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1 Definition

- Suppose $U_1, \ldots, U_m$ are subsets of V. The ***sum*** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}.$$

- Suppose $U_1, \ldots, U_m$ are subspaces of V. The sum $U_1 + \cdots + U_m$ is called a ***direct sum*** if each element of $U_1 + \cdots + U_m$ can be written uniquely as a sum $u_1 + \cdots u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

2 Theorem

2.1 Smallest containing subspace

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof

Given subspaces $U_1, \ldots, U_m$ of V, we aim to show that $U_1 + \cdots + U_m$ is a subspace of V, contains $U_1, \ldots, U_m$, and is the smallest subspace of V containing $U_1, \ldots, U_m$.

- Because $U_1, \ldots, U_m$ are subspaces of V, it is obvious that $0 \in U_1 + \cdots + U_m$ and that $U_1 + \cdots U_m$ is closed under addition and scalar multiplication.
- In each U_j, for every $u_j \in U_j$, clearly that $u_j \in U_1 + \cdots U_m$ because we can choose all the other u's as 0. Thus, each $U_j \subseteq U_1 + \cdots U_m$.
- For every subspace W of V that containing $U_1, \ldots, U_m$, we have $U_1 + \cdots + U_m \subseteq W$, for the following reason. For every $u \in U_1 + \cdots + U_m$, we can write

$$u = u_1 + \cdots + u_m,$$

where $u_1 \in U_1, \ldots, u_m \in U_m$. And because W contains $U_1, \ldots, U_m$, this means that each $u_j \in W$, and so is their sum $u_1 + \cdots + u_m$ since W is closed under addition. Thus $u \in W$.

2.2 Condition for a direct sum

Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Proof

Given subspaces $U_1, \dots, U_m$ of V, we aim to show that both directions are true.

- $\boxed{\Longrightarrow}$ Suppose that $U_1 + \cdots + U_m$ is a direct sum. If there were $u_1 \in U_1, \dots, u_m \in U_m$, not all 0, such that
$$u_1 + \cdots + u_m = 0.$$
Then we would have a different representation that adds to 0, as
$$2u_1 + \cdots + 2u_m = 0,$$
which contradicts the definition of the direct sum $U_1 + \cdots + U_m$.

- $\boxed{\Longleftarrow}$ Suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0. For every $w \in U_1 + \cdots + U_m$, we can write
$$w = u_1 + \cdots + u_m,$$
for some $u_1 \in U_1, \dots, u_m \in U_m$. To show that this representation is unique, consider that
$$w = v_1 + \cdots + v_m,$$
for some $v_1 \in U_1, \dots, v_m \in U_m$. Subtracting these two equations gives
$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$
Because $(u_1 - v_1) \in U_1, \dots, (u_m - v_m) \in U_m$, based on the assumption we have each $u_j - v_j = 0$. Thus $u_1 = v_1, \dots, u_m = v_m$; that is, there is a unique representation of w.

2.3 Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof

- $\boxed{\Longrightarrow}$ Given that U and W are subspaces of V, suppose that $U + W$ is a direct sum. Then we know that for every $u \in U$ and $w \in W$, $u + w = 0$ implies $u = w = 0$ by 2.2. Now given any $v \in U \cap W$, there exists its additive inverse $v' \in U \cap W$ such that $v + v' = 0$. And since it is also true that $v \in U$ and $v' \in W$, we have $v = 0$. Thus, $U \cap W = \{0\}$.

- $\boxed{\Longleftarrow}$ Given that U and W are subspaces of V, suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U$ and $w \in W$ such that $u + w = 0$, we want to show that $u = w = 0$. Because the additive inverse is unique, we know that $u = -w \in W$. This implies that $u \in U \cap W$, and thus $u = 0$.

2.4 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof

Given that V is finite-dimensional and U is a subspace of V, U is also finite-dimensional. Let $u_1, \ldots, u_m$ be a basis of U. The list can then be extended to be a basis of V. Let the extended list be

$$u_1, \ldots, u_m, w_1, \ldots, w_n.$$

Let $W = \operatorname{span}(w_1, \ldots, w_n)$. We aim to show that $U + W$ is a direct sum and $V = U + W$.

- For every $v \in U \cap W$, there exist scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n.$$

  Thus

$$a_1 u_1 + \cdots + a_m u_m + (-b_1) w_1 + \cdots + (-b_n) w_n = 0.$$

  Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, it must be that $a_1 = \cdots = a_m = -b_1 = \cdots = -b_n = 0$. Thus $v = 0$, which concludes that $U \cap W = \{0\}$.

- $\boxed{U + W \subseteq V}$ It is trivially true as U and W are subspaces of V.

- $\boxed{V \subseteq U + W}$ For every $v \in V$, since the list $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n.$$

  Let $u = a_1 u_1 + \cdots + a_m u_m \in U$ and $w = b_1 w_1 + \cdots + b_n w_n \in W$, we can represent v as $v = u + w$. Thus $v \in U + W$.

3 Exercise

3.1 1.C.15

Suppose U is a subspace of V. What is $U + U$?

Solution

$U + U = U$. For every $v \in U + U$, it can be written as $v = u_1 + u_2$, where $u_1, u_2 \in U$; thus $v \in U$ because addition is closed. And For every $u \in U$, it is trivial that $u \in U + U$ as it can be written as $u = u + 0$.

3.2 1.C.16

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution

Yes because addition in vector space is commutative. For every $v \in U + W$, we can write $v = u + w = w + u \in W + U$ for some $u \in U$ and $w \in W$. The other direction is true with the similar reason.

3.3 1.C.17

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?

Solution

Yes because addition in vector space is associative. For every $v \in (U_1 + U_2) + U_3$, we can write $v = (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3) \in U_1 + (U_2 + U_3)$ for some $u_1 \in U_1, u_2 \in U_2, u_3 \in U_3$. The other direction is true with the similar reason.

3.4 1.C.18

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution

For every subspaces U of V, we have $U + \{0\} = U$; thus $\{0\}$ is an additive identity. For a subspace U of V to have a additive inverse, there should be a subspace U' of V such that

$U + U' = \{0\}$. The only U satisfying this requirement is $\{0\}$, as any other U would lead to $U + U' \neq \{0\}$ for any U'.

3.5 1.C.24

A function $f : \mathbf{R} \to \mathbf{R}$ is called ***even*** if

$$f(-x) = f(x), \quad \forall x \in \mathbf{R}.$$

A function $f : \mathbf{R} \to \mathbf{R}$ is called ***odd*** if

$$f(-x) = -f(x), \quad \forall x \in \mathbf{R}.$$

Let U_e denote the set of real-valued even functions on $\mathbf{R}$ and let U_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^\mathbf{R} = U_\mathrm{e} \oplus U_\mathrm{o}$.

Solution

We aim to show that $U_\mathrm{e} + U_\mathrm{o}$ is a direct sum, and $U_\mathrm{e} + U_\mathrm{o} = \mathbf{R}^\mathbf{R}$.

- It can be shown that $U_\mathrm{e} \cap U_\mathrm{o} = \{0\}$ where 0 is the function that is identically zero; thus $U_\mathrm{e} + U_\mathrm{o}$ is a direct sum.

  Suppose to the contrary that there were non-identically-zero $f \in U_\mathrm{e} \cap U_\mathrm{o}$, then there would exist $x \in \mathbf{R}$ such that $f(x) \neq 0$. By $f \in U_\mathrm{e}$ we have $f(x) = f(-x)$; and by $f \in U_\mathrm{o}$ we have $f(x) = -f(-x)$. This leads to $f(-x) = -f(-x) \neq 0$, which is a contradiction.

- $\boxed{U_\mathrm{e} + U_\mathrm{o} \subseteq \mathbf{R}^\mathbf{R}}$

  For every $f \in U_\mathrm{e} + U_\mathrm{o}$, it is trivial that $f \in \mathbf{R}^\mathbf{R}$, as U_e and U_o are subspaces of $\mathbf{R}^\mathbf{R}$.

- $\boxed{\mathbf{R}^\mathbf{R} \subseteq U_\mathrm{e} + U_\mathrm{o}}$

  For every $f \in \mathbf{R}^\mathbf{R}$, we want to find $g \in U_\mathrm{e}$ and $h \in U_\mathrm{o}$ such that

  $$f(x) = g(x) + h(x), \quad \forall x \in \mathbf{R}.$$

  To find the value of $g(x)$ and $h(x)$ at each point $x = x_0$, we solve the following system.

  $$\begin{cases} f(x_0) = g(x_0) + h(x_0) \\ f(-x_0) = g(-x_0) + h(-x_0) = g(x_0) - h(x_0) \end{cases}$$

  $$\implies \quad g(x_0) = \frac{f(x_0) + f(-x_0)}{2}, \quad h(x_0) = \frac{f(x_0) - f(-x_0)}{2}.$$

In this way, we find the unique solution (thus also show that it is a direct sum) of $g(x)$ and $h(x)$.

3.6 2.B.8

Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

Proof

We aim to show that $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent and spans V.

- Let $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$ be scalars such that

  $$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n = 0.$$

  We can write $u = a_1 u_1 + \cdots + a_m u_m \in U$ and $w = b_1 w_1 + \cdots b_n w_n = 0$. And since $U + W$ is a direct sum, $u + w = 0$ implies $u = w = 0$. That is,

  $$a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots b_n w_n = 0.$$

  Because $u_1, \dots, u_m$ and $w_1, \dots, w_n$ are bases,

  $$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0.$$

  Thus $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent.

- For every $v \in V$, since $V = U + W$, there exist $u \in U$ and $w \in W$ such that $v = u + w$. And we can further write u and v as

  $$u = a_1 u_1 + \cdots + a_m u_m, \quad w = b_1 w_1 + \cdots b_n w_n,$$

  for some scalars $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbf{F}$. Then

  $$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n.$$

  Thus $v \in \operatorname{span}(u_1, \dots, u_m, w_1, \dots, w_n)$, which concludes that the list spans V.