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1 Definition

- The ***dimension*** of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

2 Theorem

2.1 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Proof

Suppose V is finite-dimensional. Let B_1 nad B_2 be two bases of V, with length m_1 and m_2 correspondingly.

- Since B_1 is linearly independent and B_2 spans V, we have $m_1 \leq m_2$.
- Since B_2 is linearly independent and B_1 spans V, we have $m_2 \leq m_1$.

Thus, $m_1 = m_2$.

2.2 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof

Let B_1 be a basis of V and B_2 be a basis of U, with length m_1 and m_2 correspondingly. Since B_1 spans V and B_2 is linearly independent, we have $m_1 \geq m_2$. Thus $\dim V \geq \dim U$.

2.3 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof

Let $\dim V = n$, and $v_1, \dots, v_n$ be linearly indepedent vectors in V. The list $v_1, \dots v_n$ can be extended to a basis of V. But notice that every basis of V has length n, so the extension is the trivial one. That is, the list $v_1, \dots, v_n$ is a basis.

2.4 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with $\dim V$ is a basis of V.

Proof

Let $\dim V = n$, and $v_1, \dots, v_n$ be a spanning list of V. The list $v_1, \dots v_n$ can be reduced to a basis of V. But notice that every basis of V has length n, so the reduction is the trivial one. That is, the list $v_1, \dots, v_n$ is a basis.

2.5 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof

Let

$$u_1, \dots, u_m \text{ be a basis of } U_1 \cap U_2.$$

Since $u_1, \dots, u_m$ is linearly indenpendent, it can be extended to be bases of U_1 and U_2. Let

$$\begin{aligned} u_1, \dots, u_m, v_1, \dots, v_i \quad &\text{be a basis of } U_1, \\ u_1, \dots, u_m, w_1, \dots, w_i \ &\text{be a basis of } U_2. \end{aligned}$$

We aim to show that

$$u_1, \dots, u_m, v_1, \dots, v_i, w_1, \dots, w_j$$

is a basis of $U_1 + U_2$. This will conclude that

$$\dim(U_1 + U_2) = m + i + j = (m + i) + (m + j) - m = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

- $\boxed{\text{Linear Independence}}$

Suppose that

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_iv_i + c_1w_1 + \cdots + c_jw_j = 0$$

for some scalars a's, b's, and c's in $\mathbf{F}$. Let

$$\begin{aligned} u &= a_1u_1 + \cdots + a_mu_m \in U_1 \cap U_2 \\ v &= b_1v_1 \; + \cdots + b_iv_i \quad\; \in U_1 \\ w &= c_1w_1 + \cdots + c_jw_j \;\; \in U_2 \end{aligned}$$

Then we have $u + v + w = 0$. Arranging the equation gives $v = -u - w$. Since $v \in U_1$ and $-u - w \in U_2$, we have $v \in U_1 \cap U_2$. And because $u_1, \ldots, u_m$ is a basis of $U_1 \cap U_2$, we can write v in term of this basis as

$$b_1v_1 + \cdots + b_iv_i = d_1u_1 + \cdots + d_mu_m,$$

for some scalars d's in $\mathbf{F}$. Notice that $u_1, \ldots, u_m, v_1, \ldots, v_i$ is a basis of U_1, that is, it is linearly independent, so all b's and d's equal 0. Now $u + v + w = 0$ becomes $u + w = 0$.

$$a_1u_1 + \cdots + a_mu_m + c_1w_1 + \cdots + c_jw_j = 0.$$

Similarly, $u_1, \ldots, u_m, w_1, \ldots, w_j$ is a basis of U_2, so all a's and c's equal 0. In conclusion, all a's, b's and c's equal 0, so $u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j$ is linearly independent.

- $\boxed{\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j) \subseteq U_1 + U_2}$

  For every v in $\operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)$, we can write

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_iv_i + c_1w_1 + \cdots + c_jw_j,$$

  for some scalars a's, b's, c's in $\mathbf{F}$. Because $a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_iv_i$ is in U_1 and $c_1w_1 + \cdots + c_jw_j$ is in U_2, we have $v \in U_1 + U_2$.

- $\boxed{U_1 + U_2 \subseteq \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)}$

  For every $v \in U_1 + U_2$, it can be written as

$$v = s_1 + s_2,$$

  for some $s_1 \in U_1$ and $s_2 \in U_2$. We can further represent s_1 and s_2 by their bases.

$$\begin{aligned} s_1 &= a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_iv_i \\ s_2 &= c_1u_1 + \cdots + c_mu_m + d_1w_1 + \cdots + d_jw_j \end{aligned}$$

Substitute them in, we get

$$v = (a_1 + c_1)u_1 + \cdots + (a_m + c_m)u_m + b_1v_1 + \cdots + b_iv_i + d_1w_1 + \cdots + d_jw_j$$

Thus, $v \in \text{span}(u_1, \ldots, u_m, v_1, \ldots, v_i, w_1, \ldots, w_j)$.

3 Exercise

3.1 2.C.1

Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Solution

Let $v_1, \ldots, v_m$ be a basis of U, where $m = \dim U$, then it is also a basis of V as $m = \dim V$. We claim that for any two vector spaces U and V sharing a same basis, we have $U = V$, as every vector in their spaces can be represented by that basis.

3.2 2.C.9

Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \text{span}(v_1 + w, \ldots, v_m + w) \geq m - 1.$$

Solution

We aim to find a list of linearly independent vectors of length $m - 1$ in $\text{span}(v_1 + w, \ldots, v_m + w)$. This will conclude that every basis in this subspace has length no less than $m - 1$, thus $\dim \text{span}(v_1 + w, \ldots, v_m + w) \geq m - 1$. Let

$$u_i = (v_i + w) - (v_m + w) = v_i - v_m \in \text{span}(v_1 + w, \ldots, v_m + w)$$

for $i \in \{1, \ldots, m - 1\}$. We proceed to show that $u_1, \ldots, u_{m-1}$ is linearly independent. Let $a_1, \ldots, a_{m-1}$ be scalars in $\mathbf{F}$ such that

$$a_1u_1 + \cdots + a_{m-1}u_{m-1} = 0.$$

Then substitution gives

$$a_1(v_1 - v_m) + \cdots + a_{m-1}(v_{m-1} - v_m) = 0.$$

That is,

$$a_1v_1 + \cdots + a_{m-1}v_{m-1} + (-a_1 - \cdots - a_{m-1})v_m = 0.$$

Since $v_1, \ldots, v_m$ is linearly independent, all the a's must equal 0, thus $u_1, \ldots, u_{m-1}$ is linearly independent.

3.3 2.C.14

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V. Prove that $U_1 + \cdots + U_m$ is finite-dimensional and

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

Solution

For $i \in \{1, \ldots, m\}$, let $d_i = \dim U_i$, and

$$u_{i,1},\ u_{i,2}, \ldots,\ u_{i,d_i}$$

be a basis of U_i. Then all these bases combined is a spanning list of $U_1 + \cdots + U_m$ for the following reason. For every $v \in U_1 + \cdots + U_m$ it can be written as $v = u_1 + \cdots + u_m$ for some $u_i \in U_i$. And each u_i can be further represented by its basis $u_{i,1}, u_{i,2}, \ldots, u_{i,d_i}$. In addition, the dimension of $U_1 + \cdots + U_m$ is no more than the length of this spanning list, as the spanning list can be reduced to a basis.

3.4 2.C.15

Suppose V is finite-dimensional, with $\dim V \geq 1$. Prove that there exist 1-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

Solution

Let $u_1, \ldots, u_n$ be a basis of V, then $U_1 = \text{span}(u_1), \ldots, U_n = \text{span}(u_n)$ satisfy the requirement. It is obvious that the dimension of each U_i is 1. Now we prove that $U_1 + \cdots + U_n$ is a direct sum. Suppose $w_1 \in U_1, \ldots, w_n \in U_n$ are such that

$$w_1 + \cdots + w_n = 0.$$

Each w_i can be written by its basis u_i, that is, there are scalars $a_1, \ldots, a_n$ such that

$$a_1u_1 + \cdots + a_nu_n = 0.$$

Since $u_1, \ldots, u_n$ is linearly independent, all a's equal 0. Then all w's equal 0, thus $U_1 + \cdots + U_n$ is a direct sum.

3.5 2.C.16

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Prove that $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim U_1 + \cdots + \dim U_m.$$

Solution

For $i \in \{1, \ldots, m\}$, let $d_i = \dim U_i$, and

$$u_{i,1},\ u_{i,2},\ \ldots,\ u_{i,d_i}$$

be a basis of U_i. Then all these bases combined is a spanning list of $U_1 + \cdots + U_m$ for the same reason in 3.3. Now we show that this spanning list is linearly independent when $U_1 + \cdots + U_m$ is a direct sum, so it is a basis of $U_1 \oplus \cdots \oplus U_m$. Let the following scalars a's be such that

$$\sum_{i=1}^{m} \sum_{j=1}^{d_i} a_{i,j}\, u_{i,j} = 0.$$

Let

$$w_i = \sum_{j=1}^{d_i} a_{i,j}\, u_{i,j},$$

then it is clear that $\sum_{i=1}^{m} w_i = 0$ implies each $w_i = 0$ since each $w_i \in U_i$ and $U_1 + \cdots + U_m$ is a direct sum.