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1 Definition

- A *linear combination* of a list $v_1, v_2, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m,$$

  where $a_1, \ldots, a_m \in \mathbf{F}$.

- The set of all linear combination of a list $v_1, \ldots, v_m$ of vectors in V is called the *span* of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbf{F}\}.$$

- The span of the empty list is defined to be $\{0\}$.

- If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ *spans* V.

Note

Technically we should use $(v_1, \ldots, v_m)$ to denote a list of vectors, but that would create too many parentheses, so we usually omit it. For example, we say that $v_1, \ldots, v_m$ *is* a list of vectors.

2 Exercise

2.1 Smallest subspace

The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Proof

Given a list $v_1, \ldots, v_m$ of vectors in V, we aim to show that $\operatorname{span}(v_1, \ldots, v_m)$ is a subspace of V, then that it contains $v_1, \ldots, v_m$, finally that it is the smallest subspace of V containing $v_1, \ldots, v_m$.

1. $\operatorname{span}(v_1, \ldots, v_m)$ is a subspace of V by satisfying the following conditions of subspace.

    - $\operatorname{span}(v_1, \ldots, v_m) \subseteq V$
      For all $u \in \operatorname{span}(v_1, \ldots, v_m)$,

$$u = a_1 v_1 + \cdots + a_m v_m, \quad \text{for some } a_1, \ldots, a_m.$$

And since each $v_i \in V$ and V is closed under addition and scalar multiplication, it follows that $u \in V$.

- Additive identity

$$0v_1 + \cdots + 0v_m = 0 \quad \in \operatorname{span}(v_1, \ldots, v_m).$$

- Closed under addition

For all $u, w \in \operatorname{span}(v_1, \ldots, v_m)$,

$$\begin{aligned} u + w &= (a_1 v_1 + \cdots + a_m v_m) + (c_1 v_1 + \cdots + c_m v_m) \\ &= (a_1 + c_1) v_1 + \cdots (a_m + c_m) v_m \quad \in \operatorname{span}(v_1, \ldots, v_m). \end{aligned}$$

- Closed under scalar multiplication

For all $\lambda \in \mathbf{F}$ and all $u \in \operatorname{span}(v_1, \ldots, v_m)$,

$$\begin{aligned} \lambda u &= \lambda(a_1 v_1 + \cdots + a_m v_m) \\ &= (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \quad \in \operatorname{span}(v_1, \ldots, v_m). \end{aligned}$$

2. For each $v_j \in \{v_1, \ldots, v_m\}$, v_j is clearly a linear combination of $v_1, \ldots, v_m$ (by setting all a to 0 except $a_i = 1$).

3. For every subspace U of V containing $v_1, \ldots, v_m$, U should contain all the linear combination of $v_1, \ldots, v_m$. If U did not contain some linear combination of them, U would not be a vector space since it would not be closed under addition or scalar multiplication.

   Thus, there is no subspace of V containing $v_1, \ldots, v_m$ smaller than $\operatorname{span}(v_1, \ldots, v_m)$.

2.2

Suppose v_1, v_2, v_3, v_4 spans V. Prove that the following list also spans V:

$$v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4.$$

Proof

Let $U = \operatorname{span}(v_1, v_2, v_3, v_4)$ and $U' = \operatorname{span}(v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4)$. We aim to show that $U = U'$.

- $\boxed{U \subseteq U'}$

For every $u \in U$, there exist some a_1, a_2, a_3, a_4 such that

$$\begin{aligned}
u &= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \\
&= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4
\end{aligned}$$

Thus, $u \in U'$.

- $\boxed{U \subseteq U'}$

  For every $u' \in U'$, there exist some c_1, c_2, c_3, c_4 such that

$$\begin{aligned}
u' &= c_1(v_1 - v_2) + c_2(v_2 - v_3) + c_3(v_3 - v_4) + c_4 v_4 \\
&= c_1 v_1 + (-c_1 + c_2)v_2 + (-c_2 + c_3)v_4 + (-c_3 + c_4)v_4
\end{aligned}$$

  Thus, $u' \in U$.