

# Chapter 1

## Set

### 1.1 Ring

#### 1.1.1 morphism

##### Def

Let  $A$  and  $B$  be unitary rings. We call morphism of unitary rings from  $A$  to  $B$  only mapping  $A \rightarrow B$  is a morphism of group from  $(A, +)$  to  $(B, +)$ , and a morphism of monoid from  $(A, \cdot)$  to  $(B, \cdot)$

##### Properties

- Let  $R$  be a unitary ring. There is a unique morphism from  $\mathbb{Z}$  to  $R$
- 

##### algebra

we call  $k$ -algebra any pair  $(R, f)$ , when  $R$  is a unitary ring, and  $f : k \rightarrow R$  is a morphism of unitary rings such that  $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring  $R$ , the unique morphism of unitary rings  $\mathbb{Z} \rightarrow R$  define a structure of  $\mathbb{Z}$ -algebra on  $R$  (extra:  $\mathbb{Z}$  is commutative despite  $R$  isn't guaranteed)

Notation: Let  $k$  be a commutative unitary ring,  $(A, f)$  be a  $k$ -algebra. If there is no ambiguity on  $f$ , for any  $(\lambda, a) \in k \times A$ , we denote  $f(\lambda)a$  as  $\lambda a$

##### Formal power series

reminder:  $n \in \mathbb{N}$  is possible infinite, so  $\sum_{n \in \mathbb{N}}$  couldn't be executed directly.

##### Def:

(extended polynomial actually) Let  $k$  be a commutative unitary ring. Def : Let  $T$  be a formal symbol. We denote  $k^{\mathbb{N}}$  as  $k[T]$  If  $(a_n)_{n \in \mathbb{N}}$  is an element

of  $k^{\mathbb{N}}$ , when we denote  $k^{\mathbb{N}}$  as  $k[T]$  this element is denoted as  $\sum_{n \in \mathbb{N}} a_n T^n$ . Such element is called a formal power series over  $k$  and  $a_n$  is called the Coefficient of  $T^n$  of this formal power series. Notation:

- omit terms with coefficient 0
- write  $T'$  as  $T$
- omit Coefficient those are 1;
- omit  $T^0$

Example  $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$  is written as  $1 + 2T + T^2$

Def Remind that  $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$ , define two composition laws on  $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$  form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$  is a morphism
- $(FG)H = \left( \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left( \sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left( \sum_{p,q,l=n} a_p b_q c_l \right) T^n$   
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by  $F'(T)$  or  $\mathcal{D}(F(T))$  the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$  is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote  $\exp(T) \in k[T]$  as  $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$ , which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence:  $(F_i(T))_{i \in \mathbb{N}}$  be a sequence of elements in  $k[T]$ , and  $F(T) \in k[T]$ . We say that  $(F_i(T))_{i \in \mathbb{N}}$  is a Cauchy sequence if  $\forall l \in \mathbb{N}$ , there exists  $N(l) \in \mathbb{N}$  such that  $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$ ,  $\text{ord}(F_i(T) - F_j(T)) \geq l$

## Chapter 2

# Sequences

### 2.1 Supremum and infimum

Def:

Let  $(X, \leq)$  be a partially ordered set  $A$  and  $Y$  be subsets of  $X$ , such that  $A \subseteq Y$

- If the set  $\{y \in Y \mid \forall a \in A, a \leq y\}$  has a least element then we say that  $A$  has a Supremum in  $Y$  with respect to  $\leq$  denoted by  $\sup_{(Y, \leq)} A$  this least element and called it the Supremum of  $A$  in  $Y$  (this respect to  $\leq$ )
- If the set  $\{y \in Y \mid \forall a \in A, y \leq a\}$  has a greatest element, we say that  $A$  has an infimum in  $Y$  with respect to  $\leq$ . We denote by  $\inf_{(Y, \leq)} A$  this greatest element and call it the infimum of  $A$  in  $Y$
- Observation:  $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set.

- If  $f$  is a function from  $I$  to  $X$   $\sup f$  denotes the supremum of  $f(I)$  is  $X$ .  $\inf f$  takes the same
- If  $(x_i)_{i \in I}$  is a family of element in  $X$ , then  $\sup_{i \in I} x_i$  denotes  $\sup\{x_i \mid i \in I\}$  (in  $X$ )

If moreover  $\mathbb{P}(\cdot)$  denotes a statement depending on a parameter in  $I$  then  $\sup_{i \in I, \mathbb{P}(i)} x_i$  denotes  $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let  $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$  We equip  $\mathbb{R}$  with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So  $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence  $\inf A = 0$

Example: For  $n \in \mathbb{N}$ , let  $x_n = (-1)^n \in \mathbb{R}$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let  $(X, \leq)$  be a partially ordered set,  $A, Y, Z$  be subset of  $X$ , such that  $A \subseteq Z \subseteq Y$

- If  $\max A$  exists, then it is also equal to  $\sup_{(y, \leq)} A$
- If  $\sup_{(y, \leq)} A$  exists and belongs to  $Z$ , then it is equal to  $\sup A$

$\inf$  takes the same Prop.

Let  $X, \leq$  be a partially ordered set,  $A, B, Y$  be subsets of  $X$  such that  $A \subseteq B \subseteq Y$

- If  $\sup_{(y, \leq)} A$  and  $\sup_{(y, \leq)} B$  exists, then  $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If  $\inf_{(y, \leq)} A$  and  $\inf_{(y, \leq)} B$  exists, then  $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let  $(X, \leq)$  be a partially ordered set,  $I$  be a set and  $f, g : I \rightarrow X$  be mappings such that  $\forall t \in I, f(t) \leq g(t)$

- If  $\inf f$  and  $\inf g$  exists, then  $\inf f \leq \inf g$
- If  $\sup f$  and  $\sup g$  exists, then  $\sup f \leq \sup g$

## 2.2 Interval

We fix a totally ordered set  $(X, \leq)$

Notation:

If  $(a, b) \in X \times X$  such that  $a \leq b$ ,  $[a, b]$  denotes  $\{x \in X \mid a \leq x \leq b\}$

Def:

Let  $I \subseteq X$ . If  $\forall (x, y) \in I \times I$  with  $x \leq y$ , one has  $[x, y] \subseteq I$  then we say that  $I$  is a interval in  $X$

Example:

Let  $(a, b) \in X \times X$ , such that  $a \leq b$  Then the following sets are intervals

- $]a, b[ := \{x \in X \mid a, x, b\}$
- $[a, b[ := \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let  $\Lambda$  be a non-empty set and  $(I_\lambda)_{\lambda \in \Lambda}$  be a family of intervals in  $X$ .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$  is a interval in X
- If  $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$ ,  $\bigcup_{\lambda \in \Lambda} I_\lambda$  is a interval in X

We check that  $[a, b] \subseteq I_\lambda \cup I_\mu$

- If  $b \leq x$   $[a, b] \subseteq [a, x] \subseteq I_\lambda$  because  $\{a, x\} \subseteq I_\lambda$
- If  $x \leq a$   $[a, b] \subseteq [x, b] \subseteq I_\mu$  because  $\{b, x\} \subseteq I_\mu$
- If  $a < x < b$  then  $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let  $(X, \leq)$  be a totally ordered set .I be a non-empty interval of X. If  $\sup I$  exists in X, we call  $\sup I$  the right endpoint;  $\inf$  takes the similar way.

Prop.

Let I be an interval in X.

- Suppose that  $b = \sup I$  exists.  $\forall x \in I, [x, b[ \subseteq I$
- Suppose that  $a = \inf I$  exists.  $\forall x \in I, ]a, x] \subseteq I$

Prop.

Let I be an interval in X. Suppose that I has supremum b and an infimum a in X. Then I is equal to one of the following sets  $[a, b]$   $[a, b[$   $]a, b]$   $]a, b[$

Def

let  $(X, \leq)$  be a totally ordered set .If  $\forall (x, z) \in X \times X$ , such that  $x < z$   $\exists y \in X$  such that  $x < y < z$ , then we say that  $(X, \leq)$  is thick

Prop.

Let  $(X, \leq)$  be a thick totally ordered set.  $(a, b) \in X \times X, a < b$  If I is one of the following intervals  $[a, b]; [a, b[; ]a, b]; ]a, b[$  Then  $\inf I = a$   $\sup I = b$  (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists  $x_0 \in ]a, b[$  By definition, b is an upper bound of I. If b is not the supremum of I, there exists an upper bound M of I such that  $M < b$ . Since X is thick, there is  $M' \in X$  such that  $x_0 \leq M, M' < b$  Since  $[x, b[ \subseteq ]a, b[ \subseteq I$  Hence M and M' belong to I, which conflicts with the uniqueness of supremum.

## 2.3 Enhanced real line

Def:

Let  $+\infty$  and  $-\infty$  be two symbols that are different and don not belong to  $\mathbb{R}$  We extend the usual total order  $\leq$  on  $\mathbb{R}$  to  $\mathbb{R} \cup \{-\infty, +\infty\}$  such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus  $\mathbb{R} \cup \{-\infty, +\infty\}$  become a totally ordered set, and  $\mathbb{R} = ]-\infty, +\infty[$  Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in ]-\infty, +\infty] \quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[ \quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in ]0, +\infty] \quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[ \quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$   
**ARE NOT DEFINED**

Def

Let  $(X, \leq)$  be a partially ordered set. If for any subset  $A$  of  $X$ ,  $A$  has a supremum and an infimum in  $X$ , then we say the  $X$  is order complete

Example

Let  $\Omega$  be a set  $(\mathcal{P}(\Omega), \subseteq)$  is order complete If  $\mathcal{F}$  is a subset of  $\mathcal{P}(\Omega)$ ,  $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$

Interesting tip:  $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

**AXIOM :**

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$  is order complete

In  $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any  $A \subseteq \mathbb{R} \cup -\infty, +\infty$  and  $c \in \mathbb{R}$  We denote by  $A + c$  the set  $\{a + c \mid a \in A\}$
- If  $\lambda \in \mathbb{R} \setminus \{0\}$ ,  $\lambda A$  denotes  $\{\lambda a \mid a \in A\}$
- $-A$  denotes  $(-1)A$

Prop.

For any  $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$ ,  $\sup(-A) = -\inf A$ ,  $\inf(-A) = -\sup A$  Def

We denote by  $(\mathbb{R}, \leq)$  a field  $\mathbb{R}$  equipped with a total order  $\leq$ , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$  such that  $a < b$ , one has  $\forall c \in \mathbb{R}, a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}, ab > 0$
- $\forall A \subseteq \mathbb{R}$ , if  $A$  has an upper bound in  $\mathbb{R}$ , then it has a supremum in  $\mathbb{R}$

Prop.

Let  $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

$\inf$  takes the same

Theorem:

Let  $I$  and  $J$  be non-empty sets

$$f : I \rightarrow [-\infty, +\infty], g : J \rightarrow [-\infty, +\infty]$$

$$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$$

If  $\{a, b\} \neq \{+\infty, -\infty\}$  then  $c = a + b$

$\inf$  takes the same if  $(-\infty) + (+\infty)$  doesn't happen

Corollary:

Let  $I$  be a non-empty set,  $f : I \rightarrow [-\infty, +\infty], g : J \rightarrow [-\infty, +\infty]$

$$\text{Then} \quad \sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq \left( \sup_{x \in I} f(x) \right) \left( \sup_{x \in I} g(x) \right)$$

$\inf$  takes the similar ( $\leq \rightarrow \geq$ ) (provided when the sum are defined)

## 2.4 Vector space

In this section:

$K$  denotes a unitary ring.

Let  $0$  be zero element of  $K$

$1$  be the unity of  $K$

### 2.4.1 $K$ -module

Def

Let  $(V, +)$  be a commutative group. We call left/right  $K$ -module structure: any mapping  $\Phi : K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group  $(V, +)$  equipped with a left/right  $K$ -module structure is called a left/right  $K$ -module.

Remark

Let  $K^{op}$  be the set  $K$  equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then  $K^{op}$  forms a unitary ring  
 Any left  $K^{op}$  - module is a right  $K$ -module  
 Any right  $K^{op}$  - module is a left  $K$ -module  
 $(K^{op})^{op} = K$

### Notation

When we talk about a left/right  $K$ -module  $(V, +)$ , we often write its left  $K$ -module structure as  $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

### K-vector space

If  $K$  is commutative, then  $K^{op} = K$ , so left  $K$ -module and right  $K$ -module structure are the same. We simply call them  $K$ -module structure. A commutative group equipped with a  $K$ -module structure is called a  $K$ -module. If  $K$  is a field, a  $K$ -module is also called a  $K$ -vector space

Let  $\Phi : K \times V \rightarrow V$  be a left or right  $K$ -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence  $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$   
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$  is a morphism of groups. Hence  $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$  (*is a var*)

### Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping  $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$  is a left  $K$ -module structure

The mapping  $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$  is a right  $K$ -module structure



**Remark:**

We can also write an element  $\mu$  of  $K^I$  is the form of a family  $(\mu_i)_{i \in I}$  of elements in  $K$  ( $\mu_i$  is the image of  $i \in I$  by  $\mu$ )  
Then

$$\begin{aligned} (\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I} \end{aligned}$$

**2.4.2 sub K-module****Def**

Let  $V$  be a left/right  $K$ -module. If  $W$  is a subgroup of  $V$ . Such that  $\forall a \in K, \forall x \in W \quad ax/xa \in W$ , then we say that  $W$  is left/right sub- $K$ -module of  $V$ .

**Example**

Let  $I$  be a set. Let  $K^{\oplus I}$  be the subset of  $K^I$  composed of mappings  $f : I \rightarrow K$  such that  $I_f = \{x \in I \mid f(x) \neq 0\}$  is finite. It is a left and right sub- $K$ -module of  $K^I$

In fact,  $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$   
Hence  $f - g \in K^{\oplus I}$ . So  $K^{\oplus I}$  is a subgroup of  $K^I$   
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

**2.4.3 morphism of K-modules****Def**

Let  $V$  and  $W$  be left  $K$ -module. A morphism of groups  $\phi : V \rightarrow W$  is called a morphism of left  $K$ -modules if  $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

**K-lines mapping**

If  $K$  is commutative, a morphism of  $K$ -modules is also called a  $K$ -lines mapping. We denote by  $\text{hom}_{K\text{-Mod}}(V, W)$  the set of all morphism of left- $K$ -module from  $V$  to  $W$ . This is a subgroup of  $W^V$

**Theorem**

Let  $V$  be a left  $K$ -module. Let  $I$  be a set.  
The mapping  $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I$  is a bijection where  $e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases} \quad \phi \mapsto (\phi(e_i))_{i \in I}$

**Remark:column**

In the case where  $I = 1, 2, 3, \dots, n$   $V^I$  is denoted as  $V^n$ ,  $K^I$  is denoted as  $K^n$   
 For any  $(x_1, \dots, x_n) \in V^n$ , by the theorem, there exists a unique morphism of left  
 K-modules  $\phi : K^n \rightarrow V$  such that  $\forall i \in 1, \dots, n \phi(e_i) = x_i$

We write this  $\phi$  as a column  $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$  It sends  $(a_1, \dots, a_n) \in K^n$  to  $a_1x_1 + \dots + a_nx_n$