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Part I

Set

Chapter 1

Ring

1.1 morphism

Def

Let A and B be unitary rings. We call morphism of unitary rings from A to B only mapping $A \rightarrow B$ is a morphism of group from $(A, +)$ to $(B, +)$, and a morphism of monoid from (A, \cdot) to (B, \cdot)

Properties

- Let R be a unitary ring. There is a unique morphism from \mathbb{Z} to R
-

algebra

we call k -algebra any pair (R, f) , when R is a unitary ring, and $f : k \rightarrow R$ is a morphism of unitary rings such that $\forall (b, x) \in k \times R, f(b)x = xf(b)$

Example: For any unitary ring R , the unique morphism of unitary rings $\mathbb{Z} \rightarrow R$ define a structure of \mathbb{Z} -algebra on R (extra: \mathbb{Z} is commutative despite R isn't guaranteed)

Notation: Let k be a commutative unitary ring, (A, f) be a k -algebra. If there is no ambiguity on f , for any $(\lambda, a) \in k \times A$, we denote $f(\lambda)a$ as λa

Formal power series

reminder: $n \in \mathbb{N}$ is possible infinite, so $\sum_{n \in \mathbb{N}}$ couldn't be executed directly.

Def:

(extended polynomial actually) Let k be a commutative unitary ring. Def: Let T be a formal symbol. We denote $k^{\mathbb{N}}$ as $k[T]$. If $(a_n)_{n \in \mathbb{N}}$ is an element of $k^{\mathbb{N}}$, when we denote $k^{\mathbb{N}}$ as $k[T]$ this element is denoted as $\sum_{n \in \mathbb{N}} a_n T^n$. Such

element is called a formal power series over k and a_n is called the Coefficient of T^n of this formal power series Notation:

- omit terms with coefficient 0
- write T as T
- omit Coefficient those are 1;
- omit T^0

Example $1T^0 + 2T^1 + 1T^2 + 0T^3 + \dots + 0T^n + \dots$ is written as $1 + 2T + T^2$

Def Remind that $k[T] = \{\sum_{n \in \mathbb{N}} a_n T^n \mid (a_n)_{n \in \mathbb{N}} \in k^{\mathbb{N}}\}$, define two composition laws on $k[T]$

$$\forall F(T) = a_0 + a_1 T + \dots \quad G(T) = b_0 + \dots$$

$$\text{let } F + G = (a_0 + b_0) + \dots$$

$$FG = \sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n$$

Properties:

- $(k[T], +, \cdot)$ form a commutative unitary ring.
- $k \rightarrow k[T] \quad \lambda \mapsto \lambda T$ is a morphism
- $(FG)H = \left(\sum_{n \in \mathbb{N}} \sum_{i+j=n} (a_i b_j) T^n \right) \left(\sum_{n \in \mathbb{N}} c_n T^n \right) = \sum_{n \in \mathbb{N}} \left(\sum_{p,q,l=n} a_p b_q c_l \right) T^n$
is a trick applied on integral

Derivative:

$$\text{let } F(T) \in k[T]$$

We denote by $F'(T)$ or $\mathcal{D}(F(T))$ the formal power series

$$\mathcal{D}(F) = \sum_{n \in \mathbb{N}} (n+1) a_{n+1} T^n$$

Properties:

- $\mathcal{D}(k[T], +) \rightarrow (k[T], +)$ is a morphism of groups
- $\mathcal{D}(FG) = F'G + FG'$

exp

We denote $\exp(T) \in k[T]$ as $\sum_{n \in \mathbb{N}} \frac{1}{n!} T^n$, which fulfil the differential equation

$$\mathcal{D}(\exp(T)) = \exp(T) \text{ (interesting)}$$

Cauchy sequence: $(F_i(T))_{i \in \mathbb{N}}$ be a sequence of elements in $k[T]$, and $F(T) \in k[T]$ We say that $(F_i(T))_{i \in \mathbb{N}}$ is a Cauchy sequence if $\forall l \in \mathbb{N}$, there exists $N(l) \in \mathbb{N}$ such that $\forall (i, j) \in \mathbb{N}_{\geq N(l)}^2$, $\text{ord}(F_i(T) - F_j(T)) \geq l$

Part II

Sequences

Chapter 2

Supremum and infimum

Def:

Let (X, \leq) be a partially ordered set A and Y be subsets of X , such that $A \subseteq Y$

- If the set $\{y \in Y \mid \forall a \in A, a \leq y\}$ has a least element then we say that A has a Supremum in Y with respect to \leq denoted by $\sup_{(Y, \leq)} A$ this least element and called it the Supremum of A in Y (this respect to \leq)
- If the set $\{y \in Y \mid \forall a \in A, y \leq a\}$ has a greatest element, we say that A has an infimum in Y with respect to \leq . We denote by $\inf_{(Y, \leq)} A$ this greatest element and call it the infimum of A in Y
- Observation: $\inf_{(Y, \leq)} A = \sup_{(Y, \geq)} A$

Notation:

Let (X, \leq) be a partially ordered set, I be a set.

- If f is a function from I to X $\sup f$ denotes the supremum of $f(I)$ is X . $\inf f$ takes the same
- If $(x_i)_{i \in I}$ is a family of element in X , then $\sup x_i$ denotes $\sup\{x_i \mid i \in I\}$ (in X)

If moreover $\mathbb{P}(\cdot)$ denotes a statement depending on a parameter in I then $\sup_{i \in I, \mathbb{P}(i)} x_i$ denotes $\sup\{x_i \mid i \in I, \mathbb{P}(i) \text{ holds}\}$

Example:

Let $A = \{x \in \mathbb{R} \mid 0 \leq x < 1\} \subseteq \mathbb{R}$ We equip \mathbb{R} with the usual order relation.

$$\{y \in \mathbb{R} \mid \forall x \in A, x \leq y\} = \{y \in \mathbb{R} \mid y \geq 1\}$$

So $\sup A = 1$

$$\{y \in \mathbb{R} \mid \forall x \in A, y \leq x\} = \{y \in \mathbb{R} \mid y \geq 0\}$$

Hence $\inf A = 0$

Example: For $n \in \mathbb{N}$, let $x_n = (-1)^n \in R$

$$\sup_{n \in \mathbb{N}} \inf_{k \in \mathbb{N}, k \geq n} x_k = -1$$

Proposition:

Let (X, \leq) be a partially ordered set, A, Y, Z be subset of X , such that $A \subseteq Z \subseteq Y$

- If $\max A$ exists, then it is also equal to $\sup_{(y, \leq)} A$
- If $\sup_{(y, \leq)} A$ exists and belongs to Z , then it is equal to $\sup A$

\inf takes the same Prop.

Let X, \leq be a partially ordered set, A, B, Y be subsets of X such that $A \subseteq B \subseteq Y$

- If $\sup_{(y, \leq)} A$ and $\sup_{(y, \leq)} B$ exists, then $\sup_{(y, \leq)} A \leq \sup_{(y, \leq)} B$
- If $\inf_{(y, \leq)} A$ and $\inf_{(y, \leq)} B$ exists, then $\inf_{(y, \leq)} A \geq \inf_{(y, \leq)} B$

Prop.

Let (X, \leq) be a partially ordered set, I be a set and $f, g : I \rightarrow X$ be mappings such that $\forall t \in I, f(t) \leq g(t)$

- If $\inf f$ and $\inf g$ exists, then $\inf f \leq \inf g$
- If $\sup f$ and $\sup g$ exists, then $\sup f \leq \sup g$

Chapter 3

Interval

We fix a totally ordered set (X, \leq)

Notation:

If $(a, b) \in X \times X$ such that $a \leq b$, $[a, b]$ denotes $\{x \in X \mid a \leq x \leq b\}$

Def:

Let $I \subseteq X$. If $\forall (x, y) \in I \times I$ with $x \leq y$, one has $[x, y] \subseteq I$ then we say that I is an interval in X

Example:

Let $(a, b) \in X \times X$, such that $a \leq b$. Then the following sets are intervals

- $]a, b[:= \{x \in X \mid a, x, b\}$
- $[a, b[:= \{x \in X \mid a, x, b\}$
- $]a, b] := \{x \in X \mid a, x, b\}$

Prop.

Let Λ be a non-empty set and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of intervals in X .

- $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an interval in X
- If $\bigcap_{\lambda \in \Lambda} I_\lambda \neq \emptyset$, $\bigcup_{\lambda \in \Lambda} I_\lambda$ is an interval in X

We check that $[a, b] \subseteq I_\lambda \cup I_\mu$

- If $b \leq x$ $[a, b] \subseteq [a, x] \subseteq I_\lambda$ because $\{a, x\} \subseteq I_\lambda$
- If $x \leq a$ $[a, b] \subseteq [x, b] \subseteq I_\mu$ because $\{b, x\} \subseteq I_\mu$
- If $a < x < b$ then $[a, b] = [a, x] \cup [x, b] \subseteq I_\lambda \cup I_\mu$

Def:

Let (X, \leq) be a totally ordered set. I be a non-empty interval of X . If $\sup I$ exists in X , we call $\sup I$ the right endpoint; \inf takes the similar way.

Prop.

Let I be an interval in X .

- Suppose that $b = \sup I$ exists. $\forall x \in I, [x, b] \subseteq I$
- Suppose that $a = \inf I$ exists. $\forall x \in I,]a, x] \subseteq I$

Prop.

Let I be an interval in X . Suppose that I has supremum b and an infimum a in X . Then I is equal to one of the following sets $[a, b]$ $[a, b[$ $]a, b]$ $]a, b[$

Def

let (X, \leq) be a totally ordered set. If $\forall (x, z) \in X \times X$, such that $x < z$ $\exists y \in X$ such that $x < y < z$, then we say that (X, \leq) is thick

Prop.

Let (X, \leq) be a thick totally ordered set. $(a, b) \in X \times X, a < b$ If I is one of the following intervals $[a, b]$; $[a, b[$; $]a, b]$; $]a, b[$ Then $\inf I = a$ $\sup I = b$ (for it's thick empty set is impossible)

Proof:

Since X is thick, there exists $x_0 \in]a, b[$ By definition, b is an upper bound of I . If b is not the supremum of I , there exists an upper bound M of I such that $M \neq b$. Since X is thick, there is $M' \in X$ such that $x_0 \leq M, M' < b$ Since $[x, b] \subseteq I, b \in I$ Hence M and M' belong to I , which conflicts with the uniqueness of supremum.

Chapter 4

Enhanced real line

Def:

Let $+\infty$ and $-\infty$ be two symbols that are different and don't belong to \mathbb{R} . We extend the usual total order \leq on \mathbb{R} to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$\forall x \in \mathbb{R}, -\infty < x < +\infty$$

Thus $\mathbb{R} \cup \{-\infty, +\infty\}$ becomes a totally ordered set, and $\mathbb{R} =]-\infty, +\infty[$. Obviously, this is a thick totally ordered set.

We define:

- $\forall x \in]-\infty, +\infty[\quad x + (+\infty) := +\infty \quad (+\infty) + x := +\infty$
- $\forall x \in [-\infty, +\infty[\quad x + (-\infty) := -\infty \quad (-\infty) + x := -\infty$
- $\forall x \in]0, +\infty[\quad x(+\infty) = (+\infty)x = +\infty \quad x(-\infty) = (-\infty)x = -\infty$
- $\forall x \in [-\infty, 0[\quad x(+\infty) = (+\infty)x = -\infty \quad x(-\infty) = (-\infty)x = +\infty$
- $-(+\infty) = -\infty \quad -(-\infty) = +\infty \quad (\infty)^{-1} = 0$
- $(+\infty) + (-\infty) \quad (-\infty) + (+\infty) \quad (+\infty)0 \quad 0(+\infty) \quad (-\infty)0 \quad 0(-\infty)$
ARE NOT DEFINED

Def

Let (X, \leq) be a partially ordered set. If for any subset A of X , A has a supremum and an infimum in X , then we say that X is order complete.

Example

Let Ω be a set. $(\mathcal{P}(\Omega), \subseteq)$ is order complete. If \mathcal{F} is a subset of $\mathcal{P}(\Omega)$, $\sup \mathcal{F} = \bigcup_{A \in \mathcal{F}} A$.

Interesting tip: $\inf \emptyset = \Omega \quad \sup \emptyset = \emptyset$

Axiom:

$(\mathbb{R} \cup \{-\infty, +\infty\}, \leq)$ is order complete.

In $\mathbb{R} \cup \{-\infty, +\infty\} \quad \sup \emptyset = -\infty \quad \inf \emptyset = +\infty$

Notation:

- For any $A \subseteq \mathbb{R} \cup -\infty, +\infty$ and $c \in \mathbb{R}$ We denote by $A + c$ the set $\{a + c \mid a \in A\}$
- If $\lambda \in \mathbb{R} \setminus \{0\}$, λA denotes $\{\lambda a \mid a \in A\}$
- $-A$ denotes $(-1)A$

Prop.

For any $A \subseteq \mathbb{R} \cup \{-\infty, +\infty\}$, $\sup(-A) = -\inf A$, $\inf(-A) = -\sup A$ Def

We denote by (\mathbb{R}, \leq) a field \mathbb{R} equipped with a total order \leq , which satisfies the following condition:

- $\forall (a, b) \in \mathbb{R} \times \mathbb{R}$ such that $a < b$, one has $\forall c \in \mathbb{R}$, $a + c < b + c$
- $\forall (a, b) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, $ab > 0$
- $\forall A \subseteq \mathbb{R}$, if A has an upper bound in \mathbb{R} , then it has a supremum in \mathbb{R}

Prop.

Let $A \subseteq [-\infty, +\infty]$

- $\forall c \in \mathbb{R} \quad \sup(A + c) = (\sup A) + c$
- $\forall \lambda \in \mathbb{R}_{\geq 0} \quad \sup(\lambda A) = \lambda \sup(A)$
- $\forall \lambda \in \mathbb{R}_{\leq 0} \quad \sup(\lambda A) = \lambda \inf(A)$

\inf takes the same

Theorem:

Let I and J be non-empty sets

$f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

$a = \sup_{x \in I} f(x) \quad b = \sup_{y \in J} g(y) \quad c = \sup_{(x,y) \in I \times J, \{f(x), g(y)\} \neq \{+\infty, -\infty\}} (f(x) + g(y))$

If $\{a, b\} \neq \{+\infty, -\infty\}$ then $c = a + b$

\inf takes the same if $(-\infty) + (+\infty)$ doesn't happen

Corollary:

Let I be a non-empty set, $f : I \rightarrow [-\infty, +\infty]$, $g : J \rightarrow [-\infty, +\infty]$

Then $\sup_{x \in I, \{f(x), g(x)\} \neq \{+\infty, -\infty\}} (f(x) + g(x)) \leq (\sup_{x \in I} f(x))(\sup_{x \in I} g(x))$

\inf takes the similar ($\leq \rightarrow \geq$) (provided when the sum are defined)

Chapter 5

Vector space

In this section:

K denotes a unitary ring.

Let 0 be zero element of K

1 be the unity of K

5.1 K -module

5.1.1 Def

Let $(V, +)$ be a commutative group. We call left/right K -module structure: any mapping $\Phi: K \times V \rightarrow V$

- $\forall (a, b) \in K \times K, \forall x \in V \quad \Phi(ab, x) = \Phi(a, \Phi(b, x)) / \Phi(b, \Phi(a, x))$
- $\forall (a, b) \in K \times K, \forall x \in V, \Phi(a + b, x) = \Phi(a, x) + \Phi(b, x)$
- $\forall a \in K, \forall (x, y) \in V \times V, \Phi(a, x + y) = \Phi(a, x) + \Phi(a, y)$
- $\forall x \in V, \Phi(1, x) = x$

A commutative group $(V, +)$ equipped with a left/right K -module structure is called a left/right K -module.

5.1.2 Remark

Let K^{op} be the set K equipped with the following composition laws:

- $K \times K \rightarrow K$
- $(a, b) \mapsto a + b$
- $K \times K \rightarrow K$
- $(a, b) \mapsto ba$

Then K^{op} forms a unitary ring
 Any left K^{op} - module is a right K -module
 Any right K^{op} - module is a left K -module
 $(K^{op})^{op} = K$

5.1.3 Notation

When we talk about a left/right K -module $(V, +)$, we often write its left K -module structure as $K \times V \rightarrow V \quad (a, x) \mapsto ax$

The axioms become:

$$\begin{aligned} \forall (a, b) \in K \times K, \forall x \in V \quad (ab)x &= a(bx)/b(ax) \\ \forall (a, b) \in K \times K, \forall x \in V \quad (a + b)x &= ax + bx \\ \forall a \in K, \forall (x, y) \in V \times V \quad a(x + y) &= ax + ay \\ \forall x \in V \quad 1x &= x \end{aligned}$$

5.1.4 K -vector space

If K is commutative, then $K^{op} = K$, so left K -module and right K -module structure are the same. We simply call them K -module structure. A commutative group equipped with a K -module structure is called a K -module. If K is a field, a K -module is also called a K -vector space

Let $\Phi : K \times V \rightarrow V$ be a left or right K -module structure

$$\forall x \in V, \Phi(\cdot, x) : K \rightarrow V \quad (a \in K) \mapsto \Phi(a, x)$$

is a morphism of addition groups. Hence $\Phi(0, x) = 0, \Phi(-a, x) = -\Phi(a, x)$
 $\forall a \in K, \Phi(a, \cdot) : V \rightarrow V$ is a morphism of groups. Hence $\Phi(a, 0) = 0, \Phi(a, -x) = -\Phi(a, x)$ (*is a var*)

5.1.5 Association:

$$\forall x \in K$$

$$\begin{aligned} (f(f + g) + h)(x) &= (f + g)(x) + h(x) = f(x) + g(x) + h(x) \\ (f + (g + h))(x) &= f(x) + ((g + h)(x)) = f(x) + g(x) + h(x) \end{aligned}$$

$$\text{Let } 0 : I \rightarrow K : x \mapsto 0 \quad \forall f \in K^I \quad f + 0 = f$$

$$\text{Let } -f : f + (-f) = 0$$

The mapping $K \times K^I \rightarrow K^I : (a, f) \mapsto af \quad (af)(x) = af(x)$ is a left K -module structure

The mapping $K \times K^I \rightarrow K^I : (a \in I) \mapsto ((x \in I) \mapsto f(x)a) \quad (af)(x) = af(x)$ is a right K -module structure

5.1.6 Remark:

We can also write an element μ of K^I is the form of a family $(\mu_i)_{i \in I}$ of elements in K (μ_i is the image of $i \in I$ by μ)
Then

$$\begin{aligned}(\mu_i)_{i \in I} + (\nu_i)_{i \in I} &:= (\mu_i + \nu_i)_{i \in I} \\ a(\mu_i)_{i \in I} &:= (a\mu_i)_{i \in I} \\ (\mu_i)_{i \in I} a &= (\mu_i a)_{i \in I}\end{aligned}$$

5.2 sub K-module**5.2.1 Def**

Let V be a left/right K -module. If W is a subgroup of V . Such that $\forall a \in K, \forall x \in W \quad ax/xa \in W$, then we say that W is left/right sub- K -module of V .

5.2.2 Example

Let I be a set. Let $K^{\oplus I}$ be the subset of K^I composed of mappings $f : I \rightarrow K$ such that $I_f = \{x \in I \mid f(x) \neq 0\}$ is finite. It is a left and right sub- K -module of K^I

In fact, $\forall (f, g) \in K^{\oplus I} \times K^{\oplus I} \quad I_{f-g} = \{x \in I \mid f(x) - g(x) \neq 0\} \subseteq I_f \cup I_g$
Hence $f - g \in K^{\oplus I}$ So $K^{\oplus I}$ is a subgroup of K^I
 $\forall a \in K, \forall f \in K^{\oplus I} \quad I_{af} \subseteq I_f, I_{(x \mapsto f(x)a)} \subseteq I_f$

5.3 morphism of K-modules**5.3.1 Def**

Let V and W be left K -module, A morphism of groups $\phi : V \rightarrow W$ is called a morphism of left K -modules if $\forall (a, x) \in K \times V, \phi(ax) = a\phi(x)$

5.3.2 K-linear mapping

If K is commutative, a morphism of K -modules is also called a K -linear mapping. We denote by $\text{hom}_{K\text{-Mod}}(V, W)$ the set of all morphism of left- K -module from V to W . This is a subgroup of W^V

5.3.3 Theorem

Let V be a left K -module. Let I be a set.
The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \rightarrow (\phi(e_i))_{i \in I}$ is a bijection where
$$e_i : I \rightarrow K : j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.3.4 Remark:column

In the case where $I = 1, 2, 3, \dots, n$ V^I is denoted as V^n , K^I is denoted as K^n . For any $(x_1, \dots, x_n) \in V^n$, by the theorem, there exists a unique morphism of left K -modules $\phi : K^n \rightarrow V$ such that $\forall i \in 1, \dots, n, \phi(e_i) = x_i$.

We write this ϕ as a column $\begin{pmatrix} x_1 \\ \dots \\ x_n \end{pmatrix}$. It sends $(a_1, \dots, a_n) \in K^n$ to $a_1x_1 + \dots + a_nx_n$.

5.4 kernel

5.4.1 Prop

Let G and H be groups and $f : G \rightarrow H$ be a morphism of groups

- $Im(f) \subseteq H$ is a subgroup of H
- $\ker(f) = \{x \in G \mid f(x) = e_H\}$
- f is injection iff $\ker(f) = \{e_G\}$

5.4.2 Def

$\ker(f)$ is called the kernel of f

5.4.3 Theorem

f is injection iff $\ker(f) = \{e_G\}$

5.4.4 Proof

Let e_G and e_H be neutral element of G and H respectively

- (1) Let x and y be element of G
 $f(x)f(y)^{-1} = f(x)f(y)^{-1} = f(xy^{-1}) \in Im(f)$. So $Im(f)$ is a subgroup of H
- (2) Let x and y be element of $\ker(f)$. One has $f(xy^{-1}) = f(x)f(y)^{-1} = e_H e_H^{-1} = e_H$. So $xy^{-1} \in \ker(f)$. So $\ker(f)$ is a subgroup of G .
- (3) Suppose that f is injection.
 Since $f(e_G) = e_H$ one has $\ker(f) = f^{-1}(\{e_H\}) = \{e_G\}$. Suppose that $\ker(f) = \{e_G\}$. If $f(x) = f(y)$ then $f(xy^{-1}) = f(x)f(y)^{-1} = e_H$.
 Hence $xy^{-1} = e_G \Rightarrow x = y$

5.4.5 Def

Let $(V, +)$ be a commutative group, I be a set. We define a composition law $+$ on V^I as follows

$$(x_i)_{i \in I} + (y_i)_{i \in I} := (x_i + y_i)_{i \in I}$$

Then V^I forms a commutative group

5.4.6 Remark

Let E and F be left K -modules

$\text{hom}_{K\text{-Mod}}(E, F) := \{\text{morphisms of left } K\text{-modules from } E \text{ to } F\} \subseteq F^E$ is a subgroup of F^E

In fact f and g are elements of $\text{hom}_{K\text{-Mod}}(E, F)$, then $f - g$ is also a morphism of left K -module

$$(f - g)(x + y) = f(x + y) - g(x + y) = (f(x) + f(y)) - (g(x) + g(y)) = (f(x) - g(x)) + (f(y) - g(y)) = (f - g)(x) + (f - g)(y)$$

5.4.7 Theorem

Let V be a left K -module, I be a set. The mapping $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \rightarrow V^I : \phi \mapsto (\phi(e_i))_{i \in I}$ is an isomorphism of groups, where $e_i : I \rightarrow K : j \mapsto$

$$\begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

5.4.8 Proof:

One has $(\phi + \psi)(e_i) = \phi(e_i) + \psi(e_i)$

$$\forall (\phi, \psi) \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)^2$$

$$\text{Hence } \Psi(\phi, \psi) = (\phi(e_i) + \psi(e_i))_{i \in I} = \Psi(\phi) + \Psi(\psi)$$

So Ψ is a morphism of groups

injectivity Let $\phi \in \text{hom}_{K\text{-Mod}}(K^{\oplus I}, V)$ Such that $\forall i \in I (\forall \phi \in \ker(\Psi)) \quad \phi(e_i) = 0$

$$\text{Let } a = (a_i)_{i \in I} \in K^{\oplus I} \text{ One has } a = \sum_{i \in I} a_i e_i$$

$$\text{If fact, } \forall j \in I, a_j = \sum_{i \in I, a_i \neq 0} a_i e_i(j)$$

$$\text{Thus } \phi(a) = \sum_{i \in I, a_i \neq 0} a - I\phi(e_i) = 0$$

Hence ϕ is the neutral element.

surjectivity Let $x = (x_i)_{i \in I} \in V^I$ We define $\phi_x : K^{\oplus I} \rightarrow V$ such that $\forall a = (a_i)_{i \in I} \in K^{\oplus I}, \phi_x(a) = \sum_{i \in I, a_i \neq 0} a_i x_i$

This is a morphism of left K -modules

$$\text{for all } i \in I, \phi_x(e_i) = 1x_i = x_i \text{ So } \Psi(\phi_x) = x$$

Suppose that K' is a unitary ring, and V is also equipped with a right K' -module structure, Then $\text{hom}_{K\text{-Mod}}(K^{\oplus I}, V) \subseteq V^{K^{\oplus I}}$ is a right sub- k' -module, and Ψ in the theorem is a right K' -module isomorphism

Chapter 6

Monotone mappings

6.1 Def

Let I and X be partially ordered sets, $f : I \rightarrow X$ be a mapping.

- If $\forall (a, b) \in I \times I$ such that $a < b$. One has $f(a) \leq f(b)$, then we say that f is increasing. decreasing takes similar way.
- If f is (strictly) increasing or decreasing, we say that f is (strictly) monotone.

6.2 Prop.

Let X, Y, Z be partially ordered sets. $f : X \rightarrow Y, g : Y \rightarrow Z$ be mappings

- If f and g have the same monotonicity, then $g \circ f$ is increasing
- If f and g have different monotonicities, then $g \circ f$ is decreasing

strict monotonicities takes the same

6.3 Def

Let f be a function from a partially ordered set I to another partially ordered set X . If $f|_{\text{Dom}(f)} : \text{Dom}(f) \rightarrow X$ is (strictly) increasing/decreasing then we say that f is (strictly) increasing/decreasing

6.4 Prop.

Let I and X be partially ordered sets. f be function from I to X .

- If f is increasing/decreasing and f is injection, then f is strictly increasing/decreasing
- Assume that I is totally ordered and f is strictly monotone, then f is injection

6.5 Prop

Let A be totally ordered set, B be a partially ordered set, f be an injective function from A to B

If f is increasing/decreasing, then so is f^{-1}

6.6 Def

Let X and Y be partially ordered sets. $f : X \rightarrow Y$ be a bijection. If both f and f^{-1} are increasing, then we say that f is an isomorphism of partially ordered sets.

(If X is totally, then a mapping $f : X \rightarrow Y$ is an isomorphism of partially ordered sets iff f is a bijection and f is increasing)

6.7 Prop.

Let I be a subset of \mathbb{N} which is infinite. Then there is a unique increasing bijection $\lambda_I : \mathbb{N} \rightarrow I$

6.8 Proof

6.8.1 bijection

We construct $f : \mathbb{N} \rightarrow I$ by induction as follows.

Let $f(0) = \min I$ Suppose that $f(0), \dots, f(n)$ are constructed

then we take $f(n+1) := \min(I \setminus \{f(0), \dots, f(n)\})$

Since $I \setminus \{f(0), \dots, f(n-1)\} \supseteq I \setminus \{f(0), \dots, f(n)\}$. Therefore $f(n) \leq f(n+1)$

Since $f(n+1) \notin \{f(0), \dots, f(n)\}$, we have $f(n) < f(n+1)$

Hence f is strictly increasing and this is injective

If f is not surjective, then $I \setminus \text{Im}(f)$ has a element N .

Let $m = \min\{n \in \mathbb{N} \mid N \leq f(n)\}$.

Since $N \notin \text{Im}(f)$, $N < f(m)$.

So $m \neq 0$. Hence $f(m-1) < N < f(m) = \min(I \setminus \{f(0), \dots, f(m-1)\})$

By definition, $N \in I \setminus \text{Im}(f) \subseteq I \setminus \{f(0), \dots, f(m-1)\}$,

Hence $f(m) \leq N$, causing contradiction.

6.8.2 uniqueness

exercise: Prove that $Id_{\mathbb{N}}$ is the only isomorphism of partially ordered sets from \mathbb{N} to \mathbb{N}

Chapter 7

sequence and series

Let $I \subseteq \mathbb{N}$ be a infinite subset

7.1 Def

Let X be a set. We call sequence in X parametrized by I a mapping from I to X .

7.2 Remark

If K is a unitary ring and E is a left K -module then the set of sequence E^I admits a left- K -module structure. If $x = (x_n)_{n \in I}$ is a sequence in E , we define a sequence $\sum(x) := (\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$, called the series associated with the sequence x .

7.3 Prop

$\sum : E^I \rightarrow E^{\mathbb{N}}$ is a morphism of left- K -module

7.4 proof

Let $x = (x_i)_{i \in I}$ and $y = (y_i)_{i \in I}$ be elements of E^I

$$\sum_{i \in I, i \leq n} (x_i + y_i) = (\sum_{i \in I, i \leq n} x_i) + (\sum_{i \in I, i \leq n} y_i), \lambda \sum_{i \in I, i \leq n} x_i = \sum_{i \in I, i \leq n} \lambda x_i$$

7.5 Prop

Let I be a totally ordered set . X be a partially ordered set, $f : I \rightarrow X$ be a mapping , $J \in I$ Assume that J does not have any upper bound in I

- If f is increasing ,then $f(I)$ and $f(J)$ have the same upper bounds in X
- If f is decreasing ,then $f(I)$ and $f(J)$ have the same lower bounds in X

7.6 limit

7.6.1 Def

Let $i \subseteq \mathbb{N}$ be a infinite subset. $\forall (x_i)_{n \in I} \in [-\infty, +\infty]^I$ where $[-\infty, +\infty]$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, we define:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n := \inf_{n \in I} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n := \sup_{n \in I} \left(\inf_{i \in I, i \geq n} x_i \right)$$

If $\limsup_{n \in I, n \rightarrow +\infty} x_n = \liminf_{n \in I, n \rightarrow +\infty} x_n = l$, we then say that $(x_n)_{n \in I}$ tends to l and that l is the limit of $(x_n)_{n \in I}$. If in addition $(x_n)_{n \in I} \in \mathbb{R}^I$ and $l \in \mathbb{R}$, we say that $(x_n)_{n \in I}$ converges to l

7.6.2 Remark

If $J \subseteq \mathbb{N}$ is an infinite subset, then:

$$\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \left(\sup_{i \in I, i \geq n} x_i \right)$$

$$\liminf_{n \in I, n \rightarrow +\infty} x_n = \sup_{n \in J} \left(\inf_{i \in I, i \geq n} x_i \right)$$

Therefore ,if we change the values of finitely many terms in $(x_i)_{i \in I}$ the limit superior and the limit inferior do not change.

In fact, if we take $J = \mathbb{N} \setminus \{0, \dots, m\}$, then $\inf_{n \in J}(\dots)$ and $\sup_{n \in J}(\dots)$ only depends on the values of $x_i, i \in I, i \geq m$

7.6.3 Prop

$$\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I, \quad \liminf_{n \in I, n \rightarrow +\infty} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

7.6.4 Prop

Let $(x_n)_{n \in I} \in [-\infty, +\infty]^I$

$$\begin{aligned} \forall c \in \mathbb{R} \quad & \limsup_{n \in I, n \rightarrow +\infty} (x_n + c) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + c \\ & \liminf_{n \in I, n \rightarrow +\infty} (x_n + c) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + c \\ \forall c \in \mathbb{R}_{>0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ \forall c \in \mathbb{R}_{<0} \quad & \limsup_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \liminf_{n \in I, n \rightarrow +\infty} x_n \\ & \liminf_{n \in I, n \rightarrow +\infty} (\lambda x_n) = \lambda \limsup_{n \in I, n \rightarrow +\infty} x_n \end{aligned}$$

7.6.5 Prop

Let $(x_n)_{n \in I}$ be elements in $[-\infty, +\infty]^I$. Suppose that there exists $N_0 \in \mathbb{N}$ such that $\forall n \in I, n \geq N_0$, one has $x_n \leq y_n$. Then

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

,

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

7.6.6 Theorem

Let $(x_n)_{n \in I}, (y_n)_{n \in I}, (z_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Suppose that

- $\exists N - N \in \mathbb{N}, \forall n \in I, n \geq N_0$ one has $x_n \leq y_n \leq z_n$
- $(x_n)_{n \in I}$ and $(z_n)_{n \in I}$ tend to the same limit l

Then $(y_n)_{n \in I}$ tends to l

7.6.7 Def

Let I be an infinite subset of \mathbb{N} , and $(x_n)_{n \in I}$ be a sequence in some set X . We call subsequence of $(x_n)_{n \in I}$ a sequence of the form $(x_n)_{n \in J}$, where J is an infinite subset of I

7.6.8 Prop

Let I and J be infinite subset of \mathbb{N} such that $J \subseteq I$. $\forall (x_n)_{n \in I} \in [-\infty, +\infty]^I$, one has

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n) \leq \liminf_{n \in J, n \rightarrow +\infty} x_n$$

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n) \geq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

In particular, if $(x_n)_{n \in I}$ tends to $l \in [-\infty, +\infty]$, then $(x_n)_{n \in J}$ tends to l

7.6.9 Prop

$\forall n \in \mathbb{N}$, one has

$$\liminf_{n \in J, n \rightarrow +\infty} (x_n) \geq \liminf_{n \in I, n \rightarrow +\infty} y_n$$

$$\limsup_{n \in J, n \rightarrow +\infty} (x_n) \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

7.6.10 Theorem

Let $I \subseteq \mathbb{N}$ be an infinite subset and $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$

- If the mapping $(n \in I) \mapsto x_n$ is increasing, then $(x_n)_{n \in I}$ tends to $\sup_{n \in I} x_n$
- If the mapping $(n \in I) \mapsto x_n$ is decreasing, then $(x_n)_{n \in I}$ tends to $\inf_{n \in I} x_n$

7.6.11 Notation

If a sequence $(x_n)_{n \in I} \in [-\infty, +\infty]$ tends to some $l \in [-\infty, +\infty]$ the expression $\lim_{n \in I, n \rightarrow} x_n$ denotes this limit l

7.6.12 Corollary

Let $(x_n)_{n \in I}$ be a sequence in $\mathbb{N}_{\geq 0}$. Then the series $\sum_{n \in I} x_n$ (the sequence $(\sum_{i \in I, i \leq n} x_i)_{n \in \mathbb{N}}$) tends to an element in $\mathbb{N}_{\geq 0} \cup \{+\infty\}$. It converges in \mathbb{R} iff it is bounded from above (namely has an upper bound in \mathbb{R})

7.6.13 Notation

If a series $\sum_{n \in I} x_n$ in $[-\infty, +\infty]$ tends to some limit, we use the expression $\sum_{n \in I} x_n$ to denote the limit

7.6.14 Theorem: Bolzano-Weierstrass

Let $(x_n)_{n \in I}$ be a sequence in $[-\infty, +\infty]$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\limsup_{n \in I, n \rightarrow +\infty} x_n$. There exists a subsequence of $(x_n)_{n \in I}$ that tends to $\liminf_{n \in I, n \rightarrow +\infty} x_n$.

7.6.15 Proof

Let $J = \{n \in I \mid \forall m \in I, \text{ if } m \leq n \text{ then } x_m \leq x_n\}$

If J is infinite, the sequence $(x_n)_{n \in J}$ is decreasing so it tends to $\inf_{n \in J} x_n$

$\forall n \in J$ by definition $x_n = \sup_{i \in I, i \geq n} x_i$ so $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{n \in J} \sup_{i \in I, i \geq n} x_i =$

$$\inf_{n \in J} x_n = \lim_{n \in J, n \rightarrow +\infty} x_n$$

Assume that J is finite. Let $n_0 \in I$ such that $\forall n \in J, n < n_0$. Denote by

$$l = \sup_{n \in I, n \geq n_0}$$

Let $N \in \mathbb{N}$ such that $N \geq n_0$. By definition $\sup_{i \in I, i \geq n_0} x_i \leq l$. If the strict

inequality $\sup_{i \in I, i \geq N} x_i < l$ holds, then $\sup_{i \in I, i \geq N} x_i$ is NOT an upper bound of $\{x_n \mid n \in I, n_0 \leq n < N\}$

So there exists $n \in I$ such that $n_0 \leq n < N$ such that $x_n > \sup_{i \in I, i \geq N} x_i$. We

may also assume that n is largest among elements of $I \cap [n_0, N[$ that satisfies this inequality. Then $\forall m \in I$ if $m \geq n$ then $x_m \leq x_n$. Thus $n \in J$ that contradicts the maximality of n_0 . Therefore $l = \sup_{i \in I, i \geq N} x_i$, which leads to $\limsup_{n \in I, n \rightarrow +\infty} x_n = l$

Moreover, if $m \in I, m \geq n_0$ then $m \notin J$, so $x_m < l$ (since otherwise $x_m = \sup_{i \in I, i \geq m} x_i$ and hence $m \in J$). Hence, \forall finite subset I' of $\{m \in I \mid m \geq n_0\}$

$\max_{i \in I'} x_i < l$ and hence $\exists n \in I$, such that $n > \max I'$, and $\max_{i \in I'} x_i < x_n$

We construct by induction an increasing sequence $(n_j)_{j \in \mathbb{N}}$ in I

Let n_0 be as above. Let $f : \mathbb{N} \rightarrow I_{\geq n_0}$ be a surjective mapping.

If n_j is chosen, we choose $n_{j+1} \in I$ such that $n_{j+1} > n_j, x_{n_{j+1}} > \max\{x_{f(j)}, x_{n_j}\}$

Hence the sequence $(x_{n_j})_{j \in \mathbb{N}}$ is increasing, and $\sup_{j \in \mathbb{N}} x_{n_j} \leq \sup_{j \in \mathbb{N}} x_{f(j)} = \sup_{n \in I, n \geq n_0} x_n =$

l

$$l = \sup_{n \in I, n \geq n_0}$$

So $(x_{n_j})_{j \in \mathbb{N}}$ tends to l

Chapter 8

Cauchy sequence

8.1 Def

Let $(x_n)_{n \in I}$ be a sequence in \mathbb{R}
If $\inf_{N \in \mathbb{N}} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = \lim_{N \rightarrow +\infty} \sup_{(n,m) \in I \times I, n,m \geq N} |x_n - x_m| = 0$ then
we say that $(x_n)_{n \in I}$ is a Cauchy sequence

8.2 Prop

- If $(x_n)_{n \in I} \in \mathbb{R}^I$ converges to some $l \in \mathbb{R}$, then it is a Cauchy sequence
- If $(x_n)_{n \in I}$ is a Cauchy sequence, there exists $M > 0$ such that $\forall n \in I \quad |x_n| \leq M$
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite, $(x_n)_{n \in J}$ is a Cauchy sequence.
- If $(x_n)_{n \in I}$ is a Cauchy sequence, then $\forall J \subseteq I$ infinite and $l \in \mathbb{R}$ such that $(x_n)_{n \in I}$ converges to l , then $(x_n)_{n \in J}$ converges to l too.

8.3 Theorem: Completeness of real number

If $(x_n)_{n \in I} \in \mathbb{R}^I$ is a Cauchy sequence, then it converges in \mathbb{R}

8.3.1 Proof

Since $(x_n)_{n \in I}$ is a Cauchy sequence, $\exists M \in \mathbb{R}_{>0}$ such that $-M \leq x_n \leq M \quad \forall x \in I$. So $\limsup_{n \in I, n \rightarrow +\infty} x_n \in \mathbb{R}$. By Bolzano-Weierstrass theorem. $\exists J \subseteq I$ infinite such that $(x_n)_{n \in J}$ converges to $\limsup_{n \in J, n \rightarrow +\infty} x_n \in \mathbb{R}$. Therefore $(x_n)_{n \in I}$ converges to the same limit.

8.4 Absolutely converge

We say that a series $\sum_{n \in I} x_n \in \mathbb{R}$ converges absolutely if $\sum_{n \in I} |x_n| < +\infty$

8.4.1 Prop

If a series $\sum_{n \in I} x_n$ converges absolutely, then it converges in \mathbb{R}

Chapter 9

Comparison and Technics of Computation

9.1 Def

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be sequence in \mathbb{R}

- If there exists $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$ such that $\forall n \in I_{\geq N}, |x_n| \leq M|y_n|$ then we write $x_n = O(y_n), n \in I, n \rightarrow +\infty$
- If there exists $(\epsilon_n)_{n \in I} \in \mathbb{R}^I$ and $N \in \mathbb{N}$ such that $\lim_{n \in I, n \rightarrow +\infty} \epsilon_n = 0$ and $\forall n \in I_{\geq N}, |x_n| \leq |\epsilon_n y_n|$, then we write $x_n = o(y_n), n \in I, n \rightarrow +\infty$

Example:

$$\lim_{n \rightarrow +\infty} \frac{1}{n} = 0$$

9.2 Prop.

Let I and X be partially ordered sets and $f : I \rightarrow X$ be an increasing/decreasing mapping. Let J be a subset of I . Assume that any element of I has an upper bound in J . Then $f(I)$ and $f(J)$ have the same upper/lower bounds in X

9.3 Theorem

Let I be a totally ordered set, $f : I \rightarrow [-\infty, +\infty]$ and $g : I \rightarrow [-\infty, +\infty]$ be two mappings that are both increasing/decreasing. Then the following equalities hold, provided that the sum on the right hand side of the equality is well defined.

$$\sup_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\sup_{x \in I} f(x)) + (\sup_{y \in I} g(y))$$

$$\inf_{x \in I, \{f(x), g(x)\} \neq \{-\infty, +\infty\}} = (\inf_{x \in I} f(x)) + (\inf_{y \in I} g(y))$$

9.3.1 Proof

We can assume f and g increasing. Let $a = \sup f(I), b = \sup g(I)$

Let $A = \{(x, y) \in I \times I \mid \{f(x), g(x)\} \neq \{-\infty, +\infty\}\}$

We equip A with the following order relation.

$$(x, y) \leq (x', y') \text{ iff } x \leq x', y \leq y'$$

Let $B = A \cap \Delta_I = \{(x, y) \in A \mid x = y\}$.

Consider

$$h : A \rightarrow [-\infty, +\infty] \quad h(x, y) = f(x) + g(y)$$

h is increasing.

Let $(x, y) \in A$. Assume that $x \leq y$

If $\{f(y), g(y)\} \neq \{-\infty, +\infty\}$ then $(y, y) \in B$ and $(x, y) \leq (y, y)$

If $\{f(y), g(y)\} = \{-\infty, +\infty\}$ and for $(x, y) \in A \Rightarrow f(y) = +\infty, g(y) = -\infty$. So $a = +\infty$, Hence $b > -\infty$

So $\exists z \in I$ such that $g(z) > -\infty$. We should have $y \leq z$ Hence $f(z) + g(z)$ is well defined, $(z, z) \in B$ and $(x, y) \leq (z, z)$ Similarly, if $x \geq y$, (x, y) has also an upper bound in B . Therefore: $\sup h(A) = \sup h(B)$

9.4 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I \quad \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\begin{aligned} \limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\leq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n) \\ \liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) &\geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n) \end{aligned}$$

9.4.1 Proof

$\forall n \in \mathbb{N}$, let $A_N = \sup_{n \in I, n \geq N} x_n$ $B_N = \sup_{n \in I, n \geq N} y_n$. $(A_N)_{N \in \mathbb{N}}$ and $(B_N)_{N \in \mathbb{N}}$ are decreasing, and $\limsup_{n \in I, n \rightarrow +\infty} x_n = \inf_{N \in \mathbb{N}} A_N$ $\limsup_{n \in I, n \rightarrow +\infty} y_n = \inf_{N \in \mathbb{N}} B_N$

By theorem:

$$\inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N = \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N)$$

Let $C_N = \sup_{n \in I, n \geq N} (x_n + y_n) \leq A_N + B_N$ if $A_N + B_N$ is defined.

Therefore

$$\inf_{N \in \mathbb{N}} C_N \leq \inf_{N \in \mathbb{N}, \{A_N, B_N\} \neq \{-\infty, +\infty\}} (A_N + B_N) = \inf_{N \in \mathbb{N}} A_N + \inf_{N \in \mathbb{N}} B_N$$

9.5 Prop.

Let $I \subseteq \mathbb{N}$ be an infinite subset. Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$ such that $\forall n \in I, \{x_n, y_n\} \neq \{-\infty, +\infty\}$. Then the following inequalities holds, provided that the sum on the right hand side is well defined.

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\limsup_{n \in I, n \rightarrow +\infty} x_n) + (\limsup_{n \in I, n \rightarrow +\infty} y_n)$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) \geq (\liminf_{n \in I, n \rightarrow +\infty} x_n) + (\liminf_{n \in I, n \rightarrow +\infty} y_n)$$

9.5.1 Proof

a tricky proof?:

$$\limsup_{n \in I, n \rightarrow} x_n = \limsup_{n \in I, n \rightarrow} (x_n + y_n - y_n) \leq \limsup_{n \in I, n \rightarrow} (x_n + y_n) - \liminf_{n \in I, n \rightarrow} y_n$$

to have a true proof, only need to discuss conditions with ∞

9.6 Theorem

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$. Assume that $\forall n \in I, y_n \in \mathbb{R}$ and $(y_n)_{n \in I}$ converges to some $l \in \mathbb{R}$.

Then:

$$\limsup_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\limsup_{n \in I, n \rightarrow +\infty} x_n) + l$$

$$\liminf_{n \in I, n \rightarrow +\infty} (x_n + y_n) = (\liminf_{n \in I, n \rightarrow +\infty} x_n) + l$$

9.7 Prop.

Let $(x_n)_{n \in I}$ and $(y_n)_{n \in I}$ be elements of $[-\infty, +\infty]^I$

Then:

$$\liminf_{n \in I, n \rightarrow +\infty} \max\{x_n, y_n\} = \max\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\}$$

$$\liminf_{n \in I, n \rightarrow +\infty} \min\{x_n, y_n\} = \min\{\liminf_{n \in I, n \rightarrow +\infty} x_n, \liminf_{n \in I, n \rightarrow +\infty} y_n\}$$

9.7.1 Proof

About the first inequality. Since $\max\{x_n, y_n\} \geq x_n$ and $\max\{x_n, y_n\} \geq y_n$

By the theorem of Bolzano-Weierstrass theorem, there exists an infinite subset J of I such that

$$\lim_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\} = \limsup_{n \in J, n \rightarrow +\infty} \max\{x_n, y_n\}$$

Let $J_1 = \{n \in J \mid x_n \geq y_n\}$ $J_1 = \{n \in J \mid x_n \leq y_n\}$

$J_1 \cup J_2 = J$ So either J_1 or J_2 is infinite

Suppose that J_1 is infinite, then

$$\lim_{n \in J, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J_1, n \rightarrow} \max\{x_n, y_n\} = \lim_{n \in J, n \rightarrow} x_n \leq \limsup_{n \in I, n \rightarrow +\infty} x_n$$

If J_2 is infinite

$$\limsup_{n \in I, n \rightarrow +\infty} = \lim_{n \in J_2, n \rightarrow +\infty} \max\{x_n, y_n\} \leq \limsup_{n \in I, n \rightarrow +\infty} y_n$$

9.8 Theorem

Let $(a_n)_{n \in I} \in \mathbb{R}^I$ $l \in \mathbb{R}$. The following statements are equivalent

- $(a_n)_{n \in I}$ converges to l
- $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$

9.8.1 Proof

$$|a_n - l| = \max\{a_n - l, l - a_n\}$$

$$\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = \max\left\{\left(\limsup_{n \in I, n \rightarrow +\infty} a_n\right) - l, l - \left(\liminf_{n \in I, n \rightarrow +\infty} a_n\right)\right\}$$

(1) \Rightarrow (2):

If $(a_n)_{n \in I}$ converges to l , then $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

(2) \Rightarrow (1):

If $\limsup_{n \in I, n \rightarrow +\infty} |a_n - l| = 0$, then $\limsup_{n \in I, n \rightarrow +\infty} a_n \leq l \leq \liminf_{n \in I, n \rightarrow +\infty} a_n$

Therefore: $\limsup_{n \in I, n \rightarrow +\infty} a_n = \liminf_{n \in I, n \rightarrow +\infty} a_n = l$

9.9 Remark

Let $(a_n)_{n \in I}$ be a sequence in \mathbb{R} , $l \in \mathbb{R}$

The sequence $(a_n)_{n \in I}$ converges to l iff $a_n - l = o(1), n \in I, n \rightarrow +\infty$

9.10 Calculates on $O(), o()$

9.10.1 Plus

Let $(a_n)_{n \in I}$ $(a'_n)_{n \in I}$ and $(b_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n), a'_n = O(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = O(b_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n), a'_n = o(b_n), n \in I, n \rightarrow +\infty$
then $\forall (\lambda, \mu) \in \mathbb{R}^2 \quad \lambda a_n + \mu a'_n = o(b_n), n \in I, n \rightarrow +\infty$

9.10.2 Transform

Let $(a_n)_{n \in I}$ and $(b_n)_{n \in I}$ be two sequence in \mathbb{R} If $a_n = o(b_n), n \in I, n \rightarrow +\infty$, then $a_n = O(b_n), n \in I, n \rightarrow +\infty$

9.10.3 Transition

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ and $(c_n)_{n \in I}$ be elements in \mathbb{R}^I

- If $a_n = O(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = O(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = O(b_n)$ and $b_n = o(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$
- If $a_n = o(b_n)$ and $b_n = O(c_n), n \in I, n \rightarrow +\infty$
then $a_n = o(c_n), n \in I, n \rightarrow +\infty$

9.10.4 Times

Let $(a_n)_{n \in I}, (b_n)_{n \in I}, (c_n)_{n \in I}, (d_n)_{n \in I}$ be sequences in \mathbb{R}

- If $a - N = O(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = O(b_n d_n), n \in I, n \rightarrow +\infty$
- If $a - N = o(b_n), c_n = O(d_n), n \in I, n \rightarrow +\infty$
then $a_n c_n = o(b_n d_n), n \in I, n \rightarrow +\infty$

9.11 On the limit

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be elements of \mathbb{R}^I that converges to $l \in \mathbb{R}$ and $l' \in \mathbb{R}$ respectively. Then:

- $(a_n + b_n)_{n \in I}$ converges to $l + l'$
- $(a_n b_n)_{n \in I}$ converges to ll'

9.12 Prop

Let $a \in \mathbb{R}$ Then $a^n = o(n!) \quad n \rightarrow +\infty$

9.12.1 Proof

Let $N \in \mathbb{N}$ such that $|a| < N$ For $n \in \mathbb{N}$ such that $n \geq N$

$$0 \leq \frac{|a^n|}{n!} = \frac{|a^N|}{N!} \cdot \frac{n!}{N!} \cdot \frac{|a|^{n-N}}{N^{n-N}} \leq \frac{|a^N|}{N!} \left(\frac{|a|}{N}\right)^{n-N} \rightarrow 0$$

And $0 < \frac{|a|}{N} < 1 \Rightarrow \lim_{n \rightarrow +\infty} \left(\frac{|a|}{N}\right)^n = 0$. Therefore:

$$\lim_{n \rightarrow +\infty} \frac{|a^n|}{n!} = 0$$

namely:

$$a^n = o(n!)$$

9.13 Prop

$$n! = o(n^n) \quad n \rightarrow +\infty$$

9.13.1 Proof

$$\text{Let } N \in \mathbb{N}_{\geq 1} \\ 0 \leq \frac{n!}{n^n} \leq \frac{1}{n} \Rightarrow \lim_{n \rightarrow +\infty} \frac{n!}{n^n} = 0$$

9.14 Prop

Let $(a_n)_{n \in I}, (b_n)_{n \in I}$ be the elements of \mathbb{R}^I . If the series $\sum_{n \in I} b_n$ converges absolutely and if $a_n = O(b_n) \quad n \rightarrow +\infty$ Then $\sum_{n \in I} a_n$ converges absolutely

9.14.1 Proof

By definition $\sum_{n \in I} |b_n| < +\infty$. If $|a_n| \leq M|b_n|$ for $n \in I, n \geq N$ where $N \in \mathbb{N}$. Then

$$\sum_{n \in I} |a_n| = \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |a_n| \leq \sum_{n \in I, n < N} |a_n| + \sum_{n \in I, n \geq N} |b_n| < +\infty$$

9.15 Theorem: d'Alembert ratio test

Let $(a_n)_{n \in \mathbb{N}} \in (\mathbb{R} \setminus \{0\})^{\mathbb{N}}$

- If $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely
- If $\liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$, then $\sum_{n \in \mathbb{N}} a_n$ does not converge (diverges)

9.15.1 Proof

(1)

Let $\alpha \in \mathbb{R}$ such that $\limsup_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| < \alpha < 1$, α isn't a lower bound of $\left(\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| \right)_{N \in \mathbb{N}}$
 So $\exists N \in \mathbb{N}$ such that $\sup_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right| < \alpha$ Hence for $n \geq N$ $|a_n| \leq \alpha^{n-N} |a_N|$ since

$$\frac{a_n}{a_N} = \frac{a_{N+1}}{a_N} \frac{a_{N+2}}{a_{N+1}} \dots \frac{a_n}{a_{n-1}}$$

Therefore $a_n = O(\alpha^n)$ since $\sum_{n \in \mathbb{N}} \frac{1}{1-\alpha} < +\infty$, $\sum_{n \in \mathbb{N}} a_n$ converge absolutely.

9.15.2 Lemma

If a series $\sum_{n \in \mathbb{N}} a_n \in \mathbb{R}$ converges, then $\lim_{n \rightarrow +\infty} a_n = 0$

Proof

If $\left(\sum_{i=0}^n a_i \right)_{n \in \mathbb{N}}$ converges to some $l \in \mathbb{R}$, then $\left(\sum_{i=0}^{n-1} a_i \right)_{n \in \mathbb{N}, n \geq 1}$ converges to l ,
 too. Hence $\left(a_n = \left(\sum_{i=0}^n a_i \right) - \left(\sum_{i=0}^{n-1} a_i \right) \right)_{n \in \mathbb{N}}$ converges to $l - l = 0$

9.15.3 (2)

Let $\beta \in \mathbb{R}$ such that $1 < \beta < \liminf_{n \rightarrow +\infty} \left| \frac{a_{n+1}}{a_n} \right| = \sup_{N \in \mathbb{N}} \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 So there exists $N \in \mathbb{N}$ such that $\beta < \inf_{n \geq N} \left| \frac{a_{n+1}}{a_n} \right|$
 $\forall n \in \mathbb{N}, n \geq N \quad \left| \frac{a_{n+1}}{a_n} \right| \geq \beta$
 Hence $(|a_n|)_{n \in \mathbb{N}}$ is not bounded since $|a_n| \geq \beta^{n-N} |a_N|$
 By the lemma: $\sum_{n \in \mathbb{N}} a_n$ diverges.

9.16 Prop

Let $a \in \mathbb{R}, a > 1$ Then $n = o(a^n), n \rightarrow +\infty$

9.16.1 Proof

Let $\epsilon > 0$ such that $a = (1 + \epsilon)^2$

$$a^n = (1 + \epsilon)^{2n} = (1 + \epsilon)^n (1 + \epsilon)^n \geq (1 + n\epsilon)(1 + n\epsilon) \geq \epsilon^2 n^2$$

Hence

$$n \leq \frac{a^n}{\epsilon^2 n} = o(a^n)$$

9.16.2 Corollary

Let $a > 1, t \in \mathbb{R}_{\geq 0}$ Then $n^t = o(a^n), n \rightarrow +\infty$

Proof

Let $d \in \mathbb{N}_{\geq 1}$ such that $t \leq d$ Then $n^{t-d} \leq 1$ So

$$n^t = n^d n^{t-d} = O(n^d)$$

Let $b = \sqrt[d]{a} > 1$

$$n^d = o((b^n)^d) = o(a^n)$$

Hence $n^t = o(a^n)$

9.16.3 Corollary

There exists $M \geq 1$ such that $\forall x \in \mathbb{R}, x \geq M, \ln(x) \leq x$

Proof

Let $a \in \mathbb{R}$ such that $1 < a < e$

9.17 Theorem: Cauchy root test

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{R} . Let $\alpha = \limsup_{n \rightarrow +\infty} |a_n|^{\frac{1}{n}}$

- If $\alpha < 1$, then $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.
- If $\alpha > 1$ then $\sum_{n \in \mathbb{N}} a_n$ diverges

9.17.1 Proof

(1)

Let $\beta \in \mathbb{R}, \alpha < \beta < 1$. There exists $N \in \mathbb{N}$ such that $|a_n|^{\frac{1}{n}} \leq \beta$ for $n \geq N$. That means $|a_n| = O(\beta^n)$ since $0 < \beta < 1$, $\sum_{n \in \mathbb{N}} a_n$ converges absolutely.

(2)

If $\alpha > 1$ then $\forall N \in \mathbb{N} \exists n \geq N$ such that $|a_n|^{\frac{1}{n}} \geq 1$, since otherwise $\exists N \in \mathbb{N} \forall n \geq N, |a_n|^{\frac{1}{n}} < 1$ contradiction
Hence $(|a_n|)_{n \in \mathbb{N}}$ cannot converge to 0.

Part III

Topology

Chapter 10

Absolute value and norms

10.1 Def

Let K be a field. By absolute value on K , we mean a mapping $|\cdot| : K \rightarrow \mathbb{R}_{\geq 0}$ that satisfies:

- (1) $\forall a \in K \quad |a| = 0$ iff $a = 0$
- (2) $\forall (a, b) \in K^2 \quad |ab| = |a| \cdot |b|$
- (3) $\forall (a, b) \in K^2 \quad |a + b| \leq |a| + |b|$ (triangle inequality)

10.2 Notation

\mathbb{Q} Take a prime num $p \forall \alpha \in \mathbb{Q} \setminus \{0\}$ there exists a integer $ord_p(\alpha) \frac{a}{b}$, where
 $a \in \mathbb{Z} \setminus \{0\}$
 $b \in \mathbb{N} \setminus \{0\}, p \nmid a, p \nmid b$

10.3 Prop

$$|\cdot| : \begin{matrix} \mathbb{Q} \rightarrow \mathbb{R}_{\geq 0} \\ \alpha \mapsto \begin{cases} p^{-ord_p(\alpha)} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \end{matrix}$$

is a absolute value on \mathbb{Q}

10.3.1 proof

- (1) Obviously

$$(2) \text{ If } \alpha = p^{ord_p(\alpha)} \frac{a}{b}, \beta = p^{ord_p(\beta)} \frac{c}{d} \quad p \nmid abcd$$

$$\alpha\beta = p^{ord_p(\alpha)+ord_p(\beta)} \frac{ac}{bd} \quad p \nmid ac, p \nmid bd$$

$$(3) \quad \alpha + \beta = p^{ord_p(\alpha)} \frac{a}{b} + p^{ord_p(\beta)} \frac{c}{d}$$

Assume $ord_p(\alpha) \geq ord_p(\beta)$

$$\alpha + \beta$$

$$= p^{ord_p(\beta)} \left(p^{ord_p(\alpha)-ord_p(\beta)} \frac{a}{b} + \frac{c}{d} \right)$$

$$= p^{ord_p(\beta)} \frac{p^{ord_p(\alpha)-ord_p(\beta)} ad + bc}{bd} \quad p \nmid bd$$

So

$$ord_p(\alpha + \beta) \geq ord(\beta)$$

$$\text{Hence } ord_p(\alpha + \beta) \geq \min\{ord_p(\alpha), ord_p(\beta)\}$$

$$\text{So } |\alpha + \beta|_p = p^{-ord_p(\alpha+\beta)} \leq \max\{p^{-ord_p(\alpha)}, p^{-ord_p(\beta)}\} =$$

$$\max\{|\alpha|_p, |\beta|_p\} \leq |\alpha|_p, |\beta|_p$$