Vector Space

Lan Yung-Chi

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Contents

Chapter 1

Field

1.1　Definition

A ***field*** **F** is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satifying all the properties listed below.

1. commutativity

    $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$, for all $\alpha, \beta \in$ **F**.

2. associativity

    $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$, for all $\alpha, \beta, \lambda \in$ **F**.

3. identities

    $\lambda + 0 = \lambda$ and $1\lambda = \lambda$, for all $\lambda \in$ **F**.

4. additive Inverse

    For every $\alpha \in$ **F**, there exists a unique $\beta \in$ **F** such that $\alpha + \beta = 0$.

5. multiplicative inverse

    For every $\alpha \in$ **F** with $\alpha \neq 0$, there exists a unique $\beta \in$ **F** such that $\alpha\beta = 1$.

6. distributive property

    $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$, for all $\lambda, \alpha, \beta \in$ **F**.

Note

In linear algebra we deal with fields such as **R** and **C** most of the time, so throughout this book **F** stands for either **R** or **C** unless specified.

1.2 Exercise

1.2.1 C is a field

Prove that the set of complex numbers with the following operations of addition and multiplication is a field. For the sake of simplicity, we claim that $\mathbf{R}$ is a field, and safely use calculation techniques from $\mathbf{R}$.

- A **complex number** is an ordered pair (a, b), where $a, b \in \mathbf{R}$, written as $a + b\,i$.

- The set of all complex numbers is denoted by $\mathbf{C}$:

$$\mathbf{C} = \{a + b\,i \mid a, b \in \mathbf{R}\}.$$

- Addition and multiplication on $\mathbf{C}$ are defined by

$$(a + b\,i) + (c + d\,i) = (a + c) + (b + d)\,i, \tag{1.1}$$

$$(a + b\,i)(c + d\,i) = (ac - bd) + (ad + bc)\,i; \tag{1.2}$$

with $a, b, c, d \in \mathbf{R}$.

Proof

First of all, $\{0, 1\} \subseteq \mathbf{C}$ as $0 = 0 + 0\,i$ and $1 = 1 + 0\,i$.

1. $\boxed{\text{commutativity}}$

   For all $\alpha, \beta \in \mathbf{C}$, we can write $\alpha = a + b\,i$ and $\beta = c + d\,i$ with some $a, b, c, d \in \mathbf{R}$, then

$$
\begin{aligned}
\alpha + \beta &= (a + b\,i) + (c + d\,i) \\
&= (a + c) + (b + d)\,i && \text{by (1.1)} \\
\beta + \alpha &= (c + d\,i) + (a + b\,i) \\
&= (c + a) + (d + b)\,i && \text{by (1.1)} \\
&= \alpha + \beta && \text{by comm. of } \mathbf{R}
\end{aligned}
$$

$$
\begin{aligned}
\alpha\beta &= (a + b\,i)(c + d\,i) \\
&= (ac - bd) + (ad + bc)\,i && \text{by (1.2)} \\
\beta\alpha &= (c + d\,i)(a + b\,i) \\
&= (ca - db) + (da + cb)\,i && \text{by (1.2)} \\
&= \alpha\beta && \text{by comm. of } \mathbf{R}
\end{aligned}
$$

2. $\boxed{\text{associativity}}$

For all $\alpha, \beta, \lambda \in \mathbf{C}$, write $\alpha = a + b\,i$, $\beta = c + d\,i$, $\lambda = e + f\,i$ with some $a, b, c, d, e, f \in \mathbf{R}$, then

$$
\begin{aligned}
(\alpha + \beta) + \lambda &= [(a + b\,i) + (c + d\,i)] + (e + f\,i) \\
&= [(a + c) + (b + d)\,i] + (e + f\,i) && \text{by (1.1)} \\
&= [(a + c) + e] + [(b + d) + f]\,i && \text{by (1.1)} \\
\alpha + (\beta + \lambda) &= (a + b\,i) + [(c + d\,i) + (e + f\,i)] \\
&= (a + b\,i) + [(c + e) + (d + f)\,i] && \text{by (1.1)} \\
&= [a + (c + e)] + [b + (d + f)]\,i && \text{by (1.1)} \\
&= (\alpha + \beta) + \lambda && \text{by asso. of } \mathbf{R}
\end{aligned}
$$

$$
\begin{aligned}
(\alpha\beta)\lambda &= [(a + b\,i)(c + d\,i)](e + f\,i) \\
&= [(ac - bd) + (ad + bc)\,i](e + f\,i) && \text{by (1.2)} \\
&= [(ac - bd)e - (ad + bc)f] + [(ac - bd)f + (ad + bc)e]\,i && \text{by (1.2)} \\
&= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)\,i && \text{by dist. of } \mathbf{R} \\
\alpha(\beta\lambda) &= (a + b\,i)[(c + d\,i)(e + f\,i)] \\
&= (a + b\,i)[(ce - df) + (cf + de)\,i] && \text{by (1.2)} \\
&= [a(ce - df) - b(cf + de)] + [a(cf + de) + b(ce - df)]\,i && \text{by (1.2)} \\
&= (ace - bde - adf - bcf) + (acf - bdf + ade + bce)\,i && \text{by dist. of } \mathbf{R} \\
&= (\alpha\beta)\lambda && \text{by comm. of } \mathbf{R}
\end{aligned}
$$

3. $\boxed{\text{identities}}$

For all $\lambda \in \mathbf{C}$, write $\lambda = a + b\,i$ for some $a, b \in \mathbf{R}$, then

$$
\begin{aligned}
\lambda + 0 &= (a + b\,i) + (0 + 0\,i) \\
&= (a + 0) + (b + 0)\,i && \text{by (1.1)} \\
&= a + b\,i && \text{by iden. of } \mathbf{R} \\
&= \lambda
\end{aligned}
$$

$$
\begin{aligned}
1\lambda &= (1 + 0\,i)(a + b\,i) \\
&= (1a + 0b) + (1b + 0a)\,i && \text{by (1.2)} \\
&= a + b\,i && \text{by iden. and mul. of } \mathbf{R} \\
&= \lambda
\end{aligned}
$$

4. additive inverse

For every $\alpha \in \mathbf{C}$, write $\alpha = a + bi$ for some $a, b \in \mathbf{R}$, then we can find a unique $\beta = c + di \in \mathbf{C}$ such that $\alpha + \beta = 0$ by the following manner:

$$
\begin{aligned}
\alpha + \beta &= (a + bi) + (c + di) \\
&= (a + c) + (b + d)i && \text{by (1.1)} \\
&= 0 + 0i && \text{by assumption} \\
\Rightarrow (c, d) &= (-a, -b) && \text{by add.inv. of } \mathbf{R}
\end{aligned}
$$

5. multiplicative inverse

For every $\alpha \in \mathbf{C}$, write $\alpha = a + bi$ for some $a, b \in \mathbf{R}$ with $a \neq 0$ and $b \neq 0$, then we can find a unique $\beta = c + di \in \mathbf{C}$ such that $\alpha\beta = 1$ by the following manner:

$$
\begin{aligned}
\alpha\beta &= (a + bi)(c + di) \\
&= (ac - bd) + (ad + bc)i && \text{by (1.2)} \\
&= 1 + 0i && \text{by assumption}
\end{aligned}
$$

That is, $ac - bd = 1$ and $ad + bc = 0$. By solving these two equations for c and d, we have

$$
(c, d) = \left(\frac{a}{a^2 + b^2}, \ \frac{-b}{a^2 + b^2} \right).
$$

6. distributive property

For all $\alpha, \beta, \lambda \in \mathbf{C}$, write $\alpha = a + bi$, $\beta = c + di$, $\lambda = e + fi$ with some $a, b, c, d, e, f \in \mathbf{R}$, then

$$
\begin{aligned}
\lambda(\alpha + \beta) &= (e + fi)[(a + bi) + (c + di)] \\
&= (e + fi)[(a + c) + (b + d)i] && \text{by (1.1)} \\
&= [e(a + c) - f(b + d)] + [e(b + d) + f(a + c)]i && \text{by (1.2)} \\
&= (ea + ec - fb - fd) + (eb + ed + fa + fc)i && \text{by dist. of } \mathbf{R}
\end{aligned}
$$

$$
\begin{aligned}
\lambda\alpha + \lambda\beta &= [(e + fi)(a + bi)] + [(e + fi)(c + di)] \\
&= [(ea - fb) + (eb + fa)i] + [(ec - fd) + (ed + fc)i] && \text{by (1.2)} \\
&= (ea - fb + ec - fd) + (eb + fa + ed + fc)i && \text{by (1.1)} \\
&= \lambda(\alpha + \beta) && \text{by comm. of } \mathbf{R}
\end{aligned}
$$

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Chapter 2

Vector Space

2.1 Definition

- An **addition** on a set V is a function $V \times V \to V$ that maps each pair (u, v) to an element $u + v$.

- A **scalar multiplication** on a set V over a field $\mathbf{F}$ is a function $\mathbf{F} \times V \to V$ that maps each pair (λ, v) to an element λv.

- A **vector space** is a set V over a field $\mathbf{F}$ along with an addition on V and a scalar multiplication on V over $\mathbf{F}$ such that the following properties hold:

    1. commutativity

        $u + v = v + u$, for all $u, v \in V$.

    2. associativity

        $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$, for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$.

    3. additive identity

        There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

    4. additive inverse

        For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

    5. multiplicative identity

        $1v = v$ for all $v \in V$.

    6. distributive properties

        $a(u + v) = au + av$ and $(a + b)v = av + bv$, for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

The 5-th property is not requiring the existence of 1, as 1 implicitly exists in any field $\mathbf{F}$; instead, it states that based on the given operation of scalar multiplication, 1 should be the multiplicative identity for all vectors.

- $\mathbf{F}^S$ denotes the set of functions from a set S to a field $\mathbf{F}$.

- For $f, g \in \mathbf{F}^S$, the **sum** $f + g \in \mathbf{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

  for all $x \in S$.

- For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the **product** $\lambda f \in \mathbf{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

  for all $x \in S$.

Note

$\mathbf{F}^n$ is a special case of $\mathbf{F}^S$ because a n-tuple of number in $\mathbf{F}$ can be thought of as a function from $\{1, 2, \ldots, n\}$ to $\mathbf{F}$. That is, we can think of $\mathbf{F}^n$ as $\mathbf{F}^{\{1,2,\ldots,n\}}$. Take $\mathbf{R}^3$ for example.

- $\mathbf{R}^3$ can be seen as the set of functions from the set $\{1, 2, 3\}$ to $\mathbf{R}$. Here functions are what we normally understand as vectors. That is, the function $v \in \mathbf{R}^3$ has $1, 2, 3$ map to $x_1, x_2, x_3 \in \mathbf{R}$ correspondingly, so we can write $v = (x_1, x_2, x_3)$.

- For $f, g \in \mathbf{R}^3$, the sum $f + g \in \mathbf{R}^3$ is the function defined by

$$(f + g)(x) = f(x) + g(x), \quad \text{for all } x \in \{1, 2, 3\}.$$

  That is, for all $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3) \in \mathbf{R}^3$,

$$u + v = (x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3). \tag{2.1}$$

- For $\lambda \in \mathbf{R}$ and $f \in \mathbf{R}^3$, the product $\lambda f \in \mathbf{R}^3$ is the function defined by

$$(\lambda f)(x) = \lambda f(x), \quad \text{for all } x \in \{1, 2, 3\}.$$

  That is, for all $\lambda \in \mathbf{R}$ and $v = (x_1, x_2, x_3) \in \mathbf{R}^3$,

$$\lambda v = \lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3). \tag{2.2}$$

In conclusion, this provides an definition of addition and scalar multiplication for $\mathbf{F}^S$, and we can proceed to see that $\mathbf{F}^S$ is a vector space.

2.2 Theorem

2.2.1 Unique additive identity

A vector space has a unique additive identity.

Proof

Suppose to the contrary that there were a vector space V with distinct additive identities $0, 0' \in V$, then it would create a contradiction that $0 = 0'$ as follows:

$$
\begin{aligned}
0' &= 0' + 0 && \text{by add.iden. of } V \\
&= 0 + 0' && \text{by comm. of } V \\
&= 0 && \text{by add.iden. of } V
\end{aligned}
$$

2.2.2 Unique additive inverse

Every element in a vector space has a unique additive inverse.

Proof

Suppose to the contrary that there were a vector space V with some element $v \in V$ having distinct additive inverses $w, w' \in V$, then it would create a contradiction that $w = w'$ as follows:

$$
\begin{aligned}
w &= w + 0 && \text{by add.iden. of } V \\
&= w + (v + w') && \text{by add.inv. of } V \\
&= (w + v) + w' && \text{by asso. of } V \\
&= 0 + w' && \text{by add.inv of } V \\
&= w' + 0 && \text{by comm. of } V \\
&= w' && \text{by add.iden. of } V
\end{aligned}
$$

Note

As we show that additive inverses are unique, now the following notation makes sense. Let $v, w \in V$, then

- $-v$ denotes the additive inverse of v;

- $w - v$ is defined to be $w + (-v)$.

2.2.3 The number 0

For every vector space V over a field $\mathbf{F}$, $0v = \mathbf{0}$ for all $v \in V$.

Proof

To avoid confusion, mark the additive identity in $\mathbf{F}$ as 0, and the additive identity in V as $\mathbf{0}$.

$$
\begin{aligned}
0v &= (0 + 0)v && \text{by add.iden. of } \mathbf{F} \\
&= 0v + 0v && \text{by dist. of } V \\
\Rightarrow 0v + w &= 0v + 0v + w && \text{where } w \text{ is the additive inverse of } 0v \\
\Rightarrow \mathbf{0} &= 0v + \mathbf{0} && \text{by add.inv. of } V \\
&= 0v && \text{by add.iden. of } V
\end{aligned}
$$

2.2.4 The vector 0

For every vector space V over a field $\mathbf{F}$, $a\mathbf{0} = \mathbf{0}$ for every $a \in \mathbf{F}$.

Proof

To avoid confusion, mark the additive identity in $\mathbf{F}$ as 0, and the additive identity in V as $\mathbf{0}$.

$$
\begin{aligned}
a\mathbf{0} &= a(\mathbf{0} + \mathbf{0}) && \text{by add.iden. of } V \\
&= a\mathbf{0} + a\mathbf{0} && \text{by dist. of } V \\
\Rightarrow a\mathbf{0} + w &= a\mathbf{0} + a\mathbf{0} + w && \text{where } w \text{ is the additive inverse of } a\mathbf{0} \\
\Rightarrow \mathbf{0} &= a\mathbf{0} + \mathbf{0} && \text{by add.inv. of } V \\
&= a\mathbf{0} && \text{by add.iden. of } V
\end{aligned}
$$

2.3 Exercise

2.3.1 $\mathbf{R}^3$ a is vector space

Prove that $\mathbf{R}^3$ is a vector space (with the operations (2.1) and (2.2)).

Proof

1. commutativity

For all $u, v \in \mathbf{R}^3$,

$$
\begin{aligned}
u + v &= (x_1, x_2, x_3) + (y_1, y_2, y_3) \\
&= (x_1 + y_1, x_2 + y_2, x_3 + y_3) && \text{by (2.1)} \\
v + u &= (y_1, y_2, y_3) + (x_1, x_2, x_3) \\
&= (y_1 + x_1, y_2 + x_2, y_3 + x_3) && \text{by (2.1)} \\
&= v + u && \text{by comm. of } \mathbf{R}
\end{aligned}
$$

2. $\boxed{\text{associativity}}$

For all $u, v, w \in \mathbf{R}^3$,

$$
\begin{aligned}
(u + v) + w &= \big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) + (z_1, z_2, z_3) \\
&= (x_1 + y_1, x_2 + y_2, x_3 + y_3) + (z_1, z_2, z_3) && \text{by (2.1)} \\
&= \big((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, (x_3 + y_3) + z_3\big) && \text{by (2.1)} \\
u + (v + w) &= (x_1, x_2, x_3) + \big((y_1, y_2, y_3) + (z_1, z_2, z_3)\big) \\
&= (x_1, x_2, x_3) + (y_1 + z_1, y_2 + z_2, y_3 + z_3) && \text{by (2.1)} \\
&= \big(x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), x_3 + (y_3 + z_3)\big) && \text{by (2.1)} \\
&= (u + v) + w && \text{by asso. of } \mathbf{R}
\end{aligned}
$$

For all $a, b \in \mathbf{R}$ and all $v \in \mathbf{R}^3$,

$$
\begin{aligned}
(ab)v &= (ab)(y_1, y_2, y_3) \\
&= ((ab)y_1, (ab)y_2, (ab)y_3) && \text{by (2.2)} \\
a(bv) &= a\big(b(y_1, y_2, y_3)\big) \\
&= a(by_1, by_2, by_3) && \text{by (2.2)} \\
&= (a(by_1), a(by_2), a(by_3)) && \text{by (2.2)} \\
&= (ab)v && \text{by asso. of } \mathbf{R}
\end{aligned}
$$

3. $\boxed{\text{additive identity}}$

We show the existence of zero vector by finding one. Let the zero vector be $0 = (0, 0, 0) \in \mathbf{R}^3$ such that for all $v \in \mathbf{R_3}$,

$$
\begin{aligned}
& v + 0 = v, \\
\Rightarrow\quad & (y_1 + 0, y_2 + 0, y_3 + 0) = (y_1, y_2, y_3) && \text{by (2.1)} \\
\Rightarrow\quad & (0, 0, 0) = (0, 0, 0) && \text{by add.iden. of } \mathbf{R}
\end{aligned}
$$

4. $\boxed{\text{additive inverse}}$

   We show the existence of additive inverse for every vector by finding one for each of them.
   For every $u \in \mathbf{R}^3$, let $w \in \mathbf{R}^3$ be the vector such that

   $$
   \begin{aligned}
   u + w &= 0, \\
   \Rightarrow \quad (x_1 + z_1, x_2 + z_2, x_3 + z_3) &= (0, 0, 0) && \text{by (2.1) and } 0 \in \mathbf{R}^3 \\
   \Rightarrow \quad (z_1, z_2, z_3) &= (-x_1, -x_2, -x_3) && \text{by add.inv. of } \mathbf{R}
   \end{aligned}
   $$

5. $\boxed{\text{multiplicative identity}}$

   For all $v \in \mathbf{R}^3$,

   $$
   \begin{aligned}
   1v &= 1(y_1, y_2, y_3) \\
   &= (1y_1, 1y_2, 1y_3) && \text{by (2.2)} \\
   &= (y_1, y_2, y_3) && \text{by mul.iden. of } \mathbf{R} \\
   &= v
   \end{aligned}
   $$

6. $\boxed{\text{distributive properties}}$

   For all $a \in \mathbf{R}$ and all $u, v \in \mathbf{R}^3$,

   $$
   \begin{aligned}
   a(u + v) &= a\big((x_1, x_2, x_3) + (y_1, y_2, y_3)\big) \\
   &= a(x_1 + y_1, x_2 + y_2, x_3 + y_3) && \text{by (2.1)} \\
   &= (a(x_1 + y_1), a(x_2 + y_2), a(x_3 + y_3)) && \text{by (2.2)} \\
   au + av &= a(x_1, x_2, x_3) + a(y_1, y_2, y_3) \\
   &= (ax_1, ax_2, ax_3) + (ay_1, ay_2, ay_3) && \text{by (2.2)} \\
   &= (ax_1 + ay_1, ax_2 + ay_2, ax_3 + ay_3) && \text{by (2.1)} \\
   &= a(u + v) && \text{by dist. of } \mathbf{R}
   \end{aligned}
   $$

   For all $a, b \in \mathbf{R}$ and all $v \in \mathbf{R}^3$,

   $$
   \begin{aligned}
   (a + b)v &= (a + b)(y_1, y_2, y_3) \\
   &= ((a + b)y_1, (a + b)y_2, (a + b)y_3) && \text{by (2.2)} \\
   av + bv &= a(y_1, y_2, y_3) + b(y_1, y_2, y_3) \\
   &= (ay_1, ay_2, ay_3) + (by_1, by_2, by_3) && \text{by (2.2)} \\
   &= (ay_1 + by_1, ay_2 + by_2, ay_3 + by_3) && \text{by (2.1)} \\
   &= (a + b)v && \text{by dist. of } \mathbf{R}
   \end{aligned}
   $$

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Chapter 3

Subspace

3.1 Definition

For vector space V over a field $\mathbf{F}$, a subset U of V is called a ***subspace*** of V if U is also a vector space over $\mathbf{F}$, using the same addition and scalar multiplication as on V.

3.2 Theorem

3.2.1 Conditions for a subspace

A subset U of V is a subspace of V if and only if U satisfies the following three conditions:

1. $\boxed{\text{additive identity}}$

    $0 \in U$.

2. $\boxed{\text{closed under addition}}$

    For all $u, w \in U$, $u + w \in U$.

3. $\boxed{\text{closed under scalar multiplication}}$

    For all $a \in \mathbf{F}$ and all $u \in U$, $au \in U$.

Note

If one were to carelessly exclude the first condition, thinking that it should have been implicitly satisfied by the third condition as follows:

Choose $a = 0$ and any $u \in U$, then $au = 0 \in U$ exists. (Wrong!)

While 0 always exists in a field, U might be an empty set thus u cannot be chosen. The core issue is that the 2-nd and 3-rd conditions alone fail to rule out the case $U = \varnothing$, as they only states that *all* vectors are closed under addition and scalar multiplication, which do not require the *existence* of them.

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Proof

Given a subset U of V over $\mathbf{F}$, we aim to show that both directions are true.

- $\boxed{\Longrightarrow}$ Suppose that U is a subspace of V, that is, U is a vector space over the same field. We aim to show that U satisfies the three conditions.

    - The 1-st condition is satisfied by the definition of the vector space that U has a additive identity called 0.

    - The 2-nd and 3-rd conditions are satisfied by the definitions of operations on U, as addition is the function of $U \times U \to U$ and scalar multiplication is the function of $\mathbf{F} \times U \to U$. This implies that all sums between vectors and all products between numbers and vectors, remain vectors in U.

- $\boxed{\Longleftarrow}$ Suppose that U satisfies the three conditions. We aim to show that U is a vector space over $\mathbf{F}$.

    First of all, the operations of addition $U \times U \to U$ and scalar multiplication $\mathbf{F} \times U \to U$ have to make sense. This is ensured by the 2-nd and 3-rd conditions. Now we verify the definition of the vector space.

    - The existence of additive identity is ensured by the 1-st condition.

    - For all $u \in U$, the additive inverse $(-1)u = -u$ exists in U by the 3-rd condition.

    - The other properties (i.e. commutativity, associativity, multiplicative identity, distributive properties) are automatically satisfied for U because they hold on a larger set V. This can be verified easily.

    Thus, U is a vector space and hence is a subspace of V.

3.3 Exercise

3.3.1 Continuous real functions are subspaces

The set of continuous real-valued functions on the interval $[0,1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

Proof

Let $\mathcal{C}([0,1], \mathbf{R})$ denote the set of all continuous real-valued functions on the interval $[0,1]$. Specifically,

$$\mathcal{C}([0,1], \mathbf{R}) = \{f : [0,1] \to \mathbf{R} \mid f \text{ is continuous}\}.$$

We aim to prove that $\mathcal{C}([0,1], \mathbf{R})$ is a subspace of $\mathbf{R}^{[0,1]}$. Recall that we call a function f continuous if f is continuous on all points of its domain. That is,

$$\lim_{t \to x} f(t) = f(x), \quad \forall x \in [0,1].$$

- additive identity

  $0 \in \mathcal{C}([0,1], \mathbf{R})$. The function that is identically 0 is continuous because it is constant.

- closed under addition

  Suppose $f, g \in \mathcal{C}([0,1], \mathbf{R})$. Then for all $x \in [0,1]$,

$$\begin{aligned}
\lim_{t \to x}(f+g)(t) &= \lim_{t \to x}(f(t) + g(t)) \\
&= \lim_{t \to x} f(t) + \lim_{t \to x} g(t) \\
&= f(x) + g(x) \\
&= (f+g)(x).
\end{aligned}$$

  Thus, $f + g \in \mathcal{C}([0,1], \mathbf{R})$.

- closed under scalar multiplication

  Suppose $\lambda \in R$ and $f \in \mathcal{C}([0,1], \mathbf{R})$. Then for all $x \in [0,1]$,

$$\begin{aligned}
\lim_{t \to x}(\lambda f)(t) &= \lim_{t \to x} \lambda f(t) \\
&= \lambda \lim_{t \to x} f(t) \\
&= \lambda f(x) \\
&= (\lambda f)(x).
\end{aligned}$$

  Thus, $\lambda f \in \mathcal{C}([0,1], \mathbf{R})$.

3.3.2 Differentiable real functions are subspaces

The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

Proof

Let $\mathcal{D}(\mathbf{R}, \mathbf{R})$ denote the set of all differentiable real-valued functions on $\mathbf{R}$. Specifically,

$$\mathcal{D}(\mathbf{R}, \mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} \mid f \text{ is differentiable}\}.$$

We aim to prove that $\mathcal{D}(\mathbf{R}, \mathbf{R})$ is a subspace of $\mathbf{R}^{\mathbf{R}}$. Recall that we call a function f differentiable if f is differentible on all points of its domain. That is,

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

exists for all $x \in \mathbf{R}$.

- additive identity

  $0 \in \mathcal{D}(\mathbf{R}, \mathbf{R})$. The function that is identically 0 is differentiable because the derivative of 0 is 0.

- closed under addition

  Suppose $f, g \in \mathcal{D}(\mathbf{R}, \mathbf{R})$. Then for all $x \in \mathbf{R}$,

$$\begin{aligned}
(f + g)'(x) &= \lim_{t \to x} \frac{(f + g)(t) - (f + g)(x)}{t - x} \\
&= \lim_{t \to x} \frac{f(t) + g(t) - f(x) - g(x)}{t - x} \\
&= \lim_{t \to x} \left(\frac{f(t) - f(x)}{t - x} + \frac{g(t) - g(x)}{t - x} \right) \\
&= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} + \lim_{t \to x} \frac{g(t) - g(x)}{t - x} \\
&= f'(x) + g'(x).
\end{aligned}$$

  Thus, $(f + g)'(x)$ exists for all $x \in \mathbf{R}$, then $f + g \in \mathcal{D}(\mathbf{R}, \mathbf{R})$.

- closed under scalar multiplication

  Suppose $\lambda \in R$ and $f \in \mathcal{D}(\mathbf{R}, \mathbf{R})$. Then for all $x \in \mathbf{R}$,

$$\begin{aligned}
(\lambda f)'(x) &= \lim_{t \to x} \frac{(\lambda f)(t) - (\lambda f)(x)}{t - x} \\
&= \lim_{t \to x} \frac{\lambda f(t) - \lambda f(x)}{t - x} \\
&= \lambda \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \\
&= \lambda f'(t).
\end{aligned}$$

  Thus, $(\lambda f)'(x)$ exists for all $x \in \mathbf{R}$, then $\lambda f \in \mathcal{D}(\mathbf{R}, \mathbf{R})$.

3.3.3   1.C.9 Periodic functions are not necessarily subspaces

A function $f : \mathbf{R} \to \mathbf{R}$ is called **_periodic_** if there exists a positive number p such that $f(x) = f(x + p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$?

Solution

Let $\mathcal{T}(\mathbf{R}, \mathbf{R})$ denote the set of all periodic functions on $\mathbf{R}$. Specifically,

$$\mathcal{T}(\mathbf{R}, \mathbf{R}) = \{f : \mathbf{R} \to \mathbf{R} \mid f \text{ is periodic}\}.$$

We aim to prove that $\mathcal{T}(\mathbf{R}, \mathbf{R})$ is not a subspace of $\mathbf{R}^{\mathbf{R}}$.

- **additive identity**

  $0 \in \mathcal{T}(\mathbf{R}, \mathbf{R})$. The function that is identically 0 is periodic because for any constant function $f(x) = c$, we have $f(x) = f(x + p) = c$ for any $p > 0$.

- **not closed under addition**

  Suppose $f, g \in \mathcal{T}(\mathbf{R}, \mathbf{R})$ with periods 1 and $\sqrt{2}$ correspondingly. Suppose to the contrary that their sum were periodic with period $p > 0$, then both would have to repeat after p. That is

  $$p = m \cdot 1 = n \cdot \sqrt{2}$$

  for some positive integers m and n. This would follow that

  $$\sqrt{2} = \frac{m}{n}$$

  is rational, which is a contradiction.[1]

- **closed under scalar multiplication**

  Suppose $\lambda \in \mathbf{R}$ and $f \in \mathcal{T}(\mathbf{R}, \mathbf{R})$ with period $p > 0$. Then

  $$(\lambda f)(x + p) = \lambda f(x + p) = \lambda f(x) = (\lambda f)(x).$$

  Thus, $\lambda f \in \mathcal{T}(\mathbf{R}, \mathbf{R})$.

3.3.4   1.C.10 Intersection of subspaces are a subspace

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V.

Solution

- **additive identity**

  $0 \in U_1 \cap U_2$ since $0 \in U_1$ and $0 \in U_2$.

- **closed under addition**

---

[1]Note that when their periods p_1 and p_2 are such that $p_1/p_2 \in \mathbf{Q}$, then $f + g$ is periodic.

Suppose $u, v \in U_1 \cap U_2$. Then $u + v \in U_1$ since $u, v \in U_1$ and U_1 is closed under addition; similarly, $u + v \in U_2$. Thus $u + v \in U_1 \cap U_2$.

- closed under scalar multiplication

  Suppose $a \in \mathbf{F}$ and $v \in U_1 \cap U_2$. Then $av \in U_1$ since $v \in U_1$ and U_1 is closed under scalar multiplication; similarly, $av \in U_2$. Thus $av \in U_1 \cap U_2$.

3.3.5   1.C.10 Union of subspaces

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution

Given that U_1 and U_2 are subspaces of V. We aim to prove that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

- $\boxed{\implies}$

  Suppose to the contrary that $U_1 \cup U_2$ were subspaces of V and none of them were not contained in the other. That is, $U_1 \setminus U_2 \neq \varnothing$ and $U_2 \setminus U_1 \neq \varnothing$. Let $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$. Since $u_1, u_2 \in U_1 \cup U_2$, it should have followed that $u_1 + u_2 \in U_1 \cup U_2$. Then one of the following two cases should have been true.

  (a) $u_1 + u_2 \in U_1$. Then subtraction in U_1 would give $(u_1 + u_2) - u_1 = u_2 \in U_1$, which cannot be true since $u_2 \in U_2 \setminus U_1$.

  (b) $u_1 + u_2 \in U_2$. Then subtraction in U_2 would give $(u_1 + u_2) - u_2 = u_1 \in U_2$, which also cannot be true since $u_1 \in U_1 \setminus U_2$.

  Thus, $u_1 + u_2 \notin U_1 \cup U_2$, we reach a contradiction.

- $\boxed{\impliedby}$

  Suppose that one of the subspaces is contained in the other. If $U_1 \subseteq U_2$, then $U_1 \cup U_2 = U_2$ is a subspace of V; if $U_2 \subseteq U_1$, then $U_1 \cup U_2 = U_1$ is also a subspace of V.

Chapter 4

Polynomial

4.1 Definition

- A function $p : \mathbf{F} \to \mathbf{F}$ is called a ***polynomial*** with coefficients in $\mathbf{F}$ if there exist $a_0, \ldots a_m \in \mathbf{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
  for all $z \in \mathbf{F}$.

- $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

- A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have ***degree*** m if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
  for all $z \in \mathbf{F}$.

- If p has degree m, we write $\deg p = m$.

- The polynomial that is identically 0 is said to have degree $-\infty$.

- For a nonnegative integer m, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.[1]

4.2 Theorem

4.2.1 $\mathcal{P}(\mathbf{F})$ is a vector space

With the usual operations of addition and scalar multiplication, $\mathcal{P}(\mathbf{F})$ is a vector space over $\mathbf{F}$. In other words, $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$.

---

[1]Note that we use the convention $m > -\infty$, so $0 \in \mathcal{P}_m(\mathbf{F})$ for every nonnegative integer m.

Note

Recall that $\mathbf{F}^{\mathbf{F}}$ denotes the set of functions from $\mathbf{F}$ to $\mathbf{F}$. And for $f, g \in \mathbf{F}^{\mathbf{F}}$ and $\lambda \in \mathbf{F}$, the sum $f + g \in \mathbf{F}^{\mathbf{F}}$ and the product $\lambda f \in \mathbf{F}^{\mathbf{F}}$ are defined by

$$(f + g)(z) = f(z) + g(z), \quad (\lambda f)(z) = \lambda f(z)$$

for all $z \in \mathbf{F}$.

Proof

Let's show that $\mathcal{P}(\mathbf{F})$ is a subspace of $\mathbf{F}^{\mathbf{F}}$. First of all, $\mathcal{P}(\mathbf{F})$ is a subset of $\mathbf{F}^{\mathbf{F}}$ because every polynomial is a function from $\mathbf{F}$ to $\mathbf{F}$.

- $\boxed{\text{additive identity}}$

  $0 \in \mathcal{P}(\mathbf{F})$ because 0 is a polynomial of degree $-\infty$ by definition.

- $\boxed{\text{closed under addition}}$

  For all $p, q \in \mathcal{P}(\mathbf{F})$, we can write

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m, \quad \text{for some nonnegative integer } m,$$
$$q(z) = c_0 + a_1 z + \cdots + c_n z^n, \quad \text{for some nonnegative integer } n.$$

  Without loss of generality, assume $m \geq n$, then

$$\begin{aligned}
(p + q)(z) &= p(z) + q(z) \\
&= (a_0 + a_1 z + \cdots + a_m z^m) + (c_0 + c_1 z + \cdots + c_n z^n) \\
&= (a_0 + c_0) + (a_1 + c_1)z + \cdots + (a_n + c_n)z_n + \cdots + a_m z_m
\end{aligned}$$

  Thus, $p + q \in \mathcal{P}(\mathbf{F})$.

- $\boxed{\text{closed under scalar multiplication}}$

  For all $\lambda \in \mathbf{F}$ and $p \in \mathcal{P}(\mathbf{F})$,

$$\begin{aligned}
(\lambda p)(z) &= \lambda p(z) \\
&= \lambda(a_0 + a_1 z + \cdots + a_m z^m) \\
&= \lambda a_0 + \lambda a_1 z + \cdots + \lambda a_m z^m
\end{aligned}$$

  Thus, $\lambda p \in \mathcal{P}(\mathbf{F})$.

Chapter 5

Span

5.1 Definition

- A **_linear combination_** of a list $v_1, v_2, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

  where $a_1, \ldots, a_m \in \mathbf{F}$.

- The set of all linear combination of a list $v_1, \ldots, v_m$ of vectors in V is called the **_span_** of $v_1, \ldots, v_m$, denoted $\mathrm{span}(v_1, \ldots, v_m)$. In other words,

$$\mathrm{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbf{F}\}.$$

- The span of the empty list is defined to be $\{0\}$. That is,

$$\mathrm{span}() = \{0\}.$$

- If $\mathrm{span}(v_1, \ldots, v_m)$ equals V, we say that the list $v_1, \ldots, v_m$ **_spans_** V.

5.2 Theorem

5.2.1 Smallest subspace

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof

Given a list $v_1, \ldots, v_m$ of vectors in V, we aim to show that $\mathrm{span}(v_1, \ldots, v_m)$ is a subspace of V, then that it contains $v_1, \ldots, v_m$, finally that it is the smallest subspace of V containing

$v_1, \ldots, v_m$.

1. $\text{span}(v_1, \ldots, v_m)$ is a subspace of V by satisfying the following conditions of subspace.

   - $\boxed{\text{span}(v_1, \ldots, v_m) \subseteq V}$

     For all $u \in \text{span}(v_1, \ldots, v_m)$,

     $$u = a_1 v_1 + \cdots + a_m v_m, \quad \text{for some } a_1, \ldots, a_m.$$

     And since each $v_i \in V$ and V is closed under addition and scalar multiplication, it follows that $u \in V$.

   - $\boxed{\text{additive identity}}$

     $$0 v_1 + \cdots + 0 v_m = 0 \quad \in \text{span}(v_1, \ldots, v_m).$$

   - $\boxed{\text{closed under addition}}$

     For all $u, w \in \text{span}(v_1, \ldots, v_m)$,

     $$u + w = (a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m)$$
     $$= (a_1 + c_1) v_1 + \cdots (a_m + c_m) v_m \quad \in \text{span}(v_1, \ldots, v_m).$$

   - $\boxed{\text{closed under scalar multiplication}}$

     For all $\lambda \in \mathbf{F}$ and all $u \in \text{span}(v_1, \ldots, v_m)$,

     $$\lambda u = \lambda(a_1 v_1 + \cdots + a_m v_m)$$
     $$= (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \quad \in \text{span}(v_1, \ldots, v_m).$$

2. For each $j = 1, \ldots, m$, v_j is clearly a linear combination of $v_1, \ldots, v_m$ (by setting all a's to 0 except $a_j = 1$).

3. For every subspace U of V containing $v_1, \ldots, v_m$, U should contain all the linear combination of $v_1, \ldots, v_m$. Suppose to the contrary that U did not contain some linear combination of them, U would not be a vector space since it would not be closed under addition or scalar multiplication. Thus, there is no subspace of V containing $v_1, \ldots, v_m$ smaller than $\text{span}(v_1, \ldots, v_m)$.

Chapter 6

Linear Independence

6.1 Definition

- A list $v_1, \ldots, v_m$ of vectors in V is called ***linearly independent*** if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$. In other words, for any scalars $a_1, \ldots, a_m \in \mathbf{F}$, if $a_1 v_1 + \cdots + a_m v_m = 0$, then $a_1 = \cdots = a_m = 0$.

- The empty list () is declared to be linearly independent.

- A list of vectors in V is called ***linearly dependent*** if it is not linearly independent. In other words, there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.

6.2 Theorem

6.2.1 Unique linear combination

For all $v_1, \ldots, v_m \in V$, the list $v_1, \ldots, v_m$ is linearly independent if and only if each vector in $\operatorname{span}(v_1, \ldots, v_m)$ has only one representation as a linear combination of $v_1, \ldots, v_m$.

Proof

For all $u \in \operatorname{span}(v_1, \ldots, v_m)$, we can write

$$u = a_1 v_1 + \cdots + a_m v_m, \quad \text{for some } a_1, \ldots a_m \in \mathbf{F}.$$

Let's consider another set of coefficients

$$u = c_1 v_1 + \cdots + c_m v_m, \quad \text{for some } c_1, \ldots c_m \in \mathbf{F}.$$

Subtracting the two equations gives

$$0 = (a_1 - c_1)v_1 + \cdots + (a_m - c_m)v_m.$$

- $\boxed{\Longrightarrow}$ Suppose that $v_1, \ldots, v_m$ is linearly independent, then each $a_j - c_j$ should equal 0. Thus u has only one unique linear combination of $v_1, \ldots, v_m$.

- $\boxed{\Longleftarrow}$ Suppose that u has only one unique linear combination of $v_1, \ldots, v_m$, then each $a_j = c_j$ and $v_1, \ldots, v_m$ is linearly independent. If $v_1, \ldots, v_m$ were linearly dependent, then there would exist $a_j - c_j \neq 0$, which leads to multiple representations of u.

6.2.2 Removal keeps linear independence

If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.

Proof

Given a list $v_1, \ldots, v_m$ of linearly independent vectors in V. Without loss of generality, suppose we remove some vectors at the end.[1] That is, let $I = \{1, \ldots, k\}$ be the set of indicies we keep, and $J = \{k + 1, \ldots, m\}$ be the set of indicies we remove, with $1 \leq k < m$.[2]

Suppose to the contrary that the list after removal were linearly dependent. Then there would be $a_1, \ldots a_k \in \mathbf{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_k v_k = 0.$$

By filling up with the removed vectors, we have

$$a_1 v_1 + \cdots + a_k v_k + 0 v_{k+1} + \cdots + 0 v_m = 0$$

That is, there exists a non-trivial linear combination of $v_1, \ldots, v_m$ that equals 0, which contradicts that $v_1, \ldots, v_m$ is linearly independent.

6.2.3 Linearly dependent vector causes linearly dependent list

If some vector in a list of vectors in V is a linear combination of the other vectors, then the list is linearly dependent.

---

[1]The order of vector does not matter here.
[2]For the case where we remove all vectors, $k = 0$, the empty remaining list is declared linearly independent.

Proof

Given a list $v_1, \ldots, v_m$ of vectors in V where $m > 1$. Without loss of generality, we assume that the first vector v_1 is a linear combination of the other vectors $v_2, \ldots, v_m$. Then there exist $a_2, \ldots, a_m \in \mathbf{F}$ such that

$$v_1 = a_2 v_2 + \cdots + a_m v_m$$

$$\Rightarrow -v_1 + a_2 v_2 + \cdots + a_m v_m = 0.$$

Thus, $v_1, \ldots, v_m$ is linearly dependent.

6.2.4   Linearly dependent list has a vector to blame

The converse of the previous statement is also true. Specifically, if a list of vectors is linearly dependent, then some vector in the list is a linear combination of the other vectors.

Proof

This is more tedious since we have to consider several edge cases.

- For the case where the list is of length 0, the empty list is declared to be linearly independent, which does not satisfy the condition, so we can skip it.

- For the case where the list is of length 1, we are going to show that for this list to be linearly dependent, that one vector has to be the zero vector 0. And 0 is indeed a linear combination of *the other* vectors, by definition of span() = {0}.

  Let $v_1 \in V$ be the linearly dependent list of length 1. There exists a scalar $a_1 \neq 0$ in $\mathbf{F}$ such that

  $$a_1 v_1 = 0.$$

  Thus, $v_1 = 0$.

- For the cases where the list is of length greater than 1, let the list be $v_1, \ldots, v_m$ in V where $m > 1$. Suppose the list is linearly dependent, then there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all zeros, such that

  $$a_1 v_1 + \cdots + a_m v_m = 0.$$

  Without loss of generality, assume that $a_1 \neq 0$. Then we can arrange the equation as

  $$v_1 = -\frac{a_2}{a_1} v_2 - \cdots - \frac{a_m}{a_1} v_m.$$

  Thus, we find v_1 as a linear combination of the other vectors.[3]

---

[3]Generally, each v_j with $a_j \neq 0$ is a linear combination of the other vectors since we can arrange the equation similarly.

Note

From this theorem and the previous one (6.2.3), whether a list of vector is linearly dependent could be defined as whether there is *one of the vector* being a linear combination of *the other vectors*. But it is not so precise to think like this way due to the dubious meaning of "one of the vector" and "the other vectors". In fact, we do not even consider the edge cases in 6.2.3 since it seems like the statement assumes the existence of that vector and the other vectors. In addition, this way of phrasing make one of the vector guilty, while the original definition only states that there is a non-trivial linear combination equal to 0.

6.2.5 The zero vector 0 causes linearly dependent list

Every list of vectors in V containing the zero vector is linearly dependent.

Proof

Given a list $v_1, \ldots, v_m$ of vectors in V, where $m \geq 1$, suppose one of them is the zero vector. Without loss of generality, assume v_1 is the zero vector. Then v_1 can be written as a trivial linear combination of the other vectors as

$$v_1 = 0v_2 + \cdots + 0v_m.$$

Then by 6.2.3, $v_1, \ldots, v_m$ is linearly dependent.

6.2.6 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, \ldots, m\}$ such that the following hold:

(a) $v_j \in \text{span}(v_1, \ldots, v_{j-1})$;

(b) if the j-th term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\text{span}(v_1, \ldots, v_m)$.

Proof

(a) Because the list is linearly dependent, there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Let j be the largest element of $\{1, \ldots, m\}$ such that $a_j \neq 0$. Then

$$a_1 v_1 + \cdots + a_j v_j = 0$$

$$\Rightarrow \quad v_j = -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1} \tag{6.1}$$

Thus, (a) is true.

(b) • $\boxed{\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(v_1, \ldots, v_m \setminus v_j)}$

Suppose $u \in \operatorname{span}(v_1, \ldots, v_m)$. Then there exist $c_1, \ldots, c_m \in \mathbf{F}$ such that

$$u = c_1 v_1 + \cdots + c_j v_j + \cdots + c_m v_m.$$

By replacing v_j with (6.1), we represent u as a linear combination of the list $v_1, \ldots, v_m$ without v_j. Thus $u \in \operatorname{span}(v_1, \ldots, v_m \setminus v_j)$.

• $\boxed{\operatorname{span}(v_1, \ldots, v_m \setminus v_j) \subseteq \operatorname{span}(v_1, \ldots, v_m)}$

This is trivial since for every u as a linear combination of the list $v_1, \ldots, v_m \setminus v_j$, adding $0v_j$ gives a linear combination of $v_1, \ldots, v_m$.

6.3 Exercise

6.3.1 2.A.8

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $\lambda \in \mathbf{F}$. Prove that if $\lambda \neq 0$, then $\lambda v_1, \ldots, \lambda v_m$ is linearly independent.

Proof

Suppose $\lambda \neq 0$. Let $a_1, \ldots, a_m \in \mathbf{F}$ be such that

$$a_1 \lambda v_1 + \cdots + a_m \lambda v_m = 0$$

$$\Rightarrow a_1 v_1 + \cdots + a_m v_m = 0.$$

Since $v_1, \ldots, v_m$ is linearly independent, it follows that $a_1 = \cdots = a_m = 0$. Thus $\lambda v_1, \ldots, \lambda v_m$ is linearly independent.

6.3.2 2.A.10

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_m)$.

Proof

Suppose $v_1 + w, \ldots, v_m + w$ is linearly dependent, then there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0$$

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$$\Rightarrow \quad a_1 v_1 + \cdots + a_m v_m = -(a_1 + \cdots + a_m)w$$

Since $v_1, \ldots, v_m$ is linearly independent, the linear combination on the left hand side cannot be zero, thus $a_1 + \cdots + a_m \neq 0$ and $w \neq 0$. Then we can arrange the equation as

$$w = -\frac{a_1}{a_1 + \cdots + a_m} v_1 - \cdots - \frac{a_m}{a_1 + \cdots + a_m} v_m.$$

Therefore, $w \in \mathrm{span}(v_1, \ldots, v_m)$.

6.3.3   2.A.11 Extension of linearly independent list

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Show that $v_1, \ldots, v_m, w$ is linearly independent if and only if $w \notin \mathrm{span}(v_1, \ldots, v_m)$.

Proof

Given a linearly independent list $v_1, \ldots, v_m \in V$ and $w \in V$, we aim to show that both directions are true.

- $\boxed{\Longrightarrow}$ Suppose to the contrary that $v_1, \ldots, v_m, w$ were linearly independent and $w \in \mathrm{span}(v_1, \ldots, v_m)$. Then by 6.2.3, because w is a linear combination of the other vectors, $v_1, \ldots, v_m, w$ would be linearly dependent, which is a contradiction.

- $\boxed{\Longleftarrow}$ Suppose to the contrary that $w \notin \mathrm{span}(v_1, \ldots, v_m)$ and $v_1, \ldots, v_m, w$ were linearly dependent, then there would exist $c_1, \ldots, c_m, c_w \in \mathbf{F}$, not all 0, such that

$$c_1 v_1 + \cdots + c_m v_m + c_w w = 0.$$

  - For the case where $c_w = 0$, then

$$c_1 v_1 + \cdots + c_m v_m = 0$$

  Thus, $v_1, \ldots, v_m$ would be linearly dependent, which is a contradiction.

  - For the case where $c_w \neq 0$, then we could arrange the equation so that w is a linear combination of $v_1, \ldots, v_m$, which contradicts that $w \notin \mathrm{span}(v_1, \ldots, v_m)$.

Chapter 7

Finite-dimensional

7.1 Definition

- A vector space is called ***finite-dimensional*** if some list of vectors in it spans the space.[1]

- A vector space is called ***infinite-dimensional*** if it is not finite-dimensional.

7.2 Theorem

7.2.1 $\mathbf{F}^n$ is finite-dimensional

$\mathbf{F}^n$ is a finite-dimensional vector space for every positive integer n.

Proof

For each $j = 1, \ldots, n$, define $e_j \in \mathbf{F}^n$ such that it has 1 at the j-th coordinate and 0 otherwise. We aim to show that $e_1, \ldots, e_n$ spans $\mathbf{F}^n$.

- $\boxed{\mathbf{F}^n \subseteq \operatorname{span}(e_1, \ldots, e_n)}$

  For every vector $v = (x_1, \ldots, x_n) \in \mathbf{F}^n$,

  $$(x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n.$$

  Thus $v \in \operatorname{span}(e_1, \ldots, e_n)$.

- $\boxed{\operatorname{span}(e_1, \ldots, e_n) \subseteq \mathbf{F}^n}$

  For every vector $u \in \operatorname{span}(e_1, \ldots, e_n)$, there exist $a_1, \ldots, a_n \in \mathbf{F}$ such that

  $$u = a_1 e_1 + \cdots + a_n e_n = (a_1, a_2, \ldots, a_n).$$

---

[1]Note that by definition lists have finite length.

Thus $u \in \mathbf{F}^n$.

7.2.2 $\mathcal{P}_m(\mathbf{F})$ is finite-dimensional

$\mathcal{P}_m(\mathbf{F})$ is a finite-dimensional vector space for each nonnegative integer m.

Proof

It is trivial that $\mathcal{P}_m(\mathbf{F}) = \mathrm{span}(1, z, \ldots, z^m)$, where each z^k denotes a polynomial.

7.2.3 $\mathcal{P}(\mathbf{F})$ is infinite-dimensional

$\mathcal{P}(\mathbf{F})$ is a infinite-dimensional vector space.

Proof

Suppose to the contrary that there were a spanning list of polynomials $p_1, \ldots, p_n \in \mathcal{P}(\mathbf{F})$. Let m be the higest degree of those polynomials. Then every polynomial in $\mathrm{span}(p_1, \ldots, p_n)$ would only have degree at most m. Thus z^{m+1} would not be in $\mathrm{span}(p_1, \ldots, p_n)$, which is a contradiction.

7.2.4 Length of linearly independent list $\leq$ length of spanning list

In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.[2]

Proof

Suppose $u_1, \ldots, u_m$ is linearly independent in V, and $w_1, \ldots, w_n$ spans V. We aim to prove that $m \leq n$, by observing the following procedure of adding u and removing w one at a time. At the end we will see that all u's have been added and we does not run out of w's to remove.

> Let B be the list $w_1, \ldots w_n$, and for each iteration j from 1 to m,
>
> 1. insert u_j into B such that u_j becomes the j-th element of B;
>
> 2. remove one v from B, where v is in the span of all the preceding elements.

At the end of each j-th iteration, we claim that the following loop invariants $\mathcal{I}(j)$ stay true.

(a) B spans V;

(b) B looks like this:

$$u_1, \ldots, u_j, \text{ remaining } w\text{'s} \quad (\text{total length } m)$$

---
[2]We require the vector space to be finite-dimensional so that the existence of a spanning list is granted.

Let's verify the loop invariants.

- First we aim to show that $\mathcal{I}(1)$ holds. At the start of the 1-st iteration, B is

$$w_1, \ldots, w_n \quad \text{(total length } n)$$

  After the insertion, B becomes

$$u_1, w_1, \ldots, w_n \quad \text{(total length } n+1)$$

  Because $w_1, \ldots, w_n$ spans the space, $u_1 \in V$ must be a linear combination of $w_1, \ldots, w_n$. Thus right now B is linearly dependent. And based on Linear Dependence Lemma, we can always find an element v to remove such that, v is in the span of all the preceding elements, and after the removal the span of the remaining list is still V. The invariant (a) is satisfied.

  Furthermore, the removal does not choose u_1 because if u_1 were to be chosen, it would be that $u_1 \in \text{span}() = \{0\}$, which contradicts that $u_1, \ldots, u_m$ is linearly independent.[3] Thus, one of the w's is choosen and removed. B now looks like

$$u_1, \text{ remaining } w\text{'s} \quad \text{(total length } n)$$

  The invariant (b) is satisfied.

- Suppose $\mathcal{I}(j-1)$ is true for some $j \in \{2, \ldots, m\}$, we aim to show that $\mathcal{I}(j)$ remains true. Now we have the spanning list B as

$$u_1, \ldots, u_{j-1}, \text{ remaining } w\text{'s} \quad \text{(total length } n)$$

  After the insertion, B becomes linearly dependent by 6.2.3. Now B looks like

$$u_1, \ldots, u_j, \text{ remaining } w\text{'s} \quad \text{(total length } n+1)$$

  For the removal, based on Linear Dependence Lemma, there exists an element v to be chosen, and after the removal B still spans V. The loop invariant (a) is satisfied. And because $u_1, \ldots, u_j$ is linearly independent, any u will not be choosen, as it cannot be in the span of the other u's. Thus the loop invariant (b) is satisfied.

We have shown that $\mathcal{I}(j)$ stays true throughout the whole loop, so the algorithm is valid. Therefore, in each iteration, there is always a w to be choosen and removed, so the number of w's must be at least m. That is, $m \leq n$.

---

[3]Recall that every list containing 0 is linearly dependent by 6.2.5.

7.2.5 Finite-dimensional subspaces

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof

Suppose V is finite-dimensional and U is a subspace of V. We aim to construct a spanning list B of U.

> B starts as an empty list.
>
> **Step 1**
>
> If $U = \{0\}$, then U is finite-dimensional and we are done; or else, add a nonzero vector $v_1 \in U$ into the list B.
>
> **Step j**
>
> If $U = \text{span}(B)$, then U is finite-dimensional and we are done; or else, add a nonzero vector v_j with $v_j \in U$ and $v_j \notin \text{span}(B)$ into the list B.

To ensure that the procedure is valid, we shall guarantee the existence of v when the procedure needs it.

- The first step is valid because when $U = \{0\}$ it terminates immediately, otherwise it can always find a nonzero $v_1 \in U$.

- Suppose the first $j - 1$ steps are valid, we want to ensure that the j-th step is valid. Now we observe the following claim which we shall prove at the end:

  $$U \neq \text{span}(B) \text{ if and only if there exists a nonzero vector } v_j \in U \setminus \text{span}(B).$$

  Then the j-th step is valid for the following reasons. When $U = \text{span}(B)$, the procedure terminates immediately; otherwise, the statement grants the existence of the v_j which we want to add to B.

  Since the first step is valid, and every next step if not terminating yet is valid, the algorithm is valid for every possible step j.

- Now we want to verify that the procedure always terminates eventually. After each step, we have constructed $B = v_1, \ldots, v_{j-1}$ such that no vector in B is in the span of the preceding vectors. That is, for each $i = 1, \ldots, j - 1$,

  $$v_i \notin \text{span}(v_1, \ldots, v_{i-1}).$$

  Thus B is linearly independent by Linear Dependence Lemma. And by 7.2.4, the linearly independent list B has a length upper bound because it cannot be longer than any spanning list in the finite-dimensional vector space V. And when B cannot expand anymore, this implies that we cannot find more v_j to add. With the claim, this follows that $U = \text{span}(B)$.

Claim

For every list of vectors $B = v_1, \ldots, v_{j-1}$ in U, $U \neq \text{span}(B)$ if and only if there exists a nonzero vector $v_j \in U \setminus \text{span}(B)$.

Proof

Given a list of vectors $B = v_1, \ldots, v_{j-1}$ in U. Because each $v_i \in U$, it is trivial that $\text{span}(B) \subseteq U$. Then,

(a) $\text{span}(B) \neq U$, if and only if,

(b) $\text{span}(B) \subset U$, if and only if,

(c) there exists nonzero $v_j \in U \setminus \text{span}(B)$.[4]

7.3 Exercise

7.3.1 2.A.14

Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Proof

- $\boxed{\Longrightarrow}$ Suppose that V is infinite-dimensional. We construct the sequence as the following manner.

  > **Step 1**
  >
  > Choose a nonzero vector v_1 such that $v_1 \in V$.
  >
  > **Step j**
  >
  > Choose a vector v_j such that $v_j \in V$ and $v_j \notin \text{span}(v_1, \ldots, v_{j-1})$.

  First, we can always find such v_j in each step because $\text{span}(v_1, \ldots, v_{j-1}) \subset V$. This is true since there does not exist a spanning list in V, which means $\text{span}(v_1, \ldots, v_{j-1}) \neq V$.[5]

  Then, after each step j, for every $i = 1, \ldots, j$, we have $v_i \notin \text{span}(v_1, \ldots, v_{i-1})$. Thus, the list $v_1, \ldots, v_j$ is linearly independent by the Linear Dependence Lemma.

---

[4] $v_j \neq 0$ because $0 \in \text{span}(B)$

[5] And $\text{span}(v_1, \ldots, v_{j-1}) \subseteq V$ is trivial as every v is in V.

- $\boxed{\Longleftarrow}$ Suppose that there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m. If V were to be finite-dimensional, then there would exist a spanning list with length n. But at the same time $v_1, \ldots, v_{n+1}$ is a linearly independent list longer than that spanning list, which is a contradiction.

7.3.2 2.A.15 $\mathbf{F}^\infty$ is infinite-dimensional

Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Note

Recall that $\mathbf{F}^\infty$ can be thought of as $\mathbf{F}^{\{1,2,\ldots\}}$. It is defined to be the set of all sequences of elements of $\mathbf{F}$:

$$\mathbf{F}^\infty = \{(x_1, x_2, \ldots) \mid x_j \in \mathbf{F}, \forall j \in \{1, 2, \ldots\}\}$$

Proof

Suppose to the contrary that there were a spanning list $v_1, \ldots, v_m$ of vectors in $\mathbf{F}^\infty$. For each $k \in \{1, 2, \ldots\}$, define $e_k \in \mathbf{F}^\infty$ such that it has 1 at the k-th coordinate and 0 otherwise. Then the list $e_1, \ldots, e_{m+1}$ is linearly independent and is longer than the spanning list $v_1, \ldots, v_m$, which is a contradiction.

7.3.3 2.A.16

Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

Proof

Denote the vector space as

$$\mathcal{C}([0, 1], \mathbf{R}) = \{f : [0, 1] \to \mathbf{R} \mid f \text{ is continuous}\}.$$

Suppose to the contrary that there were a spanning list $f_1, \ldots, f_m$ of functions in $\mathcal{C}([0, 1], \mathbf{R})$. But we can we have a linearly independent list $(1, x, x^2, \ldots, x^m)$ of functions in $\mathcal{C}([0, 1], \mathbf{R})$ longer than the spanning list, which is a contradiction.

Chapter 8

Basis

8.1 Definition

A ***basis*** of V is a list of vectors in V that is linearly independent and spans V.

8.2 Theorem

8.2.1 Criterion for basis

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n, \tag{8.1}$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

Proof

- $\boxed{\Longrightarrow}$ Suppose that $v_1, \ldots, v_n$ is a basis of V. For every $v \in V$, there exist $a_1, \ldots, a_n \in \mathbf{F}$ satisfying the equation (8.1) since $v_1, \ldots, v_n$ spans V. And since $v_1, \ldots, v_n$ is linearly independent, the representation is unique by 6.2.1.

- $\boxed{\Longleftarrow}$ Suppose that every $v \in V$ can be written uniquely in the form of (8.1). This clearly shows that $v_1, \ldots, v_n$ spans V. Thus, we have that every $v \in \mathrm{span}(v_1, \ldots, v_n)$ can be written uniquely in the form of (8.1), $v_1, \ldots, v_n$ is linearly independent by 6.2.1.

8.2.2 Spanning list reduced to a basis

Every spanning list in a vector space can be *reduced* to a basis of the vector space.[1]

---

[1] By *reduce* we mean removing some vectors in the list, or not removing any vector at all.

Proof

Suppose $v_1, \ldots, v_n$ is a spanning list of V. We aim to show that the following procedure reduces the list to a basis of V.

B starts as the list $v_1, \ldots, v_n$.

Step 1

    If $v_1 = 0$, remove v_1 from B; otherwise, leave B unchanged.

Step j (from 2 to n)

    If $v_j \in \text{span}(v_1, \ldots, v_{j-1})$, remove v_j from B; otherwise, leave B unchanged.

If the original spanning list is already linearly independent, then the procedure does not remove any vector, which is the correct output. Now suppose that the spanning list is linearly dependent at the beginning. By Linear Dependence Lemma, after each step the span of B does not change, thus at the end of the procedure we have $\text{span}(B) = V$. And since the procedure ensures that no vector in B is in the span of the preceding vectors, B becomes linearly independent. Therefore, the procedure correctly reduces the spanning list to a basis of V.

8.2.3   Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof

By definition, every finite-dimensional vector space V has a spanning list. And this spanning list can be reduced to a basis of V by the result of 8.2.2.

8.2.4   Linearly independent list extended to a basis

Every linearly independent list of vectors in a finite-dimensional vector space can be *extended* to a basis of the vector space.[2]

Proof

Suppose $u_1, \ldots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \ldots, w_n$ be a basis of V. Then the list

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Applying the procedure in the proof of 8.2.2 to this list produces a basis of V consisting of $u_1, \ldots, u_m$ and some w's, as none of the u's get removed because $u_1, \ldots, u_m$ is linearly independent. Thus, $u_1, \ldots, u_m$ can be extended to a basis of V.

---

[2]By *extend* we mean adding some vectors, or not adding any vector at all.

8.3 Exercise

8.3.1 2.B.1

Find all vector spaces that have exactly one basis.

Solution

The vector space $V = \{0\}$ is the only vector space that has exactly one basis, which is the empty set since span() $= \{0\}$. For every other vector spaces, suppose $w_1, w_2, \ldots, w_n$ is a basis, then $2w_1, w_2, \ldots, w_n$ is a different basis.

Chapter 9

Direct Sum

9.1 Definition

- Suppose $U_1, \ldots, U_m$ are subsets of V. The ***sum*** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}.$$

- Suppose $U_1, \ldots, U_m$ are subspaces of V. The sum $U_1 + \cdots + U_m$ is called a ***direct sum*** if each element of $U_1 + \cdots + U_m$ can be written uniquely as a sum $u_1 + \cdots u_m$, where each u_j is in U_j.

- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

9.2 Theorem

9.2.1 Smallest containing subspace

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof

Given subspaces $U_1, \ldots, U_m$ of V, we aim to show that $U_1 + \cdots + U_m$ is a subspace of V, contains $U_1, \ldots, U_m$, and is the smallest subspace of V containing $U_1, \ldots, U_m$.

- Because $U_1, \ldots, U_m$ are subspaces of V, it is obvious that $0 \in U_1 + \cdots + U_m$ and that $U_1 + \cdots U_m$ is closed under addition and scalar multiplication.

- In each U_j, for every $u_j \in U_j$, clearly that $u_j \in U_1 + \cdots U_m$ because we can choose all the other u's as 0. Thus, each $U_j \subseteq U_1 + \cdots U_m$.

- For every subspace W of V that containing $U_1, \ldots, U_m$, we have $U_1 + \cdots + U_m \subseteq W$ for the following reason. For every $u \in U_1 + \cdots + U_m$, we can write

$$u = u_1 + \cdots + u_m,$$

where $u_1 \in U_1, \ldots, u_m \in U_m$. And because W contains $U_1, \ldots, U_m$, this means that each $u_j \in W$, and so is their sum $u_1 + \cdots + u_m$ since W is closed under addition. Thus $u \in W$.

9.2.2 Condition for direct sum

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Proof

Given subspaces $U_1, \ldots, U_m$ of V, we aim to show that both directions are true.

- $\boxed{\Longrightarrow}$ Suppose to the contrary that $U_1 + \cdots + U_m$ were a direct sum and there were $u_1 \in U_1, \ldots, u_m \in U_m$, not all 0, such that

$$u_1 + \cdots + u_m = 0.$$

Then we would have a different representation that adds to the same element 0, as

$$2u_1 + \cdots + 2u_m = 0,$$

which contradicts that $U_1 + \cdots + U_m$ is a direct sum.

- $\boxed{\Longleftarrow}$ Suppose that the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0. For every $w \in U_1 + \cdots + U_m$, we can write

$$w = u_1 + \cdots + u_m,$$

for some $u_1 \in U_1, \ldots, u_m \in U_m$. To show that this representation is unique, consider another representation that
$$w = v_1 + \cdots + v_m,$$
for some $v_1 \in U_1, \ldots, v_m \in U_m$. Subtracting these two equations gives

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

Because $(u_1 - v_1) \in U_1, \ldots, (u_m - v_m) \in U_m$, based on the assumption, each $u_j - v_j$ should equal to 0. Thus $u_1 = v_1, \ldots, u_m = v_m$; that is, the representation of w is unique.

9.2.3 Condition for direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof

Given that U and W are subspaces of V. We aim to prove that both directions are true.

- $\boxed{\Longrightarrow}$ Suppose that $U + W$ is a direct sum. Then we know that for every $u \in U$ and $w \in W$, $u + w = 0$ implies $u = w = 0$ by 9.2.2. Now given any $v \in U \cap W$, there exists its additive inverse $v' \in U \cap W$ such that $v + v' = 0$. And by $v \in U$ and $v' \in W$, we have $v = v' = 0$. Thus, $U \cap W = \{0\}$.

- $\boxed{\Longleftarrow}$ Suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U$ and $w \in W$ such that $u + w = 0$, we want to show that $u = w = 0$. Because the additive inverse is unique, we know that $u = -w \in W$. This implies that $u \in U \cap W$. Thus $u = 0$, and it follows that $w = -u = 0$.

9.2.4 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof

Given that V is finite-dimensional and U is a subspace of V, U is also finite-dimensional. Let $u_1, \ldots, u_m$ be a basis of U. By 8.2.4, the list can then be extended to be a basis of V, as

$$u_1, \ldots, u_m, w_1, \ldots, w_n.$$

Let $W = \mathrm{span}(w_1, \ldots, w_n)$. We aim to show that $U + W$ is a direct sum and $V = U + W$.

- $\boxed{U \cap W = \{0\}}$

  For every $v \in U \cap W$, there exist scalars a's, b's in $\mathbf{F}$ such that

$$v = a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots + b_n w_n.$$

  Thus

$$a_1 u_1 + \cdots + a_m u_m + (-b_1) w_1 + \cdots + (-b_n) w_n = 0.$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, it must be that $a_1 = \cdots = a_m = -b_1 = \cdots = -b_n = 0$. Thus $v = 0$.

- $\boxed{U + W \subseteq V}$

  It is trivially true as U and W are subspaces of V.

- $\boxed{V \subseteq U + W}$

  For every $v \in V$, since the list $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist scalars a's, b's in $\mathbf{F}$ such that
  $$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n.$$

  Let $u = a_1 u_1 + \cdots + a_m u_m \in U$ and $w = b_1 w_1 + \cdots + b_n w_n \in W$, we can represent v as $v = u + w$. Thus $v \in U + W$.

9.3 Exercise

9.3.1 1.C.16

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution

Yes because addition in vector space is commutative. For every $v \in U + W$, we can write $v = u + w = w + u$ for some $u \in U$ and $w \in W$, thus $v \in W + U$.

9.3.2 1.C.17

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?

Solution

Yes because addition in vector space is associative. For every $v \in (U_1 + U_2) + U_3$, we can write $v = (u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$ for some $u_1 \in U_1, u_2 \in U_2, u_3 \in U_3$, thus $v \in U_1 + (U_2 + U_3)$.

9.3.3 1.C.18

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution

For every subspaces U of V, we have $U + \{0\} = U$; thus $\{0\}$ is an additive identity. For a subspace U of V to have a additive inverse, there should be a subspace U' of V such that $U + U' = \{0\}$. The only U satisfying this requirement is $\{0\}$, as any other U would lead to $U + U' \neq \{0\}$ for any U'.

9.3.4 1.C.24

A function $f : \mathbf{R} \to \mathbf{R}$ is called **even** if

$$f(-x) = f(x), \quad \forall x \in \mathbf{R}.$$

A function $f : \mathbf{R} \to \mathbf{R}$ is called **odd** if

$$f(-x) = -f(x), \quad \forall x \in \mathbf{R}.$$

Let U_{e} denote the set of real-valued even functions on $\mathbf{R}$ and let U_{o} denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = U_{\mathrm{e}} \oplus U_{\mathrm{o}}$.

Solution

We aim to show that $U_{\mathrm{e}} + U_{\mathrm{o}}$ is a direct sum, and $U_{\mathrm{e}} + U_{\mathrm{o}} = \mathbf{R}^{\mathbf{R}}$.

- $\boxed{U_{\mathrm{e}} \cap U_{\mathrm{o}} = \{0\}}$

  Suppose to the contrary that there were $f \in U_{\mathrm{e}} \cap U_{\mathrm{o}}$ such that f is not identically 0, then there would exist $x \in \mathbf{R}$ such that $f(x) \neq 0$. By $f \in U_{\mathrm{e}}$ we have $f(x) = f(-x)$; and by $f \in U_{\mathrm{o}}$ we have $f(x) = -f(-x)$. This would lead to $f(-x) = -f(-x) \neq 0$, which is a contradiction.

- $\boxed{U_{\mathrm{e}} + U_{\mathrm{o}} \subseteq \mathbf{R}^{\mathbf{R}}}$

  For every $f \in U_{\mathrm{e}} + U_{\mathrm{o}}$, it is trivial that $f \in \mathbf{R}^{\mathbf{R}}$, as U_{e} and U_{o} are subspaces of $\mathbf{R}^{\mathbf{R}}$.

- $\boxed{\mathbf{R}^{\mathbf{R}} \subseteq U_{\mathrm{e}} + U_{\mathrm{o}}}$

  For every $f \in \mathbf{R}^{\mathbf{R}}$, we want to find $g \in U_{\mathrm{e}}$ and $h \in U_{\mathrm{o}}$ such that

$$f(x) = g(x) + h(x), \quad \forall x \in \mathbf{R}.$$

  To find the value of $g(x)$ and $h(x)$ at any point $x = x_0$, we solve the following system.

$$\begin{cases} f(x_0) = g(x_0) + h(x_0) \\ f(-x_0) = g(-x_0) + h(-x_0) = g(x_0) - h(x_0) \end{cases}$$

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$$\implies \quad g(x_0) = \frac{f(x_0) + f(-x_0)}{2}, \quad h(x_0) = \frac{f(x_0) - f(-x_0)}{2}.$$

9.3.5 2.B.8

Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \ldots u_m$ is a basis of U and $w_1, \ldots, w_n$ is a basis of W. Prove that

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

is a basis of V.

Proof

We aim to show that $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent and spans V.

- Let $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ be scalars such that

$$a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n = 0.$$

We can write $u = a_1 u_1 + \cdots + a_m u_m \in U$ and $w = b_1 w_1 + \cdots b_n w_n = 0$. And since $U + W$ is a direct sum, $u + w = 0$ implies $u = w = 0$. That is,

$$a_1 u_1 + \cdots + a_m u_m = b_1 w_1 + \cdots b_n w_n = 0.$$

Because $u_1, \ldots, u_m$ and $w_1, \ldots, w_n$ are bases,

$$a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0.$$

Thus $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent.

- For every $v \in V$, since $V = U + W$, there exist $u \in U$ and $w \in W$ such that $v = u + w$. And we can further write u and v as

$$u = a_1 u_1 + \cdots + a_m u_m, \quad w = b_1 w_1 + \cdots b_n w_n,$$

for some scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$. Then

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots b_n w_n.$$

Thus $v \in \mathrm{span}(u_1, \ldots, u_m, w_1, \ldots, w_n)$, which concludes that the list spans V.

Chapter 10

Dimension

10.1   Definition

- The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.

- If V is finite-dimensional, the dimension of V is denoted by $\dim V$.

10.2   Theorem

10.2.1   Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Proof

Suppose V is finite-dimensional. Let B_1 nad B_2 be two bases of V, with length m_1 and m_2 correspondingly. By 7.2.4,

- since B_1 is linearly independent and B_2 spans V, we have $m_1 \leq m_2$;

- since B_2 is linearly independent and B_1 spans V, we have $m_2 \leq m_1$.

Thus, $m_1 = m_2$.

10.2.2   Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof

Let B_1 be a basis of V and B_2 be a basis of U, with length m_1 and m_2 correspondingly. Since B_1 spans V and B_2 is linearly independent, we have $m_1 \geq m_2$ by 7.2.4. Thus $\dim V \geq \dim U$.

10.2.3 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof

Let $\dim V = n$, and $v_1, \ldots, v_n$ be linearly indepedent vectors in V. The list $v_1, \ldots v_n$ can be extended to a basis of V by 8.2.4. But notice that every basis of V has length n, so the extension is the trivial one. That is, the list $v_1, \ldots, v_n$ is a basis.

10.2.4 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with $\dim V$ is a basis of V.

Proof

Let $\dim V = n$, and $v_1, \ldots, v_n$ be a spanning list of V. The list $v_1, \ldots v_n$ can be reduced to a basis of V by 8.2.2. But notice that every basis of V has length n, so the reduction is the trivial one. That is, the list $v_1, \ldots, v_n$ is a basis.

10.2.5 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof

Let

$$u_1, \ldots, u_m \text{ be a basis of } U_1 \cap U_2.$$

It can be extended to be basis of U_1 and U_2 by 8.2.4. Let

$$u_1, \ldots, u_m, v_1, \ldots, v_i \quad \text{be a basis of } U_1,$$
$$u_1, \ldots, u_m, w_1, \ldots, w_i \text{ be a basis of } U_2.$$

We aim to show that

$$u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j$$

is a basis of $U_1 + U_2$. This shall conclude that

$$\dim(U_1 + U_2) = m + i + j = (m + i) + (m + j) - m = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

- Linear Independence

  Suppose that

  $$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_i v_i + c_1 w_1 + \cdots + c_j w_j = 0$$

  for some scalars a's, b's, and c's in $\mathbf{F}$. Let

  $$u = a_1 u_1 + \cdots + a_m u_m \in U_1 \cap U_2$$
  $$v = b_1 v_1 + \cdots + b_i v_i \quad \in U_1$$
  $$w = c_1 w_1 + \cdots + c_j w_j \quad \in U_2$$

  Then we have $u + v + w = 0$. Arranging the equation gives $v = -u - w$. Since $v \in U_1$ and $-u - w \in U_2$, we have $v \in U_1 \cap U_2$. And because $u_1, \ldots, u_m$ is a basis of $U_1 \cap U_2$, we can write v in term of this basis as

  $$b_1 v_1 + \cdots + b_i v_i = d_1 u_1 + \cdots + d_m u_m,$$

  for some scalars d's in $\mathbf{F}$. Notice that $u_1, \ldots, u_m, v_1, \ldots, v_i$ is a basis of U_1, that is, it is linearly independent, so all b's and d's equal 0. Now $u + v + w = 0$ becomes $u + w = 0$.

  $$a_1 u_1 + \cdots + a_m u_m + c_1 w_1 + \cdots + c_j w_j = 0.$$

  Similarly, $u_1, \ldots, u_m, w_1, \ldots, w_j$ is a basis of U_2, so all a's and c's equal 0. In conclusion, all a's, b's and c's equal 0, so $u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j$ is linearly independent.

- $\boxed{\mathrm{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j) \subseteq U_1 + U_2}$

  For every v in $\mathrm{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)$, we can write

  $$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_i v_i + c_1 w_1 + \cdots + c_j w_j,$$

  for some scalars a's, b's, c's in $\mathbf{F}$. Because $a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_i v_i$ is in U_1 and $c_1 w_1 + \cdots + c_j w_j$ is in U_2, we have $v \in U_1 + U_2$.

- $\boxed{U_1 + U_2 \subseteq \mathrm{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)}$

  For every $v \in U_1 + U_2$, it can be written as $v = s_1 + s_2$, for some $s_1 \in U_1$ and $s_2 \in U_2$. We can further represent s_1 and s_2 by their bases.

  $$s_1 = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_i v_i$$
  $$s_2 = c_1 u_1 + \cdots + c_m u_m + d_1 w_1 + \cdots + d_j w_j$$

Substitute them in, we get

$$v = (a_1 + c_1)u_1 + \cdots + (a_m + c_m)u_m + b_1v_1 + \cdots + b_iv_i + d_1w_1 + \cdots + d_jw_j$$

Thus, $v \in \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)$.

10.3 Exercise

10.3.1 2.C.1

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Solution

Let $v_1, \ldots, v_m$ be a basis of U, where $m = \dim U$, then it is also a basis of V as $m = \dim V$ by 10.2.3. Observe that for any two vector spaces U and V sharing a same basis, we have $U = V$, as every vector in their spaces can be represented by that basis.

10.3.2 2.C.9

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Solution

We aim to find a list of linearly independent vectors of length $m - 1$ in $\operatorname{span}(v_1 + w, \ldots, v_m + w)$. This shall conclude that every basis in this subspace has length no less than $m - 1$ by 8.2.4, thus $\dim \operatorname{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$. Let

$$u_i = (v_i + w) - (v_m + w) = v_i - v_m \in \operatorname{span}(v_1 + w, \ldots, v_m + w)$$

for each $i = 1, \ldots, m - 1$. We proceed to show that $u_1, \ldots, u_{m-1}$ is linearly independent. Let $a_1, \ldots, a_{m-1}$ be scalars in $\mathbf{F}$ such that

$$a_1u_1 + \cdots + a_{m-1}u_{m-1} = 0.$$

Then substitution gives

$$a_1(v_1 - v_m) + \cdots + a_{m-1}(v_{m-1} - v_m) = 0.$$

$$\Rightarrow a_1 v_1 + \cdots + a_{m-1} v_{m-1} + (-a_1 - \cdots - a_{m-1}) v_m = 0.$$

Since $v_1, \ldots, v_m$ is linearly independent, all the a's must equal 0, thus $u_1, \ldots, u_{m-1}$ is linearly independent.

10.3.3 2.C.14 Dimension of sum of subspaces

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Solution

For each $i = 1, \ldots, m$, let $d_i = \dim U_i$, and

$$u_{i,1}, \ u_{i,2}, \ldots, \ u_{i,d_i}$$

be a basis of U_i. Then all these bases combined is a spanning list of $U_1 + \cdots + U_m$ for the following reason. For every $v \in U_1 + \cdots + U_m$ it can be written as $v = u_1 + \cdots + u_m$ for some $u_i \in U_i$. And each u_i can be further represented by its basis $u_{i,1}, u_{i,2}, \ldots, u_{i,d_i}$. Then, the dimension of $U_1 + \cdots + U_m$ is no more than the length of this spanning list by 8.2.2.

10.3.4 2.C.15

Suppose V is finite-dimensional, with $\dim V \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Solution

Let $u_1, \ldots, u_n$ be a basis of V, then $U_1 = \text{span}(u_1), \ldots, U_n = \text{span}(u_n)$ satisfy the requirement. It is obvious that the dimension of each U_i is 1. Now we prove that $U_1 + \cdots + U_n$ is a direct sum. Suppose $w_1 \in U_1, \ldots, w_n \in U_n$ are such that

$$w_1 + \cdots + w_n = 0.$$

Each w_i can be written by its basis u_i, that is, there are scalars $a_1, \ldots, a_n$ such that

$$a_1 u_1 + \cdots + a_n u_n = 0.$$

Since $u_1, \ldots, u_n$ is linearly independent, all a's equal 0. Then all w's equal 0, thus $U_1 + \cdots + U_n$ is a direct sum.

10.3.5   2.C.16 Dimension of direct sum of subspaces

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim U_1 + \cdots + \dim U_m.$$

Solution

For each $i = 1, \ldots, m$, let $d_i = \dim U_i$, and

$$u_{i,1}, \ u_{i,2}, \ \ldots, \ u_{i,d_i}$$

be a basis of U_i. Then all these bases combined is a spanning list of $U_1 + \cdots + U_m$ for the same reason in 10.3.3, so $U_1 + \cdots + U_m$ is finite-dimensional. Now we show that this spanning list is linearly independent when $U_1 + \cdots + U_m$ is a direct sum, so that it shall be a basis of $U_1 \oplus \cdots \oplus U_m$. Let the following scalars a's be such that

$$\sum_{i=1}^{m} \sum_{j=1}^{d_i} a_{i,j} \, u_{i,j} = 0.$$

Let

$$w_i = \sum_{j=1}^{d_i} a_{i,j} \, u_{i,j},$$

then it is clear that $\sum_{i=1}^{m} w_i = 0$ implies each $w_i = 0$ since each $w_i \in U_i$ and $U_1 + \cdots + U_m$ is a direct sum. Thus, all a's equal to 0, so the spanning list is linearly independent.