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# Part I

## Set



# Chapter 1

## product

### 1.1 direct sum

$\oplus$  is defined to be the direct product but with only finite non-zero elements.

$$\bigoplus_{i \in I} V_i \{ (x_i)_{i \in I} \in \prod_{i \in I} V_i \mid \exists J \subseteq I, I \setminus J \text{ is finite that } \forall j \in J, x_j = 0 \}$$





## Chapter 2

# Ring

### 2.1 morphism

#### Def

Let  $A$  and  $B$  be unitary rings. We call morphism of unitary rings from  $A$  to  $B$  only mapping  $A \rightarrow B$  is a morphism of group from  $(A, +)$  to  $(B, +)$ , and a morphism of monoid from  $(A, \cdot)$  to  $(B, \cdot)$

#### Properties

- Let  $R$  be a unitary ring. There is a unique morphism from  $\mathbb{Z}$  to  $R$
- 

#### algebra

we call  $k$ -algebra any pair  $(R, f)$ , when  $R$  is a unitary ring, and  $f : k \rightarrow R$  is a morphism of unitary rings such that  $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring  $R$ , the unique morphism of unitary rings  $\mathbb{Z} \rightarrow R$  define a structure of  $\mathbb{Z}$ -algebra on  $R$  (extra:  $\mathbb{Z}$  is commutative despite  $R$  isn't guaranteed)

Notation: Let  $k$  be a commutative unitary ring,  $(A, f)$  be a  $k$ -algebra. If there is no ambiguity on  $f$ , for any  $(\lambda, a) \in k \times A$ , we denote  $f(\lambda)a$  as  $\lambda a$

#### Formal power series

reminder:  $n \in \mathbb{N}$  is possible infinite, so  $\sum_{n \in \mathbb{N}}$  couldn't be executed directly.

Def:

(extended polynomial actually) Let  $k$  be a commutative unitary ring. Def: Let  $T$  be a formal symbol. We denote  $k^{\mathbb{N}}$  as  $k[T]$ . If  $(a_n)_{n \in \mathbb{N}}$  is an element of  $k^{\mathbb{N}}$ , when we denote  $k^{\mathbb{N}}$  as  $k[T]$  this element is denoted as  $\sum_{n \in \mathbb{N}} a_n T^n$ . Such

element is called a formal power series over  $k$  and  $a_n$  is called the Coefficient of  $T^n$  of this formal power series Notation:

- omit terms with coefficient 0
- write  $T^0$  as  $1$
- omit Coefficient those are 1;
- omit  $T^0$

Example  $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$  is written as  $1 + 2T + T^2$

Def Remind that  $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$ , define two composition laws on  $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$  form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$  is a morphism
- $(FG)H = \left( \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left( \sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left( \sum_{p+q+l=n} a_p b_q c_l \right) T^n$   
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by  $F'(T)$  or  $\mathcal{D}(F(T))$  the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$  is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote  $\exp(T) \in k[T]$  as  $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$ , which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence:  $(F_i(T))_{i \in \mathbb{N}}$  be a sequence of elements in  $k[T]$ , and  $F(T) \in k[T]$  We say that  $(F_i(T))_{i \in \mathbb{N}}$  is a Cauchy sequence if  $\forall l \in \mathbb{N}$ , there exists  $N(l) \in \mathbb{N}$  such that  $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$ ,  $\text{ord}(F_i(T) - F_j(T)) \geq l$

# Part II

## Sequences



## Chapter 3

# Supremum and infimum

Def:

Let  $(X, \leq)$  be a partially ordered set  $A$  and  $Y$  be subsets of  $X$ , such that  $A \subseteq Y$

- If the set  $\{y \in Y \mid \forall a \in A, a \leq y\}$  has a least element then we say that  $A$  has a Supremum in  $Y$  with respect to  $\leq$  denoted by  $\sup_{(Y, \leq)} A$  this least element and called it the Supremum of  $A$  in  $Y$  (this respect to  $\leq$ )
- If the set  $\{y \in Y \mid \forall a \in A, y \leq a\}$  has a greatest element, we say that  $A$  has an infimum in  $Y$  with respect to  $\leq$ . We denote by  $\inf_{(Y, \leq)} A$  this greatest element and call it the infimum of  $A$  in  $Y$
- Observation:  $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set.

- If  $f$  is a function from  $I$  to  $X$   $\sup f$  denotes the supremum of  $f(I)$  is  $X$ .  $\inf f$  takes the same
- If  $(x_i)_{i \in I}$  is a family of element in  $X$ , then  $\sup x_i$  denotes  $\sup\{x_i \mid i \in I\}$  (in  $X$ )

If moreover  $\mathbb{P}(\cdot)$  denotes a statement depending on a parameter in  $I$  then  $\sup_{i \in I, \mathbb{P}(i)} x_i$  denotes  $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let  $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$  We equip  $\mathbb{R}$  with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So  $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence  $\inf A = 0$

Example: For  $n \in \mathbb{N}$ , let  $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let  $(X, \leq)$  be a partially ordered set,  $A, Y, Z$  be subset of  $X$ , such that  $A \subseteq Z \subseteq Y$

- If  $\max A$  exists, then it is also equal to  $\sup_{(y, \leq)} A$
- If  $\sup_{(y, \leq)} A$  exists and belongs to  $Z$ , then it is equal to  $\sup A$

$\inf$  takes the same Prop.

Let  $X, \leq$  be a partially ordered set,  $A, B, Y$  be subsets of  $X$  such that  $A \subseteq B \subseteq Y$

- If  $\sup_{(y, \leq)} A$  and  $\sup_{(y, \leq)} B$  exists, then  $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If  $\inf_{(y, \leq)} A$  and  $\inf_{(y, \leq)} B$  exists, then  $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set and  $f, g : I \rightarrow X$  be mappings such that  $\forall t \in I, f(t) \leq g(t)$

- If  $\inf f$  and  $\inf g$  exists, then  $\inf f \leq \inf g$
- If  $\sup f$  and  $\sup g$  exists, then  $\sup f \leq \sup g$

## Chapter 4

# Interval

We fix a totally ordered set  $(X, \leq)$

Notation:

If  $(a, b) \in X \times X$  such that  $a \leq b$ ,  $[a, b]$  denotes  $\{x \in X \mid a \leq x \leq b\}$

Def:

Let  $I \subseteq X$ . If  $\forall (x, y) \in I \times I$  with  $x \leq y$ , one has  $[x, y] \subseteq I$  then we say that  $I$  is an interval in  $X$

Example:

Let  $(a, b) \in X \times X$ , such that  $a \leq b$ . Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let  $\Lambda$  be a non-empty set and  $(I_\lambda)_{\lambda \in \Lambda}$  be a family of intervals in  $X$ .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$
- If  $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ ,  $\bigcup_{\lambda \in \Lambda} I_\lambda$  is an interval in  $X$

We check that  $[a, b] \subseteq I_\lambda \cup I_\mu$

- If  $b \leq x$   $[a, b] \subseteq [a, x] \subseteq I_\lambda$  because  $\{a, x\} \subseteq I_\lambda$
- If  $x \leq a$   $[a, b] \subseteq [x, b] \subseteq I_\mu$  because  $\{b, x\} \subseteq I_\mu$
- If  $a < x < b$  then  $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let  $(X, \leq)$  be a totally ordered set.  $I$  be a non-empty interval of  $X$ . If  $\sup I$  exists in  $X$ , we call  $\sup I$  the right endpoint;  $\inf$  takes the similar way.

Prop.

Let  $I$  be an interval in  $X$ .

- Suppose that  $b = \sup I$  exists.  $\forall x \in I, [x, b] \subseteq I$
- Suppose that  $a = \inf I$  exists.  $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let  $I$  be an interval in  $X$ . Suppose that  $I$  has supremum  $b$  and an infimum  $a$  in  $X$ . Then  $I$  is equal to one of the following sets  $[a, b]$   $[a, b[$   $]a, b]$   $]a, b[$

Def

let  $(X, \leq)$  be a totally ordered set. If  $\forall (x, z) \in X \times X$ , such that  $x < z \quad \exists y \in X$  such that  $x < y < z$ , then we say that  $(X, \leq)$  is thick

Prop.

Let  $(X, \leq)$  be a thick totally ordered set.  $(a, b) \in X \times X, a < b$  If  $I$  is one of the following intervals  $[a, b]; [a, b[; ]a, b]; ]a, b[$  Then  $\inf I = a \quad \sup I = b$  (for it's thick empty set is impossible)

Proof:

Since  $X$  is thick, there exists  $x_0 \in ]a, b[$  By definition,  $b$  is an upper bound of  $I$ . If  $b$  is not the supremum of  $I$ , there exists an upper bound  $M$  of  $I$  such that  $M \neq b$ . Since  $X$  is thick, there is  $M' \in X$  such that  $x_0 \leq M, M' < b$  Since  $[x, b] \subseteq I, a, b \in I$  Hence  $M$  and  $M'$  belong to  $I$ , which conflicts with the uniqueness of supremum.



## Chapter 5

# Enhanced real line

Def:

Let  $+\infty$  and  $-\infty$  be two symbols that are different and don't belong to  $\mathbb{R}$ . We extend the usual total order  $\leq$  on  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus  $\mathbb{R} \cup \{-\infty, +\infty\}$  becomes a totally ordered set, and  $\mathbb{R} = ]-\infty, +\infty[$ . Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in ]-\infty, +\infty[ \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in ]0, +\infty[ \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$   
**ARE NOT DEFINED**

Def

Let  $(X, \leq)$  be a partially ordered set. If for any subset  $A$  of  $X$ ,  $A$  has a supremum and an infimum in  $X$ , then we say that  $X$  is order complete.

Example

Let  $\Omega$  be a set  $(\mathcal{P}(\Omega), \subseteq)$  is order complete. If  $\mathcal{F}$  is a subset of  $\mathcal{P}(\Omega)$ ,  $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$

Interesting tip:  $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

**AXIOM :**

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$  is order complete

In  $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any  $A \subseteq \mathbb{R} \cup -\infty, +\infty$  and  $c \in \mathbb{R}$  We denote by  $A + c$  the set  $\{a + c \mid a \in A\}$
- If  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\lambda A$  denotes  $\{\lambda a \mid a \in A\}$
- $-A$  denotes  $(-1)A$

Prop.

For any  $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\sup(-A) = -\inf A$ ,  $\inf(-A) = -\sup A$  Def

We denote by  $(\mathbb{R}, \leq)$  a field  $\mathbb{R}$  equipped with a total order  $\leq$ , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$  such that  $a < b$ , one has  $\forall c \in \mathbb{R}$ ,  $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ ,  $ab > 0$
- $\forall A \subseteq \mathbb{R}$ , if  $A$  has an upper bound in  $\mathbb{R}$ , then it has a supremum in  $\mathbb{R}$

Prop.

Let  $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

$\inf$  takes the same

Theorem:

Let  $I$  and  $J$  be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If  $\{a, b\} \neq \{+\infty, -\infty\}$  then  $c = a + b$

$\inf$  takes the same if  $(-\infty) + (+\infty)$  doesn't happen

Corollary:

Let  $I$  be a non-empty set,  $f : I \rightarrow [-\infty, +\infty]$ ,  $g : J \rightarrow [-\infty, +\infty]$

Then  $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

$\inf$  takes the similar ( $\leq \rightarrow \geq$ ) (provided when the sum are defined)

# Chapter 6

## Vector space

In this section:

$K$  denotes a unitary ring.

Let  $0$  be zero element of  $K$

$1$  be the unity of  $K$

### 6.1 $K$ -module

#### 6.1.1 Def

Let  $(V, +)$  be a commutative group. We call left/right  $K$ -module structure: any mapping  $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group  $(V, +)$  equipped with a left/right  $K$ -module structure is called a left/right  $K$ -module.

#### 6.1.2 Remark

Let  $K^{op}$  be the set  $K$  equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then  $K^{op}$  forms a unitary ring  
 Any left  $K^{op}$  - module is a right  $K$ -module  
 Any right  $K^{op}$  - module is a left  $K$ -module  
 $(K^{op})^{op} = K$

### 6.1.3 Notation

When we talk about a left/right  $K$ -module  $(V, +)$ , we often write its left  $K$ -module structure as  $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

### 6.1.4 $K$ -vector space

If  $K$  is commutative, then  $K^{op} = K$ , so left  $K$ -module and right  $K$ -module structure are the same. We simply call them  $K$ -module structure. A commutative group equipped with a  $K$ -module structure is called a  $K$ -module. If  $K$  is a field, a  $K$ -module is also called a  $K$ -vector space

Let  $\Phi : K \times V \rightarrow V$  be a left or right  $K$ -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence  $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$   
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$  is a morphism of groups. Hence  $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$  (*is a var*)

### 6.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping  $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$  is a left  $K$ -module structure

The mapping  $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$  is a right  $K$ -module structure

**6.1.6 Remark:**

We can also write an element  $\mu$  of  $K^I$  in the form of a family  $(\mu_i)_{i \in I}$  of elements in  $K$  ( $\mu_i$  is the image of  $i \in I$  by  $\mu$ )  
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

**6.2 sub K-module****6.2.1 Def**

Let  $V$  be a left/right  $K$ -module. If  $W$  is a subgroup of  $V$ . Such that  $\forall a \in K, \forall x \in W \quad ax/xa \in W$ , then we say that  $W$  is left/right sub- $K$ -module of  $V$ .

**6.2.2 Example**

Let  $I$  be a set. Let  $K^{\oplus I}$  be the subset of  $K^I$  composed of mappings  $f : I \rightarrow K$  such that  $I_f = \{x \in I \mid f(x) \neq 0\}$  is finite. It is a left and right sub- $K$ -module of  $K^I$

In fact,  $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$   
Hence  $f - g \in K^{\oplus I}$  So  $K^{\oplus I}$  is a subgroup of  $K^I$   
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

**6.3 morphism of K-modules****6.3.1 Def**

Let  $V$  and  $W$  be left  $K$ -module, A morphism of groups  $\phi : V \rightarrow W$  is called a morphism of left  $K$ -modules if  $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

**6.3.2 K-linear mapping**

If  $K$  is commutative, a morphism of  $K$ -modules is also called a  $K$ -linear mapping. We denote by  $\text{hom}_{K\text{-Mod}}(V, W)$  the set of all morphism of left- $K$ -module from  $V$  to  $W$ . This is a subgroup of  $W^V$

**6.3.3 Theorem**

Let  $V$  be a left  $K$ -module. Let  $I$  be a set.  
The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$  is a bijection where  
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.3.4 Remark:column

In the case where  $I = 1, 2, 3, \dots, n$   $V^I$  is denoted as  $V^n$ ,  $K^I$  is denoted as  $K^n$ . For any  $(x_1, \dots, x_n) \in V^n$ , by the theorem, there exists a unique morphism of left  $K$ -modules  $\phi : K^n \rightarrow V$  such that  $\forall i \in 1, \dots, n, \phi(e_i) = x_i$ .

We write this  $\phi$  as a column  $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$ . It sends  $(a_1, \dots, a_n) \in K^n$  to  $a_1x_1 + \dots + a_nx_n$ .

## 6.4 kernel

### 6.4.1 Prop

Let  $G$  and  $H$  be groups and  $f : G \rightarrow H$  be a morphism of groups

- $Im(f) \subseteq H$  is a subgroup of  $H$
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- $f$  is injection iff  $\ker(f) = \{e_G\}$

### 6.4.2 Def

$\ker(f)$  is called the kernel of  $f$

### 6.4.3 Theorem

$f$  is injection iff  $\ker(f) = \{e_G\}$

### Proof

Let  $e_G$  and  $e_H$  be neutral element of  $G$  and  $H$  respectively

- (1) Let  $x$  and  $y$  be element of  $G$   
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$ . So  $Im(f)$  is a subgroup of  $H$
- (2) Let  $x$  and  $y$  be element of  $\ker(f)$ . One has  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$ . So  $xy^{-1} \in \ker(f)$ . So  $\ker(f)$  is a subgroup of  $G$ .
- (3) Suppose that  $f$  is injection.  
 Since  $f(e_G) = e_H$  one has  $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$ . Suppose that  $\ker(f) = \{e_G\}$ . If  $f(x) = f(y)$  then  $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$ .  
 Hence  $xy^{-1} = e_G \Rightarrow x = y$

### 6.4.4 Def

Let  $(V, +)$  be a commutative group,  $I$  be a set. We define a composition law  $+$  on  $V^I$  as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then  $V^I$  forms a commutative group

### 6.4.5 Remark

Let  $E$  and  $F$  be left  $K$ -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$  is a subgroup of  $F^E$

In fact  $f$  and  $g$  are elements of  $\text{hom}_{K\text{-Mod}}(E, F)$ , then  $f - g$  is also a morphism of left  $K$ -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

### 6.4.6 Theorem

Let  $V$  be a left  $K$ -module,  $I$  be a set The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$  is an isomorphism of groups, where  $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

### 6.4.7 Proof:

One has  $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So  $\Psi$  is a morphism of groups

injectivity Let  $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$  Such that  $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence  $\phi$  is the neutral element.

surjectivity Let  $x = (x_i)_{i \in I} \in V^I$  We define  $\phi_x : K^{\oplus I} \rightarrow V$  such that  $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left  $K$ -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that  $K'$  is a unitary ring, and  $V$  is also equipped with a right  $K'$ -module structure, Then  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$  is a right sub- $k'$ -module, and  $\Psi$  in the theorem is a right  $K'$ -module isomorphism





## Chapter 7

# Monotone mappings

### 7.1 Def

Let  $I$  and  $X$  be partially ordered sets,  $f : I \rightarrow X$  be a mapping.

- If  $\forall (a, b) \in I \times I$  such that  $a < b$ . One has  $f(a) \leq f(b)$ , then we say that  $f$  is increasing. decreasing takes similar way.
- If  $f$  is (strictly) increasing or decreasing, we say that  $f$  is (strictly) monotone.

### 7.2 Prop.

Let  $X, Y, Z$  be partially ordered sets.  $f : X \rightarrow Y, g : Y \rightarrow Z$  be mappings

- If  $f$  and  $g$  have the same monotonicity, then  $g \circ f$  is increasing
- If  $f$  and  $g$  have different monotonicities, then  $g \circ f$  is decreasing

strict monotonicities takes the same

### 7.3 Def

Let  $f$  be a function from a partially ordered set  $I$  to another partially ordered set  $X$ . If  $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$  is (strictly) increasing/decreasing then we say that  $f$  is (strictly) increasing/decreasing

### 7.4 Prop.

Let  $I$  and  $X$  be partially ordered sets.  $f$  be function from  $I$  to  $X$ .

- If  $f$  is increasing/decreasing and  $f$  is injection, then  $f$  is strictly increasing/decreasing
- Assume that  $I$  is totally ordered and  $f$  is strictly monotone, then  $f$  is injection

## 7.5 Prop

Let  $A$  be totally ordered set,  $B$  be a partially ordered set,  $f$  be an injective function from  $A$  to  $B$

If  $f$  is increasing/decreasing, then so is  $f^{-1}$

## 7.6 Def

Let  $X$  and  $Y$  be partially ordered sets.  $f : X \rightarrow Y$  be a bijection. If both  $f$  and  $f^{-1}$  are increasing, then we say that  $f$  is an isomorphism of partially ordered sets.

(If  $X$  is totally, then a mapping  $f : X \rightarrow Y$  is an isomorphism of partially ordered sets iff  $f$  is a bijection and  $f$  is increasing)

## 7.7 Prop.

Let  $I$  be a subset of  $\mathbb{N}$  which is infinite. Then there is a unique increasing bijection  $\lambda_I : \mathbb{N} \rightarrow I$

## 7.8 Proof

### 7.8.1 bijection

We construct  $f : \mathbb{N} \rightarrow I$  by induction as follows.

Let  $f(0) = \min I$  Suppose that  $f(0), \dots, f(n)$  are constructed

then we take  $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since  $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$ . Therefore  $f(n) \leq f(n+1)$

Since  $f(n+1) \notin \{f(0), \dots, f(n)\}$ , we have  $f(n) < f(n+1)$

Hence  $f$  is strictly increasing and this is injective

If  $f$  is not surjective, then  $I \setminus \text{Im}(f)$  has a element  $N$ .

Let  $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$ .

Since  $N \notin \text{Im}(f)$ ,  $N < f(m)$ .

So  $m \neq 0$ . Hence  $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition,  $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$ ,

Hence  $f(m) \leq N$ , causing contradiction.

**7.8.2 uniqueness**

exercise: Prove that  $Id_{\mathbb{N}}$  is the only isomorphism of partially ordered sets from  $\mathbb{N}$  to  $\mathbb{N}$



# Chapter 8

## sequence and series

Let  $I \subseteq \mathbb{N}$  be a infinite subset

### 8.1 Def

Let  $X$  be a set. We call sequence in  $X$  parametrized by  $I$  a mapping from  $I$  to  $X$ .

### 8.2 Remark

If  $K$  is a unitary ring and  $E$  is a left  $K$ -module then the set of sequence  $E^I$  admits a left- $K$ -module structure. If  $x = (x_n)_{n \in I}$  is a sequence in  $E$ , we define a sequence  $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ , called the series associated with the sequence  $x$ .

### 8.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$  is a morphism of left- $K$ -module

### 8.4 proof

Let  $x = (x_i)_{i \in I}$  and  $y = (y_i)_{i \in I}$  be elements of  $E^I$

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

### 8.5 Prop

Let  $I$  be a totally ordered set .  $X$  be a partially ordered set,  $f : I \rightarrow X$  be a mapping ,  $J \in I$  Assume that  $J$  does not have any upper bound in  $I$

- If  $f$  is increasing ,then  $f(I)$  and  $f(J)$  have the same upper bounds in  $X$
- If  $f$  is decreasing ,then  $f(I)$  and  $f(J)$  have the same lower bounds in  $X$

## 8.6 limit

### 8.6.1 Def

Let  $i \subseteq \mathbb{N}$  be a infinite subset.  $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$  where  $[-\infty, +\infty]$  denotes  $\mathbb{R} \cup \{-\infty, +\infty\}$ , we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left( \inf_{i \in I, i \geq n} x_i \right)$$

If  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$ , we then say that  $(x_n)_{n \in I}$  tends to  $l$  and that  $l$  is the limit of  $(x_n)_{n \in I}$ . If in addition  $(x_n)_{n \in I} \in \mathbb{R}^I$  and  $l \in \mathbb{R}$ , we say that  $(x_n)_{n \in I}$  converges to  $l$

### 8.6.2 Remark

If  $J \subseteq \mathbb{N}$  is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} = \inf_{n \in J} \left( \sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left( \inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in  $(x_i)_{i \in I}$  the limit superior and the limit inferior do not change.

In fact, if we take  $J = \mathbb{N} \setminus \{0, \dots, m\}$ , then  $\inf_{n \in J}(\dots)$  and  $\sup_{n \in J}(\dots)$  only depends on the values of  $x_i, i \in I, i \geq m$

### 8.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

**8.6.4 Prop**

Let  $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned}
 \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \limsup_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = \left( \liminf_{n \in I, n \rightarrow +\infty} x_n \right) + c \\
 \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\
 & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n
 \end{aligned}$$

**8.6.5 Prop**

Let  $(x_n)_{n \in I}$  be elements in  $[-\infty, +\infty]^I$ . Suppose that there exists  $N_0 \in \mathbb{N}$  such that  $\forall n \in I, n \geq N_0$ , one has  $x_n \leq y_n$ . Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

**8.6.6 Theorem**

Let  $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$  one has  $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$  and  $(z_n)_{n \in I}$  tend to the same limit  $l$

Then  $(y_n)_{n \in I}$  tends to  $l$

**8.6.7 Def**

Let  $I$  be an infinite subset of  $\mathbb{N}$ , and  $(x_n)_{n \in I}$  be a sequence in some set  $X$ . We call subsequence of  $(x_n)_{n \in I}$  a sequence of the form  $(x_n)_{n \in J}$ , where  $J$  is an infinite subset of  $I$

**8.6.8 Prop**

Let  $I$  and  $J$  be infinite subset of  $\mathbb{N}$  such that  $J \subseteq I$ .  $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$ , one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if  $(x_n)_{n \in I}$  tends to  $l \in [-\infty, +\infty]$ , then  $(x_n)_{n \in J}$  tends to  $l$

### 8.6.9 Prop

$\forall n \in \mathbb{N}$ , one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

### 8.6.10 Theorem

Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$

- If the mapping  $(n \in I) \mapsto x_n$  is increasing, then  $(x_n)_{n \in I}$  tends to  $\sup_{n \in I} x_n$
- If the mapping  $(n \in I) \mapsto x_n$  is decreasing, then  $(x_n)_{n \in I}$  tends to  $\inf_{n \in I} x_n$

### 8.6.11 Notation

If a sequence  $(x_n)_{n \in I} \in [-\infty, +\infty]$  tends to some  $l \in [-\infty, +\infty]$  the expression  $\lim_{n \in I, n \rightarrow} x_n$  denotes this limit  $l$

### 8.6.12 Corollary

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{N}_{\geq 0}$ . Then the series  $\sum_{n \in I} x_n$  (the sequence  $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$ ) tends to an element in  $\mathbb{N}_{\geq 0} \cup \{+\infty\}$ . It converges in  $\mathbb{R}$  iff it is bounded from above (namely has an upper bound in  $\mathbb{R}$ )

### 8.6.13 Notation

If a series  $\sum_{n \in I} x_n$  in  $[-\infty, +\infty]$  tends to some limit, we use the expression  $\sum_{n \in I} x_n$  to denote the limit

### 8.6.14 Theorem: Bolzano-Weierstrass

Let  $(x_n)_{n \in I}$  be a sequence in  $[-\infty, +\infty]$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\limsup_{n \in I, n \rightarrow +\infty} x_n$ . There exists a subsequence of  $(x_n)_{n \in I}$  that tends to  $\liminf_{n \in I, n \rightarrow +\infty} x_n$ .



**Proof**

Let  $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If  $J$  is infinite, the sequence  $(x_n)_{n \in J}$  is decreasing so it tends to  $\inf_{n \in J} x_n$

$$\forall n \in J \text{ by definition } x_n = \sup_{i \in I, i \geq n} x_i \text{ so } \limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i = \inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$$

Assume that  $J$  is finite. Let  $n_0 \in I$  such that  $\forall n \in J, n < n_0$ . Denote by

$$l = \sup_{n \in I, n \geq n_0} x_n$$

Let  $N \in \mathbb{N}$  such that  $N \geq n_0$ . By definition  $\sup_{i \in I, i \geq n_0} x_i \leq l$ . If the strict inequality  $\sup_{i \in I, i \geq N} x_i < l$  holds, then  $\sup_{i \in I, i \geq N} x_i$  is NOT an upper bound of  $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists  $n \in I$  such that  $n_0 \leq n < N$  such that  $x_n > \sup_{i \in I, i \geq N} x_i$ . We may also assume that  $n$  is largest among elements of  $I \cap [n_0, N[$  that satisfies this inequality.

Then  $\forall m \in I$  if  $m \geq n$  then  $x_m \leq x_n$ . Thus  $n \in J$  that contradicts the maximality of  $n_0$ .

Therefore

$$l = \sup_{i \in I, i \geq N} x_i$$

, which leads to

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = l$$

Moreover, if  $m \in I, m \geq n_0$  then  $m \notin J$ , so  $x_m < l$  (since otherwise  $x_m = \sup_{i \in I, i \geq m} x_i$  and hence  $m \in J$ ). Hence,  $\forall$  finite subset  $I'$  of  $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$  and hence  $\exists n \in I$ , such that  $n > \max I'$ , and  $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence  $(n_j)_{j \in \mathbb{N}}$  in  $I$

Let  $n_0$  be as above. Let  $f : \mathbb{N} \rightarrow I_{\geq n_0}$  be a surjective mapping.

If  $n_j$  is chosen, we choose  $n_{j+1} \in I$  such that

$$n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$$

Hence the sequence  $(x_{n_j})_{j \in \mathbb{N}}$  is increasing

And

$$\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n = l$$

$$l = \sup_{n \in I, n \geq n_0} x_n$$

So  $(x_{n_j})_{j \in \mathbb{N}}$  tends to  $l$



## Chapter 9

# Cauchy sequence

### 9.1 Def

Let  $(x_n)_{n \in I}$  be a sequence in  $\mathbb{R}$   
If  $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$  then  
we say that  $(x_n)_{n \in I}$  is a Cauchy sequence

### 9.2 Prop

- If  $(x_n)_{i \in I} \in \mathbb{R}^I$  converges to some  $l \in \mathbb{R}$ , then it is a Cauchy sequence
- If  $(x_n)_{i \in I}$  is a Cauchy sequence, there exists  $M > 0$  such that  $\forall n \in I \quad |x_n| \leq M$
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite,  $(x_n)_{n \in I}$  is a Cauchy sequence.
- If  $(x_n)_{n \in I}$  is a Cauchy sequence, then  $\forall J \subseteq I$  infinite and  $l \in \mathbb{R}$  such that  $(x_n)_{n \in I}$  converges to  $l$ , then  $(x_n)_{n \in J}$  converges to  $l$  too.

### 9.3 Theorem: Completeness of real number

If  $(x_n)_{n \in I} \in \mathbb{R}^I$  is a Cauchy sequence, then it converges in  $\mathbb{R}$

#### Proof

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists M \in \mathbb{R}_{>0}$  such that  $-M \leq x_n \leq M \quad \forall x \in I$ . So  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . By Bolzano-Weierstrass theorem.  $\exists J \subseteq I$  infinite such that  $(x_n)_{n \in I}$  converges to  $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$ . Therefore  $(x_n)_{n \in I}$  converges to the same limit.

## 9.4 Absolutely converge

We say that a series  $\sum_{n \in I} x_n \in \mathbb{R}$  converges absolutely if  $\sum_{n \in I} |x_n| < +\infty$

### 9.4.1 Prop

If a series  $\sum_{n \in I} x_n$  converges absolutely, then it converges in  $\mathbb{R}$

## Chapter 10

# Comparison and Technics of Computation

### 10.1 Def

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be sequence in  $\mathbb{R}$

- If there exists  $M \in \mathbb{R}_{>0}$  and  $N \in \mathbb{N}$  such that  $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$  then we write  $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists  $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$  and  $N \in \mathbb{N}$  such that  $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$  and  $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$ , then we write  $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

### 10.2 Prop.

Let  $I$  and  $X$  be partially ordered sets and  $f : I \rightarrow X$  be an increasing/decreasing mapping. Let  $J$  be a subset of  $I$ . Assume that any elements of  $I$  has an upper bound in  $J$ . Then  $f(I)$  and  $f(J)$  have the same upper/lower bounds in  $X$

### 10.3 Theorem

Let  $I$  be a totally ordered set,  $f : I \rightarrow [-\infty, +\infty]$  and  $g : I \rightarrow [-\infty, +\infty]$  be two mappings that are both increasing/decreasing. Then the following equalities holds, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

### Proof

We can assume  $f$  and  $g$  increasing. Let  $a = \sup f(I), b = \sup g(I)$

Let  $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip  $A$  with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let  $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$ .

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

$h$  is increasing.

Let  $(x, y) \in A$ . Assume that  $x \leq y$

If  $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$  then  $(y, y) \in B$  and  $(x, y) \leq (y, y)$

If  $\{f(y), g(y)\} = \{-\infty, +\infty\}$  and for  $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$ . So  $a = +\infty$ , Hence  $b > -\infty$

So  $\exists z \in I$  such that  $g(z) > -\infty$ . We should have  $y \leq z$  Hence  $f(z) + g(z)$  is well defined,  $(z, z) \in B$  and  $(x, y) \leq (z, z)$  Similarly, if  $x \geq y$ ,  $(x, y)$  has also an upper bound in  $B$ . Therefore:  $\sup h(A) = \sup h(B)$

## 10.4 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

### Proof

$\forall n \in \mathbb{N}$ , let  $A_N = \sup_{n \in I, n \geq N} x_n$   $B_N = \sup_{n \in I, n \geq N} y_n$ .  $(A_N)_{N \in \mathbb{N}}$  and  $(B_N)_{N \in \mathbb{N}}$  are decreasing, and  $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$   $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let  $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$  if  $A_N + B_N$  is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

## 10.5 Prop.

Let  $I \subseteq \mathbb{N}$  be an infinite subset. Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$  such that  $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$ . Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n)$$

### Proof

a tricky proof ?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with  $\infty$

## 10.6 Theorem

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ . Assume that  $\forall n \in I, y_n \in \mathbb{R}$  and  $(y_n)_{n \in I}$  converges to some  $l \in \mathbb{R}$ .  
Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + l$$

## 10.7 Prop.

Let  $(x_n)_{n \in I}$  and  $(y_n)_{n \in I}$  be elements of  $[-\infty, +\infty]^I$ .  
Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\left\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\right\}$$

### Proof

About the first inequality. Since  $\max\{x_n, y_n\} \geq x_n$  and  $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset  $J$  of  $I$  such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let  $J_1 = \{n \in J \mid x_n \geq y_n\}$   $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$  So either  $J_1$  or  $J_2$  is infinite

Suppose that  $J_1$  is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If  $J_2$  is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

## 10.8 Theorem

Let  $(a_n)_{n \in I} \in \mathbb{R}^I$   $l \in \mathbb{R}$ . The following statements are equivalent

- $(a_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

### Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\{(\limsup_{n \in I, n \rightarrow +\infty} a_n) - l, l - (\liminf_{n \in I, n \rightarrow +\infty} a_n)\}$$

(1)  $\Rightarrow$  (2):

If  $(a_n)_{n \in I}$  converges to  $l$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2)  $\Rightarrow$  (1):

If  $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$ , then  $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore:  $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

## 10.9 Remark

Let  $(a_n)_{n \in I}$  be a sequence in  $\mathbb{R}$ ,  $l \in \mathbb{R}$

The sequence  $(a_n)_{n \in I}$  converges to  $l$  iff  $a_n - l = o(1), n \in I, n \rightarrow +\infty$

## 10.10 Calculates on $O()$ , $o()$

### 10.10.1 Plus

Let  $(a_n)_{n \in I}$   $(a'_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$   
then  $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$



### 10.10.2 Transform

Let  $(a_n)_{n \in I}$  and  $(b_n)_{n \in I}$  be two sequence in  $\mathbb{R}$  If  $a_n = o(b_n), n \in I, n \rightarrow +\infty$ , then  $a_n = O(b_n), n \in I, n \rightarrow +\infty$

### 10.10.3 Transition

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  and  $(c_n)_{n \in I}$  be elements in  $\mathbb{R}^I$

- If  $a_n = O(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = O(b_n)$  and  $b_n = o(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If  $a_n = o(b_n)$  and  $b_n = O(c_n), n \in I, n \rightarrow +\infty$   
then  $a_n = o(c_n), n \in I, n \rightarrow +\infty$

### 10.10.4 Times

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$  be sequences in  $\mathbb{R}$

- If  $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If  $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$   
then  $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

## 10.11 On the limit

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be elements of  $\mathbb{R}^I$  that converges to  $l \in \mathbb{R}$  and  $l' \in \mathbb{R}$  respectively. Then:

- $(a_n + b_n)_{n \in I}$  converges to  $l + l'$
- $(a_n b_n)_{n \in I}$  converges to  $ll'$

## 10.12 Prop

Let  $a \in \mathbb{R}$  then  $a^n = o(n!)$   $n \rightarrow +\infty$

### Proof

Let  $N \in \mathbb{N}$  such that  $|a| < N$   
For  $n \in \mathbb{N}$  such that  $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{|a^n - N|}{\frac{n!}{N!}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^n - N$$

And  $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$ . Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

### 10.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

#### Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

### 10.14 Prop

Let  $(a_n)_{n \in I}, (b_n)_{n \in I}$  be the elements of  $\mathbb{R}^I$ . If the series  $\sum_{n \in I} b_n$  converges absolutely and if  $a_n = O(b_n) \quad n \rightarrow +\infty$  Then  $\sum_{n \in I} a_n$  converges absolutely

#### Proof

By definition  $\sum_{n \in I} |b_n| < +\infty$ . If  $|a_n| \leq M|b_n|$  for  $n \in I, n \geq N$  where  $N \in \mathbb{N}$ . Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

### 10.15 Theorem: d'Alembert ratio test

Let  $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely
- If  $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  does not converge (diverges)

**Proof****(1)**

Let  $\alpha \in \mathbb{R}$  such that  $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$ ,  $\alpha$  isn't a lower bound of  $\left( \sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$   
 So  $\exists N \in \mathbb{N}$  such that  $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$  Hence for  $n \geq N$   $|a_n| \leq \alpha^{n-N} |a_N|$  since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore  $a_n = O(\alpha^n)$  since  $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$ ,  $\sum_{n \in \mathbb{N}} a_n$  converge absolutely.

**10.15.1 Lemma**

If a series  $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$  converges, then  $\lim_{n \rightarrow +\infty} a_n = 0$

**Proof**

If  $\left( \sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$  converges to some  $l \in \mathbb{R}$ , then  $\left( \sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$  converges to  $l$ ,  
 too. Hence  $\left( a_n = \left( \sum_{i=0}^n a_i \right) - \left( \sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$  converges to  $l - l = 0$

**10.15.2 (2)**

Let  $\beta \in \mathbb{R}$  such that  $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 So there exists  $N \in \mathbb{N}$  such that  $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$   
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$   
 Hence  $(|a_n|)_{n \in \mathbb{N}}$  is not bounded since  $|a_n| \geq \beta^{n-N} |a_N|$   
 By the lemma:  $\sum_{n \in \mathbb{N}} a_n$  diverges.

**10.16 Prop**

Let  $a \in \mathbb{R}, a > 1$  Then  $n = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $\epsilon > 0$  such that  $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

**10.16.1 Corollary**

Let  $a > 1, t \in \mathbb{R}_{\geq 0}$  Then  $n^t = o(a^n), n \rightarrow +\infty$

**Proof**

Let  $d \in \mathbb{N}_{\geq 1}$  such that  $t \leq d$  Then  $n^{t-d} \leq 1$  So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let  $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence  $n^t = o(a^n)$

**10.16.2 Corollary**

There exists  $M \geq 1$  such that  $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

**Proof**

Let  $a \in \mathbb{R}$  such that  $1 < a < e$

**10.17 Theorem: Cauchy root test**

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{R}$ . Let  $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If  $\alpha < 1$ , then  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.
- If  $\alpha > 1$  then  $\sum_{n \in \mathbb{N}} a_n$  diverges

**Proof**

(1)

Let  $\beta \in \mathbb{R}, \alpha < \beta < 1$ . There exists  $N \in \mathbb{N}$  such that  $|a_n|^{\frac{1}{n}} \leq \beta$  for  $n \geq N$ . That means  $|a_n| = O(\beta^n)$  since  $0 < \beta < 1$ ,  $\sum_{n \in \mathbb{N}} a_n$  converges absolutely.

(2)

If  $\alpha > 1$  then  $\forall N \in \mathbb{N} \exists n \geq N$  such that  $|a_n|^{\frac{1}{n}} \geq 1$ , since otherwise  $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$  contradiction  
Hence  $(|a_n|)_{n \in \mathbb{N}}$  cannot converge to 0.

## Part III

# Axiom of choice



# Chapter 11

## Preparation

### 11.1 Statement of axiom of choice

For any set  $I$  and any family  $(A_i)_{i \in I}$  of non-empty sets, there exists a mapping  $f : I \rightarrow \bigcup_{i \in I} A_i$  such that  $\forall i \in I, f(i) \in A_i$

### 11.2 Def

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$   $A$  is non-empty, there exists a least element of  $A$  then we say that  $(X, \leq)$  is a well ordered set.

### 11.3 Theorem

For any set  $X$ , there exists an order relation  $\leq$  on  $X$  such that  $(X, \leq)$  forms a well ordered set.

### 11.4 Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. If  $\forall A \subseteq X$  that is totally ordered with respect to  $\leq$ , there exists an upper bound of  $A$  inside  $X$ . Then, there exists a maximal element  $x_0$  of  $X$  ( $\forall y \in X, y > x_0$  does not hold)

### 11.5 Prop.

Let  $(X, \leq)$  be a well ordered set,  $y \notin X$ . We extend  $\leq$  to  $X \cup \{y\}$ , such that  $\forall x \in X, x < y$ . Then  $(X \cup \{y\}, \leq)$  is well ordered.

## 11.6 Proof

Let  $A \subseteq X \cup \{y\}$ ,  $A \neq \emptyset$ . If  $A = \{y\}$  then  $Y$  is the least element of  $A$ . If  $A \neq \{y\}$  then  $B = A \setminus \{y\}$  is non-empty. Let  $b$  be the least element of  $B$ . Since  $b < y$  it's also the least element of  $A$

## 11.7 Def: Initial Segment

Let  $(X, \leq)$  be a well ordered set.  $S \subseteq X$ , If  $\forall s \in S, x \in X \quad x < s$  initial  $x \in S$  ( $X_{<s} \subseteq S$ ), then we say that  $S$  is an initial segment of  $X$

If  $S$  is a initial segment such that  $S \neq X$  then we sat that  $S$  is a proper initial segment.

## 11.8 Example

$\forall x \in X \quad X_{<x} = \{s \in X \mid s < x\}$  Then  $X_{<x}$  is a proper initial segment of  $X$ .

## 11.9 Prop.

Let  $(X, \leq)$  be a well ordered set , If  $(S_i)_{i \in I}$  is a family of initial segment of  $X$ , then  $\bigcup_{i \in I} S_i$  is an initial segment of  $X$

## 11.10 Proof

$\forall s \in \bigcup_{i \in I} S_i, \exists i \in I$  such that  $s \in S_i, i \in I$  Therefore  $X_{<s} \subseteq S_i \subseteq \bigcup_{i \in I} S_i$

## 11.11 Prop

Let  $(X, \leq)$  be a well ordered set.

- (1) Let  $S$  be a proper initial segment of  $X$ ,  $x = \min(X \setminus S)$  Then  $S = X_{<x}$
- (2)  $X \rightarrow \wp(X)$   
 $x \mapsto X_{<x}$
- (3) The set of all initial segments of  $X$  forms a well ordered subset of  $(\wp(X), \subseteq)$

## 11.12 Proof

- (1)  $\forall s \in S$  if  $x \leq s$  then  $x \in S$  contradiction. Hence  $s < x$ , This shows  $S \subseteq X_{<x}$  Conversely , if  $t \in X, t \notin X \setminus S$  Hence  $t \in S$ . Hence  $X_{<x} \subseteq S$



- (2) Let  $x, y \in X, x < y$  By definition  $X_{<x} \subseteq X_{<y}$  Moreover  $x \in X_{<y} \setminus X_{<x}$  So  $X_{<x} \subsetneq X_{<y}$
- (3) Let  $\mathcal{F} \subseteq \wp(X)$  be a set of initial segments.  $\mathcal{F} \neq \emptyset$ . Then there exists  $A \subseteq X$  such that  $\mathcal{F} \setminus \{x\} = \{X_{<x} \mid x \in A\}$  If  $A = \emptyset$  then  $\mathcal{F} = \{X\}$ , and  $\{X\}$  is the least element of  $\mathcal{F}$ . Otherwise  $A \neq \emptyset$  and  $A$  has a least element  $a$ . Then by (2)  $X_{<a}$  is the least element of  $\mathcal{F}$

### 11.13 Lemma

Let  $(X, \leq)$  be a well ordered set,  $f : X \rightarrow X$  be a strictly increasing mapping. Then  $\forall x \in X, x \leq f(x)$

#### Proof

Let  $A = \{x \in X \mid f(x) < x\}$  If  $A \neq \emptyset$ , let  $a$  be the least element of  $A$ . By definition  $f(a) < a$ . Hence  $f(f(a)) < f(a)$  since  $f$  is strictly increasing. This shows  $f(a) \in A$ . But  $a$  is the least element of  $A$ ,  $f(a) < a$  cannot hold: contradiction.

### 11.14 Prop

Let  $(X, \leq)$  be a well ordered set,  $S$  and  $T$  be two initial segment of  $X$ . If  $f : S \rightarrow T$  is a bijection that's strictly increasing, then  $S = T, f = Id_S$

#### Proof

We may assume  $T \subseteq S$ . Let  $l : T \rightarrow S$  be the inclusion mapping and  $g = l \circ f : S \rightarrow S$ . Since  $g$  is strictly increasing, by the lemma,  $\forall s \in S, s \leq g(s) = f(s) \in T$ . Since  $T$  is an initial segment,  $s \in T$ . Hence  $S = T$ . Apply the lemma to  $f^{-1}$  we get  $\forall s \in S, s \leq f^{-1}(s)$  Hence  $f(s) \leq s$  Therefore  $f(s) = s$

### 11.15 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be partially ordered sets. If  $\exists f : X \rightarrow Y$  that's increasing and bijective, we say that  $(X, \leq)$  and  $(Y, \leq)$  are isomorphic

### 11.16 Def

Let  $(X, \leq)$  and  $(Y, \leq)$  be well ordered sets. If  $(X, \leq)$  is isomorphic to an initial segment of  $Y$ . We note  $X \preceq Y$  or  $Y \succeq X$ . If  $X$  is isomorphic to  $Y$ , we note  $X \sim Y$ . If  $X \preceq Y$  but  $X \not\sim Y$ , we note  $X \prec Y$  or  $Y \succ X$

### 11.17 Prop.

Let  $X$  and  $Y$  be well ordered sets. Among the following condition, one and only one holds.

$$X \prec Y \quad X \sim Y \quad X \succ Y$$

#### Proof

We construct a correspondence  $f$  from  $X$  to  $Y$ , such that  $(x, y) \in \Gamma_f$ , iff  $X_{<x} \sim Y_{<y}$   
By the last proposition of Oct. 11,  $f$  is a function.

- If  $a, b \in \text{Dom}(f)^2$ ,  $a < b$ , then  $X_{<a} \subsetneq X_{<b}$   
By definition,  $Y_{<f(b)} \sim X_{<b}$   $Y_{<f(a)} \sim X_{<a}$   
Hence  $Y_{<f(a)}$  is isomorphic to a proper initial segment of  $Y_{<f(b)}$ . Therefore  $Y_{f(a)}$  is a proper initial segment of  $Y_{<f(b)}$ . We then get  $f(a) < f(b)$ . Thus  $f$  is strictly increasing.
  - Let  $a \in \text{Dom}(f)$  Let  $x \in X, x < a$  Then  $X_{<x}$  is a initial segment of  $X_{<a} \sim Y_{<f(a)}$  Hence  $\exists y \in Y$   $X_{<x} \sim Y_{<y}$  This shows that  $x \in \text{Dom}(f)$ . Hence  $\text{Dom}(f)$  is an initial segment of  $X$ . Applying this to  $f^{-1}$ , we get :  $\text{Im}(f) = \text{Dom}(f)$  is an initial segment of  $Y$
  - Either  $\text{Dom}(f) = X$  or  $\text{Im}(f) = Y$ .  
Assume that  $x \in X \setminus \text{Dom}(f), y \in Y \setminus \text{Im}(f)$  are respectively the least elements of  $X \setminus \text{Dom}(f)$  and  $Y \setminus \text{Im}(f)$ .  
Then we get  $\text{Dom}(f) = X_{<x}, \text{Im}(f) = Y_{<y}$ .  
We obtain  $X_{<x} \sim Y_{<y}, (x, y) \in \Gamma_f$ . Contradiction
  -
- Case 1  $\text{Dom}(f) = X, \text{Im}(f) \subsetneq Y$   $X \prec Y$   
Case 2  $\text{Dom}(f) \subsetneq X, \text{Im}(f) = Y$   $X \succ Y$   
Case 3  $\text{Dom}(f) = X, \text{Im}(f) = Y$   $X \sim Y$

### 11.18 Lemma

Let  $(X, \leq)$  be a partially ordered set .  $\mathfrak{S} \subseteq \wp(X)$ . Assume that

- $\forall A \in \mathfrak{S}, (A, \leq)$  is a well-ordered set .
- $\forall (A, B) \in \mathfrak{S}^2$ , either  $A$  is an initial segment of  $B$ , or  $B$  is an initial segment of  $A$ .

Let  $Y = \bigcup_{A \in \mathfrak{S}} A$ . Then  $(Y, \leq)$  is a well ordered set, and  $\forall A \in \mathfrak{S}, A$  is an initial segment of  $Y$ .

**Proof**

- Let  $A \in \mathfrak{S}, x \in A, y \in Y, y < x$ . Since  $Y = \bigcup_{B \in \mathfrak{S}} B, \exists B \in \mathfrak{S}$ , such that  $y \in B$ . If  $y \notin A$  then  $B \not\subseteq A$ . Hence  $A$  is an initial segment of  $B$ . Hence  $y \in A$ . Contradiction
- Let  $Z \subseteq Y, Z \neq \emptyset$ . Then  $\exists A \in \mathfrak{S}, A \cap Z \neq \emptyset$ . Let  $m$  be the least element of  $A \cap Z$ . Let  $z \in Z, B \in \mathfrak{S}$ , such that  $z \in B$ . If  $z \in A$ , then  $m \leq z$ . If  $z \notin A$ , then  $A$  is an initial segment of  $B$ .

Since  $B$  is well ordered, if  $m \not\leq z$  then  $z < m$ . Since  $m \in A$ , we get  $z \in A$ . Contradiction.

Therefore,  $m$  is the least element of  $Z$ .



## Chapter 12

# Zorn's lemma

Let  $(X, \leq)$  be a partially ordered set. Suppose that any well-ordered subset of  $X$  has an upper bound on  $X$ , then  $X$  has a maximal element (a maximal element  $m$  of  $\{x \mid x > m\} = \emptyset$ )

### 12.1 Proof

Suppose that  $X$  doesn't have any maximal element.  $\forall A \in \omega. \exists f(A)$  such that  $\forall a \in A, a < f(A)$

Let

$$\omega = \{\text{well ordered subset of } X\}$$

. (guaranteed by axiom of choice)

Let  $f : \omega \rightarrow X$  such that  $f(A)$  is an upper bound of  $A \in \omega$ .

If  $A \in \omega$  satisfies

$$\forall a \in A, a = f(A_{<a})$$

, we say that  $A$  is a  $f$ -set

Let

$$\mathfrak{S} = \{f\text{-sets}\}$$

Note that

$$\emptyset \in \mathfrak{S}$$

if

$$\forall A \in \mathfrak{S}, A \cup \{f(A)\} \in \mathfrak{S}$$

In fact, if  $a \in A$ , then

$$A_{<a} = (A \cup \{f(A)\})_{<a}$$

If  $a = f(A) \notin A$  then

$$(A \cup \{f(A)\})_{<a} = A$$

Let  $A$  and  $B$  be elements of  $\mathfrak{S}$ . Let  $I$  be the union of all common initial segments of  $A$  and  $B$ . This is also a common initial segment of  $A$  and  $B$ .

If  $I \neq A$  and  $I \neq B$ , then

$$\exists(a, b) \in A \times B, I = A_{<a} = B_{<b} \quad f(I) = f(A_{<a}) = f(B_{<b})$$

. Hence

$$a = b$$

. Then  $I \cup \{a\}$  is also a common initial segment of  $A$  and  $B$ , contradiction.

By the lemma ,

$$Y := \bigcup_{A \in \mathfrak{S}} A$$

is well-ordered , and  $\forall A \in \mathfrak{S}$  is an initial segment of  $Y$ .

Since  $A$  is an initial segment of  $Y$

$$\forall a \in Y, \exists A \in \mathfrak{S} \quad a \in A \quad A_{<a} = Y_{<a}$$

. Hence

$$f(Y_{<a}) = f(A_{<a}) = a$$

. Hence

$$y \in \mathfrak{S}$$

. Thus  $Y$  is the greatest element of  $(\mathfrak{S}, \subseteq)$ . However,

$$Y \cup \{f(Y)\} \in \mathfrak{S}$$

. Hence

$$f(y) \in Y$$

.

If  $f(y)$  is not a maximal element of  $X$

$$\exists x \in X, f(y) < x$$

**Part IV**

**Topology**





# Chapter 13

## Absolute value and norms

### 13.1 Def

Let  $K$  be a field. By absolute value on  $K$ , we mean a mapping  $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$  that satisfies:

- (1)  $\forall a \in K \quad |a| = 0$  iff  $a = 0$
- (2)  $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3)  $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$  (triangle inequality)

### 13.2 Notation

$\mathbb{Q}$  Take a prime num  $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$  there exists a integer  $ord_p(\alpha) \frac{a}{b}$ , where  
 $a \in \mathbb{Z} \setminus \{0\}$   
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

### 13.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on  $\mathbb{Q}$

### Proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd$$

$$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d}$$

Assume  $ord_p(\alpha) \geq ord_p(\beta)$

$$\alpha + \beta$$

$$= p^{ord_p(\beta)} \left( p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right)$$

$$= p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd$$

So

$$ord_p(\alpha + \beta) \geq ord_p(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$$

$$\text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$$

$$\max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$

### 13.4 Def

Let  $K$  be a field and  $|\cdot|$  be an absolute value. We call  $(K, |\cdot|)$  a valued field.

## Chapter 14

# Quotient Structure

### 14.1 Def

Let  $X$  be a set and  $\sim$  be a binary relation on  $X$   
If :

- $\forall x \in X, x \sim x$
- $\forall (x, y) \in X \times X$ , if  $x \sim y$  then  $y \sim x$
- $\forall (x, y, z) \in X^3$ , if  $x \sim y, y \sim z$  then  $x \sim z$

then we say that  $\sim$  is an equivalence relation

### 14.2 equivalence class

$\forall x \in X$  we denote by  $[x]$  the set  $\{y \in X \mid y \sim x\}$  and call it the equivalence class of  $x$  on  $X$ . Let  $X/\sim$  be the set  $\{[x] \mid x \in X\}$

### 14.3 Prop.

Let  $X$  be a set and  $\sim$  be an equivalence relation on  $X$

- (1)  $\forall x \in X, y \in [x]$  on has  $[x] = [y]$
- (2) If  $\alpha$  and  $\beta$  are elements of  $X/\sim$  such that  $\alpha \neq \beta$  then  $\alpha \cap \beta = \emptyset$
- (3)  $X = \bigcup_{\alpha \in X/\sim} \alpha$

**Proof**

- (1) Let  $z \in [y]$ . Then  $y \sim z$ . Since  $y \in [x]$  one has  $x \sim y$ . Therefore,  $x \sim z$  namely  $z \in [x]$ . This proves  $[y] \subseteq [x]$ . Moreover, since  $x \sim y$ , one has  $x \in [y]$ . Hence  $[x] \subseteq [y]$ . Thus we obtain  $[x] = [y]$ .
- (2) Suppose that  $\alpha \cap \beta \neq \emptyset, y \in \alpha \cap \beta$ .  
By (1),  $\alpha = [y], \beta = [y]$ . Thus leads to a contradiction.
- (3)  $\forall x \in X \quad x \in [x]$  Hence  $x \in \bigcup_{\alpha \in X/\sim} \alpha$ . Hence  $X \subseteq \bigcup_{\alpha \in X/\sim} \alpha$ . Conversely,  
 $\forall \alpha \in X/\sim, \alpha$  is a subset of  $X$ . Hence  $\bigcup_{\alpha \in X/\sim} \alpha \subseteq X$ . Then  $X = \bigcup_{\alpha \in X/\sim} \alpha$ .

**14.4 Def**

Let  $G$  be a group and  $X$  be a set.  
We call left/right action of  $G$  on  $X$  and mapping  $G \times X \rightarrow X : (g, x) \mapsto gx / (g, x) \mapsto xg$  that satisfies:

- $\forall x \in X \quad 1x = x / x1 = x$
- $\forall (g, h) \in G^2, x \in X \quad g(hx) = (gh)x / (xg)h = x(gh)$

**14.5 Remark**

If we denote by  $G^{op}$  the set  $G$  equipped with the composition law :

$$G \times G \rightarrow G$$

$$(g, h) \mapsto hg$$

The a right action of  $G$  on  $X$  is just a left action of  $G^{op}$  on  $X$ .

**14.6 Prop**

Let  $G$  be a group and  $X$  be a set. Assume given a left action of  $G$  on  $X$ . Then the binary relation  $\sim$  on  $X$  defined as  $x \sim y$  iff  $\exists g \in G \quad y = gx$  is an equivalence relation

**14.7 Notation on Equivalence Class**

We denote by  $G/X$  the set  $X/\sim \forall x \in X$  the equivalence class of  $x$  is denoted as  $Gx/xG$  or  $orb_G(x)$  call the orbit of  $x$  under the action of  $G$

## 14.8 Proof

- $\forall x \in X \quad x = 1x$  so  $x \sim x$
- $\forall (x, y) \in X^2$  if  $y = gx$  for same  $g \in G$  then  $g^{-1}y = g^{-1}(gx) = (g^{-1}g)x = 1x = x$ . ( $y \sim x$ )
- $\forall (x, y, z) \in X^3$ , if  $\exists (g, h) \in G^2$ , such that  $y = gx$  and then  $z = h(gx) = (hg)x$  So  $x \sim z$

## 14.9 Quotient set

Let  $X$  be a set and  $\sim$  be an equivalence relation, the mapping  $X \rightarrow X/\sim$ :  
 $(x \in X) \mapsto [x]$  is called the projection mapping.

$X/\sim$  is called the quotient set of  $X$  by equivalence relation  $\sim$

### 14.9.1 Example

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . Then the mapping

$$H \times G \rightarrow G$$

$$(h, g) \mapsto hg / (h, g) \mapsto gh$$

is a left/right action of  $H$  on  $G$ . Thus we obtain two quotient sets  $H/G$  and  $G/H$

## 14.10 Def

Let  $G$  be a group and  $H$  be a subgroup of  $G$ . If  $\forall g \in G, h \in H \quad ghg^{-1} \in H$ ,  
 Then we say that  $H$  is a normal subgroup of  $G$

## 14.11 Remark

$\forall g \in G, gH = Hg$ , provided that  $H$  is a normal subgroup of  $G$ . In fact  $\forall h \in$ ,

- $\exists h' \in H$  such that  $ghg^{-1} = h'$  Hence  $gh = h'g$ . This shows  $gH \subseteq Hg$
- $\exists h'' \in H$  such that  $g^{-1}hg = h''$  Hence  $hg = gh''$ . This shows  $Hg \subseteq gH$

Thus  $gH = Hg$

## 14.12 Prop

If  $G$  is commutative, any subgroup of  $G$  is normal

### 14.13 Theorem

Let  $G$  be a group and  $H$  be a normal subgroup of  $G$ . Then the mapping

$$G/H \times H/G \rightarrow G/H$$

$$(xH, Hx) \mapsto (xy)H$$

is well defined and determine a structure of group of quotient set  $G/H$ . Moreover the projection mapping

$$\pi : G \rightarrow G/H$$

$$x \mapsto xH$$

is a morphism of groups.

#### Proof

- If  $xH = x'H, yH = y'H$  then  $\exists h_1 \in H, h_2 \in H$  such that  $x' = xh_1, y' = yh_2$ . Hence  $x'y' = xh_1yh_2 = (xy)(y^{-1}h_1y)h_2$ . For  $y^{-1}h_1y, h_2 \in H$  then  $(x'y')H = (xy)H$ . So the mapping is well defined.
- $\forall (x, y, x) \in G^3 \quad (xH)(yH \cdot zH) = xH((yx)H) = (x(yz)H) = ((xy)z)H = ((xy)H)zH = (xH \cdot yH)zH$
- $\forall x \in G \quad 1H \cdot xH = xH \cdot 1H = xH \quad x^{-1}HxH = xHx^{-1}H = 1H$
- $\pi(xy) = (xy)H = xH \cdot yH = \pi(x)\pi(y)$

### 14.14 Def

Let  $K$  be a unitary ring and  $E$  be a left  $K$ -module. We say that a subgroup  $F$  of  $(E, +)$  is a left sub- $K$ -module of  $E$  if  $\forall (a, x) \in K \times F, ax \in F$

### 14.15 Prop

Let  $K$  be a unitary ring,  $E$  be a left  $K$ -module and  $F$  be a sub- $K$ -module. Then the mapping

$$K \times (E/F) \rightarrow E/F$$

$$(a, [x]) \mapsto [ax]$$

is well defined, and defines a left- $K$ -module structure on  $E/F$ . Moreover, the projection mapping  $\pi : E \rightarrow E/F$  is a morphism of left- $K$ -modules

**Proof**

Let  $x$  and  $x'$  be elements of  $E$  such that  $[x] = [x']$ , that means:  $x' - x \in F$   
Hence  $a(x' - x) = ax' - ax \in F$  So  $[ax] = [ax']$   
Let us check that  $E/F$  forms a left  $K$ -module.

- $a([x] + [y]) = a([x + y]) = [a(x + y)] = [ax + ay] = [ax] + [ay]$
- $(a + b)[x] = [(a + b)x] = [ax + bx] = [ax] + [bx]$
- $1[x] = [1x] = [x]$
- $a(b[x]) = a[bx] = [a(bx)] = [(ab)x] = (ab)[x]$

By the provided proposition,  $\pi$  is a morphism of groups. Moreover  $\forall x \in E, a \in K$   $\pi(ax) = [ax] = a[x] = a\pi(x)$

**14.16 Def**

Let  $A$  be a unitary ring . We call two-sided ideal any subgroup  $I$  of  $(A, +)$  that satisfies :  $\forall x \in I, a \in A \quad \{ax, xa\} \subseteq I$  ( $I$  is a left and right sub- $K$ -module of  $A$ )

**14.17 Theorem**

Let  $A$  be a unitary ring and  $I$  be a two sided ideal of  $A$  . The mapping

$$(A/I) \times (A/I) \rightarrow A/I$$

$$([a], [b]) \mapsto [ab]$$

is well defined. Moreover ,  $A/I$  becomes a unitary ring under the addition and this composition law, and the projection mapping

$$A \xrightarrow{\pi} A/I$$

is a morphism of unitary ring (if is a morphism of additive groups and multiplicative monoids, namely  $\pi(a + b) = \pi(a) + \pi(b), \pi(ab) = \pi(a)\pi(b), \pi(1) = 1$ )

**Proof**

If  $a' \sim a, b' \sim b$  that means  $a' - a \in I, b' - b \in I$  then  $a'b' - ab = a'b' - a'b + a'b - ab = a'(b' - b) + (a' - a)b$ . For  $(a' - a), (b' - b) \in I$ , then  $a'b' - ab \in I$   
Therefore  $a'b' \sim ab$

### 14.17.1 Reside Class

Let  $d \in \mathbb{Z}$  and  $d\mathbb{Z} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}, n = dm\}$   $d\mathbb{Z}$  is a two sided ideal of  $\mathbb{Z}$   
 If  $m \in \mathbb{Z}$ , for any  $a \in \mathbb{Z}$   $adm = dma \in d\mathbb{Z}$

Denote by  $\mathbb{Z}/d\mathbb{Z}$  the quotient ring. The class of  $n \in \mathbb{Z}$  in  $\mathbb{Z}/d\mathbb{Z}$  is called the residue class of  $n$  modulo  $d$

If  $A$  is a commutative unitary ring, a two sided ideal of  $A$  is simply called an ideal of  $A$

### 14.18 Theorem

Let  $f : G \rightarrow H$  be a morphism of groups

- (1)  $Im(f)$  is a subgroup of  $H$
- (2)  $\ker(f) := \{x \in G \mid f(x) = 1_H\}$  is a normal subgroup of  $G$
- (3) The mapping

$$\begin{aligned} \tilde{f} : G/Ker(f) &\rightarrow Im(f) \\ [x] &\mapsto f(x) \end{aligned}$$

is well defined and is an isomorphism of groups

- (4)  $f$  is injective iff  $\ker(f) = \{1_G\}$

### Proof

- (1) Let  $\alpha$  and  $\beta$  be elements of  $Im(f)$ . Let  $(x, y) \in G^2$  such that  $\alpha = f(x), \beta = f(y)$  Then  $\alpha\beta^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$  So  $Im(f)$  is a subgroup
- (2) Let  $x$  and  $y$  be elements of  $\ker(f)$ .  
 One has  $f(xy^{-1}) = f(x)f(y)^{-1} = 1_H 1_H^{-1} = 1_H$   
 So  $xy^{-1} \in \ker f$ . Hence  $\ker f$  is a subgroup of  $G$   
 Let  $x \in \ker f, y \in G$ .  
 One has  $f(yxy^{-1}) = f(y)f(x)f(y)^{-1} = f(y)f(y)^{-1} = 1_H$  Hence  $yxy^{-1} \in \ker f$ . So  $\ker f$  is a normal subgroup
- (3) If  $x \sim y$  then  $\exists z \in \ker f$  such that  $y = xz$  Hence  $f(y) = f(x)f(z) = f(x)1_H = f(x)$  So  $f$  is well defined.  
 Moreover  $\tilde{f}([x][y]) = \tilde{f}([xy]) = f(xy) = f(x)f(y) = f([x])f([y])$  Hence  $\tilde{f}$  is a morphism of groups.  
 By definition  $Im(\tilde{f}) = Im(f)$  If  $x$  and  $y$  are elements of  $G$  such that  $f(x) = f(y)$  then  $f(xy^{-1}) = 1_H$   
 Hence  $xy^{-1} \in \ker f$  Since  $x = (xy^{-1})y$ ,  $x \sim y$  that means  $[x] = [y]$   
 Therefore  $\tilde{f}$  is injective.



- (4) If  $f$  is injective,  $\forall x \in \ker f \quad f(x) = 1_H = f(1_G)$ , so  $x = 1_G$ . Therefore  $\ker f = \{1_G\}$ .  
 Conversely, suppose that  $\ker f = \{1_G\} \quad \forall (x, y) \in G^2$  if  $f(x) = f(y)$  then  $f(x)f(y)^{-1} = 1_H$ . Hence  $xy^{-1} = 1_G, x = y$ .

## 14.19 Theorem

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Then

- (1)  $\text{Im}(f)$  is a left-sub- $K$ -module of  $F$
- (2)  $\ker(f)$  is a left-sub- $K$ -module of  $E$
- (3)  $\tilde{f} : E/\ker f \rightarrow \text{Im}(f)$  is a isomorphism of left  $K$ -modules  
 $[x] \mapsto f(x)$

### Proof

- (1)  $\forall x \in E, \quad f(ax) = af(x)$  So  $af(x) \in \text{Im}(f)$
- (2)
- (3)



# Chapter 15

## Topology

### 15.1 Def

Let  $X$  be a set. We call topology on  $X$  any subset  $\mathcal{G}$  of  $\wp(X)$  that satisfies:

- $\emptyset \in \mathcal{G}$  and  $X \in \mathcal{G}$
- If  $(u_i)_{i \in I}$  is an arbitrary family of elements in  $\mathcal{G}$ , then  $\bigcup_{i \in I} u_i \in \mathcal{G}$
- If  $u$  and  $v$  are elements of  $\mathcal{G}$ , then  $u \cap v \in \mathcal{G}$

### 15.2 Remark

If  $(u_i)_{i=1}^n$  is a finite family of elements of  $\mathcal{G}$ , then  $\bigcap_{i=1}^n u_i \in \mathcal{G}$  (by induction, this follows from (3))

#### 15.2.1 Example

$\{\emptyset, X\}$  is a topology. call the trivial topology on  $\wp(X)$  is a topology called the discrete topology.

### 15.3 Def

Let  $X$  be a set. We call metric on  $X$  any mapping  $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ , that satisfies

- $d(x, y) = 0$  iff  $x=y$
- $\forall (x, y) \in X^2, d(x, y) = d(y, x)$
- $\forall (x, y, z) \in X^3 \quad d(x, z) \leq d(x, y) + d(y, z)$  (triangle inequality)

$(X, d)$  is called a metric space

### 15.3.1 Example

Let  $X$  be a set

$$d : X^2 \rightarrow \mathbb{R}_{\geq 0}$$

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric

## 15.4 Def

Let  $(X, d)$  be a metric space. For any  $x \in X, \epsilon \in \mathbb{R}_{\geq 0}$ , let  $B(x, \epsilon) := \{y \in X \mid d(x, y) < \epsilon\}$ . We call the open ball of radius  $\epsilon$  centered at  $x$

### 15.4.1 Example

Consider  $(\mathbb{R}, d)$  with  $d(x, y) = |x - y|$ , then  $B(x, \epsilon) = ]x - \epsilon, x + \epsilon[$

## 15.5 Prop.

Let  $(X, d)$  be a metric space. Let  $\mathcal{G}_d$  be the set of  $U \subseteq X$  such that  $\forall x \in U \exists \epsilon > 0 \ B(x, \epsilon) \subseteq U$ . Then  $\mathcal{G}_d$  is a topology on  $X$ .

### Proof

- $\emptyset \in \mathcal{G}_d \quad X \in \mathcal{G}_d$
- Let  $(u_i)_{i \in I}$  be a family of elements of  $\mathcal{G}_d$ . Let  $U = \bigcup_{i \in I} u_i, \forall x \in U, \exists i \in I$  such that  $x \in u_i$ . Since  $u_i \in \mathcal{G}_d, \exists \epsilon > 0$  such that  $B(x, \epsilon) \subseteq u_i \subseteq U$ . Hence  $U \in \mathcal{G}_d$ .
- Let  $U$  and  $V$  be elements of  $\mathcal{G}_d$ . Let  $x \in U \cap V, \exists a, b \in \mathbb{R}_{\geq 0}$  such that  $B(x, a) \subseteq U, B(x, b) \subseteq V$ . Taking  $\epsilon = \min\{a, b\}$ , Then  $B(x, \epsilon) = B(x, a) \cap B(x, b) \subseteq U \cap V$ . Therefore  $U \cap V \in \mathcal{G}_d$ .

## 15.6 Def

$\mathcal{G}_d$  is called the topology induced by the metric  $d$

## 15.7 Def

We call topology space any pair  $(X, \mathcal{G})$  where  $X$  is a set and  $\mathcal{G}$  is a topology on  $X$

Given a topological space  $(X, \mathcal{G})$  If  $U \in \mathcal{G}$  then we say that  $U$  is an open subset of  $X$ . If  $F \in \wp(X)$  such that  $X \setminus F \in \mathcal{G}$ , then we say that  $F$  is closed subset of  $X$

If there exists  $d$  a metric on  $X$  such that  $\mathcal{G} = \mathcal{G}_d$  then we say that  $\mathcal{G}$  is metrizable

### 15.7.1 Example

Let  $X$  be a set . The discrete topology on  $X$  is metrizable. In fact, if  $d$  denote the metric defined as  $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$   
 $\forall x \in X \quad B(x, 1) = \{x\}$  So  $\{x\} \in \mathcal{G}_d$  Hence  $\forall A \subseteq X \quad A = \bigcup_{x \in A} \{x\} \in \mathcal{G}_d$



# Chapter 16

## Filter

### 16.1 Def

Let  $X$  be a set. We call filter if  $\mathcal{F} \subseteq \wp(X)$  that satisfies:

- (1)  $\mathcal{F} \neq \emptyset, \emptyset \notin \mathcal{F}$
- (2)  $\forall A \in \mathcal{F}, \forall B \in \wp(X), \text{ if } A \subseteq B, \text{ then } B \in \mathcal{F}$
- (3)  $\forall (A, B) \in \mathcal{F} \times \mathcal{F}, A \cap B \in \mathcal{F}$

#### 16.1.1 Example

- (1) Let  $Y \subseteq X, Y \neq \emptyset$ .  $\mathcal{F}_Y := \{A \in \wp(X) \mid Y \subseteq A\}$  is a filter, called the principal filter of  $Y$ .
- (2) Let  $X$  be an infinite set.

$$\mathcal{F}_{Fr}(X) := \{A \in \wp(X) \mid X \setminus A \text{ is infinite}\}$$

is a filter called the Fréchet filter of  $X$ .

- (3) Let  $(X, \mathcal{G})$  be a topological space,  $x \in X$ . We call neighborhood of  $x$  any  $V \in \wp(X)$  such that  $\exists U \in \mathcal{G}$ , satisfying  $x \in U \subseteq V$ . Then  $\mathcal{V} = \{\text{neighborhoods of } x\}$  is a filter.

### 16.2 Def: Filter Basis

Let  $X$  be a set.  $\mathcal{B} \subseteq \wp(X)$ . If  $\emptyset \notin \mathcal{B}$  and  $\forall (B_1, B_2) \in \mathcal{B}^2, \exists B \in \mathcal{B}$ , such that  $B \subseteq B_1 \cap B_2$ . We say that  $\mathcal{B}$  is a filter basis.

#### 16.2.1 Remark

If  $\mathcal{B}$  is a filter basis, then  $\mathcal{F}(\mathcal{B}) = \{A \subseteq X \mid \exists B \in \mathcal{B} \quad B \subseteq A\}$  is a filter

**Proof**

$\emptyset \notin \mathcal{F}(\mathcal{B}), \mathcal{F}(\mathcal{B}) \neq \emptyset$  since  $0 \neq B \subseteq \mathcal{F}(\mathcal{B})$ . If  $A \in \mathcal{F}(\mathcal{B}), A' \in \wp(X)$  such that  $A \subseteq A'$ , then  $\exists B \in \mathcal{B}$  such that  $B \subseteq A \subseteq A'$ . Hence  $A' \in \mathcal{F}(\mathcal{B})$ . If  $A_1, A_2 \in \mathcal{F}(\mathcal{B})$ , then  $\exists(B_1, B_2) \in \mathcal{B}^2$  such that  $B_1 \subseteq A_1, B_2 \subseteq A_2$ . Since  $\mathcal{B}$  is a filter basis,  $\exists B \in \mathcal{B}$  such that  $B \subseteq B_1 \cap B_2 \subseteq A_1 \cap A_2$ . Hence  $A_1 \cap A_2 \in \mathcal{F}(\mathcal{B})$ .

**16.2.2 Example**

- Let  $Y \subseteq X, Y \neq \emptyset$   
 $\mathcal{B} = \{Y\}$  is a filter basis.  $\mathcal{F}(\mathcal{B}) = \mathcal{F}_Y = \{A \subseteq X \mid Y \subseteq A\}$
- Let  $(X, \mathcal{G})$  be a topological space  $x \in X$ . If  $\mathcal{B}_x$  is a filter basis such that  $\mathcal{F}(\mathcal{B}) = \mathcal{V}_x = \{\text{neighborhood of } x\}$ , then we say that  $\mathcal{B}_x$  is a neighborhood basis of  $x$ .

**16.3 Remark**

Let  $\mathcal{B}_x$  is a neighborhood basis of  $x$  iff

- $\mathcal{B}_x \subseteq \mathcal{V}_x$
- $\forall V \in \mathcal{V}_x \quad \exists U \in \mathcal{B}_x$  such that  $U \subseteq V$
- Let  $(X, d)$  be a metric space,  $x \in X \forall \epsilon > 0$ , Let

$$B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$$

$$\overline{B}(x, \epsilon) = \{y \in X \mid d(x, y) \leq \epsilon\}$$

Then

- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$
- $\{B(x, \epsilon) \mid \epsilon > 0\}$  is a neighborhood basis of  $x$
- $\{\overline{B}(x, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  is a neighborhood basis of  $x$

**16.3.1 Example**

$\mathcal{V}_x \cap \mathcal{G}$  is a neighborhood basis of  $x$

**16.4 Def**

$V \in \wp(X)$  is called a neighborhood of  $x$  if  $\exists U \in \mathcal{G}$  such that  $x \in U \subseteq V$



## 16.5 Remark

Let  $(X, \mathcal{G})$  be a topological space,  $x \in X$  and  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Suppose that  $\mathcal{B}$  is countable. We choose a surjective mapping  $(B_n)_{n \in \mathbb{N}}$  from  $\mathbb{N}$  to  $\mathcal{B}_x$ . For any  $n \in \mathbb{N}$ , let  $A_n = B_0 \cap B_1 \cap \dots \cap B_n \in \mathcal{V}_x$ . The sequence  $(A_n)_{n \in \mathbb{N}}$  is decreasing and  $\{A_n \mid n \in \mathbb{N}\}$  is a neighborhood basis of  $x$ .

## 16.6 Extra Episode

$\wp(\mathbb{N})$  is NOT countable

Suppose that  $f : \wp(\mathbb{N}) \rightarrow \mathbb{N}$  is injective. Then  $\exists g : \mathbb{N} \rightarrow \wp(\mathbb{N})$  surjective. Taking  $A = \{n \in \mathbb{N} \mid n \notin g(n)\}$ . Since  $g$  is surjective,  $\exists a \in \mathbb{N}$  such that  $A = g(a)$ .

If  $a \in A$ , then  $a \in g(a)$ , hence  $a \notin A$

If  $a \notin A$ , then  $a \in g(a) = A$

Contradiction

## 16.7 Prop.

Let  $Y$  and  $E$  be sets,  $g : Y \rightarrow E$  be a mapping,

- If  $\mathcal{F}$  is a filter of  $Y$ , then

$$g_*(\mathcal{F}) := \{A \in \wp(E) \mid g^{-1}(A) \in \mathcal{F}\}$$

is a filter on  $E$

- If  $\mathcal{B}$  is a filter basis of  $Y$ , then

$$g(\mathcal{B}) := \{g(B) \mid B \in \mathcal{B}\}$$

is a filter of  $E$ , and  $\mathcal{F}(g(\mathcal{B})) = g_*(\mathcal{F}(\mathcal{B}))$

### Proof

- (1)  $E \in g_*(\mathcal{F})$  since  $g^{-1}(E) = Y$   
 $\emptyset \notin g_*(\mathcal{F})$  since  $g^{-1}(\emptyset) = \emptyset$

If  $A \in g_*(\mathcal{F})$  and  $A' \supseteq A$ , then  $g^{-1}(A') \supseteq g^{-1}(A) \in \mathcal{F}$ , so  $g^{-1}(A') \in \mathcal{F}$ ,  
Hence  $A' \in g_*(\mathcal{F})$

If  $A_1, A_2 \in g_*(\mathcal{F})$ . Then  $g^{-1}(A_1) \in \mathcal{F}, g^{-1}(A_2) \in \mathcal{F}$ . Hence  $g^{-1}(A_1 \cap A_2) = g^{-1}(A_1) \cap g^{-1}(A_2) \in \mathcal{F}$ . So  $A_1 \cap A_2 \in g_*(\mathcal{F})$ .

- (2) Since  $g$  is a mapping, and  $\emptyset \notin \mathcal{B}$ , we get  $\emptyset \notin g(\mathcal{B})$ , since  $\mathcal{B} \neq \emptyset, g(\mathcal{B}) \neq \emptyset$ .

Let  $B_1, B_2 \in \mathcal{B}$ , there exists  $C \in \mathcal{B}$  such that  $C \subseteq B_1 \cap B_2$ . Hence  $g(C) \subseteq g(B_1) \cap g(B_2)$ , namely  $g(\mathcal{B})$  is a filter basis.



## Chapter 17

# Limit point and accumulation point

We fix a topological space  $(X, \mathcal{G})$

### 17.1 Def

Let  $\mathcal{F}$  be a filter of  $X$  and  $x \in X$

- If  $\mathcal{V}_x \subseteq \mathcal{F}$  then we say that  $x$  is a limit point of  $\mathcal{F}$
- If  $\forall (A, V) \in \mathcal{F} \times \mathcal{V}_x, A \cap V \neq \emptyset$ , we say that  $x$  is an accumulation point of  $\mathcal{F}$

So any limit point of  $\mathcal{F}$  is necessarily a accumulation point of  $\mathcal{F}$

### 17.2 Prop

Let  $\mathcal{B}$  be a filter basis of  $X$ ,  $x \in X$ ,  $\mathcal{B}_x$  a neighborhood basis of  $x$ . Then  $x$  is an accumulation point of  $\mathcal{F}(\mathcal{B})$  iff  $\forall (B, U) \in \mathcal{B} \times \mathcal{B}_x, B \cap U \neq \emptyset$

#### Proof

##### Necessity

Since  $\mathcal{B} \subseteq \mathcal{F}(\mathcal{B})$ ,  $\mathcal{B} \subseteq \mathcal{V}_x$ , the necessity is true.

##### Sufficiency

Let  $(A, V) \in \mathcal{F}(\mathcal{B}) \times \mathcal{V}_x$ . There exist  $B \in \mathcal{B}, U \in \mathcal{B}_x$ , such that  $B \subseteq A, U \subseteq V$ . Hence  $\emptyset \neq B \cap U \subseteq A \cap V$

### 17.3 Def

Let  $Y \subseteq X, Y \neq \emptyset$ . We call accumulation point of  $Y$  any accumulation point of the principal filter  $\mathcal{F} = \{A \subseteq X \mid Y \subseteq A\}$ .

### 17.4 Def

We denote by  $\overline{Y} = \{\text{accumulation points of } Y\}$ , called the closure of  $Y$ . Note that  $x \in \overline{Y}$  iff  $\forall U \in \mathcal{B}_x, Y \cap U \neq \emptyset$

By convention  $\overline{\emptyset} = \emptyset$

### 17.5 Prop

Let  $Y \subseteq X$ . Then  $\overline{Y}$  is the smallest closed subset of  $X$  containing  $Y$ .

#### Proof

$\forall x \in X \setminus \overline{Y}$ , then there exists  $U_x = \mathcal{V} \cap \mathcal{G}$ , such that  $Y \cap U_x = \emptyset$ . Moreover,  $\forall y \in U_x, U_x \in \mathcal{V}_y \cap \mathcal{G}$ . This shows that  $\forall y \in U_x, y \notin \overline{Y}$ . Therefore  $X \setminus \overline{Y} = \bigcap_{x \in X \setminus \overline{Y}} U_x \in \mathcal{G}$

Let  $Z \subseteq X$  be a closed subset that contain  $Y$ . Suppose that  $\exists y \in \overline{Y} \setminus Z$ . Then  $U = X \setminus Z \in \mathcal{V}_y \cap \mathcal{G}$  and  $U \cap Y \subseteq U \cap Z = \emptyset$ . So  $y \notin \overline{Y}$  contradiction. Hence  $\overline{Y} \subseteq Z$ .

### 17.6 Def: dense

Let  $(X, \mathcal{G})$  be a topological space,  $Y$  a subset of  $X$ . We call  $Y$  is dense in  $X$  if

$$\overline{Y} = X$$

## Chapter 18

# Limit of mappings

### 18.1 Def

Let  $(E, \mathcal{G}_E)$  be a topological space.  $f : Y \rightarrow E$  a mapping, and  $\mathcal{F}$  be a filter of  $Y$ . If  $a \in E$  is a limit point of  $F_*(\mathcal{F})$  namely,  $\forall$  neighborhood  $V$  of  $a$ ,  $f^{-1}(V) \in \mathcal{F}$ , then we say that  $a$  is a limit of the filter  $\mathcal{F}$  by  $f$ .

### 18.2 Remark

Let  $\mathcal{B}_a$  be a neighborhood basis of  $a$ . Then  $\mathcal{V}_a \subseteq f_*(\mathcal{F})$ , iff  $\mathcal{B} \subseteq f_*(\mathcal{F})$ . Therefore,  $a$  is a limit of  $\mathcal{F}$  by  $f$  iff  $\forall V \in \mathcal{B}_a, f^{-1}(V) \in \mathcal{F}$ .

#### 18.2.1 Example

Let  $(E, \mathcal{G}_E)$  be a topological space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $x = (x_n)_{n \in I} \in E^I$ . If the Fréchet filter  $\mathcal{F}_{Fr}(I)$  has a limit  $a \in E$  by the mapping  $x : I \rightarrow E$ , we say that  $(x_n)_{n \in I}$  converges to  $a$ , denote as

$$a = \lim_{n \in I, n \rightarrow +\infty} x_n$$

### 18.3 Remark

$a = \lim_{n \in I, n \rightarrow +\infty} x_n$  iff,  $\forall U \in \mathcal{B}_a$  (where  $\mathcal{B}_a$  is a neighborhood basis of  $a$ ),  $\exists N \in \mathbb{N}$  such that  $x_n \in U$  for any  $n \in I_{\geq N}$ .

Suppose that  $\mathcal{G}_E$  is induced by a metric  $d$ .  $\{B(a, \epsilon) \mid \epsilon > 0\}, \{\overline{B}(a, \epsilon) \mid \epsilon > 0\}, \{B(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}, \{\overline{B}(a, \frac{1}{n}) \mid n \in \mathbb{N}_{\geq 1}\}$  are all neighborhood basis of  $a$ . Therefore, the following are equivalent

- $a = \lim_{n \in I, n \rightarrow +\infty} x_n$
- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \epsilon$

- $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \epsilon$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) < \frac{1}{n}$
  - $\forall k \in \mathbb{N}_{\geq 1}, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, a) \leq \frac{1}{n}$
- $(x^{-1}(B(a, \epsilon)) = \{n \in I \mid d(x_n, a) < \epsilon\})$ ? unknown position )

## 18.4 Remark

We consider the metric  $d$  on  $\mathbb{R}$  defined as

$$\forall (x, y) \in \mathbb{R}^2 \quad d(x, y) := |x - y|$$

The topology of  $\mathbb{R}$  defined by this metric is called the usual topology on  $\mathbb{R}$

## 18.5 Prop

Let  $(x_n)_{n \in I} \in \mathbb{R}^I$ , where  $I \subseteq \mathbb{N}$  is an infinite subset. Let  $l \in \mathbb{R}$ . The following statements are equivalent:

- The sequence  $(x_n)_{n \in I}$  converges to  $l$  in the topological space  $\mathbb{R}$
- $\liminf_{n \in I, n \rightarrow +\infty} x_n = \limsup_{n \in I, n \rightarrow +\infty} x_n = l$
- $\limsup_{n \in I, n \rightarrow +\infty} |x_n - l| = 0$

## 18.6 Theorem

Let  $(X, d)$  be a metric space. Let  $I \subseteq \mathbb{N}$  be an infinite subset and  $(x_n)_{n \in I}$  be an element of  $X^I$ . Let  $l \in X$ . The following statements are equivalent:

- $(x_n)_{n \in I}$  converges to  $l$
- $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$  (equivalent to  $\lim_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ )

### Proof

- (1)  $\Rightarrow$  (2) The condition (1) is equivalent to  $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \in I_{\geq N}, d(x_n, l) < \epsilon$ .  
 We then get  $\sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ . Therefore  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) < \epsilon$ . We obtain that  $\limsup_{n \in I, n \rightarrow +\infty} d(x_n, l) = 0$ .
- (2)  $\Rightarrow$  (1) Let  $\epsilon \in \mathbb{R}_{>0}$ . If  $\inf_{N \in \mathbb{N}} \sup_{n \in I_{\geq N}} d(x_n, l) = 0$ . Then  $\exists N \in \mathbb{N} \quad \sup_{n \in I_{\geq N}} d(x_n, l) < \epsilon$ .  
 Hence,  $\forall n \in I_{\geq N} d(x_n, l) < \epsilon$ . Since  $\epsilon$  is arbitrary, (\*) is true, Hence (1) is also true.

## 18.7 Prop

Let  $(X, \mathcal{G})$  be a topological space .  $Y \subseteq X, p \in \overline{Y} \setminus Y$ . Then

$$\mathcal{V}_{p,Y} := \{V \cap Y \mid V \in \mathcal{V}_p\}$$

is a filter of  $Y$ .

### Proof

$Y$  is not empty otherwise  $\overline{Y} = \emptyset$ .

- $Y = X \cap Y \in \mathcal{V}_{p,Y}$   
 $\emptyset \notin \mathcal{V}_{p,Y}$  since  $p \in \overline{Y}$
- Let  $V \in \mathcal{V}_p$  and  $A \subseteq Y$  such that  $V \cap Y \subseteq A$ . Let  $U = V \cup (A \setminus (V \cap Y)) \in \mathcal{V}_p$   
and  $U \cap Y = A \in \mathcal{V}_{p,Y}$
- Let  $U$  and  $V$  be elements of  $\mathcal{V}_p$  Let  $W = U \cap V \in \mathcal{V}_p$  Then  $W \cap Y = (U \cap Y) \cap (V \cap Y) \in \mathcal{V}_{p,Y}$

## 18.8 Def

Let  $(X, \mathcal{G}_x)$  and  $(E, \mathcal{G}_E)$  be topological spaces,  $Y \subseteq X, p \in \overline{Y} \setminus Y$ , and  $f : Y \rightarrow E$  be a mapping . If  $a$  is a limit point of  $(F_*(\mathcal{V}_{p,Y}))$ , then we say that  $a$  is a limit of  $f$  when the variable  $y \in Y$  tends to  $p$ , denoted as  $a = \lim_{y \in Y, y \rightarrow p} f(y)$

## 18.9 Remark

If  $\mathcal{B}_a$  is a neighborhood basis of  $a$ . Then  $a = \lim_{y \in Y, y \rightarrow p} f(y)$  is equivalent to  
 $\forall U \in \mathcal{B}_a \quad \exists V \in \mathcal{V}_p$  such that  $Y \cap V \subseteq f^{-1}(U) (\Leftrightarrow f(Y \cap V) \subseteq U)$

## 18.10 Prop

Let  $X$  be a set,  $\mathcal{B}$  be a filter basis,  $\mathcal{G}$  be a filter. If  $\mathcal{B} \subseteq \mathcal{G}$ , then  $\mathcal{F} \subseteq \mathcal{G}$ .

### Proof

Let  $V \in \mathcal{F}(\mathcal{B})$  By definition  $\exists U \in \mathcal{B}$  such that  $U \subseteq V$ , since  $U \in \mathcal{G}$  ( for  $\mathcal{B} \subseteq \mathcal{G}$ ) and since  $\mathcal{G}$  is a filter,  $V \in \mathcal{G}$

### 18.11 Theorem

Let  $(X, \mathcal{G}_x)$  and  $(E, \mathcal{G}_E)$  be topological spaces.  $Y \subseteq X$ ,  $p \in \overline{T} \setminus Y$ ,  $a \in E$ . We consider the following conditions.

- (i)  $a = \lim_{y \in Y, y \rightarrow p} f(y)$
- (ii)  $\forall (y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  if  $\lim_{n \rightarrow +\infty} y_n = p$  then  $\lim_{n \rightarrow \infty} f(y_n) = a$

The following statements are true

- If (i) holds, then (ii) also holds
- Assume that  $p$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

#### Proof

- (1) Let  $(y_n)_{n \in \mathbb{N}} \in Y^{\mathbb{N}}$  such that  $p = \lim_{n \rightarrow +\infty} y_n$ . For any  $U \in \mathcal{V}_p$ ,  $\exists N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}$   $y_n \in U \cap Y$ . Therefore

$$\mathcal{V}_{p,Y} \subseteq y_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

We then get

$$f_*(\mathcal{V}_{p,Y}) \subseteq f_*(y_*(\mathcal{F}_{Fr}(\mathbb{N}))) = (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

Condition (i) leads

$$\mathcal{V}_a \subseteq f_*(\mathcal{V}_{p,Y}) \subseteq (f \circ y)_*(\mathcal{F}_{Fr}(\mathbb{N}))$$

This means

$$\lim_{n \rightarrow +\infty} f(y_n) = a$$

- (2) Assume that  $p$  has a countable neighborhood basis. There exists a decreasing sequence  $(V_n)_{n \in \mathbb{N}} \in \mathcal{V}_p^{\mathbb{N}}$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $p$ .

Assume that (i) does not hold. Then there exists  $U \in \mathcal{V}_a$  such that ,

$$\forall n \in \mathbb{N} \quad V_n \cap Y \not\subseteq f^{-1}(U)$$

Take an arbitrary

$$y_n \in (V_n \cap Y) \setminus f^{-1}(U)$$

Therefore ,

$$\lim_{n \rightarrow +\infty} y_n = \emptyset$$

In fact,

$$\forall W \in \mathcal{V}_p, \exists N \in \mathbb{N} \quad V_N \subseteq W$$

Hence

$$\forall n \in \mathbb{N}_{\geq N} \quad y_n \in W$$

However  $f(y_n) \notin U$  for any  $n \in \mathbb{N}$ , so  $(f(y_n))_{n \in \mathbb{N}}$  cannot converges to  $a$ .



## 18.12 Prop.

Let  $X$  be a set. If  $(\mathcal{G}_i)_{i \in I}$  is a family of topologies on  $X$ , then  $\mathcal{G} = \bigcap_{i \in I} \mathcal{G}_i$  is a topology. In particular, for any  $\mathcal{A} \subseteq \wp(X)$ , there is a smallest topology on  $X$  that contains  $\mathcal{A}$ .

### 18.12.1 Proof

- $\forall i \in I \quad \{\emptyset, X\} \subseteq \mathcal{G}_i$  So  $\{\emptyset, X\} \subseteq \mathcal{G}$
- Let  $(u_j)_{j \in J}$  be a family of elements of  $\mathcal{G} \quad \forall j \in J, i \in I \quad u_j \in \mathcal{G}_i$  So  $\bigcup_{j \in J} u_j \in \mathcal{G}_i$  We then get  $\bigcup_{j \in J} u_j \in \mathcal{G}$
- Let  $U$  and  $V$  be elements of  $\mathcal{G} \quad \forall i \in I, \{u, v\} \subseteq \mathcal{G}_i$  So  $U \cap V \in \mathcal{G}_i$ . Therefore we get  $U \cap V \in \mathcal{G}$  Let  $\mathcal{A} \subseteq \wp(X)$  Let  $\mathcal{G}(\mathcal{A}) = \bigcap_{\mathcal{G} \subseteq \wp(X) \text{ a topology } \mathcal{A} \subseteq \mathcal{G}} \mathcal{G}$  Then  $\mathcal{G}(\mathcal{A})$  is a topology. By definition, if  $\mathcal{G}$  is a topology containing  $\mathcal{A}$ , then  $\mathcal{G}(\mathcal{A}) \subseteq \mathcal{G}$  Hence  $\mathcal{G}(\mathcal{A})$  is the smallest topology containing  $\mathcal{A}$



## Chapter 19

# Continuity

### 19.1 Def

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces  $f$  be a function from  $X$  to  $Y$ ,  $x \in \text{Dom}(f)$ . If for any neighborhood  $U$  of  $f(x)$ , there exists a neighborhood  $V$  of  $x$  such that  $f(V) \subseteq U$ . Then we say that  $f$  is continuous at  $x$ . If  $f$  is continuous at any  $x \in \text{Dom}(f)$  then we say  $f$  is continuous.

### 19.2 Remark

Let  $\mathcal{B}_{f(x)}$  be a neighborhood basis of  $f(x)$  If  $\forall U \in \mathcal{B}_{f(x)}$  there exist  $V \in \mathcal{B}_{f(x)}$  such that  $f(V) \subseteq U$ , then  $f$  is continuous at  $x$  Suppose that  $X$  and  $Y$  are metric space. Then  $f$  is continuous at  $x$  iff:

$$\forall \epsilon > 0 \exists \delta > 0 \forall y \in \text{Dom}(f) \quad d(y, x) < \delta \text{ implies } d(f(y), f(x)) < \epsilon$$

### 19.3 Theorem

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces,  $f$  be a function from  $X$  to  $Y$   $x \in \text{Dom}(f)$  Consider the following condition

- $f$  is continuous at  $x$
- $\forall (x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$ , if  $\lim_{n \rightarrow +\infty} x_n = x$ , then  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$  THEN  
(i) implies (ii) Moreover, if  $x$  has a countable neighborhood basis, then (i) and (ii) are equivalent.

### 19.4 Proof

(i)  $\Rightarrow$  (ii) Let  $(x_n)_{n \in \mathbb{N}} \in \text{Dom}(f)^{\mathbb{N}}$  that converges to  $x \forall U \in \mathcal{V}_{f(x)} \exists V \in \mathcal{V}_x, f(V) \subseteq U$  Since  $\lim_{n \rightarrow +\infty} x_n = x$ , there exists  $N \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}_{\geq N}, x_n \in V$ .

Hence  $\forall n \in \mathbb{N}_{\geq N}, f(x_n) \in f(V) \subseteq U$ . Thus  $\lim_{n \rightarrow +\infty} f(x_n) = f(x)$

(ii)  $\Rightarrow$  (i) under the hypothesis that  $x$  has countable neighborhood basis. actually we will prove  $NOT(i) \Rightarrow NOT(ii)$

Let  $(V_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $\mathcal{V}_x$  such that  $\{V_n \mid n \in \mathbb{N}\}$  forms a neighborhood basis of  $x$

If (i) does not hold, then  $\exists U \in \mathcal{V}_{f(x)} \forall n \in \mathbb{N}, f(V_n) \not\subseteq U$  Pick  $x_n \in V_n$  such that  $f(x_n) \notin U \quad \forall N \in \mathbb{N}, n \in \mathbb{N}_{\geq N}, x_n \in V_N$ . Hence  $(x_n)_{n \in \mathbb{N}}$  converges to  $x$ . However,  $f(x_n) \notin U$  for any  $n$  So  $(f(x_n))_{n \in \mathbb{N}}$  does not converges to  $f(x)$ . Therefore (ii) does not hold.

## 19.5 Prop

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y), (Z, \mathcal{G}_Z)$  be topological spaces.  $f$  be a function from  $X$  to  $Y$ ,  $g$  be a function from  $Y$  to  $Z$ . Let  $x \in \text{Dom}(g \circ f)$  If  $f$  and  $g$  are continuous at  $x$ . then  $g \circ f$  is continuous at  $x$  sectionProof Let  $U \in \mathcal{V}_{g(f(x))}$  Since  $g$  is continuous at  $f(x)$ :

$$\exists W \in \mathcal{V}_{f(x)}, g(W) \subseteq U$$

Since  $f$  is continuous at  $x$ :

$$\exists V \in \mathcal{V}_x \quad f(V) \subseteq W$$

Therefore,  $g(f(V)) \subseteq g(W) \subseteq U$  Hence  $g \circ f$  is continuous at  $x$

## 19.6 Def

Let  $(X, \mathcal{G})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{G}$ , If any element of  $\mathcal{G}$  can be written as the union of a family of sets in  $\mathcal{B}$  we say that  $\mathcal{B}$  is a topological basis of  $\mathcal{G}$

## 19.7 Prop

Let  $(X, \mathcal{G})$  be a topological space,  $\mathcal{B} \subseteq \mathcal{G}$   $\mathcal{B}$  is a topological basis iff

$$\forall x \in X, \mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$$

is a neighborhood basis of  $x$

## 19.8 Proof

$\Rightarrow$ :

$$\forall x \in X \mathcal{B}_x \subseteq \mathcal{V}_x$$

Moreover,

$$\forall U \in \mathcal{V}_x \exists V \in \mathcal{V}, x \in V \subseteq U$$

. Since  $\mathcal{B}$  is a topological basis of  $\mathcal{G}$ ,

$$\exists W \in \mathcal{B}, x \in W \subseteq V \subseteq U$$

Hence  $\mathcal{V}_x$  is generated by  $\mathcal{B}_x$

$\Leftarrow$  Let  $U \in \mathcal{G}$

$$\forall x \in U, U \in \mathcal{V}_x$$

So

$$\exists V_x \in \mathcal{B}_x \quad x \in V_x \subseteq U$$

Hence

$$U \subseteq \bigcup_{x \in U} V_x \subseteq U$$

Hence

$$U = \bigcup_{x \in U} V_x \in \mathcal{G}$$

## 19.9 Prop

Let  $(X, \mathcal{G}_X), (Y, \mathcal{G}_Y)$  be topological spaces.  $\mathcal{B}_Y$  be a topological basis of  $\mathcal{G}_Y$   
 $f : X \rightarrow Y$  be a mapping. The following conditions are equivalent:

- (1)  $f$  is continuous
- (2)  $\forall U \in \mathcal{G}_Y, f^{-1}(U) \in \mathcal{G}_X$
- (3)  $\forall U \in \mathcal{B}_Y, f^{-1}(U) \in \mathcal{G}_X$

### Proof

(1)  $\Rightarrow$  (2)

Lemma Let  $(X, \mathcal{G})$  be a topological space,  $V \in \wp(X)$ , Then  $V \in \mathcal{G}$  iff  $\forall x \in V, V$  is a neighborhood of  $x$

Proof of lemma  $\Rightarrow$  is by definition

*Leftarrow:*

$$\forall x \in V, \exists W_x \in \mathcal{G}, x \in W_x \subseteq V.$$

Hence

$$V = \bigcap_{x \in V} W_x - x \in \mathcal{G}$$

Let  $U \in \mathcal{G}_Y$

$$\forall x \in f^{-1}(U) \quad f(x) \in U$$

Hence

$$U \in \mathcal{V}_{f(x)}$$

Hence there exists an open neighborhood  $W$  of  $x$  such that  $f(W) \subseteq U$   
 Since  $f$  is a mapping ,

$$W \subseteq f^{-1}(U)$$

Therefore

$$f^{-1}(U) \in \mathcal{V}_x$$

Since  $x$  is arbitrary,

$$f^{-1}(U) \in \mathcal{G}_X$$

(2)  $\Rightarrow$  (3) For (3) is a special situation of (2), it's natural.

(3)  $\Rightarrow$  (1) Let  $x \in X$

$$\forall U \in \mathcal{B}_Y \text{ s.t. } f(x) \in U, f^{-1}(U)$$

is an open neighborhood of  $x$ , and

$$f(f^{-1}(U)) \subseteq U$$

Hence  $f$  is continuous at  $x$

## 19.10 Def

Let  $X$  be a set ,  $((Y_i, \mathcal{G}_i))_{i \in I}$  be a family of topological spaces.  $\forall i \in I$  let  $f_i : X \rightarrow Y_i$  be a mapping. We call initial topology of  $(f_i)_{i \in I}$  on  $X$  the smallest topology on  $X$  making all  $f_i$  continue

## 19.11 Remark

If  $\mathcal{G}$  is the initial topology of  $(f_i)_{i \in I}$ ,  $\forall i \in I, U_i \in \mathcal{G}_i$   $f_i^{-1}(U_i) \in \mathcal{G}$  If  $J \subseteq I$  is a finite subset,  $(U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j$  then  $\bigcap_{j \in J} f_j^{-1}(U_j) \in \mathcal{G}$

## 19.12 Prop

$$\mathcal{B} := \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ is finite } (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j \right\}$$

is a topological basis of the initial topology  $\mathcal{G}$

**Proof**

First

$$\mathcal{B} \subseteq \mathcal{G}$$

Let

$\mathcal{G}' = \{\text{subset } V \text{ of } X \text{ that can be written as the union of a family of sets in } \mathcal{B}\}$

- $\emptyset \in \mathcal{G}' \quad X \in \mathcal{B} \subseteq \mathcal{G}'$
- $\mathcal{G}'$  is stable by taking the union of any family of elements in  $\mathcal{G}'$
- If  $V_1, V_2$  are elements of  $\mathcal{G}'$ , then

$$V_1 \cap V_2 \in \mathcal{G}'$$

In fact,  $V_1, V_2$  are of the form of the union of some sets of  $\mathcal{B}$

The intersection of two elements of  $\mathcal{B}$  is still a element of  $\mathcal{B}$

$$\begin{aligned} & \left( \bigcap_{j \in J} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J'} f_j^{-1}(U'_j) \right) \\ &= \bigcap_{j \in J \cup J'} f_j^{-1}(W_j) \text{ where } W_j = \begin{cases} U_j & j \in J \setminus J' \\ U'_j & j \in J' \setminus J \\ U_j \cap U'_j & j \in J \cap J' \end{cases} \\ & \left( \bigcap_{j \in J \setminus J'} f_j^{-1}(U_j) \right) \cap \left( \bigcap_{j \in J \cap J'} f_j^{-1}(U_j) \cap f_j^{-1}(U'_j) \right) \cap \left( \bigcap_{j \in J' \setminus J} f_j^{-1}(U'_j) \right) \end{aligned}$$

So  $\mathcal{G}'$  is a topology making all  $f_i$  continuous. Hence

$$\mathcal{G} \subseteq \mathcal{G}' \subseteq \mathcal{G} \Rightarrow \mathcal{G}' = \mathcal{G}$$

**Example**

Let  $((Y_i, \mathcal{G}_i))_{i \in I}$  be topological spaces.  $Y = \prod_{i \in I} Y_i$  and

$$\begin{aligned} \pi_i : Y &\rightarrow Y_i \\ (y_j)_{j \in I} &\mapsto y_i \end{aligned}$$

The product topology on  $Y$  is by definition the initial topology of  $(\pi_i)_{i \in I}$

### 19.13 Theorem

Let  $X$  be a set,  $((Y_i, \mathcal{G}_i))_{i \in I}$  be a family of topological spaces,

$$((f_i : X \rightarrow Y_i))_{i \in I}$$

be a family of mappings and we equip  $X$  with the initial topology  $\mathcal{G}_X$  of  $(f_i)_{i \in I}$ . Let  $(Z, \mathcal{G}_Z)$  be a topological space and

$$h : Z \rightarrow X$$

be a mapping. Then  $h$  is continuous iff

$$\forall i \in I, \quad f_i \circ h \text{ is continuous}$$

#### 19.13.1 Proof

$\Rightarrow$  If  $h$  is continuous, since each  $f_i$  is continuous,  $f_i \circ h$  is also continuous.

$\Leftarrow$  Suppose that  $\forall i \in I, f_i \circ h$  is continuous. Hence

$$\forall U_i \in \mathcal{G}_i, (f_i \circ h)^{-1}(U_i) = h^{-1}(f_i^{-1}(U_i)) \in \mathcal{G}_Z$$

Let

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{G}_j \right\}$$

$\forall U \in \mathcal{B}$

$$h^{-1}(U) = \bigcap_{j \in J} h^{-1}(f_j^{-1}(U_j)) \in \mathcal{G}_Z$$

Therefore,  $h$  is continuous.

### 19.14 Remark

We keep the notation of the definition of initial topology. If  $\forall i \in I, \mathcal{B}_i$  is a topological basis of  $\mathcal{G}_i$ , then

$$\mathcal{B} = \left\{ \bigcap_{j \in J} f_j^{-1}(U_j) \mid J \subseteq I \text{ finite}, (U_j)_{j \in J} \in \prod_{j \in J} \mathcal{B}_j \right\}$$

is also a topological basis of the initial topology,



**19.14.1 Example**

Let  $((X_i, d_i))_{i \in \{1, \dots, n\}}$  be a family of metric spaces.

$$X = \prod_{i \in \{1, \dots, n\}} X_i$$

We define a mapping

$$d: (X \times X \rightarrow \mathbb{R}_{\geq 0}) \\ d: ((x_i)_{i \in \{1, \dots, n\}}, (y_i)_{i \in \{1, \dots, n\}}) \mapsto \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i)$$

$d$  is a metric on  $X$ . If  $x = (x_i)_{i \in \{1, \dots, n\}}$ ,  $y = (y_i)_{i \in \{1, \dots, n\}}$ ,  $z = (z_i)_{i \in \{1, \dots, n\}}$  are elements of  $X$ , then

$$d(x, z) = \max_{i \in \{1, \dots, n\}} d_i(x_i, z_i) \leq \max_{i \in \{1, \dots, n\}} (d_i(x_i, y_i) + d_i(y_i, z_i)) \leq d(x, y) + d(y, z)$$

Each

$$\pi_i: X \rightarrow X_i \\ \pi_i: (x_i)_{i \in \{1, \dots, n\}} \mapsto x_i$$

is continuous. Hence the product topology  $\mathcal{G}$  is contained in  $\mathcal{G}_d$

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in X$ ,  $\epsilon > 0$

$$\begin{aligned} \mathcal{B}(x, \epsilon) &= \left\{ y = (y_i)_{i \in \{1, \dots, n\}} \mid \max_{i \in \{1, \dots, n\}} d_i(x_i, y_i) < \epsilon \right\} \\ &= \prod_{i \in \{1, \dots, n\}} \mathcal{B}(x_i, \epsilon) \\ &= \bigcap_{i \in \{1, \dots, n\}} \pi_i^{-1}(\mathcal{B}(x_i, \epsilon)) \in \mathcal{G} \end{aligned}$$



## Chapter 20

# Uniform continuity and convergency

### 20.1 Def

Let  $(X, d)$  be a metric space.  $\forall A \subseteq X$ , we define

$$\text{diam}(A) := \sup_{(x,y) \in A \times A} d(x, y)$$

called the diameter of A. By convention

$$\text{diam}(\emptyset) := 0$$

If  $\text{diam}(A) < +\infty$ , we say that A is bounded

### 20.2 Remark

- If A is finite, then it's bounded
- If  $A \subseteq B$  then  $\text{diam}(A) \leq \text{diam}(B)$

### 20.3 Prop

Let  $(X, d)$  be a metric space.  $A \subseteq X, B \subseteq X, (x_0, y_0) \in A \times B$ . Then

$$\text{diam}(A \cup B) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

In particular, if A, B are bounded, then  $A \cup B$  is bounded.

**Proof**

Let  $(x, y) \in (A \cup B)^2$ . If  $\{x, y\} \subseteq A$ , then  $d(x, y) \leq \text{diam}(A)$   
 If  $\{x, y\} \subseteq B$  then  $\text{diam}(B) \geq d(x, y)$   
 If  $x \in A, y \in B$ ,

$$d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

Similarly if  $x \in B, y \in A$

$$d(x, y) \leq \text{diam}(A) + d(x_0, y_0) + \text{diam}(B)$$

**20.4 Def**

Let  $(X, d)$  be a metric space.  $I \subseteq \mathbb{N}$  be an infinite subset,  $(x_n)_{n \in I} \in X^I$ . If

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \quad \text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \epsilon$$

then we say that  $(x_n)_{n \in I}$  is a Cauchy sequence.

**20.5 Prop**

- (1) If  $(x_n)_{n \in I}$  converges, then it's a Cauchy sequence.
- (2) If  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\{x_n \mid n \in I\}$  is bounded
- (3) Suppose that  $(x_n)_{n \in I}$  is a Cauchy sequence. If there exists an infinite subset  $J$  of  $I$  such that  $(x_n)_{n \in J}$  converges to some  $x \in X$ , then  $(x_n)_{n \in I}$  converges to  $x$

**20.5.1 Proof**

- (1) trivial
- (2) trivial
- (3) Let  $\epsilon > 0, \exists N \in \mathbb{N}$

$$\text{diam}(\{x_n \mid n \in I_{\geq N}\}) \leq \frac{\epsilon}{2}$$

$$\forall n \in J_{\geq N}, d(x_n, x) \leq \frac{\epsilon}{2}$$

- Take  $n_0 \in J_{\leq N} \subseteq I_{\geq N}$

$$\forall n \in I_{\geq N} \quad d(x_n, x) \leq d(x_n, x_{n_0}) + d(x_{n_0}, x) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

Hence  $(x_n)_{n \in I}$  converges to  $x$

## 20.6 Def

Let  $(X, d_X), (Y, d_Y)$  be metric space.  $f$  be a function from  $X$  to  $Y$ . If  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta$$

implies

$$d(f(x), f(y)) \leq \epsilon$$

namely

$$\inf_{\delta > 0} \sup_{(x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

we say that  $f$  is uniformly continuous.

## 20.7 Prop

Let  $(X, d_X), (Y, d_Y)$  be metric spaces  $f$  be a function from  $X$  to  $Y$  which is uniformly continuous.

- (1) If  $I \subseteq \mathbb{N}$  is finite, and  $(x_n)_{n \in I}$  is a Cauchy sequence in  $\text{Dom}(f)^I$  then  $(f(x_n))_{n \in I}$  is Cauchy sequence
- (2)  $f$  is continuous

### 20.7.1 Proof

- (1)  $\forall \epsilon > 0, \exists \delta > 0$  such that

$$\forall (x, y) \in \text{Dom}(f)^2, d(x, y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \epsilon$$

Since  $(x_n)_{n \in I}$  is a Cauchy sequence,  $\exists N \in \mathbb{N}$  such that

$$\forall (n, m) \in I_{\geq N}^2, d_X(x_n, x_m) \leq \delta$$

Hence

$$d_Y(f(x_n), f(x_m)) \leq \epsilon$$

Therefore  $(f(x_n))_{n \in I}$  is a Cauchy sequence.

- (2) Let  $(x_n)_{n \in I}$  be a sequence in  $\text{Dom}(f)^{\mathbb{N}}$  that converges to  $x \in \text{Dom}(f)$  We define  $(y_n)_{n \in \mathbb{N}}$  as

$$y_n = \begin{cases} x & \text{if } n \text{ is odd} \\ x_{\frac{1}{2}} & \text{if } n \text{ is even} \end{cases}$$

Then  $(y_n)_{n \in \mathbb{N}}$  converges to  $x$ . Hence  $(y_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. Since  $f$  is uniformly continuous,  $(f(y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence in  $Y$ .

$$(f(y_n))_{n \in \mathbb{N}, n \text{ is odd}} = (f(x))_{n \in \mathbb{N}, n \text{ is odd}}$$

converges to  $f(x)$ . Hence  $(f(y_n))_{n \in \mathbb{N}}$  converges to  $f(x)$

## 20.8 Def

Let  $X$  be a set,  $Z \subseteq X$ ,  $(Y, d)$  be a metric space,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as their common domain of definition.

- If  $\forall x \in Z, (f_n(x))_{n \in I}$  converges to  $f(x)$ , we say that  $(f_n)_{n \in I}$  converges pointwisely to  $f$
- If

$$\lim_{n \in I, n \rightarrow +\infty} \sup_{x \in Z} d(f_n(x), f(x)) = 0$$

we say that  $(f_n)_{n \in I}$  converges uniformly to  $f$

## 20.9 Theorem

Let  $X$  and  $Y$  be metric space,  $Z \subseteq X$ ,  $I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$ ,  $f$  be functions from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is uniformly continuous

Then  $f$  is uniformly continuous.

### 20.9.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\lim_{n \in I, n \rightarrow +\infty} A_n = 0$$

$\forall (x, y) \in Z^2, n \in I$

$$\begin{aligned} & d(f(x), f(y)) \\ & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)) \\ & \leq 2A_n + d(f_n(x), f_n(y)) \end{aligned}$$

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n + \inf_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f_n(x), f_n(y)) = 0$$

Hence

$$0 \leq \inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) \leq 2A_n$$

Take  $\lim_{n \rightarrow +\infty}$ , by squeeze theorem, we get

$$\inf_{\delta > 0} \sup_{(x, y) \in Z^2, d(x, y) \leq \delta} d(f(x), f(y)) = 0$$

## 20.10 Theorem

Let  $X$  be a topological space,  $Y$  be a metric space,  $Z \subseteq X, p \in Z, I \subseteq \mathbb{N}$  infinite.  $(f_n)_{n \in I}$  and  $f$  function from  $X$  to  $Y$ , having  $Z$  as domain of definition. Suppose that:

- $(f_n)_{n \in I}$  converges uniformly to  $f$
- each  $f_n$  is continuous at  $p$

Then  $f$  is continuous at  $p$

### 20.10.1 Proof

$\forall n \in I$  let

$$A_n = \sup_{x \in Z} d(f_n(x), f(x))$$

$$\forall \epsilon > 0 \exists n \in I \quad A_n \leq \frac{\epsilon}{3}$$

Since  $f_n$  is continuous  $\exists U \in \mathcal{V}_p \quad f_n(U) \subseteq \overline{\mathcal{B}}(f_n(p), \frac{\epsilon}{3})$

$$\begin{aligned} \forall x \in U \cap Z \quad d(f(x), f(p)) & \leq d(f(x), f_n(x)) + d(f_n(x), f_n(p)) + d(f_n(p), f(p)) \\ & \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{\epsilon}{3} \end{aligned}$$

$$f(U) \subseteq \overline{\mathcal{B}}(f(p), \epsilon)$$

### 20.10.2 Def

Let  $X, Y$  be metric spaces,  $f$  be a function from  $X$  to  $Y$ ,  $\epsilon > 0$ . If

$$\forall (x, y) \in \text{Dom}(f)^2 \quad d(f(x), f(y)) \leq \epsilon d(x, y)$$

then we say that  $f$  is  $\epsilon$ -Lipschitzian

If  $\exists \epsilon > 0$  such that  $f$  is  $\epsilon$ -Lipschitzian, then it's uniformly continuous.

## 20.11 Remark

If  $f$  is Lipschitzian, then it's uniformly continuous.

## 20.12 Example

- Let  $((X_i, d_i))_{i \in I}$  be metric space.  $X = \prod_{i \in I} X_i$  where  $I$  is finite

$$\begin{aligned} X \times X & \rightarrow \mathbb{R}_{\geq 0} \\ d : d((x_i), (y_i)_{i \in I}) & = \max_{i \in I} d_i(x_i, y_i) \end{aligned}$$

$$d_i(x_i, y_i) = d_i(\pi_i(x), \pi_i(y)) \leq d(x, y)$$

Then

$$\pi_i : X \rightarrow X_i$$

is Lipschitzian. ( $\forall x = (x_i)_{i \in I}, \forall y = (y_i)_{i \in I}$ )

- Let  $(X, d)$  be a metric space

$$d : X \times X \rightarrow \mathbb{R}_{\geq 0}$$

is Lipschitzian.

$$|d(x, y) - d(x', y')| \leq 2 \max\{d(x, x'), d(y, y')\}$$



Part V

**Normed Vector Space**



## Chapter 21

# Linear Algebra

We fix a unitary ring  $K$

### 21.1 Def

Let  $M$  be a left  $K$ -module, and let  $x = (x_i)_{i \in I}$  be a family of elements of  $M$ . We define a morphism of left  $K$ -module as following:

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \quad (:= \sum_{i \in I, i \neq 0} a_i x_i) \end{aligned}$$

#### 21.1.1 Notation

$$\begin{aligned} K^{\oplus I} &:= \{(a_i)_{i \in I} \in K^I \mid \exists J \subseteq I \text{ finite, such that } a_i = 0 \text{ for } i \in I \setminus J\} \\ \varphi_x((a_i)_{i \in I} + (b_i)_{i \in I}) &= \varphi_x((a_i)_{i \in I}) + \varphi_x((b_i)_{i \in I}) \end{aligned}$$

### 21.2 Def

Let  $M$  be a left  $K$ -module,  $I$  be a set,  $x = (x_i)_{i \in I} \in M^I$ . If

$$\begin{aligned} \varphi_x : K^{\oplus I} &\rightarrow M \\ (a_i)_{i \in I} &\mapsto \sum_{i \in I} a_i x_i \end{aligned}$$

is

injective then we say  $(x_i)_{i \in I}$  is  $K$ -linearly independent

surjective then we say  $(x_i)_{i \in I}$  is system of generator

a bijection then we say  $(x_i)_{i \in I}$  is a basis of  $M$

**Example**

Let  $e_i$  be the element  $(\delta_{ij})_{j \in I}$  with

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the family

$$e = (e_i)_{i \in I} \in (K^{\oplus I})^I$$

is a basis of  $K^{\oplus I}$

**21.3 Def**

Let  $M$  be a left  $K$ -module

- If  $M$  has a basis, we say that  $M$  is a free  $K$ -module
- If  $M$  has finite system of generated  
( $\exists$  a finite set  $I$  and a family  $(x_i)_{i \in I} \in M^I$  that forms a system of generator),  
then we say that  $M$  is of finite type.

**21.4 Remark**

Let  $x = (x_i)_{i \in \{1, \dots, n\}} \in M^n$ , where  $n \in \mathbb{N}$

- $x$  is linearly independent iff

$$\forall a \in K^n \quad \sum a_i x_i = 0$$

implies

$$a = 0$$

- $x$  is a system of generator iff for any element of  $M$  can be written in the form

$$\sum b_i x_i \quad b \in K^n$$

Such expression is called a  $K$ -linear combination of  $x_1, \dots, x_n$

**21.5 Theorem**

Let  $K$  be a division ring ( $0 \neq 1$  and  $\forall k \in K \setminus \{0\}$   $k$  is invertible)

Let  $V$  be a left  $K$ -module of finite type and  $(x_i)_{i \in I}$  be a system of generators of  $V$ . Then, there exists a subset  $I$  of  $\{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ . (In particular,  $V$  is a free  $K$ -module)

**Proof**

(By induction on  $n$ )

If  $n = 0$ , then  $V = \{0\}$

In this case  $\emptyset$  is a basis of  $V$

**Induction hypothesis**

True for a system of generators of  $n - 1$  elements. Let  $(x_i)_{i \in \{1, \dots, n\}}$  be a system of generators of  $V$ . If  $(x_i)_{i \in \{1, \dots, n\}}$  is linearly independent, it's a basis. Otherwise,  $\exists (a_i)_{i \in I} \in K^n$  such that

$$(a_i, \dots, a_n) \neq 0$$

$$\sum a_i x_i = 0$$

Without loss of generality, we suppose  $a_n \neq 0$ . Then

$$x_n = -a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right)$$

Since  $(x_i)_{i \in \{1, \dots, n\}}$  is a system of generators, any elements of  $V$  can be written as

$$\begin{aligned} \sum b_i x_i &= \left( \sum_{i=1}^{n-1} b_i x_i \right) - b_n a_n^{-1} \left( \sum_{i=1}^{n-1} a_i x_i \right) \\ &= \sum_{i=1}^{n-1} (b_i - b_n a_n^{-1} a_i) x_i \end{aligned}$$

Thus  $(x_i)_{i \in \{1, \dots, n\}}$  forms a system of generators. By the induction hypothesis, there exists  $I \subseteq \{1, \dots, n\}$  such that  $(x_i)_{i \in I}$  forms a basis of  $V$ .

**21.6 Theorem**

Let  $K$  be a unitary ring and  $B$  be a left  $K$ -module.  $W$  be a left  $K$ -submodule of  $V$ . Let  $(x_i)_{i=1}^n$  be an element of  $W^n$

$$(\alpha_j)_{j=1}^l \in (V/W)^l$$

, where  $(n, l) \in \mathbb{N}^2 \forall j \in \{1, \dots, l\}$ , let  $x_{n+j}$  be an element in the equivalence class  $\alpha_j$

- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are linearly independent, then  $(x_i)_{i=1}^{n+l}$  is also linearly independent
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are system of generators of  $W$  and  $V/W$  respectively, then  $(x_i)_{i=1}^{n+l}$  is also a system of generators
- If both  $(x_i)_{i=1}^n, (\alpha_j)_{j=1}^l$  are basis, then  $(x_i)_{i=1}^{n+l}$  is also a basis

**Proof**

(1) Suppose that  $(b_i)_{i=1}^{n+l}$  such that

$$\sum_{i=1}^{n+l} b_i x_i = 0$$

Let

$$\pi : V \rightarrow V/W$$

be the projection morphism ( $\pi(x) = [x]$ )

$$0 = \pi\left(\sum_{i=1}^{n+l} b_i x_i\right) = \sum_{i=1}^{n+l} b_i \pi(x_i) = \sum_{j=1}^l b_{n+j} \pi(x_{n+j}) = \sum_{j=1}^l b_{n+j} \alpha_j$$

$$\{x_1, \dots, x_n\} \subseteq W \text{ So } \forall i \in \{1, \dots, n\}$$

$$\pi(x_i) = 0$$

Since  $(\alpha_j)_{j=1}^l$  is linearly independent,

$$b_{n+1} = \dots = b_{n+l} = 0$$

Hence

$$\sum b_i x_i = 0$$

Since  $(x_i)_{i=1}^n$  is linearly independent,

$$b_1 = \dots = b_n = 0$$

(2) Let  $y \in V$ . Then  $\pi(y) \in V/W$ . So there exists

$$(c_{n+1}, \dots, c_{n+l}) \in K^l$$

such that

$$\begin{aligned} \pi(y) &= \sum_{j=1}^l c_{n+j} \alpha_j \\ &= \sum_{j=1}^l c_{n+j} \pi(x_{n+j}) = \pi\left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \end{aligned}$$

So

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) \in W$$

$\exists c \in K^n$  such that

$$y - \left(\sum_{j=1}^l c_{n+j} x_{n+j}\right) = \left(\sum_{i=1}^n c_i x_i\right)$$

Therefore

$$y = \sum_{i=1}^{n+l} c_i x_i$$

(3) from (1)(2), proved

## 21.7 Corollary

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. If  $(x_i)_{i=1}^n$  is a linearly independent family of elements of  $V$  ( $n \in \mathbb{N}$ ), then

$$\exists l \in \mathbb{N} \quad \exists (x_{n+j})_{j=1}^l \in V_l$$

such that

$$(x_i)_{i=1}^{n+l}$$

forms a basis of  $V$

### Proof

Let  $W$  be the image of

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow V \\ (a_i)_{i=1}^n &\mapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

It's a left  $K$ -submodule of  $V$ .

Note that  $(x_i)_{i=1}^n$  forms a basis of  $W$ .

$$\begin{aligned} \varphi(x_i)_{i=1}^n : K^n &\rightarrow W \\ \varphi(x_i)_{i=1}^n(e_j) &= x_j \in W \end{aligned}$$

Moreover, since  $V$  is of finite type there exists  $d \in \mathbb{N}$  and a surjective morphism of left  $K$ -modules.

$$\psi : K^d \twoheadrightarrow V$$

Since the projection morphism

$$\pi : V \rightarrow V/W$$

is surjective.

Hence the composite morphism

$$K^d \xrightarrow[\pi \circ \psi]{\psi} V \xrightarrow{\pi} V/W$$

is surjective. Thus  $V/W$  is of finite type. There exist then a basis

$$(a_j)_{j=1}^l$$

of  $V/W$ .

Taking  $x_{n+j} \in \alpha_j$  for  $j \in \{1, \dots, l\}$ , we get a basis of  $V$ :

$$(x_i)_{i=1}^{n+l}$$

## 21.8 Def

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. We call rank of  $V$  the minimal number of elements of its basis, denote as

$$rk_K(V)$$

or simply

$$rk(V)$$

If  $K$  is a field  $rk(V)$  is also denoted as

$$dim_K(V)$$

or

$$dim(V)$$

called the dimension of  $V$ .

## 21.9 Theorem

Let  $K$  be a division ring and  $V$  be a left  $K$ -module of finite type. Let  $W$  be a left  $K$ -submodule of  $V$ .

(1)  $W$  and  $V/W$  are both of finite type, and

$$rk(V) = rk(W) + rk(V/W)$$

(2) Any basis of  $V$  has exactly  $rk(V)$  elements

## 21.10 Proof

(1) This proof is written twice. Both are kept.

10.30's Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$



In  $(\pi(x_i))_{i=1}^n$  we extract a basis of  $V/W$ , say

$$(\pi(x_i))_{i=1}^l$$

For  $j \in \{l+1, \dots, n\}$ ,

$$\exists(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

Let

$$y_j = x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Since

$$\pi(y_i) = 0$$

For any  $x \in W, \exists(a_i)_{i=1}^n \in K^n, x = \sum_{i=1}^n a_i x_i$

$$\begin{aligned} x &= \sum_{i=1}^l a_i x_i + \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \\ &= \sum_{j=l+1}^n a_j y_j + \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i \end{aligned}$$

Since

$$\pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i) = 0$$

Hence

$$x = \sum_{j=l+1}^n a_j y_j$$

Hence  $W$  is of finite type, and

$$rk(V) \geq rk(W) + rk(V/W)$$

Moreover the previous theorem shows that

$$rk(V) \leq rk(W) + rk(V/W)$$

So

$$rk(V) = rk(W) + rk(V/W)$$

11.1's By previous theorem.

$$rk(V) \leq rk(W) + rk(V/W)$$

Let  $(x_i)_{i=1}^n$  be a basis of  $V$ . Then

$$(\pi(x_i))_{i=1}^n$$

is a system of generators of  $V/W$ .

We extract a subfamily, say  $(x_i)_{i=1}^l$  such that

$$(\pi(x_i))_{i=1}^l$$

forms a basis of  $V/W$ .

For  $j \in \{1, \dots, l\}$ , there exists:

$$(b_{j,1}, \dots, b_{j,l}) \in K^l$$

such that

$$\pi(x_j) = \sum_{i=1}^l b_{j,i} \pi(x_i)$$

namely

$$y_j := x_j - \sum_{i=1}^l b_{j,i} x_i \in W$$

Let  $x \in W, \exists (a_i)_{i=1}^n \in K^n$  let  $x = \sum a_i x_i$ , then

$$\begin{aligned} x &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{j=l+1}^n a_j (y_j + \sum_{i=1}^l b_{j,i} x_i) \right) \\ &= \left( \sum_{i=1}^l a_i x_i \right) + \left( \sum_{i=1}^l \sum_{j=l+1}^n a_j b_{j,i} x_i \right) + \left( \sum_{j=l+1}^n a_j y_j \right) \\ &= \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) x_i + \sum_{j=l+1}^n a_j y_j \end{aligned}$$

and

$$0 = \pi(x) = \sum_{i=1}^l (a_i + \sum_{j=l+1}^n a_j b_{j,i}) \pi(x_i)$$

Therefore  $(y_j)_{j=l+1}^n$  is a system of generators

$$n - l \geq rk(W)$$

Hence

$$n \geq rk(W) + rk(V/W)$$

Thus

$$rk(V) \geq rk(W) + rk(V/W)$$

(2) All basis of  $V$  have  $rk(V)$  elements.

We reason by induction on  $rk(V)$

(1)

$$rk(V) = 0$$

In this case  $V = \{0\}$  The only basis of  $V$  is  $\emptyset$ . So the statement holds.

(2) Assume that there exists  $e \in V \setminus \{0\}$  such that

$$V = \{\lambda e \mid \lambda \in K\}$$

Then any basis of  $V$  is of the form

$$ae$$

where  $a \in K \setminus \{0\}$

Let  $(e_i)_{i=1}^m$  be a basis of  $V$ . We reason by induction on  $m$  to prove that

$$m = rk(V)$$

The cases where  $m = 0$  or  $1$  are proved in (1)(2) respectively. Induction hypothesis: true for a basis of  $< m$  elements

Let

$$W = \{\lambda e_i \mid \lambda \in K\}$$

Let

$$\begin{aligned} \pi : V &\rightarrow V/W \\ x &\mapsto [x] \end{aligned}$$

Then

$$(\pi(e_i))_{i=1}^m$$

forms a system of generators of  $V/W$ .

If  $(a_i)_{i=2}^m \in K^{m-1}$  such that

$$\sum_{i=2}^m a_i \pi(e_i) = 0$$

then

$$\sum_{i=2}^m a_i e_i \in W$$

Hence

$$\exists a_i \in K \quad \sum_{i=2}^m a_i e_i - a_1 e_1 = 0$$

And for  $(e_i)_{i=1}^m$  a basis of  $V$ ,

$$a_i = 0$$

Thus

$$(\pi(e_i))_{i=2}^m$$

is a basis of  $V/W$ . We then obtain that

$$rk(V/W) \leq m - 1 \leq n - 1$$

By the induction hypothesis,

$$m - 1 = rk(V/W)$$

By (2),  $rk(W) = 1$ . Hence

$$m = (m - 1) + 1 = rk(V/W) + rk(W) = rk(V)$$

## 21.11 Prop

Let  $K$  be a unitary ring and  $f : E \rightarrow F$  be a morphism of left  $K$ -modules. Let  $I$  be a set and  $(x_i)_{i \in I} \in E^I$

- If  $(x_i)_{i \in I}$  is linearly independent and  $f$  is injective, then  $(f(x_i))_{i \in I}$  is linearly independent.
- If  $(x_i)_{i \in I}$  is a system of generators and  $f$  is surjective, then  $(f(x_i))_{i \in I}$  is a system of generators.
- If  $(x_i)_{i \in I}$  is a basis and  $f$  is an isomorphism, then  $(f(x_i))_{i \in I}$  is a basis.

### 21.11.1 Proof

$$\varphi_{(f(x_i))_{i \in I}} = f \circ \varphi_{(x_i)_{i \in I}}$$

## Chapter 22

# Matrices

We fix unitary ring  $K$

### 22.1 Def

Let  $n \in \mathbb{N}$  and  $V$  be a left  $K$ -module.

For any  $(x_i)_{i=1}^n \in V^n$ , we denote by  $\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$  the morphism

$$\begin{aligned} & \phi_{(x_i)_{i=1}^n} : K^n \rightarrow V \\ (a_i)_{i=1}^n & \mapsto \sum_{i=1}^n a_i n_i \end{aligned}$$

### 22.1.1 Example

Suppose that  $V = K^p$  ( $p \in \mathbb{N}$ ) Then each  $x_i \in K^p$  is of the form  $(x_{i,1}, \dots, x_{i,p})$

Hence  $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  can be written:

$$\begin{pmatrix} x_{1,1} & \dots & x_{1,p} \\ \vdots & & \vdots \\ x_{n,1} & \dots & x_{n,p} \end{pmatrix}$$

## 22.2 Def

Let  $(n, p) \in \mathbb{N}^2$ . We call  $n$  by  $p$  matrix of coefficient in  $K$  any morphism of left  $K$ -modules from  $K^n$  to  $K^p$

### 22.2.1 Example

- Denote by  $I_n$  then identity mapping. Then  $(e_i)_{i=1}^n$  is a basis of  $K^n$  called the canonical basis of  $K^n$

$$\varphi_{(e_i)_{i=1}^n} = Id_{K^n}$$

$$\varphi_{(e_i)_{i=1}^n}((a_1, \dots, a_n)) = \sum_{i=1}^n a_i e_i = (a_1, \dots, a_n)$$

- Let  $(x_1, \dots, x_n) \in K^n$ , Denote by

$$\begin{aligned} \text{diag}(x_1, \dots, x_n) (= \varphi_{(x_i e_i)_{i=1}^n}) : K^n &\rightarrow K^n \\ (a_1, \dots, a_n) &\mapsto (a_1 x_1, \dots, a_n x_n) \end{aligned}$$

## 22.3 Def

We denote by  $M_{n,p}(K)$  the set of all  $n$  by  $p$  matrices of coefficients in  $K$ . For  $(n, p, r) \in \mathbb{N}^3$ , we define

$$\begin{aligned} M_{n,p}(K) \times M_{p,r}(K) &\rightarrow M_{n,r}(K) \\ (A, B) &\mapsto AB := B \circ A \end{aligned}$$

## 22.4 Calculate Matrices

Let  $K$  be a unitary ring, and  $V$  be a left  $K$ -module. Let  $n \in \mathbb{N}$  and

$$x = (x_1, \dots, x_n) \in V^n$$

### 22.4.1 Remind

$$\begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \varphi : (a_1, \dots, a_n) \mapsto a_1 x_1, \dots, a_n x_n \in V$$

Consider a matrix

$$A = \{a_{ij}\}_{i \in \{1, \dots, p\} \times \{1, \dots, n\}} \in M_{p,n}(K)$$

$A$  is a morphism of left  $K$ -modules from  $K^p$  to  $K^n$ . Recall that

$$A \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

is defined as

$$\varphi_x \circ A : K^p \xrightarrow{A} K^n \xrightarrow{\varphi_x} V$$

Let  $(b_1, \dots, b_n) \in K^p$

$$\begin{aligned} A((b_1, \dots, b_n)) &= \sum_{i=1}^p b_i(a_{i,1}, \dots, a_{i,n}) \\ \varphi(A((b_1, \dots, b_n))) &= \sum_{i=1}^p b_i \varphi_x((a_{i,1}, \dots, a_{i,n})) \\ &= \sum_{i=1}^p b_i(a_{i,1}x_1, \dots, a_{i,n}x_n) \end{aligned}$$

Let  $B = \{b_{ij}\}_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, r\}} : K^n \rightarrow K^r$

$$AB = \left\{ \sum_{j=1}^n a_{lj} b_{jm} \right\}_{(l,m) \in \{1, \dots, p\} \times \{1, \dots, r\}}$$





## Chapter 23

# Transpose

We fix a unitary ring  $K$

### 23.1 Def

Let  $E$  be a left- $K$ -module. Denote by

$$E^\vee := \{\text{morphisms of left } K\text{-modules } E \rightarrow K\}$$

$\forall (f, g) \in E^\vee$  let

$$\begin{aligned} f + g : E &\rightarrow K \\ x &\mapsto f(x) + g(x) \end{aligned}$$

$(E^\vee, +)$  forms a commutative group.

The neutral element is the constant mapping

$$\begin{aligned} 0 : E &\rightarrow K \\ x &\mapsto 0 \end{aligned}$$

We define

$$\begin{aligned} K \times E^\vee &\rightarrow E^\vee \\ (a, f) &\mapsto fa : x \in E \rightarrow f(x)a \end{aligned}$$

$\forall \lambda \in K$

$$\begin{aligned} (fa)(\lambda x) &= (f(\lambda f(x)))a \\ &= (\lambda f(x))a \\ &= \lambda(f(x)a) \\ &= \lambda(fa)(x) \end{aligned}$$

This mapping defines a structure of right  $K$ -module on  $E^\vee$

## 23.2 Def

Let  $E$  and  $F$  be two left  $K$ -modules.  $\varphi : E \rightarrow F$  be a morphism of left  $K$ -modules. We denote by

$$\varphi^\vee : F^\vee \rightarrow E^\vee$$

the morphism of right  $K$ -modules sending  $g \in F^\vee$  to  $g \circ \varphi \in E^\vee$ .  
Actually  $\forall a \in K$

$$g \circ \varphi(\cdot)a = g(\varphi(\cdot))a = (g(\cdot)a) \circ \varphi$$

### 23.2.1 Example

Suppose that  $E = K^n, F = K^p$

$$\varphi = \begin{pmatrix} b_{1,1} & \dots & b_{1,p} \\ \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,p} \end{pmatrix}$$

$\varphi$  sends  $(a_1, \dots, a_n)$  to  $\{\sum_{i=1}^n a_i b_{ij}\}_{j \in \{1, \dots, p\}}$ . Let  $g \in F^\vee$   $g : K^p \rightarrow K$ , then  $g$  is of the form

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}, y_i \in K$$

$g \circ \varphi$  sends  $(a_1, \dots, a_n)$  to  $\sum_{i=1}^p (\sum_{j=1}^n a_j b_{ij} y_i)$

Assume that  $K$  is commutative. We denote by

$$\iota_p : (K^p)^\vee \rightarrow K^p$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} \mapsto (x_1, \dots, x_p)$$

$$\iota_n : (K^n)^\vee \rightarrow K^n$$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto (x_1, \dots, x_n)$$

are isomorphisms of  $K$ -modules

For any morphism of K-modules  $\varphi : K^n \rightarrow K^p$ , we denote by  $\varphi^\tau$  the morphism of K-modules  $K^p \rightarrow K^n$  given by  $\iota_n \circ \varphi^\vee \circ \iota_p^{-1}$

$$\begin{array}{ccc} (K^p)^\vee & \xrightarrow{\varphi^\vee} & (K^n)^\vee \\ \cong \downarrow \iota_p & \circlearrowleft & \cong \downarrow \iota_n \\ K^p & \xrightarrow{\varphi^\tau} & K^n \end{array}$$

$\varphi^\tau$  is called the transpose of  $\varphi$

### 23.3 Prop

Let E,F,G be left K-modules.  $\varphi : E \rightarrow F, \psi : F \rightarrow G$  be morphisms of left K-modules. Then  $(\psi \circ \varphi)^\vee$  is equal to  $\varphi^\vee \circ \psi^\vee$

#### Proof

$$\forall f \in G^\vee$$

$$(\varphi^\vee \circ \psi^\vee)(f) = \varphi^\vee(f \circ \psi) = (f \circ \psi) \circ \varphi = f \circ (\psi \circ \varphi) = (\psi \circ \varphi)^\vee(f)$$

### 23.4 Corollary

Assume that K is commutative. Let  $n, p, q$  be neutral numbers.  $A \in M_{n,p}(K), B \in M_{p,q}(K)$ . Then

$$(AB)^\tau = B^\tau A^\tau$$

#### Proof

$$A^t a u = \iota_n \circ A^\vee \circ \iota_p^{-1}$$

$$B^t a u = \iota_p \circ B^\vee \circ \iota_q^{-1}$$

$$\begin{aligned} B^\tau A^\tau &= A^\tau \circ B^\tau \\ &= \iota_n \circ A^\vee \circ B^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (B \circ A)^\vee \circ \iota_q^{-1} \\ &= \iota_n \circ (AB)^\vee \circ \iota_q^{-1} \\ &= (AB)^t a u \end{aligned}$$

### 23.5 Remark

(1) For  $A \in M_{n,p}(K)$ , one has  $(A^\tau)^\tau$

(2) We have a mapping

$$\begin{aligned} E &\rightarrow (E^\vee)^\vee \\ x &\mapsto ((f \in E^\vee) \mapsto f(x)) \end{aligned}$$

This is a  $K$ -linear mapping.

If  $K$  is a field and  $E$  is of finite dimension, this is an isomorphism of  $K$ -modules.

In fact, if  $e = (e_i)_{i=1}^n$  is a basis of  $E$  over  $K$ . For  $i \in \{1, \dots, n\}$ , let

$$\begin{aligned} e_i^\vee : E &\rightarrow K \\ \lambda_1 e_1, \dots, \lambda_n e_n &\mapsto \lambda_i \end{aligned}$$

is called the dual basis of  $e$

$$\begin{array}{ccc} K^n & \xleftarrow[\iota_n]{\cong} & (K^n)^\vee \\ \varphi_e \downarrow \cong & \searrow \varphi_{e^\vee} & \downarrow \varphi_e^\vee \\ E & \xrightarrow[\cong]{} & E^\vee \end{array}$$

$(e^\vee)^\vee$  gives a basis of  $(E^\vee)^\vee$ . Hence  $E \rightarrow (E^\vee)^\vee$  is an isomorphism.

## Chapter 24

# Linear Equation

We fix a unitary ring  $K$ .

### 24.1 Def

For  $a = (a_1, \dots, a_n) \in K^n \setminus \{(0, \dots, 0)\}$ . Denote by  $j(a)$  the first index  $j \in \{1, \dots, n\}$  such that  $a_j \neq 0$ . Let  $(n, p) \in \mathbb{N}^2, A \in M_{n,p}(K)$ . We write  $A$  as a column:

$$A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \quad a^{(i)} = (a_1^{(i)}, \dots, a_n^{(i)}) \in K^p$$

We say that  $A$  is of row echelon form if,  $\forall i \in \{1, \dots, n-1\}$  one of following conditions is satisfied.

- $a^{(i+1)} = (0, \dots, 0)$
- $a^{(i)}, a^{(i+1)}$  are non-zero, and  $j(a^{(i)}) < j(a^{(i+1)})$

If in addition the following condition is satisfied

- $\forall i \in \{1, \dots, n\}$  such that  $a^{(i)} \neq (0, \dots, 0)$ , one has

$$a_{j(a^{(i)})}^{(i)} = 1$$

and

$$\forall k \in \{1, \dots, n\} \setminus \{i\} \quad a_{j(a^{(i)})}^{(k)} = 0$$

we say that  $A$  is of reduced row echelon form.

## 24.2 Prop

Suppose that  $A = \begin{pmatrix} a^{(1)} \\ \vdots \\ a^{(n)} \end{pmatrix} \in M_{n,p}(K)$  is of row echelon form. Then  $\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$  is of cardinal  $\leq p$

### Proof

Let  $k = \text{card}\{i \in \{1, \dots, n\} \mid a^{(i)} \neq (0, \dots, 0)\}$   $a^{(k+1)} = \dots = a^{(n)} = (0, \dots, 0)$  and  $j(a^{(1)}) < j(a^{(2)}) < \dots < j(a^{(k)})$  Hence

$$\{1, \dots, k\} \rightarrow \{1, \dots, p\}, i \mapsto j(a^{(i)})$$

is injection. So  $k \leq p$

## 24.3 Linear Equation

Let  $A = \{a_{ij}\}_{i \leq n, j \leq p} \in M_{n,p}(K)$ . Let  $V$  be a left  $K$ -module and  $(b_1, \dots, b_n) \in V^n$ . We consider the equation

$$A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

The set of  $(x_1, \dots, x_p) \in V^p$  that satisfies  $(*)$  is called the solution set of  $(*)$

## 24.4 Prop

Suppose that  $A$  is of reduced row echelon form. Let

$$I(A) = \{i \in \{1, \dots, n\} \mid (a_{i,1}, \dots, a_{i,p}) \neq (0, \dots, 0)\}$$

$$J_0(A) = \{1, \dots, p\} \setminus \{j((a_{i,1}, \dots, a_{i,p})) \mid i \in I(A)\}$$

- If  $\exists i \in \{1, \dots, n\} \setminus I(A)$  such that  $b_i \neq 0$  then  $(*)$  does not have any solution in  $K^n$
- Suppose that  $\forall i \in \{1, \dots, n\} \setminus I(A), b_i = 0$ . Then  $(*)$  has at least one solution. Moreover

$$V^{J_0(A)} \rightarrow V^p$$

$$(z_k)_{k \in J_0(A)} \mapsto (x_1, \dots, x_p)$$

with

$$x_j = \begin{cases} z_j, & j \in J_0(A) \\ b_i - \sum_{l \in J_0(A)} a_{i,l} z_l & j = j((a_{i,1}, \dots, a_{i,p})) \end{cases}$$

is an injective mapping, whose image is equal to the set of solution of (\*)

## 24.5 Prop

Let  $m \in \mathbb{N}, S \in M_{m,n}(K)$ . If  $(x_1, \dots, x_p) \in V^p$  is a solution of (\*), then  $(x_1, \dots, x_p)$  is a solution of  $(*)_S$ :

$$(SA) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = S \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \quad (*)$$

In the case where S is left invertible, namely there exist  $R \in M_{n,m}(K)$  such that  $RS = I_n \in M_{m,n}(K)$ . Then (\*) and  $(*)_S$  have the same solution set.

## 24.6 Def

Let  $G_n(K)$  be the set of  $S \in M_{n,n}(K)$  that can be written as  $U_1 \dots U_N$  (by convention  $S = I_n$  where  $N = 0$ ) where each  $U_i$  is of one of the following forms.

- $P_\sigma$  where  $\sigma \in \mathfrak{S}_n$
- $\text{diag}(r_1, \dots, r_n)$  where each  $r_i \in K$  is left invertible
- $S_{i,c}$  with  $i \in \{1, \dots, n\}$   $c = (c_1, \dots, c_n) \in K^n, c_i = 0$

Let  $p \in \mathbb{N}$ , we say that  $A \in M_{n,p}(K)$  is reducible by Gauss elimination if  $\exists S \in G_n(K)$  such that  $SA$  is of reduced row echelon form

## 24.7 Theorem

Assume that K is a division ring  $\forall (n, p) \in \mathbb{N}$  any  $A \in M_{n,p}(K)$  is reducible by Gauss elimination

### Proof

The case where  $n = 0$  or  $p = 0$  is trivial. We assume  $n \geq 1, p \geq 1$  We write A as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} B \quad \text{where } \lambda_i \in K, B \in M_{n,p-1}(K)$$

- If  $\lambda_1 = \dots = \lambda_n = 0$

Applying the induction hypothesis to B, for  $S \in G_n(K)$

$$SA = \begin{pmatrix} S \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} & SB \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad SB$$

- Suppose that  $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$

By permuting the rows we may assume  $\lambda_1 \neq 0$ . As K is division ring, by multiplying the first row by  $\lambda_1^{-1}$ , we may assume  $\lambda_1 = 1$ . We add  $(-\lambda_i)$  times the first row to the  $i^{th}$  row, to reduce A to the form

$$\begin{pmatrix} 1 & \mu_2 & \dots & \mu_p \\ 0 & & & \\ \vdots & C & & \\ 0 & & & \end{pmatrix} \quad \begin{array}{l} C \in M_{n-1, p-1}(K) \\ (\mu_2, \dots, \mu_p) \in K^{p-1} \end{array}$$

Applying the induction hypothesis to C, we say assume that C is of reduced row echelon form. For  $i \in \{2, \dots, k\}$  we add  $-\mu_{j(c_i)}$  times the  $i^{th}$  row of A to the first line to obtain a matrix of reduced row echelon form



## Chapter 25

# Normed Vector Space

### 25.1 Def

Let  $(X, d)$  be a metric space. If  $(x_n)_{n \in \mathbb{N}}$  is an element of  $X^{\mathbb{N}}$  such that

$$\lim_{N \rightarrow +\infty} \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

we say that  $(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence. If any Cauchy sequence in  $X$  converges, then we say that  $(X, d)$  is complete.

Let  $Cau(X, d)$  be the set of all Cauchy sequences in  $X$ . We define a binary relation  $\sim$  on  $Cau(X, d)$  as

$$(x_n)_{n \in \mathbb{N}} \sim (y_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$$

### 25.2 Prop

$\sim$  is an equivalence relation.

#### 25.2.1 Proof

$$\lim_{n \rightarrow +\infty} d(x_n, x_n) = 0$$

$$d(x_n, y_n) = d(y_n, x_n)$$

If  $(x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}}, (z_n)_{n \in \mathbb{N}}$  be elements of  $Cau(X, d)$ . For

$$0 \leq d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$$

If

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(y_n, z_n) = 0$$

then

$$\lim_{n \rightarrow +\infty} d(x_n, z_n) = 0$$

### 25.3 Def

$$\hat{X} := \text{Cau}(X, d) \setminus \sim$$

### 25.4 Def: The completion

The completion of  $(X, d)$  is defined as

$$\text{Cau}(X) / \sim$$

and is denoted as

$$\hat{X}$$

### 25.5 Theorem

The mapping

$$\begin{aligned} \hat{d} : \hat{X} \times \hat{X} &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \lim_{n \rightarrow +\infty} d(x_n, y_n) \end{aligned}$$

is well defined, and it's a metric on  $\hat{X}$

#### Proof

TO check that  $\hat{d}$  is well defined, it suffices to prove that  $\forall ([x], [y]) \in \hat{X} \times \hat{X}$ ,  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is Cauchy sequence and its limit doesn't depend on the choice of the representation  $x$  and  $y$

For  $N \in \mathbb{N}$  and  $(n, m) \in \mathbb{N}_{\geq N}$  for

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_n, y_n) - d(x_m, y_m) &\leq d(x_n, x_m) + d(y_n, y_m) \\ d(x_m, y_n) - d(x_n, y_n) &\leq d(x_n, x_m) + d(y_n, y_m) \end{aligned}$$

one has,

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m)$$

then

$$\begin{aligned} \sup_{(n, m) \in \mathbb{N}_{\geq N}} |d(x_n, y_n) - d(x_m, y_m)| &\leq \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(x_n, x_m) \right) \\ &\quad + \left( \sup_{(n, m) \in \mathbb{N}_{\geq N}} d(y_n, y_m) \right) \end{aligned}$$

Taking  $\lim_{n \rightarrow +\infty}$  we obtain that  $(d(x_n, y_n))_{n \in \mathbb{N}}$  is a Cauchy sequence.

Hence it converges in  $\mathbb{R}$ . If  $x' = (x'_n)_{n \in \mathbb{N}} \in [x], y' = (y'_n)_{n \in \mathbb{N}} \in [y]$ , thus

$$\lim_{n \rightarrow +\infty} d(x_n, x'_n) = \lim_{n \rightarrow +\infty} d(y_n, y'_n) = 0$$

$$0 \leq |d(x_n, y_n) - d(x'_n, y'_n)| \leq d(x_n, x'_n) + d(y_n, y'_n)$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\lim_{n \rightarrow +\infty} |d(x_n, y_n) - d(x'_n, y'_n)| = 0$$

So

$$\lim_{n \rightarrow +\infty} d(x_n, y_n) = \lim_{n \rightarrow +\infty} d(x'_n, y'_n)$$

In the following, we check that  $\hat{d}$  is a metric

- $\hat{d}([x], [y]) = 0$  iff  $[x] = [y]$ : trivial
- $\hat{d}([x], [y]) = \hat{d}([y], [x])$ : trivial
- $\hat{d}([x], [y]) \leq \hat{d}([x], [z]) + \hat{d}([z], [y])$ :

$$\begin{aligned} d([x], [y]) &= \lim_{n \rightarrow +\infty} \\ &\leq \lim_{n \rightarrow +\infty} (d(x_n, z_n) + d(z_n, y_n)) \\ &= \hat{d}(x, z) + \hat{d}(z, y) \end{aligned}$$

## 25.6 Remark

Let

$$\begin{aligned} i_X : X &\rightarrow \hat{X} \\ a &\mapsto [(a, a, \dots)] \end{aligned}$$

then

$$\hat{d}(i_X(a), i_X(b)) = d(a, b)$$

In particular,  $i_x$  is injective (if  $i_X(a) = i_X(b)$  then  $d(a, b) = 0$  hence  $a = b$ )

## 25.7 Prop

$i_X(X)$  is dense in  $\hat{X}$  (the closure of  $i_X(X)$  in  $\hat{X}$  is equal to  $i_X(X)$  (or to say  $\hat{X}$ ))

**Proof**

Let  $[x]$  be an equivalence class in  $\hat{X}$ . We claim that  $\forall (x_n)_{n \in \mathbb{N}} \in [x]$

$$\lim_{n \rightarrow +\infty} x_n = \lim_{n \rightarrow +\infty} i_X(x_n)$$

For any  $N \in \mathbb{N}$

$$\begin{aligned} 0 \leq \hat{d}(i_X(x_N), [x]) &= \lim_{n \rightarrow +\infty} d(x_N, x_n) \\ &\leq \sup_{(n,m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) \end{aligned}$$

Taking  $\lim_{N \rightarrow +\infty}$  we get

$$\lim_{N \rightarrow +\infty} \hat{d}(i_X(x_N), [x]) = 0$$

**25.8 Theorem**

$(\hat{X}, \hat{d})$  is a complete metric space

**Proof**

Let  $([x^{(N)}])_{N \in \mathbb{N}}$  be a Cauchy sequence in  $\hat{X}$ , where  $\forall N \in \mathbb{N}$ ,  $x^{(N)} = (x_n^{(N)})_{n \in \mathbb{N}}$  is a Cauchy sequence  
 $\forall \epsilon > 0$ ,  $\exists N_0 \in \mathbb{N}$  such that  $\forall (k, l) \in \mathbb{N}_{\geq N_0}$

$$\hat{d}([x^{(k)}], [x^{(l)}]) = \lim_{n \rightarrow +\infty} d(x_n^{(k)}, x_n^{(l)}) \leq \epsilon$$

$\forall N \in \mathbb{N}$

$$d(x_\mu^{(N)}, x_\nu^{(N)}) \leq \frac{1}{N+1}$$

for any  $(\mu, \nu) \in \mathbb{N}_{\geq \alpha(N)}$

Let  $y_N = x_{\alpha(N)}^{(N)}$ . Without loss of generality, we assume that

$$\alpha(0) \leq \alpha(1) \leq \dots$$

Let  $\epsilon > 0$  Take  $N_0 \in \mathbb{N}$  such that

$$(1) \quad \forall (k, l) \in \mathbb{N}, \quad k, l \geq N_0$$

$$\hat{d}([x^{(k)}], [x^{(l)}]) \leq \frac{\epsilon}{3}$$

$$(2)$$

$$\frac{1}{N_0 + 1} \leq \frac{\epsilon}{3}$$

Let  $(k, l) \in \mathbb{N}_{N_0}^2$ ,

$$d(y_k, y_l) = d(x_{\alpha(k)}^{(k)}, x_{\alpha(l)}^{(l)})$$

Since  $\alpha(k) \geq N_0, \forall n \in \mathbb{N}_{\geq N_0}$

$$\begin{aligned} d(y_k, y_l) &\leq d(x_{\alpha(k)}^{(k)}, x_n^{(k)}) + d(x_n^{(k)}, x_n^{(k)}) + d(x_n^{(l)}, x_{\alpha(l)}^{(l)}) \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + d(x_n^{(k)}, x_n^{(l)}) \end{aligned}$$

Taking  $\lim_{n \rightarrow +\infty}$  get

$$d(y_k, y_l) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

So  $y = (y_N)_{N \in \mathbb{N}}$  is a Cauchy sequence. We check that

$$\begin{aligned} \lim_{N \rightarrow +\infty} \hat{d}([x^{(N)}], [y]) &= 0 \\ 0 &\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, x_{\alpha(n)}^{(N)}) \\ &\leq \lim_{N \rightarrow +\infty} \frac{1}{N+1} = 0 \end{aligned}$$

$n \geq \alpha(N)$

$$\begin{aligned} d(x_n^{(N)}, y_n) &\leq d(x_n^{(N)}, y_N) + d(y_n, y_N) \\ \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(x_n^{(N)}, y_n) &\leq \limsup_{N \rightarrow +\infty} \left( \frac{1}{N+1} + \lim_{n \rightarrow +\infty} d(y_n, y_N) \right) \end{aligned}$$

Since  $y$  is Cauchy sequence

$$\leq \limsup_{N \rightarrow +\infty} \lim_{n \rightarrow +\infty} d(y_n, y_N) = 0$$

### Example

Let  $(K, |\cdot|)$  be a valued field.

$$|\cdot| : \mathbb{R}_{\geq 0}$$

- $\forall a \in K, |a| = 0$  iff  $a = 0$
- $|ab| = |a| \cdot |b|$
- $|a+b| \leq |a| + |b|$

This is a metric space with

$$d(a, b) := |a - b|$$

$\text{Cau}(K)$  forms a commutative unitary ring.

$$(a_n)_{n \in \mathbb{N}} \sim (b_n)_{n \in \mathbb{N}}$$

iff

$$\lim_{n \rightarrow +\infty} (a_n - b_n) = 0$$

Then

$$(a_n - b_n)_{n \in \mathbb{N}} \in \text{Cau}_0(K)$$

where

$$\text{Cau}_0(K) = \{\text{Cauchy sequences that converges to } 0\}$$

This is an ideal of  $\text{Cau}(K)$

Hence

$$\hat{K} = \text{Cau}(K) \setminus \text{Cau}_0(K)$$

is a quotient ring of  $\text{Cau}(K)$

$|\cdot|$  extend to  $\hat{K}$ :

$$|[(a_n)_{n \in \mathbb{N}}]| = \lim_{n \rightarrow +\infty} |a_n|$$

that forms an absolute value.

# Chapter 26

## Norms

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 26.1 Def

Let  $V$  be a vector space over  $K$  ( $K$ -module). We call seminorm on  $V$  any mapping

$$\begin{aligned} \|\cdot\| : V &\rightarrow \mathbb{R}_{\geq 0} \\ s &\mapsto \|s\| \end{aligned}$$

such that

- $\forall (a, s) \in K \times V, \|as\| = |a| \cdot \|s\|$
- $\forall (s, t) \in V \times V, \|s + t\| \leq \|s\| + \|t\|$

If additionally:

- $\forall s \in V, \|s\| = 0$  iff  $s = 0$

We say that  $\|\cdot\|$  is a norm and  $(V, \|\cdot\|)$  is normed space over  $K$ .

### 26.2 Remark

If  $\|\cdot\|$  is a norm then

$$\begin{aligned} d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (s, t) &\mapsto \|s - t\| \end{aligned}$$

sectionDef Let  $(V, \|\cdot\|)$  be a vector space over  $K$  equipped with a seminorm, and  $W$  be a vector space subspace of  $V$  (sub- $K$ -module)

- The restriction of  $\|\cdot\| : V \rightarrow \mathbb{R}_{\geq 0}$  to  $W$  forms a seminorm on  $W$ . It is a norm if  $\|\cdot\|$  is a norm.

$$\begin{aligned}\|\cdot\|_W : W &\rightarrow \mathbb{R}_{\geq 0} \\ x &\mapsto \|x\|\end{aligned}$$

- The mapping

$$\begin{aligned}\|\cdot\|_{V/W} : V/W &\rightarrow \mathbb{R}_{\geq 0} \\ \alpha &\mapsto \inf_{s \in \alpha} \|s\| \\ \|[s]\|_{V/W} &= \inf_{w \in W} \|s + w\|\end{aligned}$$

is a seminorm on  $V/W$

**Attention:** Even if  $\|\cdot\|$  is a norm,  $\|\cdot\|_{V/W}$  **might only be a seminorm**

### 26.3 Def

$\|\cdot\|_{V/W}$  is called the quotient seminorm of  $\|\cdot\|$

### 26.4 Prop

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. Then

$$N = \{s \in V \mid \|s\| = 0\}$$

forms a vector subspace of  $V$ . Moreover,  $\|\cdot\|_{V/N}$  is a norm

#### Proof

If  $(a, s) \in K \times N$  then  $\|as\| = |a| \cdot \|s\| = 0$  so  $as \in N$

If  $(s_1, s_2) \in N \times N$  then  $0 \leq \|s_1 + s_2\| \leq \|s_1\| + \|s_2\| = 0$  so  $s_1 + s_2 \in N$

#### Proof

$$\begin{aligned}\|\lambda\alpha\|_{V/W} &= \inf_{s \in \alpha} \|\lambda s\| = \inf_{s \in \alpha} |\lambda| \cdot \|s\| = |\lambda| \cdot \|\alpha\|_{V/W} \\ \|\alpha + \beta\| &= \inf_{s \in \alpha + \beta} \|s\| = \inf_{(x, y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x, y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \|\alpha\|_{V/W} + \|\beta\|_{V/W}\end{aligned}$$

Let  $\alpha \in V/N$  such that  $\|\alpha\|_{V/N} = 0$  Let  $s \in \alpha, \forall t \in N$

$$\|s + t\| \leq \|s\| + \|t\| = \|s\| = \|(s + t) + (-t)\| \leq \|s + t\| + \|-t\| = \|s + t\|$$

$$\|\alpha\|_{V/N} = \inf_{t \in N} \|s + t\| = \|s\|$$

Hence  $\|\alpha\|_{V/N} = \|s\| = 0$  We obtain that  $\alpha = N = [0]$



## 26.5 Def

Let  $(V, \|\cdot\|)$  be a vector space over  $K$ , equipped with a seminorm. For any  $x \in V$  and  $r \geq 0$ , we denote by

$$\mathcal{B}(x, r) = \{y \in V \mid \|y - x\| < r\}$$

$$\overline{\mathcal{B}}(x, r) = \{y \in V \mid \|y - x\| \leq r\}$$

## 26.6 Remark

If  $N = \{s \in V, \|s\| = 0\}$  then when  $r > 0$

$$x + N \subseteq \overline{\mathcal{B}}(x, r)$$

$$x + N \subseteq \mathcal{B}(x, r)$$

## 26.7 Def

We equip the topology such that  $\forall U \subseteq V, U$  is open iff  $\forall x \in U, \exists r_x > 0, \mathcal{B}(x, r_x) \subseteq U$

## 26.8 Prop

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorms. Let  $f : V_1 \rightarrow V_2$  be a  $K$ -linear mapping

- If  $f$  is continuous,  $\forall s \in V_1$  if  $\|s\|_1 = 0$  then  $\|f(s)\|_2 = 0$
- If there exists  $C > 0$  such that  $\forall x \in V_1, \|f(x)\|_2 \leq C\|x\|_1$  then  $f$  is continuous.

The converse is true

when  $|\cdot|$  is non-trivial

or  $V_2/\{y \in V_2 \mid \|y\|_2 = 0\}$  is of finite type

### Proof

- (1) Lemma If  $(V, \|\cdot\|)$  is a vector space over  $K$ , equipped with a seminorm, then

$$N_{\|\cdot\|} := \{s \in V \mid \|s\| = 0\}$$

is closed.

Proof of lemma Let  $s \in V \setminus N_{\|\cdot\|}$  Then  $\|s\| > 0$ . Let  $\epsilon = \frac{\|s\|}{2}$ ,  $\forall x \in \mathcal{B}(s, \epsilon)$

$$\|x\| \geq \|s\| - \|s - x\| \geq \|s\| - \epsilon = \epsilon > 0$$

So

$$\mathcal{B}(s, \epsilon) \subseteq V \setminus N_{\|\cdot\|}$$

– Then  $f^{-1}(N_{\|\cdot\|_2})$  is closed.

Note that

$$0 \in f^{-1}(N_{\|\cdot\|_2})$$

hence

$$\overline{\{0\}} \subseteq f^{-1}(N_{\|\cdot\|_2})$$

$$\forall x \in N_{\|\cdot\|_1}, \forall \epsilon > 0$$

$$x + N_{\|\cdot\|_1} \subseteq \mathcal{B}(x, \epsilon)$$

and

$$0 \in \mathcal{B}(x, \epsilon)$$

Therefore  $x \in \overline{\{0\}}$

(2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence of  $V_1$  that converges to some  $x \in V_1$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \|f(x_n) - f(x)\|_2 &= \limsup_{n \rightarrow +\infty} \|f(x_n - x)\| \\ &\leq \limsup_{n \rightarrow +\infty} C \|x_n - x\|_1 \\ &= C \limsup_{n \rightarrow +\infty} \|x_n - x\| \\ &= 0 \end{aligned}$$

So  $(f(x_n))_{n \in \mathbb{N}}$  converges to  $f(x)$ . Hence  $f$  is continuous at  $x$

Assume that  $|\cdot|$  is non-trivial and  $f$  is continuous. Then

$$f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

is an open subset of  $V_1$  containing  $0 \in V_1$

So there exists  $\epsilon > 0$  such that

$$\{x \in V_1 \mid \|x\|_1 \leq \epsilon\} \subseteq f^{-1}(\{y \in V_2 \mid \|y\|_2 < 1\})$$

namely  $\forall x \in V_1$  if  $\|x\|_1 < \epsilon$  then  $\|f(x)\|_2 < 1$

Since  $|\cdot|$  is nontrivial,  $\exists a \in K, 0 < |a| < 1$  We prove that  $\forall x \in V_1$

$$\|f(x)\|_2 \leq \frac{1}{\epsilon|a|} \|x\|_1$$

If  $\|x\|_1 = 0$  by (1) we obtain

$$\|f(x)\|_2 = 0$$

Suppose that  $\|x\|_1 > 0$  then  $\exists n \in \mathbb{Z}$  such that

$$\begin{aligned} \|a^n x\|_1 &= |a|^n \|x\|_1 \\ &< \epsilon \leq \\ &\|a^{n-1} x\|_1 = |a|^{n-1} \|x\|_1 \end{aligned}$$

Thus

$$\|f(a^n x)\|_2 < 1$$

Hence

$$\begin{aligned} \|f(x)\|_2 &< \frac{1}{|a|^n} = \frac{1}{|a|^{n-1}} \frac{1}{|a|} \\ &\leq \frac{1}{\epsilon} \|x\|_1 \frac{1}{|a|} = \frac{\|x\|_1}{\epsilon|a|} \end{aligned}$$

## 26.9 Def: Operator Seminorm

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with seminorm. We say that a  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  is bounded if there exists  $C > 0$  that

$$\forall x \in V_1 \quad \|f(x)\|_2 \leq C\|x\|_1$$

For a general  $K$ -linear mapping  $f : V_1 \rightarrow V_2$  we denote

$$\|f\| := \begin{cases} \sup_{x \in V_1, \|x\|_1 > 0} \left( \frac{\|f(x)\|_2}{\|x\|_1} \right) & \text{if } f(N_{\|\cdot\|_1} \subseteq N_{\|\cdot\|_2}) \\ +\infty & \text{if } f(N_{\|\cdot\|_1} \not\subseteq N_{\|\cdot\|_2}) \end{cases}$$

$f$  is bounded iff

$$\|f\| < +\infty$$

$\|f\|$  is called the operator seminorm of  $f$

We denote by  $\mathcal{L}(V_1, V_2)$  the set of all bounded  $K$ -linear mappings from  $V_1$  to  $V_2$

## 26.10 Prop

$\mathcal{L}(V_1, V_2)$  is a vector subspace of  $\text{Hom}_K(V_1, V_2)$ . Moreover  $\|\cdot\|$  is a seminorm on  $\mathcal{L}(V_1, V_2)$

### Proof

Let  $f, g$  be elements of  $\mathcal{L}(V_1, V_2)$

$$\begin{aligned} \|f + g\| &= \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x) + g(x)\|_2}{\|x\|_1} \\ &\leq \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2 + \|g(x)\|_2}{\|x\|_1} \\ &\leq \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|f(x)\|_2}{\|x\|_1} \right) + \left( \sup_{x \in V_1, \|x\|_1 \neq 0} \frac{\|g(x)\|_2}{\|x\|_1} \right) \\ &\leq +\infty \end{aligned}$$

Hence  $f + g \in \mathcal{L}(V_1, V_2)$

Let  $\lambda \in K$ ,  $\lambda f : x \mapsto \lambda f(x)$

$$\begin{aligned}\|\lambda f\| &= \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|\lambda f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \sup_{x \in V_1, \|x\|_1 > 0} \frac{\|f(x)\|_2}{\|x\|_1} \\ &= |\lambda| \|f\| < +\infty\end{aligned}$$

### 26.11 Remark

Let  $f \in \mathcal{L}(V_1, V_2)$ . Suppose that  $\exists x \in V_1$  such that  $f(x) \neq 0$ . Since

$$f(x) \notin N_{\|\cdot\|_2} = \{0\}$$

we obtain

$$\|x\|_1 = 0$$

Thus

$$\|f\| \geq \frac{\|f(x)\|_2}{\|x\|_1} > 0$$

Therefore  $\|\cdot\|$  is a norm

### 26.12 Def

Let  $(V, \|\cdot\|)$  be a normed vector space. If  $V$  is complete with respect to the metric

$$\begin{aligned}d : V \times V &\rightarrow \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \|x - y\|\end{aligned}$$

then we say that  $(V, \|\cdot\|)$  is a Banach space.

### 26.13 Theorem

Let  $(V_1, \|\cdot\|_1)$  and  $(V_2, \|\cdot\|_2)$  be vector spaces over  $K$ , equipped with semi-norm. If  $(V_2, \|\cdot\|_2)$  is a Banach space, then

$$(\mathcal{L}(V_1, V_2), \|\cdot\|)$$

is a Banach space

**Proof**

Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $\mathcal{L}(V_1, V_2)$ .  
 $\forall x \in V_1$ , the mapping

$$(f \in \mathcal{L}(V_1, V_2)) \mapsto f(x)$$

is  $\|x\|_1$ -Lipschitzian mapping:

$$\|f(x) - g(x)\|_2 = \|(f - g)(x)\|_2 \leq \|f - g\| \|x\|_1$$

So  $(f_n(x))_{n \in \mathbb{N}}$  is a Cauchy sequence, for  $V_2$  is complete, that converges to some  $g(x) \in V_2$ . Then we obtain a mapping  $g : V_1 \rightarrow V_2$ . We prove that  $g$  is an element of  $\mathcal{L}(V_1, V_2)$

- $\forall (x, y) \in V_1^2$

$$g(x, y) = \lim_{n \rightarrow +\infty} f_n(x + y) = \lim_{n \rightarrow +\infty} f_n(x) + f_n(y)$$

$$\begin{aligned} \|f_n(x) + f_n(y) - g(x) - g(y)\| &\leq \|f_n(x) - g(x)\| + \|f_n(y) - g(y)\| \\ &= o(1) + o(1) = o(1), (n \rightarrow +\infty) \end{aligned}$$

So

$$\lim_{n \rightarrow +\infty} f_n(x) + f_n(y) = g(x) + g(y)$$

- $\forall x \in V_1, \lambda \in K$

$$g(\lambda x) = \lim_{n \rightarrow +\infty} f_n(\lambda x) = \lim_{n \rightarrow +\infty} \lambda f_n(x)$$

$$\|\lambda f_n(x) - \lambda g(x)\| = |\lambda| \cdot \|f_n(x) - g(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g(\lambda x) = \lambda g(x)$

- $\forall x \in V_1$

$$\|g(x)\| = \lim_{n \rightarrow +\infty} \|f_n(x)\| \leq (\lim_{n \rightarrow +\infty} \|f_n\|) \cdot \|x\|$$

(because  $\forall (a, b) \in V_2^2 \quad \|a\| - \|b\| \leq \|a - b\|$ ) Then

$$\|f_n(x)\| - \|g_n(x)\| \leq \|f_n(x) - g_n(x)\| = o(1) (n \rightarrow +\infty)$$

So  $g \in \mathcal{L}(V_1, V_2)$

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall (n, m) \in \mathbb{N}_{\geq N}, \|f_n - f_m\| \leq \epsilon$$

$\forall x \in V_1$

$$\|(f_n - f_m)(x)\| \leq \epsilon \cdot \|x\|$$

Taking  $\lim_{n \rightarrow +\infty}$  we get

$$\|(f_n - g)(x)\| \leq \epsilon \|x\|$$

So  $\forall n \in \mathbb{N}, n \geq N$

$$\|f_n - g\| \leq \epsilon$$



## Chapter 27

# Differentiability

In this chapter we fix a field  $K$  and an absolute value  $|\cdot|$  on  $K$ . We assume that  $(K, |\cdot|)$  forms a complete metric space with respect to the metric:

$$\begin{aligned} K \times K &\rightarrow \mathbb{R}_{\geq 0} \\ (a, b) &\mapsto |a - b| \end{aligned}$$

### 27.1 Def

Let  $X$  be a topological space and  $p \in X$ . Let  $K$  be a complete valued field and  $(E, \|\cdot\|)$  be a normed vector space over  $K$ .

Let  $f : X \rightarrow E$  be a mapping and  $g : X \rightarrow \mathbb{R}_{\geq 0}$  be a non-negative mapping.

- We say that

$$f(x) = O(g(x)) \text{ } x \rightarrow p$$

if there is a neighborhood  $V$  of  $p$  in  $X$  and a constant  $C > 0$  such that  $\forall x \in V$

$$\|f(x)\| \leq Cg(x)$$

- We say that

$$f(x) = o(g(x)) \text{ } x \rightarrow p$$

if there exists a neighborhood  $V$  of  $p$  in  $X$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that

$$\lim_{x \in V, x \rightarrow p} \epsilon(x) = 0$$

which is equivalent to

$$\forall \delta > 0, \exists \text{ neighborhood } U \text{ of } p \text{ } U \subseteq V \text{ and } \forall x \in U, 0 \leq \epsilon(x) \leq \delta$$

and  $\forall x \in V$

$$\|f(x)\| \leq \epsilon(x)g(x)$$

## 27.2 Def

Let  $E$  and  $F$  be normed vector space over  $K$ .  $U \subseteq E$  be an open subset,  $f : U \rightarrow F$  be a mapping and  $p \in U$ . If there exists  $\varphi \in \mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

we say that  $f$  is differentiable at  $p$ , and  $\varphi$  is the differential of  $f$  at  $p$ . Suppose that  $|\cdot|$  is not trivial.  $\varphi(x - p)$  also written as

$$d_p f$$

### Reminder

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|) \quad x \rightarrow p$$

means there exists an open neighborhood  $V$  of  $p$  with  $V \subseteq U$  and a mapping  $\epsilon : V \rightarrow \mathbb{R}_{\geq 0}$  such that  $\lim_{x \rightarrow p} \epsilon(x) = 0$  and that  $\forall x \in V$

$$\|f(x) - f(p) - \varphi(x - p)\| \leq \epsilon(x) \cdot \|x - p\|$$

## 27.3 Prop

If  $f$  is differentiable at  $p$ , then its differential at  $p$  is unique

### Proof

Suppose that there exists  $\varphi$  and  $\psi$  in  $\mathcal{L}(E, F)$  such that

$$f(x) = f(p) + \varphi(x - p) + o(\|x - p\|)$$

$$f(x) = f(p) + \psi(x - p) + o(\|x - p\|)$$

then

$$(\varphi - \psi)(x - p) = o(\|x - p\|)$$

$\forall \delta > 0$

$$\|\varphi - \psi\| = \sup_{y \in E \setminus \{0\}} \frac{\|\varphi - \psi\|}{\|y\|} = \sup_{y \in E \setminus \{0\}, \|y\| \leq \delta} \frac{\|(\varphi - \psi)(y)\|}{\|y\|}$$

Therefore

$$\begin{aligned} \|\varphi - \psi\| &= \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \frac{\|\varphi - \psi\| (y - p)}{\|y - p\|} \\ &\leq \inf_{\delta > 0} \sup_{y \in E, 0 < \|y - p\| \leq \delta} \epsilon(y) \\ &= \limsup_{y \rightarrow p} \epsilon(y) = 0 \end{aligned}$$



## 27.4 Example

### 27.4.1

$$f : U \rightarrow F : f(x) = y_0 \quad \forall x \in U$$

$$\forall p \in U$$

$$f(x) - f(p) = 0 = 0 + o(\|x - p\|)$$

Hence  $\forall x \in E$

$$d_p(f(x)) = 0$$

### 27.4.2

Let  $f \in \mathcal{L}(E, F)$

$$f(x) - f(p) = f(x - p)$$

Hence  $d_p f = f$

### 27.4.3

$$A : E \times E \rightarrow E$$

$$(x, y) \mapsto x + y$$

Let  $E$  be a normed space. Then  $\forall (p, q) \in E \times E$

$$d_{(p,q)} A = A$$

### 27.4.4

$$m : K \times E \rightarrow E$$

$$(\lambda, x) \mapsto \lambda x$$

Let  $(a, p) \in K \times E$

$$\begin{aligned} \lambda x - ap &= \lambda x - ax + ax - ap \\ &= (\lambda - a)x + a(x - p) \\ &= (\lambda - a)p + a(x - p) + (\lambda - a)(x - p) \end{aligned}$$

- when  $(\lambda, x) \rightarrow (a, p)$

$$\begin{aligned} \|(\lambda - a)(x - p)\| &= |\lambda - a| \cdot \|x - p\| \\ &= o(\max\{|\lambda - a|, \|x - p\|\}) \end{aligned}$$

- The mapping

$$((\mu, y) \in K \times E) \mapsto \mu p + ay \in E$$

is a  $K$ -linear mapping.

$$\begin{aligned}
- & (\mu_1 + \mu_2)p + a(y_1 + y_2) = (\mu_1 p + ay_1) + (\mu_2 p + ay_2) \\
- & b\mu p + a(by) = b(\mu p + ay) \\
- & \|\mu p + ay\| \leq |\mu| \|p\| + |a| \|y\| \\
& \leq \max\{|\mu|, \|y\|\}(|a| + \|p\|)
\end{aligned}$$

Hence  $m$  is differentiable and  $\forall (\mu, y) \in K \times E$

$$d_{(a,p)}m(\mu, y) = \mu p + ay$$

## 27.5 Theorem:Chain rule

Let  $E, F, G$  be normed vector spaces,  $U \subseteq E, V \subseteq F$  be open subsets.

Let  $f : U \rightarrow F, g : V \rightarrow G$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . Assume that  $f$  is differentiable at  $p$  and  $g$  differentiable at  $f(p)$ . Then  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

### Proof

Let  $x \in U$ . By definition

$$\begin{aligned}
f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\
f(x) - f(p) &= O(\|x - p\|)
\end{aligned}$$

and

$$\begin{aligned}
(g \circ f)(x) &= g(f(p)) + d_{f(p)}g(f(x) - f(p)) + o(\|f(x) - f(p)\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p) + o(\|x - p\|)) + o(\|x - p\|) \\
&= g(f(p)) + d_{f(p)}g(d_p f(x - p)) + o(\|x - p\|)
\end{aligned}$$

So  $g \circ f$  is differentiable at  $p$  and

$$d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

## 27.6 Prop

Let  $n$  be a positive integer. Let  $(F_i)_{i \in \{1, \dots, n\}}$  be normed vector spaces over  $K$ . Let  $U \subseteq E$  be an open subset,  $p \in U$ .

$\forall i \in \{1, \dots, n\}$  let  $f_i : U \rightarrow F_i$  be a mapping. Let

$$f : U \rightarrow F = \prod F_i$$

be the mapping that sends  $x \in U$  to  $(f_i(x))_{i \in \{1, \dots, n\}}$ . We equip  $F$  with the norm  $\|\cdot\|$  defined as :

$$\|(y_i)_{i \in \{1, \dots, n\}}\| = \max_{i \in \{1, \dots, n\}} \|y_i\|$$

Then  $f$  is differentiable at  $p$  iff each  $f_i$  is differentiable at  $p$ . Moreover, when this happen, one has

$$\forall x \in E \quad d_p f(x) = (d_p f_i(x))_{i \in \{1, \dots, n\}}$$

### Proof

$\Leftarrow$  Suppose that  $(f_i)_{i \in \{1, \dots, n\}}$  are differentiable at  $p$

$$\begin{aligned} f(x) - f(p) &= (f_i(x) - f_i(p))_{i \in \{1, \dots, n\}} \\ &= (d_p f_i(x - p))_{i \in \{1, \dots, n\}} + o(\|x - p\|) \end{aligned}$$

Therefore  $f$  is differentiable at  $p$  and

$$d_p f(\cdot) = (d_p f_i(\cdot))_{i \in \{1, \dots, n\}}$$

$\Rightarrow$  Let

$$\begin{aligned} \pi_i : F &\rightarrow F_i \\ (x_i)_{i \in \{1, \dots, n\}} &\mapsto x_i \end{aligned}$$

is a bounded linear mapping, one has  $\|\pi_i\| \leq 1$  because

$$\|x_i\| \leq \max_{i \in \{1, \dots, n\}} \|x_i\| = \|(x_i)_{i \in \{1, \dots, n\}}\|$$

$\pi_i$  is differentiable at  $p$  then  $\pi_i \circ f = f_i$  is differentiable at  $p$

## 27.7 Def

Let  $U$  be an open subset of  $K$  and  $(F, \|\cdot\|)$  be a normed vector space. If  $f : U \rightarrow F$  is a mapping that is differentiable at some  $p \in U$ . We denote by  $f'(p)$  the element

$$d_p f(1) \in F$$

called the derivative of  $f$  at  $p$

## 27.8 Corollary

Let  $U$  and  $V$  be open subsets of  $K$ ,  $(F, \|\cdot\|)$  be a normed vector space over  $K$ .  $f : U \rightarrow K$ ,  $g : V \rightarrow F$  be mappings such that  $f(U) \subseteq V$ . Let  $p \in U$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$  then

$$(g \circ f)'(p) = f'(p)g'(f(p))$$

By definition

$$\begin{aligned} d_p(g \circ f)(1) &= d_{f(p)}g(d_P(f)(1)) \\ &= d_{f(p)}g(f'(p)) \\ &= d_{f(p)}g(f'(p) \cdot 1) \\ &= f'(p) \cdot d_{f(p)}g(1) \\ &= f'(p)g'(f(p)) \end{aligned}$$

Let  $E$  and  $F$  be normed vector spaces,  $U \subseteq E$  an open subset.  $f : U \rightarrow L$  and  $g : U \rightarrow F$  be mappings and  $p \in U$ . If both  $f, g$  differentiable at  $p$  then

is also differentiable at  $p$  and

$$\forall l \in E \quad d_p(fg)(l) = f(p)d_p f(l) + g(p)d_p f(l)$$

Consider

$$\begin{aligned} m : K \times F &\rightarrow F \\ (a, y) &\rightarrow ay \end{aligned}$$

We have shown  $m$  is differentiable and

$$d_{a,y}m(b,z) = by = az$$

$fg$  is the following composite:

$$U \begin{array}{c} \xrightarrow{\quad h \quad} \\ \searrow \scriptstyle fg \\ \end{array} K \times F \xrightarrow{\quad m \quad} F$$

$$x \mapsto (f(x), g(x)) \mapsto f(x)g(x)$$

$$\begin{aligned} d_p(fg)(l) &= d_p(m \circ h)(l) \\ &= d_{h(p)}m(d_ph(l)) \\ &= d_{(f(p), g(p))}m(d_pf(l), d_pg(l)) \\ &= f(p)d_pg(l) + d_pf(l)g(p) \end{aligned}$$

## 27.10 Corollary

Let  $U$  be an open subset of  $K$ ,  $f, g$  be mappings from  $U$  to  $K$  and to a normed space  $F$  respectively. If  $f, g$  are differentiable at  $p \in U$  then

$$(fg)'(p) = d_p(fg)(1) = d_p f(1)g(p) + f(p)d_p g(1) = f'(p)g(p) + f(p)g'(p)$$

### Example

$$\begin{aligned} f_n : K &\rightarrow K \\ x &\mapsto x^n \end{aligned}$$

is differentiable at any  $x \in K$

$$f'_n(x) = nx^{n-1}$$

### Proof

$f_1 : K \rightarrow K$  is differentiable  $\forall x \in K$

$$d_x f_1 = f_1$$

If  $f'_n(x) = nx^{n-1}$  then

$$\begin{aligned} f'_{n+1}(x) &= (f_n f_1)'(x) \\ &= f_n(x)f'_1(x) + f'_n(x)f_1(x) \\ &= x^n + x'_n(x) = x^n + nx^{n-1} \\ &= (n+1)x^n \end{aligned}$$

and

$$\begin{aligned} d_x f_n(1) &= l d_x f_n(1) \\ &= nx^{n-1} \end{aligned}$$

## 27.11 Prop

Let  $E, F, G$  be normed vector spaces.  $U \subseteq E$  be an open subset,  $\varphi \in \mathcal{L}(F, G)$ ,  $p \in U$  if  $f : U \rightarrow E$  is differentiable at  $p$  then so is  $\varphi \circ f$ . Moreover

$$d_p(\varphi \circ f) = \varphi \circ d_p(f)$$

### Proof

$\varphi$  is differentiable at  $f(p)$  nad  $d_{f(p)}\varphi = \varphi$

### 27.12 Corollary

Let  $E$  and  $F$  be normed vector spaces  $U \subseteq E$  be an open subset,  $p \in U$ . Let  $f : U \rightarrow F$  and  $g : U \rightarrow F$  be mappings that are differentiable at  $p$ ,  $(a, b) \in K \times K$ . Then  $af + bg$  is differentiable at  $p$  and

$$d_p(af + bg) = ad_p f + bd_p g$$

#### Proof

$af + bg$  is composite:

$$U \xrightarrow{h} K \times F \xrightarrow{m} F$$

$ay+bz$

$$x \longmapsto (f(x), g(x)) \longmapsto af(x) + bg(x)$$

$$\begin{aligned} \|ay + bz\| &\leq |a| \cdot \|y\| + |b| \cdot \|z\| \\ &\leq (|a| + |b|) \max\{\|y\|, \|z\|\} \end{aligned}$$

### 27.13 Def: Equivalence of Norms

Let  $E$  be a vector space over  $K$  and  $\|\cdot\|_1, \|\cdot\|_2$  be norms on  $E$ . We say that  $\|\cdot\|_1$  and  $\|\cdot\|_2$  are equivalent if there exist constants  $C_1, C_2 > 0$  such that  $\forall s \in E$

$$C_1 \|s\|_1 \leq \|s\|_2 \leq C_2 \|s\|_1$$

### 27.14 Prop

If  $\|\cdot\|_1, \|\cdot\|_2$  are equivalent, then

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are bounded linear mappings. Moreover  $\|\cdot\|_1, \|\cdot\|_2$  defines the same topology on  $E$ .

#### Proof

$$\|s\|_2 \leq C_2 \|s\|_1 \leq C_1^{-1} \|s\|_2$$

So the linear mappings are bounded. Hence

$$Id_E : (E, \|\cdot\|_1) \rightarrow (E, \|\cdot\|_2)$$

$$Id_E : (E, \|\cdot\|_2) \rightarrow (E, \|\cdot\|_1)$$

are continuous. So  $\forall$  open subset  $U$  of  $(E, \|\cdot\|_2)$

$$Id_E^{-1}(U) = U$$

is open in  $(E, \|\cdot\|_1)$ . Conversely if  $V$  is open in  $(E, \|\cdot\|_1)$  then

$$V = Id_E^{-1}(V)$$

is open in  $(E, \|\cdot\|_2)$

## 27.15 Remark

If  $\|\cdot\|_1, \|\cdot\|_2$  are two norms on  $E$  that define the same topology on  $E$ , then they are equivalent (under the assumption that  $|\cdot|$  is not trivial)

## 27.16 Prop

Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be normed vector spaces  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$  be norms on  $E$  and  $F$  that are equivalent to  $\|\cdot\|_E, \|\cdot\|_F$  respectively. Let  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping.

Let  $p \in U$  Then  $f$  is differentiable at  $p$  with respect to  $\|\cdot\|_E$  and  $\|\cdot\|_F$  iff it's differentiable with respect to  $\|\cdot\|'_E$  and  $\|\cdot\|'_F$

Moreover the differentiable of  $f$  at  $p$  is not changed in the change of norms from  $(\|\cdot\|_E, \|\cdot\|_F)$  to  $(\|\cdot\|'_E, \|\cdot\|'_F)$

### Proof

$$U \xrightarrow{Id_U} U \xrightarrow{f} F \xrightarrow{Id_F} F$$

$f$

$$(E, \|\cdot\|'_E) \quad (E, \|\cdot\|_E) \quad \|\cdot\|_F \quad \|\cdot\|'_F$$

$$\begin{aligned} d'_p f &= d_{f(p)} Id_F \circ d_p f \circ d_p Id_U \\ &= Id_F \circ d_p f \circ Id_E \\ &= d_p f \end{aligned}$$

$$d'_p f : (E, \|\cdot\|'_E) \rightarrow (F, \|\cdot\|'_F)$$

## 27.17 Theorem

Let  $V$  be a finite dimensional vector space over  $K$ . Then all norms on  $V$  are equivalent. Moreover  $V$  is complete with respect to any norm on  $V$ .

**Proof**

Let  $(e_i)_{i=1}^n$  be a basis of  $V$  (linear independent system of generators) The mapping:

$$V \rightarrow \mathbb{R}_{\geq 0}$$

$$\sum_{i \in \{1, \dots, n\}} a_i e_i \mapsto \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

is a norm on  $V$

Let  $\|\cdot\|$  be another norm on  $V$ . One has

$$\left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\| \leq \sum_{i \in \{1, \dots, n\}} |a_i| \|e_i\|$$

$$\leq \left( \sum_{i \in \{1, \dots, n\}} \|e_i\| \right) \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

We reason by induction that there exists  $C > 0$  such that

$$\max_{i \in \{1, \dots, n\}} \{|a_i|\} \leq C \left\| \sum_{i \in \{1, \dots, n\}} a_i e_i \right\|$$

The case where  $n = 0$  is trivial.

$n=1$

$$\|a_1 e_1\| = |a_1| \|e_1\| \quad |a_1| = \|e_1\|^{-1} \cdot \|a_1 e_1\|$$

Induction hypothesis true for vector spaces of dimension  $< n$

Let

$$W = \left\{ \sum_{i \in \{1, \dots, n-1\}} a_i e_i \mid (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1} \right\}$$

equipped with  $\|\cdot\|$  restricted to  $W$

The induction hypothesis shows that  $W$  is complete. Hence it's closed in  $V$ . Let  $Q = V/W$  and  $\|\cdot\|_Q$  be the quotient norm on  $Q$  that's defined as

$$\forall \alpha \in Q \quad \|\alpha\|_Q = \inf_{s \in \alpha} \|s\|$$

– If  $s \in V \setminus W$ ,  $\exists \epsilon > 0$  such that

$$\overline{B}(s, \epsilon) \cap W = \emptyset$$

$\forall t \in W$ ,

$$s + t \notin \overline{B}(0, \epsilon)$$

since otherwise

$$-t \in W \cap \overline{B}(s, \epsilon)$$

Therefore

$$\|[s]\|_Q = \inf_{i \in W} \|s + t\| \geq \epsilon > 0$$



–  $\forall \lambda \in K$

$$\begin{aligned}\|\lambda \alpha\|_Q &= \inf_{s \in \alpha} \|\lambda s\| = |\lambda| \\ \inf_{s \in \alpha} \|s\| &= |\lambda| \cdot \|\alpha\|_Q\end{aligned}$$

–

$$\begin{aligned}\|\alpha + \beta\|_Q &= \inf_{s \in \alpha + \beta} \|s\| \\ &= \inf_{(x,y) \in \alpha \times \beta} \|x + y\| \\ &\leq \inf_{(x,y) \in \alpha \times \beta} (\|x\| + \|y\|) \\ &= \inf_{x \in \alpha} \|x\| + \inf_{y \in \beta} \|y\|\end{aligned}$$

Applying the induction hypothesis then we obtain the existence of some  $A > 0$  such that  $\forall (a_i)_{i \in \{1, \dots, n-1\}} \in K^{n-1}$

$$\max_{i \in \{1, \dots, n-1\}} \{|a_i|\} \leq A \left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\|$$

Take

$$s = \sum_{i \in \{1, \dots, n\}} a_i e_i \in V$$

Let  $\alpha = [s] = a_n [e_n] \in Q$

$$\left\| \sum_{i \in \{1, \dots, n-1\}} a_i e_i \right\| = \|s - a_n e_n\| \leq \|s\| + |a_n| \cdot \|e_n\| \leq \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

$$\|\alpha\|_Q = |a_n| \|[e_n]\|_Q = |a_n| \inf_{t \in W} \|e_n + t\|$$

Take  $e'_n \in V$  such that  $[e'_n] = [e_n]$  and  $\|e'_n\| \leq \|[e_n]\|_Q + \epsilon$

Note that  $(e_1, \dots, e_{n-1}, e'_n)$  forms also basis of  $V$  over  $K$ . Hence by replacing  $e_n$  by  $e'_n$  we may assume that  $\|e_n\| \leq \|[e_n]\|_Q + \epsilon$

$s = a_n e_n + t \in V$  with  $t \in W$

$$\|s\| \geq \|a_n e_n\|_Q = |a_n| \|[e_n]\|_Q \geq B^{-1} |a_n| \cdot \|e_n\|$$

– If  $\|a_n e_n\| < \frac{1}{2} \|t\|$

$$\|s\| \geq \|t\| - \|a_n e_n\| > \frac{1}{2} \|t\| \geq \frac{1}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

– If  $\|a_n e_n\| \geq \frac{1}{2} \|t\|$

$$\|s\| \geq B^{-1} |a_n| \cdot \|e_n\| \geq \frac{B^{-1}}{2} \|t\| \geq \frac{B^{-1}A}{2} \max_{i \in \{1, \dots, n-1\}} \{|a_i|\}$$

We take  $C = \max\{B^{-1} \|e_n\|, \frac{A}{2}, \frac{B^{-1}A}{2}\}$  Then

$$\|s\| \geq C \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Another proof

completeness Under the norm  $\max_{i \in \{1, \dots, n\}}$ , a sequence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  is a Cauchy sequence iff  $\forall i \in \{1, \dots, n\}$   $(a_i^{(k)})_{k \in \mathbb{N}}$  is a Cauchy sequence. Since  $K$  is complete each  $(a_i^{(k)})_{k \in \mathbb{N}}$  converges to some  $a_i \in K$  Hence  $(a_i^{(k)} e_i)_{k \in \mathbb{N}, i \in \{1, \dots, n\}}$  converges.

## 27.18 Prop

Let  $(E, \|\cdot\|_E), (F, \|\cdot\|_F)$  be normed vector spaces over  $K$ . Assume that  $E$  is finite dimensional. Then any  $K$ -linear mapping  $\varphi : E \rightarrow F$  is bounded.

### Proof

Let  $(e_i)_{i=1}^n$  be a basis of  $E$ . For any two norms on  $E$  are equivalent.  
 $\forall (a_1, \dots, a_n) \in K$

$$\left\| \sum_{i=1}^n a_i e_i \right\|_E = \max_{i \in \{1, \dots, n\}} \{|a_i|\}$$

Then for any  $s = \sum_{i=1}^n a_i e_i$

$$\|\varphi(s)\|_F = \left\| \sum_{i=1}^n a_i e_i \right\| \leq \sum_{i=1}^n |a_i| \|\varphi(e_i)\| \leq \left( \sum_{i=1}^n \|\varphi(e_i)\|_F \right) \|s\|_E$$

## 27.19 Theorem

Let  $E, F$  be normed vector spaces over a complete valued field,  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping. If  $f$  is differentiable at  $p$  then  $f$  is continuous at  $p$

### Proof

$$\begin{aligned} f(x) &= f(p) + d_p f(x - p) + o(\|x - p\|) \\ &= f(p) + O(\|x - p\|) \\ &= f(p) + o(1) \quad x \rightarrow p \\ &\Rightarrow \lim_{x \rightarrow p} f(x) = f(p) \end{aligned}$$

## Chapter 28

# Compactness

### 28.1 Def: cover

Let  $X$  be a topological space,  $Y \subseteq X$  we call open cover of  $Y$  any family  $(U_i)_{i \in I}$  open subset of  $X$  such that

$$Y \subseteq \bigcup_{i \in I} U_i$$

If  $I$  is finite set, we say that  $(U_i)_{i \in I}$  is a finite open cover. If  $J \subseteq I$  such that

$$Y \subseteq \bigcup_{j \in J} U_j$$

then we say that  $(U_j)_{j \in J}$  is a sub cover of  $(U_i)_{i \in I}$

### 28.2 Def: compact

If any open cover of  $Y$  has a finite subcover, we say that  $Y$  is quasi-compact. If in addition  $X$  is Hausdorff, namely  $\forall (x, y) \in X \times X$  with  $x \neq y \exists$  open neighborhoods  $U$  and  $V$  of  $x$  and  $y$  such that  $U \cap V = \emptyset$ , we say that  $Y$  is compact

### 28.3 Def

Let  $X$  be a set and  $\mathcal{F}$  be a filter on  $X$ . If there does not exist any filter  $\mathcal{F}'$  of  $X$  such that  $\mathcal{F} \subsetneq \mathcal{F}'$ , then we say that  $\mathcal{F}$  is an ultrafilter.

**Zorn's lemma** implies that  $\forall \mathcal{F}_0$  of  $X$  there exist an ultrafilter  $\mathcal{F}$  if  $X$  containing  $\mathcal{F}_0$

## 28.4 Prop

Let  $\mathcal{F}$  be a filter on a set  $X$ . The following statements are equivalent.

- (1)  $\mathcal{F}$  is an ultrafilter
- (2)  $\forall A \subseteq X$  either  $A \in \mathcal{F}$  or  $X \setminus A \in \mathcal{F}$
- (3)  $\forall (A, B) \in \wp(X)^2$  if  $A \cap B \in \mathcal{F}$  then  $A \in \mathcal{F}$  or  $B \in \mathcal{F}$

### Proof

- (1)  $\Rightarrow$  (2) Suppose that  $A \in \wp(X)$  such that  $A \notin \mathcal{F}$  and  $X \setminus A \notin \mathcal{F} \forall B \in \mathcal{F}$  one has

$$B \cap A \neq \emptyset$$

since otherwise  $B \subseteq X \setminus A$  and hence  $X \setminus A \in \mathcal{F}$  contradiction.

- (2)  $\Rightarrow$  (3) Suppose that  $B \notin \mathcal{F}$  then  $X \setminus B \in \mathcal{F}$

$$(A \cup B) \cap (X \setminus B) = A \setminus B \in \mathcal{F}$$

So  $A \in \mathcal{F}$

- (3)  $\Rightarrow$  (1) Suppose that  $\mathcal{F}'$  is a filter such that  $\mathcal{F} \subsetneq \mathcal{F}'$  Take  $A \in \mathcal{F}' \setminus \mathcal{F}$  Then by  $X = A \cup (X \setminus A) \in \mathcal{F}$  Hence

$$X \setminus \mathcal{F} \subseteq \mathcal{F}' \quad \emptyset = A \cap (X \setminus A) \in \mathcal{F}'$$

which is impossible.

## 28.5 Theorem

Let  $(X, \mathcal{G})$  be a topological space . The following are equivalent

- (1)  $X$  is quasi-compact
- (2) Any filter of  $X$  has an accumulation point
- (3) Any ultrafilter of  $X$  is converges.

### Proof

- (1)  $\Rightarrow$  (2) Assume that a filter  $\mathcal{F}$  of  $X$  does not have any accumulation point.  $\forall x \in X \exists A_x \in \mathcal{F} \exists$  open neighborhood  $V_x$  of  $x$  such that  $A_x \cap V_x = \emptyset$  Since  $X = \bigcup_{x \in X} V_x$  there is

$$\{x_1, \dots, x_n\} \subseteq X$$

such that

$$X = \bigcup_{i=1}^n V_{x_i}$$

$$\text{Take } B = \bigcap_{i=1}^n A_{x_i} \in \mathcal{F}$$

$$B \cap X = B = \emptyset$$

Since  $\forall i \ B \cap V_x = \emptyset$  contradiction.

- (2)  $\Rightarrow$  (3) Let  $\mathcal{F}$  be an ultrafilter of  $X$ . By (2) there exist  $x \in X$  such that  $\mathcal{F} \cup \mathcal{V}_x$  generates a filter  $\mathcal{F}'$  Since  $\mathcal{F}$  is an ultrafilter  $\mathcal{F} = \mathcal{F}'$  and hence  $\mathcal{V}_x \subseteq \mathcal{F}$
- (3)  $\Rightarrow$  (1) Let  $(U_i)_{i \in I}$  be an open cover of  $X$  we suppose that this have no finite subcover.  $\forall i \in I$  let

$$F_i = X \setminus U_i$$

For any  $J \subseteq I$  finite

$$F_J = \bigcap_{j \in J} F_j = X \setminus \bigcup_{j \in J} U_j \neq \emptyset$$

Let  $\mathcal{F}$  be the smallest filter on  $X$  that contains

$$\{\mathcal{F}_J \mid J \subseteq I \text{ finite}\}$$

Let  $\mathcal{F}'$  be ultrafilter containing  $\mathcal{F}$ . It has a limit point  $x$  There exist  $i \in I$  such that  $x \in U_i$ . Since  $U_i$  is a neighborhood of  $x$  and  $\mathcal{V}_x \subseteq \mathcal{F}'$  we get  $U_i \in \mathcal{F}'$  This is impossible since  $F_i \in \mathcal{F}'$

## 28.6 Theorem

Let  $(X, d)$  be a metric space. The following statements are equivalent:

- (1)  $X$  is complete and  $\forall \epsilon > 0 \ \exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

- (2)  $X$  is compact

### Proof

- (1)  $\Rightarrow$  (2) Let  $\mathcal{F}$  be an ultrafilter Let  $\epsilon > 0$  and  $\{x_1, \dots, x_n\} \subseteq X$  such that

$$X = \bigcup_{i=1}^n \mathcal{B}(x_i, \epsilon)$$

There exists some  $i \in \{1, \dots, n\}$  such that  $\mathcal{B}(x_i, \epsilon) \in \mathcal{F}$  That means  $\mathcal{F}$  is a Cauchy filter (namely  $\forall \delta > 0 \ \exists A \in \mathcal{F}$  of diameter  $\leq \delta$ ) Since  $X$  is complete  $\mathcal{F}$  has a limit point. So  $\mathcal{F}$  is compact.

(2)  $\Rightarrow$  (1) Let  $\epsilon > 0$  One has

$$X = \bigcup_{x \in X} \mathcal{B}(x, \epsilon)$$

Since  $X$  is compact  $\exists X_\epsilon \subseteq X$  finite such that

$$X = \bigcup_{x \in X_\epsilon} \mathcal{B}(x, \epsilon)$$

$\mathcal{F}$  is an ultrafilter

$$\Leftrightarrow \forall A \subseteq X \ A \in \mathcal{F} \text{ or } X \setminus A \in \mathcal{F}$$

$$\Leftrightarrow \forall y \in \mathcal{F} \text{ if } y = A \cup B \text{ either } A \in \mathcal{F} \text{ or } B \in \mathcal{F}$$

$$\Leftrightarrow \forall Y \in \mathcal{F} \text{ if } Y = A_1 \cup A_2 \cup \dots \cup A_n \ \exists i \in \{1, \dots, n\}, A_i \in \mathcal{F}$$

Let  $\mathcal{F}$  be a Cauchy filter Let  $x \in X$  be an accumulation point of  $\mathcal{F}$   
 $\forall \epsilon > 0 \ \exists A \in \mathcal{F}$  with diameter  $\leq \frac{\epsilon}{2}$  Note that  $A \cup \mathcal{B}(x, \frac{\epsilon}{2}) \neq \emptyset$  Take  
 $y \in A \cap \mathcal{B}(x, \frac{\epsilon}{2}) \ \forall z \in A$

$$\begin{aligned} d(x, z) &\leq d(x, y) + d(y, z) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore  $A \subseteq \mathcal{B}(x, \epsilon)$  So  $\mathcal{B}(x, \epsilon) \in \mathcal{F}$  This implies  $\nu_x \subseteq \mathcal{F}$

## 28.7 Lemma

Let  $(X, d)$  be a metric space

- (1) Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$
- (2)  $X$  is complete iff any Cauchy filter of  $X$  has a limit point

### Proof

(1)

- Let  $\mathcal{F}$  be a Cauchy filter on  $X$ . Any accumulation point of  $\mathcal{F}$  is a limit point of  $\mathcal{F}$

- (2) Suppose that  $X$  is complete. Let  $\mathcal{F}$  be a Cauchy filter.  $\forall n \in \mathbb{N}_{\geq 1}$  let  $A_n \in \mathcal{F}$  such that  $\text{diam}(A_n) \leq \frac{1}{n}$  Take  $x_n \in \bigcap_{k=1}^n A_k \in \mathcal{F}$  Then  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  is a Cauchy sequence since  $\forall \epsilon > 0$  if we take  $N \in \mathbb{N}$  with  $\frac{1}{N} \leq \epsilon$  then  $\forall (n, m) \in \mathbb{N}_{\geq N} \ d(x_n, x_m) \leq \frac{1}{N}$  Hence  $(x_n)_{n \in \mathbb{N}_{\geq 1}}$  converges to some  $x \in X$  Note that  $x$  is an limit point of  $\mathcal{F}$  since  $\forall \epsilon > 0 \ \exists n \in \mathbb{N}$  with  $A_n \subseteq \mathcal{B}(x, \epsilon)$  It suffices to take  $n$  such that  $\frac{1}{n} < \frac{\epsilon}{2}$

$\Leftarrow$  Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $X$ . Let

$$\mathcal{F} = \{A \subseteq X \mid \exists N \in \mathbb{N}, \{x_N, x_{N+1}, \dots\} \subseteq A\}$$

This is a Cauchy filter on  $X$  since

$$\lim_{N \rightarrow +\infty} \text{diam}\{x_N, x_{N+1}, \dots\} = 0$$

Hence  $\mathcal{F}$  has a limit point  $x \in X$  By definition  $\forall U \in \mathcal{V}_x \exists N \in \mathbb{N}$

$$\{x_N, x_{N+1}, \dots\} \subseteq U$$

$$\text{So } x = \lim_{n \rightarrow +\infty} x_n$$

## 28.8 Prop

Let  $f : X \rightarrow Y$  be a continuous mapping of topological spaces. If  $A \subseteq X$  is quasi-compact then  $f(A) \subseteq Y$  is also quasi-compact.

### Proof

Let  $(V_i)_{i \in I}$  be an open cover of  $f(A)$  Then

$$(f^{-1}(V_i))_{i \in I}$$

is an open cover of  $A$  So  $\exists J \subseteq I$  such that

$$A \subseteq \bigcup_{j \in J} f^{-1}(V_j)$$

This implies

$$f(A) \subseteq \bigcup_{j \in J} V_j$$

So  $f(A)$  is quasi-compact.

## 28.9 Prop

Let  $X$  be a topological space and  $A \subseteq X$  be a quasi-compact subset. For any closed subset  $F$  of  $X$   $A \cap F$  is quasi-compact.

### Proof

Let  $(U_i)_{i \in I}$  be an open cover of  $A \cap F$ . Then

$$A \subseteq \left( \bigcup_{i \in I} U_i \right) \cup (X \setminus F)$$

Since  $A$  is quasi-compact there exist  $J \subseteq I$  finite such that

$$A \subseteq \left( \bigcup_{j \in J} U_j \right) \cup (X \setminus F)$$

Hence  $A \cap F \subseteq \bigcup_{j \in J} U_j$

### 28.10 Prop

Let  $X$  be a Hausdorff topological space. Any compact subset  $A$  of  $X$  is closed.

#### Proof

Let  $x \in X \setminus A$   $\forall y \in A, \exists$  open subsets  $U_y$  and  $V_y$  such that  $y \in U_y, x \in V_y$  and  $U_y \cap V_y = \emptyset$  Since  $A \subseteq \bigcup_{y \in A} U_y$   $\exists \{y_1, \dots, y_n\} \subseteq A$  such that

$$A \subseteq \bigcup_{i=1}^n U_{y_i}$$

Let

$$U = \bigcup_{i=1}^n U_{y_i} \quad V = \bigcap_{i=1}^n V_{y_i}$$

These are open subset Moreover  $A \subseteq U, x \in V$  and  $U \cap V = \bigcup_{i=1}^n (U_{y_i} \cap V) = \emptyset$

In particular  $x \in V \subseteq X \setminus A$  So  $X \setminus A$  is open

### 28.11 Prop

Let  $X$  be a Hausdorff topological space and  $A$  and  $B$  be compact subsets of  $X$  such that  $A \cap B = \emptyset$  Then there exists open subsets  $U$  and  $V$  such that

$$A \subseteq U, B \subseteq V \text{ and } U \cap V = \emptyset$$

#### proof

We have seen in the proof of the previous proposition that  $\forall x \in B, \exists U_x, V_x$  open such that  $A \subseteq U_x, x \in V_x$  and  $U_x \cap V_x = \emptyset$  Since

$$B \subseteq \bigcup_{x \in B} V_x$$

$\exists \{x_1, \dots, x_m\} \subseteq B$  such that

$$B \subseteq \bigcup_{i=1}^m V_{x_i}$$



We take

$$U = \bigcap_{i=1}^m U_{x_i} \quad V = \bigcup_{i=1}^m U_{x_i} V_{x_i}$$

One has

$$A \subseteq U, B \subseteq V \quad U \cap V = \emptyset$$

## 28.12 Theorem

Let  $(X, \mathcal{G})$  be a Hausdorff topological space. If  $(A_n)_{n \in \mathbb{N}}$  is a sequence of non-empty compact subsets of  $X$  such that

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$$

Then

$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$$

### Proof

Suppose that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

then

$$A_0 \subseteq \bigcup_{n \in \mathbb{N}} (X \setminus A_n)$$

Since  $A_0$  is compact,  $\exists N \in \mathbb{N}$  such that

$$\begin{aligned} A_0 &\subseteq \bigcup_{n=0}^N (X \setminus A_n) \\ &= X \setminus \bigcap_{n=0}^N A_n \\ &= X \setminus A_N \end{aligned}$$

So

$$A_N = \emptyset$$

## 28.13 Def

Let  $(X, \tau)$  be a topological space. If any sequence in  $X$  has a convergent subsequence, we say that  $X$  is sequentially compact.

**Example**

By Bolzano-Weierstrass, any bounded sequence in  $\mathbb{R}$  has a convergent subsequence. So any bounded and closed subset of  $\mathbb{R}$  is sequentially compact.

**Note**

bounded and closed together implies sequentially compact.

**28.14 Theorem**

Let  $(X, d)$  be a metric space. Then the following statements are equivalent:

- (1)  $(X, d)$  is compact
- (2)  $(X, d)$  is sequentially compact

**Proof**

- (1)  $\Rightarrow$  (2) Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$ . Assume that no subsequence of  $(x_n)_{n \in \mathbb{N}}$  converges in  $X$ . For any  $p \in X$  there exists  $\epsilon_p > 0$  such that

$$\{n \in \mathbb{N} : d(p, x_n) < \epsilon\}$$

is finite.

Otherwise we can construct a strictly increasing sequence  $(n_k)_{k \in \mathbb{N}}$  such that

$$d(p, x_{n_k}) \leq \frac{1}{k}$$

For  $X$  is compact  $\exists (p_i)_{i \in \{1, \dots, n\}}$

$$X \subseteq \bigcup_{i=1}^n \mathcal{B}(p_i, \epsilon_{p_i})$$

then

$$\mathbb{N} = \bigcup_{i=1}^n \{n \in \mathbb{N} : d(p_i, x_n) \leq \epsilon_{p_i}\}$$

is finite. Contradiction.

- (2)  $\Rightarrow$  (1)

prove  $(X, d)$  is complete Let  $(x_n)_{n \in \mathbb{N}}$  be a Cauchy sequence. For it's sequentially compact it contains a convergent subsequence. Therefore by a fact proved that its subsequences  $(x_{k_n})_{n \in \mathbb{N}}$  must converges to the same limit.

So  $(X, d)$  is complete

If  $X$  is not covered by finitely many balls of radius  $\epsilon$  we can construct a sequence  $(x_{k_n})_{n \in \mathbb{N}}$  such that

$$x_{n+1} \in X \setminus \bigcup_{k=0}^n \mathcal{B}(x_k, \epsilon)$$

then any subsequence of this sequence is not Cauchy, then not convergent.

## 28.15 Def

Let  $X$  be a Hausdorff topological space. If for any  $x \in X$  there exist a compact neighborhood  $\mathcal{C}_x$  we say that  $X$  is locally compact.

### Example

$\mathbb{R}$  is locally compact.

## 28.16 Prop

Assume that  $(K, |\cdot|)$  is a locally compact non-trivial valued field. Let  $(E, \|\cdot\|)$  be a finite dimensional normed  $K$ -vector space. A subset  $Y \subseteq E$  is compact iff it's closed and bounded.

### Proof

$\Rightarrow$  Let  $Y \subseteq X$  be compact. Then for  $Y$  is Hausdorff,  $Y$  is closed. Moreover

$$Y \subseteq \bigcup_{n \in \mathbb{N}_{\geq 1}} \mathcal{B}(0, n)$$

We can find finitely many positive integers

$$n_1 \leq \dots \leq n_k$$

such that

$$Y \subseteq \bigcup_{i=1}^k \mathcal{B}(0, n_i)$$

$\Rightarrow Y$  is bounded.

$\Leftarrow$  We prove sequentially compact by a theorem proved before.

Let  $(e_i)_{i=1}^d$  be a basis of  $E$ . Again we assume

$$\left\| \sum_{i=1}^d a_i e_i \right\| = \max_{i \in \{1, \dots, d\}} \{|a_i|\}$$

Then any sequence could be written as

$$(x_n)_{n \in \mathbb{N}} = \left( \sum_{i=1}^d a_i^{(n)} e_i \right)_{n \in \mathbb{N}}$$

Since  $Y$  is bounded for any  $i \in \{1, \dots, d\}$  the sequence  $(a_i^{(n)})$  is bounded. In particular we find  $M > 0$  such that  $\forall i \in \{1, \dots, n\}$

$$|a_i^{(n)}| < M$$

Since  $(K, |\cdot|)$  is locally compact, there exists a compact set  $\mathcal{C} = \mathcal{C}_0 \subseteq K$  that's a neighborhood of 0. Let  $\epsilon > 0$

$$\overline{\mathcal{B}}(0, \epsilon) \subseteq \mathcal{C}$$

Since  $K$  is not trivially valued, then exists  $a \in K$  such that

$$|a| \geq \frac{M}{\epsilon}$$

Then

$$\overline{\mathcal{B}}(0, M) \subseteq a\mathcal{C}$$

$\mathcal{C} \subseteq K$  is compact. We have the  $K$ -linear mapping

$$\begin{aligned} K &\rightarrow K \\ y &\mapsto ay \end{aligned}$$

is bounded, then continuous. Hence  $a\mathcal{C}$  is compact. So

$$\overline{\mathcal{B}} \subseteq a\mathcal{C}$$

is a closed subspace of a compact. So it's compact, additionally sequentially compact.

Therefore we can find  $(I_i)_{i=1}^d$  are infinite subsets of  $\mathbb{N}$  with

$$I_1 \supseteq \dots \supseteq I_d$$

such that  $(a_j)_{j \in I_i}^{(n)}$  converges to some  $a_i \in K$ . It follows that our original

sequence has a convergent subsequence converges to  $\sum_{i=1}^d a_i e_i$ .

So  $Y$  is sequentially compact.

## 28.17 Theorem

Let  $X$  be a topological space and  $f : X \rightarrow \mathbb{R}$  be a continuous mapping. If  $Y \subseteq X$  is a quasi-compact subset, then there exists  $a \in Y$  and  $b \in Y$  such that  $\forall x \in Y$

$$f(a) \leq f(x) \leq f(b)$$

Namely the restriction of  $f$  to  $Y$  attains its maximum and minimum.

**Proof**

$f(Y) \subseteq \mathbb{R}$  is a non-empty compact subset since  $Y$  is quasi-compact and  $\mathbb{R}$  is Hausdorff. Moreover, since  $\mathbb{R}$  is locally compact. SO  $f(Y)$  is bounded and closed.

Note that there exists sequences  $(\alpha_n)_{n \in \mathbb{N}}$  and  $(\beta_n)_{n \in \mathbb{N}}$  in  $f(Y)$  that tends to  $\sup f(Y)$  and  $\inf f(Y)$  respectively. Since  $f(Y)$  is closed,  $\sup f(Y), \inf f(Y)$  belongs to  $f(Y)$ . So  $f(Y)$  has a greatest and a least element.



## Chapter 29

# Mean Value Theorems

### 29.1 Rolle Theorem

Let  $a, b$  be real numbers such that  $a < b$ . Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping that is differentiable on  $]a, b[$ . If  $f(a) = f(b)$  then  $\exists t \in ]a, b[$  such that

$$f'(t) = 0$$

#### Proof

Since  $[a, b]$  is closed and bounded then it's compact,  $f$  attains its maximum and minimum. Let  $M = \max f([a, b]), m = \min f([a, b]), l = f(a) = f(b)$

If  $M \neq l \exists t \in ]a, b[$  such that  $f(t) = M$

$$f(t+x) = f(t) + f'(t)x + o(|x|)$$

$$f(t-x) = f(t) - f'(t)x + o(|x|)$$

$$0 \leq (f(t+x) - f(t))(f(t-x) - f(t))$$

$$= -f'(t)^2 x^2 + o(|x|^2)$$

$$0 \leq -f'(t)^2 + o(1) \quad x \rightarrow 0$$

Taking the limit when  $x \rightarrow 0$  we get  $f'(t)^2 = 0$

If  $m \neq l$  then any  $t \in ]a, b[$  such that  $f(t) = m$  verifies  $f'(t) = 0$

If  $m = l = M$   $f$  is constant, so  $\forall t \in ]a, b[, f'(t) = 0$

### 29.2 Mean value theorem(Lagrange)

Let  $a, b$  be real numbers  $a < b$ ,  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous mapping differentiable on  $]a, b[$ , then  $\exists t \in ]a, b[$  such that

$$f(b) - f(a) = f'(t)(b - a)$$

**Proof**

Let  $g : [a, b] \rightarrow \mathbb{R}$  be defined as

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then  $g(a) = f(a)$   $g(b) = f(a)$  then apply Rolle Theorem to  $g$  we get the proof.

**29.3 Mean value inequality**

Let  $a, b$  be real numbers such that  $a < b$   $(E, \|\cdot\|)$  be a normed vector space over  $\mathbb{R}$   $f : [a, b] \rightarrow E$  be a continuous mapping such that  $f$  is differentiable on  $]a, b[$  Then

$$\|f(b) - f(a)\| \leq \left( \sup_{x \in ]a, b[} \|f'(x)\| \right) (b - a)$$

**Proof**

Suppose that

$$\sup_{x \in ]a, b[} \|f'(x)\| < +\infty$$

Let  $M \in \mathbb{R}$  such that

$$M > \sup_{x \in ]a, b[} \|f'(x)\|$$

Let

$$J = \{x \in [a, b] \mid \forall y \in [a, x], \|f(y) - f(a)\| \leq M(y - a)\}$$

By definition  $J$  is an interval containing  $a$ , so  $J$  is of form  $[a, c[$  or  $[a, c]$  Since  $f$  is continuous by taking a sequence  $(c_n)_{n \in \mathbb{N}}$  in  $[a, b[$  that converges to  $c$  we obtain

$$\begin{aligned} \|f(c) - f(a)\| &= \lim_{n \rightarrow +\infty} \|f(c_n) - f(a)\| \\ &\leq \lim_{n \rightarrow +\infty} M(c_n - a) \\ &= M(c - a) \end{aligned}$$

Hence  $c \in J$  namely  $J = [a, c]$

$c > a$  We will prove that  $c = b$  by contradiction

Suppose that  $c < b$   $\forall h \in ]0, b - c[$

$$\begin{aligned} \|f(c + h) - f(c)\| &= \|h \cdot f'(c) + o(h)\| \\ &\leq \|f'(c)\| h + o(h) \end{aligned}$$

Since  $M > \|f'(c)\|$ ,  $\exists h_0 > 0$  such that  $\forall 0 < h < h_0$

$$\|f(c + h) - f(c)\| \leq Mh$$



Hence

$$\begin{aligned}\|f(c+h)f(c)\| &\leq \|f(c+h) - f(c)\| + \|f(c) - f(a)\| \\ &\leq M(c_h - c + c - a) \\ &= M(c + h - a)\end{aligned}$$

So  $c + h_0 \in J$  Contradiction. Thus

$$\|f(b) - f(a)\| \leq M(b - a)$$

for any  $M > \sup_{x \in ]a, b[} \|f'(x)\|$  since  $M$  is arbitrary the expected inequality holds.

$c = a$  In general, we apply the particular case (fis-extendable to a differentiable mapping at  $a$ ) to  $[\frac{a+b}{2}, b]$  and  $[a, \frac{a+b}{2}]$  to get

$$\begin{aligned}\left\|f(b) - f\left(\frac{a+b}{2}\right)\right\| &\leq C \frac{b-a}{2} \\ \left\|f\left(\frac{a+b}{2}\right) - f(a)\right\| &\leq C \frac{b-a}{2}\end{aligned}$$

with  $C = \sup_{x \in ]a, b[} \|f'(x)\|$

**Remark** If  $f$  is defined on an open neighborhood of  $a$  and is differentiable at  $a$  the the same arguments hold without the assumption

## 29.4 Theorem

Let  $I$  be an interval in  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a continuous mapping, then  $f(I)$  is an interval.

### Proof

Let  $x \neq y$  be two elements of  $f(I)$  Let  $a, b$  elements of  $I$  such that  $x = f(a)$   $y = f(b)$  without loss of generality, we assume  $a < b$   
Let  $z \in \mathbb{R}$  such

$$(z - x)(z - y) \leq 0$$

We construct by induction three sequences  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}, (c_n)_{n \in \mathbb{N}}$  such that

- $a_0 = a, b_0 = b, c_0 = \frac{a+b}{2}$
- If  $a_n, b_n, c_n$  are constructed, satisfying

$$c_n = \frac{1}{2}(a_n + b_n)$$

$$(z - f(a_n))(z - f(b_n)) \leq 0$$

we let

$$\begin{aligned} (a_{n+1}, b_{n+1}) &= (a_n, c_n) & \text{if } (z - f(a_n))(z - f(c_n)) \leq 0 \\ (a_{n+1}, b_{n+1}) &= (c_n, b_n) & \text{if } (z - f(a_n))(z - f(c_n)) > 0 \\ & & ((z - f(c_n))(z - f(b_n)) \leq 0) \end{aligned}$$

$$c_{n+1} = \frac{a_{n+1} + b_{n+1}}{2}$$

The sequence  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$  are increasing and decreasing respectively and bounded, hence converges to some  $l, m \in [a, b]$

Note that

$$|b_n - a_n| = \frac{1}{2^n} |b - a| \rightarrow 0 (n \rightarrow +\infty)$$

So  $l = m$ , by  $(z - f(a_n))(z - f(b_n)) \leq 0$  we obtain by letting  $n \rightarrow +\infty$

$$(z - f(l))^2 \leq 0$$

So  $z = f(l)$

## 29.5 Theorem(Heine)

Let  $I$  be an open interval of  $\mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be a differentiable mapping. Then  $f'(I)$  is an interval.

### Proof

Let  $(a, b) \in I^2$  such that  $a < b$ . Consider the following mappings:

$$\begin{aligned} g : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(x) - f(a)}{x - a} & x \neq a \\ f'(a) & x = a \end{cases} \\ h : [a, b] &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} \frac{f(b) - f(x)}{b - x} & x \neq b \\ f'(b) & x = b \end{cases} \end{aligned}$$

$g, h$  are continuous  $(\frac{f(x) - f(a)}{x - a} = f'(a) + o(1) \text{ as } x \rightarrow a)$

So  $g([a, b])$  and  $h([a, b])$  are intervals. Moreover, by mean value theorem,

$$g([a, b]) \subseteq f'(I)$$

$$h([a, b]) \subseteq f'(I)$$

So

$$\{f'(a), f'(b)\} \subseteq g([a, b]) \cup h([a, b]) \subseteq f'(I)$$

Note that  $g(b) = h(a)$  so

$$g([a, b]) \cup h([a, b])$$

is an interval. Hence  $f'(I)$  is an interval.

## Chapter 30

# Fixed Point Theorem

### 30.1 Def

Let  $X$  be a set and  $T : X \rightarrow X$  be a mapping. If  $x \in X$  satisfies  $T(x) = x$  we say that  $x$  is a fixed point of  $T$ .

### 30.2 Def

Let  $(X, d)$  be a metric space and  $T : X \rightarrow X$  be a mapping. If  $\exists \epsilon \in [0, 1[$  such that  $T$  is  $\epsilon$ -Lipschitzian then we say that  $T$  is a contraction.

### 30.3 Fixed Point Theorem

Let  $(X, d)$  be a COMPLETE non-empty metric space, and  $T : X \rightarrow X$  be a contraction. Then  $T$  has a unique fixed point. Moreover,  $\forall x_n \in X$  if we let

$$x_{n+1} = T(x_n), x_0 \in X$$

then  $(x_n)_{n \in \mathbb{N}}$  converges to the fixed point.

#### Proof

If  $p$  and  $q$  are two fixed point of  $T$ , then

$$d(p, q) = d(T(p), T(q)) \leq \epsilon d(p, q)$$

So  $d(p, q) = 0$ .

Let

$$x_{n+1} = T(x_n), x_0 \in X$$

$\forall n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \leq \epsilon^n d(x_0, x_1)$$

$$d(T(x_{n-1}), T(x_n)) \leq \epsilon d(x_{n-1}, x_n)$$

For any  $N \in \mathbb{N}$ ,  $\forall (n, m) \in \mathbb{N}_{\geq N}^2$   $n < m$

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \epsilon^n d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \\ &\leq \frac{\epsilon^n}{1 - \epsilon} d(x_0, x_1) \end{aligned}$$

So

$$\lim_{N \rightarrow +\infty} \sup_{(n, m) \in \mathbb{N}_{\geq N}^2} d(x_n, x_m) = 0$$

$(x_n)_{n \in \mathbb{N}}$  is a Cauchy sequence, hence converges to some  $p \in X$

$$d(T(p), p) = \lim_{n \rightarrow +\infty} d(T(x_n), x_n) = 0$$

since  $d : X^2 \rightarrow \mathbb{R}_{\geq 0}$  is continuous.

## Part VI

# Higher differentials



## Chapter 31

# Multilinear mapping

Let  $K$  be a commutative cenitary ring.

### 31.1 Def

Let  $n \in \mathbb{N}$ ,  $V_1, \dots, V_n, W$  be  $K$ -modules. We call  $n$ -linear mapping from  $V_1 \times \dots \times V_n$  to  $W$  any mapping  $f : V_1 \times \dots \times V_n \rightarrow W$  such that  $\forall i \in \{1, \dots, n\} \forall (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in V_1 \times \dots \times V_{i-1} \times V_{i+1} \times \dots \times V_n$  the mapping

$$\begin{aligned} f(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) : V_i &\rightarrow W \\ x_i &\mapsto f(x_i) \end{aligned}$$

is a morphism of  $K$ -modules

We denote by  $Hom^{(n)}(V_1 \times \dots \times V_n, W)$  the set of all  $n$ -linear mappings from  $V_1 \times \dots \times V_n$  to  $W$ .

### 31.2 Example

$$\begin{aligned} K \times K &\rightarrow K \\ (a, b) &\mapsto ab \end{aligned}$$

is a 2-linear mapping (bilinear mapping)

### 31.3 Remark

$$Hom^{(0)}(\{0\}, W) := W \text{ (by convention)}$$

$$Hom^{(1)}(V_1, W) = Home(V_1, W) = \{\text{morphism of } K\text{-module from } V_1 \text{ to } W\}$$

### 31.4 Prop

Suppose that  $n \geq 2$  For any  $i \in \{1, \dots, n-1\}$

$$\begin{aligned} \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) &\xrightarrow{\Phi} \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n)) \\ f &\mapsto ((x_1, \dots, x_i) \mapsto ((x_{i+1}, \dots, x_n) \mapsto f(x_1, \dots, x_n))) \end{aligned}$$

is a bijection

### Proof

The inverse of  $\Phi$  is given by

$$g \in \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \mapsto (((x_1, \dots, x_n) \in V_1 \times \dots \times V_n) \mapsto g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n))$$

### 31.5 Remark

$\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$  is a sub-K-module of  $W^{V_1 \times \dots \times V_n}$  and  $\Phi$  is an isomorphism of K-modules.



## Chapter 32

# Operator norm of Multilinear field

Let  $(K, |\cdot|)$  be a complete valued field

### 32.1 Def

Let  $V_1 \times \dots \times V_n$  and  $W$  be normed vector spaces over  $K$ . We define

$$\|\cdot\| : \text{Hom}^{(n)}(V_1 \times \dots \times V_n, W) \rightarrow [0, +\infty]$$

as

$$\|f\| := \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

If  $\|f\| < \infty$  we say that  $f$  is bounded. We denote by  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  the set of bounded  $n$ -linear mappings from  $V_1 \times \dots \times V_n$  to  $W$ .

### 32.2 Theorem

For any  $i \in \{1, \dots, n-1\}$ ,  $\forall f \in \mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \forall (x_1, \dots, x_i) \in V_1 \times \dots \times V_i$  the  $(n-i)$ -linear mapping

$$\begin{aligned} f(x_1, \dots, x_i, \cdot) : V_{i+1} \times \dots \times V_n &\rightarrow W \\ (x_{i+1}, \dots, x_n) &\mapsto f(x_1, \dots, x_n) \end{aligned}$$

belongs to  $\mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)$ . Moreover

$$\|f\| = \sup_{(x_1, \dots, x_n) \in V_1 \times \dots \times V_n, x_1 \dots x_n \neq 0} \frac{\|f(x_1, \dots, x_n)\|}{\|x_1\| \cdots \|x_n\|}$$

**Proof**

$$\forall (x_{i+1}, \dots, x_n) \in V_{i+1} \times \dots \times V_n$$

$$\begin{aligned} \|f(x_1, \dots, x_n)\| &\leq \|f\| \|x_1\| \dots \|x_n\| \\ &= (\|f\| \|x_1\| \dots \|x_i\|) \|x_{i+1}\| \dots \|x_n\| \end{aligned}$$

So

$$\|f(x_1, \dots, x_i, \cdot)\| \leq \|f\| \|x_1\|, \dots, \|x_i\|$$

If we define

$$\|f\|' := \sup_{(x_1, \dots, x_i) \in V_1 \times \dots \times V_i, x_1 \dots x_i \neq 0} \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|}$$

then

$$\|f\|' \leq \|f\|$$

**32.3 Corollary**

- (1)  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  is a vector subspace of  $\text{Hom}^{(n)}(V_1 \times \dots \times V_n, W)$
- (2)  $\|\cdot\|$  is a norm on  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$
- (3)  $\forall i \in \{1, \dots, n\}$

$$\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W) \xrightarrow{\Phi} \mathcal{L}^{(n)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$$

is a K-linear isomorphism that preserves operator norms.

$$\|f\| = \|\Phi(f)\|$$

**32.3.1 Proof**

Conversely  $\forall (x_1, \cdot, x_n) \in V_1 \times \dots \times V_n$  such that  $x_1 \dots x_n \neq 0$

$$\|f(x_1, \dots, x_n)\| \leq \|f(x_1, \dots, x_i, \cdot)\| \|x_{i+1}\| \dots \|x_n\|$$

Hence

$$\frac{f(x_1, \dots, x_n)}{\|x_1\| \dots \|x_n\|} \leq \frac{\|f(x_1, \dots, x_i, \cdot)\|}{\|x_1\| \dots \|x_i\|} \leq \|f\|'$$

Taking sup, we get

$$\|f\| \leq \|f\|'$$

We reason by induction on  $n$

$n = 1$

$$\mathcal{L}^{(1)}(V_1, W) = \mathcal{L}(V_1, W)$$

$i \in \{1, \dots, n-1\}$  Suppose that the corollary is true for  $m$ -linear mappings with  $m < n$ . We consider the following diagram of mapping

To show that  $\mathcal{L}^{(n)}(V_1 \times \dots \times V_n, W)$  is a vector subspace, it suffices to check that  $\forall g \in \mathcal{L}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W))$  one has  $\|\Phi^{-1}(g)\| = \|g\| < +\infty$

$$\begin{aligned} \mathcal{L}^{(i)}(V_{i+1} \times \dots \times V_n, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \mathcal{L}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \\ &\subseteq \text{Hom}^{(i)}(V_1 \times \dots \times V_i, \text{Hom}^{(n-i)}(V_{i+1} \times \dots \times V_n, W)) \end{aligned}$$

For any  $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n$

$$\begin{aligned} \|\Phi^{-1}(g)(x_1, \dots, x_n)\| &= \|g(x_1, \dots, x_i)(x_{i+1}, \dots, x_n)\| \\ &\leq \|g(x_1, \dots, x_i)\| \|x_{i+1}\| \cdots \|x_n\| \\ &\leq \|g\| \|x_1\| \cdots \|x_i\| \|x_{i+1}\| \cdots \|x_n\| \end{aligned}$$

Therefore

$$\|\Phi^{-1}(g)\| \leq \|g\| = \|\Phi^{-1}(g)\|$$



## Chapter 33

# Higher differentials

We fix a complete non-trivial valued field  $(K, |\cdot|)$  and normed  $K$ -vector space  $E$  and  $F$ .

### 33.1 Def

Let  $U \subseteq E$  be an open subset and  $f : U \rightarrow F$  be a mapping

- (1) If  $f$  is continuous, we say that  $f$  is of class  $C^0$  and  $f$  is 0-times differentiable
- (2) If  $f$  is differentiable on an open neighborhood  $V \subseteq U$  of some point  $p \in U$  and

$$\begin{aligned} df : V &\rightarrow \mathcal{L}(E, F) \\ x &\mapsto d_x f \end{aligned}$$

is  $n$ -times differentiable at  $p$ , then we say that  $f$  is  $(n+1)$ -times differentiable at  $p$ . If  $f$  is  $(n+1)$ -times differentiable at any point  $p \in U$ , we denote by

$$D^{n+1}f : U \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

the mapping that sends  $x \in U$  to the image of  $D^n(df)(x)$  by the  $K$ -linear bijection

$$\mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \rightarrow \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

$$df : U \rightarrow \mathcal{L}(E, F)$$

$$D^n(df) : U \rightarrow \mathcal{L}^{(n)}(E^n, \mathcal{L}(E, F)) \xrightarrow{\Phi} \mathcal{L}^{(n+1)}(E^{n+1}, F)$$

If  $D^{n+1}f$  is continuous, we say that  $f$  is of class  $C^{n+1}$  ( $n \geq 0$ ) (Any mapping  $f : U \rightarrow F$  is considered as 0-times differential  $D^0f := f$ )

### 33.2 Remark

If  $f$  is  $n$ -times differentiable  $\forall i \in \{1, \dots, n-1\}$   
 $\forall p \in U, (h_1, \dots, h_n) \in E^n$  one has

$$D^i(D^{n-i}f)(p)(h_1, \dots, h_i)(h_{i+1}, \dots, h_n) = D^n f(p)(h_1, \dots, h_n)$$

$$D^{n-i}f : U \rightarrow \mathcal{L}^{(n-i)}(E^{n-i}, F)$$

$$D^i(D^{n-i}f) : \quad U \xrightarrow{\quad} \mathcal{L}^{(i)}(E^i, \mathcal{L}^{(n-i)}(E^{n-i}, F)) \quad U \rightarrow$$

$$\quad \quad \quad \searrow D^n f \quad \quad \quad \updownarrow \cong \quad \quad \quad \mathcal{L}^{(n)}(E^n, F)$$

### 33.3 Theorem

Assume that  $(K, |\cdot|) = (\mathbb{R}, |\cdot|)$   
 Let  $f : U \rightarrow F$  be a mapping that is  $(n+1)$ -times differentiable on  $U$ . Let  
 $p \in U$  and  $h \in E$  such that  $p + th \in U \forall t \in [0, 1]$  Then

$$\left\| f(p+h) - f(p) - \sum_{k=1}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq$$

$$\left( \sup_{t \in ]0, 1[} \frac{(1-t)^n}{n!} \|D^{n+1} f(p+th)\| \right) \cdot \|h\|^{n+1}$$

(Taylor-Lagrange formula)

### 33.4 Prop(Gronwall inequality)

Let  $F$  be a normed vector space over  $\mathbb{R}$   $(a, b) \in \mathbb{R}^2, a < b$  Let  $f : [a, b] \rightarrow F$   
 and  $g : [a, b] \rightarrow \mathbb{R}$  be continuous mappings that are differentiable on  $]a, b[$

Suppose that  $\forall t \in ]a, b[$

$$\|f'(t)\| \leq g'(t)$$

then

$$\|f(b) - f(a)\| \leq g(b) - g(a)$$

#### Proof

Let  $c \in ]a, b[$  Let  $\epsilon > 0$  Let

$$J = \{t \in [c, b] \mid \forall s \in [c, t], \|f(s) - f(c)\| \leq g(s) - g(c)\}$$

By definition  $J$  is an interval.

Since  $f, g$  are continuous,  $J$  is a closed interval, hence  $J$  is of the form  $[c, t]$ .  
If  $t < b$  then for  $h > 0$  Sufficiently small.

$$f(t+h) - f(t) = hf'(t) + o(h)$$

$$g(t+h) - g(t) = hg'(t) + o(h)$$

$$\exists \delta > 0 \forall h \in [0, \delta]$$

$$\|f(t+h)\| \leq \|f'(t)\| \cdot h + \frac{\epsilon}{2}h$$

$$g(t+h) - g(t) \geq g'(t)h - \frac{\epsilon}{2}h$$

So

$$\|f(t+h) - f(t)\| \leq g(t+h) - g(t) + \epsilon h$$

Moreover

$$\|f(t) - f(c)\| \leq g(t) - g(c) + \epsilon(t - c)$$

$\Rightarrow$

$$\|f(t+h) - f(c)\| \leq g(t+h) - g(c) + \epsilon(t+h-c)$$

$\Rightarrow$

$$J \supseteq [c, t+\delta]$$

Contradiction, hence

$$\|f(b) - f(c)\| \leq g(b) - g(c) + \epsilon(b - c)$$

For the same reason

$$\|f(c) - f(a)\| \leq g(c) - g(a) + \epsilon(c - a)$$

Hence

$$\|f(b) - f(a)\| \leq g(b) - g(a) + \epsilon(b - a)$$

Since  $\epsilon > 0$  is arbitrary

$$\|f(b) - f(c)\| \leq g(b) - g(c)$$

Mean value theorem:

$$g(t) = (\sup(\|f'(\cdot)\cdot\|))$$

### 33.5 Theorem

Let  $n \in \mathbb{N}$ ,  $E, F$  be normed vector spaces over  $\mathbb{R}$   $U \subseteq E$  open and  $f : U \rightarrow F$  be a mapping that is  $(n+1)$ -times differentiable. Let  $p \in U$  and  $h \in E$ . Assume that  $\forall \epsilon \in [0, 1], p + th \in U$

Let

$$M = \sup_{t \in ]0, 1[} \|D^{n+1}f(p+th)\|$$

Then

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h) \right\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

If  $E = \mathbb{R}$  Then the formula become

$$\left\| f(p+h) - \sum_{k=0}^n \frac{1}{k!} f^{(k)}(p) h^k \right\| \leq \frac{M}{(n+1)!} |h|^{n+1}$$

### Proof

Consider  $\phi : [0, 1] \rightarrow F$

$$\phi(t) = \sum_{k=0}^n \frac{(1-t)^k}{k!} D^k f(p+th)(h, \dots, h)$$

$$\phi(1) = f(p+h)$$

$$\phi(0) = \sum_{k=0}^n \frac{1}{k!} D^k f(p)(h, \dots, h)$$

$$\begin{aligned} \phi'(t) &= \sum_{k=0}^n \frac{(1-t)^k}{k!} D^{k+1} f(p+th)(\underbrace{h, \dots, h}_{k+1 \text{ copies}}) - \sum_{k=1}^n \frac{(1-t)^{k-1}}{(k-1)!} D^k f(p+th)(h, \dots, h) \\ &= \frac{(1-t)^n}{n!} D^{n+1} f(p+th)(h, \dots, h) \end{aligned}$$

then

$$\|\phi'(t)\| \leq M \frac{(1-t)^n}{n!} = (-M \frac{(1-t)^{n+1}}{(n+1)!})'$$

By Gronwall inequality,

$$\|\phi(1) - \phi(0)\| \leq \frac{M}{(n+1)!} \|h\|^{n+1}$$

### 33.6 Def

Let  $n \in \mathbb{N}$   $E_1, \dots, E_n$  and  $F$  be normed vector spaces over a complete non-trivial valued field  $(K, |\cdot|)$  Let  $U \in E_1 \times \dots \times E_n$  be an open subset.  $p = (p_1, \dots, p_n) \in U$   $i \in \{1, \dots, n\}$ ,  $f : U \rightarrow F$  If there exists an open neighborhood  $U_i$  of  $p_i$  in  $E_i$  such that

$$\begin{aligned} U_i &\rightarrow F \\ x_i &\mapsto f(p_1, \dots, p_{i-1}, x_i, p_{i+1}, \dots, p_n) \end{aligned}$$

is well defined and is differentiable at  $p_i$

We denote by  $\frac{\partial f}{\partial x_i}(p)$  the differential of this mapping  $U_i \rightarrow F$  and say that  $f$  admits the  $i^{th}$  partial differentials at  $p$



### 33.7 Prop

Suppose that  $(K, |\cdot|)$  and  $f$  has all partial differentials on  $U$  and

$$\frac{\partial f}{\partial x_i} : U \rightarrow \mathcal{L}(E_i, F)$$

is continuous for any  $i \in \{1, \dots, n\}$  Then  $f$  is of class  $C^1$  and  $\forall h = (h_1, \dots, h_n) \in E_1 \times \dots \times E_n$

$$\forall p \in U \quad d_p(h) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p)(h_i)$$

#### Proof

By induction, it suffices to treat the case where  $n = 2$   
 $\forall \epsilon > 0 \exists \delta > 0$

$$\forall (h, k) \in E_1 \times E_2 \quad \max\{|h|, |k|\} \leq \delta$$

one has

$$\left\| \frac{\partial f}{\partial x_i}(a + h, b + k) - \frac{\partial f}{\partial x_i}(a, b) \right\| \leq \epsilon \text{ (by continuity of } \frac{\partial f}{\partial x_i} \text{)}$$

Consider the mapping  $\phi : [0, 1] \rightarrow F$

$$\phi(t) = f(a + h, b + tk) - f(a + h, b) - t \underbrace{\frac{\partial f}{\partial x_2}(a + h, b)(k)}_{\in \mathcal{L}(E_2, F)}$$

$$\begin{aligned} \|\phi'(t)\| &= \left\| \frac{\partial f}{\partial x_2}(a + h, b + tk)(k) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \\ &\leq 2\epsilon \|k\| \end{aligned}$$

$$\|\phi(1) - \phi(0)\| \leq 2\epsilon \|k\|$$

then

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| \leq 2\epsilon \|k\|$$

So

$$\left\| f(a + h, b + k) - f(a + h, b) - \frac{\partial f}{\partial x_2}(a + h, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

$f$  has 1<sup>st</sup> partial differential

$$\left\| f(a+h, b) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

by continuity of  $\frac{\partial f}{\partial x_i}$

$$\left\| \frac{\partial f}{\partial x_2}(a+h, b)(k) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

take the sum of above three statements, we get:

$$\left\| f(a+h, b+k) - f(a, b) - \frac{\partial f}{\partial x_1}(a, b)(h) - \frac{\partial f}{\partial x_2}(a, b)(k) \right\| = o(\max\{\|h\|, \|k\|\})$$

### 33.8 Theorem

Let  $E, F$  be normed vector spaces over  $\mathbb{R}$   $U \subseteq E$  open  $(f_n)_{n \in \mathbb{N}}$  be a sequence of differentiable mapping from  $U$  to  $F$  Let  $g : U \rightarrow \mathcal{L}(E, F)$  Suppose that

- (1)  $(df_n)_{n \in \mathbb{N}}$  converges uniformly to  $g$
- (2)  $(f_n)_{n \in \mathbb{N}}$  converges pointwisely to some mapping  $f : U \rightarrow F$

Then  $f$  is differentiable and  $df = g$

#### Proof

Let  $p \in U, \forall (m, n) \in \mathbb{N}^2, \forall x \in \mathcal{B}(p, r) \in U (r > 0)$

$$\|f_n(x) - f_m(x) - (f_n(p) - f_m(p))\| \leq (\sup_{\xi \in U} \|d_\xi f_m - d_\xi f_n\|) \cdot \|x - p\| \quad (\text{mean value inequality})$$

Take  $\lim_{m \rightarrow +\infty}$  we get:

$$\|(f_n(x) - f(x)) - (f_n(p) - f(p))\| \leq \epsilon_n \|x - p\|$$

where  $\epsilon_n = \sup_{\xi \in U} \|d_\xi f_m - g\|$ .

So

$$\begin{aligned} \|f(x) - f(p) - g(p)(x-p)\| &\leq \|(f(x) - f_n(x)) - (f(p) - f_n(p))\| \\ &\quad + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| \\ &\quad + \|d_p f_n(x-p) - g(p)(x-p)\| \\ &\leq \epsilon_n \|x-p\| + \|f_n(x) - f_n(p) - d_p f_n(x-p)\| + \epsilon_n \|x-p\| \\ \limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} &\leq 2\epsilon_n \end{aligned}$$

Take  $\lim_{n \rightarrow +\infty}$  we get:

$$\limsup_{x \rightarrow p} \frac{\|f(x) - f(p) - g(p)(x-p)\|}{\|x-p\|} = 0$$

# Chapter 34

## Permutations

### 34.1 Def

Let  $X$  be a set. We denote with  $\mathfrak{S}_X$  the set of all bijections from  $X$  to itself. The elements of  $\mathfrak{S}_X$  are called permutations if the set  $X$  is finite. If  $x_1, \dots, x_n \in X$  are distinct elements then

$$(x_1, \dots, x_n) \in \mathfrak{S}_X$$

such that

$$x_i \mapsto x_{i+1}$$

$$x_n \mapsto x_1$$

this is called an  $n$ -cycle. A 2-cycle is called a transposition.

#### 34.1.1 Example

$$X = \{1, \dots, 7\}$$

$$1 \mapsto 4$$

$$2 \mapsto 1$$

$$3 \mapsto 2$$

$$(2\ 3)(4\ 2\ 1) = 4 \mapsto 3$$

$$5 \mapsto 5$$

$$6 \mapsto 6$$

$$7 \mapsto 7$$

$$= (1\ 4\ 3\ 2)$$

### 34.2 Def

We denote with

$$orb_\sigma(x) = \{\underbrace{\sigma \circ \dots \circ}_{n\text{-times}} \quad n \in \mathbb{N}\}$$

$$x \in X, \sigma \in \mathfrak{S}_X$$

### 34.3 Prop

If  $\text{orb}_\sigma(x)$  is a finite set of  $d$  elements, then one has

$$\sigma^d(x) = x \quad \text{orb}_\sigma(x) = \{x, \sigma(x), \dots, \sigma^{d-1}(x)\}$$

moreover

$$\sigma^{-1}(x) \in \text{orb}_\sigma(x)$$

#### 34.3.1 Proof

The set

$$\{(n, m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

is not empty. Let

$$d' := \min\{m - n \mid (n, m) \in \mathbb{N}^2, n < m, \sigma^n(x) = \sigma^m(x)\}$$

therefore  $x, \sigma(x), \dots, \sigma^{d'-1}(x)$  are all distinct.

Now use the each deass division

$$h = qd' + r \quad r < d'$$

$$\sigma^h(x) = \sigma^r(x) \quad 0 \leq r < d'$$

then

$$d' \geq d$$

and for

$$\{x, \sigma(x), \dots, \sigma^{d'-1}(x)\} \subseteq \text{orb}_\sigma(x)$$

$\Rightarrow$

$$d' \leq d$$

then

$$d' = d$$

### 34.4 Remark

Let  $Y \subseteq X$ , then we have a homomorphism of groups:

$$\begin{aligned} \mathfrak{S}_Y &\rightarrow \mathfrak{S}_X \\ \sigma &\mapsto \left( x \mapsto \begin{cases} \sigma(x) & \text{if } x \in Y \\ x & \text{if } x \in X \setminus Y \end{cases} \right) \end{aligned}$$

If  $Y$  and  $Z$  are subset of  $X$

$$Y \cap Z = \emptyset, \sigma \in \mathfrak{S}_Y, \tau \in \mathfrak{S}_Z$$

then

$$\sigma \circ \tau = \tau \circ \sigma$$

If  $X$  is finite with  $n$  elements  $\mathfrak{S}_X = S_n$  permutation group of  $n$  elements.

### 34.5 Theorem

Let  $X$  be a finite set and let  $\sigma \in \mathfrak{S}_X$  then exist  $d \in \mathbb{N}$  and  $(n_1, \dots, n_d) \in \mathbb{N}_{\geq 2}^d$  and pairwise disjoint subsets  $X_1, \dots, X_d$  of  $X$  of cardinalities  $n_1, \dots, n_d$ , together with  $n_i$ -cycle  $\tau_i$  of  $X_i$  such that

$$\sigma = \tau_1 \circ \dots \circ \tau_d$$

In other words. Any permutation can be decomposed in composition of finitely many cycles on disjoint subsets.

#### Proof

By induction on the cardinality of  $X$ .

The case  $\sigma = id_X$  is trivial. ( $d = 0$ ) So the case when  $N = 0, 1$  is clear.

Assume  $N \geq 2$ . Take  $x \in X$  such that  $\sigma(x) \neq x$  and let  $X_1 = orb_\sigma(x)$   
 $Y = X \setminus X_1 \forall y \in Y$  we have  $\sigma(y) \in Y$  (because if  $\sigma(y) \in X$  by the previous proposition  $\sigma(y) \in X_1$ )

Let  $\tau = \sigma|_Y \in \mathfrak{S}_Y$  Use the induction hypothesis, we get  $X_2, \dots, X_d$  of cardinalities  $n_2, \dots, n_d$  and  $n_i$ -cycle  $\tau_i$  such that

$$\tau = \tau_2 \circ \dots \circ \tau_d$$

Consider  $\tau_1 = \sigma|_{X_1}$  then  $\tau_1$  is a  $n_1$ -cycle of  $X_1$

$\Rightarrow$

$$\sigma = \tau_1 \circ \tau_2 \circ \dots \circ \tau_d$$

#### 34.5.1 Remark

This theorem say that the groups of permutation si generated by cycles.

### 34.6 Corollary

Let  $X$  be a finite set. Then  $\mathfrak{S}_X$  is generated by transpositions.

**Proof**

Note that

$$(x_1, \dots, x_n) = (x_1, x_2) \circ (x_2, \dots, x_n)$$

By induction

$$(x_1, \dots, x_n) = (x_1, x_2) \circ \dots \circ (x_{n-1}, x_n)$$

**34.6.1 Remark**

The decomposition of transposition is unique.

**34.7 Def**

Let  $\tau \in \mathfrak{S}_n := G_{\{1, \dots, n\}}$  is called adjacent if  $\tau$  is of the form  $(j, j+1)$  for  $j = 1, \dots, n-1$

**34.8 Corollary**

$\mathfrak{S}_n$  is generated by adjacent transposition.

**34.8.1 Proof**

Note that

$$(i, j) = (i, i+1) \circ (i+1, i+2) \circ \dots \circ (j-1, j) \circ (j-2, j-1) \circ \dots \circ (i+2, i+1)$$

**Some other information on  $\mathfrak{S}_n$**

**34.9 Caybey Theorem**

Any finite group can be embedded (injective morphism) in a  $\mathfrak{S}_n$  for some  $n \in \mathbb{N}$

**Proof**

Let  $G$  be a finite group and  $n = \text{card}(G)$ . Let

$$\begin{aligned} \varphi : G &\rightarrow \mathfrak{S} \\ g &\mapsto l_g \end{aligned}$$

be the mapping sends  $g \in G$  to  $l_g(x) = gx, \forall x \in G$

### 34.10 Theorem

Let  $X$  be a finite set. Assume that  $\sigma \in \mathfrak{S}_X$  can be written as

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

where  $\tau_1$  is transposition.

We put

$$\text{sgn}(\sigma) := (-1)^\sigma$$

This is a well-defined function. Moreover  $\text{sgn}$  is a morphism from  $\mathfrak{S}_X$  to  $(\{-1, 1\}, \times)$

### Proof

Let's define the mapping:

$$\begin{aligned} \phi : \mathfrak{S}_n &\rightarrow \mathbb{Q}^\times \\ \sigma &\mapsto \prod_{(i,j) \in \{1, \dots, n\}^2, i < j} \frac{\sigma(i) - \sigma(j)}{i - j} \end{aligned}$$

To show that  $\phi$  is a morphism of groups. Let

$$\theta = \{U \in \wp(\{1, \dots, n\}) \mid \text{card}(U) = 2\}$$

$$\begin{aligned} \phi(\sigma \circ \tau) &= \prod_{(i,j) \in \theta} \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{i - j} \\ &= \left( \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{\tau(i) - \tau(j)} \right) \times \left( \prod_{(i,j) \in \theta} \frac{\tau(i) - \tau(j)}{i - j} \right) \\ &= \phi(\sigma)\phi(\tau) \end{aligned}$$

When  $\tau$  is a transposition,  $\phi(\tau) = -1$ . Therefore

$$\phi(\sigma) = \prod_{i=1}^d \phi(\tau_i)$$

since

$$\sigma = \tau_1 \circ \cdots \circ \tau_d$$

### 34.11 Remark

Let  $A_n \subsetneq \mathfrak{S}_n$  such that

$$A_n = \{\sigma \in \mathfrak{S}_n \mid \text{sgn}(\sigma) = 1\}$$

is an alternating symmetric group.

### 34.12 Exercise

Let  $X$  be a set of cardinality  $n$ . Let  $\sigma : X \rightarrow \{1, \dots, n\}$  be a bijection. Prove that

$$\begin{aligned} \phi : \mathfrak{S}_X &\rightarrow \mathfrak{S}_n \\ \tau &\mapsto \sigma^{-1} \circ \tau \circ \sigma \end{aligned}$$

is an isomorphism.

### 34.13 Symmetric of multilinear mapping

We fix a commutative unitary ring  $K$  and  $K$ -modules  $E, F$

### 34.14 Def: Symmetric and Alternating

symmetric Let  $n \in \mathbb{N}$  and  $f \in \text{Hom}^{(n)}(E^n, F)$ . If for any  $\sigma \in \mathfrak{S}_n$  one has  $\forall x \in E^n$

$$f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

Then we say  $f$  is symmetric

alternating If for any  $(i, j) \in \{1, \dots, n\}^2$  such that  $i \neq j$  and any  $(x_1, \dots, x_n) \in E^n$  such that  $x_i = x_j$

$$f(x_1, \dots, x_n) = 0$$

then we say that  $f$  is alternating.

### 34.15 Prop

Suppose that  $f \in \text{Hom}^{(n)}(E^n, F)$  is alternating, then  $\forall (x_1, \dots, x_n) \in E^n$ ,  $\sigma \in \mathfrak{S}_n$

$$f(x_1, \dots, x_n) = \text{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

#### Proof

By corollary 34.8, it's enough to prove the proposition for adjacent transitions. Let  $i \in \{1, \dots, n-1\}$  then

$$\begin{aligned} 0 &= f(x_1, \dots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \dots, x_n) \\ &= f(x_1, \dots, x_{i-1}, x_i, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, x_{i+2}, \dots, x_n) \\ &\quad + f(x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_n) \end{aligned}$$

The adjacent transition  $\sigma$  is  $(i, i+1)$



### 34.16 Def:

$Hom_s$  and  $Hom_a$

We denote with  $Hom_s^{(n)}(E^n, F)$  and  $Hom_a^{(n)}(E^n, F)$  the set of symmetric and alternating  $n$ -linear mappings from  $E$  to  $F$ .

$Hom_s^{(n)}(E^n, F)$  and  $Hom_a^{(n)}(E^n, F)$  are sub-K-modules of  $Hom^{(n)}(E^n, F)$  and when  $n = 1$ , by convention

$$Hom_s^{(1)}(E, F) = Hom_a^{(1)}(E, F) = Hom(E, F)$$

### 34.17 Reminder

Let  $E, F$  be two normed vector spaces over  $\mathbb{R}$   $f : E \rightarrow F$  is differentiable (twice)

$$\begin{aligned} df : E &\rightarrow \mathcal{L}(E, F) \\ D^2d : E &\rightarrow \mathcal{L}(E, \mathcal{L}(E, F)) \\ A &\mapsto ((x, y) \rightarrow A(x)(y)) \end{aligned}$$

### 34.18 Theorem(Schwarz)

$U \subseteq E$  is an open set,  $f : U \rightarrow F$  is a function of class  $C^n$ . Then for any  $p \in U$

$$D^n f(p) \in \mathcal{L}^n(E^n, F)$$

is symmetric

#### Proof

By induction and by the fact that permutation are decomposed in transpositions, we can reduce to prove only the case  $n = 2$

$$d_{p+u}f - d_p f = D^2 f(p)(u, \cdot) + o(u)$$

$\forall \epsilon > 0, \exists \delta > 0$  such that  $0 < \|u\| < \delta$ , then

$$\|d_{p+u}f - d_p f - D^2 f(p)(u, \cdot) + o(u)\| \leq \epsilon \|u\|$$

For any  $x \in \mathcal{B}(p, \frac{\epsilon}{2})$  let's introduce the following function

$$\varphi(x) = f(x+k) - f(x) - D^2 f(p)(k, x)$$

We use the mean value inequality on  $\varphi$

$$\begin{aligned} &\|\varphi(p+h) - \varphi(p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - D^2 f(p)(k, p+h) - f(p+k) - f(p) - D^2 f(p)(k, p)\| \\ &= \|f(p+h+k) + f(p) - f(p+h) - f(p+k) - D^2 f(p)(k, h)\| \\ &\leq \left( \sup_{t \in [0,1]} \|d_{p+th} \varphi\| \right) \|h\| \end{aligned}$$

$$\|d_{p+th}(\varphi)\| = \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\|$$

add and subtract  $d_p f, D^2f(p)(th, \cdot)$  then by triangle inequality

$$\begin{aligned} & \|d_{p+th+k}f - d_{p+th}f - D^2f(p)(k, \cdot)\| \\ & \leq \|d_{p+th+k}(f) - d_p f - D^2f(p)(k + th, \cdot)\| \\ & \quad + \|d_{p+th}f - d_p f - D^2f(p)(th, \cdot)\| \\ & \leq \epsilon \|th + k\| + \epsilon(th) \\ & \leq 2\epsilon(\|h\| + \|k\|) \end{aligned}$$

then

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - D^2f(p)(k, h)\| \\ & = o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

exchange the role of  $h, k$  then we get

$$\begin{aligned} & \|f(p + h + k) + f(p) - f(p + k) - f(p + h) - f(p + k) - D^2f(p)(h, k)\| \\ & \leq o(\max\{\|h\|, \|k\|\}^2) \end{aligned}$$

then

$$\underbrace{\|D^2f(p)(k, h) - D^2f(p)(h, k)\|}_{\text{bilinear function}} = o(\underbrace{\max\{\|h\|, \|k\|\}^2}_{\text{quachetic}})$$

this implies that the LHS is 0

### 34.19 Def

Let  $E, F$  be normed vector spaces over a complete value field  $(K, |\cdot|)$  let  $U \subseteq E, V \subseteq F$  be open subsets and  $f : U \rightarrow V$  is a bijection.

- (1) If  $f$  and  $f^{-1}$  are both continuous we say that  $f$  is a homeomorphism
- (2) If  $f$  and  $f^{-1}$  are both of class  $C^n$  we say that  $f$  is a  $e^n$ -diffeomorphism

If (2) is true for any  $n \in \mathbb{N}$  we say that  $f$  is a  $C^\infty$ -diffeomorphism

### 34.20 Prop

Let  $E, F$  be two normed Banach spaces. Let  $I(E, F) \in \mathcal{L}(E, F)$  be the set of linear continuous and invertible mappings such that  $\text{norm}\varphi^{-1} \leq +\infty$ . Then  $I(E, F)$  is open in  $\mathcal{L}(E, F)^\vee$  Moreover the mapping

$$\begin{aligned} I(E, F) & \rightarrow I(F, E) \\ \phi & \mapsto \varphi^{-1} \end{aligned}$$

is a  $e^1$ -diffeomorphism

**Proof**

Let  $\varphi \in I(E, F)$  we want to show that

$$\varphi - \psi \in I(E, F)$$

for  $\psi \in \mathcal{E}, \mathcal{F}$  such that  $\|\psi\| < \frac{1}{\|\varphi^{-1}\|}$  Notice that

$$\varphi - \psi = \varphi \circ (Id_E - \varphi^{-1} \circ \psi)$$

Since

$$\|\varphi^{-1}\psi\| \leq \|\varphi^{-1}\| \|\psi\| < 1$$

This means that the series

$$\sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n}$$

is absolutely convergent in  $\mathcal{L}(E, E)$  This series is the inverse of  $(Id_E - \varphi^{-1}\psi)$

$$(Id_E - \varphi^{-1}\psi) \circ \sum_{n=0}^{N-1} (\varphi^{-1} \circ \psi) \overset{\text{composite n times}}{\widehat{\circ n}} = Id_E - (\varphi^{-1} \circ \psi)^{\circ N}$$

take  $\lim_{N \rightarrow +\infty}$ , then

$$(\varphi - \psi)^{-1} = \sum_{n \in \mathbb{N}} (\varphi^{-1} \circ \psi)^{\circ n} \circ \varphi^{-1}$$

and

$$(\varphi - \psi)^{-1} = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

replace the inverse with  $i$

$$i(\varphi - \psi) - i(\varphi) = \varphi^{-1} + \varphi^{-1} \circ \psi \circ \varphi^{-1} + o(\|\psi\|)$$

then

$$d_\varphi i(\psi) = i(\varphi) \circ (-\psi) \circ i(\varphi)$$

so  $i$  is differentiable. Moreover  $i$  and  $i^{-1}$  are continuous.

**Remark**

By induction we can show that  $i$  is a  $C^{+\infty}$ -diffeomorphism

**34.21 Prop**

Let  $n \in \mathbb{N} \cup \{\infty\}$  Let  $E, F, G$  be normed vector spaces over a complete valued field  $(K, |\cdot|)$   $U \subseteq E, V \subseteq F$  be open sets.  $f : U \rightarrow V, g : V \rightarrow G$  be mappings of class  $C^n$ , then  $g \circ f$  also of class  $C^n$

### 34.21.1 Proof

The case where  $n = 0$  is known

Denote by

$$\begin{aligned}\Phi : \mathcal{L}(E, F) \times E &\rightarrow F \\ (\beta, \alpha) &\mapsto \beta \circ \alpha\end{aligned}$$

$\Phi$  is a bounded bilinear mapping

$$\|\Phi(\beta, \alpha)\| \leq \|\beta\| \cdot \|\alpha\|$$

Suppose that  $n \geq 1$  and the statement is true for mappings of class  $C^{n-1}$   $g \circ f$  is differentiable.

$$\forall p \in U \quad d_p(g \circ f) = d_{f(p)}g \circ d_p f$$

$$D^1(g \circ f) : U \rightarrow \mathcal{L}(E, G)$$

$$D^1 = \Phi \circ (D^1 g \circ f, D^1 f)$$

$$\begin{aligned}(D^1 g \circ f, D^1 f) : U &\rightarrow \mathcal{L}(F, G) \times \mathcal{L}(E, F) \\ p &\mapsto (d_{f(p)}g, d_p f)\end{aligned}$$

$$d_{\beta_0, \alpha_0} \Phi(\beta, \alpha) = \beta_0 \circ \alpha + \beta \circ \alpha_0$$

$$\begin{aligned}D^1 \Phi : \mathcal{L}(F, G) \times \mathcal{L}(E, F) &\rightarrow \mathcal{L}(\mathcal{L}(F, G) \times \mathcal{L}(E, F), \mathcal{L}(E, G)) \\ (\alpha_0, \beta_0) &\mapsto ((\alpha, \beta) \mapsto \beta_0 \circ \alpha + \beta \circ \alpha_0)\end{aligned}$$

Since  $g, f$  are of class  $C^n$   $D^1 f, D^1 g$  are of class  $C^{n-1}$  Thus, by induction hypothesis,

$$(D^1 g \circ f, D^1 f)$$

is of class  $C^{n-1}$  Since  $\Phi$  is of class  $C^\infty$ , we obtain that

$$D^1(g \circ f)$$

is of class  $C^{n-1}$  then

$$g \circ f$$

is of class  $C^n$

### 34.22 Prop

Let  $E$  and  $F$  be Banach space over a complete valued field  $(K, |\cdot|)$ .  $U$  and  $V$  be open subsets of  $E$  and  $F$  respectively.  $n \in \mathbb{N} \cup \{\infty\}$  and  $f : U \rightarrow V$  be a bijection. If  $f$  is of class  $C^n$ , then  $f^{-1}$  is differentiable, then  $f^{-1}$  is of class  $C^n$

**Proof**

$$f \circ f^{-1} = Id_V$$

$$\forall y \in V$$

$$d_y(f \circ f^{-1}) = d_{f^{-1}(p)}f \circ d_yf^{-1} = Id_F$$

For  $x \in U, y = f(x)$

$$d_y(f \circ f^{-1}) = d_xf \circ d_yf^{-1} = Id_F$$

$$d_x(f^{-1} \circ f) = d_yf \circ d_xf^{-1} = Id_E$$

So

$$d_yf^{-1} - (d_xf)^{-1}$$

that is

$$D^1f^{-1} = \iota \circ (D^1f \circ f^{-1})$$

where

$$\begin{aligned} \iota : I(E, F) &\rightarrow I(F, E) \\ \phi &\mapsto \phi^{-1} \end{aligned}$$

Suppose that  $f^{-1}$  is of class  $C^{n-1}$  then

$$D^1f^{-1} = \iota D^1f \circ f^{-1}$$

is of class  $C^{n-1}$

**34.23 Local Inversion Theorem**

Let  $E$  and  $F$  be Banach space over  $\mathbb{R}$   $U \in E$  open,  $f : U \rightarrow F$  be a mapping of class  $C^n$  and  $a \in U$ . Suppose that  $d_a f \in I(E, F)$  ( $d_a f$  is invertible and of bounded inverse) Then there exists open neighborhoods  $V$  and  $W$  of  $a$  and  $f(a)$  respectively, such that

- $V \subseteq U$  and  $f(V) \subseteq W$
- The restriction of  $f$  to  $V$  defines a bijection from  $V$  to  $W$
- 

$$(f|_V)^{-1}W \rightarrow V$$

is of class  $C^n$

**34.23.1 Proof**

For  $y \in F$  consider the mapping:

$$\begin{aligned}\phi_y : U &\rightarrow F \\ x &\mapsto x - (d_a f)^{-1}(f(x) - y)\end{aligned}$$

$f(x) = y$  iff  $\phi_y(x) = x$  i.e.  $x$  is a fix point of  $\phi_y$   $\phi_y$  is of class  $C^1$  and

$$d_x \phi_y(v) = v - d_a f^{-1}(d_x f(v))$$

$\forall v$

$$d_a \phi_y^{(v)} = 0$$

By the continuity of  $D^1 f$  there exists  $r > 0$  such that

$$\overline{\mathcal{B}}(a, r) \subseteq U$$

and  $\forall y \in F, \forall x \in \overline{\mathcal{B}}(a, r)$

$$\|d_x \phi_y\| \leq \frac{1}{2}$$

By the mean value inequality.  $\forall (x_1, x_2) \in \overline{\mathcal{B}}(a, r)$

$$\|\phi_y(x_1) - \phi_y(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|$$

Hence  $\phi_y$  is contraction.

By the boundedness of  $(d_a f)^{-1} \exists \delta > 0$  such that

$$\forall y \in \overline{\mathcal{B}}(f(a), \delta) \quad \|(d_a f)^{-1}(f(a) - y)\| \leq \frac{r}{2}$$

Then  $\forall x \in \overline{\mathcal{B}}(a, r) \quad y \in \overline{\mathcal{B}}(f(a), \delta)$

$$\begin{aligned}\|\phi_y(x) - a\| &\leq \|\phi_y(x) - \phi_y(a)\| + \|\phi_y(a) - a\| \\ &\leq \frac{1}{2} \|x - a\| + \frac{r}{2} \\ &\leq \frac{r}{2} + \frac{r}{2} = r\end{aligned}$$

$\phi_y(\overline{a}, \overline{r}) \in \overline{\mathcal{B}}(a, r)$ . By the fixed point theorem

$$\exists g : \overline{\mathcal{B}}(f(a), \delta) \rightarrow \overline{\mathcal{B}}(a, r)$$

sending  $y$  to the fixed point of  $\phi_y$  Let  $W = \mathcal{B}(f(a), g)$ , then

$$g|_W : W \rightarrow V$$

is the inverse of  $f|_V : V \rightarrow W$  Hence  $f^{-1}(W) = V$  is open.

In the following, we prove that  $g$  is of class  $C^n$  on an open neighborhood of  $f(a)$ . By reducing  $V$  and  $W$ , we may assume that  $\forall x \in V$

$$d_x f \in I(E, F)$$

Let  $x_0 \in V$   $y_0 = f(x_0)$   $x_0 = g(y_0)$

$$y - y_0 = f(g(y)) - f(g(y_0)) = d_{x_0} f(g(y) - g(y_0)) + o(\|g(y) - g(y_0)\|)$$

So

$$g(y) - g(y_0) = (d_x f)^{-1}(y - y_0) + o(\|g(y) - g(y_0)\|)$$

Thus leads to

$$g(y) - g(y_0) = O(\|y - y_0\|)$$

$(\exists \epsilon > 0 \quad (1 - \epsilon) \|g(y) - g(y_0)\| \leq \|d_{x_0} f\|^{-1}$  when  $\|y - y_0\|$  is sufficiently small)

So

$$d_{y_0} g = (d_x f)^{-1}$$

By the previous proposition,  $g$  is of class  $C^n$





**Part VII**

**Integration**



## Chapter 35

# Integral operators

We fix a set  $\Omega$  and a vector subspace  $S$  of  $\mathbb{R}^\Omega$  over  $\mathbb{R}$ . We suppose that  $\forall (f, g) \in S^2$

$$\begin{aligned} f \wedge g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \min\{f(\omega), g(\omega)\} \end{aligned}$$

belongs to  $S$

### 35.1 Prop

(1)  $\forall (f, g) \in S^2$

$$\begin{aligned} f \vee g : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto \max\{f(\omega), g(\omega)\} \end{aligned}$$

$$f \vee g \in S$$

(2)  $\forall f \in S$

$$\begin{aligned} |f| : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto |f(\omega)| \end{aligned}$$

$$|f| \in S$$

### Proof

(1)

$$f \vee g = f + g - f \wedge g$$

(2)

$$|f| = f \vee (-f)$$

### 35.2 Def

We call integral operator on  $S$  any  $\mathbb{R}$ -linear mapping  $I : S \rightarrow \mathbb{R}$  that satisfies the following conditions:

- (1) If  $f \in S$  is such that  $\forall \omega \in \Omega, f(\omega) \geq 0$  then  $I(f) \geq 0$
- (2) If  $(f_n)_{n \in \mathbb{N}}$  is a decreasing sequence of elements in  $S$  such that  $\forall \omega \in \Omega \lim_{n \rightarrow +\infty} f_n(\omega) = 0$  then

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

$$(\forall \omega \in \Omega, n \in \mathbb{N}, f_n(\omega) \geq f_{n+1}(\omega))$$

### 35.3 Example

- (1)  $\Omega = \mathbb{R}$   $S$ =vector subspace of  $\mathbb{R}^{\mathbb{R}}$  generated by mappings of the form  $\mathbb{1}_{]a,b]}$   $(a, b) \in \mathbb{R}^2, a < b$

$$\mathbb{1}_{]a,b]} = \begin{cases} 1, x \in ]a, b] \\ 0, else \end{cases}$$

Any element of  $S$  is of the form

$$\sum_{i=1}^n \lambda_i \mathbb{1}_{]a_i, b_i]}$$

$I : S \rightarrow \mathbb{R}$  is defined as

$$I\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (b_i - a_i)$$

More generally if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is increasing and right continuous ( $\forall x \in \mathbb{R}, \lim_{\epsilon > 0, \epsilon \rightarrow 0} \varphi(x + \epsilon) = \varphi(x)$ ) We define

$$I_\varphi : S \rightarrow \mathbb{R}$$

$$I\left(\sum_{i=1}^n \lambda \mathbb{1}_{]a_i, b_i]}\right) = \sum_{i=1}^n \lambda_i (\varphi(b_i) - \varphi(a_i))$$

- (2) (Radon measure)

Let  $\Omega$  be a quasi-compact topological space

$$S = C^0(\Omega) := \{f : \Omega \rightarrow \mathbb{R} \text{ continuous}\}$$

Let  $I : S \rightarrow \mathbb{R}$   $\mathbb{R}$ -linear, such that  $\forall f \in S, f \geq 0$  one has  $I(f) \geq 0$

### 35.4 Dini's theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $C^0(\Omega)$ , that converges pointwisely to some  $f \in C^0(\Omega)$  Then  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$

#### Proof

Let  $g_n = f_n - f > 0$  Fix  $\epsilon > 0 \forall n \in \mathbb{N}$  let

$$U_n = \{\omega \in \Omega \mid g_n(\omega) < \epsilon\}$$

is open

Moreover

$$\bigcup_{n \in \mathbb{N}} U_n = \Omega \quad (U_0 \subseteq U_1 \subseteq \dots)$$

Since  $\Omega$  is quasi-compact,  $\exists N \in \mathbb{N}, \Omega = U_N$  Therefore  $\forall n \in \mathbb{N}, n \geq N, \forall \omega \in \Omega$

$$g_n(\omega) < \epsilon$$

Consequence. If  $(f_n)_{n \in \mathbb{N}} \in S^{\mathbb{N}}$  is decreasing and converges pointwisely to 0, then

$$\|f_n\|_{\sup} := \sup_{\omega \in \Omega} |f_n(\omega)|$$

converges to 0 when  $n \rightarrow +\infty \forall n \in \mathbb{N}$

$$f_n \leq \|f_n\|_{\sup} \cdot \mathbb{1}_{\Omega}$$

So

$$0 \leq I(f_n) \leq \|f_n\|_{\sup} I(\mathbb{1}_{\Omega}) \rightarrow 0 \quad (n \rightarrow +\infty)$$

(If  $f \leq g$  then  $g - f \geq 0$  so  $I(g - f) = I(g) - I(f) \geq 0 \Rightarrow I(g) \geq I(f)$ )

### 35.5 Def

We call  $\sigma$ -algebra any subset  $\mathcal{A}$  of  $\wp(\Omega)$  that satisfies the following conditions:

- $\emptyset \in \mathcal{A}$
- If  $A \in \mathcal{A}$  then  $\Omega \setminus A \in \mathcal{A}$
- If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  then  $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$

Given a  $\sigma$ -algebra  $\mathcal{A}$  on  $\Omega$ , we mean by measure on  $(\Omega, \mathcal{A})$  any mapping  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  such that :

- $\mu(\emptyset) = 0$
- If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$  such that  $A_i$  are pairwise disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$



## Chapter 36

# Riemann integral

### 36.1 Def

Let  $\Omega$  be a non-empty set and  $S$  be a vector subspace of  $\mathbb{R}^\Omega$ . If  $\forall (f, g) \in S^2, f \wedge g \in S$ , we say that  $S$  is a Riesz space.

In this section, we fix a Riesz space and an integral operator  $I : S \rightarrow \mathbb{R}$

### 36.2 Def

For any  $f : \Omega \rightarrow \mathbb{R}$  let

$$I^*(f) := \inf_{\mu \in S, \mu \geq f} I(\mu)$$

$$I_*(f) := \sup_{l \in S, l \leq f} I(l)$$

If  $I^*(f) = I_*(f)$  then we say that  $f$  is I-Riemann integral, and denote by  $I(f)$  the value  $I^*(f)$  (or  $I_*(f)$ )

### 36.3 Theorem

The set  $\mathcal{R}$  of all I-Riemann integral mappings form a vector space of  $\mathbb{R}^\Omega$  that contains  $S$ . Moreover,  $I : \mathcal{R} \rightarrow \mathbb{R}$  is an  $\mathbb{R}$ -linear mapping extending  $I : S \rightarrow \mathbb{R}$

#### Proof

$$\forall h \in S$$

$$I^*(h) = I_*(h) = I(h)$$

So  $h \in \mathcal{R}$

Let  $(f_1, f_2) \in \mathcal{R}$ . If  $(\mu_1, \mu_2) \in S^2, \mu_1 \geq f_1, \mu_2 \geq f_2$  then

$$\mu_1 + \mu_2 \in S, \mu_1 + \mu_2 \geq f_1 + f_2$$

Hence

$$I(\mu_1) + I(\mu_2) \geq I^*(f_1 + f_2)$$

Take the infimum with respect to  $(\mu_1, \mu_2)$  we get

$$I^*(f_1) + I^*(f_2) \geq I^*(f_1 + f_2)$$

Similarly

$$I_*(f_1) + I_*(f_2) \leq I_*(f_1 + f_2)$$

Hence

$$I^*(f_1 + f_2) = I_*(f_1 + f_2) = I(f_1) + I(f_2)$$

Let  $f : \Omega \rightarrow \mathbb{R}$  be a mapping,  $\lambda \in \mathbb{R}_{>0}$

$$I^*(\lambda f) = \inf_{\mu \in S, \mu \geq \lambda f} I(\mu) = \inf_{\nu \in S, \nu \geq f} I(\lambda \nu) = \lambda I^*(f)$$

Similarly

$$I_*(\lambda f) = \lambda I_*(f)$$

Hence if  $f \in \mathcal{R}$  then  $\lambda f \in \mathcal{R}$  and  $I(\lambda f) = \lambda I(f)$

$$I^*(-f) = \inf_{\mu \in S, \mu \geq -f} I(\mu) = \inf_{l \in S, l \leq f} I(-l) = - \sup_{l \in S, l \leq f} I(l) = -I_*(f)$$

Similarly

$$I_*(-f) = -I^*(f)$$

Hence if  $f \in \mathcal{R}$  then  $-f \in \mathcal{R}$  and  $I(-f) = -I(f)$



## Chapter 37

# Daniell integral

We fix an integral operator  $I : S \rightarrow \mathbb{R}$

### 37.1 Prop

#### 37.1.1

Let  $(f_n)_{n \in \mathbb{N}}$  be an increasing sequence in  $S$  that converges pointwisely to some  $f \in S$ . Then

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

**Proof**

Let  $g_n = f - f_n \in S$   $(g_n)_{n \in \mathbb{N}}$  is decreasing and converges pointwisely to 0. Then

$$\lim_{n \rightarrow +\infty} I(g_n) = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f)$$

#### 37.1.2

Let  $(f_n)_{n \in \mathbb{N}}$  be an increasing sequence in  $S$ ,  $f \in S$  If  $f \leq \lim_{n \rightarrow +\infty} f_n$ , then

$$I(f) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

**Proof**

$$f = \lim_{n \rightarrow +\infty} f \wedge f_n$$

So

$$I(f) = \lim_{n \rightarrow +\infty} I(f \wedge f_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.2 Def

Let

$$S^\uparrow = \left\{ f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\} \mid \begin{array}{l} \exists (f_n)_{n \in \mathbb{N}} \in S^\mathbb{N} \text{ increasing such that} \\ f = \lim_{n \rightarrow +\infty} f_n \text{ pointwisely} \end{array} \right\}$$

### 37.3 Prop

Let  $f, g$  be elements of  $S^\uparrow$  such that  $f \leq g$ . Let  $(f_n)_{n \in \mathbb{N}}$  and  $(g_m)_{m \in \mathbb{N}}$  be increasing sequences in  $S$  such that  $f = \lim_{n \rightarrow +\infty} f_n, g = \lim_{m \rightarrow +\infty} g_m$ . Then

$$\lim_{n \rightarrow +\infty} I(f_n) \leq \lim_{m \rightarrow +\infty} I(g_m)$$

#### Proof

For any  $m \in \mathbb{N}$

$$f_m \leq f \leq g$$

Hence

$$I(f_m) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Taking  $\lim_{m \rightarrow +\infty}$  we get

$$\lim_{m \rightarrow +\infty} I(f_m) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.4 Corollary

Let  $f \in S^\uparrow$ . If  $(f_n)_{n \in \mathbb{N}}$  and  $(\tilde{f}_n)_{n \in \mathbb{N}}$  be increasing sequence in  $S$  such that

$$f = \lim_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} \tilde{f}_n$$

then

$$\lim_{n \rightarrow +\infty} I(f_n) = \lim_{n \rightarrow +\infty} I(\tilde{f}_n)$$

We denote by  $I(f)$  the limit  $\lim_{n \rightarrow +\infty} I(f_n)$

Thus we obtain a mapping  $I : S^\uparrow \rightarrow \mathbb{R} \cup \{+\infty\}$  such that

- If  $(f_n)_{n \in \mathbb{N}} \in S^\mathbb{N}$  is increasing then

$$I\left(\lim_{n \rightarrow +\infty} f_n\right) = \lim_{n \rightarrow +\infty} I(f_n)$$

- If  $(f, g) \in S^{\uparrow 2}$   $f \leq g$  then  $I(f) \leq I(g)$
- If  $(f, g) \in S^{\uparrow 2}$  then  $f + g \in S^\uparrow$  and

$$I(f + g) = I(f) + I(g)$$

- If  $f \in S^\uparrow, \lambda \geq 0$  then  $\lambda f \in S^\uparrow$  and  $I(\lambda f) = \lambda I(f)$

### 37.5 Prop

Let  $(f_n)_{n \in \mathbb{N}} \in (S^\uparrow)^\mathbb{N}$  be an increasing sequence and  $f = \lim_{n \rightarrow +\infty} f_n$ . Then

$$f \in S^\uparrow$$

and

$$I(f) = \lim_{n \rightarrow +\infty} I(f_n)$$

#### Proof

For  $k \in \mathbb{N}$  let  $(g_{k,m})_{m \in \mathbb{N}} \in S^\mathbb{N}$  be an increasing sequence such that

$$f_k = \lim_{m \rightarrow +\infty} g_{k,m}$$

For  $n \in \mathbb{N}$  let  $h_n = g_{0,n} \vee \cdots \vee g_{n,n} \in S$  The sequence  $(h_n)_{n \in \mathbb{N}}$  is increasing. Moreover

$$f_n \geq k_n \geq g_{k,n} \quad (k \leq n)$$

Hence

$$f_n \geq h_n$$

Taking  $\lim_{n \rightarrow +\infty}$  we get  $\forall k \in \mathbb{N}$

$$f = \lim_{n \rightarrow +\infty} f_n \geq \lim_{n \rightarrow +\infty} h_n \geq \lim_{n \rightarrow +\infty} g_{k,n} = f_k$$

Taking  $\lim_{k \rightarrow +\infty}$  we get

$$f = \lim_{n \rightarrow +\infty} h_n$$

Hence  $f \in S^\uparrow$  and

$$I(f) = \lim_{n \rightarrow +\infty} I(h_n) \leq \lim_{n \rightarrow +\infty} I(f_n)$$

Conversely,  $\forall n \in \mathbb{N}, f \geq f_n$  Hence

$$I(f) \geq \lim_{n \rightarrow +\infty} I(f_n)$$

### 37.6 Def

Let  $S^\downarrow = \{-f \mid f \in S^\uparrow\}$  We extend  $I$  to  $I : S^\downarrow \rightarrow \mathbb{R}U - \infty$  by letting  $I(-f) := -I(f)$  for  $f \in S^\uparrow$

### 37.7 Prop

Let  $(f, g) \in (S^\uparrow \cup S^\downarrow)^2$  If  $f \leq g$  then

$$I(f) \leq I(g)$$

**Proof**

It suffices to treat the cases where  $(f, g) \in S^\uparrow \times S^\downarrow$  and  $(f, g) \in S^\uparrow \times S^\downarrow$

If  $(f, g) \in S^\uparrow \times S^\downarrow$  then  $-f \in S^\downarrow$  and hence  $g - f \in S^\uparrow, g - f \geq 0$  In both cases,

$$0 \leq I(g - f) = I(g) + I(-f) = I(g) - I(f)$$

**37.8 Def**

Let  $f : \Omega \rightarrow \mathbb{R}$  be a mapping. We define

$$\bar{I}(f) := \inf_{\mu \in S^\uparrow, \mu \geq f} I(\mu) \leq \inf_{\mu \in S, \mu \geq f} I(\mu) = I^*(f)$$

$$\underline{I}(f) := \sup_{\mu \in S^\downarrow, \mu \leq f} I(\mu) \geq \sup_{\mu \in S, \mu \leq f} I(\mu) = I_*(f)$$

If  $\bar{I}(f) = \underline{I}(f)$  then we say that  $f$  is  $I$ -integrable (in the sense of Daniell)

**37.9 Remark**

If  $f$  is I-integrable in the sense of Riemann, then it is I-integrable in sense of Daniell

**37.10 Daniell Theorem**

The set  $L^1(I)$  of all I-integrable mappings forms a vector subspace of  $\mathbb{R}$ . Moreover

- $\forall (f, g) \in L^1(I) \ f \wedge g \in L^1(I)$
- $I : L^1(I) \rightarrow \mathbb{R}$  is an integral operator extending  $I : S \rightarrow \mathbb{R}$

**Proof**

Let  $(f_1, f_2) \in L^1(I)^2$  let  $(l_1, l_2) \in S^{\downarrow 2}, l_1 \leq f_1, l_2 \leq f_1$  Let  $(\mu_1, \mu_2) \in S^{\uparrow 2}, f_1 \leq \mu_1, f_2 \leq \mu_2$

We have

$$l_1 + l_2 \leq f_1 + f_2 \leq \mu_1 + \mu_2$$

Taking the supremum with respect to  $(l_1, l_2)$ , we get

$$I(f_1) + I(f_2) (= \underline{I}(f_1) + \underline{I}(f_2)) \leq \underline{I}(f_1 + f_2)$$

Taking the infimum with respect to  $(\mu_1, \mu_2)$ , we get

$$\bar{I}(f_1 + f_2) \leq I(f_1) + I(f_2)$$

Then

$$\bar{I}(f_1 + f_2) = \underline{I}(f_1 + f_2)$$

So  $f_1 + f_2 \in L^1(I)$  and  $I(f_1 + f_2) = I(f_1) + I(f_2)$

Similarly, if  $f \in L^1(I), \lambda \geq 0$  then

$$\begin{aligned} \underline{I}(\lambda f) &= \sup_{l \leq \lambda f, l \in S^\downarrow} I(l) \\ &= \sup_{l \leq f, l \in S^\downarrow} I(\lambda l) \\ &= \lambda \underline{I}(f) = \lambda I(f) \end{aligned}$$

$$\bar{I}(\lambda f) = \lambda \bar{I}(f) = \lambda I(f)$$

So  $\lambda f \in L^1(I)$  and  $I(\lambda f) = \lambda I(f)$

Moreover, if  $f \in L^1(I), \mu \in S^\uparrow, l \in S^\downarrow, l \leq f \leq \mu$  then

$$-\mu \in S^\downarrow, -l \in S^\uparrow, -\mu \leq -f \leq -l$$

Hence

$$\bar{I}(-f) = -\underline{I}(f) = -I(f) \quad \underline{I}(-f) = -\bar{I}(f) = -I(f)$$

So  $-f \in L^1(I)$  and  $I(-f) = -I(f)$

We proved that  $\forall (f_1, f_2) \in L^1(I)^2$

$$f_1 \wedge f_2 \in L^1(I)$$

Let  $(f_1, f_2) \in L^1(I)^2$ , for any  $\epsilon > 0 \exists (l_1, l_2) \in S^\downarrow{}^2, (\mu_1, \mu_2) \in S^\uparrow{}^2$  such that

$$l_1 \leq f_1 \leq \mu_1 \quad l_2 \leq f_2 \leq \mu_2$$

such that

$$I(\mu_1 - l_1) \leq \frac{\epsilon}{2} \quad I(\mu_2 - l_2) \leq \frac{\epsilon}{2}$$

One has  $l_1 \wedge l_2 \leq f_1 \wedge f_2 \leq \mu_1 \wedge \mu_2$

$$\mu_1 \wedge \mu_2 - l_1 \wedge l_2 \leq (\mu_1 - l_1) + (\mu_2 - l_2)$$

$$\left( \begin{array}{l} \text{If } \mu_1(\omega) \leq \mu_2(\omega), l_1 \leq l_1(\omega) \\ LHS = \mu_1(\omega) - l_1(\omega) \\ RHS = \mu_1(\omega) - l_2(\omega) + \mu_2(\omega) - l_1(\omega) \geq \mu_1(\omega) - l_2(\omega) \end{array} \right)$$

### 37.11 Beppo Levi Theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a monotone sequence of elements of  $L_1(I)$ , which converges pointwisely to some  $f : \Omega \rightarrow \mathbb{R}$  If  $(I(f_n))_{n \in \mathbb{N}}$  converges to a real number  $\alpha$  Then  $f \in L^1(I)$  and  $I(f) = \alpha$

**Proof**

Assume that  $(f_n)_{n \in \mathbb{N}}$  is increasing. Moreover, by replacing  $f_n$  by  $f_n - f_0$  we may assume that  $f_0 = 0$

Let  $\epsilon > 0 \forall n \in \mathbb{N}$  let  $\mu_n \in S^\uparrow$  such that  $f_n - f_{n-1} \leq \mu_n$  and

$$I(f_n - f_{n-1}) \geq I(\mu_n) - \frac{\epsilon}{2}$$

the existence

$$I(f_n - f_{n-1}) = \inf_{\mu \in S^\uparrow, \mu \geq f_n - f_{n-1}} I(\mu)$$

If  $\forall \mu \in S^\uparrow, \mu \geq f_n - f_{n-1}$  one has

$$I(\mu) > I(f_n - f_{n-1}) + \frac{\epsilon}{2}$$

then

$$I(f_n - f_{n-1}) + \frac{\epsilon}{2} \leq I(f_n - f_{n-1})$$

contradiction.

Thus

$$f_n = \sum_{k=1}^n (f_k - f_{k-1}) \leq \mu_1 + \cdots + \mu_n$$

and

$$I(f_n) \geq \sum_{k=1}^n (I(\mu_k) - \frac{\epsilon}{2^k}) \geq I(\mu_1) + \cdots + I(\mu_n) - \epsilon$$

Let  $\mu = \mu_1 + \cdots + \mu_n + \cdots \in S^\uparrow$

$$I(\mu) = \sum_{n \in \mathbb{N}} I(\mu_n)$$

One has  $\mu \geq f$

$$\lim_{n \rightarrow +\infty} \geq I(\mu) - \epsilon \geq \bar{I}(f) - \epsilon$$

Similarly, one can choose  $l_n \in S^\downarrow, l_n \leq f_n, I(l_n) \geq I(f_n) - \epsilon$

$$\liminf_{n \rightarrow +\infty} I(l_n) \geq \alpha - \epsilon$$

Note that  $l_n \leq f_n \leq f$ , so

$$\alpha - \epsilon \leq \liminf_{n \rightarrow +\infty} I(l_n) \leq \underline{I}(f)$$

Thus

$$\alpha - \epsilon \leq \underline{I}(f) \leq \bar{I}(f) \leq \alpha + \epsilon$$

Let  $\epsilon \rightarrow 0$  we get

$$\bar{I}(f) = \underline{I}(f) = \alpha$$

### 37.12 Fatou's Lemma

Let  $(f_n)_{n \in \mathbb{N}} \in L^1(I)^\mathbb{N}$ . Assume that there is  $g \in L^1(I)$  such that

$$\forall n \in \mathbb{N} \quad f_n \geq g$$

If  $\liminf_{n \rightarrow +\infty} f_n$  is a mapping from  $\Omega$  to  $\mathbb{R}$  and  $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$ , then  $\liminf_{n \rightarrow +\infty} f_n \in L^1(I)$  and

$$I(\liminf_{n \rightarrow +\infty} f_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

#### Proof

For any  $n \in \mathbb{N}$ , let

$$g_n = \lim_{k \rightarrow +\infty} (f_n \wedge f_{n+1} \wedge \cdots \wedge f_{n+k})$$

Then

$$\liminf_{n \rightarrow +\infty} f_n = \lim_{n \rightarrow +\infty} g_n$$

For any  $k$  one has

$$f_n \wedge \cdots \wedge f_{n+k} \geq g$$

Hence

$$I(f_n) \geq \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \geq I(g)$$

By the theorem of Beppo Levi,

$$g_n \in L^1(I) \text{ and } I(g_n) = \lim_{k \rightarrow +\infty} I(f_n \wedge \cdots \wedge f_{n+k}) \leq I(f_n)$$

Note that  $(g_n)_{n \in \mathbb{N}}$  is increasing and  $\liminf_{n \rightarrow +\infty} I(f_n) < +\infty$ . Hence

$$\lim_{n \rightarrow +\infty} I(g_n) = \liminf_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n) < +\infty$$

By the theorem of Beppo Levi,

$$\lim_{n \rightarrow +\infty} g_n \in L^1(I)$$

and

$$I(\liminf_{n \rightarrow +\infty} f_n) = I(\lim_{n \rightarrow +\infty} g_n) = \lim_{n \rightarrow +\infty} I(g_n) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

### 37.13 Lebesgue dominated convergence theorem

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence in  $L^1(I)$  that converges pointwisely to some  $f : \Omega \rightarrow \mathbb{R}$ . Assume that there exists  $g \in L^1(I)$  such that  $\forall n \in \mathbb{N}, |f_n| \leq g$ . Then  $f \in L^1(I)$  and  $I(f) = \lim_{n \rightarrow +\infty} I(f_n)$ .

**Proof**

Apply Fatou's lemma to  $(f_n)_{n \in \mathbb{N}}$  and  $(-f_n)_{n \in \mathbb{N}}$  to get

$$I(f) \leq \liminf_{n \rightarrow +\infty} I(f_n)$$

and

$$\begin{aligned} I(-f) &\leq \liminf_{n \rightarrow +\infty} I(-f_n) \\ &= \limsup_{n \rightarrow +\infty} I(f_n) \\ &\leq \limsup_{n \rightarrow +\infty} I(f_n) \leq I(f) \end{aligned}$$

**37.14 Notation**

Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be an increasing and right continuous mapping. Let  $S$  be the vector subspace of  $\mathbb{R}^{\mathbb{R}}$  generated by  $\mathbb{1}_{[a,b]}$  with  $(a,b) \in \mathbb{R}^2, a < b$ . For any  $f \in L^1(I_\varphi)$   $I_\varphi(f)$  is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

For any subset  $A$  of  $\mathbb{R}$  if  $\mathbb{1}_A f \in L^1(I)$  then

$$\int_A f(x) d\varphi(x) \text{ denotes } \int_{\mathbb{R}} \mathbb{1}_A(x) f(x) d\varphi(x) = I(\mathbb{1}_A f)$$

If  $(a,b) \in \mathbb{R}^2, a < b$

$$\int_a^b f(x) d\varphi(x) \text{ denotes } \int_{[a,b]} f(x) d\varphi(x)$$

$$\int_b^a f(x) d\varphi(x) \text{ denotes } - \int_{[a,b]} f(x) d\varphi(x)$$

If  $\varphi(x) = x$  for any  $x \in \mathbb{R}$  we replace  $d\varphi(x)$  by  $dx$ .



## Chapter 38

# Semialgebra

### 38.1 Notation

Let  $A, (A_i)_{i \in I}$  be sets the notation.

$$A = \bigsqcup_{i \in I} A_i$$

denotes:

- $(A_i)_{i \in I}$  is a pairwise disjoint
- $A = \bigcup_{i \in I} A_i$

### 38.2 Def

Let  $\Omega$  be a set. We call semialgebra on  $\Omega$  any  $\mathcal{C} \subseteq \wp(\Omega)$  that verifies:

- $\emptyset \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, A \cap B \in \mathcal{C}$
- $\forall (A, B) \in \mathcal{C}^2, \exists (C_i)_{i=1}^n$  a finite family of elements in  $\mathcal{C}$  such that  $B \setminus A = \bigsqcup_{i=1}^n C_i$

#### 38.2.1 Example

$$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$$

### 38.3 Def

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ . The set

$$\{A \in \wp(\Omega) \mid \exists n \in \mathbb{N}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigsqcup_{i=1}^n A_i\}$$

is called the algebra generated by  $\mathcal{C}$

### 38.4 Prop

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ .  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Then:

- $\emptyset \in \mathcal{A}$
- $\forall (A, B) \in \mathcal{A}^2, A \cap B \in \mathcal{A}, B \setminus A \in \mathcal{A}, A \cup B \in \mathcal{A}$

#### Proof

By definition,  $\emptyset \in \mathcal{A}, \mathcal{C} \subseteq \mathcal{A}$ . Moreover, if  $A$  and  $B$  are elements of  $\mathcal{A}$  such that  $A \cap B = \emptyset$  then  $A \cup B \in \mathcal{A}$ . Let  $A = \bigsqcup_{i=1}^n A_i$  and  $B = \bigsqcup_{i=1}^n B_i$  be elements of  $\mathcal{A}$  then

$$A \cap B = \bigsqcup_{(i,j) \in \{1, \dots, n\}^2} (A_i \cap B_j)$$

Hence  $A \cap B \in \mathcal{A}$  Finally

$$A \cup B = (A \cap B) \sqcup (A \setminus B) \sqcup (B \setminus A) \in \mathcal{A}$$

### 38.5 Prop

Let  $\mathcal{C}$  be a semialgebra on  $\Omega$ .  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Let  $S$  be the  $\mathbb{R}$ -vector subspace of  $\mathbb{R}^\Omega$ ,  $I : S \rightarrow \mathbb{R}$  be an  $\mathbb{R}$ -linear mapping generated by mappings of the form  $\mathbb{1}_A, A \in \mathcal{C} (f \in S, f = \sum \lambda_i \mathbb{1}_{A_i})$   
Assume that

$$\forall (f, g) \in S^2, f \leq g \text{ one has } I(f) \leq I(g)$$

Then  $I$  is an integral operator iff, for any decreasing sequence  $(A_n)_{n \in \mathbb{N}}$  in  $\mathcal{A}^\mathbb{N}$  such that  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ , one has

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

### 38.5.1 Proof

$$\forall A \in \mathcal{A}, \exists (A_i)_{i=1}^n \in \mathcal{C}^n, A = \bigcup_{i=1}^n A_i \text{ so } \mathbb{1}_A = \sum_{i=1}^n \mathbb{1}_{A_i} \in S$$

Lemma  $\forall (f, g) \in S^2, f \wedge g \in S$

$\Rightarrow$  Suppose that  $I$  is an integral operator  $(\mathbb{1}_{A_n})_{n \in \mathbb{N}}$  is a decreasing sequence in  $S$  and

$$\lim_{n \rightarrow +\infty} \mathbb{1}_{A_n} = 0$$

Hence

$$\lim_{n \rightarrow +\infty} I(\mathbb{1}_{A_n}) = 0$$

$\Leftarrow$  Let  $(f_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $S$  that converges pointwisely to 0. Let

$$B = \{\omega \in \Omega \mid f_0(\omega) > 0\} \in \mathcal{A} \quad M = \max\{f_0(\omega) \mid \omega \in \Omega\}$$

- For any  $\epsilon > 0$  let

$$A_n^\epsilon = \{\omega \in \Omega \mid f_n(\omega) \geq \epsilon\} \in \mathcal{A}$$

Moreover, since  $\lim_{n \rightarrow +\infty} f_n(\omega) = 0, \bigcap_{n \in \mathbb{N}} A_n^\epsilon = \emptyset$

$$f(\omega) = \begin{cases} \lambda_i & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \in \Omega \setminus \bigcap_{i=1}^n A_i \end{cases}$$

$(\forall f \in S, \exists (A_i)_{i=1}^n \text{ pairwise disjoint and } (\lambda_i)_{i=1}^n \in \mathbb{R} \text{ } f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i})$

Note that

$$0 \leq f_n \leq \epsilon \mathbb{1}_B + M \mathbb{1}_{A_n^\epsilon}$$

So

$$0 \leq I(f_n) \leq \epsilon I(\mathbb{1}_B) + M I(\mathbb{1}_{A_n^\epsilon})$$

which leads to

$$\limsup_{n \rightarrow +\infty} I(f_n) \leq \epsilon I(\mathbb{1}_B) \quad \forall \epsilon > 0$$

So

$$\lim_{n \rightarrow +\infty} I(f_n) = 0$$

### 38.5.2 Example

Let  $\Omega = \mathbb{R}$  and  $\mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a \leq b\}$   
 $\mathcal{A}$  be algebra generated by  $\mathcal{C}$ .  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  increasing, right continuous.  $S$  be  $\mathbb{R}$ -vector subspace generated by  $\mathbb{1}_{]a, b]}, (a, b) \in \mathbb{R}, a \leq b$

$$\begin{aligned} I_\varphi : S &\rightarrow \mathbb{R} \\ \mathbb{1}_{]a, b]} &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

Lemma  $\forall \epsilon > 0, A \in \mathcal{A}, A \neq \emptyset, \exists B \in \mathcal{A},$

$$\emptyset \neq \overline{B} \subseteq A \text{ and } I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) \leq \epsilon$$

#### Proof

We first consider the the case where  $A \in \mathcal{C}, A = ]a, b], a < b$  By the right continuous of  $\varphi, \exists ]a', b[$  such that  $\varphi(a') - \varphi(a) \leq \epsilon$ . Let  $B = ]a', b], \overline{B} = [a', b] \subseteq ]a, b]$ .

$$\begin{aligned} I_\varphi(\mathbb{1}_B) &= \varphi(b) - \varphi(a') \\ I_\varphi(\mathbb{1}_A) &= \varphi(b) - \varphi(a) \\ I_\varphi(\mathbb{1}_A) - I_\varphi(\mathbb{1}_B) &= \varphi(a') - \varphi(a) \leq \epsilon \end{aligned}$$

In general

$$A = \bigsqcup_{i=1}^n A_i$$

with  $A_i \in \mathcal{C} \forall i \in \{1, \dots, n\}, \exists B_i \in \mathcal{C}$

$$\emptyset \neq \overline{B_i} \subseteq A_i \quad I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \frac{\epsilon}{n}$$

Let  $B = \bigsqcup_{i=1}^n B_i$  then

$$I(\mathbb{1}_A) - I(\mathbb{1}_B) = \sum_{i=1}^n I(\mathbb{1}_{A_i}) - I(\mathbb{1}_{B_i}) \leq \epsilon$$

### 38.6 Theorem

$I_\varphi$  is an integral operator

#### Proof

Let  $(A_n)_{n \in \mathbb{N}}$  be a decreasing sequence in  $\mathcal{A}$  such that

$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset$$

Let  $\epsilon > 0$  For any  $n \in \mathbb{N}$  let  $B_n \in \mathcal{A}$  such that

$$\overline{B_n} \subseteq A_n \text{ and } I_\varphi(A_n) - I_\varphi(B_n) \leq \frac{\epsilon}{2^n}$$

Note that  $\overline{B_n}$  is compact. For any  $n \in \mathbb{N}$  let

$$\begin{aligned} C_n &= \bigcap_{i=1}^n B_i \\ &\subseteq \bigcap_{i=1}^n \overline{B_i} \end{aligned}$$

Since  $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$ ,  $\bigcap_{n \in \mathbb{N}} \overline{B_n} = \emptyset$

So

$$I_\varphi(\mathbb{1}_{A_n}) \leq \frac{\epsilon}{2^n} + \frac{\epsilon}{2^{n-1}} \cdots \frac{\epsilon}{2} \leq \epsilon$$

Thus

$$\lim_{n \rightarrow +\infty} I_\varphi(\mathbb{1}_{A_n}) = 0$$

Let  $\Omega$  be a set  $\mathcal{C}$  be a semialgebra on  $\Omega$  and  $\mathcal{A}$  be the algebra generated by  $\mathcal{C}$ . Let  $S$  be the  $\mathbb{R}$ -vector subspace of  $\mathbb{R}^\Omega$  generated by mappings of the form  $\mathbb{1}_A, A \in \mathcal{C}$

## 38.7 Prop

For any  $f \in S, \exists (A_i)_{i=1}^n \in \mathcal{A}^n$  pairwise disjoint and  $(\lambda_i)_{i=1}^n$  such that

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}$$

### Proof

$f$  is of the form

$$f = \sum_{j=1}^m a_j \mathbb{1}_{B_j} \quad B_j \in \mathcal{C}$$

We reason by induction on  $m$ . For any  $I \subseteq \{1, \dots, m\}$  let

$$B_I = \bigcap_{j \in I} B_j \cap \bigcap_{j \in \{1, \dots, m\} \setminus I} (\Omega \setminus B_j)$$

Then  $(B_I)_{I \subseteq \{1, \dots, m\}}$  are pairwise disjoint.

Moreover, if  $I = \emptyset, B_I \in \mathcal{A}$

$$B_i = \bigsqcup_{I \subseteq \{1, \dots, m\}, i \in I} B_I$$

Hence

$$f = \sum_{U \subseteq \{1, \dots, m\}} \left( \sum_{j \in U} a_j \mathbb{1}_{B_j} \right)$$

### 38.8 Corollary

(1) If  $f \in S$  then

$$f \wedge 0 \in S$$

(2) If  $(f, g) \in S^2$  then

$$f \wedge g = (f - g) \wedge 0 + g \in S$$

#### Proof

We intend to define

$$I_\mu\left(\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i}\right)$$

as

$$\sum_{i=1}^n \lambda_i I_\mu(\mathbb{1}_{A_i})$$

for  $A_i \in \mathcal{C}$ . We need to check that if  $f \in S$  is written as

$$f = \sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} = \sum_{j=1}^m \xi_j \mathbb{1}_{B_j}$$

then

$$\sum_{i=1}^n \xi_i I_\mu(\mathbb{1}_{A_i}) = \sum_{j=1}^m \xi_j I_\mu(\mathbb{1}_{B_j})$$

so

$$0 = \sum_{i=1}^n \xi_i \mathbb{1}_{A_i} - \sum_{j=1}^m \xi_j \mathbb{1}_{B_j}$$

It suffices to prove that if

$$\sum_{i=1}^n \xi_i \mathbb{1}_{A_i} = 0$$

then

$$\sum_{j=1}^m \xi_j \mathbb{1}_{B_j} = 0$$

For  $I \subseteq \{1, \dots, n\}$  let

$$A_I = \{\omega \in \Omega \mid \forall i \in I, \omega \in A_i, \forall i \in \{1, \dots, n\} \setminus I, \omega \in \Omega \setminus A_i\}$$

$A_I \in \mathcal{A}$  when  $I \neq \emptyset$

### 38.9 Lemma

Let  $B \in \mathcal{A}$  If

$$B = \bigsqcup_{i=1}^n B_i = \bigsqcup_{j=1}^m C_j$$

with  $B_i, C_j \in \mathcal{C}$ , then

$$\sum_{i=1}^n \mu(B_i) = \sum_{j=1}^m \mu(C_j)$$

In particular, we can extend  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  to  $\mu : \mathcal{A} \rightarrow \mathbb{R}_{\geq 0}$  such that  $\forall D_1, \dots, D_n$  in  $\mathcal{A}$  disjoint

$$\mu(D_1 \cup \dots \cup D_n) = \sum_{i=1}^n \mu(D_i)$$

#### 38.9.1 Proof

$$\begin{aligned} B_i &= \bigsqcup_{j=1}^m (B_i \cap C_j) & \mu(B_i) &= \sum_{j=1}^m \mu(B_i \cap C_j) \\ \sum_{i=1}^n \mu(B_i) &= \sum_{i=1}^n \sum_{j=1}^m \mu(B_i \cap C_j) \\ &= \sum_{j=1}^m \sum_{i=1}^n \mu(B_i \cap C_j) \\ &= \sum_{j=1}^m \mu(C_j) \end{aligned}$$

Back to the proof

$$0 = \sum_{i=1}^n a_i \mathbb{1}_{A_i} = \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \left( \sum_{i \in I} a_i \right) \mathbb{1}_{A_i}$$

hence

$$\begin{aligned} \sum_{i \in I} a_i &= 0 \\ 0 &= \sum_{i=1}^n a_i \mu(A_i) \\ &= \sum_{i=1}^n a_i \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \mu(A_i) \\ &= \sum_{\emptyset \neq I \subseteq \{1, \dots, n\}} \mu(A_i) \sum_{i \in I} a_i \end{aligned}$$





## Chapter 39

# Integral function

### 39.1 Setting

Let  $\Omega$  be a set.  $S \subseteq \mathbb{R}^\Omega$  be  $\mathbb{R}$ -vector subspace,  $\forall (f, g) \in S^2, f \wedge g \in S$   
 $I : S \rightarrow \mathbb{R}$  integral operator.

### 39.2 Prop

Suppose that  $\mathbb{1}_\Omega \in L^1(I)^\uparrow$  The set

$$\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I)^\uparrow\}$$

is a  $\sigma$ -algebra on  $\Omega$

Moreover, if we denote by  $\mu : \mathcal{G} \rightarrow \mathbb{R}_{\geq 0}$  the mapping define as

$$\mu(A) := I(\mathbb{1}_A)$$

then  $\mu$  satisfies :

$\forall (A_n)_{n \in \mathbb{N}} \in \mathcal{G}^\mathbb{N}$  that's is pairwise disjoint, then

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

#### 39.2.1 Proof

(1)

$$\emptyset \in \mathcal{G}$$

since

$$0 = \mathbb{1} \in L^1(I)^\uparrow$$

(2) If A and B are elements of  $\mathcal{G}$ ,  $A \subseteq B$ , then

$$\mu(A) \leq \mu(B)$$

so

$$\mathbb{1}_B - \mathbb{1}_{B \setminus A} \in L^1(I)^\uparrow \Rightarrow B \setminus A \in \mathcal{G}$$

(3) If  $(A, B) \in \mathcal{G}^2$

$$\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B - \mathbb{1}_{A \cap B} \in L^1(I)^\uparrow$$

So  $A \cup B \in \mathcal{G}$

If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}$ ,  $A = \bigcup_{n \in \mathbb{N}} A_n$  then

$$\mathbb{1}_A = \lim_{n \rightarrow +\infty} \mathbb{1}_{A_1 \cup \dots \cup A_n} \in L^1(I)^\uparrow \Rightarrow A \in \mathcal{G}$$

$$\begin{aligned} \underbrace{I(\mathbb{1}_{\sum_{n \in \mathbb{N}} A_n})}_{\mu(\bigcup_{n \in \mathbb{N}} A_n)} &= \lim_{n \rightarrow +\infty} \mathbb{1}_{A_0 \cup \dots \cup A_n} \\ &= \lim_{n \rightarrow +\infty} \sum_{i=0}^n \mathbb{1}_{A_i} \in L^1(I)^\uparrow \\ &= - \sum_{n \in \mathbb{N}} \mu(A_n) \end{aligned}$$

## Chapter 40

# Limit and Differential of Integrals with Parameters

Let  $\Omega$  be a set.  $S \subseteq \mathbb{R}^\Omega$  be  $\mathbb{R}$ -vector subspace such that  $\forall (f, g) \in S^2, f \wedge g \in S$ . Let  $I : S \rightarrow \mathbb{R}$  be an integral operator.

### 40.1 Theorem

Let  $X$  be a topological space,  $p \in X$ ,  $f : \Omega \times X \rightarrow \mathbb{R}$  be a mapping,  $g \in L^1(I)$ . Suppose that

(1)  $\forall \omega \in \Omega$

$$\begin{aligned} f(\omega, \cdot) : \Omega &\rightarrow \mathbb{R} \\ x &\mapsto f(\omega, x) \end{aligned}$$

is continuous at  $p$

(2)  $\forall x \in X$

$$\begin{aligned} f(\cdot, x) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, x) \end{aligned}$$

belongs to  $L^1(I)$  and  $\forall \omega \in \Omega \quad |f(\omega, x)| \leq g(\omega)$

(3)  $p$  has a countable neighborhood basis in  $X$

Then

$$(x \in X) \mapsto I(f(\cdot, x))$$

is continuous at  $p$

**Proof**

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $X$  that converges to  $p$ . For any  $n \in \mathbb{N}$  let  $f_n : \Omega \rightarrow \mathbb{R}$ ,  $f_n(\omega) := f(\omega, x_n)$ . One has  $|f_n| \leq g$ . Moreover  $\forall \omega \in \Omega$

$$\lim_{n \rightarrow +\infty} f_n(\omega) = \lim_{n \rightarrow +\infty} f(\omega, x_n) = f(\omega, p)$$

Hence, by dominate convergence theorem

$$\lim_{n \rightarrow +\infty} I(f_n) = I(f(\cdot, p))$$

**40.2 Theorem**

Let  $J$  be an open interval in  $\mathbb{R}$ .  $f : \Omega \times J \rightarrow \mathbb{R}$  be a mapping.  $g \in L^1(I)$ . Assume that

$$(1) \quad \forall \omega \in \Omega$$

$$f(\omega, \cdot) : J \rightarrow \mathbb{R}$$

is differentiable (we denote by  $\frac{\partial f}{\partial t}(\omega, t)$  its derivative at  $t$ )  
and  $\forall t \in J$

$$\left| \frac{\partial f}{\partial t}(\omega, t) \right| \leq g(\omega)$$

$$(2) \quad \forall t \in J$$

$$\begin{aligned} f(\cdot, t) : \Omega &\rightarrow \mathbb{R} \\ \omega &\mapsto f(\omega, t) \end{aligned}$$

belongs to  $L^1(I)$

Then

$$\begin{aligned} \varphi : J &\rightarrow \mathbb{R} \\ t &\mapsto I(f(\cdot, t)) \end{aligned}$$

is differentiable and

$$\varphi'(t) = I\left(\frac{\partial f}{\partial t}(\cdot, t)\right) = \frac{d}{dt} I(f(\cdot, t))$$

**Proof**

Let  $a \in J$  and  $(t_n)_{n \in \mathbb{N}}$  be a sequence in  $J \setminus \{a\}$  such that

$$\lim_{n \rightarrow +\infty} t_n = a$$

. Then

$$\frac{\varphi(t_n) - \varphi(a)}{t_n - a} = I\left(\frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a}\right)$$

$\forall \omega \in \Omega$

$$\left| \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} \right| \leq g(\omega) \text{ (by mean value theorem)}$$

and

$$\lim_{n \rightarrow +\infty} \frac{f(\cdot, t_n) - f(\cdot, a)}{t_n - a} = \frac{\partial f}{\partial t}(\omega, a)$$

Hence

$$\lim_{n \rightarrow +\infty} \frac{\varphi(t_n) - \varphi(a)}{t_n - a} = \frac{d}{dt} I(f(\cdot, t))$$



# Chapter 41

## Measure theory

### 41.1 Def

We call measure space any pair  $(E, \mathcal{E})$ , where  $E$  is a set and  $\mathcal{E}$  is a  $\sigma$ -algebra on  $E$ .

### 41.2 Prop

Let  $\Omega$  be a set. And  $(\mathcal{G}_i)_{i \in I}$  be a family of  $\sigma$ -algebras on  $\Omega$ . Then  $\bigcap_{i \in I} \mathcal{G}_i$  is a  $\sigma$ -algebra

#### Proof

- $\forall i \in I$

$$\emptyset \in \mathcal{G}_i$$

Hence

$$\emptyset \in \bigcap_{i \in I} \mathcal{G}_i$$

- If  $A \in \bigcap_{i \in I} \mathcal{G}_i$  then  $\forall i \in I$   $A \in \mathcal{G}_i$  Hence  $\forall i \in I$

$$\Omega \setminus A \in \mathcal{G}_i$$

So

$$\Omega \setminus A \in \bigcap_{i \in I} \mathcal{G}_i$$

- Let  $(A_n)_{n \in \mathbb{N}} \in (\bigcap_{i \in I} \mathcal{G}_i)^{\mathbb{N}}$ . For any  $i \in I$

$$(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_i^{\mathbb{N}}$$

So

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \mathcal{G}_i$$

so

$$\bigcap_{n \in \mathbb{N}} (A_n) \in \bigcap_{i \in I} \mathcal{G}_i$$

### 41.3 Def

Let  $\mathcal{C} \subseteq \wp(\Omega)$ . We denote by  $\sigma(\mathcal{C})$  the intersection of all  $\sigma$ -algebras on  $\Omega$  containing  $\mathcal{C}$ . It's the smallest  $\sigma$ -algebra containing  $\mathcal{C}$

### 41.4 Example

- Let  $(X, \mathcal{G})$  be a topological space.  $\sigma(\mathcal{G})$  is called the Borel  $\sigma$ -algebra of  $X$
- On  $[-\infty, +\infty]$  the following  $\sigma$ -algebras are the same:

$$g_1 = \sigma(\{[a, +\infty] \mid a \in \mathbb{R}\})$$

$$g_2 = \sigma(\{]a, +\infty[ \mid a \in \mathbb{R}\})$$

$$g_3 = \sigma(\{[-\infty, a] \mid a \in \mathbb{R}\})$$

$$g_4 = \sigma(\{[-\infty, a[ \mid a \in \mathbb{R}\})$$

Moreover

$$\mathcal{B} = \{A \subseteq \mathbb{R} \mid A \in g_1\}$$

is equal to the Borel  $\sigma$ -algebra of  $\mathbb{R}$

proof  $\forall a \in \mathbb{R}$

$$[a, +\infty] = \bigcap_{n \in \mathbb{N}_{\geq 1}} ]a - \frac{1}{n}, +\infty[ \in g_2 \quad \Rightarrow g_1 \in g_2$$

$$]a, +\infty[ = [-\infty, +\infty] \setminus [-\infty, a] \in g_3 \quad \Rightarrow g_2 \in g_3$$

$$[-\infty, a] = \bigcap_{n \in \mathbb{N}_{\geq 1}} [-\infty, a + \frac{1}{n}[ \in g_4 \quad \Rightarrow g_3 \in g_4$$

$$[-\infty, a[ = [-\infty, +\infty] \setminus [a, +\infty] \in g_1 \quad \Rightarrow g_4 \in g_1$$

$$\sigma(\{]a, b[ \mid a < b, (a, b) \in \mathbb{R}^2\}) = \text{Borel } \sigma\text{-algebra of } \mathbb{R}$$

$J \subseteq \mathbb{R}$  open We define  $\sim$  a binary relation on  $J$  such that  $x \sim y \Leftrightarrow$  there exists an interval  $I$  such that  $\{x, y\} \subseteq I \subseteq J$

Any equivalence class of  $\sim$  is a non-empty open interval.

$$]a, b[ = [a, +\infty] \cup [-\infty, b[$$

Hence Borel  $\sigma$ -algebra of  $\mathbb{R} \subseteq \{A \subseteq \mathbb{R} \mid A \in g_1\}$



## 41.5 Def

Let  $f : X \rightarrow Y$  be a mapping of sets.

- For any  $\mathcal{C}_Y \subseteq \wp(Y)$  we denote by

$$f^{-1}(\mathcal{C}_Y) := \{f^{-1}(B) \mid B \in \mathcal{C}_Y\}$$

- For any  $\mathcal{C}_X \subseteq \wp(X)$  we denote by

$$f_*(\mathcal{C}_X) := \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{C}_X\}$$

## 41.6 Prop

Let  $f : X \rightarrow Y$  be a mapping.

- (1) If  $\mathcal{G}_Y$  is a  $\sigma$ -algebra on  $Y$  then  $f^{-1}(\mathcal{G}_Y)$  is a  $\sigma$ -algebra on  $X$
- (2) If  $\mathcal{G}_X$  is a  $\sigma$ -algebra on  $X$  then  $f_*(\mathcal{G}_X)$  is a  $\sigma$ -algebra on  $Y$

### Proof

(1)

$$\emptyset = f^{-1}(\emptyset) \in f^{-1}(\mathcal{G}_Y)$$

$$\forall B \in \mathcal{G}_Y$$

$$X \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$$

If  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}_Y^{\mathbb{N}}$ ,  $A = \bigcup_{n \in \mathbb{N}} A_n$ ,  $A_n \in \mathcal{G}_Y$ , then

$$\bigcup_{n \in \mathbb{N}} f^{-1}(A_n) = f^{-1}(A) \in f^{-1}(\mathcal{G}_Y)$$

(2)

$$f^{-1}(\emptyset) = \emptyset \in \mathcal{G}_X$$

so

$$\emptyset \in f_*(\mathcal{G}_X)$$

$$\forall B \in f_*(\mathcal{G}_X)$$

$$f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \in \mathcal{G}_X$$

so

$$Y \setminus B \in f_*(\mathcal{G}_X)$$

$$\forall (B_n)_{n \in \mathbb{N}} \in f_*(\mathcal{G}_X)^{\mathbb{N}}, B = \bigcup_{n \in \mathbb{N}} B_n$$

$$f^{-1}(B) = \bigcap_{n \in \mathbb{N}} f^{-1}(B_n)$$

So  $B \in f_*(\mathcal{G}_X)$

### 41.7 Def

Let  $(X, \mathcal{G}_X)$  and  $(Y, g_Y)$  be measurable spaces,  $f : X \rightarrow Y$  be a mapping. If  $f^{-1}(g_Y) \subseteq \mathcal{G}_X$  or equivalently  $g_Y \subseteq f_s(\mathcal{G}_X)$  (or  $\forall B \in g_Y, f^{-1}(B) \in \mathcal{G}_X$ ) then we say that  $f$  is measurable.

### 41.8 Prop

Let  $(X, \mathcal{G}_X), (Y, g_Y), (Z, g_Z)$  be measurable spaces.  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be measurable mappings. Then  $g \circ f$  is measurable.

#### Proof

$$\forall B \in g_Z$$

$$(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$$

and

$$g^{-1}(B) \in g_Y$$

so

$$f^{-1}(g^{-1}(B)) \in \mathcal{G}_X$$

### 41.9 Def

Let  $\Omega$  be a set  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.  $f = (f_i)_{i \in I}$  where  $f_i : \Omega \rightarrow E_i$  is a mapping. We denote by  $\sigma(f)$  the  $\sigma$ -algebra  $\sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i))$ . It's the smallest  $\sigma$ -algebra on  $\Omega$  making all  $f_i$  measurable.

### 41.10 Prop

We keep the notation of the above definition. For any  $i \in I$ , let  $\mathcal{C} \subseteq \wp(E_i)$  such that  $\sigma(\mathcal{C}_i) = \mathcal{E}_i$ . Then

$$\sigma(f) = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$$

#### Proof

Let  $g = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$ . By definition

$$g \subseteq \sigma(f)$$

For any  $i \in I$ ,  $f_{i,*}(\bigcup_{i \in I} f_i^{-1}(\mathcal{C}_i))$  is a  $\sigma$ -algebra on  $\Omega$  containing  $\mathcal{C}_i$ . So

$$\mathcal{E} \subseteq f_{i,*}(\sigma(f_i^{-1}(\mathcal{C}_i)))$$

which leads to

$$f_i^{-1}(C_i) \subseteq \sigma(f_i^{-1}(C_i)) \subseteq g$$

Hence

$$\bigcup_{i \in I} f_i^{-1}(C_i) \subseteq g$$

$\Rightarrow$

$$\sigma(f) \subseteq g$$

$$(f_{i,*}(\mathcal{A}) = \{B \subseteq E_i \mid f_i(B) \in \mathcal{A}\})$$

### 41.11 Corollary

Let  $(X, \mathcal{G}_X), (Y, g_Y)$  be measurable spaces.  $f : X \rightarrow Y$  be a mapping.  $\mathcal{C}_Y \subseteq g_Y$  such that  $g_Y = \sigma(\mathcal{Y})$  Then  $f$  is measurable iff

$$\forall B \in \mathcal{C}_Y \quad f^{-1}(B) \in \mathcal{G}_X$$

**Proof**

$$\sigma(f) = \sigma(f^{-1}(\mathcal{C}_Y))$$

$f$  is measurable iff  $\sigma(f) \subseteq \mathcal{G}_X$

### 41.12 Example

Let  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.

$$E = \prod_{i \in I} E_i$$

$\forall i \in I$

$$\pi_i : E \rightarrow E_i$$

$$(x_j)_{j \in I} \mapsto x_i$$

We denote by  $\bigotimes_{i \in I} \mathcal{E}_i$  the  $\sigma$ -algebra  $\sigma((\pi_i)_{i \in I})$

### 41.13 Prop

Let  $X$  be a set  $((E_i, \mathcal{E}_i))_{i \in I}$  be a family of measurable spaces.  $(\Omega, g)$  be a measurable space.  $f = (f_i : X \rightarrow E_i)_{i \in I}$  be a mappings,  $\varphi : \Omega \rightarrow X$  be a mapping. Then

$$\varphi : (\Omega, g) \rightarrow (X, \sigma(f))$$

is measurable iff

$$\forall i \in I \quad f_i \circ \varphi : (\Omega, g) \rightarrow (E_i, \mathcal{E}_i) \text{ is measurable.}$$

**Proof**

- $\Rightarrow$  If  $\varphi$  is measurable, since each  $f_i$  is measurable, one has  $f_i \circ \varphi$  is measurable.  
 $\Leftarrow$  If  $f_i \circ \varphi$  is measurable,  $\forall B \in \mathcal{E}_i$

$$(f_i \circ \varphi)^{-1}(B) = \varphi^{-1}(f_i^{-1}(B)) \in g$$

Hence

$$\varphi^{-1}\left(\bigcup_{i \in I} f_i^{-1}(B)\right) \subseteq g$$

Since

$$\sigma(f) = \sigma\left(\bigcup_{i \in I} f_i^{-1}(\mathcal{E}_i)\right)$$

$\varphi$  is measurable.

**41.14 Example**

Let  $(\Omega, \mathcal{G})$  be a measurable space

- $\forall A \in \mathcal{G} \quad \mathbb{1}_A : \Omega \rightarrow \mathbb{R}$  is measurable. For any  $U \subseteq \mathbb{R}$

$$\mathbb{1}_A^{-1}(U) = A \text{ or } \Omega \setminus A \text{ or } \Omega \text{ or } \emptyset$$

- If  $X$  and  $Y$  be topological spaces.  $f : X \rightarrow Y$  be a continuous mapping, then  $f$  is measurable with respect to Borel  $\sigma$ -algebra. In fact,  $\forall V \subseteq Y$  open  $f^{-1}(V) \subseteq X$  open.
- Let  $(\Omega, \mathcal{G})$  be a measurable space. If  $f : \Omega \rightarrow \mathbb{R}, g : \Omega \rightarrow \mathbb{R}$  are measurable then  $f + g, fg, f \wedge g, f \vee g, |f|$  are measurable.
- Let  $(f_n)_{n \in \mathbb{N}}$  be a family of measurable mappings from  $\Omega$  to  $[-\infty, +\infty]$

$$f = \sup_{n \in \mathbb{N}} f_n \quad (f(\omega) = \sup_{n \in \mathbb{N}} f_n(\omega))$$

Then  $f$  is measurable.

Similarly  $\inf_{n \in \mathbb{N}} f_n$  is measurable.

In fact, for any  $a \in \mathbb{R}$

$$\{\omega \in \Omega \mid f(\omega) > a\} = \bigcup_{n \in \mathbb{N}} \{\omega \in \Omega \mid f_n(\omega) > a\}$$

## Chapter 42

# Measure

### 42.1 Def

Let  $\Omega$  be a set.  $\mathcal{C}$  be a semi-algebra on  $\Omega$ .  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  be a mapping. If  $\forall n \in \mathbb{N}, \forall (A_i)_{i=1}^n \in \mathcal{C}^n$  pairwise disjoint, with  $A = \bigcup_{i=1}^n A_i$  one has

$$\mu(A) = \sum_{i=1}^n \mu(A_i)$$

we say that  $\mu$  is additive.

Let

$S =$  vector subspace of  $\mathbb{R}^\Omega$  generated by  $(\mathbb{1}_A)_{A \in \mathcal{C}}$

Then

$$I_\mu : S \rightarrow \mathbb{R}$$
$$\sum_{i=1}^n \lambda_i \mathbb{1}_{A_i} \mapsto \sum_{i=1}^n \lambda_i \mu(A_i)$$

is well defined. If  $I_\mu$  is an integral operator, we say that  $\mu$  is  $\sigma$ -additive.

### 42.2 Def

Let  $(\Omega, \mathcal{G})$  be a measurable space.  $\mu : \mathcal{G} \rightarrow [0, +\infty]$  be a mapping. If  $\mu(\emptyset) = 0$  and if for any  $(A_n)_{n \in \mathbb{N}} \in \mathcal{G}^{\mathbb{N}}$  pairwise disjoint.

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$$

we say that  $\mu$  is a measure.

$(\Omega, \mathcal{G}, \mu)$  is called a measure space.

### 42.3 Def

If  $\exists (A_n)_{n \in \mathbb{N}}$  such that  $\Omega = \bigcup_{n \in \mathbb{N}} A_n$  and  $\mu(A_n) < +\infty$  then  $\mu$  is said to be  $\sigma$ -finite.

### 42.4 Carathéodory Theorem

Let  $\Omega$  be a set,  $\mathcal{C}$  be a semi-algebra on  $\Omega$ ,  $\mu : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  be a  $\sigma$ -additive mapping. Assume that there is a sequence  $(A_n)_{n \in \mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$  such that  $\Omega = \bigcup_{n \in \mathbb{N}} A_n$ . Then  $\mu$  extends to a  $\sigma$ -finite measure on  $\sigma(\mathcal{C})$

#### Proof

Let  $S \subseteq \mathbb{R}^{\Omega}$  be the vector subspace generated by  $(\mathbb{1}_A), A \in \mathcal{C}$ . Let  $\mathcal{G} = \{A \subseteq \Omega \mid \mathbb{1}_A \in L^1(I_{\mu})^{\uparrow}\}$  then  $\mathcal{G}$  is a  $\sigma$ -algebra containing  $\mathcal{C}$ . Hence  $\sigma(\mathcal{C}) = \mathcal{G}$ . Moreover,  $(A \in \mathcal{G}) \mapsto I_{\mu}(\mathbb{1}_A)$  is a measure on  $\mathcal{G}$  which is  $\sigma$ -finite.

### 42.5 Example

$\Omega = \mathbb{R}, \mathcal{C} = \{]a, b] \mid (a, b) \in \mathbb{R}^2, a < b\}$   $\sigma(\mathcal{C}) = \text{Borel } \sigma\text{-algebra}$   $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  increasing and right continuous.

$$\begin{aligned} \mu_{\varphi} : \mathcal{C} &\rightarrow \mathbb{R}_{\geq 0} \\ ]a, b] &\mapsto \varphi(b) - \varphi(a) \end{aligned}$$

is  $\sigma$ -additive.

Hence  $\mu_{\varphi}$  extends to a measure:

$$\sigma(\mathcal{C}) \rightarrow [0, +\infty]$$

called the Stieltjes measure. In the particular case where  $\varphi(x) = x$  ( $\forall x \in \mathbb{R}$ )  $\mu_{\varphi}$  is called a Lebesgue measure.

### 42.6 Def

Let  $(\Omega, \mathcal{G}, \mu)$  be a measure space. Then

$$\{A \in \mathcal{G} \mid \mu(A) < +\infty\}$$

is a semialgebra.  $\sigma(\mathcal{C}) = \mathcal{G}$  and  $\mu|_{\mathcal{C}}$  is  $\sigma$ -additive.

•

$$\mu(A_0) = \sum_{n \in \mathbb{N}} \mu(B_n) < +\infty$$

$$\sum_{k \geq n} \mu(B_k) = \mu(A_n) \rightarrow 0 \quad (n \rightarrow +\infty)$$

We denote by  $L^1(\Omega, \mathcal{G}, \mu)$  the set of measurable mappings  $f : \Omega \rightarrow \mathbb{R}$  that belongs to  $L^1(I_\mu)$ . For  $f \in L^1(\Omega, \mathcal{G}, \mu)$

$$I_\mu(f)$$

is denoted as

$$\int_{\Omega} f(\omega) \mu(d\omega)$$

### 42.6.1 Particular case

If  $\Omega = \mathbb{R}$   $\mu = \mu_\varphi$  Stieltjes measure.

$$\int_{\mathbb{R}} f(x) \mu_\varphi(dx)$$

is denoted as

$$\int_{\mathbb{R}} f(x) d\varphi(x)$$

## 42.7 Prop

Let  $(\Omega, \mathcal{G}, \mu)$  be a  $\sigma$ -finite measure space.  $f : \Omega \rightarrow \mathbb{R}$  is measurable. If

$$\exists g \in L^1(\Omega, \mathcal{G}, \mu), g \leq f$$

then

$$f \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

### Proof

By replacing  $f$  by  $f - g$ , we may assume that  $g = 0$  Consider first the case where

$$f = \mathbb{1}_B, B \in \mathcal{G}$$

Let  $(A_n)_{n \in \mathbb{N}}$  be a increasing sequence in  $\mathcal{G}$ ,  $\mu(A_n) < +\infty$ ,  $\bigcup_{n \in \mathbb{N}} A_n = \Omega$  Then

$$\mathbb{1}_B = \lim_{n \in \mathbb{N}} \mathbb{1}_{B \cap A_n} \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

For general  $f \geq 0$

$$f = \lim_{n \rightarrow +\infty} f_n \in L^1(\Omega, \mathcal{G}, \mu)^\uparrow$$

where

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega \in \Omega \mid \frac{k}{2^n} \leq f(\omega) < \frac{k+1}{2^n}\}} + n \mathbb{1}_{\{\omega \in \Omega \mid f(\omega) \geq n\}}$$

### 42.8 Corollary

Let  $f : \Omega \rightarrow \mathbb{R}$  be a measurable mapping. Then

$$f \in L^1(\Omega, \mathcal{G}, \mu)$$

iff

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

#### Proof

$\Rightarrow$  One has  $f \in L^1(I_{\mu})$ . Hence  $|f| \in L^1(I_{\mu})$  So  $I_{\mu}(|f|) < +\infty$

$\Leftarrow$  Suppose that

$$\int_{\Omega} |f(\omega)| \mu(d\omega) < +\infty$$

Since  $f \vee 0$  and  $-(f \wedge 0)$  belongs to  $L^1(\Omega, \mathcal{G}, \mu)^+$  and  $f \vee 0 \leq |f|$ ,  $-(f \wedge 0) \leq |f|$  so

$$f \vee 0 \text{ and } -(f \wedge 0) \in L^1(\Omega, \mathcal{G}, \mu)$$

Hence

$$f = f \vee 0 + f \wedge 0 \in L^1(\Omega, \mathcal{G}, \mu)$$



## Chapter 43

# Fundamental theorem of calculus

### 43.1 Theorem

Let  $J$  be an open interval in  $\mathbb{R}$   $x_0 \in J$   $f : J \rightarrow \mathbb{R}$  be a continuous mapping.

(1)  $\forall (a, b) \in J^2, a < b$

$$\begin{aligned} \mathbb{1}_{]a, b]} : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto f(x) \quad \text{if } x \in ]a, b] \\ &\mapsto 0 \quad \text{if } x \notin ]a, b] \end{aligned}$$

belongs to  $L^1(\mathbb{R}, \mathcal{B}, \mu)$  ( $\mathcal{B}$  is Borel  $\sigma$ -algebra,  $\mu$  is Lebesgue measure)

(2) Let  $F : J \rightarrow \mathbb{R}$   $F(x) := \int_{x_0}^x f(t)dt$ . Then  $F$  is differentiable on  $J$  with  $F'(x) = f(x), \forall x \in J$

### 43.2 Corollary

If  $G : J \rightarrow \mathbb{R}$  is a mapping such that  $G' = f$  then  $\forall (a, b) \in J^2, a < b$

$$G(b) - G(a) = \int_a^b f(t)dt$$

#### 43.2.1 Proof

(1)  $f$  is bounded on  $[a, b]$  Hence

$$\int_{\mathbb{R}} \mathbb{1}_{]a, b]}^{(x)} |f(t)^{(x)}| dx < +\infty$$

- (2) Let  $x \in J, h > 0$  such that  $[x, x+h] \subseteq J$ ,  $f$  is uniformly continuous on  $[x, x+h]$   
 For  $0 < t \leq h$

$$\inf_{[x, x+t]} f \leq \frac{F(x+t) - F(x)}{t} = \frac{1}{t} \int_x^{x+t} f(s) ds \leq \sup_{[x, x+t]} f$$

Since  $f$  is continuous

$$\liminf_{t \rightarrow 0} \inf_{[x, x+t]} f = \limsup_{t \rightarrow 0} \sup_{[x, x+t]} f = f(x)$$

So

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Similarly

$$\lim_{t > 0, t \rightarrow 0} \frac{F(x+t) - F(x)}{t} = f(x)$$

Hence

$$F'(x) = f(x)$$

### Application

- Let  $F$  and  $G$  be two mapping of class  $C^1$  from  $J$  to  $\mathbb{R}$ . Then  $FG$  is of class  $C^1$  and

$$FG'(x) = F'(x)G(x) + F(x)G'(x)$$

Let  $f = F', g = G'$ , then  $\forall (a, b) \in J^2, a, b$

$$\int_a^b f(t)G(t)dt = F(b)G(b) - F(a)G(a) - \int_a^b F(t)g(t)dt$$

- Let  $\varphi : I \rightarrow J$  be a mapping of class  $C^1$ , where  $I$  is open interval. Let  $F : J \rightarrow \mathbb{R}$  be a mapping of class  $C^1$ .

$$(F \circ \varphi)'(x) = F'(\varphi(x))\varphi'(x)$$

- Hence  $\forall (\alpha, \beta) \in I^2, \alpha < \beta$

$$\int_{\alpha}^{\beta} F(\varphi(t))\varphi'(t)dt = F(\varphi(\beta)) - F(\varphi(\alpha))$$

## Chapter 44

### $L^p$ space

#### 44.1 Def

We fix a measure space  $(\Omega, \mathcal{G}, \mu)$  the set of measurable mappings  $f : \Omega \rightarrow \mathbb{R}$  such that

$$\|f\|_{L^p} := \left( \int_{\Omega} |f(\omega)|^p \mu(dx) \right)^{\frac{1}{p}} < +\infty$$

Lemma Let  $(p, q) \in \mathbb{R}_{\geq 1}^2$  such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

for any  $(a, b) \in \mathbb{R}_{\geq 0}^2$

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

Proof We may assume that  $(a, b) \in \mathbb{R}_{\geq 0}$

$$\frac{a^p}{p} + \frac{b^q}{q} = \frac{1}{p} \exp(p \ln a) + \frac{1}{q} \exp(q \ln b) \geq \exp(\ln a + \ln b) = ab$$

$$\begin{aligned} \int_{\Omega} |\varphi(x)\psi(x)| \mu(dx) &\leq \frac{\int_{\Omega} |\varphi(x)|^p \mu(dx)}{p \|f\|_{L^p}^p} + \frac{\int_{\Omega} |\psi(x)|^q \mu(dx)}{q \|g\|_{L^q}^q} \\ &= \frac{1}{p} + \frac{1}{q} = 1 \end{aligned}$$

#### 44.2 Hölder inequality

Let  $f : \Omega \rightarrow \mathbb{R}$  and  $g : \Omega \rightarrow \mathbb{R}$  be measurable mappings. One has

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

**Proof**

Take

$$\varphi = \frac{f}{\|f\|_{L^p}}, \psi = \frac{g}{\|g\|_{L^q}}$$

then

$$|\varphi(x)\psi(x)| \leq \frac{|\varphi(x)|^p}{p} + \frac{|\psi(x)|^q}{q}$$

**44.3 Corollary**

Let  $p \geq 1$   $\forall (f, g) \in L^p(\Omega, \mathcal{G}, \mu)$

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$$

**Proof**

Apply Hölder inequality to  $f(f + g)^{p-1}$  and  $g(f + g)^{p-1}$

# Part VIII

## tensor



# Chapter 45

## tensor product

Let  $R$  be a commutative ring with unity

### 45.1 Theorem

Let  $M$  and  $N$  be two  $R$ -modules. Then exists an  $R$ -module denoted by  $M \otimes_R N$  and a bilinear mapping

$$t : M \times N \rightarrow M \otimes_R N$$

having the following properties:

- (1) For any  $R$ -module  $P$  and any bilinear mapping  $s : M \times N \rightarrow P$ . There exists a unique linear mapping  $f_s : M \otimes_R N \rightarrow P$  such that  $s = f_s \circ t$

$$\begin{array}{ccc} M \times N & \xrightarrow{s} & P \\ \downarrow t & \nearrow f_s & \\ M \otimes_R N & & \end{array}$$

- (2) If  $T, t'$  is another couple that satisfies (1) with  $s \mapsto g_s$  then there exists a unique isomorphism

$$T \cong M \otimes_R N$$

### Proof

- (2) note that the the morphisms on  $R$ -module category are just linear mapping.

$$\begin{array}{ccc} M \times N & \xrightarrow{t'} & M \otimes_R N \\ \downarrow t & \nearrow f_{t'} & \\ T & \nearrow g_t & \end{array}$$

$$(f_{t'} \circ g_t) \circ t' = f_{t'} \circ t' = t$$

It means that we have the following structure

$$f_{t'} \circ g_t = id_{M \otimes_R N}$$

$$g_t \circ f_{t'} = id_T$$

Then isomorphic.

(1) let  $\mathcal{F}$  be the free  $R$ -module generated by  $M \times N$

$$\mathcal{F} = \left\{ \sum_{finite} a_{ij}(m_i, n_i) : a_{ij} \in R, m_i \in M, n_i \in N \right\}$$

let  $\mathcal{G}$  be the  $R$ -submodule generated by the elements of the following shape  
 $m, m' \in M \quad n, n' \in N \quad \mathfrak{z} \in R$

$$(m + m', n) - (m, n) - (m', n)$$

$$(m, n + n') - (m, n) - (m, n')$$

$$(\mathfrak{z}m, n) - \mathfrak{z}(m, n)$$

$$(m, \mathfrak{z}n) - \mathfrak{z}(m, n)$$

$$M \otimes_R N := \mathcal{F} / \mathcal{G}$$

## 45.2 Def

$$f_s(\mathcal{G} + (m, n)) := s(m, n)$$

Extend this mapping to linearity. This makes the diagram commutative. It's clearly the unique mapping

## 45.3 Def

The  $R$ -module  $M \otimes_R N$  constructed above is called the tensor product of  $M$  and  $N$ . An element of  $M \otimes_R N$  is called tensor. We denote

$$t(m, n) := m \otimes n$$

and any elements of this form is called pure tensor.

## 45.4 Remark

Pure tensors generate  $M \otimes_R N$ . In particular any tensor can be written as sum of pure tensors.

Example

$$0 = (m + m') \otimes n - m \otimes n - m \otimes n'$$



## 45.5 Corollary

The mapping  $s \mapsto f_s$  defined above gives an isomorphism

$$\mathcal{L}(M, N; P) \cong \mathcal{L}(M \otimes_R N, P)$$

for any  $R$ -module  $P$

### Proof

surjection Take  $\varphi \in \mathcal{L}(M \otimes_R N, P)$ , the  $\varphi \circ t$  is clearly bilinear ( $\in \mathcal{L}(M, N; P)$ ), so  $\varphi = f_{\varphi \circ t}$ . This shows surjectivity.

injection if  $0 \neq s = f_s \circ t \Rightarrow f_s \neq 0$ , hence

## 45.6 exercise

### 45.6.1

show that

$$M \otimes_R N \cong N \otimes_R M$$

### 45.6.2

show that

$$(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$$

so we can remove parenthesis and write

$$M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$$

(call it the  $n$ -fold tensor product of  $M_1, \dots, M_n$ )

### 45.6.3

show that  $M_1 \otimes_R M_2 \otimes_R \cdots \otimes_R M_n$  factorizes the multi-linear mappings, and

$$\mathcal{L}(M_1, \dots, M_n; P) \cong \mathcal{L}(M_1 \otimes_R \cdots \otimes_R M_n, P)$$

$$\begin{array}{ccc}
 M_1 \times M_2 & \xrightarrow{s} & P \\
 \downarrow t & \nearrow f_s & \\
 M_1 \otimes_R M_2 & & \\
 \downarrow \wr & & \\
 M_1 \otimes_R \cdots \otimes_R M_n & & 
 \end{array}$$

We have the general definition of tensor products for  $R$ -modules. But we're interested in the case  $R = K$  when  $K$  is a field.  $\mathcal{L}$  denotes:  $V_1 \otimes \cdots \otimes V_n$

$$\mathcal{L}(V_1, \dots, V_n; K) \cong (V_1 \otimes \cdots \otimes V_n)^\vee$$

This is the pervious corollary  $f \sim P = K$

## 45.7 Lemma

Let  $V_1, \dots, V_n$  be  $K$ -vector spaces of finite dimension  $d_i > 0$  let

$$e_{i1}, \dots, e_{id_i}$$

be a basis for  $V_i$ . Let's define the following functions.

$$\begin{aligned} \varphi_{i_1, \dots, i_n} : V_1 \times \cdots \times V_n &\rightarrow K \\ (v_1, \dots, v_n) &\mapsto \prod_{j=1}^n e_{ji_j}^\vee(V_j) \end{aligned}$$

Then the set  $\{\varphi_{i_1, \dots, i_n}\}$  is a basis for  $\mathcal{L}(V_1, \dots, V_n; K)$

### 45.7.1 Proof

We do the proof for  $n = 2$ . Then the general case follows by induction.

$$\begin{aligned} V_1 &= \langle e_1, \dots, e_m \rangle & m &= d_1 \\ V_2 &= \langle \omega_1, \dots, \omega_n \rangle & n &= d_2 \end{aligned}$$

This special our  $\varphi_{i_1, \dots, i_n}$  are denoted by

$$\xi_{ij}(x, y) = e_i^\vee(x) \omega_j^\vee(y)$$

Let's show that  $\xi_{ij}$  is a generating set

$\varphi \in \mathcal{L}(V_1, V_2; K)$  such that  $\varphi(e_i, \omega_i) := A_{ij} \in K$

$$\begin{aligned} \varphi(x, y) &= \varphi\left(\sum \alpha_i e_i, \sum \beta_j \omega_j\right) \\ &= \sum \alpha_i \beta_j \varphi(e_i, \omega_j) \\ &= \sum \alpha_i \beta_j A_{ij} \\ &= \sum A_{ij} e_i^\vee(x) \omega_j^\vee(y) \\ &= \sum A_{ij} \xi_{ij}(x, y) \end{aligned}$$

we prove that  $\xi_{ij}$  are linearly independent

$$\sum A_{ij} \xi_{ij}(x, y) = 0 \quad \forall (x, y) \in V_1 \times V_2$$

Evaluate in

$$(x, y) = (e_i, \omega_i) \Rightarrow A_{ij} = 0 \quad \forall i, j$$

## 45.8 Prop

Assume that  $V_1, \dots, V_n$  are vector spaces and  $V_i$  has basis:  $\{e_{i1}, \dots, e_{id_i}\}$  then

$$B = \{e_{1i_1} \otimes \dots \otimes e_{ni_n}, 1 \leq i_j \leq d_j\}$$

is a basis for  $V_1 \otimes \dots \otimes V_n$ . In particular,  $V_1 \otimes \dots \otimes V_n$  has dimension  $\prod_{i=1}^n d_i$

### Proof

Again we assume  $n = 2, m = d_1, n = d_2$

$$V_1 = \langle e_1, \dots, e_m \rangle \quad V_2 = \langle \omega_1, \dots, \omega_n \rangle$$

We know that

$$\begin{aligned} \mathcal{L}(V_1, V_2; P) &\cong (V_1 \otimes V_2)^\vee \\ s &\mapsto f_s \end{aligned}$$

Recall that

$$\begin{aligned} \xi_{ij}(x, y) &= e_i(x)w_j(y) \\ f_{\xi_{ij}}(x \otimes y) &= \xi_{ij}(x, y) = e_i^\vee(x)w_j^\vee(y) \\ f_{\xi_{ij}}(e_k \otimes w_l) &= \begin{cases} 1 & \text{if } (i, j) = (k, l) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

It follows that  $\{e_k \otimes w_l\}_{k,l}$  is a basis of  $V_1 \otimes V_2$

## 45.9 tensor product and duality

### 45.9.1 product

Let  $V_1, \dots, V_n$  be vector spaces as above. Then

$$(V_1^\vee \otimes \dots \otimes V_n^\vee) \cong (V_1 \otimes \dots \otimes V_n)^\vee$$

### Proof

Define

$$\begin{aligned} V_1^\vee \times \dots \times V_n^\vee &\rightarrow \mathcal{L}(V_1, \dots, V_n; K) (\cong (V_1 \otimes \dots \otimes V_n)^\vee) \\ (\varphi_1, \dots, \varphi_n) &\mapsto [(v_1, \dots, v_n) \mapsto \prod \varphi_i(v_i)] \end{aligned}$$

This mapping is multi-linear. It describes by the property of tensor product to a map

$$\begin{aligned} V_1^\vee \otimes \dots \otimes V_n^\vee &\rightarrow \mathcal{L}(V_1, \dots, V_n; K) (\cong (V_1 \otimes \dots \otimes V_n)^\vee) \\ \varphi_1 \otimes \dots \otimes \varphi_n &\mapsto [(v_1, \dots, v_n) \mapsto \prod \varphi_i(v_i)] \end{aligned}$$

By Prop 45.8 these two space have the same  $\dim \prod d_i$ . It enough to show that the mapping is surjective. Let's do it for  $n = 2$  (keep the same notation as above). Take  $\xi_{ij}$

$$\xi_{ij}(x, y) = e_i^\vee(x)w_j^\vee(y) = F(e_i^\vee \otimes w_j^\vee)(x, y)$$

### 45.9.2 duality

Let  $V$  and  $W$  be vector spaces of finite dimension. Then

$$\mathcal{L}(V, W) \cong V^\vee \otimes W^\vee$$

#### Proof

$$\begin{aligned} s : V^\vee \times W &\rightarrow \mathcal{L}(V, W) \\ (\varphi, \omega) &\mapsto [\sigma \mapsto \varphi(\sigma)\omega] \end{aligned}$$

Let's check that  $s$  is bilinear. (note that  $\varphi(\sigma) \in K$ )

$$((\varphi + \psi)(\sigma))\omega = (\varphi(\sigma) + \psi(\sigma))\omega = \varphi(\sigma)\omega + \psi(\sigma)\omega$$

$$\varphi(\sigma)(\omega + \omega') = \varphi(\sigma)\omega + \varphi(\sigma)\omega'$$

Thus map  $s$  is then bilinear. So it induces  $f_s : V^\vee \otimes W \rightarrow \mathcal{L}(V, W)$ . We have to show that this is the required isomorphism.

Let  $\{v_1^\vee, \dots, v_m^\vee\}$  be a basis for  $V^\vee$ , and let  $\{w_1, \dots, w_n\}$  be a basis for  $W$ . Let's see what happens to

$$f_s(v_i^\vee \otimes w_j) = [v_i^\vee \mapsto v_i^\vee(w_j)]w_j = \delta_{ij}w_j$$

Consider the matrix associated to  $f_s$  with respect to the basis.

$$\begin{array}{ccc} (e_1, e_n)E & \xrightarrow{F} & P(p_1, \dots, p_m) \\ \uparrow \scriptstyle b_1 & & \uparrow \scriptstyle b_2 \\ K^n & \xrightarrow{M_F} & K^m \end{array}$$

Call this matrix  $M_{ab}$

$$M_{ab} = \begin{cases} 1 & \text{if } (a, b) = (j, i) \\ 0 & \text{otherwise} \end{cases}$$

The matrices of this form are a basis of  $\mathcal{L}(K^n, K^m) \cong \mathcal{L}(V, W)$

And important case of this prop is when  $V = W$ :

$$\mathcal{L}(V; V) \cong V^\vee \otimes V$$

More in general

$$\begin{aligned}\mathcal{L}(V, W) &\xrightarrow{\cong} V^\vee \otimes W \\ f &\mapsto \sum a_{ij} \sigma_i^\vee \otimes w_j\end{aligned}$$

note that  $\sigma_i^\vee \otimes w_j$  is a basis.

For instance  $V = W$

$$id_V = \mathcal{L}(V, V) \mapsto \sum_i \sigma_i^\vee \otimes \sigma_i$$

### 45.9.3 Exercise

Let  $M, N, P$   $R$ -modules. Show that

$$\mathcal{L}(M \otimes_R N; P) \cong \mathcal{L}(M; \mathcal{L}(N; P))$$

## 45.10 Def

We want to define the tensor product of linear mappings. let  $M_1, M_2, N_1, N_2$  be  $R$ -modules and let  $f_i : M_i \rightarrow N_i$  be linear mappings. Then we define

$$\begin{aligned}f_1 \otimes f_2 : M_1 \otimes M_2 &\rightarrow N_1 \otimes N_2 \\ m_1 \otimes m_2 &\mapsto f_1(m_1) \otimes f_2(m_2)\end{aligned}$$

This is a linear mapping

$$\begin{array}{ccc} M_1 \times M_2 & \xrightarrow{f_1 \times f_2} & N_1 \times N_2 \\ \downarrow & & \downarrow \\ M_1 \otimes M_2 & \xrightarrow{f_1 \otimes f_2} & N_1 \otimes N_2 \end{array}$$

## 45.11 Extension of scalars

Let  $\varphi : R \rightarrow S$  be a commutative unitary ring homomorphism. Let  $M$  be a  $R$ -module. Goal is to give to  $M$  also a structure of  $S$ -module "conveyed by  $\varphi$ "

Note that  $S$  has a structure of  $R$ -module  $s \in S, r \in R$

$$rs := \varphi(r)s$$

Now take the tensor product  $M \otimes_R S$ . Now we give a structure of  $S$ -module to  $M \otimes_R S$ .

Take  $s \in S$

$$s(\underbrace{m \otimes s'}_{\in M \otimes_R S}) := m \otimes ss'$$

note that  $ss'$  is a multi in  $S$  and we cannot product  $sm$ .

Notice we've a mapping

$$\begin{aligned} i : M &\rightarrow M \otimes_R S \\ m &\mapsto m \otimes s \end{aligned}$$

Be careful, in general the mapping  $i$  is NOT injective.

Example  $R = \mathbb{Z}$   $S = \mathbb{Z}/2\mathbb{Z}$   $\alpha : \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$   $M = \mathbb{Z}[X]$

$$i(2X) = 2X \otimes 1 = 2(X \otimes 1) = X \otimes \alpha(2) \cdot 1 = X \otimes 0 = 0$$

## 45.12 Prop

Let  $K \subseteq L$  be a field extension and let  $V$  be a  $K$ -vector space. Moreover let's denote  $V_L = V \otimes_K L$ . If  $\{e_i\}_{i=1}^n$  is a basis of  $V$  then  $\{e_i \otimes 1\}_{i=1}^n$  is a  $L$ -basis of  $V_L$ . ( $V_L$  has the same dim of  $V$ )

### Proof

The set  $\{e_i \otimes 1\}_{i=1}^n$  generates  $V_L$  if fact

$$\sigma \otimes l = \left( \sum \underbrace{\alpha_i}_{K} e_i \otimes \underbrace{l}_L \right) = \sum l \alpha_i (e_i \otimes 1)$$

We have to show that the elements are linearly independent.

$$0 = \sum \alpha_i (e_i \otimes 1) = \sum e_i \otimes \alpha_i \quad \alpha_i \in L$$

( $L$  is a  $K$ -vec space)

Define the mapping with  $\lambda_i \in K$

$$\begin{aligned} b_i : V \times L &\rightarrow L \\ \left( \sum \lambda_i e_i, \beta \right) &\mapsto \lambda_i \beta \end{aligned}$$

This mapping is bilinear. It induces a mapping

$$f_i = f_{b_i} \left( \sum \lambda_i e_i \right) \otimes \beta \mapsto \lambda_i \beta$$

Note that  $f_i(e_j \otimes \beta) = \delta_{ij} \beta$

•

$$f_i \left( \sum_j e_j \otimes \alpha_j \right) = \alpha_i$$

But

$$0 = f_i(0) = f_i \left( \sum_j e_j \otimes \alpha_j \right) = \alpha_i \quad \forall i$$

### 45.13 Remark

As a consequence we have that the mapping  $i : V \rightarrow V_L$  (mapping of  $K$ -vet spaces) is injective.

### 45.14 Exercise

Show that

$$\begin{aligned} V \otimes_K K &\cong V \\ \sigma \otimes a &\mapsto as \end{aligned}$$

### 45.15 Exactness of the tensor product

fix a  $R$ -module  $N$  and consider:

$$_-\otimes N : M \mapsto M \otimes_R N$$

for any  $R$ -module  $M$ .

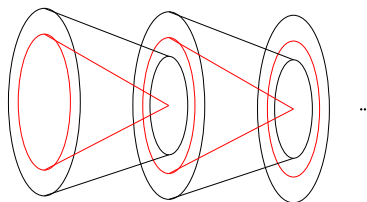
Moreover for any linear mapping  $(f : M \rightarrow P) \rightsquigarrow f \otimes id_N : M \otimes_R N \rightarrow P \otimes_R N$   
This association sends  $id_M$  to  $id_{M \otimes_R N}$  and moreover is well defined with respect to the composition

$$f \circ g \mapsto (f \circ g) \otimes id_N = (f \otimes id_N) \circ (g \otimes id_N)$$

### 45.16 Def

A sequence of  $R$ -modules is a diagram of the following form (also called complex of  $R$ -modules)

$$M_1 \xrightarrow{d^1} M_2 \xrightarrow{d^2} \dots$$



$M_i$  is an  $R$ -module,  $d^i$  is linear mapping and

$$\text{Ker}(d^{i+1}) \supseteq \text{Im}(d^i)$$

Thus we also have:

$$d^{i+1} \circ d^i = 0$$

The diagram is called exact if

$$\text{Ker}(d^{i+1}) = \text{Im}(d^i)$$

take a morphism  $f : M \rightarrow N$  then

- $f$  is injective iff

$$0 \rightarrow M \xrightarrow{f} N$$

is exact

- $f$  is surjective iff

$$M \xrightarrow{f} N \rightarrow 0$$

is exact

The first theorem of homomorphism

$$\bar{f} : M/\text{Ker}(f) \xrightarrow{\cong} \text{Im}(f)$$

can be written as an exact sequence

$$0 \rightarrow \text{Ker}(f) \xrightarrow{i} M \xrightarrow{f} \text{Im}(f) \rightarrow 0$$

More in general sequence like

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

are called short exact sequences.

## 45.17 Prop

$N$  is a  $R$ -module

$$- \otimes_R N : \forall M \quad M \mapsto M \otimes_R N$$

$$f : M \rightarrow P$$

$$f \otimes id_N : M \otimes_R N \rightarrow P \otimes_R N$$

Assume that we have a short exact (also complex) sequence of  $R$ -modules

$$0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$$

Then we apply  $- \otimes_R N$  to whole sequence

$$0 \rightarrow M_1 \otimes_R N \xrightarrow{f \otimes id_N} M_2 \otimes_R N \xrightarrow{g \otimes id_N} M_3 \otimes_R N \rightarrow 0$$

is a complex sequence if it's exact then we call  $N$  a flat  $R$ -module. One significant example's that the free module is flat.



**45.17.1 Example**

$$\begin{array}{ccc}
 0 \rightarrow \mathbb{Z} & \xrightarrow{\mu} & \mathbb{Z} \\
 & & x \mapsto 2x \\
 & & y \mapsto 2\mathbb{Z} + y
 \end{array}
 \quad
 \begin{array}{c}
 \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \rightarrow 0
 \end{array}$$

Now apply  $-\otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccc}
 0 \rightarrow \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} & \xrightarrow{\mu \otimes id} & \mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \\
 x \otimes (2\mathbb{Z} + y) \mapsto & (2x) \otimes (2\mathbb{Z} + y) & \\
 z \otimes (2\mathbb{Z} + y) \mapsto & (2\mathbb{Z} + z)(2\mathbb{Z} + y) &
 \end{array}$$

and

$$\begin{aligned}
 2x \otimes (2\mathbb{Z} + y) &= 2(x \otimes 2\mathbb{Z} + y) \\
 &= x \otimes (2\mathbb{Z} + 2y) \\
 &= x \otimes 2\mathbb{Z} \\
 &= 0
 \end{aligned}$$

which is not injective, thus above isn't exact.

**45.18 Exercise(important)**

If  $R = N$  then  $-\otimes_R N$  (where  $N$  is a finite dim vec space) is exact. Hint: use the basis.



## Chapter 46

# Tensor algebra

Fix a vec space  $V$  (over  $K$ ) of finite dimension

### 46.1 Def

We denote

$$\begin{aligned} T_p^q &:= (V^\vee)^{\otimes p} \otimes V^{\otimes q} \quad p, q \in \mathbb{N} \\ &= \underbrace{V^\vee \otimes \cdots \otimes V^\vee}_{p \text{ times}} \otimes \underbrace{V \otimes \cdots \otimes V}_{q \text{ times}} \end{aligned}$$

An element of  $T_p^q(V)$  is called a tensor of type  $(p, q)$  (or a mixed tensor which is  $p$ -covariant and  $q$ -contravariant)

Let's denote:

$$T(V) := \bigoplus_{q \in \mathbb{N}} T_0^q(V)$$

itemize some item in it:

$$\begin{aligned} T_0^0(V) &= K \\ T_1^0(V) &= V^\vee \\ T_0^1(V) &= V \\ T_1^1(V) &= V^\vee \otimes V \cong \mathcal{L}(V; V) \\ T_2^0(V) &= V^\vee \otimes V^\vee \cong (V \otimes V)^\vee \cong \mathcal{L}(V, V; K) \end{aligned}$$

If you have a  $R$ -module  $M$ , then

$$\bigoplus_{n=0}^{\infty} M = \{(m_1, \dots, m_n, \dots) : m_i \in M \text{ all but finite many } m_i = 0\}$$

On  $T(V)$  we have following operation:

$$\begin{aligned} T_0^l(V) \times T_0^q(V) &\rightarrow T_0^{l+q}(V) \\ ((x_1 \otimes \cdots \otimes x_l), (y_1 \otimes \cdots \otimes y_q)) &\mapsto x_1 \otimes \cdots \otimes x_l \otimes y_1 \otimes \cdots \otimes y_q \end{aligned}$$

With this operation  $T(V)$  becomes a  $K$ -algebra. It called the tensor algebra associated to  $V$

## 46.2 exterior product

Let  $W$  be the two sided ideal of  $T(V)$  generated by the element of the type  $x \otimes x$

$$W = \left\{ \sum_{i(finite)} (y_1 \otimes \cdots \otimes y_{m_i}) \otimes (x_i \otimes x_i) \otimes (z_1 \otimes \cdots \otimes z_{n_i}) \right\}$$

With  $x_j, y_j, z_j \in V$  and  $n_i, m_j \in \mathbb{N}$

## 46.3 Def

The quotient algebra

$$\bigwedge(V) := T(V)/W$$

is a  $K$ -algebra, which called the exterior algebra of  $V$

$$\begin{aligned} \pi : T(V) &\rightarrow \bigwedge(V) \\ x_1 \otimes \cdots \otimes x_n &\mapsto x_1 \wedge \cdots \wedge x_n \end{aligned}$$

This def is try to transform  $\otimes$  to  $\wedge$

## 46.4 Notation

$$\bigwedge(V) = \bigoplus_{n \in \mathbb{N}} \bigwedge^n(V)$$

$$\bigwedge^n(V) := T_0^n(V) / (W \cap T_0^n(V))$$

this is called  $n$ -fold exterior product

## 46.5 Prop

Let  $\sigma \in \mathfrak{S}_n$  then

$$x_1 \wedge \cdots \wedge x_n = \text{sgn}(\sigma) x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(n)}$$

**Proof**

Since any permutation can be written as the product of adjacent transpositions, it's enough to do the proof for  $\sigma = (i, i+1)$

$$\begin{aligned} 0 &= (x_i + x_{i+1}) \wedge (x_i + x_{i+1}) \\ &= (x_i \wedge x_i) + (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) + (x_{i+1} \wedge x_{i+1}) \\ &= (x_i \wedge x_{i+1}) + (x_{i+1} \wedge x_i) \end{aligned}$$

**46.6 Def**

Let  $E$  be an  $R$ -module and  $f : E^n \rightarrow M$  a mapping. We say that the pair  $(M, f : E^n \rightarrow M)$  satisfies the universal property for the  $n^{\text{th}}$ -exterior power if

- $M$  is an  $R$ -module,  $f : E^n \rightarrow M$  is an  $n$ -linear mapping s.t.

$$\forall i \in \{1, \dots, n-1\}$$

if

$$x_i = x_{i+1}$$

then

$$f(x_1, \dots, x_n) = 0$$

(alternating  $n$ -linear mapping)

- If  $P$  is an  $R$ -module and  $\varphi : E^n \rightarrow P$  is an alternating mapping, then

$$\exists! \Phi : M \rightarrow P \text{ s.t. } \Phi \circ f = \varphi$$

**46.7 Def**

$V$  is a  $K$ -vct space. A multi-linear mapping:

$$\varphi : V \times \dots \times V \rightarrow W$$

is called skew-symmetric(alternating) if

$$\varphi(x_1, \dots, x_n) = 0 \text{ when } \exists i \neq j : x_i = x_j$$

**46.8 Prop**

Let  $V$  be a vct space. For any alternating multi-linear mapping

$$s : \underbrace{V \times \dots \times V}_{n \text{ times}} \rightarrow M$$

when  $M$  is another vct space, there exists a unique linear mapping

$$g_s : \bigwedge^n(V) \rightarrow M$$

such that the following diagram commutes

$$\begin{array}{ccc} V^n & \xrightarrow{s} & M \\ \downarrow t & \nearrow f_s & \\ T_0^n(V) & & \\ \downarrow & \nearrow g_s & \\ \bigwedge^n(V) & & \end{array}$$

**Proof**

$$g_s(\sigma_1 \wedge \cdots \wedge \sigma_n) := s(\sigma_1, \cdots, \sigma_n)$$

check the diagram is commutative

$$\mathcal{F}(V^n) \xrightarrow{t} T_0^n(V) \longrightarrow \bigwedge^n(V)$$

$$\{(\sigma_1, \cdots, \sigma_n)\} \longmapsto \{\sigma \otimes \cdots \otimes \sigma_n\} \longmapsto \{\sigma_1 \wedge \cdots \wedge \sigma_n\}$$

## 46.9 Remark/exercise

The couple  $\bigwedge^n V$  with

$$V^n \rightarrow \bigwedge^n(V)$$

that satisfies Prop 46.8 is unique to unique isomorphism

## 46.10 Prop

Let  $V$  be a vct space of dimension  $n$  with a basis  $\{e_1, \cdots, e_n\}$ . Then  $\bigwedge^k(V)$  is a vct space with a basis given by

$$\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

In particular,  $\bigwedge^k(V)$  has dimension  $\binom{n}{k}$

### 46.10.1 Proof

$\mathcal{B}$  is clearly a generating set. The different part is to show that  $\mathcal{B}$  is made of linearly independent elements.

$$I = \{i_1, \dots, i_k\}$$

with  $1 \leq i_1 < \dots < i_k \leq n$ , define

$$\begin{aligned} \varphi_I : V^n &\rightarrow K \\ (e_{j_1}, \dots, e_{j_n}) &\mapsto \begin{cases} \text{sgn}(t) & \text{if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

$\varphi_I$  is multilinear and alternating (skew-symm), hence it induce a linear mapping

$$\begin{aligned} g_{\varphi_I} = \overline{\varphi_I} : \bigwedge^k(V) &\rightarrow K \\ (e_{j_1} \wedge \dots \wedge e_{j_k}) &\mapsto \begin{cases} \text{sgn}(t) & \text{if } \exists \tau \in S_I \quad \tau(j_m) = i_m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

With  $\sigma \in \bigwedge^n(V)$ , assume that

$$0 = \sigma = \sum_{1 \leq j_1 < \dots < j_k \leq n} \lambda_{j_1, \dots, j_k} e_{j_1} \wedge \dots \wedge e_{j_k}$$

By linearity

$$0 = \overline{\varphi_I}(\sigma) = \pm \lambda_I$$

Do it for any positive  $I$  this shows that any  $\lambda_{j_1, \dots, j_k}$  is zero.





## Chapter 47

# Determinant

### 47.1 Def

Let  $V$  be a vct space of dimension  $n$ , then

$$\det(V) = \bigwedge^n(V)$$

is called the determinant of  $V$ . It is a vct space of dimension  $1 = \binom{n}{n}$  and a basis is given by

$$e_1 \wedge \cdots \wedge e_n$$

when  $\{e_1, \dots, e_n\}$  is a basis of  $V$ .

#### 47.1.1 Proof

Let  $f \in \mathcal{L}(V; V)$  then consider

$$\begin{aligned} \tilde{f} : V^k &\rightarrow \bigwedge^k V \\ (v_1, \dots, v_n) &\mapsto f(v_1) \wedge \cdots \wedge f(v_n) \end{aligned}$$

This is multilinear and alternating. Therefore it induces a mapping

$$\begin{aligned} g_{\tilde{f}} = \bigwedge^k f : \bigwedge^k(V) &\rightarrow \bigwedge^k(V) \\ v_1 \wedge \cdots \wedge v_k &\mapsto f(v_1) \wedge \cdots \wedge f(v_k) \end{aligned}$$

Since  $\det(V)$  has  $\dim 1$

$$\det(f) : \sigma_1 \wedge \cdots \wedge \sigma_n \mapsto \underbrace{\det_f}_{\in K}(v_1 \wedge \cdots \wedge v_n) = f(v_1) \wedge \cdots \wedge f(v_n)$$

By abuse of notation we identity

$$\det(f) = \det_f$$

## 47.2 Prop

$f \in \mathcal{L}(V; V)$  is invertible iff  $\det(f) \neq 0$

### 47.2.1 Proof

$f$  is not invertible iff  $\{f(e_1), \dots, f(e_n)\}$  is not a basis.  
iff there's a non-trivial linear combination

$$\sum_n \lambda_i f(e_i) = 0$$

After relabelling the  $e_i$  we can assume

$$f(e_i) = \sum_{i \geq 2} \mu_i f(e_i)$$

$$\begin{aligned} \det(f)(e_1 \wedge \dots \wedge e_n) &= \det_f \cdot (e_1 \wedge \dots \wedge e_n) \\ &= \left( \sum_{i \geq 2} \mu_i f(e_i) \right) \wedge f(e_1) \wedge \dots \wedge f(e_n) \\ &= \sum_{i \geq 2} \mu_i (f(e_1) \wedge f(e_2) \wedge \dots \wedge f(e_n)) \\ &= 0 \end{aligned}$$

## 47.3 Prop

$$\det(f \circ g) = \det(f) \cdot \det(g)$$

### Proof

$$\begin{aligned} \det(f \circ g) &= (f \circ g)(e_1) \wedge \dots \wedge (f \circ g)(e_n) \\ &= f(g(e_1)) \wedge \dots \wedge f(g(e_n)) \\ &= (\det f)(g(e_1) \wedge \dots \wedge g(e_n)) \\ &= \det f \cdot \det g(e_1 \wedge \dots \wedge g(e_n)) \end{aligned}$$

## 47.4 Prop

The determinant of  $f$  is equal to the determinant of any matrix that represents  $f$  with respect to a fixed basis. This doesn't depend on the choice of the basis.

**Proof**

Fix a basis  $\{e_1, \dots, e_n\}$  of  $V$ . Then

$$\begin{array}{ccc}
 \begin{array}{c} v_i \\ \uparrow \\ e_i \end{array} & \begin{array}{ccc} V & \xrightarrow{f} & V \\ \cong \uparrow \mathcal{B} & & \cong \uparrow \mathcal{B} \\ K^n & \xrightarrow{A_f} & K^n \end{array} & \Longrightarrow & \begin{array}{ccc} \det(V) & \xrightarrow{\det(f)} & \det(V) \\ \uparrow \bigwedge^n b & & \uparrow \bigwedge^n b \\ \det(K^n) & \xrightarrow{\det(A_f)} & \det(K^n) \end{array}
 \end{array}$$

$A_f^{(v_1, \dots, v_n)}$  is the matrix associated to  $f$  with respect to the basis  $\{v_1, \dots, v_n\}$  suppose that  $f(v_i) = \xi_{ij}v_j$ . One we can see

$$A_f = \mathcal{B}^{-1} \circ f \circ \mathcal{B}(e_i)$$

$$\begin{aligned}
 \det(A_f) &= ((a_{11}, a_{12}, \dots, a_{1n}) \wedge (0, a_{22}, a_{23}, \dots, a_{2n}) \wedge \dots \wedge (0, 0, \dots, 1)) \\
 &= \mathcal{B}^{-1}(f(\mathcal{B}(a_{11}, a_{12}, \dots, a_{1n}))) \wedge \dots \wedge \mathcal{B}^{-1}(f(\mathcal{B}(0, 0, \dots, 1))) \\
 &= \xi_{1j}(0, \dots, a_{1j}, \dots, a_{1n}) \wedge \dots \wedge \xi_{nj}(0, \dots, a_{nj}, \dots, a_{nn})
 \end{aligned}$$

(Einstein notation used for  $j$ ) We actually done the thing like

$$\begin{vmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ 0 & a_{22} \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & 0 \cdots & a_{nn} \end{vmatrix}$$

compare the result with

$$\det(f)(v_1 \wedge \dots \wedge v_n) = \xi_{1j}(v_1) \wedge \dots \wedge \xi_{nj}(v_n)$$

We could find that

$$\det(A_f) = \det(f)$$

Then we got

$$\det(A) = \prod_{i=1}^n a_{ii}$$

**47.5 Prop**

If one column of  $A$  can be expressed as a linear combination of other columns of  $A$ , then

$$\det(A) = 0$$

The columns are images of  $\{e_1, \dots, e_n\}$ , means that  $A(e_1), \dots, A(e_n)$  are linearly dependent. Then  $A$  is not an isomorphism, thus  $\det(A) = 0$ . If we exchange two columns of  $A$ , then  $\det(A)$  changes sign.

### 47.6 Prop

Let  $(a_{ij})$  be a matrix of dimension  $n \times n$ . Then

$$\det(A) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

#### Proof

Let  $\{v_1, \dots, v_n\}$  be the columns of  $A$ ,  $v_i = A(e_i)$

$$\begin{aligned} \det(A)(e_1 \wedge \dots \wedge e_n) &= \sigma_1 \wedge \dots \wedge \sigma_n \\ &= \left( \sum_i a_{i1} e_i \right) \wedge \dots \wedge \left( \sum_i a_{in} e_i \right) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \prod_i a_{i\sigma(i)} \cdot e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(n)} \\ &= \left( \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i\sigma(i)} \right) e_1 \wedge \dots \wedge e_n \end{aligned}$$

### 47.7 Corollary

$$\det A = \det A^T$$

#### Proof

$$A^T = (\alpha_{ij}), A = (a_{ij}) \quad \forall i, j \quad a_{ij} = \alpha_{ji}$$

$$\begin{aligned} \det A^T &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n \alpha_{i\sigma(i)} \\ &= \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i)i} \\ &\stackrel{j=\sigma(i)}{=} \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma^{-1}) \prod_{i=1}^n a_{j\sigma^{-1}(j)} \\ &\stackrel{\text{sgn}(\sigma)=\text{sgn}(\sigma^{-1})}{=} \det A \end{aligned}$$

### 47.8 Prop

If you fix some basis on  $V$  and  $W$ , then  $A_f$  is the matrix associated to  $f^T$  is  $A_f^T$

## 47.9 ?

Fix  $A$  of dimension of  $n \times n$ . Apply Gauss reduction and we get  $A'$  a upper-triangle.

By the properties listed above

$$|\det A| = |\det A'|$$

But on  $A'$  the det is just the product of elements on then diagonal

Second method to compare the determinant is to use Gauss reduction and keep track of the row/column exchanges.

## 47.10 Def

Fix  $A = (a_{ij})(i, j) \in \{1, \dots, n\}^2$ . Denote with  $A_{[i,j]}$  the matrix obtained removing the  $i^{th}$  row and  $j^{th}$  column of  $A$ .

## 47.11 Laplace expansion of the determinant

Let  $A = (a_{ij})$  then

$$\begin{aligned} \det A &= \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{[i,j]} \\ &= \sum_{i=1}^n (-1)^{i+j} a_{ij} \det A_{[i,j]} \end{aligned}$$

## Proof

TEDIOUS

$$\begin{array}{ccc} K^n & \xrightarrow{A} & K^n \\ t_j \uparrow & & \downarrow p_i \\ K^{n-1} & \xrightarrow{A_{[i,j]}} & K^{n-1} \end{array}$$

$\{e'_1, \dots, e'_n\}$  is a standard basis of  $K^n$   
 $\{e_1, \dots, e_n\}$  is a standard basis of  $K^{n-1}$   $p_i$  is the mapping that forgets about the  $i$ -th row.

$$p_i = (x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, \widehat{x_i}, \dots, x_n) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

$$\tau_j(e_i) = \begin{cases} e'_i & \text{if } i < j \\ e_i & \text{if } i \geq j \end{cases}$$

You can check that the above diagram is commutative. Now take  $\bigwedge^{n-1}$  of the diagram

$$\begin{array}{ccc} \bigwedge^{n-1} K^n & \xrightarrow{\bigwedge^{n-1} A} & \bigwedge^{n-1} K^n \\ \bigwedge^{n-1} t_j \uparrow & & \bigwedge^{n-1} p_i \downarrow \\ \det(K^{n-1}) & \xrightarrow{\det(A_{[i,j]})} & \det(K^{n-1}) \end{array}$$

$$\begin{aligned} \det(A)(e'_1, \dots, e'_n) &= (-1)^{i-1} \det(A)(e'_i \wedge e'_1 \wedge \dots \wedge \widehat{e}_i \wedge \dots \wedge e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge A(e'_1) \wedge \dots \wedge A(\widehat{e}_i) \wedge \dots \wedge A(e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \bigwedge^{n-1} A(e'_1, \dots, \widehat{e}_i, \dots, e'_n) = (*) \end{aligned}$$

Let

$$\begin{aligned} \pi_j : K^n &\rightarrow K^n \\ (x_1, \dots, x_n) &\mapsto (0, \dots, x_j, \dots, 0) \end{aligned}$$

Then

$$A = \sum_i (\pi_j \circ A)$$

It means that

$$\begin{aligned} (*) &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A)(e'_1, \dots, \widehat{e}_i, \dots, e'_n) \\ &= (-1)^{i-1} A(e'_i) \wedge \sum_j \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \dots, e_{n-1}) \\ &= \sum_{k,j} \left( (-1)^{i-1} a_{kj} e'_k \wedge \bigwedge^{n-1} (\pi_j \circ A \circ \tau_i)(e_1, \dots, e_{n-1}) \right) = (**) \end{aligned}$$

But  $\pi_j(\cdot)$  is always collinear of  $e_j$ , so when  $k = j$ , the element in the sum is zero. We can remove the items that  $k = j$

$$\begin{aligned} \rho_k &:= \tau_k \circ p_k : K^n \rightarrow K^n \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n) \end{aligned}$$

$$\pi_k = id_{K^n} - \rho_k \text{ and } \sum_{j \neq k} \pi_j = \rho_k$$

Then

$$\begin{aligned} (**) &= \sum_k (-1)^{i-1} a_k i e'_k \wedge \bigwedge^{n-1} \tau_k \circ \bigwedge^{n-1} (p_k \circ A \circ \tau_i)(e_1 \wedge \dots \wedge e_{n-1}) \\ &= (***) \end{aligned}$$

But by the diagram

$$\bigwedge_{k=1}^{n-1} (p_k \circ A \circ \tau_i) = \det A_{[i,k]}$$

$$\bigwedge_{k=1}^{n-1} \tau_k(e_1 \wedge \cdots \wedge e_{n-1}) = e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n$$

Thus

$$\begin{aligned} (* * *) &= \sum_k (-1)^{i-1} a_{ki} \det(A_{[k,i]})(e'_k \wedge e'_1 \wedge \cdots \wedge \widehat{e_k} \wedge \cdots \wedge e'_n) \\ &= \sum_k (-1)^{i+k} a_{ki} \det(A[k,i]) e'_1 \wedge \cdots \wedge e'_n \end{aligned}$$





## Chapter 48

# The Structure of Linear Mappings

### 48.1 Theorem

Let  $f : V \rightarrow W$  be a linear mapping between vct spaces of finite and same dim. Then:

- 1 there exists decomposition  $V = V_0 \oplus V_1$  and  $W = W_1 \oplus W_2$  such that  $V_0 = \ker f$  and  $f$  includes an isomorphism between  $V_1$  and  $W_1$  (namely  $f|_{V_1}$ )
- 2 There exists basis in  $V$  and  $W$  s.t. the associated matrix  $A_f = a_{ij}$  satisfies  $\forall 1 \leq i \leq r, \exists r \leq n$  have  $a_{ii} = 1$  and have  $a_{ij} = 0$  elsewhere
- 3 Let  $A$  be a  $m \times n$  matrix Then there exists two square matrices (with  $\det \neq 0$ )  $B$  and  $C$  of dim  $m \times m$  and  $n \times n$  and a num  $r \leq \min(m, n)$  s.t.  $BAC$  has the form in (2) Moreover the number  $r$  is unique  $r = \text{rank}(A)$

### 48.2 Def

Let  $F : V \rightarrow V$  be a linear mapping. A subspace  $V_0 \subseteq V$  is said to be an invariant subspace of  $F$  if  $F(V_0) \subseteq V_0$

### 48.3 Def

A linear mapping  $f : V \rightarrow V$  (finite dim) is diagonalizable if the following equivalent conditions are satisfied

- 1  $V$  decomposes as a direct sum of one-dimensional invariant subspace of  $f$
- 2 There exists a basis of  $V$ , in which the matrix  $A_f$  is diagonal.

**Proof of equivalence**

$2 \Rightarrow 1$  Assume that in the base  $\{v_1, \dots, v_n\}$ , we have  $A_f = \begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$  by the

familiar diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \uparrow b & & \uparrow b \\ K^n & \xrightarrow{A_f} & K^n \end{array}$$

$$f(v_i) = b \circ A_f(e_i) = b(\lambda_i e_i) = \lambda_i v_i \in \langle v_i \rangle$$

So

$$V = \langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle$$

$1 \Rightarrow 2$  Assume that  $V = \langle v_1 \rangle \oplus \dots \oplus \langle v_n \rangle$ , where  $f(\langle v_i \rangle) \subseteq \langle v_i \rangle$ , then  $\{v_1, \dots, v_n\}$  forms a basis of  $V$

Consider the previous diagram

$$A(e_1) = b^{-1} \circ f \circ b(e_i) = b^{-1}(f(v_i)) = b^{-1}(\lambda_i v_i) = \lambda_i e_i$$

**48.3.1 Example**

Take

$$A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad A = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

$A$  is not diagonalizable.

**48.4 Def**

Let  $L$  be a one-dimensional invariant subspace of  $f : V \rightarrow V$ . Then  $f|_L = x \mapsto \lambda x$  with a scalar  $\lambda \in K$ . Such  $\lambda$  is called eigenvalue of  $f$ . A non-zero vector  $v \in V$  is called an eigenvector of  $V$  if  $\langle v \rangle$  is an invariant subspace of  $f$

**48.5 Remark**

$$\{\text{eigenvectors}\} \longrightarrow \{\text{Set of invariant subspaces of dim 1}\} \longrightarrow K$$

$$v \langle v \rangle \longmapsto \text{eigenvalue}$$

This mapping is generally NOT injective. If  $V$  is an eigenvector, then  $\mu v$  is also an eigenvector,  $\forall \mu \in K$

## 48.6 Remark/exercise

Assume that  $f$  is diagonalizable and  $A_f$  is a diagonal matrix that represents  $f$ . Then  $A_f$  is unique up to permutation of the columns in the diagonal.

$$V = \langle v_1 \rangle \oplus \cdots \oplus \langle v_n \rangle = \langle v_{\sigma(1)} \rangle \oplus \cdots \oplus \langle v_{\sigma(n)} \rangle \quad \sigma \in \mathfrak{S}_n$$

## 48.7 Def

V a vector space over  $K$   $\dim(V) = n$ ,  $f \in \mathcal{L}(V; V)$  let  $A_f$  be an associated matrix (in any basis) the mapping

$$\begin{aligned} P : K &\rightarrow K \\ t &\mapsto \det(tI_n - A_f) \end{aligned}$$

This is a polynomial in  $K[t]$  (with degree  $n$ )

## 48.8 Lemma

$P(t)$  is a monic polynomial of degree  $n$

### Proof

$$P(t) = \det(tI_n - A_f) = \sum_{\sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^n (t\delta_{i\sigma(i)} - A_{i\sigma(i)})$$

The only item giving  $t^n$  is when  $\sigma = id$

## 48.9 Theorem

Use the notations introduced before

- 1  $P(t)$  doesn't depend on  $A_f$  (if you change basis,  $P(t)$  does not change)
- 2 Any eigenvalue of  $f$  is a root of  $P(t)$ . Conversely any  $K$ -root of  $P(t)$  is an eigenvalue of  $f$

### Proof

- 1 Put  $A = A_f$  and  $A'$  be another representation of  $f$ . Then  $A' = B^{-1}AB$  where  $B$  invertible  $n \times n$  matrix.

$$\begin{aligned} \det(tI_n - A') &= \det(tI_n - B^{-1}AB) \\ &= \det(B^{-1}(tI_n)B - B^{-1}AB) \\ &= \det(B^{-1}(tI_n - A)B) \\ &= \det(tI_n - A) \end{aligned}$$

2 Let  $\lambda \in K$  be a  $K$ -root of  $P(t)$ , then

$$\det(\lambda I_n - A_f) = 0 = P(\lambda)$$

$\lambda I_n - A_f$  is not invertible,  $\exists v \neq 0 \in \ker(\lambda I_n - A_f)$  s.t.

$$A_f(\sigma) = \lambda \sigma$$

then  $\sigma$  is an eigenvector

Vice versa if  $\sigma \neq 0, f(\sigma) = \lambda \sigma, \sigma \in \ker(\lambda I_n - A_f), \det(\lambda I_n - A_f) = 0 = P(\lambda)$

### 48.10 Def

The polynomial  $P(t)$  will be denoted by  $P_f(t)$ . It's called the characteristic polynomial of  $f$

### 48.11 Corollary

If  $P_f(t)$  splits with no repeated roots, then  $f$  is diagonalizable.

#### Proof

Natural

$$\langle \sigma_1 \rangle, \dots, \langle \sigma_n \rangle$$

are all different then

$$V = \langle \sigma_1 \rangle \oplus \dots \oplus \langle \sigma_n \rangle$$

### 48.12 Remark

The inverse version does not hold.

### 48.13 Def: Jordan block

A matrix of form

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & & & \\ 0 & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \lambda \end{pmatrix} \in M_{r \times r}(K) \quad r \geq 1$$

is called a Jordan block (element  $\lambda \in K$  is  $J_1(\lambda)$ )

### 48.14 Def: Jordan matrix

A Jordan matrix is a matrix of form

$$J = \begin{pmatrix} J_{r_1}(\lambda_1) & & \cdots & \\ & J_{r_2}(\lambda_2) & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix}$$

### 48.15 Example

Let  $V_n(\lambda)$  be the vector space of complex functions:

$$F(x) := e^{\lambda x} f(x)$$

where  $\lambda \in \mathbb{C}$ ,  $f \in \mathbb{C}[x] \leq n-1$

Verify that  $V_n(\lambda)$  is a vector space of dim  $n$

$$\begin{aligned} \frac{d}{dx}(e^{\lambda x} f(x)) &= \lambda e^{\lambda x} f(x) + e^{\lambda x} f'(x) \\ &= e^{\lambda x} (\lambda f(x) + f'(x)) \end{aligned}$$

$\frac{d}{dx} \in \mathcal{L}(V_n(\lambda); V_n(\lambda))$  Consider

$$v_{i+1} = \frac{x^i}{i!} e^{\lambda x}$$

Show that  $\{v_0, \dots, v_{n-1}\}$  forms a basis of  $V_n(\lambda)$

$$\begin{aligned} \frac{d}{dx} v_{i+1} &= \lambda v_{i+1} + \frac{x^{i-1}}{(i-1)!} e^{\lambda x} \\ &= \lambda v_{i+1} + v_i \end{aligned}$$

Then

$$A_{\frac{d}{dx}} = \begin{pmatrix} \lambda & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \lambda \end{pmatrix} = (J_n(\lambda))^T$$

### 48.16 Def

Let  $a_0 + a_1 t + \dots + a_n t^n = Q(t) \in K[t]$ , then for  $f \in \mathcal{L}(V; V)$  we define

$$Q(f) := a_0 \text{id}_V + a_1 f + a_2 f^{\circ 2} + \dots + a_n f^{\circ n}$$

Remark From now on we write

$$f^{\circ k} = f^k$$

these are operations in  $\mathcal{L}(V; V)$ ,  $+$ ,  $\circ$

we say that  $Q$  annihilates  $f$  if  $Q(f) = 0$

**48.17 Prop**

Let  $f \in \mathcal{L}(V; V)$ . There exists a polynomial  $Q \in K[t] \setminus \{0\}$  that annihilates  $f$  (i.e.  $Q(f) = 0$ )

**Proof**

$$\dim(\mathcal{L}(V; V)) = n^2$$

Hence the mapping  $\underbrace{id_V, f^2, \dots, f^{n^2}}_{n^2+1 \text{ mappings}} \in \mathcal{L}(V; V)$  are linear dependent. There exists a non-trivial linear comb:

$$\lambda_0 id_V + \lambda_1 f + \dots + \lambda_{n^2} f^{n^2} = 0$$

So, take

$$Q(t) = \lambda_0 + \lambda_1 t + \dots + \lambda_{n^2} t^{n^2}$$

This show that  $Q \neq 0$  and  $Q(f) = 0$

**Remark**

The proof of this proposition also gives the degree of a polynomial that annihilates ( $\leq n^2$ )

**48.18 Def**

Let  $m(t) \in K[t] \setminus \{0\}$  be a monic polynomial of minimal degree that annihilates  $f \in \mathcal{L}(V; V)$ . Then  $m(t)$  is called minimal polynomial of  $f$

And by prop above (48.17),  $m(t)$  exists.

**48.19 Prop**

If  $m(t)$  is minimal polynomial of  $f$ , then  $m(t)$  is unique.

**Proof**

Assume that  $m_1(t)$  is another minimal polynomial of  $f$ . Then  $m - m_1(t) \in K[t]$

$$(m - m_1)(f) = m(f) - m_1(f) = 0 - 0 = 0$$

Now  $m$  and  $n$  are both monic, so

$$\deg(m - m_1) < \deg(m) = \deg(m_1)$$

$m - m_1$  is a polynomial of  $\deg < \deg(m)$  that annihilates  $f$ , thus

$$m - m_1 = 0 \in K[t]$$

## Notation

From now we denote the minimal polynomial of  $f$  by  $m_f$

## Question

$f \in \mathcal{L}(V; V)$  we have  $P_f, m_f \in K[t]$ .

What is the relationship between  $P_f$  and  $m_f$ ?

## 48.20 Prop

Let  $Q \in K[t] \setminus \{0\}$  be a polynomial that annihilates  $f$ . Then  $m_f \mid Q$

## Proof

Let

$$Q(t) = m_f(t) \cdot s(t) + \mathfrak{z}(t)$$

such that  $\deg(\mathfrak{z}) < \deg(m_f)$ . So

$$0 = Q(f) = m_f(f)s(f) + \mathfrak{z}(f) = 0 + \mathfrak{z}(f) \Rightarrow \mathfrak{z}(f) = 0$$

But since  $m_f$  is the minimal polynomial of  $f$ , then

$$\mathfrak{z}(t) = 0$$

## 48.21 Def

Let  $A$  be a matrix of dim  $n \times n$  and

$$M_{ij} := (-1)^{i+j} \det(A_{[i,j]}) \quad \forall (i, j) \in \{1, \dots, n\}^2$$

In this expression

$$\det(A_{[i,j]})$$

is called the  $(i, j)$ -monic of  $A$ .

Then we define

$$\text{Adj}(A) := (M_{ij})^T$$

called adjugate matrix of  $A$

## 48.22 Prop

$$\text{Adj}(A) \cdot A = A \cdot \text{Adj}(A) = \det(A) \cdot I_n$$

**Proof**

use Laplace expansion.

**48.23 Theorem: Cayley-Hamilton Theorem**

The characteristic polynomial  $P_f$  annihilates  $f$

Consequence:  $m_f \mid P_f$

**Proof**

Let  $A = A_f$  any matrix that represents  $f$ . Consider

$$B := \text{Adj}(tI_n - A)$$

$B$  is a matrix with coefficient in  $K[t]$  ( $B \in M_{n \times n}(K[t])$ )

Then

$$(tI_n - A) \cdot B = \det(tI_n - A) \cdot I_n = P_f(t) \cdot I_n$$

We can decompose  $B$  in the following way

$$B = \sum_{i=0}^{n-1} t^i B_i \quad B_i \in M_{n \times n}(K)$$

We have at most  $n-1$ , because the coefficient of  $B$  have degree at most  $n-1$   
(Any entry  $\text{Adj}$  is a det of a matrix of dim  $(n-1) \times (n-1)$ )

$$\begin{aligned} P_f(t)I_n &= (tI_n - A) \cdot \sum_{i=0}^{n-1} t^i B_i \\ &= \left( \sum_{i=0}^{n-1} tI_n \cdot t^i B_i \right) - \left( \sum_{i=0}^{n-1} A \cdot t^i B_i \right) \\ &= \sum_{i=0}^{n-1} t^{i+1} B_i - \sum_{i=0}^{n-1} A \cdot t^i B_i \\ &= t^n B_{n-1} + \sum_{i=0}^{n-1} t^i (B_{i-1} - AB_i) - AB_0 \end{aligned}$$

Recall that  $P_f(t) \cdot I_n = t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_1 t I_n + c_0 I_n$

$$\begin{aligned} &t^n I_n + c_{n-1} t^{n-1} I_n + \cdots + c_0 I_n \\ &= \cdots \\ &= t^n B_{n-1} + \sum_{i=1}^{n-1} t^i (B_{i-1} - AB_i) - AB_0 \end{aligned}$$



Then we can compare the coefficients:

$$\begin{aligned} B_{n-1} &= I_n \\ B_{i-1} - AB_i &= c_i I_n \quad 1 \leq i \leq n-1 \\ -AB_0 &= c_0 I_n \end{aligned}$$

Multiply by  $A^i$   $0 \leq i \leq n$

$$A^n B_{n-1} + \sum_{i=1}^{n-1} (A^i B_{n-1} - A^{i+1} B) - AB_0 = A^n + c_{n-1} A^{n-1} + \cdots + c_1 A + c_0 I_n$$

Now the LHS we have a telescopic sum and got

$$0 = P_f(A) \Leftrightarrow 0 = P_f(f)$$

## 48.24 Example

(a)  $m_f$  and  $P_f$  are in general different. let  $f = id_V$ , ( $dim V = n$ )

$$P_f(t) = (t-1)^n \quad m_f(t) = t-1$$

(b) Assume  $f : V \rightarrow V$  ( $dim V = \mathfrak{z}$ ) and  $A_f = J_{\mathfrak{z}}(\lambda)$ . Then

$$P_f(t) = (t-\lambda)^{\mathfrak{z}}$$

Moreover

$$J_{\mathfrak{z}}(\lambda) = \lambda I_{\mathfrak{z}} + J_{\mathfrak{z}}(0)$$

and

$$J_{\mathfrak{z}}(0)^k = \begin{pmatrix} \overbrace{0 \cdots 0}^{k+1} & & \\ & \ddots & \\ & & 1 \\ & & \vdots \\ & & 0 \end{pmatrix}$$

if  $k \geq \mathfrak{z}$ ,  $J_{\mathfrak{z}}(0)^k = 0$

$$(J_{\mathfrak{z}}(\lambda) - \lambda I_n)^k = (\lambda I_{\mathfrak{z}} - J_{\mathfrak{z}}(0) - \lambda I_{\mathfrak{z}})^k = J_{\mathfrak{z}}(0)^k \neq 0$$

if  $0 \leq k \leq \mathfrak{z}-1$

We know that  $m_f \mid (t-\lambda)^{\mathfrak{z}}$  (by Cayley-Hamilton),  $m_f$  must be of the type

$$m_f = (t-\lambda)^k$$

But the only possibility is  $k = \mathfrak{z}$ , thus

$$m_f = P_f$$

### 48.25 Theorem

Let  $f \in \mathcal{L}(V; V)$  when  $V$  is a vector space of dim  $n$ , over an algebraically closed field.

Then

- (1)  $f$  can be represented by a Jordan matrix
- (2) This above matrix is unique up to permutation of the Jordan blocks

(Note that a field  $K$  is algebraically closed if any non-zero polynomial has a root in  $K$ )

### 48.26 Def

Let  $f \in \mathcal{L}(V; V)$  and let  $\lambda \in K$ . A vector  $w \in V \setminus \{0\}$  is called a root vector of  $f$  corresponding to  $\lambda$ , if there exists  $\tau \in \mathbb{N}$  s.t.

$$(f - \lambda id_V)^\tau(w) = 0$$

#### Remark

Eigenvector are root vectors (corresponding to their eigenvalues) take  $\tau = 1$

#### Remark

Let  $J_\tau(\lambda)$  be a Jordan block. Then any  $\sigma \in V$  is a root vector of  $f$  corresponding to  $\lambda$ . In fact:

$$(J_\tau(\lambda) - \lambda I_n)^m = 0 \quad \text{if } m \geq \tau$$

### 48.27 Prop

Let  $K$  be an algebraically closed field. Let  $\lambda_1, \dots, \lambda_k$  be all of distinct eigenvalues of  $f$  ( $k \geq 1$ ), then

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

#### Proof

Since  $K$  is algebraically closed, then

$$P_f(t) = \prod_{i=1}^k (t - \lambda_i)^{r_i} \in K[t]$$

Consider

$$F_i(t) := P_f(t) \cdot (t - \lambda_i)^{-r_i} \in K[t]$$

Then we define

$$f_i := F_i(f) \in \mathcal{L}(V; V), V_i = \text{Im} f_i$$

### Setp 1

We want to prove that

$$(f - \lambda_i \text{Id}_V)^{\circ r_i}(V_i) = 0 \Leftrightarrow V_i \subseteq V(\lambda_i)$$

which got from

$$(f - \lambda_i \text{Id}_V)^{\circ r_i} \circ (f_i) = (t - \lambda_i)^{r_i}(f) \circ F_i(f) = P_f(f) = 0$$

### Step 2

We want to prove that

$$V = \bigoplus_{i=1}^k V_i$$

Since the polynomials  $F_i(t)$  are coprime, then

$$\exists G_i(t) \in K[t] \text{ s.t. } \sum_{i=1}^k F_i(t)G_i(t) = 1$$

Let  $f$  substitute for  $t$

$$\sum_{i=1}^k F_i(f)G_i(f) = \text{Id}$$

take  $v \in V$

$$\sum_{i=1}^k f_i \circ G_i(f)(v) = v$$

$$\begin{array}{ccccccc}
 & & & \bigoplus_{i=1}^k V_i & & & \\
 & & \nearrow & & \nwarrow & & \\
 & V_1 & & & & V_{k-1} & V_k \\
 & \uparrow & \nearrow & & \nwarrow & \uparrow & \uparrow \\
 & f_1 \circ G_1(f) & f_2 \circ G_2(f) & & f_{k-1} \circ G_{k-1}(f) & f_k \circ G_k(f) & \\
 & \uparrow & \uparrow & & \uparrow & \uparrow & \\
 V & = & V & = & \dots & = & V & = & V
 \end{array}$$

$i$  is the inclusion mapping.

**Step 3**

We want to show that

$$V_i \cap \left( \sum_{j \neq i} V_j \right) = \{0\}$$

Let  $v$  be a vector in this intersection. Then by calculation,

$$(f - \lambda_i)^{r_i}(v) = 0$$

$$F_i(f)(v) = \prod_{j \neq i} (f - \lambda_j) Id^{or_i}(v) = 0$$

Now  $(t - \lambda_i)^{r_i}$  and  $F_i(t)$  are coprime. Then there exists  $G_1(t)$  and  $G_2(t)$  such that:

$$G_1(t)(t - \lambda_i)^{t_i} + G_2(t)F_i(t) = 1$$

substitute  $f$  instead of  $t$  by

$$G_1(f) \circ (f - \lambda_i Id_V)^{or_i} + G_2(f) \circ F_i(f) = Id_V$$

Then apply to  $v = \sum_{j \neq i} v_j, v_j \in V_j$

$$G_1(f) \circ (f - \lambda_i Id_V)^{or_i}(v) + G_2(f) \circ F_i(f)(v) = v = 0$$

**Step 4**

We want to show that

$$V_i = V(\lambda_i)$$

By step 1 we get

$$V_i \subseteq V(\lambda_i)$$

Take  $v \in V(\lambda_i)$ , write it as

$$v = v'(\in V(\lambda_i)) + v''(\in \bigotimes_{j \neq i} V_j)$$

By step 3,

$$v'' = v - v' \in V(\lambda_i)$$

Use same trick, substitute  $f$  for  $t$  and calculate in  $v''$

$$v'' = 0$$

**48.28 Def**

Let  $f \in \mathcal{L}(V; V)$ . Then  $f$  is said to be nilpotent if there exists  $t \in \mathbb{N}$  that  $f^t = 0$

**48.29 Lemma**

Let  $f$  be a nilpotent mapping, then

$$\text{Ker}(f) = \{\text{set of eigenvectors of } f\} \cup \{0\}$$

**Proof**

Let  $v \in \text{Ker}(f)$  then  $v$  is an eigenvector with eigenvalue  $= 0$

Let  $v$  be an eigenvector, then  $\forall m \geq r$

$$0 = f^m(v) = f^{m-1}(f(v)) = f^{m-1}(\lambda v) = \lambda^m v \Rightarrow \lambda^m = 0 \Rightarrow \lambda = 0$$

**48.30 Lemma**

Let  $f$  be a nilpotent mapping, then  $\text{Ker}(f) \neq \{0\}$

**Proof**

Let  $\tau$  be the minimal integer s.t.  $f^\tau = 0$  then

$$f^{\tau-1}(V) \subseteq \text{Ker}(f)$$

but  $f^{\tau-1}(V) \neq \{0\}$  because of the minimality of  $\tau$

**Remark**

Another way to prove is that  $Q(t) = t^\tau$  annihilates  $f$ . So  $m_p = t^{\tau'}, \tau' \leq \tau$   
 Note that 0 is a root of  $m_f$ , by Cayley-Hamilton theorem, 0 is an eigenvalue  
 $f(x) = 0 \cdot x = 0$  for some  $x \neq 0$

**48.31 Jordan matrix of form  $J_\tau(0)$** 

Recall that

$$J_\tau(0)^k = 0 \text{ if } k \geq \tau$$

Then the Jordan matrix

$$\begin{pmatrix} J_{\tau_1}(0) & & \\ & J_{\tau_2}(0) & \\ & & \ddots \end{pmatrix}$$

Are nilpotent mappings since each block is nilpotent. Take one block

$$J_\tau(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} = \begin{cases} e_1 \mapsto 0 \\ e_2 \mapsto 1 \\ \vdots \\ e_\tau \mapsto e_{\tau-1} \end{cases}$$

We represent the action of a Jordan block on a basis as the following diagram

$$\underbrace{e_\tau \rightarrow e_{\tau-1} \rightarrow e_{\tau-2} \rightarrow \cdots \rightarrow e_1 \rightarrow 0}_{\text{lenth of the block}(\tau)}$$

$e_1$  is the one which mapped to 0 (thus an eigenvector)

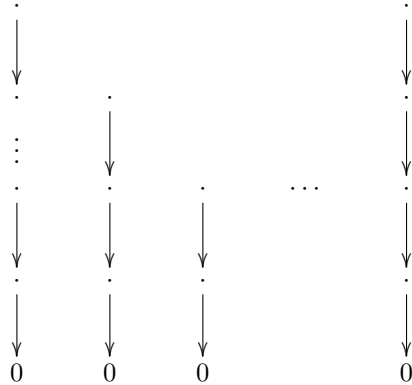
Given  $f \in \mathcal{L}(V; V)$  if we find a basis on which  $f$  acts as in the previous diagram. Then we have found a Jordan basis made of blocks of the type " $J_\tau(0)$ "

### 48.32 Theorem

Let  $f \in \mathcal{L}(V; V)$  be a nilpotent mapping, then there exists a Jordan basis for  $f$  that gives a Jordan matrix made of blocks of the type  $J_\tau(0)$

#### Proof

We need to find a basis that induces a diagram of the type  $\mathcal{D}$  :(dots in the diagram are basis)



(Last line of dots naturally be eigenvectors, others on chain are called general eigenvectors, the num of (general) eigenvectors is the scale of the Jordan block of this eigenvalue)

We work by induction on  $\dim(V)$ . If  $\dim(V) = 1$ , then

$$f = \mu(\cdot), f^\tau = 0 \quad \mu^\tau v = 0 \quad \forall v \Rightarrow \mu = 0$$

But  $0 = J_1(0)$ . Assume that the theorem is true for  $\dim(V) < n$  Let

$$V_0 = \text{Ket } f = \{\text{the set of eigenvalues}\} \cup \{0\}$$

Since  $f$  is nilpotent

$$\dim(V_0) \geq 1$$

. Therefore

$$\dim(V/V_0) < n$$

So define the following mapping

$$\begin{aligned} \bar{f}: V/V_0 &\rightarrow V/V_0 \\ \bar{\sigma} = V_0 + \sigma &\mapsto V_0 + f(v) = \overline{f(v)} \\ \bar{f} \cdot \bar{\sigma} &\mapsto \overline{f(v)} \end{aligned}$$

is nilpotent We use the induction hypothesis

We have a Jordan basis for  $\bar{f}$ , so we have elements  $\bar{\sigma}_1, \dots, \bar{\sigma}_m \in V/V_0$  that give a diagram  $\mathcal{D}$ :

Now left  $\bar{\sigma}_i$  to some element  $\sigma_i \in V$  choose  $\sigma_i \in V$  s.t  $\sigma + V_0 = \bar{\sigma}_i$  Now start applying  $f$  to these elements  $\sigma_i \neq 0$

$$v_i \rightarrow f(v_i) \rightarrow \dots \rightarrow f^{b_i-1}(v_i) \rightarrow f^{b_i}(v_i)$$

When  $b_i$  is the first integer such that

$$\bar{f}^{b_i}(\bar{v}_i) = 0$$

This means that

$$f^{b_i}(v_i) \in V_0$$

hence  $f^{b_i}(\sigma_i)$  is an eigenvalue for ? Consider now the vector subspace generated by  $f^{b_1}(v_1), f^{b_2}(v_2), \dots, f^{b_m}(v_m)$

$$\langle f^{b_1}(v_1), \dots, f^{b_m}(v_m) \rangle \subseteq V_0$$

Extract a basis and complete to a basis of  $V_0$ . The new vectors are denoted by  $u_1, \dots, u_t$

We want to prove that the elements of  $\mathcal{D}$  form a basis of  $V$

1

The elements of  $\mathcal{D}$  generate  $V$  let  $\sigma \in V$

$$\bar{\sigma} = \sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} \bar{f}^j(\bar{v}_i)$$

Now I use the properties of  $\bar{f}$

$$\begin{aligned} \bar{f}(\bar{v}_i) &= \overline{f(v_i)} \\ \bar{f}(\bar{f}(v)) &= \overline{f(f(v))} \end{aligned}$$

then

$$\bar{\sigma} = \overline{\sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} f^j(v_i)}$$

which gives

$$\sigma - \sum_{i=1}^m \sum_{j=0}^{b_i-1} a_{ij} f^j(v_i) \in V_0$$

this finishes. We know that

$$V_0 = \langle f^{b_1}(v_1), \dots, f^{b_m}(v_m), u_1, \dots, u_t \rangle$$

## 2

We need to prove that the elements of  $\mathcal{D}$  are linearly independent

a We show that the elements of the bottom row are linearly independent

$$\sum_{i=1}^m a_i f^{b_i}(v_i) + \sum_{i=1}^t c_i u_t = 0$$

This is a non-trivial linear comb.

The first observation is that  $b_i = 0$ . Because if  $b_j \neq 0$

$$u_j = \frac{\sum_{i=1}^m a_i f^{b_i}(v_i)}{\sum_{i=1}^t c_i}$$

But  $u_1, \dots, u_t$  were an extension of a basis. So

$$0 = \sum_{i=1}^m a_i f^{b_i}(v_i) = f\left(\sum_{i=1}^m a_i f^{b_i-1}(v_i)\right) \Rightarrow \left(\sum_{i=1}^m a_i f^{b_i-1}(v_i)\right) \in V_0$$

It means that

$$\sum_{i=1}^m a_i \bar{f}^{b_i}(v_i) = 0 \Rightarrow a_i = 0 \forall i$$

b If there is a non-trivial linear comb that equals to 0. For elements of  $\mathcal{D}$ .

We can write it as linear comb of elements of the last row

$$f\left(\sum_{i=1}^m \sum_{j=1}^{b_i} a_{ij} f^j(v_i)\right) + \sum_{i=1}^t c_i u_t = 0$$

By applying  $f$  many times we get a linear comb of elements of the last row.

By point a, finished.

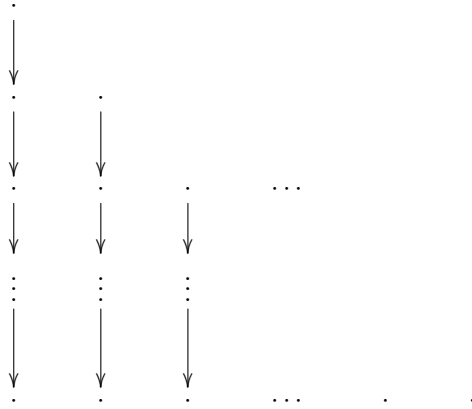


### 48.33 Prop

The Jordan matrix that represents a nilpotent mapping  $f \in \mathcal{L}(V)$  is unique to permutations of the blocks.

#### Proof

Recall that a Jordan basis of  $f$  is given by diagram  $\mathcal{D}$  of the type



These columns are ordered in a decreasing height on them, recalling that the height of a column is the dimension of a Jordan block. In the proof of existence of Jordan basis, the diagram was constructed as a lift of  $\mathcal{D}$

Focus on the last row. The elements of last row generates  $V_0 = \ker f$  and moreover, they are linearly independent. Then the length of the last row is exactly  $\dim(V_0)$ , which is independent of the choice of basis.

Viewing the penultimate row, this corresponds to the last row of the diagram  $\overline{\mathcal{D}}$ . So if we work by induction, we done the proof:

All the rows have length independent of the choice of basis.

#### Remark

$$\ker(f^{\circ 3})/\ker(f^{\circ 2}) \rightarrow \ker(f^{\circ 2})/\ker(f) \rightarrow \ker f = V_0$$

### 48.34 Lemma

Let  $f \in \mathcal{L}(V)$ ,  $\lambda$  be an eigenvalue of  $f$ . Then there exists  $r \in \mathbb{N}$  s.t.

$$\forall v \in V(\lambda) \quad (f - \lambda Id)^{or}(v) = 0$$

**Proof**

Take a basis  $\{v_1, \dots, v_n\}$  of  $V(\lambda)$ . By definition, we have  $(r_1, \dots, r_n)$  such that  $\forall i$   $r_i$  is the least integer that

$$\forall v \in V \quad (f - \lambda Id)^{or}(v) = 0$$

Take  $r = \max\{r_i\}$ , then proved by calculation.

**48.35 Theorem**

Let  $K$  be an algebraically closed field. Let  $f \in \mathcal{L}(V)$ . Then  $f$  admits a Jordan basis (namely there exists a basis s.t.  $A_f$  is a Jordan matrix).

**Proof**

Since  $K$  is algebraically closed, by Prop 48.27

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

where  $\lambda_i$  are distinct eigenvalues of  $f$

Recall that  $V(\lambda_i)$  is the set of root vectors for  $\lambda_i$  and 0

Consider  $f|_{V(\lambda_i)} = g, \lambda_i = \lambda$ . Only need to prove the theorem for  $g$

$$(g - \lambda Id) : V(\lambda) \rightarrow V(\lambda)$$

This function is nilpotent on  $V(\lambda)$  by definition. By lemma 48.34, we have some  $J_{g-\lambda Id}$  made of blocks of the type  $J_{g-\lambda Id}(0)$

Take the matrix and restrict to  $J_r(0)$

$$g - \lambda Id = BJ_r(0)B^{-1}$$

One see that

$$\lambda Id + BJ_r(0)B^{-1} = B\lambda IdB^{-1} + BJ_r(0)B^{-1} = B(\lambda Id + J_r(0))B^{-1}$$

Uniqueness follows the uniqueness of  $J_r(0)$

## Chapter 49

# Jordan Matrix

To find relations between Jordan matrix and diagonal representations

### 49.1 Def

Let  $\lambda$  be an eigenvalue of  $f \in \mathcal{L}(V)$

$$E(\lambda) := \ker(f - \lambda Id)$$

This  $E(\lambda)$  is called the eigenspace of  $\lambda$

$$mult(\lambda)_{geo} = \dim(E(\lambda))$$

is called the geometric multiplicity of  $\lambda$

Moreover

$$mult(\lambda)_{alg} = \max \{k \in \mathbb{N} \mid (t - \lambda)^k \mid P_f(t)\}$$

is called the algebraic multiplicity of  $\lambda$

### 49.2 Prop

Let  $K$  be algebraically closed. Then  $\forall \lambda$  eigenvalues of  $f$

$$mult(\lambda)_{geo} \leq mult(\lambda)_{alg}$$

**Proof**

$$V = \bigoplus_{i=1}^k V(\lambda_i)$$

Take  $\lambda = \lambda_i$ . Let  $J_f$  be the Jordan matrix of  $f$ . Then

$$\det J_f = \det f$$

so

$$P_f(t) = \prod_i (t - \lambda_i)^{\dim(V(\lambda_i))} \Rightarrow \dim(V(\lambda)) = \text{mult}(\lambda)_{alg}$$

### 49.3 Corollary

Let  $K$  be an algebraically closed field. Let  $f \in \mathcal{L}(V)$ .  $f$  is diagonalizable iff

$$\forall \lambda_i \quad \text{mult}(\lambda_i)_{geo} = \text{mult}(\lambda_i)_{alg}$$

# Chapter 50

## Inner Product

### 50.1 Def

Two matrices  $G, G' \in M_{n \times n}(K)$  are said conjugate if  $\exists A \in \mathcal{Q}_{n \times n}(K)$  s.t.  
 $G = G'^T$

#### Exercise

Verify that this is an equivalence relation

### 50.2 Def

Let  $V$   $n$ -dimensional vector space over  $K$  ( $K = \mathbb{R}$  or  $K = \mathbb{C}$ ),  $g \in \mathcal{L}(V, V; K)$  is said a bilinear form. Choose a basis  $\{v_1, \dots, v_n\}$  of  $V$ . The matrix

$$G = (g(v_i, v_j))_{ij} \in M_{n \times n}(K)$$

is called the Gram matrix of  $g$  with respect to  $\{v_1, \dots, v_n\}$

By bilinearity,  $G$  determinant uniquely  $g$

$$x = \sum \alpha_i v_i \rightarrow x = \sum \alpha_i e_i \quad x = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

$x, y \in V$

$$g(x, y) = g\left(\sum x_i v_i, \sum y_j v_j\right) = \sum_{i,j} x_i y_j g(v_i, v_j) = x^T G y$$

On the other hand, given a basis  $\{v_1, \dots, v_n\}$  and  $G \in M_{n \times n}(K)$  the mapping:

$$\begin{aligned} V \times V &\rightarrow K \\ (x, y) &\mapsto x^T G y \end{aligned}$$

this is a bilinear form and the associated Gram matrix is exactly  $G$

Fix a couple  $(V, \{v_1, \dots, v_n\})$  we have defined a bijection.

$$\begin{array}{ccc} \mathcal{L}(V, V; K) & \xrightarrow{\cong} & K \\ g & \mapsto & G \end{array}$$

What happens if  $g$  is fixed but we change basis. We have also  $\{v'_1, \dots, v'_n\}$

$$\begin{array}{ccccc} & & v_i & & \\ & \nearrow & & \nwarrow & \\ e_i & & K^n & \xleftarrow{A} & K^n & \\ & \nwarrow & & \nearrow & \\ & & v'_i & & \\ & \nwarrow & & \nearrow & \\ & & e_i & & \end{array}$$

$$A = b^{-1} \circ (b') \quad (b')^{-1}(x) = x'$$

then  $A$  satisfies

$$Ax' = x$$

so

$$g(x, y) = x^T G y = (Ax')^T G (Ay') = (x')^T (A^T G A)(y')$$

The new Gram matrix with respect to the basis  $\{v'_1, \dots, v'_n\}$  is  $A^T G A$

### 50.3 Prop

There exists a surjection:

$$\mathcal{L}(V, V; K) \rightarrow M_{n \times n}(K) / \sim_{conj}$$

### Proof

Recall

$$\begin{array}{ccccc} \mathcal{L}(V, V; K) & \rightarrow & \mathcal{L}(T_0^2(V); K) & \rightarrow & \mathcal{L}(V; V^\vee) \\ g & \mapsto & g_s & \mapsto & [x \mapsto g_s(x \otimes -)] = \tilde{g} \end{array}$$

### 50.4 Def

Given  $g \in \mathcal{L}(V, V; K)$  we can define several other bilinear mappings:

$$\begin{array}{ccc} g_p : V \times V & \rightarrow & K \\ (x, y) & \mapsto & g(y, x) \end{array}$$

$$\begin{array}{ccc} \overline{g_p} : V \times V & \rightarrow & K \\ (x, y) & \mapsto & \overline{g(y, x)} = \overline{g_p(x, y)} \end{array}$$

If  $K = \mathbb{R}$  then  $g_p = \overline{g_p}$

## 50.5 Def

A bilinear form  $g$  is said

Symmetric if  $g = g_p$

Symplectic (skew-symmetric) if  $g = -g_p$

hermitian if  $g = \overline{g_p}$

(if  $K = \mathbb{R}$  symmetric  $\neq$  hermitian)

### 50.5.1 Example

$$\begin{array}{ccc} K^n \times K^n & \rightarrow & K \\ (x, y) & \mapsto & x^T y \end{array}$$

is symmetric

$$\begin{array}{ccc} K^2 \times K^2 & \rightarrow & K \\ (v_1, v_2) & \mapsto & \det(v_1 \mid v_2) \end{array}$$

is skew-symmetric

$$\begin{array}{ccc} \mathbb{C}^n \times \mathbb{C}^n & \rightarrow & \mathbb{C} \\ (x, y) & \mapsto & x^T \overline{y} \end{array}$$

is hermitian

## 50.6 Def

$g \in \mathcal{L}(V, V; K)$  is an inner product of  $V$ , if  $g$  is either symmetric, symplectic or hermitian.

And  $(V, g)$  is called an inner space.

(note that  $g = -\overline{g_p}$  is complicated)

## 50.7 Def

Let  $(V, g)$  be an inner product space. Two vectors  $v_1, v_2 \in V$  are said orthogonal (with respect to  $g$ ) if  $g(v_1, v_2) = 0$ . Two subspace  $V_1, V_2 \subseteq V$  are orthogonal if  $g(v_1, v_2) = 0 \ \forall v_1 \in V_1, v_2 \in V_2$  ( $g(V_1, V_2) = 0$ )

### Exercise

Show the following

- If  $g$  is symmetric

$$G = G^T$$

- If  $g$  is symplectic

$$G = -G^T$$

- If  $g$  is hermitian

$$G = \overline{G^T}$$

## 50.8 Def

Let  $(V_g)$  be an inner product space the kernel of  $g$

$$\ker(g) := \{v \in V \mid g(v, w) = 0 \ \forall w \in V\}$$

Moreover  $g$  is said non-degenerated if

$$\ker(g) = \{0\}$$

## 50.9 Remark

Note that  $\ker(g) = \ker(\tilde{g})$  when

$$\tilde{G} \in \mathcal{L}(V; V^\vee)$$

$$\tilde{g}_x = 0 \Leftrightarrow g(x, y) = 0 \ \forall y \in V$$

This implies that  $\ker(g)$  is a linear subspace of  $V$



# Chapter 51

## Differential Forms in $\mathbb{R}^n$

### 51.0.1 Notation

$$a|_p := (p, a)$$

### 51.1 Def

Let  $p \in \mathbb{R}^n$  be a fixed point

$$\mathbb{R}_p^n := \{p\} \times \mathbb{R}^n$$

$$(p, a) \in \mathbb{R}_p^n, a \in \mathbb{R}^n$$

$$(p, a) + (p, b) = (p, a + b)$$

$$\alpha(p, a) = (p, \alpha a) \quad \alpha \in \mathbb{R}$$

With these operation  $\mathbb{R}_p^n$  is a vector space, which is called the tangent space of  $\mathbb{R}^n$  at  $p$ .

The dual space is

$$(\mathbb{R}_p^n)^\vee = \{p\} \times (\mathbb{R}^n)^\vee$$

A basis of  $\mathbb{R}_p^n$  is denoted by

$$(e_1|_p, \dots, e_n|_p)$$

$\bigsqcup_p \mathbb{R}_p^n$  is called the tangent bundle of  $\mathbb{R}^n$

We have a projection mapping:

$$\begin{aligned} \bigsqcup_p \mathbb{R}_p^n &\xrightarrow{\pi} \mathbb{R}^n \\ (p, a) &\mapsto p \end{aligned}$$

and

$$\begin{aligned}\mathbb{R}^n \times \mathbb{R}^n &\cong \bigsqcup_p \mathbb{R}_p^n \\ (p, a) &\leftarrow (p, a)\end{aligned}$$

Take  $\{e_1|_p, \dots, e_n|_p\}$  as a basis of  $\mathbb{R}_p^n$ . The dual basis is denoted by

$$\{dx_1|_p, \dots, dx_n|_p\} = \{(e_1|_p)^\vee, \dots, (e_n|_p)^\vee\} \in (\mathbb{R}_p^n)^\vee$$

$$\begin{aligned}dx_i|_p : \mathbb{R}_p^n &\rightarrow \mathbb{R} \\ v = \left(\sum \alpha_i e_i|_p\right) &\mapsto \alpha_i\end{aligned}$$

$$\frac{\partial x_i}{\partial x_j} = dx_i|_p(e_j|_p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Recalled the wedge algebra:

$$\bigwedge (\mathbb{R}_p^n)^\vee := T(\mathbb{R}_p^n)^\vee / I = \bigoplus_{k \in \mathbb{N}} \bigwedge^k (\mathbb{R}_p^n)^\vee$$

Consider

$$\bigwedge^k (\mathbb{R}_p^n)^\vee$$

what's a basis of this vector space?

$$\{dx_{i_1}|_p \wedge \dots \wedge dx_{i_k}|_p \mid 1 \leq i_1 < \dots < i_k \leq n\}$$

and

$$\dim(\bigwedge^k (\mathbb{R}_p^n)^\vee) = \binom{n}{k}$$

Proved.

## 51.2 Do Carmo Differential forms

### 51.3 Def

An exterior  $k$ -form in  $\mathbb{R}^n$  is a mapping:

$$\begin{aligned}\omega : \mathbb{R}^n &\rightarrow \bigsqcup_p \bigwedge^k (\mathbb{R}_p^n)^\vee \\ p &\mapsto \omega(p)\end{aligned}$$

that's a section of the projection  $\pi$

$$(\pi \circ \omega = id_{\mathbb{R}}) = (\omega(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee)$$

$$\omega(p) = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k}(p) dx_{i_1} |_p \wedge \dots \wedge dx_{i_k} |_p \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

Note that

$$\begin{array}{ccc} \bigsqcup_p \bigwedge^k (\mathbb{R}_p^n)^\vee & \xrightarrow{\pi} & \mathbb{R}^n \\ f |_p & \mapsto & p \\ \omega \leftrightarrow & \{a_{i_1}, \dots, a_{i_k}\} & \end{array}$$

if all  $a_{i_j}$  are of class  $C^m(\mathbb{R})$  the  $\omega$  is called a  $C^m$ -differential  $k$ -form. If  $m = +\infty$  *omega* is called a smooth  $k$ -form.

## 51.4 Notation

$$\omega = \sum_I a_I dx_I$$

where  $I = (i_1, \dots, i_k)$

### Example

take  $n = 4$

1-form

$$\begin{aligned} \omega &= a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + a_4 dx_4 \\ \omega(p) &= a_1(p) dx_1 |_p + a_2(p) dx_2 |_p + a_3(p) dx_3 |_p + a_4(p) dx_4 |_p \end{aligned}$$

2-form

$$\begin{aligned} \omega &= a_{12} dx_1 \wedge dx_2 + a_{13} dx_1 \wedge dx_3 + a_{14} dx_1 \wedge dx_4 \\ &\quad + a_{23} dx_2 \wedge dx_3 + a_{24} dx_2 \wedge dx_4 + a_{34} dx_3 \wedge dx_4 \end{aligned}$$

## 51.5 Notation

When  $k = 0$  a 0-form of class  $C^m$ -differential 0-form is  $f \in C^m(\mathbb{R}^n)$

$$C^m(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{R} \text{ of class } C^m\}$$

## 51.6 Notation

$$\Omega_{(m)}^k(\mathbb{R}^n) := \{\text{set of } C^m\text{-diff } k\text{-forms}\}$$

$$\Omega_{(m)}^0(\mathbb{R}^n) = C^m(\mathbb{R}^n)$$

$m$  could be omitted if no confusion.

### 51.7 Prop

$\Omega_{(m)}^k(\mathbb{R}^n)$  is a module over  $\Omega_{(m)}^0(\mathbb{R}^n)$

#### Proof

$$\omega, \eta \in \Omega^k(\mathbb{R}^n)$$

$$(\omega + \eta)(p) = \omega(p) + \eta(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

$$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n)$$

$$f\omega \in \Omega^k(\mathbb{R}^n) \quad (f\omega)(p) = f(p)\omega(p) \in \bigwedge^k (\mathbb{R}_p^n)^\vee$$

### 51.8 Def

$f : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable then

$$df|_p : \mathbb{R}_p^n \rightarrow \mathbb{R}_{f(p)} \cong \mathbb{R}$$

$$df|_p \in (\mathbb{R}_p^n)^\vee$$

$$df|_p = \sum_{i=1}^n f_i(p) dx_i|_p$$

because

$$\{dx_1|_p, \dots, dx_n|_p\}$$

is a basis of  $(\mathbb{R}_p^n)^\vee$

By  $df$  then  $f_i$  are the partial derivatives of  $f$ . This means that  $df$  is a differential 1-form.

Moreover,

$$F : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

differential, then

$$F = (F_1, \dots, F_m)$$

when  $F_i : \mathbb{R}^n \rightarrow \mathbb{R}$  differential.

$$dF|_p : \mathbb{R}_p^n \rightarrow \mathbb{R}_{f(p)}^m$$

$$dF_i|_p = dx_i|_{f(p)} (dF|_p) = d(x_i \circ F)|_p$$

$$dF_i|_p : \mathbb{R}_p^n \xrightarrow{dF|_p} \mathbb{R}_p^m \xrightarrow{dx_i|_{f(p)}} \mathbb{R}$$

and

$$\begin{aligned} dx_i|_p : \mathbb{R}_p^n &\rightarrow \mathbb{R} \\ v = \sum \alpha_i e_i|_p &\mapsto \alpha_i \end{aligned}$$

where  $e_i|_p = (p, (0, \dots, \underbrace{1}_{i\text{-th}}, 0, \dots))$

Recall that if  $V$  is a vector space, then

$$T(V) = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$$

This is a  $K$ -module with the multiplication:

$$\begin{aligned} V^{\otimes n} \times V^{\otimes m} &\rightarrow V^{\otimes n+m} \\ (x_1 \otimes \dots \otimes x_n, y_1 \otimes \dots \otimes y_m) &\mapsto x_1 \otimes \dots \otimes x_n \otimes y_1 \otimes \dots \otimes y_m \end{aligned}$$

From  $T(V)$  we construct

$$\begin{aligned} \bigwedge(V) &= T(V)/I \\ T(V) &\rightarrow \bigwedge(V) \\ x_1 \otimes \dots \otimes x_n &\mapsto x_1 \wedge \dots \wedge x_n \end{aligned}$$

therefore also in  $\bigwedge(V)$  we have the multiplication that makes  $\bigwedge(V)$  a  $K$ -algebra

$$\begin{aligned} \bigwedge^k(V) &\rightarrow \bigwedge^l(V) \\ (x_1 \wedge \dots \wedge x_k, y_1 \wedge \dots \wedge y_l) &\mapsto x_1 \wedge \dots \wedge x_k \wedge y_1 \wedge \dots \wedge y_l \end{aligned}$$

We define now a wedge product on  $\Omega(\mathbb{R}^n)$

$$\begin{aligned} \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) &\rightarrow \Omega^{k+l}(\mathbb{R}^n) \\ (\omega, \eta) &\mapsto \omega \wedge \eta \end{aligned}$$

take  $\omega = \sum_I a_I dx_I$  and  $\eta = \sum_J b_J dx_J$

$$\omega \wedge \eta := \sum_{IJ} a_i b_J dx_{IJ}$$

where

$$IJ := (i_1, \dots, i_k, j_1, \dots, j_l)$$

with  $I = (i_1, \dots, i_k)$  and  $J = (j_1, \dots, j_l)$

### Example

$$\omega = x_1 dx_1 + x_2 dx_2 + x_3 dx_3 \in \Omega^1(\mathbb{R}^3)$$

$$\eta = x_1 dx_1 \wedge dx_2 + dx_1 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$$

$$\omega \wedge \eta = (x_1 x_3 - x_2) dx_1 \wedge dx_2 \wedge dx_3$$

### 51.9 Prop

Take  $\omega \in \Omega^k(\mathbb{R}^n), \eta \in \Omega^l(\mathbb{R}^n), \varphi \in \Omega^s(\mathbb{R}^n)$ , then

$$(1) \quad (\omega \wedge \eta) \wedge \varphi = \omega \wedge (\eta \wedge \varphi)$$

$$(2) \quad (\omega + \eta) = (-1)^{kl}(\eta \wedge \omega)$$

$$(3) \quad \text{Take } \theta \in \Omega^k(\mathbb{R}^n) \\ \omega \wedge (\varphi + \theta) = \omega \wedge \varphi + \omega \wedge \theta$$

### Proof

Exercise

Try to do this. Consequence of the properties of  $\wedge$  for vector spaces.

### 51.10 Def

Now we have

$$\Omega(\mathbb{R}^n) = \bigoplus_{k \in \mathbb{N}} \Omega^k(\mathbb{R}^n)$$

a  $\mathbb{R}$ -algebra with the  $\wedge$ -product

And it's also a  $\Omega^0(\mathbb{R}^n)$  module and  $\Omega^0(\mathbb{R}^n)$ -algebra

### 51.11 Remark

$$f \in \Omega^0(\mathbb{R}^n), \omega \in \Omega^k(\mathbb{R}^n) \\ f \wedge \omega = f\omega$$

### 51.12 Def: Pullback of forms

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a mapping of  $C^r$ , then it induces a mapping

$$f^* : \Omega_{(x)}^k(\mathbb{R}^m) \rightarrow \Omega_{(x)}^k(\mathbb{R}^n) \\ \omega \mapsto f^*\omega$$

and

$$f^*(\omega)(p)(v_1, \dots, v_k) = \omega(f(p))(df|_p(v_1), \dots, df|_p(v_k))$$

recalling

$$df|_p : \mathbb{R}^n \rightarrow \mathbb{R}_{f(p)}^m \Rightarrow df|_p(v_i) \in \mathbb{R}_{f(p)}^n$$

### 51.13 Prop

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a differentiable mapping.  $\omega, \eta \in \Omega^k(\mathbb{R}^n)$  and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  a differentiable mapping. ( $g \in \Omega^0(\mathbb{R}^m)$ ) Then

(1)

$$f^*(\omega + \eta) = f^*(\omega) + f^*(\eta)$$

(2)

$$f^*(g\omega) = f^*g^*f^*(\omega)$$

where  $f^*g := g \circ f$

(3) If  $\omega_1, \dots, \omega_k$  are 1-forms in  $\mathbb{R}^m$ , then

$$f^*(\omega_1 \wedge \dots \wedge \omega_k) = f^*(\omega_1) \wedge \dots \wedge f^*(\omega_k)$$

### Proof

(1)

$$\begin{aligned} f^*(\omega + \eta)(p)(v_1, \dots, v_k) &= (\omega + \eta)(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= \omega(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &\quad + \eta(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (f^*\omega)(p)(v_1, \dots, v_k) + (f^*\eta)(p)(v_1, \dots, v_k) \end{aligned}$$

(2)

$$\begin{aligned} f^*(g\omega) &= g\omega(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (g \circ f)(p)(f^*\omega)(p)(v_1, \dots, v_k) \end{aligned}$$

(3)

$$\begin{aligned} (f_1 f_2)(x) &= f_1(x) f_2(x) \\ f^*(\omega_1 \wedge \dots \wedge \omega_k)(p)(v_1, \dots, v_k) &= (\omega_1 \wedge \dots \wedge \omega_k)(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= \omega_1(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \wedge \\ &\quad \dots \wedge \omega_k(f(p))(df|_p(v_1), \dots, df|_p(v_k)) \\ &= (f^*(\omega_1))(p)(v_1) \wedge \dots \wedge (f^*(\omega_k))(p)(v_k) \end{aligned}$$

General fact

$$\begin{aligned} f_1, \dots, f_k : \quad & V \rightarrow V \\ f_1 \wedge \dots \wedge f_k : \quad & \bigwedge^k V \rightarrow \bigwedge^k V \\ (v_1, \dots, v_k) \mapsto & f_1(v_1) \wedge \dots \wedge f_k(v_k) \\ g^{\otimes n} : V^{\otimes n} \rightarrow & V^{\otimes n} \\ (v_1, \dots, v_n) \mapsto & g(v_1) \otimes \dots \otimes g(v_n) \end{aligned}$$

Let's see what happens in terms of coordinates:

$$\begin{aligned} f : \mathbb{R}^n & \rightarrow \mathbb{R}^m \\ (x_1, \dots, x_n)^T & \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))^T \end{aligned}$$

$$\Omega = \sum_I a_I dy_I \in \Omega^k(\mathbb{R}^m)$$

$$f^* \omega = \sum_I f^*(a_I) (f^* dy_{i_1}) \wedge \dots \wedge (f^* dy_{i_k})$$

Note that

$$(f^* dy_i)(v) = dy_i(df(v)) = d(y_i \circ f)(v) = (df_i)(v)$$

then

$$f^* \omega = \sum_I a_I (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) df_{i_1} \wedge \dots \wedge df_{i_k}$$

### 51.14 Remark

$U \subseteq \mathbb{R}^n$  open then consider  $\Omega^k(U) \subseteq \Omega^k(\mathbb{R}^n)$

#### Example

$$\omega = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy \in \Omega^1(\mathbb{R}^2 \setminus \{(0, 0)\}) (= U)$$

$$V = \{(r, \theta) \in \mathbb{R}^2 : r > 0, 0 \leq \theta \leq 2\pi\}$$

$$\begin{aligned} f : V & \rightarrow U \\ (r, \theta)^T & \mapsto f(r, \theta) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} \end{aligned}$$

Let's compute  $f^* \omega$

$$df_1 = \cos \theta dr - r \sin \theta d\theta$$

$$df_2 = \sin \theta dr + r \cos \theta d\theta$$

$$f^* \omega = -\frac{r \sin \theta}{r^2} (\cos \theta dr - r \sin \theta d\theta) + \frac{r \cos \theta}{r^2} (\sin \theta dr + r \cos \theta d\theta) = d\theta$$

### 51.15

$U \subseteq \mathbb{R}^n$  an open subset

$$\Omega_{(m)}^k(U)$$



this is a module over  $\Omega_{(m)}^0(U)$  Moreover,  $\omega \in \Omega^k(U), \eta \in \Omega^l(U)$

$$\omega \wedge \eta \in \Omega^{k+l}(U)$$

$$f : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \underbrace{\mathbb{R}^m}_{\subseteq \mathbb{R}^m} \text{ } f \text{ is of class } C^{m+1}$$

$$f^*\omega \in \Omega_{(m)}^k U()$$

$df$  is a one-form

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

where  $\frac{\partial f}{\partial x_i} = a_i : \mathbb{R}^n \rightarrow \mathbb{R}$  differentiable

## 51.16 Prop

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a differentiable mapping. Then

(1) for any two forms in  $\mathbb{R}^m$

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*(\eta))$$

(2) for  $g : \mathbb{R}^p \rightarrow \mathbb{R}^n$  differentiable

$$(f \circ g)^*\omega = g^*(f^*\omega)$$

### Proof

1

$$(y_1, \dots, y_m) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)) \in \mathbb{R}^m, (x_1, \dots, x_n) \in \mathbb{R}^n$$

$$\omega = \sum_I a_I dy_I \quad \eta = \sum_J b_J dy_J$$

$$f^*(\omega \wedge \eta) = f^*\left(\sum_{IJ} a_I b_J dy_I \wedge dy_J\right)$$

$$\begin{aligned} (\text{by def of pullback}) &= \sum_{IJ} a_I(f_1, \dots, f_m) b_J(f_1, \dots, f_m) df_I \wedge df_J \\ &= \left(\sum_I a_I(f_1, \dots, f_m) df_I\right) \wedge \left(\sum_J b_J(f_1, \dots, f_m) df_J\right) \\ &= (f^*\omega) \wedge (f^*(\eta)) \end{aligned}$$

**2**

$$\begin{aligned}
(f \circ g)^* \omega &= \sum_I a_I((f \circ g)_1, \dots, (f \circ g)_m) d(f \circ g)_I \\
&= \sum_I a_I(f_1(g_1, \dots, g_n), \dots, f_m(g_1, \dots, g_n)) df_I(dg_1, \dots, dg_n) \\
&= g^*(f^* \omega)
\end{aligned}$$

**51.17**

The differential of a function is a one-form

$$\begin{array}{ccc}
\underbrace{f} & \rightsquigarrow & \underbrace{df} \\
\text{0-form} & & \text{1-form}
\end{array}$$

We went to generalize this to any (exterior) differentials

$$\begin{array}{ccc}
d : \Omega_{(m)}^k(U) & \rightarrow & \Omega_{(m)}^{k+1}(U) \\
\omega & \mapsto & d\omega \\
\sum_I a_I dx_I & \mapsto & \sum_I da_I \wedge dx_I
\end{array}$$

where  $a_I \in C^m(U)$ ,  $da_I = \sum \frac{\partial a_I}{\partial x_i} dx_i$

**51.18 Example**

$$\omega = xyz dx + yz dy + (x + z) dz$$

$$\begin{aligned}
d\omega &= d(xyz) \wedge dx + d(yz) \wedge dy + d(x + z) \wedge dz \\
&= (yz dx + xz dy + xy dz) \wedge dx + (z dy + y dz) \wedge dy + (x dz) \wedge dz \\
&= -xz dx \wedge dy - xy dx \wedge dz - y dy \wedge dz + dx \wedge dz \\
&= -xz \underline{dx \wedge dy} + (1 - xxy) \underline{dx \wedge dz} - y \underline{dy \wedge dz}
\end{aligned}$$

**51.19 Prop**

$$\forall \omega_1, \omega_2 \in \Omega^k(U), \eta \in \Omega^l(U)$$

(1)

$$d(\omega_1 + \omega_2) = d(\omega_1) + d(\omega_2)$$

(2)

$$d(\omega_1 \wedge \omega_2) = d(\omega_1) \wedge \eta + (-1)^k \omega \wedge d\eta$$

$$(3) \quad d(d\omega) = 0 \quad (d^2\omega = 0)$$

$$(4) \quad f : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \underbrace{V}_{\subseteq \mathbb{R}^m}$$

$$d(f^*\omega) = f^*(d\omega)$$

### Proof

(1) Exercise

$$(2) \quad \omega = \sum_I a_I dx_I, \eta = \sum_J b_J dx_J; \quad \omega \wedge \eta = \sum_{IJ} a_I b_J dx_I \wedge dx_J$$

$$\begin{aligned} d(\omega \wedge \eta) &= \sum_{IJ} d(a_I b_J) \wedge dx_I \wedge dx_J \\ &= \left( \sum_{IJ} b_J da_I \wedge dx_I \wedge dx_J \right) + \left( \sum_{IJ} a_I db_J \wedge dx_I \wedge dx_J \right) \\ &= d\omega \wedge \eta + (-1)^k \sum_{IJ} a_I dd x_I \wedge b_J \wedge dx_J \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta \end{aligned}$$

(3) First assume  $\omega = f \in \Omega^0(U)$

$$\begin{aligned} d(df) &= d\left(\sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j\right) \\ &= \sum_{j=1}^n d\left(\frac{\partial f}{\partial x_j} \wedge dx_j\right) \\ &= \sum_{j=1}^n \left(\sum_{i=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i \wedge dx_j\right) \\ &= 0 \end{aligned}$$

Notice that  $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

By (1) we can prove for  $\omega = a_I dx_I, a_I \neq 0$ , from (2) we have

$$d\omega = da_I \wedge dx_I + a_I d^2 x_I$$

But

$$d^2 x_I = d(1 \cdot dx_I) = d1 \wedge dx_I = 0$$

Hence

$$\begin{aligned} d^2\omega &= d(d\omega) \\ &= d(da_I \wedge dx_I) \\ &= 0 \end{aligned}$$

(4) As above let's prove it for  $\omega = g \in \Omega^0(U)$

$$\begin{aligned} g : \mathbb{R}^m & \rightarrow \mathbb{R} \\ (y_1, \dots, y_m) & \mapsto g(y_1, \dots, y_m) \end{aligned}$$

$$\begin{aligned} f^*(dg) &= f^*\left(\sum_{i=1}^m \frac{\partial g}{\partial y_i} dy_i\right) \\ &= \sum_{i,j} \frac{\partial g}{\partial y_i} \frac{\partial f}{\partial x_j} dx_j \\ &= \sum_j \frac{\partial(g \circ f)}{\partial x_j} dx_j \\ &= d(g \circ f) \\ &= d(f^*g) \end{aligned}$$

Now let's do the proof for  $\omega \in \Omega^k(U), \omega = \sum_I a_I dx_I$

$$\begin{aligned} d(f^*g) &= d\left(f^*\left(\sum_I a_I dx_I\right)\right) \\ (\text{by prop of } f^*) &= d\left(\sum_I f^*a_I \wedge f^*dx_I\right) \\ (by(1)) &= \sum_I d(f^*a_I \wedge f^*dx_I) \\ (\text{use(2)}) &= \sum_I f^*(da_I) \wedge f^*dx_I \\ (\text{prop of } f^*) &= f^*\left(\sum_I da_I \wedge dx_I\right) \\ &= f^*(d\omega) \end{aligned}$$

## 51.20

$$df : p \mapsto df_p$$

is a differential form.

## 51.21 Def?

$$D_h f(p) := \lim_{t \rightarrow 0} \frac{f(p+th) - f(p)}{h} = df_p(h)$$

# Chapter 52

## Line integral

### 52.1 Def

$$\omega = \sum_i a_i dx_i \in \Omega^1_m(U), U \subseteq \mathbb{R}^n$$

$$\gamma : [a, b] \rightarrow U^n$$

a parametric curve

$$f : [a, b] \rightarrow \mathbb{R}$$

of class  $C^1$

$$\gamma : t \mapsto (t, f(t)) = \text{Graph of } f$$

this is a parametric curve piecewise of class  $C^1$ :  $\exists t_0 = a < t_1 < \dots < t_k = b$   
such that

$$\gamma_j := \gamma|_{]t_j, t_{j+1}[}$$

is of class  $C^1$

$$\gamma_j(]t_k, t_{k+1}[) \rightarrow \mathbb{R}^n$$

we can define  $\gamma_j^* \omega$  this is one form in  $\Omega^1(]t_k, t_{k+1}[)$   
if  $\gamma_j(t) = (x_1(t), \dots, x_n(t))$  then

$$\gamma_j^* \omega = \sum_{i=1}^n a_i(x_1(t), \dots, x_n(t)) \frac{dx_i}{dt} dt \quad x_i(t) = \frac{dx_i}{dt}?$$

### 52.2 Def: Path integral

Let  $\gamma$  and  $\omega$  be as above.

$$\int_{\gamma} \omega := \sum_i \int_{t_k}^{t_{k+1}} \gamma_j^* \omega$$

this is the integral of  $\omega$  along the parametric curve  $\gamma$  with

$$\gamma = t \mapsto (x_1(t), \dots, x_n(t))$$

where  $x_i(t) = \frac{dx_i}{dt}$

### 52.3 What's this in physics?

Fix  $\gamma(t), \gamma'(t) = (\frac{dx_1}{dt}, \dots, \frac{dx_n}{dt})$  = the tangent vector of  $\gamma$  in  $\gamma(t)$  then

$$\int_{t_k}^{t_{k+1}} \gamma_j^* \omega = \int_{t_k}^{t_{k+1}} \langle a \circ \gamma_j, \gamma_j' \rangle dt$$

where  $a = (a_1, \dots, a_n), a_i : \mathbb{R}^n \rightarrow \mathbb{R}$

## Chapter 53

# Complement of measure theory

### 53.1 Def( $\sigma$ -finite)

Let  $(X, \Sigma_X, \mu)$  be a measure space. We say that it's  $\sigma$ -finite if there exists a sequence  $\{E_n\}_{n \in \mathbb{N}}$  of measurable sets. (namely  $E_n \in \Sigma_X$ ) such that

$$X = \bigcup_{n \in \mathbb{N}} E_n \text{ and } \mu(E_n) < +\infty, \forall n \in \mathbb{N}$$

### 53.2 Example( $\mathbb{R}$ , Norel $\sigma$ -algebra, Lebesgue measure)

this is  $\sigma$ -finite

$$\lambda([-n, n]) = 2n < +\infty$$

### 53.3 Notation

Take sets  $A \subseteq X \times Y$  For  $x \in X$ , we define

$$A_x := \{u \in Y \mid (x, u) \in A\}$$

called a **vertical section** of  $A$  or  $x$ -fiber of  $A$

For  $y \in Y$  we define

$$A_y := \{x \in X \mid (x, y) \in A\}$$

called a **horizontal section** of  $A$ , or  $y$ -fiber of  $A$

### 53.4 Def

Let  $X$  be a set. then  $\mathcal{D} \subseteq \wp(X)$  is a **Dynkin system** if

- $X \in \mathcal{D}$  and  $\emptyset \in \mathcal{D}$
- $\forall D \in \mathcal{D} \quad X \setminus D \in \mathcal{D}$
- If  $\{D_n\}_{n \in \mathbb{N}}$  is a sequence in  $\mathcal{D}$  of pairwise disjoint sets, then

$$\bigsqcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$$

### Remark

A  $\sigma$ -algebra is a Dynkin system

### 53.5 Def

Let  $(\mathcal{G} \subseteq \wp(X))$  then  $\delta(\mathcal{G}) \subseteq \wp(X)$  is called the Dynkin system generated by  $\mathcal{G}$  if

- $\mathcal{G} \subseteq \delta(\mathcal{G})$
- If  $\mathcal{D}$  is a Dynkin system containing  $\mathcal{G}$ , then  $\delta(\mathcal{G}) \subseteq \mathcal{D}$

### Exercise

$\delta(\mathcal{G})$  exists and it's unique.

### 53.6 Prop

If  $\mathcal{D}$  is a Dynkin system closed under the intersection, then it's a  $\sigma$ -algebra, namely

$$\forall (D, E) \in \mathcal{D}^2, D \cap E \in \mathcal{D} \Rightarrow \forall \{D_n\}_{n \in \mathbb{N}} \in \mathcal{D}^{\mathbb{N}} \quad \bigcup_{n \in \mathbb{N}} D_n \in \mathcal{D}$$

### Proof

We have to show that  $\mathcal{D}$  is closed under any countable union. Let  $\{D_n\}_{n \in \mathbb{N}}$  be any sequence in  $\mathcal{D}$ , let

$$E_n = D_n \cap \bigcap_{m < n} X \setminus D_m$$

and we know that

$$\bigcup_{k \in \mathbb{N}} E_k = \bigcup_{k \in \mathbb{N}} D_k \supseteq D_n \cap D_m \quad \forall n, m$$



### 53.7 Prop

Let  $X$  be a set and let  $\mathcal{G} \subseteq \wp(X)$ . Assume that  $\mathcal{G}$  is closed under the finite intersection. Then

$$\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$$

#### Proof

Prove  $\delta(\mathcal{G}) \subseteq \sigma(\mathcal{G})$   
trivial

Prove  $\sigma(\mathcal{G})$  is a  $\sigma$ -algebra, which gives that  $\delta(\mathcal{G}) \supseteq \sigma(\mathcal{G})$   
Let

$$\delta_D = \{E \subseteq X \mid E \cap D \in \delta(\mathcal{G})\}$$

Verify that  $\forall D \in \mathcal{G}, \delta_D$  is a Dynkin system:

- Since  $X \cap DS \in \mathcal{G} \Rightarrow X \in \delta_D$
- Take  $E \in \delta_D$

$$(X \setminus E) \cap D = X \setminus ((E \cap D) \cup (X \setminus D))$$

Where  $E \cap D \in \delta(\mathcal{G})$  (since  $E \in \delta_D$ ),  $X \setminus \in \delta(\mathcal{G})$  (by def)

Hence

$$X \setminus E \in \delta_D$$

- Let  $\{E_n\}$  be elements in  $\delta_D$  which are pairwise disjoint, then

$$\left(\bigcup_{n \in \mathbb{N}} E_n\right) \cap D = \bigcup_{n \in \mathbb{N}} (E_n \cap D)$$

Then  $\forall G \in \mathcal{G}$

$$\delta(\mathcal{G}) \subseteq \delta_G$$

since  $\delta(\mathcal{G})$  is the smallest Dynkin system containing  $\mathcal{G}$  and  $\forall G \in \mathcal{G} \mathcal{G} \subseteq \delta_G$   
since  $\mathcal{G}$  is closed under finite intersection. By definition

$$\forall H \in \delta(\mathcal{G}), H \cup G \in \delta(\mathcal{G})$$

### 53.8 Lemma

Let  $(X, \Sigma_X)$  be a measurable space. Then the mapping 0 is measurable

**Proof**

By def

$$0_*(\Sigma_X) = \{B \subseteq \mathbb{R} \mid 0^{-1}(B) \in \Sigma_X\}$$

Since either  $0^{-1}(B) = \emptyset$  or  $0^{-1}(B) = X$ , then

$$0_*(\Sigma_X) = \wp(\mathbb{R})$$

Hence

$$\mathcal{B}(\mathbb{R}) \subseteq 0_*(\Sigma_X) = \wp(\mathbb{R})$$

**53.9 Theorem**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. Then  $\forall E \in \Sigma_X \otimes \Sigma_Y$ , the functions

$$\begin{aligned} f_E : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu(E_x) \\ g_E : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto \mu(E_y) \end{aligned}$$

are respectively  $\Sigma_X$ -measurable and  $\Sigma_Y$ -measurable

**Proof**

We first cope with special ones that  $\nu$  is finite ( $\mu(Y) < +\infty$ ) Let

$$F = \{E \in \Sigma_X \otimes \Sigma_Y \mid f_E \text{ is measurable}\}$$

We want to have  $F = \Sigma_X \otimes \Sigma_Y$  Only to show  $\Sigma_X \otimes \Sigma_Y \subseteq F$  by definition of product measure

Let  $S_1 \in \Sigma_X, S_2 \in \Sigma_Y$

$$(S_1 \times S_2)_x = \begin{cases} S_2 & \text{if } x \in S_1 \\ 0 & \text{if } x \notin S_1 \end{cases} \quad f_{S_1 \times S_2}(x) = \nu(S_1 \times S_2) = \nu(S_2)_{\chi_{S_1}}(x)$$

?

$f_{S_1 \times S_2}$  is measurable.

Now show that  $F$  is a Dynkin system:

- $X \times Y \in F$
- Let  $D \in F$ , we want to show that

$$(X \times Y) \setminus D \in F$$

Note that

$$((X \times Y) \setminus D)_x = Y \setminus D_x$$

then

$$\begin{aligned}
 f_{(X \times Y \setminus D)}(x) &= \nu((X \times Y) \setminus D)_x \\
 &= \nu(Y \setminus D_x) \\
 &= \nu(Y) - \nu(D_x) \\
 &= \nu(Y) - f_D(x)
 \end{aligned}$$

Which means that  $f_{(X \times Y \setminus D)}$  is measurable.

- Let  $\{D_n\}_{n \in \mathbb{N}}$  be a sequence of disjoint sets such that  $D_n \in F$ . ( $f_{D_n}$  is measurable)  $D = \bigcup_{n \in \mathbb{N}} D_n$

$$\begin{aligned}
 f_D(x) &= \nu(D_x) \\
 &= \nu\left(\bigcup_{n \in \mathbb{N}} D_n\right) \\
 &= \sum_{n \in \mathbb{N}} \nu(D_n) \\
 &= \sum_{n \in \mathbb{N}} f_{D_n}(x)
 \end{aligned}$$

Hence  $F$  is a Dynkin system.

Consider

$$\mathcal{G} = \{S_1 \times S_2 \mid S_1 \in \Sigma_X, S_2 \in \Sigma_Y\} \subseteq F$$

Moreover,  $\mathcal{G}$  is closed under the intersection.

$$(S_1 \times T_1) \cap (S_2 \times T_2) = (S_1 \cap S_2) \times (T_1 \cap T_2)$$

So  $\delta(\mathcal{G})$  is  $\sigma$ -algebra. By proposition 53.7

$$\delta(\mathcal{G}) = \sigma(\mathcal{G}) = \Sigma_X \otimes \Sigma_Y \subseteq F$$

Secondly, for general  $\nu$ , since  $\nu$  is  $\sigma$ -finite, there exists

$$Y = \bigcup_{n \in \mathbb{N}} Y_n \quad \nu(Y_n) < +\infty$$

As above

$$F_0 = Y_0, F_n = Y_n \setminus \bigcup_{k \in \mathbb{N}} Y_k, \nu(F_n) < +\infty$$

$\{F_n\}$  are disjoint, measurable, of finite measure and  $Y = \bigcup_n F_n$

For all  $n$  we define a measure  $\nu^{(n)}$  on  $Y_n$

$$\nu^{(n)}(E) := \nu(E \cap F_n)$$

Notice that

$$\nu^{(n)}(Y) = \nu(Y \cap F_n) = \nu(F_n) < +\infty$$

Hence we have

$$\begin{aligned} f_E^{(n)} : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \nu^{(n)}(E_x) \end{aligned}$$

By step 1,  $f_E^{(n)}$  is measurable.  
 $\forall E, n$

$$\begin{aligned} f_E^{(n)}(x) &= \nu(E_x) \\ &= \nu(E_x \cap Y) \\ &= \nu\left(E_x \cap \bigcup_n F_n\right) \\ &= \nu\left(\bigcup_n E_x \cap F_n\right) \\ &= \sum_n \nu(E_x \cap F_n) = \sum \nu^{(n)}(E_x) \\ &= \sum_n (f_E^{(n)}(x)) \end{aligned}$$

It follows that  $f_E$  is measurable

Then we need prove that  $F \supseteq \Sigma_X \times \Sigma_Y$  and  $F$  is  $\sigma$ -algebra, so  $F \supseteq \sigma(\Sigma_X \times \Sigma_Y) = \Sigma_X \otimes \Sigma_Y$

–  $E_\mu \in \Sigma_X$  and  $E_\epsilon \in \Sigma_Y$

$$(E_\mu \times E_\epsilon)_x = \begin{cases} E_\epsilon & \text{if } x \in E_\mu \\ \emptyset & \text{otherwise} \end{cases} \in \Sigma_X$$

– exercise

### 53.10 Prop

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces.  $\forall E \in \Sigma_X \otimes \Sigma_Y$  the functions:

$$\begin{aligned} \rho_X(E) &:= \int_X f_E(x) d\mu(x) \\ \rho_Y(E) &:= \int_Y g_E(y) d\nu(y) \end{aligned}$$

Define two measure on the measurable spaces  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  such that

$$\rho_X(S_1 \times S_2) = \rho_Y(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

**Proof**

We already know that  $f_E$  and  $g_E$  are measurable. So the integral makes sense. Only needs to prove for  $\rho_X$

- Since  $f_E \geq 0$  and  $g_E \geq 0$ , then  $\rho_X(E) \geq 0 \ \forall E \in \Sigma_X \otimes \Sigma_Y$
- $\rho_X(\emptyset) = \int_X \nu(\emptyset) d\mu(x) = 0$
- Assume that  $\{E_n\}_{n \in \mathbb{N}}$  is a sequence in  $\Sigma_X \otimes \Sigma_Y$  of disjoint subsets,

$$\begin{aligned} \rho_X\left(\bigsqcup_{n \in \mathbb{N}} E_n\right) &= \int_X \nu\left(\bigsqcup_{n \in \mathbb{N}} (E_n)_x\right) d\mu(x) \\ &= \int_X \sum_{n \in \mathbb{N}} \nu(E_n)_x d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \int_X \nu(E_n)_x d\mu(x) \\ &= \sum_{n \in \mathbb{N}} \rho_X(E_n) \end{aligned}$$

•

$$\begin{aligned} \rho_X(S_1 \times S_2) &= \int_X \nu(S_1 \times S_2)_x d\mu(x) \\ &= \int_X \nu(S_2) \mathbb{1}_{S_1}(x) d\mu(x) \\ &= \nu(S_2) \mu(S_1) \end{aligned}$$

**53.11 Prop**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. Any measure  $\eta$  on  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  that satisfies

$$\eta(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

is  $\sigma$ -finite

**Proof**

$$\nu(E_n \times F_m) = \mu(E_n)\nu(F_m) < +\infty$$

**53.12 Prop**

Let  $(X, \Sigma_X)$  be a measurable space and assume that  $\mathcal{G} \subseteq \wp(X)$  such that  $\Sigma = \sigma(\mathcal{G})$

Moreover, assume that  $\mathcal{G}$  satisfies the following conditions:

- (1) It's closed under finite intersection.
- (2) There exists a sequence  $\{G_n\}_{n \in \mathbb{N}}$  in  $\mathcal{G}$  such that  $\{G_m\} \uparrow X$  (namely  $G_i \subseteq G_{i+1}$  and  $\bigcup_n G_n = X$ )

Let  $\mu$  and  $\nu$  be two measure on  $(X, \Sigma)$  such that

- (a)  $\forall G \in \mathcal{G} \quad \mu(G) = \nu(G)$
- (b)  $\forall n \in \mathbb{N} \quad \mu(G_n) = \nu(G_n)$

Then  $\mu = \nu$

### Proof

Define

$$\mathcal{D}_n = \{E \in \Sigma \mid \mu(G_n \cap E) = \nu(G_n \cap E)\} \subseteq \Sigma$$

We show that  $\mathcal{D}_n$  is a Dynkin system  $\forall n$

- $G_n \cap X = G_n$
- Assume that  $D \in \mathcal{D}_n$ 

$$\begin{aligned} \mu(G_n \cap (X \setminus D)) &= \mu(G_n \setminus D) \\ &= \mu(G_n) - \mu(G_n \cap D) \quad (\text{here use the fact that } \mu(G_n) < +\infty) \\ &= \nu(G_n) - \nu(G_n \cap D) \\ &= \nu(G_n \cap (X \setminus D)) \end{aligned}$$
- Take  $\{D_m\}_{m \in \mathbb{N}}$  in  $\mathcal{D}_n$  of pairwise choice

$$\begin{aligned} \mu(G_n \cap \bigcup_m D_m) &= \mu(\bigcup_m (G_n \cap D_m)) \\ &= \sum_m \mu(G_n \cap D_m) \\ &= \sum_m \nu(G_n \cap D_m) \\ &= \nu(G_n \cap \bigcup_m D_m) \end{aligned}$$

Combining (1) and (a)  $\mathcal{G} \subseteq \mathcal{D}_n$ . By prop 53.7, consider

$$\delta(\mathcal{G}) = \sigma(\mathcal{G}) = \Sigma$$

Moreover, since  $\mathcal{G} \subseteq \mathcal{D}_n$  and  $\mathcal{D}_n$  a Dynkin system

$$\delta(\mathcal{G}) \subseteq \mathcal{D}_n$$

We get

$$\Sigma = \mathcal{D}_n$$

Since  $\bigcup_n G_n \cap E = E \cap \bigcup_n G_n = E$

$$\mu(E) = \lim_{x \rightarrow +\infty} \mu(G_n \cap E) = \lim_{x \rightarrow +\infty} \nu(G_n \cap E) = \nu(E)$$

### 53.13 Theorem

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be  $\sigma$ -finite measure spaces. There exists a unique  $\sigma$ -finite measure  $\mu \times \nu$  on  $(X \times Y, \Sigma_X \otimes \Sigma_Y)$  such that

$$\mu \times \nu(S_1 \times S_2) = \mu(S_1)\nu(S_2) \quad \forall (S_1, S_2) \in \Sigma_X \times \Sigma_Y$$

and moreover, we have

$$(\mu \times \nu)(E) = \int_X f_E d\mu = \int_Y g_E d\nu$$

### 53.14 Corollary

On  $\mathbb{R}^n$ , we can define a unique measure  $\lambda^{(n)}$  as product of the Lebesgue measure on  $\mathbb{R}$ . This is called the Lebesgue measure on  $\mathbb{R}^n$

#### Proof

Assume that  $\eta$  and  $\eta'$  are two measures on the product satisfies the equation. Let  $\mathcal{G} = \Sigma_1 \times \Sigma_2$ ,  $\sigma(\mathcal{G}) = \Sigma_1 \otimes \Sigma_2$ . And  $\mathcal{G}$  is stable under finite intersection.

Since  $\mu$  and  $\nu$  are  $\sigma$ -finite, we can find some  $\{E_n\} \uparrow X$  and  $\{F_m\} \uparrow Y$  such that  $\mu(E_n) < +\infty, \nu(F_m) < +\infty$

$$X \times Y = \bigcup_{n,m} E_n \times F_m$$

We can find some ordering of the couple

$$X \times Y = \bigcup_{i_k} E_{i_k} \times F_{i_k}$$

$$\{E_{i_k} \times F_{i_k}\} \uparrow X \times Y$$

$$G_k := E_{i_k} \times F_{i_k}$$

By the equal in conditions  $\forall k$

$$\eta(G_k) = \eta'(G_k)$$

We apply prop53.12 to get

$$\eta = \eta' = \mu \times \nu$$

By prop53.11

$$\mu \times \nu$$

is  $\sigma$ -finite

And  $\mu \times \nu$  exists by Prop53.10

### 53.15 Monotone convergence theorem

Let  $(X, \Sigma_X, \mu)$  be a measure space.  $f : X \rightarrow \mathbb{R}_{\geq 0}$  be a measurable function. Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of measurable functions

$$f_n : X \rightarrow \mathbb{R}_{\geq 0}$$

such that  $f_i < f_j \quad \forall i < j$  and

$$\lim_{n \rightarrow +\infty} f_n(x) = f(x)$$

almost everywhere in  $X$  ( $\forall x \in X \setminus Z$  when  $Z \in \Sigma, \mu(Z) = 0$ ) Then

$$\int_X f d\mu = \lim_{n \rightarrow +\infty} \int_X f_n d\mu$$

#### Proof

dominated convergence theorem  $\Rightarrow$  monotone convergence theorem.

### 53.16 Recall

Product measure on  $\mathbb{R}^n$ . This is the unique measure on  $\lambda^n$  that exact is the naive product measure on rectangles.

$$\begin{aligned} \Sigma_{\mathbb{R}^n} &= \mathcal{B}(\mathbb{R}) \otimes \cdots \otimes \mathcal{B}(\mathbb{R}) \\ \Leftrightarrow \\ \mathcal{B}(\mathbb{R}^n); \lambda^n(\prod_i [a_i, b_i]) &= \prod_i \lambda[a_i, b_i] \end{aligned}$$

### 53.17 Def

$$\mathcal{O}^n = \{\text{set of open sets of } \mathbb{R}^n\}$$

$$\mathcal{C} = \{\text{set of closed sets of } \mathbb{R}^n\}$$

$$\mathcal{R}^n = \{\text{set of compact sets of } \mathbb{R}^n\}$$

$$\mathcal{J}_{ha}^n = \{\text{set of all half-open rectangles in } \mathbb{R}^n\}$$

$$\mathcal{J}_{ha, rat}^n = \{\text{set of all half-open rectangles of } \mathbb{R}^n, \text{ with rational end points}\}$$

### 53.18 Prop

$$\mathcal{B}(\mathbb{R}^n) = \sigma(\mathcal{O}^n) = \sigma(\mathcal{C}^n) = \sigma(\mathcal{R}^n) = \sigma(\mathcal{J}_{ha}^n) = \sigma(\mathcal{J}_{ha, rat}^n)$$



**Proof**

Exercise

**53.19 Recall**

Let  $(X, \Sigma_X)$  and  $(Y, \Sigma_Y)$  be measurable spaces. Moreover, assume that

$$\Sigma_Y = \sigma(\mathcal{G})$$

where  $\mathcal{G} \subseteq \wp(X)$ .

A function  $f : X \rightarrow Y$  is measurable iff

$$\forall S \in \mathcal{G} \quad f^{-1}(S) \in \Sigma(X)$$

**Hint**

$$\mathcal{M} := \{B \subseteq Y \mid f^{-1}(B) \in \Sigma_X\} \subseteq \wp(Y)$$

show that this is a  $\sigma$ -algebra

**53.20 Corollary**

$f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , if  $f$  is continuous, the  $f$  is measurable with respect to the Lebesgue measure.

**53.21 Def: Push-forward measure**

Let  $(X, \Sigma_X, \mu)$  be a measure space, and let  $(Y, \Sigma_Y)$  be a measurable space. If  $f : X \rightarrow Y$  is a measurable function, then define:

$$f_*\mu(E) = \mu(f^{-1}(E)) \quad \forall E \in \Sigma_Y$$

This is a measure on  $Y$ , called the push forward of  $\mu$  through  $f$

**53.22 Prop**

Let  $p \in \mathbb{R}$  and let  $E \in \mathcal{B}(\mathbb{R}^n)$ , then

$$\lambda^n(E + p) = \lambda^n(E)$$

note that

$$E + p = \{x + p \mid x \in E\}$$

**Proof**

$$p = (p_1, \dots, p_n)$$

Consider the translation

$$\begin{aligned} \tau_p : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto x - p \end{aligned}$$

this is continuous, so measurable. We consider

$$\lambda_p^n := \tau_{p*} \lambda^n$$

let's show that  $\lambda_p^n = \lambda^n$

$$\begin{aligned} \lambda_p^n \left( \prod_{i=1}^n [a_i, b_i] \right) &\stackrel{\text{by def of } f_*}{=} \lambda^n (\tau_p^{-1} \left( \prod_{i=1}^n [a_i, b_i] \right)) \\ &= \lambda^n \left( \prod_{i=1}^n [a_i + p_i, b_i + p_i] \right) \\ &= \prod_{i=1}^n (b_i - a_i) \end{aligned}$$

By the uniqueness of the product measure, we have

$$\lambda_p^n = \lambda^n$$

**53.23 Lemma**

Let  $f : \Omega \rightarrow \mathbb{R}_{\geq 0}$  be a mapping. Then there exists an increasing sequence  $\{f_n\}_{n \in \mathbb{N}}$  such that converges pointwisely to  $f$

**Proof**

$$f_n = \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{\{\omega \in \Omega \mid \frac{k}{2^n} \leq f(\omega) \leq \frac{k+1}{2^n}\}} + n \mathbb{1}_{\{\omega \in \Omega \mid f(\omega) \geq 0\}}$$

**53.24 Fubini-Tobelli Theorem**

Let  $(X, \Sigma_X, \mu)$  and  $(Y, \Sigma_Y, \nu)$  be two  $\sigma$ -finite measure spaces. Let  $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$  be the product space. Let  $f : X \times Y \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$  be a measurable function. Then

$$\begin{aligned} \int_{X \times Y} |f| \, d(\mu \times \nu) &= \int_X \left( \int_Y |f(x, y)| \, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left( \int_X |f(x, y)| \, d\mu(x) \right) d\nu(y) \end{aligned}$$

**Proof**

We can assume that  $f \geq 0$  and

$$\begin{aligned} \forall x \in X \quad f_x : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto f(x, y) \\ \forall y \in Y \quad f_y : X &\rightarrow \mathbb{R} \cup \{+\infty\} \end{aligned}$$

**Step 1:  $f_x$  and  $f_y$  are measurable**

Let's do it for  $f_y$ . We have to show that for  $D \in \mathcal{B}(\mathbb{R})$  where  $f_y^{-1}(D) \in \Sigma_X$

$$f_y^{-1}(D) = \{x \in X \mid f(x, y) \in D\} = \{x \in X \mid (x, y) \in f^{-1}(D)\} = (f^{-1}(D))_y$$

We have shown that if  $E$  is measurable, the  $E_y$  is measurable.

**Step 2**

Consider the functions

$$\begin{aligned} G : Y &\rightarrow \mathbb{R} \cup \{+\infty\} \\ y &\mapsto \int_X f(x, y) d\mu(x) \\ F : X &\rightarrow \mathbb{R} \cup \{+\infty\} \\ x &\mapsto \int_Y f(x, y) d\nu(y) \end{aligned}$$

we want to prove that they're both measurable

Let do this for  $G$ . Assume that  $f = \chi_E$  for  $E \in \Sigma_X \otimes \Sigma_Y$

$$\begin{aligned} (\chi_E)_y(x) &= \chi_E(x, y) = 1 \\ &\Leftrightarrow (x, y) \in E \\ &\Leftrightarrow x \in E_y \\ &\Leftrightarrow \chi_{E_y}(x) = 1 \end{aligned}$$

This chain of implications shows that

$$(\chi_E)_y = \chi_{E_y}$$

Hence

$$G(y) = \int_X (\chi_E)_y d\mu = \int_X \chi_{E_y} d\mu = \mu(E_y)$$

And we have proved that such functions are measurable

Now assume

$$f = \sum_{i=1}^n a_k \chi_{E_k} \quad E_k \in \Sigma_X \otimes \Sigma_Y, \quad a_k \in \mathbb{R}_{\geq 0}$$

and

$$f_y = \sum_{i=1}^n a_k \chi_{E_k \cap E_y}$$

then

$$\begin{aligned} G(y) &= \int_X f_y d\mu \\ &= \sum_{k=1}^n a_k \int_X \chi_{(E_k)_y} d\mu \\ &= \sum_{k=1}^n a_k \mu((E_k)_y) \\ &\Rightarrow G \text{ is measurable} \end{aligned}$$

Now assume  $f$  measurable

By lemma 53.23,  $\exists \{f_n\}$  increasing sequence such that converges pointwisely to  $f$

Moreover,  $\{(f_n)_y\}_{n \in \mathbb{N}}$  (all simple functions) converges to  $f_y$  too. Consider

$$\begin{aligned} g_n : Y &\rightsquigarrow \mathbb{R} \\ y &\mapsto \int_X (f_n)_y d\mu \end{aligned}$$

Since  $f_n$  are simple. By the previous claim in step 2, we know that  $g_n$  is measurable. And since  $Im(g_n) \subseteq \mathbb{R}$

$$G(y) = \lim_{n \rightarrow +\infty} g_n(y) = \sup_{n \rightarrow \infty} g_n(y)$$

$G$  is measurable.

### Step 3

First we show that the equation in theorem holds for  $f = \mathbb{1}_E$ . By prop 53.10

$$\int_X \left( \int_Y f_x d\nu \right) d\mu = \int_X \nu(E_x) d\mu = (\mu \times \nu)(E)$$

while

$$\int_{X \times Y} f d(\mu \times \nu) = \int_{X \times Y} \mathbb{1}_E d(\mu \times \nu) = (\mu \times \nu)(E)$$

By two equations above:

$$\int_X \left( \int_Y f_x d\nu \right) d\mu = \int_{X \times Y} f d(\mu \times \nu)$$

Second we then prove for ant measurable  $f \geq 0$

There exists a increasing sequence  $\{f_n\}_{n \in \mathbb{N}}$  of simple non-negative functions converges pointwisely to  $f$ . Then define

$$g_n(y) = \int_X (f_n)_y d\mu$$

Note that

$$\int_{X \times Y} f d\mu \times \nu = \int_Y \left( \int_X f_y d\mu \right) d\nu = \int_Y g_n d\nu$$

take the limits

$$\begin{aligned} \int_{X \times Y} f d(\mu \times \nu) &= \int_{X \times Y} \lim_{n \rightarrow +\infty} f_n d(\mu \times \nu) \\ &= \int_Y \lim_{n \rightarrow +\infty} g_n d\nu \\ &= \int_Y \lim_{n \rightarrow +\infty} \left( \int_X (f_n)_y d\mu \right) d\nu \\ &= \int_Y \left( \int_X f_y d\mu \right) d\nu \end{aligned}$$

## 53.25 Corollary

Fubini-Tobelli holds for  $f \in L^1(X \times, \Sigma_X \otimes \Sigma_Y, \mu \times \nu)$

### 53.25.1 Proof

Apply the theorem to  $f \vee 0$  and  $-(f \wedge 0)$ , since

$$f = f \vee 0 - (-(f \wedge 0)) = f \vee 0 + f \wedge 0$$

## 53.26 Remark

Deny of the corollary neither hold nor make sense. The integral gives either  $+\infty$  or  $-\infty$

## 53.27 Remark

If  $X = Y = \mathbb{R}$  and  $\Sigma = \mathcal{B}(\mathbb{R})$  For  $f : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f \in L^1(I_{\lambda^2})$ , you can find a rectangle  $E \subseteq R = [a, b] \times [c, d]$  And define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

Then apply Fubini-Tobelli theorem.

**Example**

- $E = \{(x, y) \in \mathbb{R} : 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq x\}$

$$\iint_E \sim (x + y) dx dy$$

•

$$\begin{aligned} \iint_E \sin(x + y) dx dy &= \int_0^{\frac{\pi}{2}} \int_0^x \sin(x + y) dy dx \\ &= \int_0^{\frac{\pi}{2}} -\cos(x + y) \Big|_0^x dx \\ &= \int_0^{\frac{\pi}{2}} -\cos(2x) + \cos(x) dx \\ &= \left( -\frac{\sin(2x)}{2} + \sin(x) \right) \Big|_0^{\frac{\pi}{2}} \\ &= 1 \end{aligned}$$

**53.28 Notation**

$U \subseteq \mathbb{R}^n$  is an open set.  $C_c^0(U)$  denotes the set of continuous functions  $f : U \rightarrow \mathbb{R}$  that have compact support

$$\text{Supp}(f) := \{x \in U \mid f(x) \neq 0\}$$

**53.29 Remark**

Functions of  $C_c^0(U)$  are measurable

Let  $g : \underbrace{U}_{\subseteq \mathbb{R}^n} \rightarrow \mathbb{R}^m$  be a differentiable function. Then the Jacobian of  $g$  is the matrix

$$J_g(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{pmatrix}$$

where

$$g(x_1, \dots, x_n) = \begin{pmatrix} g_1(x_1, \dots, x_n) \\ \vdots \\ g_m(x_1, \dots, x_n) \end{pmatrix}$$

The Jacobian is related to the differential of  $g$ . In fact

$$\begin{aligned} dg_p : \mathbb{R}_p^n &\rightarrow \mathbb{R}_{g(p)}^m \\ \sigma &\mapsto J_g|_p(\sigma) \end{aligned}$$

### 53.30 Theorem(Change of variables for the Lebesgue integral)

Let  $V \subseteq \mathbb{R}^n$  be an open set, and let  $\varphi : V \rightarrow \mathbb{R}^n$  be a  $C^1$ -differ morphism, then

$$\int_{\varphi(V)} f d\lambda^n = \int_V (f \circ \varphi) |\det J_\varphi| d\lambda^n \quad \forall f \in C_c^0(\varphi(V))$$

#### Proof

### 53.31 Remark

The theorem can be generalized to a bigger classes of functions. In fact, it possible to show:

It holds whenever one of the two integrals exists  
(Zorich II)

### 53.32 Compute integrals in $\mathbb{R}^n$

#### 53.32.1 Example

$$f(x, y) = \frac{1}{1+x^2+y^2}$$

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \sqrt{3}x; 1 < x^2 + y^2 < 4\} \int_A f dx dy = ?$$

Use polar coordinates

$$\begin{aligned} \varphi : [0, +\infty[ \times [0, 2\pi[ &\rightarrow \mathbb{R}^2 \\ (\rho, \theta) &\mapsto (\rho \cos \theta, \rho \sin \theta) \end{aligned}$$

$\varphi$  is  $C^1$ -differentiable.

use the theorem

$$\begin{aligned} \int_A f(x, y) dx dy &= \int_{\tilde{A}} (f \circ \varphi) |\det J_\varphi| d\rho d\theta \\ &= \int_0^{\frac{\pi}{3}} \int_1^2 \frac{\rho}{1+\rho^2} d\rho d\theta \\ &= \int_0^{\frac{\pi}{3}} \left[ \frac{1}{2} \ln(1+\rho^2) \right]_1^2 d\theta \\ &= \frac{\pi}{6} \ln\left(\frac{5}{2}\right) \end{aligned}$$

### 53.33 Def

$\omega = \sum a_i x_i \in \Omega^n(U)$   $\gamma : [a, b] \rightarrow U$  is piecewise of class  $C^1$  Then we have defined  $\int_\gamma \omega$  The fact that  $\gamma$  is differentiable is important thus we need  $\gamma^* \omega$

Let

$$\varphi : \tau = [c, d] \rightarrow t = [a, b]$$

is a  $C^1$ -diffeomorphism. We say that  $\varphi$  preserves the orientation if  $\varphi' > 0$ , we say that  $\varphi$  reverses the orientation if  $\varphi' < 0$

Assume it preserves orientation

$$\begin{aligned} \int_{\gamma} \omega &= \int_a^b \left( \sum_i a_i(\gamma(t)) \cdot \frac{dx_i}{dt} \right) dt \\ &= \gamma(t) = (x_1(t), \dots, x_n(t)) \\ &= \int_a^b \left( \sum_i a_i(\gamma(\varphi(\tau))) \frac{dx_i}{d\tau} \underbrace{\frac{\tau}{t}}_{= |J_{\varphi^{-1}}|} \right) dt \\ \text{the change of variables} &= \int_c^d \sum_i a_i(\gamma(\varphi(\tau))) \frac{dx_i}{d\tau} d\tau \\ &= \int_{\gamma \circ \varphi} \end{aligned}$$

We call  $\gamma \circ \varphi$  a reparameterization of the curve  $\gamma$ , with the  $C^1$ -differ  $\varphi$ . If  $\varphi$  preserves the orientation, then

$$\int_{\gamma} \omega = \int_{\gamma \circ \varphi} \omega$$

if reverse

$$\int_{\gamma} \omega = - \int_{\gamma \circ \varphi} \omega$$

### 53.34 Def

$$\omega = \sum_{i=1}^n a_i dx_i \in \Omega^1(U)$$

- We say that  $\omega$  is **closed** if  $d\omega = 0$
- We say that  $\omega$  is **exact** in  $V \subseteq U$  if there exists a mapping  $f : V \rightarrow \mathbb{R}$  s.t.  $\omega = df$  in  $V$

### Goal:

to relate the notions of exact forms/closed forms/integrals along curves.



### 53.35 Def

Let  $X$  be topological space.  $U \in X$  is connected if cannot be written as disjoint union of non-empty open sets.

Equally,  $U \in X$  is connected if

$$U = A \sqcup B \Rightarrow A = \emptyset \text{ or } B = \emptyset$$

### 53.36 Lemma

Let  $U \subseteq \mathbb{R}^n$  be a connected open set. Then any two points of  $U$  can be joined by a piecewise  $C^1$ -curve.

#### Proof

Take  $a \in U$  let  $H \subseteq U$  the set of points that can be joined to  $a$  with a piecewise  $C^1$ -curve. Let  $K = U \setminus H$ .

Take  $x \in H$  then  $\exists \mathcal{B}(x, \epsilon) \subseteq U$  since  $U$  is open

Any two points in  $\mathcal{B}(x, \epsilon)$  can be jointed with a segment. Take any  $y \in \mathcal{B}(x, \epsilon)$ , this  $y$  can be joined to  $a$  with a piecewise  $C^1$ -curve.

This means that  $H$  is open. Similarly  $K$  is also open. Since  $U = H \sqcup K$  and for  $U$  connected  $H = U$

### 53.37 Notation

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be a curve, we define:

$$(-\gamma)(t) = \gamma(a + b - t)$$

as reserved curve of  $\gamma$

$$\int_{-\gamma} \omega = - \int_{\gamma} \omega$$

### 53.38 Def

If  $\gamma_1 : [a, b] \rightarrow U$  and  $\gamma_2 : [b, c] \rightarrow U$  are two curves and  $\gamma_1(b) = \gamma_2(b)$ , we define

$$\begin{aligned} \gamma_1 \sqcup \gamma_2 : [a, c] &\rightarrow U \\ t &\mapsto \begin{cases} \gamma_1(t) & \text{if } t \in [a, b] \\ \gamma_2(t) & \text{if } t \in [b, c] \end{cases} \end{aligned}$$

One has

$$\int_{\gamma_1 \sqcup \gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\gamma_2} \omega$$

### 53.39 Theorem

The following statements are equivalent:

- 1  $\omega$  is exact in a connected open set  $V \subseteq U$
- 2  $\int_\gamma \omega$  depends only on the end-point of  $\gamma$  ( $\forall \gamma$  in  $V$ )
- 3  $\int_\gamma \omega = 0$  for all closed curves  $\gamma$  in  $V$

#### Proof

1  $\rightarrow$  2 Since  $\omega = df$

$$\int_\gamma \omega = \int_\gamma df = f(\gamma(b)) - f(\gamma(a))$$

2  $\rightarrow$  3 Take a closed curve such that  $\gamma(a) = \gamma(b) = p \in V$ . Take the curve  $\gamma' = \{p\}$

$$\int_\gamma \omega = \int_{\{p\}} \omega = 0$$

3  $\rightarrow$  2 consider  $\gamma \sqcup -\gamma$ , trivial

2  $\rightarrow$  1 Fix  $p \in V$  For any  $(p+t) \in V$  by the lemma 53.36, there exists a curve  $\gamma_x$  piecewise  $C^1$  that connects  $p$  and  $(p+t)$   
Let

$$\begin{aligned} f: V &\rightarrow \mathbb{R} \\ x &\mapsto \int_{\gamma_x} \omega \end{aligned}$$

Take  $\omega = \sum_{i=1}^n a_i dx_i$ , one knows that

$$\inf_{x \in [p_i, p_i+t_i]} a_i(x) \leq \int_{p_i}^{p_i+t_i} a_i dx_i \leq \sup_{x \in [p_i, p_i+t_i]} a_i(x)$$

Since  $a_i$  are continuous

$$\lim_{t_i \rightarrow 0} \inf_{x \in [p_i, p_i+t_i]} a_i(x) = \lim_{t_i \rightarrow 0} \sup_{x \in [p_i, p_i+t_i]} a_i(x) = a_i(p_i)$$

This means that for any curve  $\gamma$  with end points  $p$  and  $p+t$

$$\int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

Hence

$$o(\|t\|) + d_p f(t) = f(p+t) - f(p) = \int_\gamma \omega = \omega(p)(t) + o(\|t\|)$$

which ends the proof.

### 53.40 Poincaré Lemma

Let  $\omega$  be a 1-form. Then  $d\omega = 0$  iff

$$\forall p \in U \quad \exists V \in \mathcal{V}_p, f : V \rightarrow \mathbb{R} \in C^1(\mathbb{R})$$

such that

$$df = \omega$$

#### Proof

Assume that  $\omega$  is locally exact, then locally

$$\omega = df$$

Assume that  $U = \bigcup_{\alpha} V_{\alpha}$ ,  $\omega$  is exact,  $d$  on  $V_{\alpha}$  is  $d(df) = 0$

Assume that  $d\omega = 0, \forall p = (p_1, \dots, p_n) \in U$ , consider an open ball  $\mathcal{B}_p$  centered on  $p$  and all contained in  $V$

Fix an  $x \in \mathcal{B}_p$  Consider  $B(t) = p + t(x - p), t \in [0, 1]$  Let

$$f(x) = \int_{B(t)} \omega = \int_0^1 a_i(B(t))(x_i - p_i) dt$$

We put  $V = B(p)$  and show that  $df = \omega$

Note that

$$\begin{aligned} 0 = d\omega &= \sum_i d \wedge dx_i = \sum_i \left( \sum_j \frac{\partial a_i}{\partial x_j} dx_j \right) dx_i = \sum_{ij} \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \\ \frac{\partial a_i}{\partial x_j} &= \frac{\partial a_j}{\partial x_i} \end{aligned}$$

Consider  $x_i = x_1$

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \int_0^1 \left( \frac{\partial a_1}{\partial x_1} t(x_1 - p_1) + a_1 + \sum_{i>1} \frac{\partial a_i}{\partial x_1} t(x_i - p_i) \right) dt \\ &= \int_0^1 \left( \frac{dt}{d} (a_1 B(t)) t + a_1 \right) dt \\ &= \int_0^1 \frac{dt}{d} (a_1 \circ B(t) \cdot t) dt \\ &= a_1 \circ B(1) \\ &= a_1(x) \end{aligned}$$

When locally exact implies exact depends on the topological properties of  $U$

### 53.41 Notation

For any mapping  $\gamma : [a, b] \rightarrow U$

- $\gamma$  is called a closed curve if  $\gamma(a) = \gamma(b)$  and  $\gamma$  is a curve
- $\gamma$  is called a path if  $\gamma$  is of class  $C^0$
- $\gamma$  is called a loop if  $\gamma$  is a closed path

We try to integrate forms along paths.

### 53.42 Def

Let  $\omega$  be a closed 1-form  $\gamma : [a, b] \rightarrow U$  be a path. We derive a partition of  $[a, b]$ :

$$0 \leq t_0 < t_1 < \cdots < t_k < t_{k+1} < b$$

such that  $\gamma_i := \gamma|_{[t_i, t_{i+1}]}$  form a finite open covering

$$\bigcup_i B_i \supseteq \gamma([a, b])$$

We may assume that  $\gamma_i \subseteq B_i$ . From Poincare lemma, we find a function  $f_i : B_i \rightarrow \mathbb{R}$  s.t. on  $B_i$

$$df_i = \omega$$

We define the integral along path  $\gamma$  as

$$\int_{\gamma} \omega := \sum_i f_i(t_{i+1}) - f_i(t_i)$$

### 53.43 Prop

The def of path integral is unique and coincides with the usual definition 52.2

#### Proof

Let  $\mathcal{P} = \{t_1, \dots, t_n\}$  be a partition of  $[a, b]$  and  $\mathcal{P}'$  be a refinement of  $\mathcal{P}$ , namely  $\mathcal{P} \in \mathcal{P}'$ . We know that  $\exists t' \in \mathcal{P}' \setminus \mathcal{P}$  s.t.  $t' \in ]t_i, t_{i+1}[$ ,  $\gamma(t') \subseteq B_i$

Compute the integral with the partition  $\mathcal{P}'$ , in the summation:

$$f_i(\gamma(t_{i+1})) - f_i(\gamma(t')) + f_i(\gamma(t')) + c - f_i(\gamma(t_i)) - c$$

This means that if  $\mathcal{P}' \subseteq \mathcal{P}$  then

$$\int_{\gamma}^{(\mathcal{P})} \omega = \int_{\gamma}^{(\mathcal{P}')} \omega$$

For any two partition  $\mathcal{P}_1$  and  $\mathcal{P}_2$ , find the common refinement

$$\int_{\gamma}^{(\mathcal{P}_1)} \omega = \int_{\gamma}^{(\mathcal{P}_1 \cup \mathcal{P}_2)} \omega = \int_{\gamma}^{(\mathcal{P}_2)} \omega$$

### 53.44 Def

Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow U$  be two paths. A **homotopy** between  $\gamma_0$  and  $\gamma_1$  is a continuous mapping:

$$\begin{aligned} H : [a, b] \times [0, 1] &\rightarrow U \\ (s, t) &\mapsto H(s, t) \end{aligned}$$

such that

- $H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$
- $H(a, \cdot) \equiv \gamma_0(a) = \gamma_1(a) \quad H(b, \cdot) \equiv \gamma_0(b) = \gamma_1(b)$

### 53.45 Def: Lebesgue number

Let  $(X, \rho)$  be a metric space and  $\mathcal{U} = \{U_i\}$  be an open covering  $X$

A **Lebesgue number**  $\delta = \delta_{\mathcal{U}}$  (of the open covering  $\mathcal{U}$ ) is a non-negative number that:

If  $Z \subseteq X$  is a subset with  $\text{diam}(Z) < \delta$ , then  $Z \subseteq U_j$  for some  $U_j \in \mathcal{U}$

#### Remark

- $\delta' < \delta$  is also a Lebesgue number
- In principle, a Lebesgue number  $\delta$  can be 0

### 53.46 Lemma

If  $X$  is compact, then for any open covering there exists a positive Lebesgue number.

#### Proof

Let  $\mathcal{U}$  be an open covering. Since  $X$  is compact, then

$$\mathcal{U} \supseteq \{U_1, \dots, U_n\}$$

If one of  $U_i$  is equal to  $X$ , then any  $\delta > 0$  is a Lebesgue number. So we can assume that  $\forall i \in \{1, \dots, n\}$  the set

$$C_i := X \setminus U_i$$

is non-empty. let the mapping  $f$  be:

$$\begin{aligned} f : X &\rightarrow \mathbb{R} \\ x &\mapsto \frac{1}{n} \sum_{i=1}^n d(x, C_i) \end{aligned}$$

$f$  is continuous on a compact set. Hence it attains a minimum. This minimum is not 0 because  $d(x, C_i) > 0$  for some  $i$

Let  $\delta = \min_{x \in X} f(x)$ , we show that  $\delta$  is a Lebesgue number

Let  $Y \subseteq X$  s.t.  $\text{diam}(Y) < \delta$ . Take  $x_0 \in Y$ , then

$$Y \subseteq \mathcal{B}(x_0, \delta)$$

Since  $f(x_0) \geq \delta$ , then there exists  $i$  such that

$$d(x_0, C_i) \geq \delta$$

(otherwise  $f(x_0) = \frac{1}{n} \sum_{i=1}^n d(x_0, C_i) < \frac{1}{n} n\delta = \delta$ )

this means

$$(X \setminus U_i) \cap Y = \emptyset$$

Hence  $Y \subseteq U_i$

### Exercise

Being homotopic is an equivalence relation.

## 53.47 Theorem(homotopy invariance of the integrals)

Ler  $\omega$  be a closed form on an open set  $U$ . Let  $\gamma_0, \gamma_1$  be homotopy paths in  $U$ , then

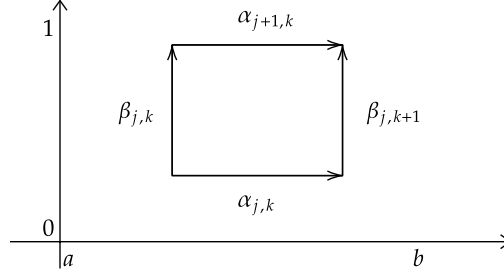
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

### Proof

Since  $\omega$  is closed, it's locally exact (by Poincare lemma 53.40) Let  $H : [a, b] \times [0, 1] \rightarrow U$  be the homotopy between  $\gamma_0$  and  $\gamma_1$ . Let  $\mathcal{B} = \{B_i\}$  be a open cover of  $\text{Im}(H) \subseteq U$  made of finite many open balls (since compact) where  $\omega$  is locally exact in each.

Consider  $W_i = H^{-1}(B_i)$  We have covered the departure set  $\mathcal{D}_H$  with  $\{W_i\}$ . Since  $\mathcal{D}_H$  is compact, we can choose a Lebesgue number  $\delta > 0$  for the covering  $\{W_i\}$  (by lemma 53.36)

Divide  $\mathcal{D}_H$  into rectangles  $\{R_j k\}$  having diameter  $< \delta$



The border of the rectangles are loops. We divide the loops in this way:

$$\partial R_{j,k} = \alpha_{j,k} \sqcup \beta_{j,k+1} \sqcup (-\alpha_{j+1,k}) \sqcup (-\beta_{j,k})$$

$$H\partial R_{j,k} = H\alpha_{j,k} \sqcup H\beta_{j,k+1} \sqcup H(-\alpha_{j+1,k}) \sqcup H(-\beta_{j,k})$$

$H\partial R_{j,k}$  is closed curve contained in some  $B_i$ , but in such balls  $\omega$  is exact, then  $\int_{\partial R_{j,k}} \omega = 0$ . Do this  $\forall j, k$

$$0 = \sum_{j,k} \int_{\partial R_{j,k}} \omega = \sum_{j,k} \left( \int_{H\alpha_{j,k}} \omega + \int_{H\beta_{j,k+1}} \omega - \int_{H\alpha_{j+1,k}} \omega - \int_{H\beta_{j,k}} \omega \right)$$

Moreover, if we do this for some particular  $j, k$

$$0 = \int_{H(a,0) \rightsquigarrow (b,0)} \omega + \int_{H(b,0) \rightsquigarrow (b,1)} \omega - \int_{H(a,1) \rightsquigarrow (b,1)} \omega - \int_{H(a,0) \rightsquigarrow (a,1)} \omega$$

Since  $H(a,0) \rightsquigarrow (b,0)$  and  $H(a,1) \rightsquigarrow (b,1)$  are points, then

$$\int_{H(a,0) \rightsquigarrow (a,1)} \omega = \int_{H(b,0) \rightsquigarrow (b,1)} \omega$$

Proved





## Chapter 54

# Winding Numbers

### 54.1 Def: Free Homotopy

Let  $\gamma_0, \gamma_1 : [a, b] \rightarrow U$  be two loops (namely  $\gamma(a) = \gamma(b)$ )

A **free homotopy** between  $\gamma_0$  and  $\gamma_1$  is a continuous mapping:

$$\begin{array}{ccc} H : [a, b] \times [0, 1] & \rightarrow & U \\ (s, t) & \mapsto & H(s, t) \end{array}$$

such that

- 

$$H(\cdot, 0) = \gamma_0 \quad H(\cdot, 1) = \gamma_1$$

- For any fixed  $t_0$

$$H(\cdot, t_0)$$

is a loop

### 54.2 Notation

A path  $\gamma : [a, b] \rightarrow I$  is said simple if  $\gamma|_{]a, b[}$  is injective (No self-cross this is)

### 54.3 Jordan Theorem

Let  $\gamma$  be a simple loop  $\gamma : [a, b] \rightarrow U$ , then  $\mathbb{R}^2 \setminus \gamma([a, b])$  consists exactly of two connected components. One of this is bounded (interior), the other one unbounded (exterior). Moreover  $\gamma([a, b])$  is the boundary of two components.

**Proof**

Exercise?

Let  $S^1 \subseteq \mathbb{R}$  be the unit circle. Let  $c : I \rightarrow S^1$  be a closed curve. ( $0 \in I$ )

We want to measure the "net" turns of  $c$  around the region.

$$\begin{aligned} c(t) &= (x(t), y(t)), x^2 + y^2 = 1 \quad \forall t \in I \\ &\Rightarrow 2(xx' + yy') = 0 \end{aligned}$$

Consider  $\varphi_0 \in [0, 2\pi[$  such that

$$\cos(\varphi_0) = x(0), \sin(\varphi_0) = y(0)$$

Define the function

$$\begin{aligned} \varphi : I &\rightarrow [0, 2\pi] \\ t &\mapsto \varphi_0 + \int_0^t (xy' - yx') dt \end{aligned}$$

$\varphi$  is called an angle function of  $c$ , namely is of class  $C^0$  and  $c(t) = (\cos(\varphi(t)), \sin(\varphi(t)))$

And

$$\begin{aligned} F(t) &= (x(t) - \cos \varphi(t))^2 + (y(t) - \sin \varphi(t))^2 \quad t \in I \cap [0, 2\pi] \\ F'(t) &= -y \sin \varphi (yy' + xx') - x \cos \varphi (xx' + yy') = 0 \end{aligned}$$

This implies  $F(t) = C$  a constant. However

$$F(0) = (x(0) - x(0))^2 + (y(0) - y(0))^2 = 0$$

So  $C = 0$  It means that

$$\begin{cases} \cos(\varphi(t)) = x(t) \\ \sin(\varphi(t)) = y(t) \end{cases} \quad \forall t \in I$$

**54.4 Def**

Let  $c : [a, b] \rightarrow S^1$  be a closed curve. Let  $\varphi$  be the angular function of  $c$ . We define the winding number of  $c$  as:

$$n(c) = \frac{1}{2\pi}(\varphi(b) - \varphi(a))$$

Since  $c$  is a closed curve,  $n(c) \in \mathbb{Z}$

**54.5 Def**

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  be a closed curve. ( $\gamma_p + \rho(t)c(t)$ ), when  $c(t) \in S^1$

$$\gamma(t) = p + \rho(t)(\cos(\theta(t)) + \sin(\theta(t)))$$

Then we define the winding number of  $\gamma$  at  $p$

$$n_p(\gamma) := n(c)$$

## 54.6 Prop

Let  $\gamma = p + \rho(t)c(t)$  be a closed curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  then

$$n_p(\gamma) = \frac{1}{2\pi i} \int_C \omega_0$$

where

$$\omega_0 = -\frac{y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy$$

### Proof

$$\begin{aligned} \frac{1}{2\pi} \int_C \omega_0 &= \frac{1}{2\pi} \int_a^b (x(t)y'(t) - y(t)x'(t)) dt \\ &= \frac{\varphi(b) - \varphi(a)}{2\pi} \\ &= n(c) \end{aligned}$$

## 54.7 Prop

Let  $\gamma_0, \gamma_1 : [0, b] \rightarrow \mathbb{R}^2 \setminus \{p\}$  be two closed curves. Then they're freely homotopic iff

$$n_p(\gamma_0) = n_p(\gamma_1)$$

### Proof

$\Rightarrow$  Follows from the invariance of the integral of  $\omega_0$  along free homotopic loops. (We use the previous prop)

$\Leftarrow$  Assume  $\gamma_0 = p + \rho_0 c_0(t)$ ;  $\gamma_1 = p + \rho_1 c_1(t)$ . The angle functions of  $c_0$  and  $c_1$  are  $\varphi_0$  and  $\varphi_1$  respectively. Consider

$$\varphi(s, t) = (1 - t)\varphi_0(s) + t\varphi_1(s)$$

$$(s, t) \in [0, b] \times [0, 1]$$

$$H(s, t) = (\cos(\varphi(s, t)), \sin(\varphi(s, t)))$$

To claim this is a free homotopy between  $c_0$  and  $c_1$ , we have to check that any curve in the homotopy is closed.

$$\begin{aligned} \varphi(b, t) - \varphi(0, t) &= (1 - t)\varphi_0(b) + t\varphi_1(b) - (1 - t)\varphi_0(0) - t\varphi_1(0) \\ &= (1 - t)(\varphi_0(b) - \varphi_0(0)) + t(\varphi_1(b) - \varphi_1(0)) \\ &= 2\pi(1 - t)n_p(\gamma_0) + 2\pi t n_p(\gamma_1) \\ &= 2\pi n_p(\gamma_0) \end{aligned}$$

So we have proved that  $c_0$  and  $c_1$  are freely homotopic. What we anticipate is

$$\gamma_i \sim_{hom} \hat{\gamma}_i = p + c_i \sim_{hom} c_i$$

Now we only need to prove the first homotopic.

$$H(s, t) = p + \frac{\rho_1(s)}{(1-t) + t|\rho_1(s)|} c_i(s)$$

### 54.8 Def

Let  $F : U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a differential mapping. We say that  $p \in U$  is a zero of  $F$  if  $F(p) = 0$ . If then exists a neighborhood  $V$  of  $p$  such that  $V$  contains no zero of  $F$  other than  $p$ , then  $p$  is called isolated zero.

If  $p$  is a zero of  $F$  and  $dF|_p$  is non singular at  $p$ , then we say that  $p$  is a simple zero.

### 54.9 Remark

By the inverse function then  $F$  is one to one in a neighborhood of a simple zero. Hence a simple zero if isolated.

$$F(x, y) = (f(x, y), g(x, y))$$

$D \subseteq U$  is a closed disk, with boundary  $\partial D = C$  Assume that  $C$  doesn't contain zeros of  $F$ . Consider the form

$$\theta = \frac{f dg - g df}{f^2 + g^2} \in \Omega^1(U \setminus \{(x, y) : F(x, y) = 0\})$$

### 54.10 Def

The index of  $F$  in  $D$ , is defined as

$$n(F, D) := \frac{1}{2\pi} \int_C \theta$$

See that  $\theta = F^* \omega_0$ ,  $\omega_0 = \frac{-y dx + x dy}{x^2 + y^2}$

$$\begin{aligned} n(F, D) &= \frac{1}{2\pi} \int_C \theta \\ &= \frac{1}{2\pi} \int_C F^* \omega_0 \\ &= \frac{1}{2\pi} \int_{F \circ C} \omega_0 \\ &= (\text{winding number of } F \circ C \text{ at the center of } FD) \end{aligned}$$

### 54.11 Remark

$$n(F, D) = \frac{1}{2\pi} \int_C \theta = \frac{1}{2\pi} \int_{F \circ C} \omega_0$$

### 54.12 Prop

If  $n(F, D) \neq 0$  then  $\exists q \in D$  s.t.  $F(q) = 0$

#### Proof

Assume that such  $q$  doesn't exist. Let  $p$  be the center of  $D$

$$H(s, t) = F((1 - t)C(s) + t \cdot p)$$

then  $H$  is a free homotopy between  $F \circ s \mapsto p$  and  $\circ C$

$$0 = \frac{1}{2\pi} \int_F \circ C \omega_0 = n(F, D)$$

Contradiction

### 54.13 Def

A simple zero  $p$  of  $F$  is said **positive** if  $\det(d_p F) > 0$ , otherwise is said **negative** (what's =0?)

### 54.14 Kronecker Index Theorem

Assume that  $F; U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$  has only finite simple zeros in a disk  $D \subseteq U$  and none of them in  $\partial D$ . Then

$$n(F, D) = P - N$$

where  $P$  is the number of positive simple zeros and  $N$  is the number of negative simple zeros.

#### Lemma

Assume that  $F$  has a simple zero  $p \in D \subseteq U$  then  $n(F, D) = \pm 1$  corresponding to  $\det(d_p F) > 0$  or  $\det(d_p F) < 0$

**Proof**

After translating, we assume that  $p = (0, 0)$  By definition of differential

$$F(q) = Tq + R(q) |q| \quad q \rightarrow 0 \quad \lim_{q \rightarrow 0} R(q) = 0$$

Consider the mapping

$$\begin{aligned} H : [0, b] \times [0, 1] &\rightarrow \mathbb{R}^2 \\ (s, t) &\mapsto Tq + (1 - t)R(q) |q| \end{aligned}$$

If we show that for  $D$  small enough,  $H(q, t) \neq 0, \forall q \in D, t \in [0, 1]$  (\*) Then  $H(C(s), t)$  is a free homotopy between  $F_0 \circ C$  and  $T \circ C$  (Exercise) Then

$$n(F, D) = n(T, D)$$

which one should prove is  $n(T, D) = 1$

For the part (\*), since  $p$  is non-singular,  $c = \frac{1}{\|T^{-1}\|} > 0$ , then

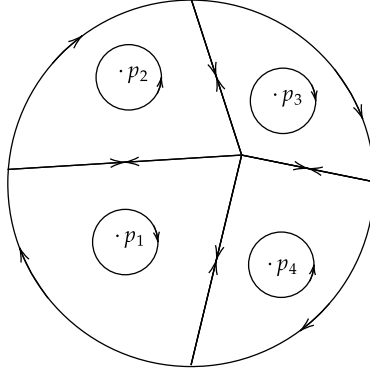
$$\|q\| = \|T^{-1}Tq\| \leq \|T^{-1}\| \|Tq\| = \frac{\|Tq\|}{c}$$

Take  $\epsilon > 0$  such that  $\forall p \in D$  (disk of radius  $< \epsilon$ )  $\|R(q)\| \leq \frac{c}{2}$  Then if  $q \in D_\epsilon \setminus \{0, 0\}$

$$\begin{aligned} \|H(q, t)\| &= \|Tq + (1 - t)R(q) |q|\| \\ &\geq \|Tq\| - (1 - t) \|R(q)\| \|q\| \\ &\geq c \|q\| - \frac{(1 - t)c}{2} \|q\| \\ &\geq c \|q\| - \frac{c}{2} \|q\| > 0 \end{aligned}$$

**Proof**

Let  $p_1, \dots, p_k$  be the zeros of  $F$  in  $D$ . Since these zeros are isolated, we choose a set of balls  $\{B_i\}_{i=1}^k$  containing  $\{p_i\}_{i=1}^k$  respectively. Then cut the circle into  $k$  sections correspondingly.



One can easily construct the homotopy for these balls and sections.  
By lemma

$$\int_{F \circ C} \omega_0 = \sum_{i=1}^k \int_{FB_i} \omega_0 = \sum_{i=1}^k \text{sgn}(\det d_{p_i} F) = P - N$$

**54.15 Lemma**

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$

1 If  $\gamma \in C^1([a, b])$  then

$$\int_a^b \gamma'(t) dt = \gamma(b) - \gamma(a)$$

2 For any  $c \in \mathbb{R}^n$ :

$$\left\langle c, \int_a^b \gamma(t) dt \right\rangle = \int_a^b \langle c, \gamma(t) \rangle dt$$

3

$$\left\| \int_a^b \gamma(t) dt \right\| \leq \int_a^b \|\gamma(t)\| dt$$

**Proof**

1 Is consequence of the fundamental theorem of calculus computation

2

$$\begin{aligned}
\left\langle c, \int_a^b \gamma(t) dt \right\rangle &= \sum_{i=1}^n c_i \int_a^b \gamma_i(t) dt \\
&= \int_a^b \sum_i c_i \gamma_i(t) dt \\
&= \int_a^b \langle c, \gamma(t) \rangle dt
\end{aligned}$$

3 We apply (2) with  $c = \int_a^b \gamma(t) dt \in \mathbb{R}^n$ 

$$\begin{aligned}
\left\| \int_a^b \gamma(t) dt \right\| &= \left\langle \int_a^b \gamma(\tau) d\tau, \int_a^b \gamma(t) dt \right\rangle \\
(\text{by (2)}) &= \int_a^b \left\langle \int_a^b \gamma(\tau) d\tau, \gamma(t) \right\rangle dt = (*)
\end{aligned}$$

Apply Cauchy-Schwartz inequality

$$(*) \leq \int_a^b \left\| \int_a^b \gamma(\tau) d\tau \right\| \|\gamma(t)\| dt = \left\| \int_a^b \gamma(\tau) d\tau \right\| \int_a^b \|\gamma(t)\| dt$$

Divide by  $\left\| \int_a^b \gamma(\tau) d\tau \right\|$ **54.16 Def**Let  $\mathcal{P} = \{t = t_a, t_1, \dots, t_n = b\}$ ,  $p_i = \gamma(t_i)$ 

$$l_{\mathcal{P}}(\gamma) = \sum_{i=0}^n \|p_{i+1} - p_i\|$$

The length of  $\gamma$  is

$$l(\gamma) := \sup_p \{l_p(\gamma)\}$$

If  $l(\gamma) < +\infty$ , then path  $\gamma$  is said rectifiable.**54.16.1 Example**

Consider  $\gamma : [0, 1] \rightarrow \mathbb{R}^2$   $\gamma_1(t) = t$   $\gamma_2(t) = \begin{cases} t \sin \frac{\pi}{2t} & \text{if } t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

$$\mathcal{P}_n \{0, \frac{1}{2n+1}, \frac{1}{2n-1}, \dots, \frac{1}{3}, 1\}$$



Recall that  $\sin((j - \frac{1}{2})\pi) = (-1)^{j+1}$ , then for  $i > 0$ ,

$$\begin{aligned} p_i = \gamma(t_i) &= \left( \frac{1}{2j-1}, \frac{1}{2j-1}(-1)^{j+1} \right) \\ \|p_i - p_j\| &= \sqrt{\left(\frac{2}{4j^2-1}\right)^2 + \left(\frac{4j}{4j^2-1}\right)^2} \\ &= \sqrt{\frac{4(4j^2+1)}{(4j^2-1)^2}} \\ &\geq \sqrt{\frac{4(4j^2-1)}{(4j^2-1)^2}} \\ &= \sqrt{\frac{4}{4j^2-1}} \\ &\geq \frac{1}{j} \end{aligned}$$

then

$$l_p(\gamma) \geq \sum_{j=1}^n \frac{1}{j} \Rightarrow l(\gamma) = +\infty$$

## 54.17 Prop

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  be of class  $C^1$ , then  $\gamma$  is rectifiable and

$$l(\gamma) = \int_a^b \|\gamma'(t)\| dt$$

moreover  $l(\gamma)$  doesn't depend on the parametrization of  $\gamma$

### Proof

Let  $\mathcal{P}$  be a partition of  $[a, b]$ .

$$\|\gamma(t_{j+1}) - p_j\| = \left\| \int_{t_i}^{t_{j+1}} \gamma'(t) dt \right\| \leq \int_{t_i}^{t_{j+1}} \|\gamma'(t)\| dt$$

So

$$\begin{aligned} l_{\mathcal{P}}(\gamma) &= \sum_{j=0}^{n-1} \|\gamma(t_{j+1}) - \gamma(t_j)\| \\ &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt \\ &= \int_a^b \|\gamma'(t)\| dt \end{aligned}$$

take sup over  $\mathcal{P}$

$$l(\gamma) \leq \int_a^b \|\gamma'(t)\| dt$$

which means  $l(\gamma)$  is finite.

Let's prove the other inequality:

$\gamma'(t)$  is continuous on  $[a, b]$  then is uniformly continuous (Cantor-Itener Theorem):  $\forall \epsilon > 0 \exists \delta = \delta(\epsilon)$  such that if  $t, s \in [a, b]$  and  $|t - s| < \delta$ . Then  $\|\gamma'(t) - \gamma'(s)\| < \epsilon$

Choose now a partition  $\mathcal{P}$  such that  $|t_{j+1} - t_j| < \delta \forall i$  then if  $t \in [t_i, t_{i+1}]$  then  $\|\gamma'(t) - \gamma'(t_j)\| < \epsilon$

$$\begin{aligned} \|\gamma'(t)\| - \|\gamma'(t_j)\| &\leq \|\gamma'(t) - \gamma'(t_j)\| \\ &\leq \|\gamma'(t) - \gamma'(t_j)\| \\ &\leq \epsilon \end{aligned}$$

Hence

$$\begin{aligned} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt &\leq \|\gamma'(t_j)\| (t_{j+1} - t_j) + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t_j) + \gamma'(t_j)) dt \right\| + (t_{j+1} - t_j)\epsilon \\ &\leq \left\| \int_{t_j}^{t_{j+1}} \gamma'(t) dt \right\| + \left\| \int_{t_j}^{t_{j+1}} (\gamma'(t) - \gamma'(t_j)) dt \right\| + (t_{j+1} - t_j)\epsilon \\ &\leq \underbrace{\|\gamma(t_{j+1}) - \gamma(t_j)\|}_{\text{lemma 1}} + \underbrace{\int_{t_j}^{t_{j+1}} \|\gamma'(t) - \gamma'(t_j)\| dt}_{\text{lemma 2}} + (t_{j+1} - t_j)\epsilon \\ &\leq \|\gamma(t_{j+1}) - \gamma(t_j)\| + 2(t_{j+1} - t_j)\epsilon \end{aligned}$$

Hence

$$\begin{aligned} \int_a^b \|\gamma'(t)\| dt &\leq \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} \|\gamma'(t)\| dt \\ &\leq \sum_{j=0}^{n-1} \|\gamma(t_{j+1}) - \gamma(t_j)\| + 2(t_{j+1} - t_j)\epsilon \\ &= l_{\mathcal{P}}(\gamma) + 2\epsilon(b - a) \leq l(\gamma) + 2\epsilon(b - a) \end{aligned}$$

Take limit for  $\epsilon \rightarrow 0$

Now the formula proved. We'll show that  $l(\gamma)$  doesn't depends on the parametrization. Let  $\varphi : [\alpha, \beta] \rightarrow [a, b]$  is of class  $C^1$ ,  $\varphi'(\tau) \neq 0, \forall \tau \in [\alpha, \beta]$  Let

$$\tilde{\gamma} = \gamma \circ \varphi$$

might as while assume  $\varphi' > 0$

$$\begin{aligned} l(\gamma) &= \int_a^b \|\gamma'(t)\| dt = \int_\alpha^\beta \|\gamma'(\varphi(\tau))\| \varphi'(\tau) d\tau \\ &= \int_\alpha^\beta \|\tilde{\gamma}'(\tau)\| d\tau = l(\tilde{\gamma}) \end{aligned}$$

Same for  $\varphi' < 0$

## 54.18 Corollary(exercise)

If  $\gamma$  is a curve (piecewise  $C^1$ ), then  $\gamma$  is rectifiable and the length is the sum of the length of it's  $C^1$  pieces.

## 54.19 Def

A  $C^1$ -curve is **regular** if  $\gamma'(t) \neq 0$  for any  $t \in [a, b]$  A piecewise  $C^1$ -path (curve) is regular if all its pieces are regular

## 54.20 Def

$$N := \frac{T}{\|T\|}$$

is **normal vector** of  $T$

## 54.21 Def

Let  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  a  $C^1$  curve; Let  $l$  be the length of  $\gamma$  (by theorem proved  $l(\gamma) < +\infty$ ) Let's define the following function:

$$s(t) := \int_a^t \|\gamma'(u)\| du$$

$s(t)$  is the length of  $\gamma|_{[a,t]}$  The function  $\|\gamma'(u)\|$  is continuous, hence

$$s'(t) = \|\gamma'(t)\|$$

Now assume that  $\gamma$  is  $C^1$  and **regular** ( $\gamma'(t) \neq 0, \forall t \in [a, b]$ ), then  $s'(t) > 0$

So  $s : [a, b] \rightarrow [0, l]$  is a  $C^1$ -diffeomorphism, the inverse is

$$t : [0, l] \rightarrow [a, b]$$

$$\frac{dt}{ds} = \frac{1}{\|\gamma'(t)\|}$$

We reparameterize  $\gamma$  with  $t$  and get

$$\tilde{\gamma}(s) = (\gamma \circ t)(s)$$

$\tilde{\gamma} : [0, l] \rightarrow \mathbb{R}^n$  we say that  $\tilde{\gamma}$  is the reparameterization of  $\gamma$  with respect to its **curvilinear coordinate**  $s(t)$

## 54.22 Def

$f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$   $f$  is a  $C^{(k)}$ -differ if

- $f$  is of class  $C^{(k)}$
- $f$  is bijection, and the inverse is  $C^{(k)}$

## 54.23 Def

In general

$$\gamma : [a, b] \rightarrow \mathbb{R}^n \rightsquigarrow \tilde{\gamma} : [0, l] \rightarrow \mathbb{R}^n$$

regular and  $C^1$

$$\frac{d\tilde{\gamma}}{ds} = \frac{d\gamma}{dt} \frac{dt}{ds} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

$$\left\| \frac{d\tilde{\gamma}}{ds} \right\| = 1$$

$$T(t) := \frac{d\tilde{\gamma}}{ds} = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

tangent: (vector)  $\rightarrow$  vector of norm 1

$$0 = \frac{d}{dt} \|T(t)\|^2 = \frac{d}{dt} \langle T(t), T(t) \rangle = 2 \langle T(t), T'(t) \rangle \Leftrightarrow T'(t) \perp T(t)$$

use the fact that in  $\mathbb{R}^n$ ,  $u, v : \mathbb{R} \rightarrow \mathbb{R}^n$  differentiable

$$\frac{d}{dt} \langle u(t), v(t) \rangle = \left\langle \frac{du}{dt}, v(t) \right\rangle + \left\langle u(t), \frac{dv}{dt} \right\rangle$$

then

$$\begin{aligned} \frac{d^2\tilde{\gamma}}{ds^2} &= \frac{d}{ds} \left( \frac{d\tilde{\gamma}}{ds} \right) \\ &= \frac{d}{ds} (T(t)) \\ &= \frac{dT}{dt} \frac{dt}{ds} \\ &= \frac{T'(t)}{\|\gamma'(t)\|} \end{aligned}$$

$N(t) = \frac{d^2\tilde{\gamma}}{ds^2} / \left\| \frac{d^2\tilde{\gamma}}{ds^2} \right\|$ . If  $n = 2$  Along the curve we have a 'moving' canonical basis of

$$\begin{aligned}\mathbb{R}_{\gamma(t)}^2 &= \{T(t), N(t)\} \\ &= \{\alpha T(t) + \beta N(t) \mid \alpha, \beta \in \mathbb{R}\}\end{aligned}$$

$\{T(t), N(t)\}$  is a orthonormal basis of  $\mathbb{R}_{\gamma(t)}^2$



## Chapter 55

# Curvilinear integral

Let  $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  measurable and  $\gamma : [a, b] \rightarrow \mathbb{R}^n$  regular

### 55.1 Def

The curvilinear integral of  $f$  along  $\gamma$  is defined as:

$$\int_{\gamma} f(t) ds = \int_a^b f(\gamma(t)) s'(t) dt = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt$$

where  $s'(t)$  is curvilinear coordinate.

### 55.2 Prop

Let  $f, g : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  be two measurable functions  $\gamma, \gamma_1, \gamma_2$  (piecewise) regular  $C^1$  curves, then

1  $\alpha, \beta \in \mathbb{R}$

$$\int_{\gamma} (\alpha f + \beta g) ds = \alpha \int_{\gamma} f ds + \beta \int_{\gamma} g ds$$

2 If  $\gamma_a \sqcup \gamma_2$  make sense

$$\int_{\gamma_a \sqcup \gamma_2} f ds = \int_{\gamma_a} f ds + \int_{\gamma_2} f ds$$

3

$$\int_{\gamma_1} f ds = \int_{-\gamma_1} f ds$$

4  $\int_{\gamma} f ds$  doesn't depend on the reparameterization of  $\gamma$

**Proof**

Exercise

**55.3 Geometry**

Let  $V$  be a finite dim vector space over  $K$  ( $K = \mathbb{R}, \mathbb{C}$ ). Let  $g : V \times V \rightarrow K$  be a bilinear function. Choose a basis  $\{\sigma_1, \dots, \sigma_n\}$  we have the Gram matrix  $G = (g(\sigma_i, \sigma_j))_{i,j}$

$$g(x, y) = x^T G y$$

what happens when we change basis:

$$\begin{array}{ccc} & x \in V & \\ b \nearrow & & \nwarrow b' \\ x \in K^n & \xleftarrow{A} & x' \in K^n \end{array}$$

$$g(x, y) = x^T G y = (x'^T)(A^T G A) y'$$

$$\{v_1, \dots, v_n\} \quad G \rightsquigarrow A^T G A$$

**55.4 Def**

Two matrices  $G, G' \in M_{n \times n}(K)$  are said **congruent** if  $\exists A \in GL_n(K)$  (namely  $A$  invertible) such that  $G' = A^T G A$

$$\begin{array}{ccc} \mathcal{L}(V, V; K) & \xrightarrow{\cong} & \mathcal{L}(T_0^2(V); K) \xrightarrow{\cong} \mathcal{L}(V, V^\vee) \\ g \mapsto & & g_s \mapsto [x \mapsto g_s(x \otimes -)] \end{array}$$

In particular

$$\begin{array}{ccc} \tilde{g} : V \rightarrow V^\vee & & \\ x \mapsto & [y \mapsto g(x, y)] & \end{array}$$

- $g : V \times V \rightarrow K$
- $\begin{array}{ccc} g_p : V \times V & \rightarrow & K \\ (x, y) & \mapsto & g(y, x) \end{array}$
- $\begin{array}{ccc} \overline{g_p} : V \times V & \rightarrow & K \\ (x, y) & \mapsto & \overline{g(y, x)} \end{array}$



### 55.5 Def

A bilinear form  $g$  is

- **symmetric** if  $g = g_p$
- **symplectic** if  $g = -g_p$
- **hermitian** if  $g = -\overline{g_p}$

### 55.6 Def

$(V, g)$  is a vector space with an inner product. Two vector  $v_1, v_2 \in V$  are said **orthogonal** (w.r.t (with respect to)  $g$ ) if

$$g(v_1, v_2) = 0$$

Two subspaces  $V_1, V_2 \in V$  are **orthogonal** (w.r.t  $g$ ) if

$$g(v_1, v_2) = 0 \quad \forall v_1, v_2 \in V$$

### 55.7 Def

The kernel of an inner product  $g$  is

$$\ker g := \{\sigma \in V \mid g(\sigma, w) = 0, \forall w \in V\}$$

$g$  is said non-degenerate if

$$\ker g = \{0\}$$

### 55.8 Remark

$\ker g = \ker \tilde{g}$  where  $\tilde{g}$  a linear mapping  $\tilde{g} : V \rightarrow V^\vee$

### 55.9 Prop

Let  $g$  be a bilinear form on  $V$  and let  $G$  be the Gram matrix of  $g$  w.r.t the basis  $\{\sigma_1, \dots, \sigma_n\}$ . Then the matrix of  $\tilde{g}$  w.r.t. the basis  $\{\sigma_1, \dots, \sigma_n\}$  and  $\{\sigma_1^\vee, \dots, \sigma_n^\vee\}$  is  $G^T$

$$\begin{array}{ccc} V & \xrightarrow{\tilde{g}} & V^\vee \\ \uparrow b & & \uparrow b^\vee \\ K^n & \xrightarrow{G^T} & K^n \end{array}$$

**Proof**

$$\begin{aligned}
((b^\vee)^{-1} \circ \tilde{g} \circ b)(e_i) &= (b^\vee)^{-1} \tilde{g}(v_i) \\
&= (b^\vee)^{-1} g(v_i, -) \\
&= g_{v_i}(\cdot) : V \rightarrow K
\end{aligned}$$

$$\begin{aligned}
g_{v_i}(x) &= g(v_i, \sum_j x_j v_j) \\
&= \sum_j x_j G_{ij} \\
&= \sum_j G_{ij} v_j^\vee(x)
\end{aligned}$$

$$\text{so } g_{v_i} = \sum_j G_{ij} v_j^\vee$$

$$\begin{aligned}
(b^\vee)^{-1}(g_\vee(\cdot)) &= (b^\vee)^{-1}(\sum_j G_{ij} v_j^\vee) \\
&= \sum_j G_{ij} e_j \\
&= \text{i-th row of } G
\end{aligned}$$

**55.10 Prop**

- $g$  is symmetric iff  $G = G^T$
- $g$  is symplectic iff  $G = -G^T$
- $g$  is hermitian iff  $G = \overline{G^T}$

**Proof**

Exercise

**55.11 Prop**

Let  $g$  be an inner product on  $V$ , Then  $g$  is non-degenerate iff

$$\det G \neq 0$$

(w.r.t. any choice of basis)

**Proof**

Let  $G$  and  $G'$  be two Gram matrices of  $g$ .

$$G' = A^T G A$$

for  $A$  invertible

$$\det G' = (\det A)^2 \det G$$

$$\det G' \neq 0 \Leftrightarrow \det G \neq 0$$

Fix a basis  $\{v_1, \dots, v_n\}$

$$\ker g \cong \ker G^t$$

by the fact that  $\ker g = \ker \tilde{g}$ . So  $\ker g \neq 0$  iff

$$\det G = \det G^T \neq 0$$

**55.12 Def:isometry**

$$(V, g) \xrightarrow{f} (W, g')$$

a morphism  $f$  of vector space with inner product is **isometry** if

$$g(x, y) = g'(f(x), f(y))$$

**55.13 Def:isometric**

$V \xrightarrow{\cong} W$  up to isomorphism.

Then  $(V, g)$  and  $(W, g')$  are **isometric** if there are two isometry

$$\begin{aligned} f : (V, g) &\rightarrow (W, g') \\ f' : (W, g') &\rightarrow (V, g) \end{aligned}$$

such that

$$f \circ f' = f' \circ f = Id$$



## Chapter 56

# Complex conjugate vector space

### 56.1 Def: complex conjugate vector space

Let  $V$  be a vector space over  $K = \mathbb{C}$ . The **complex conjugate**  $\bar{V}$  is the same set of  $V$ . The sum on  $\bar{V}$  is the same of  $V$ , but we define

$$\alpha * v := \bar{\alpha}v \quad \alpha \in \mathbb{C}$$

### 56.2 Def: Semilinear

If  $V$  and  $W$  are two complex vector sapce, then a **semilinear mapping** is a mapping  $f : V \rightarrow W$  such that

- $f(v_1 + v_2) = f(v_1) + f(v_2)$
- $f(\alpha v) = \alpha * f(v) = \bar{\alpha}f(v)$

So a semilinear mapping is a linear mapping:  $f : V \rightarrow W$

For sesquilinear forms, the theory is similar to the theory of bilinear forms.

$$g \rightsquigarrow G(\text{fix a basis}) \quad g(x, y) = xG\bar{y}$$

If you change basis, then the Gram matrix changes in the following way:

$$G \rightsquigarrow A^T G \bar{A}$$

If  $g$  is bilinear

$$g \rightsquigarrow \tilde{g} : V \rightarrow V^\vee$$

and

$$g \rightsquigarrow \tilde{g} : V \rightarrow \overline{V}^\vee$$

linear if  $g$  is sesquilinear ( $\tilde{g} : V \rightarrow V^\vee$  is semilinear)

**56.3 Def**

A sesquilinear form  $g : V \times \overline{V} \rightarrow K$  is **hermitian** if

$$g(x, y) = \overline{g(y, x)}$$

And note that inner product is any of symmetric symplectic or hermitian.

## Chapter 57

# Classification (up to isometry) of vector spaces of small dim

Let  $(V, g)$  be vector space over  $K (= \mathbb{R}, \mathbb{C})$  with inner product.

### 57.1 $\dim V = 1$ and $g$ is symmetric

choose  $v \in V \setminus \{0\}$  if  $g(v, v) = 0$ , then  $g$  is degenerated  $\Rightarrow g = 0$   
If  $g$  is non-deg (non-degenerate)  $\exists v$  s.t.  $g(v, v) = a \neq 0$

$$\forall x \in K \quad g(xv, xv) = ax^2$$

Any  $v$  s.t.  $g(v, v) = a \neq 0$  induce a set

$$\mathcal{C}(v) := \{ax^2 : x \in K^*\}$$

this is an element in  $K^*/\{x^2 \mid x \in K^*\}$

#### 57.1.1 Prop

Let  $(V_1, g_1), (V_2, g_2)$  be two vector spaces of dim 1 s.t.  $g_1$  and  $g_2$  are symmetric. Then  $(V_1, g_1)$  and  $(V_2, g_2)$  are isometric iff

$$\exists v_1 \in V_1, v_2 \in V_2 \text{ s.t. } \mathcal{C}_{g_1}(v_1) = \mathcal{C}_{g_2}(v_2)$$

#### Proof

$\Rightarrow f : V_1 \rightarrow V_2$  be an isometry,  $v_1 \in V \setminus \{0\}$ ,  $v_2 := f(v_1)$

$$g_2(v_2, v_2) = g_2(f(v_1), f(v_1)) = g_1(v_1, v_1)$$

$\Rightarrow$

$$C_{g_1}(v_1) = C_{g_2}(v_2)$$

$\Leftarrow$  Assume  $\exists v_1, v_2 : C_{g_1}(v_1) = C_{g_2}(v_2)$  then  $\forall a \in K^*$

$$g_1(v_1, v_1) = ax^2 \quad g_2(v_2, v_2) = ay^2$$

Then let

$$\begin{aligned} f : V_1 &\rightarrow V_2 \\ v_1 &\mapsto y^{-1}xv_2 \end{aligned}$$

for dim 1 this can def whole mapping. So

$$g_2\left(\frac{x}{y}v_2, \frac{xx}{y}v_2\right) = \frac{x^2}{y^2}g(v_2, v_2) = ax^2 = g(v_1, v_1)$$

For  $K = \mathbb{R}$   $\mathbb{R}^*/\{x^2 \mid x \in \mathbb{R}^*\} \cong \{\pm 1\}$

$$\begin{aligned} f : \mathbb{R}^* &\rightarrow \{\pm 1\} \\ x &\mapsto \text{sgn}(x) \end{aligned}$$

For  $K = \mathbb{C}$   $\mathbb{C}^* \cong \{x^2 \mid x \in \mathbb{C}\}$ ,  $\mathbb{C}^*/\{x^2 \mid x \in \mathbb{C}^*\} = \{[1]\}$

### 57.1.2 Theorem

$(V, g)$  has dim 1,  $g$  symmetric. Then  $(V, g)$  is isometric to one of the following

- $K = \mathbb{R}$

$$(\mathbb{R}, g(x, y) = xy) \quad (\mathbb{R}, g(x, y) = -xy) \quad (\mathbb{R}, g(x, y) = 0)$$

- $K = \mathbb{C}$

$$(\mathbb{C}, g(x, y) = xy) \quad (\mathbb{C}, g(x, y) = 0)$$

### Proof

Follows form last prop

## 57.2 dim $V = 1$ $g$ is hermitian

Again  $g$  degenerate  $\Rightarrow g = 0$  We use that same reason as above.  $v \in V :$   
 $g(v, v) = a \neq 0, \forall a \in \mathbb{C}^*$

$$g(av, av) = \|a\|^2 g(v, v)$$

So any element  $v \in V \setminus \{0\}$  s.t.  $g(v, v) = a$  induces a coset in  $\mathbb{C}^*/\mathbb{R}_{>0}$

Inside  $\mathbb{C}^*$ ,  $\mathbb{R}_{>0}$  is a (mult) subgroup



For any  $z \in \mathbb{C}^*$  can be written uniquely as  $z = re^{i\theta}$ , hence

$$\begin{aligned} \mathbb{C}^* &\cong \mathbb{R}_{>0} \times S^1 \rightarrow S^1 \\ z &\mapsto (r, e^{i\theta}) \mapsto e^{i\theta} \end{aligned}$$

The kernel is  $\mathbb{R}_{>0}$  and  $S^1 \cong \mathbb{C}^*/\mathbb{R}_{>0}$ .

But  $g$  is hermitian, so

$$g(v, v) \in \mathbb{R}$$

It follows that the coset

$$\{\|a\|^2 g(v, v) \mid a \in \mathbb{C}^*\} \in (\mathbb{C}^*/\mathbb{R}_{>0}) \cap (\mathbb{R}/\mathbb{R}_{>0}) \cong \{\pm 1\}$$

We repeat the proposition before with  $g$  hermitian and the following theorem

### 57.2.1 Theorem

$(V, g)$  if  $\dim 1$ , with  $g$  hermitian. Then  $(V, g)$  is isometry to one of the following

$$(\mathbb{C}, g(x, y) = x\bar{y}) \quad (\mathbb{C}, g(x, y) = -x\bar{y}) \quad (\mathbb{C}, g(x, y) = 0)$$

## 57.3 $\dim V = 1$ $g$ symplectic

With  $\dim 1 \forall v_1, v_2 \in V$  can be write as  $v_1 = ae, v_2 = be$  with  $e \in K^*$

$$g(v_1, v_2) = ab \cdot g(v, v) = 0$$

### 57.3.1 Theorem

$(V, g)$  of  $\dim 1$ ,  $g$  symplectic, then

$$(V, g) \cong (K, g = 0)$$

## 57.4 $\dim V = 2$ $g$ symplectic

Assume that  $g$  is degenerated, then  $\exists x \in V$  s.t.  $g(x, y) = 0, \forall y \in V$

Extend  $x$  to a basis  $\{x, x'\}$  of  $V$

$$g(ax + a'x', bx + b'x') = ab \cdot g(x, x) + ab' \cdot g(x, x') - a'b \cdot g(x, x') + a'b' \cdot g(x', x') = 0$$

So when  $g$  degenerated  $g = 0$

Take  $g$  non-degenerated  $\exists v_1, v_2 \in V$  s.t.  $g(v_1, v_2) = a \neq 0$ .

For  $g(a^{-1}v_1, v_2) = a^{-1}a = 1$ , we may assume that  $a = 1$

Let's show that  $v_1, v_2$  are linearly independent. Assume by contraction:

$$v_1 = \lambda v_2$$

$$1 = g(v_1, v_2) = g(\lambda v_2, v_2) = \lambda \cdot g(v_2, v_2) = 0$$

$\Rightarrow \{v_1, v_2\}$  is a basis of  $V$ . Then

$$\alpha_1\beta_2 - \alpha_2\beta_1 = g(\alpha_1v_1 + \alpha_2v_2, \beta_1v_1 + \beta_2v_2) = (\alpha_1, \alpha_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \stackrel{(?)}{=} \begin{pmatrix} \overline{\beta_1} \\ \overline{\beta_2} \end{pmatrix}$$

**57.4.1 Theorem**

$(V, g)$  is dim 2,  $g$  symplectic. Then  $(V, g)$  is isometric to one of the following

$$(K^2, g(x, y) = 0) \quad (K^2, g(x, y) = x_1y_2 - x_2y_1)$$

## Chapter 58

# Compliment

### 58.1 Def: non-degenerate

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a subspace. We say that  $V_0$  is non-degenerate if  $g|_{V_0}$  is non-degenerate.

Moreover,  $V_0$  is isotropic if  $g|_{V_0} = 0$

#### Remark

Isotropic means degenerate

#### Example

1  $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_1 - x_2y_2$$

symmetric non-degenerate.  $V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$

2  $(\mathbb{R}^2, g)$

$$g(x, y) = x_1y_2 + x_2y_1$$

$$V_0 = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

### 58.2 Def

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a subspace. The orthogonal complement of  $V_0$  is define as

$$V_0^\perp := \{v \in V \mid g(v, v_0) = 0, \forall v_0 \in V_0\}$$

### 58.3 Prop

Let  $(V, g)$  be a inner product space. Let  $V_0 \subseteq V$  be a non-degenerate subspace. Then

$$V = V_0 \oplus V_0^\perp$$

#### Proof

Let  $\{v_0, \dots, v_n\}$  be a basis of  $V_0$  ( $\dim V_0 = r \leq \dim V = n$ ) Extend this to a basis of  $V$

$$\{v_1, \dots, v_r, v_{r+1}, \dots, v_n\}$$

Let  $G$  be the Gram matrix of  $g$  with respect to such case:

$$G = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) & \cdots & g(v_1, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) & \cdots & g(v_r, v_n) \\ \vdots & & \vdots & & \vdots \\ g(v_n, v_1) & \cdots & g(v_n, v_r) & \cdots & g(v_n, v_n) \end{pmatrix}$$

while  $G_0 \in M_{r,n}(K)$  is defined as a part of  $G$

$$G_0 = \begin{pmatrix} g(v_1, v_1) & \cdots & g(v_1, v_r) \\ \vdots & & \vdots \\ g(v_r, v_1) & \cdots & g(v_r, v_r) \end{pmatrix}$$

Consider the dimension:

$$\begin{aligned} \dim V_0^\perp &= \dim \ker G_0 \\ &= n - \text{rank} G_0 \\ &= n - \text{rank} G_0^T \end{aligned}$$

Since  $V_0$  is non-degenerated.  $G_0^T$  is non-degenerated with  $\text{rank} = \dim V_0 = r$

$$\dim V_0 + \dim V_0^\perp = \dim V$$

We finally have to show that this sum is a direct sum:

If  $x \in V_0 \cap V_0^\perp$ , then  $g(x, V_0) = 0$  since  $x \in V_0^\perp$  and  $g_0(x, V_0) = 0$  implies  $x = 0$  since  $g_0$  is non-degenerate.

### 58.4 Theorem

Let  $(V, g)$  be an finite dimensional inner product space. If both  $V_0$  and  $V_0^\perp$  are non-degenerate, then  $(V^\perp)^\perp = V_0$

**Proof****Remark**

$V_0 \subseteq (V_0^\perp)^\perp$  always holds, regardless of degenerate and dimension.

**58.5 Theorem**

Let  $(V, g)$  be a finite dimensional inner product space. Then There exists a decomposition

$$V = V_1 \oplus \cdots \oplus V_n$$

such that  $\{V_i\}_{i=1}^n$  are pairwise orthogonal and

- 1 They are 1-dim if  $g$  is symmetric or hermitian
- 2 They are 1-dim but degenerated or 2-dim non-degenerate if  $g$  is symplectic.

**Proof**

We work by induction on  $n = \dim V$

$n = 1$

trivial

$n \geq 2$

If  $g = 0$ , there is nothing to prove.

$g$  symplectic There exists  $x$  and  $x'$  such that  $g(x, x') = 0$  and  $\langle x, x' \rangle = V_0$ . Since  $V_0$  is non-degenerate, then apply induction on  $V_0^\perp$

$g$  symmetric/hermitian It's enough to show that there exists  $V_0 \subseteq V$  of dim 1, which is non-degenerate.



## Chapter 59

# Signature

Now we discuss the uniqueness of such decomposition

### 59.1 Def

Let  $(V, g)$  be an inner product space with  $\dim V = 1$ . Moreover, assume that  $g$  is symmetric or hermitian. We say that  $(V, g)$  is **positive** if  $(V, g)$  is isometry to either  $(\mathbb{R}, g(x, y) = xy)$  or  $(\mathbb{C}, g(x, y) = x\bar{y})$

We say that  $(V, g)$  is **negative** if  $(V, g)$  is isometry to either  $(\mathbb{R}, g(x, y) = -xy)$  or  $(\mathbb{C}, g(x, y) = -x\bar{y})$

### 59.2 Notation

By theorem 58.5, we can count the number of positive subspace of any inner product space.

- $r_0 := \dim \ker g$
- $r_+ :=$  the number of positive subspaces
- $r_- :=$  the number of negative subspaces

### 59.3 Def

Let  $(V, g)$  be an inner product space.

- 1 If  $g$  is real symmetric or, hermitian, then  $(r_0, r_+, r_-)$  is signature of  $V$
- 2 If  $g$  is symplectic or complex symmetric, then  $(\dim V, r_0)$  is the signature of  $V$

### 59.4 Theorem

Let  $(V, g)$  and  $(V', g')$  be two inner product spaces, with  $g, g'$  that are either (both) symplectic or complex symmetric.

Then  $(V, g)$  and  $(V', g')$  are isometric iff

$$(n, r_0) = (n', r'_0)$$

#### Proof

$\Rightarrow$

trivial

$\Leftarrow$

Use orthogonal decomposition

$$V = \bigoplus_{i=1}^n V_i$$

We rearrange the subspaces as  $V_0, \dots, V_r, v_{r+1}, \dots, V_n$  such that  $V_i \subseteq \ker g, \forall 1 \leq i \leq r$

The remaining is shown as below:

$$(V_1, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_1, g')$$

$\vdots$

$\vdots$

$$(V_r, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_r, g')$$

$$(V_{r+1}, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_{r+1}, g')$$

$\vdots$

$\vdots$

$$(V_n, g) \xleftarrow{\cong} (\mathbb{R}, 0) \xleftarrow{\cong} (V'_n, g')$$



## 59.5 Theorem

Let  $(V, g)$  and  $(V', g')$  be two inner product spaces, with  $g, g'$  that are either (both) hermitian or real symmetric.

Then  $(V, g)$  and  $(V', g')$  are isometric iff

$$(r_0, r_+, r_-) = (r'_0, r'_+, r'_-)$$

### Proof

exercise



## Chapter 60

# Orthonormal

### 60.1 Def

Let  $(V, g)$  be an inner product space. The basis  $\{v_1, \dots, v_n\}$  is said **orthogonal** if  $g(v_i, v_j) = 0, \forall i \neq j$

Moreover,  $g$  is said **orthonormal** if  $g(v_i, v_i) \in \{0, -1, 1\}, \forall i$

### Remark

If  $g$  is hermitian or symmetric. We can always find an orthonormal basis from an orthogonal basis.

### 60.2 Def

Let  $V$  be a vector space over  $K$  ( $\text{char} K \neq 2$ ). A **quadratic form** on  $V$  is a mapping  $q : V \rightarrow K$  such that

- $q(\alpha v) = \alpha^2 q(v) \forall \alpha \in K, v \in V$
- $f = (u, v) \mapsto q(u + v) - q(u) - q(v)$  is bilinear

### Remark

Any symmetric bilinear form  $h : V^2 \rightarrow K$  is a quadratic form. Given a quadratic form  $q : V \rightarrow K$ , we can define a symmetric bilinear form

$$h_p(u, v) = \frac{1}{2} (q(u + v) - q(u) - q(v))$$

### 60.3 Gram-Schmidt algorithm

Let  $(V, g)$  be an inner product space with  $g$  symmetric or hermitian. Let  $\{v'_1, \dots, v'_n\}$  be a basis of  $V$  such that  $V_i = \langle v'_1, \dots, v'_i \rangle \forall i \in \{1, \dots, n\}$  is non-degenerate.

Then there exists an orthogonal basis  $\{v_1, \dots, v_n\}$  such that  $V_i = \langle v_1, \dots, v_i \rangle \forall i \in \{1, \dots, n\}$  is non-degenerate.

#### Proof

We construct the  $v_i$  by induction.  $v_1 = v'_1$  If  $v_1, \dots, v_{i-1}$  have been already constructed, then  $v_i$  must be of the form:

$$v_i = v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j$$

We have  $\langle v_1, \dots, v_{i-1} \rangle = \langle v'_1, \dots, v'_{i-1} \rangle$  by induction hypothesis, then

$$\langle v_1, \dots, v_{i-1} \rangle = \langle v'_1, \dots, v'_{i-1}, v_i \rangle = \langle v'_1, \dots, v'_i \rangle$$

We have to show that  $v_i$  are orthogonal

$$\begin{aligned} g \left( v'_i - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} v_j, v_k \right) &= g(v'_i, v_k) - \sum_{j=1}^{i-1} \frac{g(v'_i, v_j)}{g(v_j, v_j)} g(v_j, v_k) \\ &= g(v'_i, v_k) - \frac{g(v'_i, v_k)}{g(v_k, v_k)} g(v_k, v_k) \\ &= 0 \end{aligned}$$

## Chapter 61

# Euclidean and Unitary Spaces

### 61.1 Def:Euclidean vector space

A euclidean vector space is a finite dimensional inner product space over  $\mathbb{R}$   $(E, g)$  with  $g$  symmetric and positive definite ( $g(x, x) > 0, \forall x \neq 0$ )

We denote

$$\langle x, y \rangle := g(x, y)$$

#### Remark

Any non-zero subspace of  $(E, \langle \cdot, \cdot \rangle)$  is non-degenerated:

Take  $V \subseteq E$  if  $\exists x \in V \setminus \{0\}$  s.t.  $\langle x, V \rangle = 0$  then

$$\langle x, x \rangle = 0$$

The signature of  $E$  is of the type  $(0, \mathfrak{z}_+, \mathfrak{z}_-)$  denoted by  $(P, Q)$  ( $P = \mathfrak{z}_+, Q = \mathfrak{z}_-$ )

An euclidean space is a normed vector space

$$\|x\| = \sqrt{\langle x, x \rangle}$$

( $\langle \cdot, \cdot \rangle$  positive defined required)

Any euclidean vector admits an orthonormal basis  $\{v_1, \dots, v_n\}, \langle v_i, v_i \rangle = 1$

So orthonormal means

$$\|v_1\| = 1$$

In a euclidean space we have a distance

$$d(x, y) = \|x - y\|$$

It's a topological metric space

### 61.2 Prop

A euclidean space  $(E, \langle, \rangle)$  of dim  $n$  is isometric to  $(\mathbb{R}^n, \underbrace{\langle, \rangle}_{\text{usual scalar product}})$

#### Proof

$\{v_1, \dots, v_n\}$  is an orthonormal basis of  $E$ . Consider the mapping

$$f : v_i \mapsto e_i = (0, \dots, \underbrace{1}_{i\text{-th}}, \dots, 0)$$

Let  $x = \sum \alpha_i v_i, y = \sum \beta_i v_i$  then

$$\langle x, y \rangle = \sum \alpha_i \beta_i$$

and  $f(x) = \sum \alpha_i e_i, f(y) = \sum \beta_i e_i$ , so

$$\langle f(x), f(y) \rangle = \sum \alpha_i \beta_i = \langle x, y \rangle$$

Thus  $f$  an isometry.

### 61.3 Remark

Cauchy-Schwartz inequality

$$\langle x, y \rangle \leq \|x\| \|y\|$$

Triangle inequality

$$\|x + y\| \leq \|x\| + \|y\|$$

### 61.4 Pythagoras's Theorem

If  $x_1, \dots, x_k$  are pairwise orthogonal, then

$$\left\| \sum_{i=1}^k x_i \right\|^2 = \sum_{i=1}^k \|x_i\|^2$$

### 61.5 Def:Angles

By Cauchy-Schwartz inequality:

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \|y\|} \leq 1$$

Then there exists a element  $\phi \in [0, \pi]$  such that

$$\cos \phi = \frac{\langle x, y \rangle}{\|x\| \|y\|}$$

$\phi$  is defined as the angle between  $x$  and  $y$

Notice that  $\phi$  is not an oriented angle.

## 61.6 Notation

Let  $U, V \subseteq E$  be two subspace, then

$$d(U, V) := \inf\{\|u - v\| \mid u \in U, v \in V\}$$

## 61.7 Def

$V \subsetneq E$  a vector subspace,  $x \in E \setminus \{0\}$ . Then  $E = V \oplus V^\perp$  (proved) Then we write (uniquely)

$$x = x_0 + x'_0$$

where  $x_0 \in V, x'_0 \in V^\perp$ .

Then  $x_0$  is called the orthogonal projection of  $x$  on  $V$ ,  $x'_0$  is called the orthogonal projection of  $x$  on  $V^\perp$

## 61.8 Prop

Use the above notation:

$$d(x, V) = \|x'_0\|$$

### Proof

$$\forall y \in V$$

$$\|x - y\|^2 = \|x_0 + x'_0 - y\|^2 = \|x_0 - y\|^2 + \|x'_0\|^2$$

then

$$\|x - y\| \geq \|x'_0\|$$

$\Rightarrow$

$$d(x, V) \geq \|x'_0\|$$

The equality is achieved iff  $y = x_0$ .

$$d(x, V) = \|x'_0\|$$

### 61.9 Prop

Use the previous notations. Assume that  $m = \dim V$ ,  $V \subseteq E$ ,  $\{v_1, \dots, v_m\}$  ( $m \leq n = \dim E$ ) is an orthonormal basis of  $V$ . Then

$$x_0 = \sum_{i=1}^m \langle x, v_i \rangle v_i$$

#### Proof

Consider  $y = x_0 - \sum_{i=1}^m \langle x, v_i \rangle v_i$ ,  $y \in V$ . Moreover

$$\begin{aligned} \langle y, v_j \rangle &= \left\langle x_0 - \sum_{i=1}^m \langle x, v_i \rangle v_i, v_j \right\rangle \\ &= \left\langle x - \sum_{i=1}^m \langle x, v_i \rangle v_i, v_j \right\rangle \\ &= \langle x, v_j \rangle - \langle x, v_j \rangle \\ &= 0 \end{aligned}$$

$y$  is orthonormal to any  $v_j \Rightarrow \langle y, V \rangle = 0 \Rightarrow y \in V^\perp$  so

$$y \in V \cap V^\perp = \{0\}$$

### 61.10 Relationship with calculus

$(E, \langle, \rangle) = (\mathbb{R}^n, \langle, \rangle)$  on  $\mathbb{R}^n$  we have the notion of volumes

$$\text{vol}(B) := \lambda^n(B)$$

where  $B$  is a Borel set.

A  $n$ -dimensional parallelepiped is :

$$P_n = \{t_1 v_1 + \dots + t_n v_n \mid t_i \in [0, 1] \forall i\}$$

Consider a linear mapping

$$\begin{aligned} A_{P_n} &= A : \mathbb{R}^n \rightarrow \mathbb{R}^n \\ x &\mapsto Ax \end{aligned}$$

where  $A \in \mathcal{M}_{n \times n}(\mathbb{R})$ ,  $A = (v_1 \mid \dots \mid v_n)$

$A$  is invertible iff  $\{v_1, \dots, v_n\}$  is a basis. Let  $\prod_n = [0, 1]^n$  then

$$A\left(\prod_n\right) = P_n$$



If  $A$  invertible

$$\begin{aligned}
 \text{vol}(P_n) &= \lambda^n(P_n) \\
 &= \int_{A(\Pi_n)} \chi_{P_n} d\lambda^n \\
 \text{change of variables} &= \int_{A(\Pi_n)} |\det A| d\lambda^n \\
 &= |\det A| \\
 \text{by the prop of det} &= \sqrt{\det A^T A}
 \end{aligned}$$

## 61.11 Prop

$$\text{vol}(P_n) = \sqrt{\det G}$$

### Proof

The Gram matrix

$$G = (\langle v_i, v_j \rangle_{ij})$$

Notice that  $\det G = 0$  iff  $\{v_1, \dots, v_n\}$  is not a basis. Let  $A = (v_1 \mid \dots \mid v_n)$  then  $G = A^T A$

$$\begin{aligned}
 \det G &= (\det A)^2 \\
 &= \det A^T A \\
 &= \text{vol}(P_n)
 \end{aligned}$$



## Chapter 62

# Unitary Space

### 62.1 Def

A complex inner vector space  $(H, h)$  where  $h$  is hermitian and positive define then it's called **unitary space**

As in the Euclidean space, we have orthonormal basis and define a norm, then a distance.

$$\|x\| = \sqrt{h(x, x)}$$

### 62.2 Decomplexification

Let  $V$  be a complex vector space of dimension  $n$ . We resist the module structure  $\mathbb{C} \times V \rightarrow V$  to  $\mathbb{R} \times V \rightarrow V$

The  $\mathbb{R}$ -vector space denoted by  $V_{\mathbb{R}}$  has the same vector of  $V$ . For any  $\mathbb{C}$ -linear mapping  $f : V \rightarrow W$ , the module induces a mapping

$$f_{\mathbb{R}} \rightarrow W_{\mathbb{W}}$$

### Exercise

Verify that  $(\cdot)_{\mathbb{R}}$  is a functor

### 62.3 Theorem

Let  $V$  be a complex vector of dim  $n$

- Let  $V$  be a complex vector basis of  $V$ , then  $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$  is a real basis of  $V_{\mathbb{R}}$

- $f : V \rightarrow W$  is a linear mapping. Assume that it's metric representation with respect to the basis  $\{v_1, \dots, v_n\}$  of  $V$  and  $\{w_1, \dots, w_n\}$  of  $W$  is

$$A = B + iC$$

where  $B, C \in \mathbb{R}$ . Then the metric representation of  $f_{\mathbb{R}}$  with respect to the basis  $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$  and  $\{w_1, \dots, w_n, iw_1, \dots, iw_n\}$  is

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

### Proof

1

Take  $v \in V$

$$\begin{aligned} v &= \sum_{k=1}^n a_k v_k \\ &= \sum_{k=1}^n (\Re a_k + \Im a_k) v_k \\ &= \sum_{k=1}^n (\Re a_k) v_k - \sum_{k=1}^n (\Re i a_k) i v_k \end{aligned}$$

generate. Assume  $\sum b_k v_k + \sum c_k i v_k = 0$ . But  $\{v_k\}$  is a basis, implying

$$b_k + i c_k = 0 \Rightarrow b_k = c_k = 0$$

linearly independent.

2

Decompose  $V = \bigoplus_{k=1}^n V_k$  and  $W = \bigoplus_{k=1}^n W_k$  into 1 dimensional orthogonal subspaces.

Take  $k, k'$  consider

$$\begin{aligned} f_{k,k'} : V_k &\rightarrow W_{k'} \\ v_k &\mapsto \alpha w_{k'} \end{aligned}$$

Take

$$\begin{aligned} f_{k,k',\mathbb{R}} : V_{k,\mathbb{R}} &\rightarrow W_{k',\mathbb{R}} \\ v_k &\mapsto (\Re \alpha) w_{k'} + (\Im \alpha) i w_{k'} \\ i v_k &\mapsto -(\Im \alpha) w_{k'} + (\Re \alpha) i w_{k'} \\ A_{f_{k,k',\mathbb{R}}} &= \begin{pmatrix} \Re \alpha & -\Im \alpha \\ \Im \alpha & \Re \alpha \end{pmatrix} \end{aligned}$$

Rearrange the whole mapping, we get

$$A = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

## 62.4 Corollary

Let  $f : V \rightarrow V$  be a  $\mathbb{C}$ -linear mapping. Then

$$\det f_{\mathbb{R}} = \det f \overline{\det f}$$

### Proof

Assume  $A = B + iC$  is the matrix representation of  $f$

$$\begin{aligned} \det f_{\mathbb{R}} &= \det \begin{pmatrix} B & -C \\ C & B \end{pmatrix} \\ &= \det \begin{pmatrix} B + iC & -C + iB \\ C & B \end{pmatrix} \\ &= \det \begin{pmatrix} B + iC & 0 \\ C & B - iC \end{pmatrix} \\ &= \det(B + iC) \cdot \det(B - iC) \\ &= \det A \cdot \overline{\det A} \end{aligned}$$



## Chapter 63

# Complexification

### 63.1 Def: Complex structure

Let  $W$  be a real vector space of dim  $n$ . Consider  $J : W \rightarrow W$  a linear mapping such that  $J^{\circ 2} = -Id$ . Then  $J$  is called the **complex structure** of  $W$ . Then couple  $(W, J)$  is a vector space with a complex structure.

#### Example

$$\begin{aligned} J : V_{\mathbb{R}} &\rightarrow V_{\mathbb{R}} \\ v &\mapsto iv \\ iv &\mapsto -v \end{aligned}$$

### 63.2 Theorem

Let  $(W, J)$  be a real vector space with a complex structure. Then on  $W$  we introduces the following complex module:

$$(a + bi)w := aw + bJ(w)$$

We obtain a complex vector space  $W$  such that  $(W)_{\mathbb{R}} = W$

### 63.3 Corollary

If  $(W, J)$  is a vector space with a complex structure, then  $\dim W$  is even. Assume that if it's even, then it's possible to find a basis on  $W$  such that  $J$  is represented by

$$\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

Consider a orthonormal basis on  $H\{v_1, \dots, v_n\}$ .  $G_h : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is the Gram matrix of  $h$  with respect to  $\{v_1, \dots, v_n\}$

$$G_h = B_i C$$

Now

$$G_{\mathbb{R}} = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

Then  $G_{\mathbb{R}}$  defines an inner product and  $(H_{\mathbb{R}}, \langle, \rangle)$  is Euclidean.

## 63.4 Notation

Now fix a inner product space. Then

$$h(x, y) = a(x, y) + ib(x, y)$$

when  $a, b : V \times V \rightarrow \mathbb{R}$

## 63.5 Prop

In the above notation the following structure holds:

- 1  $a(x, y)$  and  $b(x, y)$  are inner products on  $V_{\mathbb{R}}$ , with  $a$  symmetric and  $b$  skew-symmetric. In addition:

$$a(ix, iy) = a(x, y) \quad b(ix, iy) = b(x, y)$$

In other word  $a, b$  are invariant by the multiplication by  $i$  Invariance w.r.t. the complex structure of  $H_{\mathbb{R}}$

- 2 The following relations hold.

$$a(x, y) = b(ix, y) \quad b(x, y) = -a(ix, y)$$

- 3 Any pair of  $J$ -invariant bilinear forms on  $V_{\mathbb{R}}$   $a, b : V \times V \rightarrow \mathbb{R}$  that are symmetric and symplectic, respectively and s.t. (2) is satisfied. Define an hermitian inner product

$$h(x, y) := a(x, y) + ib(x, y)$$

Moreover  $h$  is positive definite iff  $a$  is positive definite.



**Proof****1**

First to check is

$$\begin{aligned}
h(x, y) &= h(y, x) \\
\Rightarrow a(x, y) + ib(x, y) &= a(y, x) - ib(y, x) \\
\Rightarrow \begin{cases} a(x, y) = a(y, x) & \text{symm} \\ b(x, y) = -b(y, x) & \text{skew-symm} \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
h(ix, iy) &= i\bar{i}h(x, y) = h(x, y) \\
\Rightarrow a(ix, iy) + ib(ix, iy) &= a(x, y) + ib(x, y) \Rightarrow \begin{cases} a(ix, iy) = a(x, y) \\ b(ix, iy) = b(x, y) \end{cases}
\end{aligned}$$

**2**

$$\begin{aligned}
h(ix, y) &= ih(x, y) \\
\Rightarrow a(ix, y) + ib(ix, y) &= ia(x, y) - b(x, y) \\
\Rightarrow \begin{cases} a(ix, y) = -b(x, y) \\ b(ix, y) = a(x, y) \end{cases}
\end{aligned}$$

**3**

$$\begin{aligned}
ih(x, y) &= ia(x, y) - b(x, y) \\
(2) &= ib(ix, y) + a(ix, y) \\
&= h(ix, y)
\end{aligned}$$

We have linearity in the first variable, then positive/negative define follows from

$$b(x, x) = -b(x, x) \Rightarrow b(x, x) = 0$$

so

$$h(xx, x, ) = a(x, x)$$

**63.6 Complex Cauchy-Schwartz inequality** $(H, h)$  is a unitary space with finite dim. Then the inequality

$$|h(x, y)| \leq \|x\| \|y\|$$

holds iff  $x$  and  $y$  are propositional ( $x = ty$ )

**Proof**

When  $x = 0$ , obvious.  $x \neq 0, \forall t \in \mathbb{R}$

$$\begin{aligned} 0 &\leq \|tx + y\|^2 \\ &= t^2 \|x\|^2 + 2t\Re(h(x, y)) + \|y\|^2 \end{aligned}$$

This is a quadratic inequality. The equality has at most one solution when  $y = -tx$ . In general the discriminant of the quadratic polynomial in  $t \leq 0$

$$(\Re h(x, y))^2 \leq \|x\|^2 \|y\|^2$$

Now take

$$h(x, y) = |h(x, y)| e^{i\theta}$$

Then

$$\Re(h(e^{-i\theta}, y)) = |h(x, y)|$$

Hence

$$\Re(h(e^{-i\theta}, y))^2 = |h(x, y)|^2 \leq \|e^{-i\theta}\|^2 \|y\|^2$$

**63.7 Corollary:Complex triangle inequality**

$$\|x + y\| \leq \|x\| + \|y\|$$

**63.8 Def:Angle for unitary space**

For

$$0 \leq \frac{|h(x, y)|}{\|x\| \cdot \|y\|}$$

$$\exists! \phi \in [0, \frac{\pi}{2}]$$

$$\cos \phi = \frac{|h(x, y)|}{\|x\| \cdot \|y\|}$$

The physical application is that  $\phi$  can be considered as probability.

**63.9 Prop**

Let  $(V, g)$  be an inner product space with a non-degenerate inner product hermitian or real symm and  $f : V \rightarrow V$  be a linear mapping. Then the following statements are equivalent:

0  $f$  is isometry

1  $g(f(x), f(x)) = g(x, x) \forall x \in V$

- 2 Let  $\{v_1, \dots, v_n\}$  be a basis for  $V$  and let  $G$  be a Gram matrix of  $g$  w.r.t. such basis. If  $A$  is the matrix of  $f$  w.r.t.  $\{v_1, \dots, v_n\}$ , then

$$A^T G A = G \text{ or } A^T G \bar{A} = G$$

- 3  $f$  transforms orthonormal basis into orthonormal basis
- 4 If the signature of  $g$  is  $(p, q)$  ( $\mathbf{z}_0 = 0$ ), then the matrix of  $f$  w.r.t. any orthonormal basis  $\{v_1, \dots, v_p, v_{p+1}, \dots, v_{p+q}\}$  where

$$(v_i, v_i) = \begin{cases} 1 & \text{if } i \leq p \\ -1 & \text{if } p < i \leq p + q \end{cases}$$

satisfies the following property:

symm case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} A = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

hermitian case

$$A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \bar{A} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$$

## Proof

$0 \Leftrightarrow 1$

$\Rightarrow$  trivial

realsymm  $q_g(x) = g(x, x)$

$$g = \frac{1}{2} (q(x+y) - q(x) - q(y)) \quad (*)$$

From (\*) it's evident that if  $f$  preserves  $q$ , then it also preserves  $g$ .

hermitian  $q(x) = g(x, x)$

$$\Re g(x, y) = \frac{1}{2} (q(x, y) - q(x) - q(y))$$

and

$$\Im g(x, y) = -\Re g(ix, y)$$

by the proposition proved. So if  $f$  preserves the quadratic form, then it also preserves  $g$ .

**0 $\Leftrightarrow$ 2**

$\mathcal{B} = \{v_1, \dots, v_n\}$  is orthonormal. let  $\mathcal{B}' = \{f(v_1), \dots, f(v_n)\}$ .

$\Rightarrow$  Since  $f$  is an isometry then the Gram matrix is the same. We also proved that if we change the Gram matrix changes in the following way

$$G \rightsquigarrow A^T G A \Rightarrow G = A^T G A$$

(real symm case)

$\Leftarrow$

$$\begin{aligned} g(x, y) &= x^T G y \\ &= x^T A^T G A y \\ &= f(x)^T G f(y) \\ &= g(f(x), f(y)) \end{aligned}$$

The same proof holds in hermitian case.(taking care of the conjugation)

**3 and 4**

they are just special cases of the others(Exercise)

## Chapter 64

# Special operators

### 64.1 Def

$(E, \langle, \rangle)$  is a Euclidean space, then an isometry  $f : E \rightarrow E$  is called an **orthogonal operator**

$(H, h)$  a unitary space, then an isometry  $f : H \rightarrow H$  is said a **unitary operator**

### 64.2 Corollary

Orthogonal and unitary operators have the following properties:  
w.r.t. an (all) orthonormal basis, they are represented by a matrix  $U$

$$UU^T = I_n \text{ or } UU^\dagger = I_n$$

### 64.3 Def: orthogonal matrices

$$O(n) := \{A \in GL_n(\mathbb{R}) \mid AA^T = I_n\}$$

### 64.4 Def: unitary matrices

$$U(n) := \{A \in GL_n(\mathbb{C}) \mid AA^\dagger = I_n\}$$

#### Remark

By the corollary 64.2, we have

- $O(n)$  is the set of orthogonal operators of  $(\mathbb{R}^n, \langle, \rangle)$
- $U(n)$  is the set of unitary operators of  $(\mathbb{C}^n, \langle, \rangle)$

**Remark**

$$\begin{array}{ccc} O(n) & \subseteq GL_n(\mathbb{R}) \subseteq & M_{n,n}(\mathbb{R}) \\ \text{isometry} & & \text{endomorphism} \end{array}$$

Take  $T \in O(n)$ ,  $TT^T = I_n$

$$(\det T)^2 = 1 \Rightarrow \det T = \pm 1$$

**64.5 Notation**

$$\begin{aligned} SL_n(\mathbb{R}) &:= \{A \in GL_n(\mathbb{R}) \mid \det A = 1\} \\ SO(n) &:= \{A \in O_n \mid \det A = 1\} \end{aligned}$$

## Chapter 65

# Classification of operators

$$U(1) = \{a \in \mathbb{C} \mid a\bar{a} = 1\} = \{e^{i\phi} \mid \phi \in \mathbb{R}\}$$
$$O(1) = \{1, -1\} = U(1) \cap \mathbb{R}$$

Let's study  $O(2)$  and classify all its elements

$O(n)/SO(n) = \{\pm 1\}$   $SO(n)$  is a normal subgroup of  $O(n)$  of index 2, namely

$$\#(O(n)/SO(n)) = 2$$

Take  $T \in O(2)$ , we have two cases:  $\begin{cases} T \in SO(2) \\ T \in O(2) \setminus SO(2) \end{cases}$

### 65.0.1 $T \in SO(2)$

Assume  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , and  $TT^T = Id_2$

$$\begin{cases} \det T = ad - bc = 1 \\ a^2 + b^2 = 1 \\ c^2 + d^2 = 1 \\ ac + bd = 0 \end{cases}$$

This implies  $\exists \alpha$  unique up to add by  $2k\pi$  s.t.

$$a = \cos \alpha \quad b = \sin \alpha$$

We have shown that

$$SO(2) = \left\{ \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \mid \alpha \in [0, 2\pi[ \right\}$$

Note that  $T$  doesn't have eigenvalues for  $\alpha \neq 0, \pi$

**65.0.2**  $T \in O(2) \setminus SO(2)$ 

$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the reflection with respect to the line  $y = 0$

$$TA \in SO(2)$$

since

$$\det(TA) = \det T \cdot \det A = 1$$

By the previous reasoning,

$$T = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & -\cos \alpha \end{pmatrix}$$

The set  $O(2) \setminus SO(2)$  represents reflections.

Consider the mapping

$$\begin{aligned} U(1) &\xrightarrow{\cong} SO(2) \\ e^{i\phi} &\mapsto \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \end{aligned}$$

$$O(2) \cong U(1) \cup (O(2) \setminus SO(2))$$

**Remark**

Reflections are diagonalizable with eigenvalues  $\{\pm 1\}$  and the corresponding eigenvectors are orthogonal.

**65.1 Theorem**

- 1 Let  $(H, h)$  be a unitary space. A linear mapping  $f : H \rightarrow H$  is unitary iff it's diagonalizable in an orthonormal basis and with eigenvalues in  $S^1$
- Let  $(E, \langle, \rangle)$  be a Euclidean space. A linear mapping  $f : E \rightarrow E$  is orthogonal iff in some orthonormal basis  $f$  is represented by matrix:

$$\begin{pmatrix} R(\phi_1) & & & & \\ & \ddots & & & \\ & & R(\phi_n) & & \\ & & & Id_1 & \\ & & & & \ddots \\ & & & & & Id_m \end{pmatrix}$$

where

$$R(\phi_i) = \begin{pmatrix} \cos \phi_i & -\sin \phi_i \\ \sin \phi_i & \cos \phi_i \end{pmatrix} \quad \phi_i \in [0, 2\pi[$$

- 3 The eigenvectors of orthogonal/unitary operators corresponding to different eigenvalues are orthogonal.



**Proof****1**

$\Leftarrow$  Assume that  $f = ADA^\dagger$ ,  $D = \text{diag} \lambda_1, \dots, \lambda_n$  with  $\lambda_i \bar{\lambda}_i = 1$ . Then  $f$  is represented by a unitary matrix  $D$

$\Rightarrow$  Assume  $f$  unitary. Let  $L_\lambda$  be the eigenspace of  $\lambda$ . We can write  $H = L_\lambda \oplus L_\lambda^\perp$ .  $L_\lambda$  is a invariant subspace of  $f$ , that's,  $f|_{L_\lambda}$  is the scalar multiplication by  $\lambda$ . Any orthonormal basis of  $L_\lambda$  is a basis of eigenvector for  $f|_{L_\lambda}$ . So if we show that  $L_\lambda^\perp$  is  $f$ -invariant, we done the proof, for we can prove the theorem for  $f|_{L_\lambda}$  by induction.

Take  $v \in L_\lambda^\perp$ ,  $\forall v_0 \in L_\lambda$ , then

$$h(f(v), v_0) = 0$$

because

$$h(f(v), v_0) = h(f(v), f(\lambda^{-1}v_0)) = \bar{\lambda}^{-1}h(f(v), v_0) = 0$$

**2**

$\Leftarrow$  Directly check because it's easy to see that  $A$  is orthogonal.

$\Rightarrow$  The cases where  $E$  is of dim 1 or 2 are right for the declaration of  $O(1), O(2)$  Assume that  $\dim E \geq 3$

a  $f$  have at least one real eigenvalue.  $E = L_\lambda \oplus L_\lambda^\perp$  and the proof is the same with (1)

b  $f$  have no real eigenvalues. We have two complex conjugate eigenvalues  $\lambda$  and  $\bar{\lambda}$ . (We already know that  $\lambda = e^{i\theta}$  by (1))

Let  $v$  and  $\bar{v}$  two eigenvectors of  $\lambda$  and  $\bar{\lambda}$  respectively. This means that  $\langle v, \bar{v} \rangle$  is 2-dim.  $v, \bar{v}$  are already  $f$ -invariant.

$$f(\alpha v + \beta \bar{v}) = \alpha \lambda v + \beta \bar{\lambda} \bar{v}$$

$f$  has a 2-dim invariant subspace. And if  $E = L_0 \oplus L_0^\perp$ , the theorem is right for the classification of elements of  $O(2)$

As above it's enough to show that

$$f(L_0^\perp) \subseteq L_0^\perp$$

3 Let  $\lambda_1 \neq \lambda_2$  and  $f(v_i) = \lambda_i v_i$

$$\langle v_1, v_2 \rangle = \langle f(v_1), f(v_2) \rangle = \lambda_1 \bar{\lambda}_2 \langle v_1, v_2 \rangle$$

since  $\lambda_1 \neq \lambda_2$

$$\lambda_1 \bar{\lambda}_2 \neq 1 \Rightarrow \langle v_1, v_2 \rangle = 0$$



**Part IX**

**Fourier Series**



## Chapter 66

# Fourier Coefficient

Goal We have an infinite dim vector space (usually a space of function) we want to express the elements as combinations of "orthogonal" vectors. (w.r.t. some nice inner product)

### 66.1 Def: Orthogonal and Orthonormal System

$V$  is a vector space over  $\mathbb{R}$  or  $\mathbb{C}$ ,  $\langle, \rangle$  is an inner product which either symm or hermitian.. Moreover,  $\langle, \rangle$  is non-degenerate and positive define.

A set of vectors  $\{l_k \mid k \in I\}$  (where  $I$  be the set of indexes) is said to be an orthogonal system if

$$\langle l_j, l_k \rangle = 0 \text{ iff } j \neq k$$

Moreover  $\{l_k\}$  is an orthonormal system if

$$\langle l_j, l_k \rangle = \delta_{jk}$$

### 66.2 Prop

Let  $\{l_k\}$  be an orthogonal system, then  $\{l_k\}$  is a set of non-zero linearly independent vectors.

#### Proof

trivial

### 66.3 Prop

The inner product  $\langle, \rangle$  is continuous (w.r.t. the Euclidean topology in the co-domain, and the product topology on the domain, where on  $V$  we put the

topology induced by  $\langle, \rangle$ )

If  $\{f_k\}$  is orthogonal system and  $x \in V = \sum_{k=1}^{\infty} x_k l_k$ , then  $\forall y \in V$

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x_k, y \rangle$$

If  $\{f_k\}$  is orthonormal system and  $x = \sum_{k=1}^{\infty} x_k l_k$ ,  $y = \sum_{k=1}^{\infty} y_k l_k$ , then

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k \bar{y}_k$$

or

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$$

### Proof

Follow from the Cauchy-Schwartz inequality (both real/complex)

$$\langle x, y \rangle = \sum_{k=1}^n \langle v_i, y \rangle + \left\langle \sum_{k=n+1}^{\infty} v_i, y \right\rangle$$

then use (1). Since

$$\lim_{n \rightarrow +\infty} \sum_{k=n+1}^{\infty} v_i = 0$$

This is an application of (2) keeping in mind that  $\{l_k\}$  is orthonormal.

## 66.4 Corollary: Pythagoras

1 If  $\{v_k\}$  is an orthogonal system and  $v = \sum_{k=1}^{\infty} v_k$ , then

$$\|v\|^2 = \sum_i |v_i|^2$$

2 If  $\{l_k\}$  is an orthonormal system and  $x = \sum_{k=1}^{\infty} x_k l_k$ , then

$$\|x\|^2 = \sum_i |x_i|^2$$

### Proof

Application of the pervious prop.

## 66.5 Def: Fourier coefficient

Let  $\{l_k\}$  be an orthogonal system in  $V$ . Assume that  $x = \sum_{k=1}^{\infty} x_k l_k$ , then

$$x_k := \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle}$$

is called a **Fourier coefficient** of  $x$  in  $\{l_k\}$

Consider  $x \in V$ , then the Fourier series of  $x$  in  $\{l_k\}$  is

$$x \sim \sum_{k=1}^{\infty} x_k l_k$$

We don't know whether it converges to  $x$  yet.

## 66.6 Main example

$$V = L^2(X, \mathbb{K}) / \sim$$

where  $X \in \mathbb{K}^n$  a measurable subset (w.r.t.  $\lambda^n$ ),  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$

$$L^2(X, \mathbb{K}) := \{f : X \rightarrow \mathbb{K} \mid \int_X |f|^2 dx < +\infty\}$$

We define a equivalence relation on  $L^2(X, \mathbb{K})$  by

$$f \sim g \text{ iff } \lambda^n(\{x \in X \mid f(x) \neq g(x)\}) = 0$$

we identify two functions equal if they're equal on almost everywhere (only diff on set that measures zero) From now on, we write elements in  $V$  simply by representatives.

We define an inner product on  $V$ :

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\rightarrow \mathbb{K} \\ (f, g) &\mapsto \int_X f \bar{g} d\lambda^n \end{aligned}$$

(Well defined w.r.t.  $\sim$ ?) Check that  $\int_X f \bar{g} d\lambda^n$  is well defined? Recall that

$$\|f(x)\overline{g(x)}\| = \|f(x)\| \|g(x)\|$$

Then inequality:

$$\begin{aligned} \|f(x)\| \|g(x)\| &\leq \frac{1}{2}(\|f(x)\|^2 + \|g(x)\|^2) \\ \Leftrightarrow 0 &\leq (\|f(x)\| + \|g(x)\|)^2 \end{aligned}$$

We always have  $\|f(x)\| \|g(x)\| \leq \frac{1}{2}(\|f(x)\|^2 + \|g(x)\|^2)$ . It follows that  $\langle f, g \rangle$  is well-defined. It is easy to show that  $\langle \cdot, \cdot \rangle$  is hermitian. The only non-trivial thing is:

$$0 = \langle f, f \rangle \Leftrightarrow f(x) = 0 \text{ almost everywhere}$$

This means that if we don't use  $\sim$  that we cannot say  $\langle \cdot, \cdot \rangle$  is positive defined.

Consider the following Integral:

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} dx = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$$

So  $\{x \mapsto e^{imx} \mid m \in \mathbb{Z}\}$  is an orthogonal system for  $V = L^2([-\pi, \pi], \mathbb{C}) / \sim$ . To make it orthonormal, consider

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{imx} \mid m \in \mathbb{Z} \right\}$$

If you want to replace  $[-\pi, \pi]$  by  $[-a, a]$ , consider:

$$\left\{ \frac{1}{\sqrt{2\pi}} e^{\frac{imx}{a}} \mid m \in \mathbb{Z} \right\}$$

This is an orthonormal system for  $L^2([-a, a], \mathbb{C}) / \sim$ . In the real case, consider the following integrals:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos(nx) \sin(nx) dx = 0$$

$$\int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx = \begin{cases} 0 & \text{if } m \neq n \text{ or } mn = 0 \\ \pi & \text{if } m = n \neq 0 \end{cases}$$

It follows that  $\{1, \cos(nx), \sin(mx) \mid (m, n) \in \mathbb{N}^2\}$  is an orthogonal system for  $L^2([-\pi, \pi], \mathbb{R})$

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k(f) \sin(kx)$$



$$\begin{cases} a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx \\ b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx \end{cases}$$

For instance if  $f = Id$  then  $a_k = 0, b_k = (-1)^{k+1} \frac{2}{k}$



## Chapter 67

# Convergence

Always assume that the orthogonal system countable this chapter

### 67.1 Prop

Take  $x \in V$  and let

$$\bar{x} = \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle}{\langle l_x, l_x \rangle} l_k$$

Then if we write  $x = \bar{x} + h$  then  $h$  is orthogonal to  $\bar{x}$  and  $h$  is orthogonal to the topological closure of  $\langle \{l_k\} \rangle$

### Proof

Since the inner product is continuous, it's enough to show that  $\langle h, l_m \rangle = 0 \forall m$

$$\begin{aligned} \langle h, l_m \rangle &= \langle x, l_m \rangle - \left\langle \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle}{\langle l_x, l_x \rangle} l_k, l_m \right\rangle \\ &= \langle x, l_m \rangle - \frac{\langle x, l_m \rangle}{\langle l_x, l_x \rangle} \langle l_m, l_m \rangle \\ &= 0 \end{aligned}$$

### Remark

By Pythagoras Theorem 66.4, since  $x = \bar{x} + h$

$$\|x\|^2 = \|\bar{x}\|^2 + \|h\|^2 \geq \|\bar{x}\|^2$$

If we write that inequality with respect to the Fourier coefficients, we get **Bessel's inequality**

Note that

$$\begin{aligned}\|x\|^2 &= \sum_k \left| \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle} \right|^2 \langle l_k, l_k \rangle = \sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle} \\ \sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle} &\leq \|x\|^2 \quad (\text{Bessel's inequality})\end{aligned}$$

So far we have assumed that the Fourier series converges to prove Bessel's inequality. But we DON'T NEED this assumption (exercise)

## 67.2 Theorem

Assume  $\{l_k\}$  is orthonormal. Let  $x_k = \langle x, l_k \rangle$ . If  $V$  is complete. then  $\sum_k x_k l_k$  converges.

### Proof

Only to prove that  $\sum_k x_k l_k$  is Cauchy. Bessel's inequality implies that

$$\sum_k |\langle x, l_k \rangle|^2 \leq \|x\|^2$$

By Pythagoras  $\forall m, n \in \mathbb{N}, \epsilon > 0$  have

$$\|x_m l_m + \cdots + x_n l_n\|^2 = |x_m|^2 + \cdots + |x_n|^2 < \sqrt{\epsilon}$$

### Remark

In the proof we assumed  $\{l_k\}$  orthogonal. But this is not essential.

**We have studied the existence of the limit  $x$ . What about the relation between  $\bar{x}$  and  $x$**

## 67.3 Prop

Let  $\{l_k\}$  be an orthogonal system. Take  $x \in V$  and assume that

$$V \ni \bar{x} = \sum_{k=1}^{\infty} \frac{\langle x, l_k \rangle \langle l_k, l_k \rangle}{\langle l_k, l_k \rangle}$$

Then for any  $y = \sum_k d_k l_k$  ( $d_k \in \mathbb{F}$ ) it holds that:

$$\|x - \bar{x}\| \leq \|x - y\|$$

The equality is true iff  $\bar{x} = y$

**Proof**

$$\begin{aligned}\|x - y\|^2 &= \|x - \bar{x} + (\bar{x} - y)\|^2 \\ &= \|h\|^2 + \|\bar{x} - y\|^2\end{aligned}$$

**67.4 Def**

A family of vectors  $\mathcal{F} = \{x_\alpha \mid \alpha \in A\}$  in a normed vector space  $V$  is **complete** in a subset  $E \subseteq V$  if every vector  $x \in E$  can be approximated with arbitrary accuracy by a **finite** linear combination of elements in  $\mathcal{F}$

**Another statement**

Let  $L = \{\mathcal{F}\}$ , then  $\mathcal{F}$  is complete in  $E$  if  $E \subseteq \bar{L}$

**Example**

$V = L^2([a, b], \mathbb{F})/\sim$  and the family  $\mathcal{F} = \{x^k \mid k \in \mathbb{N}\}$ , then  $\mathbb{F}$  is complete in  $V$

**67.5 Weierstrass Approximation Theorem**

Let  $f \in \mathcal{C}([a, b])$ . For any  $\epsilon > 0$ , there exists a polynomial  $p \in \mathbb{F}[x]$  such that for any  $x \in [a, b]$ , we have

$$|f(x) - p(x)| < \epsilon$$

In fact

$$\|f - p\| = \sqrt{\int_a^b |f - p|^2 d\lambda} < \epsilon \sqrt{b - a}$$

**Proof**

check here

**67.6 Prop**

Let  $V$  be a complete vector space over  $\mathbb{F}$  with inner product  $\langle, \rangle$  hermitian or real symm, and positive definite and non-degenerate.

Moreover,  $\{l_k\}$  is an orthogonal system at most countable. Then the following conditions are equivalent:

- 1  $\{l_k\}$  is complete in  $E \subseteq V$

2 For any  $x \in E$ , we have  $x = \sum_k \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle} l_k$

3 Any vector  $x \in E$  satisfies

$$\|x\|^2 = \sum_k \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle}$$

### Proof

(1)  $\Rightarrow$  (2)

Take  $x \in V$ . Since  $V$  is complete,  $\bar{x} = \sum_k x_k l_k$ ,  $x_k := \frac{\langle x, l_k \rangle}{\langle l_k, l_k \rangle}$   
 $\forall \epsilon > 0$ ,  $\exists$  a finite linear combination  $\alpha_1 l_1 + \cdots + \alpha_m l_m$  s.t.

$$\|x - (\alpha_1 l_1 + \cdots + \alpha_m l_m)\| < \epsilon$$

By prop 67.3

$$0 \leq \|x - \bar{x}\| \leq \|x - (\alpha_1 l_1 + \cdots + \alpha_m l_m)\| < \epsilon$$

(2)  $\Rightarrow$  (3)

Pythagoras

(3)  $\Rightarrow$  (1)

$$\begin{aligned} x - \sum_k \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle} l_k &= h_n + \sum_{k>n} \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle} l_k \\ \left\langle x - \sum_{k=1}^n x_k l_k, \sum_{k=1}^n x_k l_k \right\rangle &= \left\langle h_n + \sum_{k>n} \frac{|\langle x, l_k \rangle|}{\langle l_k, l_k \rangle} l_k, \sum_{k=1}^n x_k l_k \right\rangle = 0 \end{aligned}$$

(By prop 67.6)

Apply Pythagoras

$$\left\| x - \sum_{k=1}^n x_k l_k \right\|^2 = \|x\|^2 - \left\| \sum_{k=1}^n x_k l_k \right\|^2 = \|x\|^2 - \sum_{k=1}^n \frac{|\langle x, l_k \rangle|^2}{\langle l_k, l_k \rangle}$$

The RHS can be arbitrary small. That ends the proof.

# Chapter 68

## 68.1 Def: Hamal basis

A countable family of vectors  $\{b_k\}$  is a **Hamal basis** of  $V$  if any  $v \in V$  there exists a unique sequence  $\{\alpha_k\}$  in  $\mathbb{K}$  with  $\alpha_k = 0$  for all but finitely many  $k$  s.t.

$$v = \sum_k \alpha_k b_k$$

(In this def we don't need to use the topological properties of  $V$ )

## 68.2 Def: Schauder basis

A countable family of vectors  $\{b_k\}$  is a **Schauder basis** for  $V$  if for any  $v \in V$  there exists a unique sequence  $\{\alpha_k\}$  such that

$$v = \sum_k \alpha_k b_k \quad (\text{as convergent series})$$

A Hamal basis is a Schauder basis(? to prove an element in basis can't be represented by others). In particular, a Schauder basis is a complete family of vectors (in  $E = V$ )

In pervious, we've proved that if  $\{l_k\}$  is an orthogonal complete system (in  $E = V$ ) with  $V$  complete, then any  $x \in V$  can be written as

$$x = \sum_k \alpha_k l_k$$

when  $\alpha_k$  are the Fourier coefficients.

In general it's FALSE that a complete family  $\{b_k\}$  is a Schauder ( $x \in \overline{\langle \{b_k\} \rangle}$ )

## Example

$V = \mathcal{C}([-1, 1], \mathbb{R}) \subseteq L^2([-1, 1], \mathbb{R})$  we induce on  $V$  the inner product from  $L^2([-1, 1], \mathbb{R})$ . We've proved that  $\{x^k \mid k \geq 0\}$  is a complete family in  $V$  Now we show that it's not a Schauder basis (in particular we show that not any element  $v \in V$  can be expressed as convergent series)

**68.2.1**

We will show that if  $\sum_k \alpha_k x^k$  converges in the  $L^2$ -norm, then it converges pointwisely in  $] -1, 1[$ . In fact

$$\|\alpha_k x^k\|_2^2 = \int_{-1}^1 (\alpha_k x^k)^2 dx = \alpha_k^2 \frac{2}{k+1}$$

$\|\alpha_k x^k\|_2^2 \rightarrow 0$ , so it follows that for  $k$  big enough

$$\alpha_k^2 < (2k+1) \frac{\epsilon}{2} < 2k+1$$

In other words for  $k \geq N$  then  $\alpha_k < \sqrt{2k+1}$  Now take  $x \in ]0, 1[$  Consider

$$\sum_{k \geq N} \alpha_k x^k \leq \sum_{k \geq N} (2k+1)^{\frac{1}{2}} x^k$$

Use the ratio test

$$\lim_{k \rightarrow +\infty} \frac{x^{k+1}(2k+3)^{\frac{1}{2}}}{x^k(2k+1)^{\frac{1}{2}}} = x < 1$$

It means that the series converges, because  $\sum_{k \geq N} \alpha_k x^k$  converges.

By symmetry do the same in  $] -1, 0[$  It follows  $\sum_k \alpha_k x^k$  converges pointwisely  $\forall x \in ] -1, 1[$

Call  $\varphi : ] -1, 1[ \rightarrow \mathbb{R}$  the limit of such sequence. This means that the radius of convergence for  $\sum_k \alpha_k x^k$  is  $\mathbb{R} > 1$ . It means that  $\sum_k \alpha_k x^k$  converges uniformly in any  $[a, b] \subseteq ] -1, 1[$  to  $\varphi|_{[a,b]}$

Assume that  $f \stackrel{L^2}{=} \sum_k \alpha_k x^k$  so on  $] -1, 1[$   $f = \varphi|_{]-1, 1[}$  But  $\varphi$  is uniform limit of a power series, then  $\varphi$  is a  $C^\infty(]-1, 1[)$  function. (if  $f$  is the  $L^2$ -limit of  $\sum_k \alpha_k x^k$ ) then  $f$  is of class  $C^\infty$  in  $[-1, 1]$  Therefore it's enough to take  $g \in V \setminus (C^\infty(]-1, 1[))$  such function cannot be a  $L^2$ -limit then

$$\{x^l \mid x \geq 0\}$$

is not a Schauder basis.

$V = L^2([-\pi, \pi], \mathbb{K})$  if  $\mathbb{K} = \mathbb{R}$   $\{1, \cos kx, \sin kx \mid k \in \mathbb{N}_{\geq 1}\}$  an orthogonal system (trigonometric system)  $f \in V$

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k(f) \cos kx + b_k(f) \sin kx$$

and the Bessel's inequality:

$$\frac{|a_0(f)|^2}{2} + \sum_{k=1}^{\infty} |a_k(f)|^2 + |b_k(f)|^2 \leq (=) \frac{1}{\pi} \int_{-\pi}^{\pi} |f|^2 d\lambda$$



Question: Consider

$$\sum_k \frac{\sin(kx)}{\sqrt{k}}$$

this converges pointwisely on  $\mathbb{R}$  Is this a Fourier series of some  $f \in V$ ?

No for Bessel inequality failed:

$$\sum_k \left(\frac{1}{\sqrt{k}}\right)^2 = \sum_k \frac{1}{k} \quad \text{diverges}$$

What we want to prove in remaining part is

1  $L^2([-\pi, \pi], \mathbb{K})$  is complete

2  $\{1, \cos kx, \sin kx \mid k \in \mathbb{N}_{\geq 1}\}$  is a complete family in  $E = V$

Then conclude that any  $f \in V$

$$f \stackrel{L^2}{=} \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx + b_k \sin kx$$

Parseval equality



# Chapter 69

## 69.1 Gaol

what we want to prove here are:

**1**

$L^2([-\pi, \pi[, \mathbb{K})$  is complete as topological vector space.

**2**

$\{1, \cos kx, \sin kx \mid k \in \mathbb{N}_{\geq 1}\}$  is a complete family.

## 69.2 The first one

For (1) we prove more.

Let  $(X, \mu)$  be a measure space In particular

$$X = \Omega \subseteq \mathbb{R}^n$$

when  $p = 2$ ,  $L^2(X, \mu)$  is an inner product space.

$$\langle f, g \rangle = \int_X f \bar{g} d\mu$$

Recall  $L^p(X, \mu)$  a normed vector space:

$$L^p(X, \mu) = \{f : X \rightarrow \mathbb{K} \mid \int_X |f|^p d\mu < +\infty\}$$

where  $p \in \mathbb{R}_{\geq 1}$ , and the norm on it:

$$\|f\|_p := \left( \int_X |f|^p d\mu \right)^{\frac{1}{p}}$$

### 69.3 Def

$f : X \setminus \{x_0\} \rightarrow [0, +\infty[$  we say that  $f$  is **unbounded** at  $x_0$  if  $\forall U \ni x_0, M > 0$

$$\exists x \in U \text{ s.t. } f(x) > M$$

### 69.4 Def: extend by periodicity

Let  $f : [-\pi, \pi[ \rightarrow \mathbb{R}$  extend this func by periodicity.

$$\tilde{f} = f(x - 2k\pi) \quad k \in \mathbb{Z}$$

### 69.5 Things used

- $\mathbb{K}$  is complete
- Holden inequality
- Monotone convergence theorem
- Fatou lemma

### 69.6 Theorem

$L^p(X, \mu)$  is complete w.r.t the topology induced by  $\|\cdot\|_p$

#### Proof

We need to show that any Cauchy sequence is convergent.

Consider  $\{f_n\} \subseteq L^p(X, \mu)$  Cauchy sequence. Choose  $n_1 < \dots < n_j$  such that

$$\|f_n - f_m\|_p \leq 2^{-2j} \text{ for } m, n \geq n_j := n_j(\epsilon)$$

Let  $g_j := f_{n_j}$ ;  $h_k := g_{k+1} - g_k$  Now we will study  $\{h_k\}_{k \in \mathbb{N}}$   
Notice that:

$$\int_X |h_k|^p d\mu = \|h_k\|^p = \|g_{k+1} - g_k\|^p \leq 2^{-2pk} \quad (*)$$

#### Step 1

We show that  $\sum_{k=1}^{\infty} h_k(x)$  converges pointwisely almost everywhere in  $X$

Let  $a_k := |h_k(x)| 2^{\frac{k}{p}}$ ;  $b_k := 2^{-\frac{k}{p}}$

$$\begin{aligned} \sum_{k=1}^{\infty} |h_k(x)| &= \sum_{k=1}^{\infty} a_k b_k \\ (\text{by Hölder}) &\leq \left( \sum_{k=1}^{\infty} |a_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{\infty} |b_k|^q \right)^{\frac{1}{q}} \\ &= \left( \sum_{k=1}^{\infty} 2^k |h_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{\infty} 2^{-\frac{kq}{p}} \right)^{\frac{1}{q}} \end{aligned}$$

then

$$\int_X \left( \sum_{k=1}^{\infty} |h_k| \right)^p d\mu \leq C^p \int_X \sum_{k=1}^{\infty} 2^k |h_k|^p d\mu$$

Use the monotone convergence theorem on the sequence

$$S_N(x) := \sum_{k=1}^N 2^k |h_k(x)|^p$$

then

$$\begin{aligned} C^p \int_X \sum_{k=1}^{\infty} 2^k |h_k|^p d\mu &\stackrel{(*)}{\leq} C^p \sum_{k=1}^{\infty} 2^k 2^{-2pk} \\ &= C^p \sum_{k=1}^{\infty} 2^{-k(2p-1)} < +\infty \end{aligned}$$

what we want is that

$$\int_X \left( \sum_{k=1}^{\infty} |h_k| \right)^p d\mu < +\infty \Rightarrow \sum_{k=1}^{\infty} |h_k(x)| < +\infty$$

### Lemma

$f : X \rightarrow [-\infty, +\infty]$  measurable. If  $\int_X |f|^p < +\infty$ . Then  $|f|^p$  is bounded almost everywhere

### Proof

$$E_n = \{x \in X \mid |f(x)|^p > n\}$$

$$C \geq \int_X |f|^p d\mu \geq \int_{E_n} |f|^p d\mu \geq n\mu(E_n)$$

$$n\mu(E_n) < C \Rightarrow \mu(E_n) \leq \frac{C}{n} \text{ and } \lim_{n \rightarrow +\infty} \mu(E_n) = 0$$

Let  $E = \bigcap_{n=1}^{\infty} E_n = \{x \in X \mid |f(x)|^p = +\infty\}$

$$\mu(E) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow +\infty} \mu(E_n) = 0$$

This lemma means that  $\{\sum h_k(x)\}$  absolutely converges almost everywhere. Since  $\mathbb{K}$  is complete, then  $\{\sum h_k(x)\}$  converges almost everywhere

### 69.6.1 Step 2: Conclusion

Define

$$f(x) := \begin{cases} \underbrace{g_1(x) + \sum_{k=1}^{\infty} h_k(x)}_{= \lim_{n \rightarrow +\infty} g_n(x)} & \text{if series converge} \\ 0 & \text{otherwise} \end{cases}$$

Fix an index  $k$

$$\begin{aligned} & 2^{-2kp} \stackrel{(*)}{\liminf}_{j \rightarrow +\infty} \int_X |g_j(x) - g_k(x)|^p d\mu \\ (\text{Fatou's lemma}) & \geq \int_X \liminf_{j \rightarrow +\infty} |g_j(x) - g_k(x)|^p d\mu \\ & = \end{aligned}$$

Notice that for  $k = 1$

$$f - g_1 \in L^p(X, \mu)$$

so

$$f = (f - g_1) + g_1 \in L^p(X, \mu)$$

we want to show

$$f_n \xrightarrow{L^p} f$$

for all  $n \geq n_k$

$$\|f_n - f\|_p \leq \underbrace{\|f_n - g_k\|_p}_{\text{by } (*) \leq 2^{-2k}} + \underbrace{\|g_k - f\|_p}_{\leq 2^{-2k}} \leq 2^{-2k+1}$$

this means

$$f_n \xrightarrow{L^2} f$$

In particular  $L^2([-\pi, \pi[, \mathbb{K})$  is complete.

## Chapter 70

# Pointwise Convergence of Fourier Series

We want to prove  $\{1, \cos kx, \sin kx \mid k \in \mathbb{N}_{\geq 1}\}$  is a complete family.  
We need to study the pointwise convergence of Fourier series

$$\sum_k \frac{\langle v, l_k \rangle}{\langle l_k, l_k \rangle} l_k \rightarrow f(x)$$

let

$$T_n(x) = \sum_{k=-n}^n \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left( \sum_{k=-n}^n e^{ik(x-t)} \right) dt$$

the Goal is to write the Fourier coefficients in a "better".

### 70.1 Def: Dirichlet kernel

$$D_n(x) := \sum_{k=-n}^n e^{iku} = \frac{\sin(n + \frac{u}{2})}{\sin \frac{u}{2}}$$

this is called **Dirichlet kernel**, which has the prop

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(u) du = \frac{1}{\pi} \int_0^{\pi} D_n(u) du = 1$$

Back to  $T_n$  putting  $u = x - t$

$$\begin{aligned} T_n(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) D_n(u) du \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-u) \frac{\sin(n + \frac{u}{2})}{\sin \frac{u}{2}} du \end{aligned}$$

Now use that  $D_n$  is an even function

$$T_n(x) = \frac{1}{2\pi} \int_0^\pi (f(x-u) + f(x+u)) D_n(u) du = \frac{1}{2\pi} \int_0^\pi (f(x-u) + f(x+u)) \frac{\sin(n + \frac{u}{2})}{\sin \frac{u}{2}} du$$

## 70.2 Riemann-Lebesgue's Lemma

Let  $f :: [a, b] \rightarrow \mathbb{R}$  be an integrable function. Then

$$\lim_{\lambda \rightarrow +\infty} \int_a^b f(x) e^{i\lambda x} dx = 0$$

### Proof

#### Step 1

$f$  is a simple function, namely  $f(x) = \sum_{k=1}^n c_k \mathbb{1}_{[x_k, x_{k+1}[}(x)$

$$\int_a^b f(x) e^{i\lambda x} dx = \sum_{k=1}^n c_k \int_{x_k}^{x_{k+1}} e^{i\lambda x} dx$$

Note that

$$\begin{aligned} \int_{x_k}^{x_{k+1}} e^{i\lambda x} dx &= \int_{x_k}^{x_{k+1}} \cos \lambda x dx + i \int_{x_k}^{x_{k+1}} \sin \lambda x dx \\ &= \frac{1}{\lambda} \sin \lambda x \Big|_{x_k}^{x_{k+1}} + \frac{i}{\lambda} \cos \lambda x \Big|_{x_k}^{x_{k+1}} \\ &= \frac{e^{i\lambda x_{k+1}} - e^{i\lambda x_k}}{\lambda} \\ &\rightarrow 0 \end{aligned}$$

#### Step 2

Since  $f$  is integrable, then  $f$  can be approximated arbitrarily by simple functions (lemma 53.23). Then for any  $\epsilon > 0$ ,  $\exists \phi : [a, b] \rightarrow \mathbb{R}$  simple, s.t.

$$\int_a^b |f(x) - \phi(x)| dx < \epsilon$$



thus

$$\begin{aligned}
 \left| \int_a^b f(x) e^{i\lambda x} dx \right| &= \left| \int_a^b (f(x) - \phi(x)) e^{i\lambda x} dx + \int_a^b \phi(x) e^{i\lambda x} dx \right| \\
 &\leq \left| \int_a^b (f(x) - \phi(x)) e^{i\lambda x} dx \right| + \left| \int_a^b \phi(x) e^{i\lambda x} dx \right| \\
 &\leq \int_a^b |f(x) - \phi(x)| dx + \left| \int_a^b \phi(x) e^{i\lambda x} dx \right| \\
 &\rightarrow 0
 \end{aligned}$$

### 70.3 Corollary

$$\begin{aligned}
 \lim_{\lambda \rightarrow +\infty} \int_a^b f(x) \cos \lambda x dx &= 0 \\
 \lim_{\lambda \rightarrow +\infty} \int_a^b f(x) \sin \lambda x dx &= 0
 \end{aligned}$$

Go back to Fourier series

$$\begin{aligned}
 T_{n,f}(x) &= T_n(x) \\
 &= \sum_{k=-n}^n \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt \right) e^{ikx} \\
 &= \frac{1}{2\pi} \int_0^{\pi} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du
 \end{aligned}$$

Let  $\delta \in ]0, \pi[$

$$\begin{aligned}
 &\frac{1}{2\pi} \int_0^{\pi} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du \\
 &\leq \frac{1}{2\pi} \int_0^{\delta} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du \\
 &+ \left| \frac{1}{2\pi} \int_{\delta}^{\pi} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du \right|
 \end{aligned}$$

But

$$\left| \frac{1}{2\pi} \int_{\delta}^{\pi} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du \right| \leq \left| \frac{1}{2\pi} \int_{\delta}^{\pi} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du \right|$$

By Riemann-Lebesgue's Lemma 70.2

$$T_n(f) = \frac{1}{w\pi} \int_0^{\delta} (f(x-u) + f(x+u)) \frac{\sin(b + \frac{u}{2})}{\sin \frac{u}{2}} du + o(1) \quad n \rightarrow +\infty$$

## 70.4 Localization Principle

Let  $f, g \in L^2([-\pi, \pi], \mathbb{K})$ . If  $f, g$  coincide in a neighborhood of  $x_0 \in ]-\pi, \pi[$  ( $f = g$ ), the Fourier series

$$f \sim \sum_{-\infty}^{+\infty} c_k(f) e^{i\lambda_k x} \quad g \sim \sum_{-\infty}^{+\infty} c_k(g) e^{i\lambda_k x}$$

*either* both diverges *or* both converges. Moreover if they converges at  $x_0$ , then their limits are the same (NOT to be  $f(x_0) = g(x_0)$ )

### Proof

exercise, hint: use equation right above the theorem.

## 70.5 Def: Dini's Condition

Let  $U_x^0 = [-\delta, x[ \cup ]x, \delta[$  and  $f : U_x^0 \rightarrow \mathbb{C}$ . We say that  $f$  satisfies **Dini's Condition** at  $x$  if

- $f(x_-)$  and  $f(x_+)$  exists and finite
- $\exists > 0$  s.t.

$$\int_0^\epsilon \left| \frac{(f(x-t) - f(x_-)) + (f(x+t) - f(x_+))}{t} \right| dt < +\infty$$

### Example

Let  $f : U_2 \rightarrow \mathbb{C}$  be a continuous function satisfying the Holder inequality

$$|f(x+t) - f(x)| \leq M |t|^\alpha \quad M > 0, \alpha \in ]0, 1]$$

Moreover

$$\left| \frac{f(x+t) - f(x)}{t} \right| \leq \frac{M}{|t|^{1-\alpha}}$$

So condition (2) is satisfied. Dini's conditions are satisfied at  $x$ . Moreover,  $f : U_2^0 \rightarrow \mathbb{C}$ ,  $f(x_-)$  and  $f(x_+)$  exists and finite.

Assume that

$$|f(x+t) - f(x_+)| \leq M t^\alpha$$

$$|f(x-t) - f(x_-)| \leq M t^\alpha$$

Also in this case Dini's conditions at  $x$  are satisfied

## 70.6 Theorem: pointwise convergence of Fourier series

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a periodic function of period  $2\pi$ , such that  $f$  is integrable in  $[-\pi, \pi]$ . If  $f$  satisfies the Dini's condition at  $x \in \mathbb{R}$ , then its Fourier series converges at  $x$  and

$$\sum_{-\infty}^{+\infty} c_k(f) e^{i\lambda x} = \frac{f(x_-) + f(x_+)}{2}$$

### Proof

$$T_n(x) - \frac{f(x_-) + f(x_+)}{2} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x-t) - f(x_-) + f(x+t) - f(x_+)}{2 \sin \frac{t}{2}} \sin(n + \frac{1}{2})t dt$$

Now for  $t \rightarrow 0^+$ ,  $2 \sin \frac{t}{2} \sim t$ , so use then Localization Principle 70.4, the Dini's conditions 70.5 and Riemann-Lebesgue's Lemma 70.2 to conclude that

$$\left( T_n(x) - \frac{f(x_-) + f(x_+)}{2} \right) \xrightarrow{n \rightarrow +\infty} 0$$

### Example

Consider  $f = Id_{[-\pi, \pi]}$  and extend this by periodicity. Its Fourier series is

$$\sum_{-\infty}^{+\infty} 2 \frac{(-1)^{k+1}}{k} \sin kx$$

Use Theo. above

$$\sum_{-\infty}^{+\infty} 2 \frac{(-1)^{k+1}}{k} \sin kx = \begin{cases} x & \text{if } |x| < \pi \\ 0 & \text{if } |x| = \pi \end{cases}$$

Consider  $f : \mathbb{R} \rightarrow \mathbb{C}$  of period  $2\pi$ . Let  $T_0(x), \dots, T_n(x)$  be the trigonometric polynomial of  $f$  of order  $0, \dots, n$ .

Let

$$\sigma_n(x) := \frac{T_0(x) + \dots + T_n(x)}{n+1}$$

Recall the expression of  $T_i(x)$  in terms of the Dirichlet kernel  $D_n(x)$

$$T_i(x) = \frac{1}{2\pi} \int_0^\pi f(x-u) D_n(u) du$$

So we have

$$\sigma_n(x) = \frac{1}{2} \int_{-\pi}^\pi f(x-t) F_n(t) dt$$

where

$$F_n(t) = \frac{1}{n+1} (D_0(t) + \cdots + D_n(t))$$

which is called the Fejer's kernel

## 70.7 Lemma

$$\sum_{k=0}^n \sin\left(k + \frac{1}{2}\right)t = \frac{\sin^2\left(\frac{n+1}{2}t\right)}{\sin^2\frac{t}{2}}$$

$$F_n(t) = \frac{\sin^2\frac{n+1}{2}t}{(n+1)\sin^2\frac{t}{2}}$$

What happens when  $t = 2k\pi$ ? Use Taylor's expansion:

$$F_n(t) = \frac{\left(\frac{(n+1)}{2}t + o(t)\right)^2}{(n+1)\left(\frac{t}{2} + o(t)\right)^2} \quad t \rightarrow 0$$

$$F_n(t) = n+1$$

So we can extend  $F_n$  at all points  $2l\pi$  by putting  $F_n(2k\pi) = n+1$

## 70.8 Def: approximated identity(delta function)

A family of functions  $\{K_n\}_{n \in \mathbb{N}}$  with  $K_n : \mathbb{R} \rightarrow \mathbb{R}$  is called a **approximated identity** if

- $\frac{1}{2\pi} \int_{-\infty}^{\infty} K_n(t) dt = 1 \quad \forall n \geq 0$
- $K_n(t) \geq 0, \forall t \in \mathbb{R}, n \geq 0$
- For any  $\delta > 0$

$$\lim_{n \rightarrow +\infty} \int_{|t| > \delta} K_n(t) dt = 0$$

## 70.9 Prop

Consider

$$\delta_n(x) = \begin{cases} \frac{1}{2\pi} F_n(x) & \text{if } |x| \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

Then  $\{\delta_n\}$  is an approximated identity

**Proof****2**

obvious

**1**

$$\int_{\mathbb{R}} \delta_n d\lambda = \frac{1}{2\pi(n+1)} \sum_{k=0}^n \int_{\mathbb{R}} D_k(t) d\lambda = 1$$

**3**

$$0 \leq \int_{-\infty}^{-\delta} \delta_n(t) dt = \int_{\delta}^{+\infty} \delta_n(t) dt \leq \frac{1}{2\pi(n+1)} \int_{\delta}^{\pi} \frac{1}{\sin^2 \frac{t}{2}} dt = \frac{2\delta}{2\pi(n+1) \tan \frac{\delta}{2}} \xrightarrow{n \rightarrow +\infty} 0$$

**70.10 Fejer Theorem**

Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  continuous and with period of  $2\pi$  and integrable in  $[-\pi, \pi]$ . Then  $\sigma_{f,n}$  converges uniformly to  $f$

**Proof**

We want to prove that

$$\lim_{n \rightarrow +\infty} \sup_{x \in [-\pi, \pi]} |\sigma_n(x) - f(x)| = 0$$

Since  $f$  is continuous in a compact set, it's uniformly continuous in  $[-\pi, \pi]$  then  $\forall \epsilon > 0 \exists \delta > 0$  s.t.

$$|f(x-t) - f(x)| < \epsilon \quad \forall |t| < \delta$$

Now by theorem 70.6

$$\begin{aligned} \sigma_n(x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) F_n(t) dt - f(x) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x-t) - f(x)) F_n(t) dt \end{aligned}$$

Now split the integral into two parts:

Near  $t = 0$   $|t| < \epsilon$  For  $f$  continuous and by prop 70.9

$$\int_{|t| < \epsilon} (f(x-t) - f(t)) F_n(t) dt \leq \epsilon \int_{|t| < \epsilon} F_n(t) dt \leq \epsilon 2\pi$$

Away from  $0$   $t \in [\delta, \pi]$  By 70.9 proof 3

$$\int_{t \in [\delta, \pi]} (f(x-t) - f(t)) F_n(t) dt \leq 2 \sup_{[-\pi, \pi]} f \int_{t \in [\delta, \pi]} F_n(t) dt \rightarrow 0$$

### 70.11 Weierstrass approximation

Let  $f : [-\pi, \pi] \rightarrow \mathbb{C}$  be a continuous functions s.t.  $f(-\pi) = f(\pi)$ . Then such function can be approximated uniformly by  $\sigma_n$  arbitrarily

#### Proof

Since  $f(-\pi) = f(\pi)$ , when we extend this function by periodicity, we will get an continuous function over  $\mathbb{R}$ . Then we can apply Fejer's theorem