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1 Definition

- A ***linear map*** from V to W is a function $T : V \to W$ with the following properties:

  additivity

  $T(u + v) = Tu + Tv$ for all $u, v \in V$;

  homogeneity

  $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbf{F}$ and all $v \in V$.

- The set of all linear maps from V to W is denoted as $\mathcal{L}(V, W)$.

- Let the symbol 0 denote the function that maps each element of some vector space to the additive identity of another vector space. That is, $0 \in \mathcal{L}(V, W)$ is defined by

$$0v = 0,$$

  where the 0 on the left is a function from V to W, and the 0 on the right is the additive identity in W.

- The ***identity map***, denoted I, is the function on some vector space that maps each element to itself. That is, $I \in \mathcal{L}(V, V)$ is defined by

$$Iv = v.$$

2 Theorem

2.1 Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Proof

We first show the existence of such linear map, then we prove its uniqueness.

- Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n,$$

where c's are arbitrary scalars in $\mathbf{F}$. Then it is easy to see that $Tv_j = w_j$ for each j, by taking $c_j = 1$ and all the other c's equal to 0.

Let's verify that $T \in \mathcal{L}(V, W)$.

- $\boxed{T : V \to W \text{ is a function}}$

  For every $v \in V$, $v = c_1 v_1 + \cdots + c_n v_n$ for some scalars c's as $v_1, \ldots, v_n$ is a basis of V, so Tv exists for all $v \in V$. Then $c_1 w_1 + \cdots + c_n w_n \in W$ as addition is closed, so $T(V) \in W$.

- $\boxed{\text{additivity}}$

  Suppose $u, v \in V$, with $u = a_1 v_1 + \cdots + a_n v_n$ and $v = b_1 v_1 + \cdots + b_n v_n$,

$$\begin{aligned} T(u+v) &= T((a_1+b_1)v_1 + \cdots + (a_n+b_n)v_n) \\ &= (a_1+b_1)w_1 + \cdots + (a_n+b_n)w_n \\ &= a_1 w_1 + \cdots + a_n w_n + b_1 w_1 + \cdots + b_n w_n \\ &= Tu + Tv. \end{aligned}$$

- $\boxed{\text{homogeneity}}$

  Suppose $v \in V$ and $\lambda \in \mathbf{F}$, with $v = c_1 v_1 + \cdots + c_n v_n$,

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda(c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda T v. \end{aligned}$$

• To show the uniqueness, let $S \in \mathcal{L}(V, W)$ be such that $Sv_j = w_j$ for each $j = 1, \ldots, n$. For every $v \in V$ with $v = c_1 v_1 + \cdots + c_n v_n$, by the additivity and homogeneity of S,

$$\begin{aligned} Sv &= S(c_1 v_1 + \cdots + c_n v_n) \\ &= c_1 S v_1 + \cdots + c_n S v_n \\ &= c_1 w_1 + \cdots + c_n w_n, \end{aligned}$$

which is identical to the T we defined above.

2.2 Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof

By additivity of T, we have

$$T(0) = T(0+0) = T(0) + T(0).$$

Adding the additive inverse of $T(0)$ on both side then gives $0 = T(0)$.

3 Exercise

3.1 3.A.3

Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

Solution

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbf{F}^n$, that is, e_k has 1 on the k-th coordinate, and 0 on the other ones. We define the scalars $A_{1,k}, \ldots, A_{m,k}$ to be the entries of Te_k; specifically, for $k = 1, \ldots, n$,

$$Te_k = (A_{1,k}, \ldots, A_{m,k}).$$

Then for all $v = (x_1, \ldots, x_n) \in \mathbf{F}^n$, it can be written as $v = x_1e_1 + \cdots + x_ne^n$. Thus

$$\begin{aligned}
T(x_1, \ldots, x_n) &= Tv \\
&= T(x_1e_1 + \cdots + x_ne_n) \\
&= x_1Te_1 + \cdots + x_nTe_n \\
&= x_1(A_{1,1}, \ldots, A_{m,1}) + \cdots + x_n(A_{1,n}, \ldots, A_{m,n}) \\
&= (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)
\end{aligned}$$

3.2 3.A.4

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Solution

Let $a_1, \ldots, a_m$ be scalars in $\mathbf{F}$ such that

$$a_1 v_1 + \cdots + a_m v_m = 0.$$

Since T is a function,

$$T(a_1 v_1 + \cdots + a_m v_m) = T(0),$$

which gives

$$a_1 T v_1 + \cdots + a_m T v_m = 0.$$

And because $Tv_1, \ldots, Tv_m$ is linearly independent, it follows that all the a's equal to 0. Thus $v_1, \ldots v_m$ is linearly independent.

3.3 3.A.7

Show that every linear map from a 1-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Solution

Let u be a basis of V. Since $Tu \in V$, it can be written by the basis u. Let $\lambda \in \mathbf{F}$ such that

$$Tu = \lambda u.$$

Then for every $v \in V$, it can be written as $v = au$ for some $a \in \mathbf{F}$, thus

$$Tv = T(au) = aTu = a\lambda u = \lambda v.$$

3.4 3.A.10

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$. Define $T : V \to W$ by

$$Tv = \begin{cases} Sv, & \text{if } v \in U, \\ 0, & \text{if } v \in V \setminus U. \end{cases}$$

Prove that T is not a linear map on V.

Solution

Suppose $v \in U$ and $v' \in V \setminus U$, then it can be shown that $v + v' \in V \setminus U$. Thus

$$T(v+v') = 0, \quad Tv + Tv' = Sv + 0 = Sv,$$

which invalidates the additivity of a linear map.

3.5 3.A.11

Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Solution

Let $u_1, \ldots, u_m$ be a basis of U, and it can be extended as $u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n}$ to be a basis of V. Define $w_1, \ldots, w_{m+n}$ by

$$w_i = \begin{cases} Su_i, & \text{if } i = 1, \ldots, m \\ 0, & \text{if } i = m+1, \ldots, m+n \end{cases}$$

Then by 2.1, there exists $T \in \mathcal{L}(V, W)$ such that $Tu_i = w_i$ for each $i = 1, \ldots, m+n$. And it remains true that $Tu = Su$ for all $u \in U$, as $u = a_1u_1 + \cdots + a_mu_m$ for some scalars a's in $\mathbf{F}$, then

$$\begin{aligned} Tu &= T(a_1u_1 + \cdots + a_mu_m) \\ &= a_1Tu_1 + \cdots + a_mTu_m \\ &= a_1w_1 + \cdots + a_mw_m \\ &= a_1Su_1 + \cdots + a_mSu_m \\ &= S(a_1u_1 + \cdots + a_mu_m) \\ &= Su. \end{aligned}$$

Note

Let's take more effort to explicitly write out the formula of T. Let U' be the subspace of V such that $V = U \oplus U'$, then $u_{m+1}, \ldots, u_{m+n}$ is a basis of U'. For every $v \in V$, it can be written as

$$v = u + u' = a_1u_1 + \cdots + a_mu_m + a_{m+1}u_{m+1} + \cdots + a_{m+n}u_{m+n},$$

for some unique $u \in U$ and $u' \in U'$, and for some scalars a's in $\mathbf{F}$. Then

$$\begin{aligned}
Tv &= T(u + u') \\
&= Tu + Tu' \\
&= Su + a_{m+1}Tu_{m+1} + \cdots + a_{m+n}Tu_{m+n} \\
&= Su + a_{m+1}w_{m+1} + \cdots + a_{m+n}w_{m+n} \\
&= Su + 0 \\
&= Su.
\end{aligned}$$

This is different from 3.4 because when $v \in V \setminus U$, Tv does not snap it to 0. Instead, by writing $v = u + u'$, we can see that it only snaps u' to 0 but preserves u as Su. For comparison, suppose $v \in U$ and $v' \in V \setminus U$, then we can write $v' = u + u'$. Thus

$$T(v + v') = Sv + Su, \quad Tv + Tv' = Sv + Su.$$

3.6 3.A.13

Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $w \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(L, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

Solution

By the contraposition of 3.2, since $v_1, \ldots, v_m$ is linearly dependent, it follows that $Tv_1, \ldots, Tv_m$ is a linearly dependent list in W. Then let $w_1, \ldots, w_m$ be linearly independent in W, we can show that no $T \in \mathcal{L}(L, W)$ exists for the requirement as follows. Let $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, be such that

$$a_1v_1 + \cdots + a_mv_m = 0.$$

Since T is a function,

$$T(a_1v_1 + \cdots + a_mv_m) = T(0).$$

Then

$$a_1Tv_1 + \cdots + a_mTv_m = 0.$$

If it were true that $Tv_k = w_k$ for each $k = 1, \ldots, m$, then it would imply that $w_1, \ldots, w_m$ is linearly dependent, which would be a contradiction.