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1 Definition

For vector space V over a field $\mathbf{F}$ and subset U in V,

$$U \text{ is a } \textit{subspace} \text{ of } V \text{ if } U \text{ is also a vector space over } \mathbf{F},$$

with the same operations of addition and scalar multiplication as on V.

2 Conditions

A subset U of V is a subspace of V iff U satisfies the following three conditions:

1. **Additive Identity**

   $0 \in U$.

2. **Closed Under Addition**

   For all $u, w \in U$, $u + w \in U$.

3. **Closed Under Scalar Multiplication**

   For all $a \in \mathbf{F}$ and all $u \in U$, $au \in U$.

Note about additive identity

If one were to carelessly exclude the first condition, thinking that it should have been implicitly satisfied by the third condition as follows:

$$\text{Choose } a = 0 \text{ and any } u \in U, \text{ then } au = 0 \in U \text{ exists. (Wrong!)}$$

While this may good as 0 always exists in a field, U might be an empty set thus u cannot be chosen. The issue is that the second and third conditions alone fail to rule out the case $U = \varnothing$, as they only states that *all* vectors are closed under addition and scalar multiplication, not the *existence* of them.

Note about definition

The three conditions are not the definition of subspaces; instead, it shall be proved that they are equivalent to the definition of subspaces.

Proof

- $\boxed{\Longrightarrow}$ Given a subset U of V, if U is a subspace of V, then U satisfies the three conditions.

  Suppose that U is a subspace of V, that is, U is the vector space over the same field.

  - The 1-st condition is satisfied by the definition of the vector space that U has a additive identity 0.
  - The 2-nd and 3-rd conditions are satisfied by the definitions of operations on U, as addition is the function of $U \times U \to U$ and scalar multiplication is the function of $\mathbf{F} \times U \to U$. This implies that all sums between vectors and all products between numbers and vectors, are both vectors in U.

- $\boxed{\Longleftarrow}$ Given a subset U of V, if U satisfies the three conditions, then U is a subspace of V.

  Suppose that a subset U of V satisfies the three conditions, we aim to show that U is a vector space over the same field.

  First of all, the operations of addition $U \times U \to U$ and scalar multiplication $\mathbf{F} \times U \to U$ have to make sense. This is ensured by the 2-nd and 3-rd conditions.

  Let's move on to the properties of vector spaces.

  - The existence of additive identity is ensured by the 1-st condition.
  - For all $u \in U$, the additive inverse $(-1)u = -u$ exists in U by the 3-rd condition.
  - The other properties (i.e. commutativity, associativity, multiplicative identity, distributive properties) are automatically satisfied for U because they hold on a larger set V. This can be verified easily.

  Thus, U is a vector space and hence is a subspace of V.