# Flood Analysis using Bivariate Normal Distribution

# MTH-357/657 Final Project

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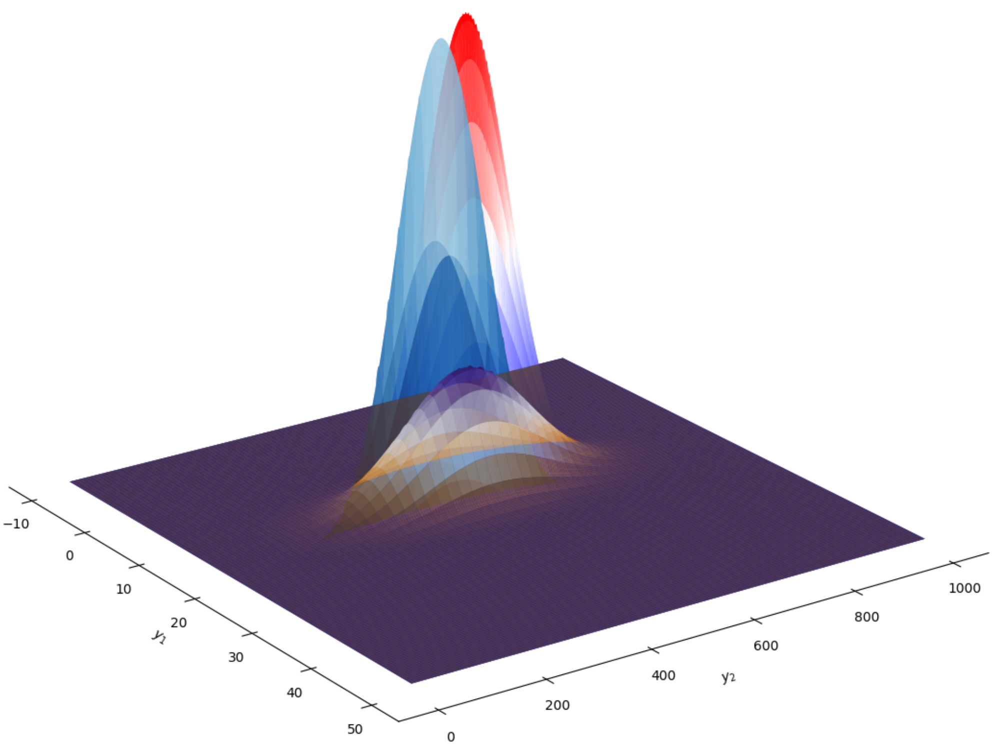


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# Chapter 1

# Introduction

A flood is the resulting behavior of a water source as after the the capacity is exceeded and the discharge of the river overflows onto land. Flooding is caused when discharge increases. We analyse the relationship between flood peaks and volumes, which follows a bivariate normal distribution [SB87]. A flood peak is the highest elevation reached by a flood wave, [Mince]. Flood discharge is the volume of water passing through a unit of space per unit time (flux).
Let us define the continuous random variable $Y_1$ to represent the flood peak. Further define $Y_2$ as the flood volume. Both $Y_1$ and $Y_2$ are measured in terms of flow rates; the amount of volume per time. Using the baseline settings stated in Table 1 in [SB87], we provide initial data for the following three settings.

Table 1.1: Initial Data $Y_1$

| | Mean ($\frac{m^3}{s}$) | Standard Deviation ($\frac{m^3}{s}$) |
|---|---|---|
| Baseline | 14.71 | 1.14 |
| Setting 2 | 20 | 1.14 |
| Setting 3 | 14.71 | 0.5 |

Table 1.2: Initial Data $Y_2$

| | Mean ($\frac{\text{day}\, m^3}{s}$) | Standard Deviation ($\frac{\text{day}\, m^3}{s}$) |
|---|---|---|
| Baseline | 510.20 | 84.14 |
| Setting 2 | 400 | 84.14 |
| Setting 3 | 510.20 | 120 |

# Chapter 2

# Background Theory

The bivariate normal probability density function $f(y_1, y_2)$ is defined as follows

$$f(y_1, y_1) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)}, \tag{2.0.1}$$

where $y_1, y_2 \in \mathbf{R}$, $\mu_1 \in \mathbf{R}$, and $\sigma_1 > 0$ are the mean and standard deviation of $Y_1$, respectively. Further, $\mu_2 \in \mathbf{R}$ and $\sigma_2 > 0$ are the mean and standard deviation of $Y_2$, respectively.
We define the following integral,

$$\int e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \operatorname{erf}\left(\sqrt{a}x\right) dx, \tag{2.0.2}$$

$$\int e^{-\frac{1}{2}x^2} dx = \frac{\sqrt{2\pi}}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right), \tag{2.0.3}$$

$$\int xe^{-ax^2} dx = -\frac{1}{2}\frac{1}{a} e^{-ax^2} \tag{2.0.4}$$

$$\int xe^{-\frac{1}{2}x^2} dx = -e^{-\frac{1}{2}x^2} \tag{2.0.5}$$

where erf is the Gauss error function. We provide two identities without proof;

$$\operatorname{erf}(-z) = -\operatorname{erf}(z) \tag{2.0.6}$$

$$\operatorname{erf}(\infty) = 1. \tag{2.0.7}$$

## 2.1 Marginal Density Functions

Let us begin by computing the marginal PDFs for the variables $Y_1$ and $Y_2$. The marginal PDF for $Y_1$ is defined as

$$f(y_1) = \int_{\text{Domain}(Y_2)} f(y_1, y_2) dy_2 \tag{2.1.1}$$

$$= \int_{\mathbf{R}} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)} dy_2 \tag{2.1.2}$$

$$f(y_1) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \int_{\mathbf{R}} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} dy_2 \tag{2.1.3}$$

$$\tag{2.1.4}$$

Letting $a = \frac{1}{2}$ and defining $x$ such that

$$x = \frac{y_2 - \mu_2}{\sigma_2} \implies \sigma_2 \, dx = dy_2. \tag{2.1.5}$$

Using 2.0.2, we write 2.1.1 as

$$f(y_1) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \int_{\mathbf{R}} \sigma_2 e^{-ax^2} dx \tag{2.1.6}$$

$$= \frac{\sigma_2}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \left[\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}}} \operatorname{erf}\left(\frac{1}{\sqrt{2}}x\right)\right]_{-\infty}^{\infty} \tag{2.1.7}$$

$$= \frac{\sqrt{2\pi}}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \Big[\operatorname{erf}(\infty) - \operatorname{erf}(-\infty)\Big] \tag{2.1.8}$$

$$= \frac{\sqrt{2\pi}}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \Big[\operatorname{erf}(\infty) + \operatorname{erf}(\infty)\Big] \tag{2.1.9}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2}. \qquad \square \tag{2.1.10}$$

Thus, the marginal density function for $Y_1$ is

$$f(y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2}. \tag{2.1.11}$$

Similarly, the marginal density function for $Y_2$ is defined by

$$f(y_2) = \int_{\text{Domain}(Y_1)} f(y_1, y_2) dy_1 \tag{2.1.12}$$

$$= \int_{\mathbf{R}} \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right)} dy_1 \tag{2.1.13}$$

$$f(y_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \int_{\mathbf{R}} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 \tag{2.1.14}$$

$$\tag{2.1.15}$$

Letting $a = \frac{1}{2}$ and defining $x$ such that

$$x = \frac{y_1 - \mu_1}{\sigma_1} \implies \sigma_1\, dx = dy_1. \tag{2.1.16}$$

Using 2.0.2, we write 2.1.12 as

$$\begin{aligned}
f(y_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \int_{\mathbf{R}} \sigma_1 e^{-ax^2} dx & (2.1.17)\\
&= \frac{\sigma_1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \left[\frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2}}} \operatorname{erf}\left(\frac{1}{\sqrt{2}}x\right)\right]_{-\infty}^{\infty} & (2.1.18)\\
&= \frac{\sqrt{2\pi}}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \Big[\operatorname{erf}(\infty) - \operatorname{erf}(-\infty)\Big] & (2.1.19)\\
&= \frac{\sqrt{2\pi}}{2\pi\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \Big[\operatorname{erf}(\infty) + \operatorname{erf}(\infty)\Big] & (2.1.20)\\
&= \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2}. \qquad \square & (2.1.21)
\end{aligned}$$

Thus, the marginal density function for $Y_2$ is

$$f(y_2) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2}. \tag{2.1.22}$$

### 2.1.1 Discussion

We now compare the marginal density functions for each of the three cases defined 1.

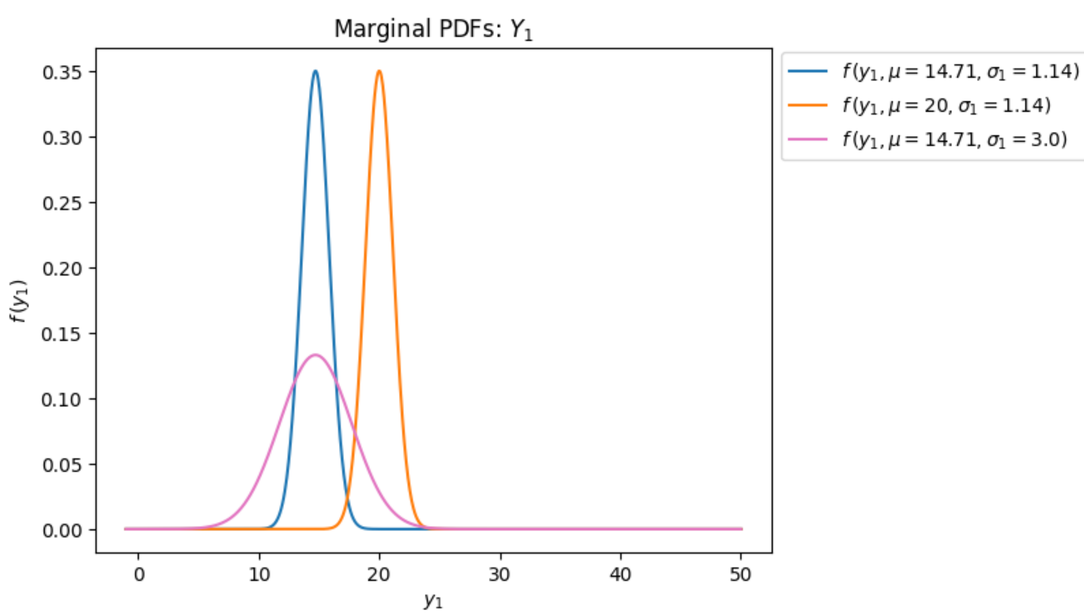


Figure 2.1: Marginal PDFs of $Y_1$

Using the baseline initial data, shown by the blue curve in Figure 2.1, the PDF of $Y_1$ is centered at the baseline mean, $\mu = 14.71$, and achieves a maximum at $f(y_1 = 14.71) = 0.349$. In the second case, where we shift the mean to $\mu = 20$, we note the shape of the curve stays the same, but its center horizontally shifts in the positive direction from the baseline mean to $\mu = 20$. When we return the mean to the baseline, but change the standard deviation $\sigma$, we note the curve centered at the baseline mean has a lower peak, at $f(y_1) = 0.132$.

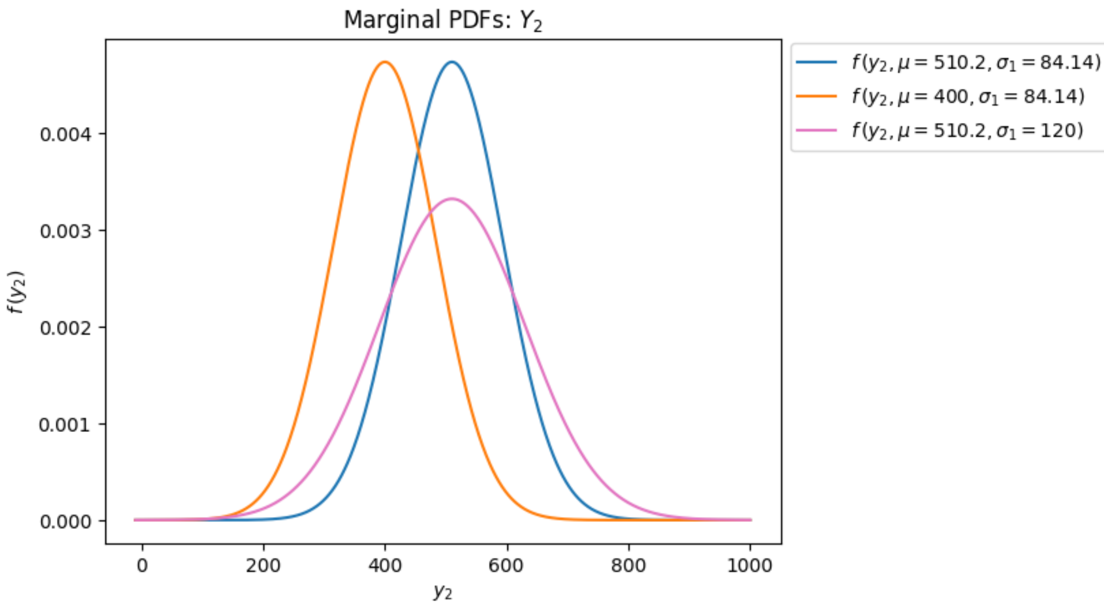


Figure 2.2: Marginal PDFs of $Y_2$

## 2.2 Marginal Cumulative Density Functions

The cumulative distribution function, CDF, for the continuous random variable $y_1$, $F(y_1)$ is defined as

$$F(y_1) = \int_{-\infty}^{y_1} f(x_1) dx_1 \tag{2.2.1}$$

where $f(x_1)$ is the marginal density function for $Y_1$, 2.1.11. Inserting this density function into 2.2.1,

$$F(y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{y_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} dx_1. \tag{2.2.2}$$

Let $\xi = \frac{x_1-\mu_1}{\sigma_1}$, which gives $\sigma_1 \, d\xi = dx_1$, we rewrite 2.2.2 as

$$\begin{aligned}
F(y_1) &= \frac{\sigma_1}{\sqrt{2\pi}\sigma_1} \int_{-\infty}^{x_1=y_1} e^{-\frac{1}{2}\xi^2} d\xi && (2.2.3)\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi=\frac{y_1-\mu_1}{\sigma_1}} e^{-\frac{1}{2}\xi^2} d\xi && (2.2.4)\\
&= \frac{\sqrt{2\pi}}{2} \frac{1}{\sqrt{2\pi}} \left[ \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)\right]_{-\infty}^{\frac{y_1-\mu_1}{\sigma_1}} && (2.2.5)\\
&= \frac{1}{2}\left[\operatorname{erf}\left(\frac{y_1-\mu_1}{\sqrt{2}\sigma_1}\right) + 1\right]. \qquad \square && (2.2.6)
\end{aligned}$$

Thus, the CDF for $y_1$ is

$$F(y_1) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{y_1 - \mu_1}{\sqrt{2}\sigma_1}\right)\right]. \tag{2.2.7}$$

A similar computation as 2.2.7 gives

$$F(y_2) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{y_2 - \mu_2}{\sqrt{2}\sigma_2}\right)\right]. \tag{2.2.8}$$

### 2.2.1 Discussion

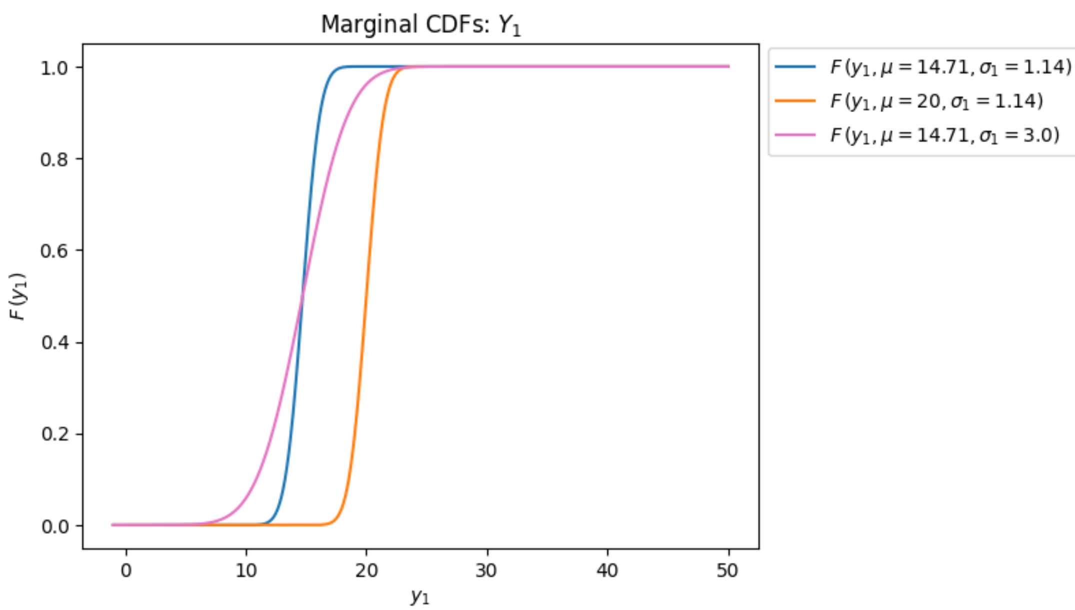


Figure 2.3: Marginal CDFs of $Y_1$

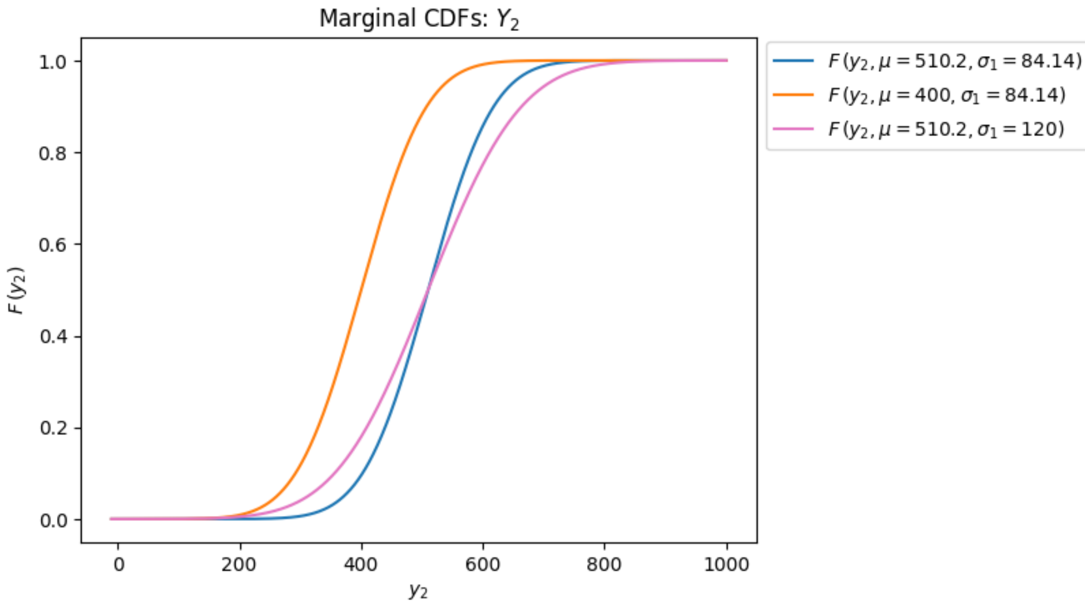


Figure 2.4: Marginal CDFs of $Y_2$

## 2.3 Expected Values

In order to compute the expected values of $Y_1$ and $Y_2$, we can

$$E(Y_1) = \int_{\mathbf{R}} y_1 f(y_1) \tag{2.3.1}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} \int_{\mathbf{R}} y_1 e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1. \tag{2.3.2}$$

Using the fact that $y_1 = (y_1 - \mu_1) + \mu_1$, we can decompose 2.3.1 as

$$E(Y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \int_{\mathbf{R}} (y_1 - \mu_1) e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} + \mu_1 e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 \tag{2.3.3}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} \cdot \left\{ \int_{\mathbf{R}} (y_1 - \mu_1) e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 + \mu_1 \int_{\mathbf{R}} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \right\} \tag{2.3.4}$$

$$E(Y_1) = \frac{1}{\sqrt{2\pi}\sigma_1} \cdot \left\{ \mathscr{I}_1 + \mathscr{I}_2 \right\} \tag{2.3.5}$$

Let us begin with the first integral

$$\mathscr{I}_1 = \int_{\mathbf{R}} (y_1 - \mu_1) e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 \tag{2.3.6}$$

Implementing the $u - sub$,

$$u = -\frac{1}{2}\left(\frac{y_1 - \mu_1}{\sigma_1}\right)^2 \implies du = \frac{(y_1 - \mu_1)}{-\sigma_1^2} \implies -\sigma_1^2 du = (y_1 - \mu_1) dy_1. \tag{2.3.7}$$

Inserting 2.3.7 into 2.3.6,

$$\mathscr{I}_1 = \int_{\mathbf{R}} -\sigma_1^2 e^u du \tag{2.3.8}$$

$$= -\sigma_1^2 e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} |_{-\infty}^{\infty} \tag{2.3.9}$$

$$= 0 \tag{2.3.10}$$

Computing the second integral

$$\mathscr{I}_2 = \mu_1 \int_{\mathbf{R}} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 \tag{2.3.11}$$

$$= \mu_1 \frac{1}{2}\sigma_1\sqrt{2\pi}\left[\operatorname{erf}(\frac{y_1}{\sqrt{2}})\right]_{-\infty}^{\infty} \tag{2.3.12}$$

$$= \mu_1 \cdot \sqrt{2\pi}\sigma_1 \tag{2.3.13}$$

$$\tag{2.3.14}$$

Combing the results 2.3.6 and 2.3.11, we find

$$E(Y_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\left\{0 + \mu_1\sqrt{2\pi}\sigma_1\right\} \tag{2.3.15}$$

$$= \mu_1. \qquad \square \tag{2.3.16}$$

Thus the expected value of $Y_1$ is

$$E(Y_1) = \mu_1. \tag{2.3.17}$$

### 2.3.1 Discussion

For our $Y_1$ baseline (setting 1a), a flood peak with a mean flow rate $\mu_1 = 14.71m^3/s$ with standard deviation $\sigma_1 = 1.14m^3/s$, has a flow rate of $14.71m^3/s$ on average. Additionally, in setting 1b, a flood peak with a mean flood rate $\mu_1 = 20m^3/s$ with standard deviation $\sigma_1 = 1.14$, has a flow rate of $20m^3/s$ on average. Similarly, in setting 1c a flood peak with a mean flow rate $\mu_1 = 14.71m^3/s$ with standard deviation $\sigma_1 = 0.5m^3/s$, has a flow rate of $14.71m^3/s$ on average. In order to derive the expected value of $Y_2$, we can follow the work in 2.3.1 by replacing $y_1$ with $y_2$, $\mu_1$ with $\mu_2$, and $\sigma_1$ with $\sigma_2$.

## 2.4 Moment Generating Functions

The moment generating function of $Y_1$, $M_{Y_1}(t)$, is defined by

$$M_{Y_1}(t) = E(e^{tY_1}) \tag{2.4.1}$$

$$= \int_{\mathbf{R}} e^{t\,y_1} \cdot f(y_1)\, dy_1 \tag{2.4.2}$$

where $f(y_1)$ is the marginal density function for $y_1$, 2.1.11. Inserting the marginal density function into 2.4.1,

$$M_{y_1}(t) = E(e^{ty_1}) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\int_{\mathbf{R}} e^{t\,y_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 \tag{2.4.3}$$

Let us begin by simplifying the arguments of the exponential term; $ty_1 - \frac{1}{2}\frac{(y_1-\mu_1)^2}{\sigma_1^2}$. By using the substitution $u = \frac{y_1-\mu_1}{\sigma_1}$, it follows that $y_1 = \sigma_1 u + \mu_1$. Further, the differentials become $\sigma_1 du = dy_1$. It follows that

$$ty_1 - \frac{1}{2}\frac{(y_1-\mu_1)^2}{\sigma_1^2} = t(\sigma_1 u + \mu_1) - \frac{1}{2}u^2 \tag{2.4.4}$$

$$= -\frac{1}{2}u^2 + (\sigma_1 t)u + (t\mu_1) \tag{2.4.5}$$

Now that the argument is a polynomial in $u$, we complete the square by

$$-\frac{1}{2}u^2 + (\sigma_1 t)u + (t\mu_1) = -\frac{1}{2}[u^2 - (2\sigma_1 t)u] + t\mu_1 \tag{2.4.6}$$

$$= -\frac{1}{2}[u^2 - (2\sigma_1 t)u + (\sigma_1 t)^2 - (\sigma_1 t)^2] + t\mu_1 \tag{2.4.7}$$

$$= -\frac{1}{2}[(u-\sigma_1 t)^2 - (\sigma_1 t)^2] + t\mu_1 \tag{2.4.8}$$

$$= -\frac{1}{2}(u-\sigma_1 t)^2 + \frac{1}{2}(\sigma_1 t)^2 + t\mu_1 \tag{2.4.9}$$

$$= -\frac{1}{2}(u-\sigma_1 t)^2 + \kappa \tag{2.4.10}$$

where the constant $\kappa = \frac{1}{2}(\sigma_1 t)^2 + t\mu_1$. Using the substitution $\zeta = u - \sigma_1 t$, with $d\zeta = du$, we arrive at the final form of the argument;

$$ty_1 - \frac{1}{2}\frac{(y_1-\mu_1)^2}{\sigma_1^2} = -\frac{1}{2}\zeta^2 + \kappa. \tag{2.4.11}$$

Inserting 2.4.11 into 2.4.3, it follows that

$$M_{y_1}(t) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\int_{\mathbf{R}} e^{t\,y_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 = \frac{1}{\sqrt{2\pi}\sigma_1}\int_{\mathbf{R}} e^{-\frac{1}{2}u^2+(\sigma_1 t)u+(t\mu_1)}\sigma_1 du \quad = \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} e^{-\frac{1}{2}\zeta^2+\kappa}\, d\zeta \tag{2.4.12}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\pi}}{2} e^{\kappa}\left[\operatorname{erf}\left(\frac{\zeta}{\sqrt{2}}\right)\right]_{-\infty}^{\infty} \tag{2.4.13}$$

$$= e^{\kappa} \tag{2.4.14}$$

$$= e^{\frac{1}{2}\sigma_1^2 t^2 + \mu_1 t}. \quad \square \tag{2.4.15}$$

Thus, the MGF of $Y_1$ is

$$M_{Y_1}(t) = e^{\frac{1}{2}\sigma_1^2 t^2 + \mu_1 t}. \tag{2.4.16}$$

Let us now consider the MGF of $Y_2$, $M_{Y_2}(t)$. In order to compute this, we can swap $y_1, \mu_1, \sigma_1$ with their counterparts $y_2, \mu_2, \sigma_2$. It follows that

$$M_{Y_2}(t) = e^{\frac{1}{2}\sigma_2^2 t^2 + \mu_2 t}. \tag{2.4.17}$$

## 2.5 Conditional Density Functions

The conditional PDF, $f\,(y_1|y_2)$, is given by

$$\begin{aligned}
f\,(y_1|y_2) &= \frac{f\,(y_1, y_2)}{f(y_2)} \\
&= \frac{1}{\sqrt{2\pi}\,\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \cdot \sqrt{2\pi}\,\sigma_2\, e^{\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \\
f\,(y_1|y_2) &= \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2}. \qquad \square \qquad (2.5.1)
\end{aligned}$$

for $y_1 \in \mathbf{R}$, and 0 otherwise. Similarly, the conditional PDF, $f\,(y_2|y_1)$, is given by

$$\begin{aligned}
f\,(y_2|y_1) &= \frac{f\,(y_1, y_2)}{f(y_1)} \\
&= \frac{1}{\sqrt{2\pi}\,\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} \cdot \sqrt{2\pi}\,\sigma_1\, e^{\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \\
f\,(y_2|y_1) &= \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2}. \qquad \square \qquad (2.5.2)
\end{aligned}$$

for $y_2 \in \mathbf{R}$, and 0 otherwise.

## 2.6 Conditional Cumulative Density Functions

We now find the conditional CDF of $Y_1$, given $Y_2 = y_2$; $F(y_1|Y_2 = y_2)$. By definition,

$$F(y_1|Y_2 = y_2) = \int_{-\infty}^{y_1} f(y_1|y_2)dy_1, \tag{2.6.1}$$

$$\tag{2.6.2}$$

Additionally, we can also set up the conditional CDF of $Y_2$, given $Y_1 = y_1$; $F(y_2|Y_1 = y_1)$. By definition,

$$F(y_2|Y_1 = y_1) = \int_{-\infty}^{y_2} f(y_2|y_1)dy_2, \tag{2.6.3}$$

$$\tag{2.6.4}$$

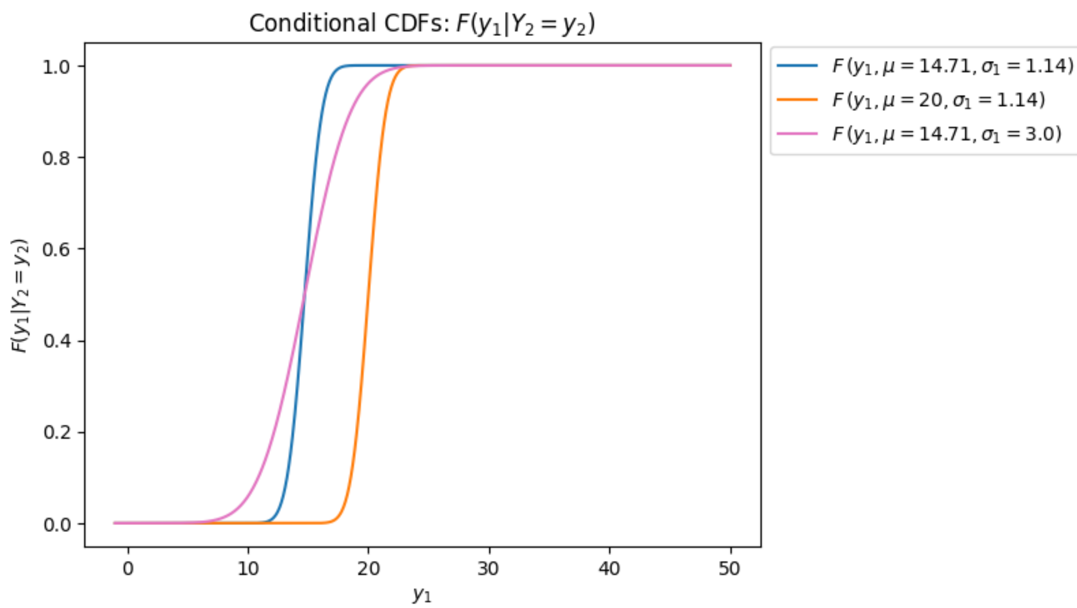


Figure 2.5: Conditional CDFs $F(y_1|Y_2 = y_2)$

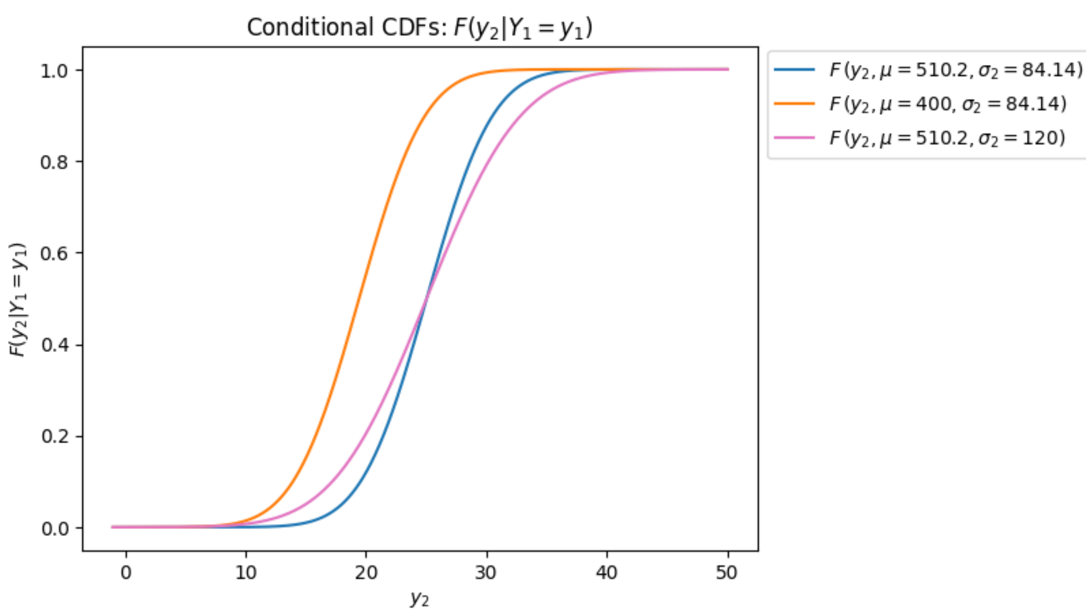


Figure 2.6: Conditional CDFs $F(y_2|Y_1 = y_1)$

## 2.7 Correlation, Covariance, and Independence

In order to investigate the independence of the random variables $Y_1$ and $Y_2$, we begin by computing their covariance. This covariance is defined as

$$Cov(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1) \cdot E(Y_2), \tag{2.7.1}$$

where $E$ denotes the expected value of its argument. Let us find $E(Y_1Y_2)$. By definition,

$$E(Y_1Y_2) = \int_{\mathbf{R}} \int_{\mathbf{R}} y_1 y_2 f(y_1, y_2) dy_1\, dy_2 \tag{2.7.2}$$

where $f(y_1, y_2)$ is the joint PDF 2.0.1. It follows that

$$E(Y_1Y_2) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{\mathbf{R}} y_2 e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} dy_2 \int_{\mathbf{R}} y_1 e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1. \tag{2.7.3}$$

Let $u = \frac{y_1-\mu_1}{\sigma_1^2}$ such that $\sigma_1 du = dy_1$. Further, $y_1 = \sigma_1 u + \mu_1$. Then,

$$\int_{\mathbf{R}} y_1 e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} dy_1 = \int_{\mathbf{R}} (\sigma_1 u + \mu_1) e^{-\frac{1}{2}u^2} \sigma_1 du \tag{2.7.4}$$

$$= \int_{\mathbf{R}} \sigma_1^2 u e^{-\frac{1}{2}u^2} du + \int_{\mathbf{R}} \sigma_1 \mu_1 e^{-\frac{1}{2}u^2} du \tag{2.7.5}$$

$$= \sigma_1^2 \left[ -e^{-\frac{1}{2}u^2} \right]_{-\infty}^{\infty} + \sigma_1\mu_1 \left[ \frac{\sqrt{2\pi}}{2} \operatorname{erf}\left( \frac{u}{\sqrt{2}} \right) \right]_{-\infty}^{\infty} \tag{2.7.6}$$

$$= 0 - \sqrt{2\pi}\mu_1\sigma_1. \tag{2.7.7}$$

Thus 2.7.3 becomes,

$$E(Y_1Y_2) = \frac{-\mu_1}{\sqrt{2\pi}\sigma_2} \int_{\mathbf{R}} y_2 e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} dy_2. \tag{2.7.8}$$

It follows by a similar computation, replacing $y_1$ with $y_2$, that

$$\int_{\mathbf{R}} y_2 e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2} dy_2 = -\sqrt{2\pi}\mu_2\sigma_2. \tag{2.7.9}$$

Hence, 2.7.8 becomes

$$E(Y_1Y_2) = \frac{-\mu_1}{\sqrt{2\pi}\sigma_2} \cdot -\sqrt{2\pi}\mu_2\sigma_2 \tag{2.7.10}$$

$$= \mu_1\mu_2. \qquad \square \tag{2.7.11}$$

Thus, the covariance is

$$Cov(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1) \cdot E(Y_2) \tag{2.7.12}$$

$$= \mu_1\mu_2 - \mu_1\mu_2 = 0, \tag{2.7.13}$$

hence $Y_1$ and $Y_2$ are uncorrelated. Since the covariance is simply the product of $Y_1$'s and $Y_2$'s means subtracted from itself, regardless of the setting the covariance is 0. Note that this implies the correlation $\rho = Cov(Y_1Y_2)/(\sigma_1\sigma_2) = 0$.

Let us now investigate the independence of $Y_1$ and $Y_2$. We say that these random variables are independent if we can write the joint PDF 2.0.1 as

$$f(y_1, y_2) = \varphi(y_1)\psi(y_2), \tag{2.7.14}$$

where $\varphi$ and $\psi$ are nonnegative functions of $y_1$ and $y_2$, respectively. Let us define

$$\varphi(y_1) = \frac{1}{2\pi\sigma_1} e^{-\frac{1}{2}\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2} \tag{2.7.15}$$

$$\psi(y_2) = \frac{1}{\sigma_2} e^{-\frac{1}{2}\left(\frac{y_2-\mu_2}{\sigma_2}\right)^2}. \tag{2.7.16}$$

It follows by 2.7.14, that $Y_1$ and $Y_2$ are independent, regardless of the values for $\mu$ and $\sigma$.

## Chapter 3

# 3D Models and Results

# Chapter 4

# Discussion and Conclusions

Our reference paper [SB87], as well as other literature, see [GSC98], shows a correlation between flood peak and volume. By using the proposed model 2.0.1, we note that the correlation coefficient, $\rho$, is set to 0, hence the model will force a lack of correlation, disagreeing with the current literature.
All plots are made in Python with the Lightburn outlines done in $R$.
Both of authors did not take a photo in WakerSpace, but hope the attendence and results satisfy the project requirements.

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