \gamma : [a, b] \to S$ curve s.t. $\gamma(a) = z, \gamma(b) = w$

6.2 Theorem

Let $U \subseteq \mathbb{C}$ open set. U is connected $\Leftrightarrow$ path connected.

Proof

Let $U \subseteq \mathbb{C}$ open $\forall z \in U$ let

$$V_z := \{\text{points} w \in U \text{ s.t.} \exists \text{curve connecting } z, w\}$$

Since every open disk is path connected, V_z is open, $U \setminus V_z$ is open.

$\Rightarrow$

Assume U not path connected. Then $\exists z \in U$ s.t. $V_z \neq U$. Let $V_1 := V_z, V_2 := U \setminus V_z$. Then V_1, V_2 are non-empty open disjoint sets, then U not connected. Contradiction.

$\Leftarrow$

Assume U not connected. We can write

$$U = V_1 \cup V_2$$

V_1, V_2 non-empty and disjoint open sets. Let $z \in V_1, w \in V_2$ and $\gamma : [a,b] \to U$ curve $\gamma(a) = z, \gamma(b) = w$. Let

$$I_1 := \gamma^{-1}V_1 \quad I_2 := \gamma^{-1}V_2$$

I_1, I_2 open non-empty and disjoint and $[a,b] \cup I_1, I_2$, telling $[a,b]$ not connected. Contradiction.

6.2.1 Remark

This conclusion isn't true in general when U is not open. Consider

$$S : \{iy \mid |y| \leq 1\} \cup \{x + i\sin\frac{1}{x} \mid 0 < x \leq 1\}$$

S closed. Try to prove:

- S connected
- S not path connected

6.3 Def:homotopy

Let $U \subseteq \mathbb{C}$ be an open set. Let $\gamma_0 : [a,b] \to U$, $\gamma_1 : [a,b]$ be two curves. A homotopy between γ_0 and γ_1 is a continuous mapping

$$H : [0,1] \times [a,b] \to U$$

s.t.

$$H(0,t) = \gamma_0(t) \quad H(1,t) = \gamma_1(t)$$

and $\forall s \in [0,1]$

$$H(s,a) = \gamma_0(a) \quad H(s,b) = \gamma_0(b)$$

We call γ_0, γ_1 are homotopic if $\exists$ such a mapping.

6.4 Def

Let $U \subseteq \mathbb{C}$ be a connected open set. U is called simply connected if $\forall$ two curves in U with same starting and end pts are homotopic.

Chapter 7

Complex value function and holomorphic function

7.1 Def:Complex valued function

$U \subseteq \mathbb{C}$ a open set. **Complex value function** is a mapping $f : U \to \mathbb{C}$ We can view

$$f = u(x, y) + i(x, y)$$

via $\mathbb{C} \cong \mathbb{R}^2, z = x + iy$. We say that f is differentiable if u, v are differentiable. In particular, $\exists$ partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

For $z = x + iy \in U$, define:

$$\frac{\partial f}{\partial x}(z) := \frac{\partial u}{\partial x}(z) + i\frac{\partial v}{\partial x}(z)$$

7.2 Def: Differential form

Let $U \subseteq \mathbb{C}$ open. A differential form is a formal sum $g\mathrm{d}x + h + \mathrm{d}y$, where $g, h : U \to \mathbb{C}$ complex valued function.

Let $f : U \to \mathbb{C}$ differentiable. Define

$$\mathrm{d}f := \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

7.3 Prop

Let $f, g : U \to \mathbb{C}$ differentiable

Linearity

$$\mathrm{d}(f+g) = g\mathrm{d}f + f\mathrm{d}g$$

Leibniz rule

$$\mathrm{d}(fg) = \mathrm{d}fg + g\mathrm{d}f$$

$z : \mathbb{C} \to \mathbb{C}, \overline{z} : \mathbb{C} \to \mathbb{C}$, then

$$\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y, \mathrm{d}\overline{z} = \mathrm{d}x - i\mathrm{d}y$$

$\Rightarrow$

$$\mathrm{d}x = \frac{1}{2}(\mathrm{d}z + \mathrm{d}\overline{z}) \quad \mathrm{d}y = \frac{1}{2i}(\mathrm{d}z - \mathrm{d}\overline{z})$$

$\Rightarrow$

$$\mathrm{d}f = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})\mathrm{d}z + \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})\mathrm{d}\overline{z}$$

This motivates

$$\partial f := \frac{\partial f}{\partial z}\mathrm{d}z$$

$$\frac{\partial f}{\partial z} := \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$$

Similarly

$$\overline{\partial f} := \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}$$

$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$$

7.4 Def:Holomorphic functions

Let $U \subseteq \mathbb{C}$ open $f : U \to \mathbb{C}$. Let $z \in U$ we say f is complex differentiable at z if

$$\lim_{u \to z} \frac{f(u) - f(z)}{u - z} = f'(z)$$

exists.

Geometrically: in the tangent space level, f acts not just like a $\mathbb{R}$-linear mapping, but also a $\mathbb{C}$-linear mapping.

If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-linear, then f is complex differentiable iff $\exists a \in \mathbb{C}$, s.t.

$$f(z) = az$$

7.5 Def

Let $U \subseteq \mathbb{C}$ open. $f : U \to \mathbb{C}$ is called holomorphic if f is complex differentiable at every point in U.

7.6 Lemma

$U \subseteq \mathbb{C}$ open. $z \in U$, $f : U \to \mathbb{C}$, THen f is complex differentiable at z iff f is real differentiable and satisfies the following Cauchy-Riemann equations:

$$\frac{\partial f}{\partial \overline{z}}(z) = 0$$

Proof

$\Leftarrow$

We can write $f(w) - f(z) = A(w - z)_o(|w - z|)$, where A is a real 2×2 matrix. In the coordinate $z = x + iy, f = u + iv$

$$A = \begin{pmatrix} \frac{\partial u}{\partial x}(z) & \frac{\partial u}{\partial y}(z) \\ \frac{\partial v}{\partial x}(z) & \frac{\partial v}{\partial y}(z) \end{pmatrix}$$

C-R equation:

$$\frac{\partial f}{\partial \overline{z}} \Leftrightarrow \begin{cases} \frac{\partial u}{\partial x}(z) & = \frac{\partial v}{\partial y}(z) \\ \frac{\partial u}{\partial y}(z) & = -\frac{\partial v}{\partial x}(z) \end{cases}$$

Define

$$b := \frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z) \in \mathbb{R}$$

$$c := -\frac{\partial u}{\partial y}(z) = \frac{\partial v}{\partial x}(z) \in \mathbb{R}$$

Let $a := b + ci \in \mathbb{C}$, then

$$A(z) = az$$

we can write

$$f(u) - f(z) = a(w - z) + o(|w - z|)$$

$\Rightarrow f'(z)$exists.

$\Rightarrow$

Trivial.

7.7 Corollary

$U \subseteq \mathbb{C}$ open set. $f : U \to \mathbb{C}$ f is holomorphic on U iff f is real differentiable on U and the C-R equation

$$\frac{\partial f}{\partial \overline{z}}(z) = 0 \quad \text{holds } \forall z \in U$$

Chapter 8

Conformal matrix

Let $A \in M_{2\times 2}(\mathbb{R})$ be a matrix. $A : \mathbb{R}^2 \to \mathbb{R}^2$ linear mapping. The inner product $v = (x_1, y_2), w = (x_2, y_2)$

$$\langle v, w \rangle := x_1 x_2 + y_1 y_2$$

$z, w \in \mathbb{C}$

$$\langle z, w \rangle = \Re(z\overline{w})$$

Let $J : \mathbb{R}^2 \to \mathbb{R}^2, z \mapsto \overline{z}$ be the complex conjugation matrix. $\forall z, w \in \mathbb{C}$

$$\langle Jz, Jw \rangle = \langle z, w \rangle$$

reflexction w.r.t. real axis.

8.1 Def

A is called a rotation matrix if

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

($\Leftrightarrow$ A present inner product and $\det A > 0$)

8.2 Prop

A matrix A is given by $z \mapsto az, a \in \mathbb{C} \Leftrightarrow \exists \rho \geq 0$ and a rotation matrix R_θ s.t. $A = \rho R_\theta$

$$\begin{cases} \rho = |a| \\ \theta = \arg z \end{cases}$$

8.3 Polar decomposition

Any $A \in M_{2\times 2}(\mathbb{R})$ can be written as

$$A = R_\theta P$$

or

$$A = JR_\theta P$$

, where R_θ is a rotation matrix, P is positive, semi-definite matrix.

8.4 Def

$A \in M_{2\times 2}(\mathbb{R})$ is called conformal if A preserves the angle between two vectors i.e. $\forall z, w \in \mathbb{C}$

$$\frac{\langle Az, Aw\rangle}{|Az|\,|Aw|} = \frac{\langle z, w\rangle}{|z|\,|w|}$$

8.5 Remark

Conformal matrix is invertible. A circle in $\mathbb{C}$ is given by

$$\{z \in \mathbb{C} \mid |z - z_0| = r\} \quad r > 0, z_0 \in \mathbb{C}$$

8.6 Prop

Let $A \in M_{2\times 2}(\mathbb{R})$ s.t. $\det A > 0$
(such matrix are called orientation preserving)
Then the following conditions are equivalent:

1 A is conformal

2 A is given by $z \mapsto az, a \in C$

3 A maps a circle to a circle

Proof

Since $\det A > 0$ by polar decomposition

$$A = \begin{cases} R_\theta P \\ JR_\theta P \end{cases} \Rightarrow A = R_\theta P$$

R_θ rotation, P positive semi-definite.
It suffices to prove the prop for P.

$$P = R_\beta D R_\beta^{-1}$$

where $D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$, $\lambda_1, \lambda_2 > 0$, and R_β rotation.

It suffice to prove the prop for D. In this case (2)$\Leftrightarrow$ $\lambda_1 = \lambda_2$ It suffices to show (1)$\Rightarrow$ $\lambda_1 = \lambda$ and (3)$\Rightarrow$ $\lambda_1 = \lambda_2$

We first show (1)$\Rightarrow$ $\lambda_1 = \lambda_2$:

$D(1) = \lambda_1, D(1+i) = \lambda_1 + \lambda_2 i$,

$$D = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

. D conformal $\Rightarrow$

$$\frac{\langle Dz, Dw \rangle}{|Dz|\,|Dw|} = \frac{\langle z, w \rangle}{|z|\,|w|}$$

Take $z = 1, w = 1 + i$

$$\frac{\langle \lambda_1, \lambda_1 + \lambda_2 i \rangle}{\lambda_1 \sqrt{\lambda_1^2 + \lambda_2^2}} = \frac{1, 1+i}{\sqrt{2}}$$

Hence

$$\frac{\lambda_1^2}{\lambda_1 \sqrt{\lambda_1^2 + \lambda_2^2}} = \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $\lambda_1 = \lambda_2$

Next (3)$\Rightarrow$ $\lambda_1 = \lambda_2$. Assume D maps circle to circle. Consider $\partial D(0, \sqrt{2})$. Then the image of $\partial D(0, \sqrt{2})$ is a circle, which is central symmetry w.r.t. $D(0) = 0$. So the image is a circle centred at 0. Consider pts $\sqrt{2}, 1 + i \in \partial D(0, \sqrt{2})$. Since

$$D(\sqrt{2}) = \lambda_1 \sqrt{2} \quad D(1+i) = \lambda_1 + \lambda_2 i$$

we have

$$\left|\lambda_1 \sqrt{2}\right| = |\lambda_1 + \lambda_2 i|$$

$\Rightarrow \lambda_1 = \lambda_2$

Chapter 9

Power series

Let $n \in \mathbb{Z}$ one can verify

$$\begin{aligned} z^n : &\mathbb{C} &&\to \mathbb{C} \\ &z &&\mapsto z^n \end{aligned} \quad n \geq 0$$

$$\begin{aligned} z^{-n} : &\mathbb{C} \setminus \{0\} &&\to \mathbb{C} \\ &z &&\mapsto z^{-n} \end{aligned} \quad n \geq 0$$

we have

$$\frac{\partial z^n}{\partial z} = nz^{n-1} \quad \frac{\partial z^n}{\partial \overline{z}} = 0$$

$$\frac{\partial \overline{z}^{-n}}{\partial z} = 0 \quad \frac{\partial \overline{z}^n}{\partial \overline{z}} = nz^{n-1}$$

so z^n holomorphic but $\overline{z}$ not holomorphic.

In following we fix $n \in \mathbb{Z}$, $z^n : \mathbb{C} \to \mathbb{C}$, differentiable

9.1 Def

Let $z_0 \in \mathbb{C}$ A **power series** centered at z_0 is of the form

$$S = \sum_{n=0}^{+\infty} a_n (z - z_0)^n \quad a_n \in \mathbb{C}$$

Let $z \in \mathbb{C}$. We say that S converges at z if the number series $\sum\limits_{n=0}^{+\infty} a_n(z - z_0)^n$ converges, otherwise called diverges at z

9.2 Def

Let $K \subseteq \mathbb{C}$, $z_0 \in K$ and $S = \sum\limits_{n=0}^{+\infty} a_n(z - z_0)^n$ is a power series. We say S is uniformly convergent on K is $S(z)$ converges uniformly to a function on K

9.3 Cauchy's criteria

S uniformly convergent on K iff $(\forall \epsilon > 0)(\exists N)(\forall n > m \geq N)(\forall z \in K)$

$$\left| \sum_{k=m}^{n} a_k (z - z_0)^k \right| < \epsilon$$

9.4 Corollary: Dominated convergence

If $(\forall n)(\exists M_n \in \mathbb{R}_{\geq 0})(|a_n(z - z_0)^n| \leq M_n)$

$$\sum_{n=0}^{+\infty} a_n (z - z_0)^n < +\infty$$

then S uniformly convergent on K.

9.5 Abel Theorem

Let $S = \sum_{n=0}^{+\infty} a_n (z - z_0)^n$ be a power series converges at $z \neq z_0$ Let $R := |z' - z_0|$. $\forall 0 < r < R$ S uniformly converges on the closed disk $\overline{D}(z_0, r)$

9.6 Def:convergent radius

Let S be a power series

$$R := \sup \{|z - z_0| \mid S \text{ converges at } z\} \in [0, +\infty]$$

9.7 Prop

Let Ω be the convergent set of S. Then $\exists! D$ disk s.t. $\Omega \subseteq \overline{D}$ and $D \subseteq \Omega$

9.8 Lemma

Let S be a power series, R be the convergent radius of S. Then

$$\frac{1}{R} = \limsup_{n \to +\infty} |a_n|^{\frac{1}{n}}$$

9.9 Theorem

Let S be a power series with convergent radius R. Then on $D(z_0, R)$, S is holomorphic, and

$$f'(z) = \sum_{n=1}^{+\infty} n a_n (z - z_0)^{n-1}$$

Proof

- $$\limsup_{n\to+\infty}|a_n|^{\frac{1}{n}}=\limsup_{n\to+\infty}|na_n|^{\frac{1}{n-1}}$$

  The series f' exists

- $\frac{\partial f}{\partial \bar{z}}=0$ complex differentiable.
- uniformly convergence $\Rightarrow$ f' is the derivative of limit function of f.

9.10 Prop

Convergent radius of (S_1+S_2)(S_1 and S_2 shares the same center)

$$R\geq\min\{R_1,R_2\}$$

Chapter 10

Line integrals on piecewise smooth or rectifiable curves

10.1 Def

A **smooth curve** $\gamma = \gamma_1 + i\gamma_2 : [a, b] \to \mathbb{C}$ is a curve such that the derivatives

$$\gamma'(t) := \gamma_1'(t) + i\gamma_2'(t)$$

exist and form a continuous function.

A curve is called **piecewise smooth** if we can decompose

$$[a, b] = \bigcup_{j=1}^{n} [a_j, b_j]$$

as finite union of closed intervals, where $I_j \cap I_{j+1}$ is a single point s.t. $\gamma\mid_{I_j}$ is smooth on each intervals.

10.2 Def

Let $z, w \in \mathbb{C}$ and L be the line segment in $\mathbb{C}$ connecting z, w. Then $\gamma : [0, 1] \to \mathbb{C} :=\ t \mapsto tz + (1 - t)w$ is a smooth curve s.t.

$$Im\gamma = \gamma([a, b]) =: L$$

Given finitely many points $\{z_1, \cdots, z_n\} \in \mathbb{C}$, we can connect them one by one using line segments. This gives us a piecewise smooth curve. Those curves are **piecewise linear curves**

By definition, a curve $\gamma : [a, b] \to \mathbb{C}$ is called **piecewise linear** if we can decompose $[a, b]$ as finite union of closed intervals s.t. γ is $\mathbb{R}$-linear on each closed sub-intervals.

10.3 Def

Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve. Let $f : Im\gamma \to \mathbb{C}$ be a continuous function. We define the **line integral** of f along γ as

$$\int_\gamma f \mathrm{d}z := \int_a^b f(\gamma(t))\gamma'(t)\mathrm{d}t$$

To introduce the weakest condition for curves s.t. integration theory makes sense, we first define the **length** of a curve:

$$l(\gamma) := \sup_{a \le t_0 \le \cdots \le t_N \le b} \sum_{j=1}^N |\gamma(t_j) - \gamma(t_{j-1})| \ \in [0, +\infty]$$

10.4 Def

A curve $\gamma : [a,b] \to \mathbb{C}$ is called **rectifiable** if

$$l(\gamma) < +\infty$$

10.5 Prop

Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable curve. Let

$$f : \gamma([a,b]) \to \mathbb{C}$$

be a continuous function. Then the line integral

$$\int_\gamma f \mathrm{d}z$$

exists in the sense for every $\epsilon > 0$ s.t. if $a \le t_0 \le \cdots \le t_N \le b$ satisfies $|t_j - t_{j-1}| < \delta,\ t_j^* \in [t_{j-1}, t_j]$, then

$$\left| \sum_{i=1}^N f(\gamma(t_j^*))(\gamma(t_j) - \gamma(t_{j-1})) - \int_\gamma f \mathrm{d}z \right| < \epsilon$$

We call $a \le t_0 \le \cdots \le t_N \le b$ a **partition** of $[a,b]$. If $|t_j - t_{j-1}| < \delta$ we call the size of this partition is at most δ.

Proof

It suffices to show for every $\epsilon > 0$, there exists $\delta > 0$ s.t. for any two partitions $a \le t_0 \le \cdots \le t_N \le b$ and $a \le s_0 \le \cdots \le s_N \le b$ with size at most δ, for $t_j^* \in [t_{j-1}, t_j], s_j^* \in [s_{j-1}, s_j]$, we have

$$\left| \sum_{i=1}^N f(\gamma(t_j^*))(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{i=1}^N f(\gamma(s_j^*))(\gamma(s_j) - \gamma(s_{j-1})) \right| < \epsilon$$

From the triangle inequality, and from the fact that any two partitions have a common refinement, it suffices to prove this under the additional assumption that the second partition is a refinement of the first. This means that there's an increasing sequence $0 = m_0 < \cdots < m_N = M$ s.t.

$$s_{m_j} = t_j \quad 0 \leq j \leq N$$

Then the above difference is equal to

$$\left| \sum_{j=1}^{N} E_j \right|$$

where

$$E_j := f(\gamma(t_j^*))(\gamma(t_j) - \gamma(t_{j-1})) - \sum_{k=m_{j-1}+1}^{m_j} f(\gamma(s_k^*))(\gamma(s_k) - \gamma(s_{k-1}))$$

we can rearrange E_j as

$$E_j = \sum_{k=m_{j-1}+1}^{m_j} (f(\gamma(t_j^*) - f(\gamma(s_k^*))))(\gamma(s_k) - \gamma(s_{k-1}))$$

Since $\gamma : [a, b] \to \mathbb{C}$ is uniformly continuous, $l(\gamma) < +\infty$, for δ small enough,

$$\left| f(\gamma(t_j^*)) - f(\gamma(s_k^*)) \right| < \frac{\epsilon}{l(\gamma)}$$

By the triangle inequality,

$$|E_j| \leq \frac{\epsilon}{l(\gamma)} \sum_{k=m_{j-1}+1}^{m_j} |\gamma(s_k) - \gamma(s_{k-1})|$$

Summing on j and, again, by triangle inequality

$$\left| \sum_{j=1}^{N} E_j \right| \leq \sum_{j=1}^{N} |E_j| \leq \epsilon$$

which finishes the proof.

10.6 Lemma

Let $U \subseteq \mathbb{C}$ open, $f : U \to \mathbb{C}$ continuous. Let $\gamma : [a, b] \to U$ rectifiable curve. Then there exists a sequence of piecewise linear curves $\gamma_n : [a, b] \to U$ s.t. $(\gamma_n) \to \gamma$ uniformly and $\gamma_n(a) = \gamma(a), \gamma_n(b) = \gamma(b)$, moreover:

$$\lim_{n \to +\infty} \int_{\gamma_n} f \mathrm{d}z = \int_{\gamma} f \mathrm{d}z$$

10.7 Def

Let $\gamma : [a, b] \to \mathbb{C}$ be a curve. Let $\phi : [c, d] \to [a, b]$ a homeomorphism s.t. $\phi(c) = a,\ \phi(d) = b$(this implies ϕ is strictly increasing). A **reparametrization** of γ by ϕ is the curve $\gamma \circ \phi$.

We can also define the reverse of γ by

$$-\gamma : [-b, -a] \to \mathbb{C} := t \mapsto \gamma(-t)$$

We have $-\gamma(-b) = \gamma(b)$ and $-\gamma(-a) = \gamma(a)$

10.8 Prop

Let γ be a rectifiable curve and $\gamma \circ \phi$ be a reparametrization curve. Let $f : Im\gamma \to \mathbb{C}$ be a continuous function. Then

- $$\int_\gamma f\mathrm{d}z = \int_{\gamma\circ\phi} f\mathrm{d}z$$
- $$-\int_\gamma f\mathrm{d}z = \int_{-\gamma} f\mathrm{d}z$$

10.9 Prop

One can easily check the following properties with the def of line integral on curves:

- Linearity: If $a, b \in \mathbb{C}$, then
$$\int_\gamma (af + bg)\mathrm{d}z = a\int_\gamma f\mathrm{d}z + b\int_\gamma g\mathrm{d}z$$
- Additivity: Let $\gamma_1 : [a, b] \to \mathbb{C}, \gamma_2 : [b, c] \to \mathbb{C}$ be two rectifiable curves s.t. $\gamma_1(b) = \gamma_2(b)$. Then one can define $\gamma_1 + \gamma_2 := [a, c] \to \mathbb{C}$. Then
$$\int_{\gamma_1+\gamma_2} f\mathrm{d}z = \int_{\gamma_1} f\mathrm{d}z + \int_{\gamma_2} f\mathrm{d}z$$
- Absolutely value inequality:
$$\left|\int_\gamma f\mathrm{d}z\right| \le \int_\gamma |f|\,\mathrm{d}z$$
- Monotonicity: Let $f : Im\gamma \to \mathbb{C}$ be continuous. If f is increasing on $Im\gamma$, then
$$\int_\gamma f\mathrm{d}s \ge 0$$
where $\mathrm{d}s = \mathrm{d}\,|z| = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$

10.10 Lemma: Uniform convergence

Let γ be a rectifiable curve. Let f_n be a sequence of continuous functions on $Im\gamma$ s.t. f_n converges to f uniformly. Then f is continuous on $Im\gamma$ and

$$\lim_{n\to+\infty}\int_\gamma f_n \mathrm{d}z = \int_\gamma f \mathrm{d}z$$

10.11 Fundamental Theo. of Calculus

$U \subseteq \mathbb{C}$ open $f, g : U \to \mathbb{C}$ continuous. Assume $\exists F$ s.t.

$$\frac{\partial F}{\partial x} = f \quad \frac{\partial F}{\partial y} = g$$

Let $\gamma : [a, b] \to U$ rectifiable curve,

$$\int_\gamma \mathrm{d}x + g\mathrm{d}y = F(\gamma(b)) - F(\gamma(a))$$

Proof

By lemma 10.6(actually we need a slightly generalized version of this lemma for $\int_\gamma f\mathrm{d}x + g\mathrm{d}y$), it suffices to show the result if $\gamma = \gamma_1 + i\gamma_2$ is a piecewise linear curve. For such γ, it's in particular piecewise smooth.

Define

$$G(t) := F \circ \gamma(t)$$

then

$$G'(t) = f(\gamma(t))\gamma_1'(t) + g(\gamma(t))\gamma_2'(t)$$

By fundamental theorem of calculus10.11

$$\begin{aligned}\int_\gamma f\mathrm{d}x + g\mathrm{d}y &:= \int_a^b (f(\gamma(t))\gamma_1'(t) + g(\gamma(t))\gamma_2'(t))\mathrm{d}t \\ &= \int_a^b G'(t)\mathrm{d}t \\ &= G(b) - G(a) \\ &= F(\gamma(b)) - F(\gamma(a))\end{aligned}$$

10.12 Corollary

Let $f : U \to \mathbb{C}$ continuous, assume $\exists\ F : U \to \mathbb{C}$, s.t.

$$\frac{\partial F}{\partial x} = f \quad \frac{\partial F}{\partial y} = if$$

then

$$\int_\gamma f\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

Proof

We have

$$\int_\gamma f \mathrm{d}z = \int_\gamma f \mathrm{d}x + i \int_\gamma f \mathrm{d}y$$

then apply the Theorem above

10.13 Remark

In Corollary and Theorem above, since the partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous, F is real differentiable. In Theorem, the condition $\frac{\partial F}{\partial x} = f$, $\frac{\partial F}{\partial y} = if$ implies the Cauchy-Riemann equation $\frac{\partial F}{\partial \bar{z}} = 0$ holds, hence F is actually holomorphic.

10.14 Remark

F is differentiable C-R equation

$$\frac{\partial F}{\partial x} = f \quad \frac{\partial F}{\partial y} = if$$

$\Rightarrow$ F is holomorphic.

10.15 Def

A curve is **closed** if

$$\gamma(a) = \gamma(b)$$

10.16 Corollary

Let $n \in \mathbb{Z}, n \neq -1$. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed rectifiable curve. If $n \leq -2$ assume further $0 \notin Im\gamma$. Then

$$\int_\gamma z^n \mathrm{d}z = 0$$

Proof

Since

$$\left(\frac{z^{n+1}}{n+1}\right)' = z^n$$

apply 10.13 we get the result

Chapter 11

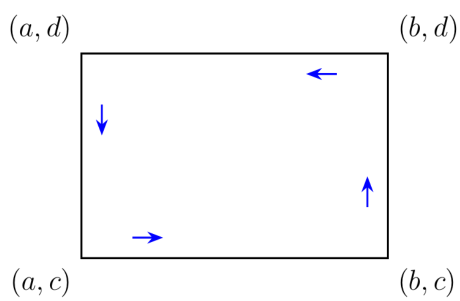
Cauchy's Integral Theorem

11.1 Def

A curve $\gamma : [a, b] \to \mathbb{C}$ is called **simple** if γ is injective.
Let R be an rectangle defined by inequalities

$$a < x < b \quad c < y < d$$

Let ∂R be the positively oriented simple closed curve. The following is an example of positively oriented ∂R



We say a function is holomorphic in $\overline{R}$ if there's an open neighborhood of U, $\overline{R} \subseteq U$ s.t. f is holomorphic in U.

11.2 Goursat Theorem

Let R be an rectangle and f be a holomorphic function in $\overline{R}$, then

$$\int_{\partial R} f \mathrm{d}z = 0$$

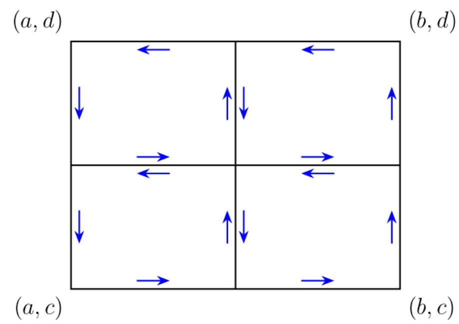
Proof

For a rectangle R, define

$$\eta(R) := \int_{\partial R} f \mathrm{d}z$$

Divide R into four congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$, then

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)})$$



At most one small rectangle $R^{(k)}$ satisfies

$$\left|\eta(R^{(k)})\right| \geq \frac{1}{4} |\eta(R)|$$

for some $1 \leq k \leq 4$. We denote this rectangle by R_1. Continue this procession for R_1, we get a sub-rectangle $R_2 \subseteq R_1$. By induction, we get sequence of nested rectangles R_n s.t.

$$|\eta R(_n)| \geq 4^{-n} |\eta(R)|$$

The rectangles $R_n \to z_0 \in \overline{R}$

$\forall \epsilon > 0$, choose $\delta > 0$ s.t. f is holomorphic in $D(z_0, \delta)$, and $\forall z \in D(z_0, \delta)$:

$$|f(z) - f(z_0) - (z - z_0)f'(z_0)| \leq \epsilon |z - z_0|$$

For n large, $\overline{R_n} \subseteq D(z_0, \delta)$. By Corollary10.16, we have

$$\int_{\partial R_n} \mathrm{d}z = 0$$

and

$$\int_{\partial R_n} z \mathrm{d}z = 0$$

Hence

$$\begin{aligned}|\eta(R_n)| &= \left|\int_{\partial R_n} f(z) - f(z_0) - (z - z_0)f'(z_0)\mathrm{d}z\right| \\ &\leq \int_{\partial R_n} |f(z) - f(z_0) - (z - z_0)f'(z_0)|\, \mathrm{d}z \\ &\leq \epsilon \int_{\partial R_n} |z - z_0|\, \mathrm{d}s \\ &\leq \epsilon d_n L_n\end{aligned}$$

where d_n is the diameter of R_n, L_n is the perimeter of ∂R_n. If d is the diameter of R, and L is the perimeter of ∂R, then

$$d_n = 2^{-n}d \quad L_n = 2^{-n}L$$

Hence we have

$$|\eta(R_n)| \leq \epsilon 4^{-n}\mathrm{d}L$$

By our assumption that

$$|\eta(R_n)| \geq 4^{-n}\, |\eta(R)|$$

we have

$$|\eta(R)| \leq \epsilon dL$$

Since ϵ is arbitrary, we have $\eta(R) = 0$, which ends the proof.

11.3 Cauchy's integral theorem on a disk

Let $D \subseteq \mathbb{C}$ be a disk, $f : D \to \mathbb{C}$ a holomorphic function, γ a closed rectifiable curve in D, then

$$\int_\gamma f\mathrm{d}z = 0$$

Proof

By theorem10.13, it suffice to show there exists a function $F : D \to \mathbb{C}$, s.t.

$$\frac{\partial F}{\partial x} = f \quad \frac{\partial F}{\partial y} = if$$

Without loss of generality we assume $D = \mathbb{D}$. Let $w = x + iy$, and γ_w be the piecewise linear curve $\gamma_w(0) = x$ and $\gamma_w(x) = x + iy$. Define

$$F(w) := \int_{\gamma_w} f\mathrm{d}z$$

This implies $\frac{\partial F}{\partial y}(w) = if(w)$. Actually, by the def of parital derivatives:

$$\begin{aligned}\frac{\partial F}{\partial y}(w) &:= \lim_{\epsilon \to 0} \frac{F(w+\epsilon i) - F(w)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\int_{\gamma_\epsilon} f \mathrm{d}z}{\epsilon} \\ &= if(w)\end{aligned}$$

where γ_w is the linear curve $\gamma_w(x+iy) = x+iy+i\epsilon$. Let $\tilde{\gamma_w}$ be the piecewise linear curve that $\tilde{\gamma_w}(0) = iy$ and $\tilde{\gamma_w}(iy) = x+iy$. By theorem11.2,we have

$$F(w) = \int_{\tilde{\gamma_w}} f \mathrm{d}z$$

This implies $\frac{\partial F}{\partial x}(w) = f(w)$ by repeating the same argument for $\frac{\partial F}{\partial y}(w)$, which finishes the proof.

(Note that the equality above is not a definition, it's actually equivalent to Goursat's theorem)

Chapter 12

Winding number

Let $\gamma : [a, b] \to \mathbb{C}$ be a closed curve. Let $w \notin I_m\gamma$.
Define $u : [a, b] \to S^1$

$$U(t) := \frac{\gamma(t) - w}{|\gamma(t) - w|}$$

Let

$$\begin{aligned} \pi : \mathbb{R} &\to S^1 \\ s &\mapsto e^{2\pi i s} \end{aligned}$$

$\Rightarrow \exists$ lift of u,

$$\tilde{u} : [a, b] \to \mathbb{R}$$

continuous and

$$\pi \circ \tilde{u} = u$$

such choice of $\tilde{u}$ is not unique, but if $\tilde{u}_1, \tilde{u}_2$ are two lift of $u \Rightarrow$

$$\tilde{u}_1 - \tilde{u}_2 = l \in \mathbb{Z}$$

12.1 Def

The winding number

$$n(\gamma, w) := \tilde{u}(b) - \tilde{u}(a)$$

12.2 prop

1 $n(\gamma, w) \in \mathbb{Z}$

2 $n(\gamma, w)$ take the same value when w in a connected comp of $\mathbb{C} \setminus I_m\gamma$

3 Let Ω_∞ be the unbounded component of $\mathbb{C} \setminus Im\gamma$

Proof

1

Since γ closed $\Rightarrow$

$$e^{i2\pi\tilde{u}(b)} = e^{i2\pi\tilde{u}(a)}$$

$\Rightarrow$

$$\tilde{u}(b) - \tilde{u}(a) \in \mathbb{Z}$$

2

For a fixed γ $n(\gamma, \cdot)$ is a continuous function w.r.t. w. SSince Ω connected,

$$n(\gamma, \cdot) : \Omega \to \mathbb{Z}$$

the image $n(\gamma, \cdot)$ is connected in $\mathbb{Z}$, which implies single point, so $n(\gamma, \cdot)$ is a constant mapping.

3

For $w \in \Omega_\infty$ $n(\gamma, \cdot) \to 0$ when $w \to \infty$, but $n(\gamma, w) \in \mathbb{Z}$ so

$$n(\gamma, \cdot) = 0 \ \forall w \in \Omega_\infty$$

12.3 Corollary

$\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ is not simply connected.

12.4 Prop

(Winding number is homotopy invariant) Given two closed curves.

$$\gamma_0 : [a, b] \to \mathbb{C} \quad \gamma_1[a, b] \to \mathbb{C}$$

$H : [a, b] \times [0, 1] \to \mathbb{C} \setminus \{w\}$ be a homotopy. Then

$$n(\gamma_0, w) = n(\gamma_1, w)$$

Proof

Let $U : [a, b] \times [0, 1] \to S^1$

$$U(t, s) := \frac{H(t, s) - w}{|H(t, s) - w|}$$

Then $\exists\ \tilde{U} : [a, b] \times [0, 1] \to \mathbb{R}$ continuous s.t.

$$\pi \circ \tilde{U} = U$$

where$\pi : \mathbb{R} \to S^1$ is the universal cover. Then $\forall s \in [0,1]$

$$n(\gamma_s, w) = \tilde{U}(b, s) - \tilde{U}(a, s)$$

Since $\tilde{U}$ is continuous, $n(\gamma_s, w) \in \mathbb{Z}$, so $n(\gamma_s, w)$ is a constant $\forall s \in [0,1] \Rightarrow$

$$n(\gamma_0, w) = n(\gamma_1, w)$$

12.5 Lemma: Winding number formula for rectifiable curves

$\gamma : [a,b] \to \mathbb{C}$ rectifiable, closed curve, let $w \notin Im\gamma$

$$n(\gamma, w) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z - w} \mathrm{d}z$$

Proof

Assume γ is piecewise smooth. Consider $h : [a,b] \to \mathbb{C}$

$$h(t) := \frac{1}{2\pi i} \int_a^t \frac{\gamma'(s)}{\gamma(s) - w} \mathrm{d}s$$

h piecewise smooth.

$$h'(t) = \frac{1}{2\pi i} \frac{\gamma'(t)}{\gamma(t) - w}$$

and

$$\left(e^{-2\pi i h(t)}(\gamma(t) - w)\right)' = 0$$

outside a finite set. So

$$e^{2\pi i h(t)} = \frac{\gamma(t) - w}{\gamma(a) - w} \quad \forall t \in [a,b]$$

On the other hand, $\tilde{u} : [a,b] \to \mathbb{R}$

$$e^{2\pi i \tilde{u}(t)} = \frac{\gamma(t) - w}{|\gamma(a) - w|}$$

Let $v(t) := \ln |\gamma(t) - w|$, then we have:

$$e^{2\pi i \tilde{u}(t) + v(t)} = \gamma(t) - w$$

$\Rightarrow \exists$ constant $\alpha \in \mathbb{C}$ s.t.

$$2\pi i h(t) = 2\pi i \tilde{u}(t) + \alpha$$

then we have

$$n(\gamma, w) := \tilde{u}(b) - \tilde{u}(a) = h(b) - h(a)$$

12.6 Jordan Curve Theorem

γ a closed curve, γ is simple(no self crossing: if $\forall t \neq s \in (a,b)$, have $\gamma(t) \neq \gamma(s)$)

Let γ a simple closed curve. Then $\mathbb{C} \setminus Im\gamma$ has two connected components.

Chapter 13

Cauchy's integral formula

13.1 Lemma

(A generalization of Gousat's theorem) Let R be an rectangle, and let $E \subseteq R$ be a finite set, let U be an open set s.t. $\overline{R} \subseteq U$ let $f : U \setminus E \to \mathbb{C}$ holomorphic, moreover

$$\lim_{z \to \xi} (z - \xi) f(z) = 0 \ \forall \xi \in E$$

then

$$\in_{\partial R} f \mathrm{d}z = 0$$

13.1.1 Proof

We can decompose R into small rectangles s.t small rectangle contains one pt in E, We can assume $E = \{\xi\}$ $\forall \epsilon > 0$, we can decompose R into 9 rectangle s.t. in the center rectangle R_0

$$|f(z)| \leq \frac{\epsilon}{|z - \xi|} \quad \forall z \in \partial R_0$$

By original Gousat's theorem

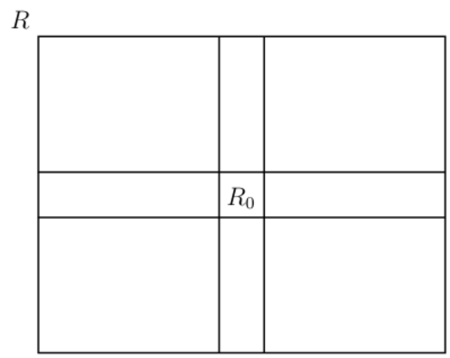
$$\int_{\partial R} f \mathrm{d}z = \int_{\partial R_0} f \mathrm{d}z$$

Hence

$$\left| \int_{\partial R} f \mathrm{d}z \right| \leq \epsilon \int_{\partial R_0} \frac{1}{|z - \xi|} \mathrm{d}z \leq 8\epsilon$$

Since ϵ can be arbitrarily small

$$\int_{\partial R} f \mathrm{d}z = 0$$



13.2 Corollary

Let $D \subseteq \mathbb{C}$ disk let $E \subseteq D$ a finite set. Let $f : D \to \mathbb{C}$ continuous holomorphic in $D \setminus E$ moreover

$$\lim_{z \to \xi} (z - \xi) f(z) = 0 \quad \forall \xi \in E$$

Let γ be a closed rectifiable curve in D, then

$$\int_\gamma f \mathrm{d}z = 0$$

13.3 Cauchy integral formula on a disk

Let $D \subseteq \mathbb{C}$ disk. Let $f : D \to \mathbb{C}$ holomorphic let $a \in D$, let γ be a closed rectifiable curve s.t. $a \notin Im\gamma$, Then

$$n(\gamma, a) f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} \mathrm{d}z$$

Proof

Define function $g : D \setminus \{a\} \to \mathbb{C}$

$$g(z) := \frac{f(z) - f(a)}{z - a}$$

g is holomorphic on $D \setminus \{a\}$, g can be continuously extend to D.

Then by the previous Corollary, since

$$\lim_{z \to a} (z - a) g(z) := \lim_{z \to a} f(z) - f(a) = 0$$

then

$$\int_\gamma g(z) \mathrm{d}a = 0$$

This implies

$$n(\gamma, a)f(a) := \frac{1}{2\pi i}\int_\gamma \frac{f(a)}{z-a}\mathrm{d}z \overset{\text{coro}}{=} \frac{f(z)}{z-a}\mathrm{d}z$$

13.4 Corollary

Let $D \subseteq \mathbb{C}$ disk. Let $f : \overline{D} \to \mathbb{C}$ be a holomorphic functions. Then

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(w)}{w-z}\mathrm{d}w \quad \forall z \in D$$

Proof

$\forall z \in D$, we have $n(\partial D, z) = 1$. Let D' be a slightly larger disk with the same center, $\overline{D} \subseteq D$, s.t. f is holomorphic in D'. Apply Theorem13.3, and end the proof.

13.5 Theorem

Holomorphic functions is locally a power series, in particular, it's C^∞

13.6 Theorem

Let γ be a rectifiable curve, $\phi : Im\gamma \to \mathbb{C}$ continuous. For $z \notin Im\gamma$, define

$$f(z) := \int_\gamma \frac{\phi(w)}{w-z}\mathrm{d}w$$

Then $\forall z_0 \in \mathbb{C} \setminus Im\gamma$ let $\gamma := dist(z_0, Im\gamma)$ f is a power series on $D(z_0, r)$

$$f(z) = \sum_{n=0}^{\infty}\left(\int_\gamma \frac{\phi(w)}{(w-z_0)^n}\mathrm{d}w\right)(z-z_0)^n$$

In particular, f is holomorphic.

Proof

Fix z_0,$w \in Im\gamma$, consider function $\frac{\phi(w)}{w-z}$ (as func of z). It has a power series expansion on $D(z_0, r)$

$$\frac{\phi(w)}{w-z} = \sum_{n=0}^{\infty} \frac{\phi(w)}{(w-z_0)^n}(z-z_0)^n$$

The above formula holds for every $w \in Im\gamma$ and $z \in D(z_0, r)$ Define $g_n : Im\gamma \to \mathbb{C}$(function of w)

$$g_n(w) := \sum_{k=0}^{n} \frac{\phi(w)}{(w-z_0)^{n+1}}(z-z_0)^k$$

We are going to show g_n converges uniformly to $\frac{\phi(w)}{w-z}$ on $Im\gamma$. Let $M := \max_{w \in Im\gamma} |\phi(w)|$ and $r' := |z - z_0| < r$

$$\begin{aligned}
\left| g_n(w) - \frac{\phi(w)}{w-z} \right| &= \left| \sum_{k=n}^{\infty} \frac{\phi(w)}{(w-z_0)^{k+1}} (z-z_0)^k \right| \\
&\leq M \sum_{k=n}^{\infty} \left| \frac{(z-z_0)^k}{(w-z_0)^{k+1}} \right| \\
&\leq M \sum_{k=n+1}^{\infty} (\frac{r'}{r})^k \frac{1}{r'}
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_\gamma \frac{\phi(w)}{w-z} &= \lim_{n \to +\infty} \int_\gamma \frac{\phi(w)}{(w-z_0)^n} \mathrm{d}w \\
&= \lim_{n \to \infty} \left(\int_\gamma \frac{\phi(w)}{(w-z_0)^n} \mathrm{d}w (z-z_0)^n \right) \\
&= \sum_{k=0}^{\infty} (\int_\gamma \frac{\phi(w)}{(w-z_0)^n} \mathrm{d}w)(z-z_0)^n
\end{aligned}$$

For function in $\mathbb{R}$, differentiable does not imply infinitely differentiable, but in $\mathbb{C}$:

13.7 Corollary

Let $D \subseteq \mathbb{C}$ be a disk. Let $f : \overline{D} \to \mathbb{C}$ be a holomorphic function. Then f is equal to a power series on D.

Proof

By Corollary13.4, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w-z} \mathrm{d}w$$

Apply Theorem13.6 and get the result.

13.8 Morera Theorem

Let $U \subseteq \mathbb{C}$ be an open set. Let $f : U \to \mathbb{C}$ be a continuous function. Assume for every closed rectifiable curve γ in U, we have

$$\int_\gamma f \mathrm{d}z = 0$$

Then f is holomorphic in U.

Proof

(Holomorphic function is locally a power series)

If suffice to show the result on each connected component of U. So we can assume Y is connected. Fix $z_0 \in U$, $\forall z \in U$, let γ_z be any rectifiable curve connecting z_0 and z, define

$$F(z) := \int_{\gamma_z} f \mathrm{d}z$$

This def is independent of the choice of γ_z be our assumption that

$$\int_{\gamma} f \mathrm{d}z = 0$$

We have $F' = f$, which is holomorphic in U. Since f is the derivative of a holomorphic function, by Corollary13.8, f is holomorphic in U.

13.9 Formula for higher derivatives

Let $D := D(z_0, r) \subseteq \mathbb{C}$ be a disk. Let $f : \overline{D} \to \mathbb{C}$ be a holomorphic function. Then the n-th derivative at z_0 satisfies

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w$$

Proof

Apply Theorem13.6 to the formula:

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z_0} \mathrm{d}w$$

we get

$$f(z) = \sum_{n=0}^{\infty} \left(\int_{\partial D} \frac{f(w)}{(w - z_0)^n} \mathrm{d}w \right) (z - z_0)^n$$

By Tayler expansion:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

We get the result by comparing the coefficient.

13.10 Cauchy's estimate

Let $D := D(z_0, r) \subseteq \mathbb{C}$ be a disk. Let $f : \overline{D} \to \mathbb{C}$ be a holomorphic function. Let

$$M := \max_{z \in \partial D} |f(z)|$$

then

$$\left| f^{(n)} \right| \leq M n! r^{-n}$$

Proof

By Theorem13.9

$$\begin{aligned}\left| f^{(n)}(z_0) \right| &= \left| \frac{n!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w - z_0)^{n+1}} \mathrm{d}w \right| \\ &\leq \left| \frac{n!}{2\pi i} \int_{\partial D} M r^{-(n+1)} \mathrm{d}s \right| \\ &= M n! r^{-n}\end{aligned}$$

13.11 Liouvile's theorem

Let $f : \mathbb{C} \to \mathbb{C}$ be bounded holomorphic function. Then f is a constant.

Proof

Assume $|f(z)| \leq M$ on $\mathbb{C}$. Let $R > 0$ and $z_0 \in \mathbb{C}$. Aplly Theorem13.10 to $D(z_0, R)$, we get

$$|f'(z_0)| \leq M R^{-1}$$

Let $R \to +\infty$ we get $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$ This implies f is a constant, by the fundamental theorem of calculus.

13.12 Fundamental Theorem of Algebra

Let f be a polynomial with complex coefficient, $\deg f \geq 1$. Then f has at least one root in $\mathbb{C}$.

Proof

Assume by contradiction that f has no roots in $\mathbb{C}$. Then $g := \frac{1}{f} : \mathbb{C} \to \mathbb{C}$ is a holomorphic function. Since $\deg f \geq 1$,

$$|f(z)| \to +\infty \quad |z| \to \infty$$

This implies g is a bounded holomorphic function. By theorem13.11, g is a constant, which implies f is a constant, contradiction.

13.13 Theorem

(Local uniform limit of holomorphic functions is holomorphic)

Let $U \subseteq \mathbb{C}$ be an open set. Let $f_n : U \to \mathbb{C}$ be a sequence of holomorphic functions, converges uniformly to a function f on any compact subset $K \subseteq U$. Then f is holomorphic in U.

Proof

Let D be a disk s.t. $\overline{D} \subseteq U$. By Cauchy integral formula13.4, we have $\forall z \in D$

$$f_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(w)}{w - z} \mathrm{d}w$$

Since $f_n \to f$ uniformly on $\overline{D}$, we have

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(w)}{w - z} \mathrm{d}w$$

This implies f holomorphic in D, by Theorem13.6