

1 Definition

- A function $p : \mathbf{F} \rightarrow \mathbf{F}$ is called a *polynomial* with coefficients in \mathbf{F} if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in \mathbf{F} .
- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have *degree* m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$.

- If p has degree m , we write $\deg p = m$.
- The polynomial that is identically 0 is said to have degree $-\infty$.
- For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in \mathbf{F} and degree at most m .

Note that we use the convention $m > -\infty$, which means that the polynomial 0 is in $\mathcal{P}_m(\mathbf{F})$ for every nonnegative integer m .

2 Exercise

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbf{F})$ is a vector space over \mathbf{F} . In other words, $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^\mathbf{F}$.

Note

Recall that $\mathbf{F}^\mathbf{F}$ denotes the set of functions from \mathbf{F} to \mathbf{F} . And for $f, g \in \mathbf{F}^\mathbf{F}$ and $\lambda \in \mathbf{F}$, the sum $f + g \in \mathbf{F}^\mathbf{F}$ and the product $\lambda f \in \mathbf{F}^\mathbf{F}$ are defined by

$$(f + g)(z) = f(z) + g(z), \quad (\lambda f)(z) = \lambda f(z)$$

for all $z \in \mathbf{F}$.

Proof

Let's show that $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$.

- $\mathcal{P}(\mathbf{F}) \subseteq \mathbf{F}^{\mathbf{F}}$ because a polynomial is a function from \mathbf{F} to \mathbf{F} .
- $0 \in \mathcal{P}(\mathbf{F})$ by definition of the degree $-\infty$ polynomial.
- For all $p, q \in \mathcal{P}(\mathbf{F})$, we can write

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad \text{for some nonnegative integer } m,$$

$$q(z) = c_0 + c_1 z + \cdots + c_n z^n, \quad \text{for some nonnegative integer } n.$$

Without loss of generality, assume $m \geq n$, then

$$\begin{aligned} (p+q)(z) &= p(z) + q(z) \\ &= (a_0 + a_1 z + \cdots + a_m z^m) + (c_0 + c_1 z + \cdots + c_n z^n) \\ &= (a_0 + c_0) + (a_1 + c_1)z + \cdots + (a_n + c_n)z_n + \cdots + a_m z_m \\ &\in \mathcal{P}(\mathbf{F}). \end{aligned}$$

Thus, $\mathcal{P}(\mathbf{F})$ is closed under addition.

- For all $\lambda \in \mathbf{F}$ and $p \in \mathcal{P}(\mathbf{F})$,

$$\begin{aligned} (\lambda p)(z) &= \lambda p(z) \\ &= \lambda(a_0 + a_1 z + \cdots + a_m z^m) \\ &= \lambda a_0 + \lambda a_1 z + \cdots + \lambda a_m z^m \\ &\in \mathcal{P}(\mathbf{F}) \end{aligned}$$

Thus, $\mathcal{P}(\mathbf{F})$ is closed under scalar multiplication.