Definition

Given two functions $f : \mathbb{N} \to \mathbb{R}$ and $g : \mathbb{N} \to \mathbb{R}$,

$$\begin{aligned} f(n) \in O(g(n)) \quad &\iff \quad \exists\, c \in \mathbb{R}^+ \text{ and } n_0 \in \mathbb{N} \text{ such that} \\ &\qquad\;\; 0 \le f(n) \le c\, g(n), \quad \forall n \ge n_0, \\ f(n) \in o(g(n)) \quad &\iff \quad \forall\, c \in \mathbb{R}^+, \quad \exists\, n_0 \in \mathbb{N} \text{ such that} \\ &\qquad\;\; 0 \le f(n) < c\, g(n), \quad \forall n \ge n_0. \end{aligned}$$

Claim

While the textbook uses "$\le$" for big-O and "$<$" for little-o, this distinction is NOT generally mathematical but rather stylistic. The choose to use different inequality signs serves to highlight the difference in growth rates. In reality, the two signs are **mostly interchangeable**, and the key distinction lies in the quantification of the constant c.

Statement

We define another set of notations with different inequality signs as follow,

$$\begin{aligned} f(n) \in O'(g(n)) \quad &\iff \quad \exists\, c' \in \mathbb{R}^+ \text{ and } n_0 \in \mathbb{N} \text{ such that} \\ &\qquad\;\; 0 \le f(n) < c'\, g(n), \quad \forall n \ge n_0, \\ f(n) \in o'(g(n)) \quad &\iff \quad \forall\, c' \in \mathbb{R}^+, \quad \exists\, n_0 \in \mathbb{N} \text{ such that} \\ &\qquad\;\; 0 \le f(n) \le c'\, g(n), \quad \forall n \ge n_0. \end{aligned}$$

And we aim to prove that $O(g(n)) = O'(g(n))$ and $o(g(n)) = o'(g(n))$, given any $g : \mathbb{N} \to \mathbb{R}$. (Spoiler: actually not *every* g satisfies, as we shall see that there is a pitfall.)

Proof of Big-O

1. $\boxed{O(g(n)) \subseteq O'(g(n))}$

   Given any function $f : \mathbb{N} \to \mathbb{R}$ with $f \in O(g(n))$, we have positive constants c and n_0 such that for all integers $n \geq n_0$,
   $$0 \leq f(n) \leq c\, g(n).$$
   This means that for all real number $c' > c$, and assume that $g(n) \neq 0$, we have
   $$0 \leq f(n) \leq c\, g(n) < c'\, g(n).$$
   Thus, we have witnesses c' and n_0 that satisfy $f(n) \in O'(g(n))$.

2. $\boxed{O' g(n) \subseteq O(g(n))}$

   This is rather trivial since $f(n) < c\, g(n)$ implies $f(n) \leq c\, g(n)$.

Therefore, $O(g(n) = O'(g(n)))$, and whether we use "$\leq$" or "$<$" does not matter.

Proof of Little-o

1. $\boxed{o(g(n)) \subseteq o'(g(n))}$

   Similarly, $f(n) < c\, g(n)$ implies $f(n) \leq c\, g(n)$.

2. $\boxed{o'(g(n)) \subseteq o(g(n))}$

   Given any $f : \mathbb{N} \to \mathbb{R}$ with $f(n) \in o'(g(n))$, we have that for all positive real numbers c', there exists n_0 such that for all integers $n \geq n_0$,
   $$0 \leq f(n) \leq c'\, g(n).$$
   Then, given any real numbers $c > 0$, we can always find $c' \in (0, c)$ such that
   $$0 \leq f(n) \leq c'\, g(n) < c\, g(n),$$
   assuming that $g(n) \neq 0$. Thus, for any $c > 0$, we have n_0 that satisfy $f(n) \in o(g(n))$.

Therefore, $o(g(n)) = o'(g(n))$, and whether we use "$\leq$" or "$<$" does not matter.

Conclusion

We have shown that swapping the strict and non-strict inequalities in the definitions of the asymptotic classes $O(\,\cdot\,)$ and $o(\,\cdot\,)$ does not change the sets of functions:

$$O(g(n)) = O'(g(n)), \quad o(g(n)) = o'(g(n)).$$

The decisive feature for big-O is the *existential* quantifier ($\exists\, c > 0$); for little-o, it is the *universal* quantifier ($\forall\, c > 0$). Whether the bound is written with "$\leq$" or "$<$" is mostly a matter of convention.

However, our proof assumes that $g(n) \neq 0$ eventually. In the rare case where $g(n)$ becomes zero asymptotically—i.e., $g(n) = 0$ for all sufficiently large n—the definitions diverge:

$$O(g(n)) = \{f(n) \mid f(n) \text{ is asymptotically zero}\}, \quad O'(g(n)) = \varnothing,$$

$$o(g(n)) = \varnothing, \quad o'(g(n)) = \{f(n) \mid f(n) \text{ is asymptotically zero}\}.$$

Such behavior is not encountered in typical algorithmic or analytic settings, so this edge case can be safely ignored in practice.

Thus, the common choice to use "$\leq$" in big-O (emphasizing a *bound*) and "$<$" in little-o (emphasizing a *strict* vanishing rate) serves only as a helpful visual cue; mathematically, the two forms are interchangeable under standard assumptions.