A Wigner-Eckart theorem for group equivariant convolution kernels

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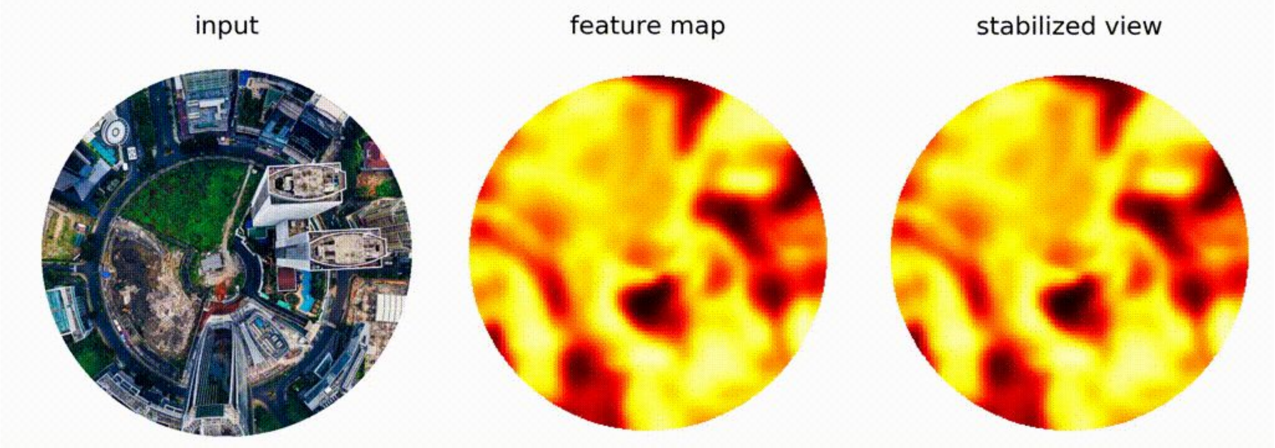
Rotation Equivariant Feature Maps



Visualisation by Gabriele Cesa

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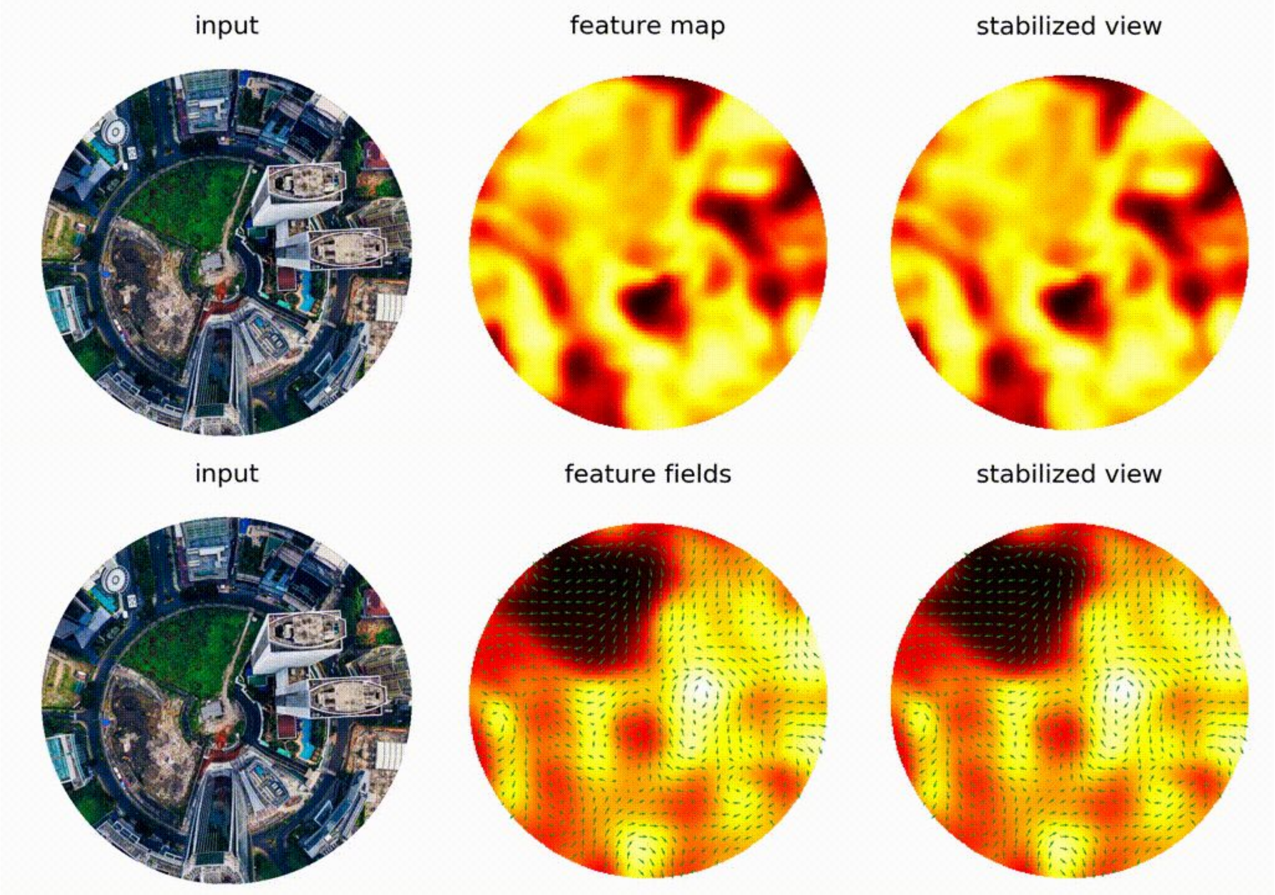

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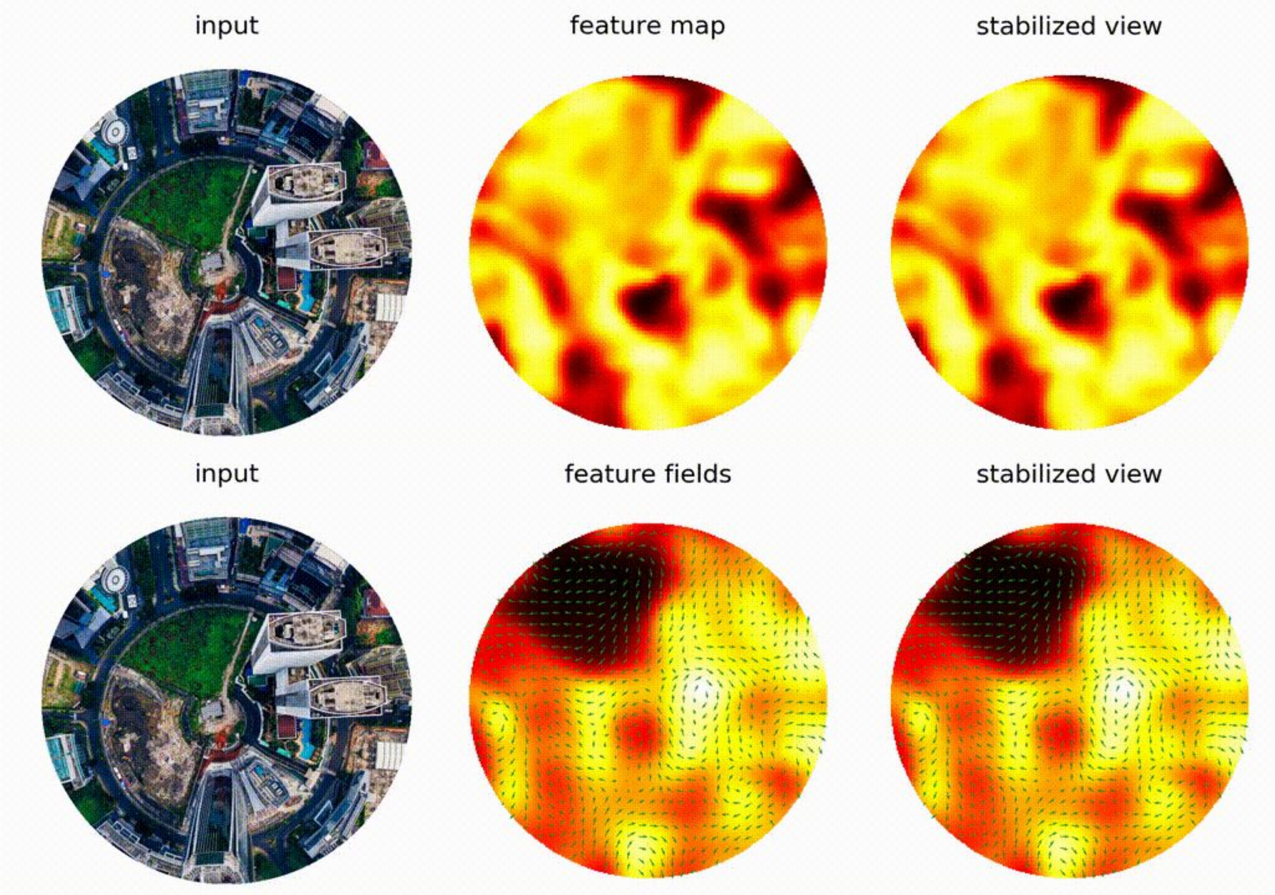

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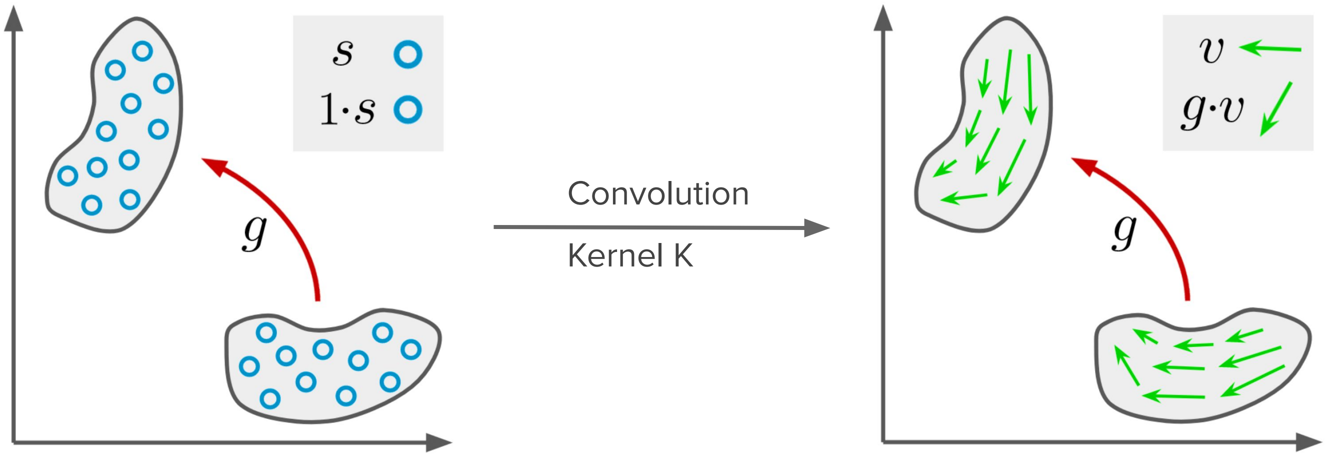
The G-Steerability Constraint



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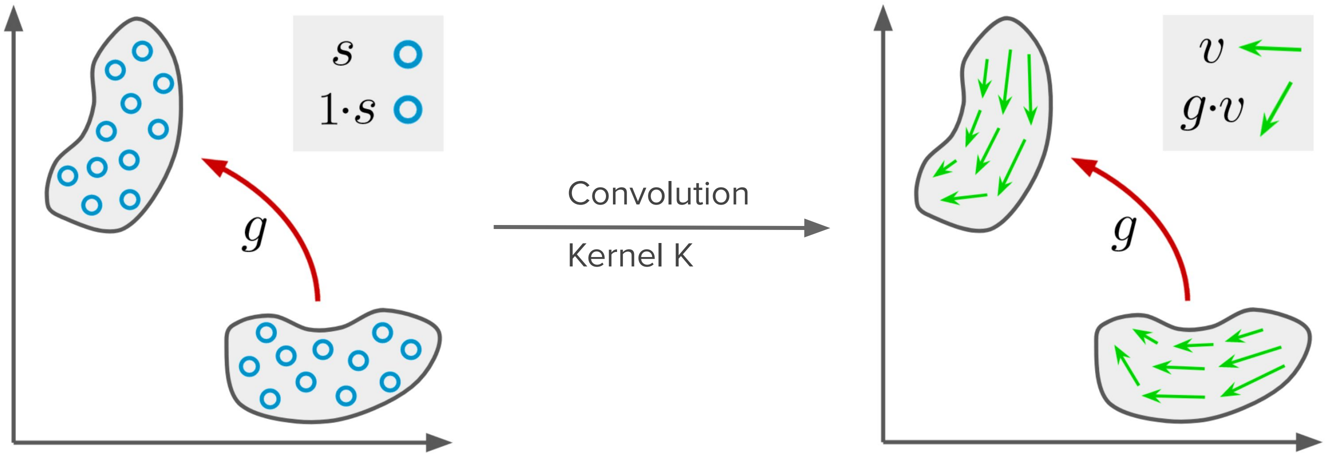

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Quantum operators - Wigner-Eckart theorem

Spherical tensor operators:

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$2j+1$ operators $T_j^m, m = -j, \dots, j$

satisfying symmetry constraint $\sum_{n=-j}^{j} D_j^{mn}(g)\, T_j^n \ = \ U(g)^\dagger T_j^m U(g) \quad \forall g \in \mathrm{SO}(3)$

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can we generalize this result to the matrix elements $\langle JM|K(x)|ln\rangle$ of G-steerable kernels?

Wigner-Eckart theorem for G-steerable Kernels

Wigner-Eckart for steerable kernels = explain dof’s in matrix elements of steerable kernels

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$$\underbrace{\langle JM|K(x)|ln\rangle}_{\text{kernel matrix elements}} = \sum_{j\in\widehat{G}}\sum_{i=1}^{m_j}\sum_{s=1}^{[J(jl)]}\sum_{m=1}^{[j]}\sum_{M'=1}^{[J]} \underbrace{\langle JM\big|c_{jis}\big|JM'\rangle}_{\text{endomorphisms}} \cdot \underbrace{\langle s, JM'\big|jm;ln\rangle}_{\text{Clebsch-Gordan}} \cdot \underbrace{\langle i, jm\big|x\rangle}_{\text{harmonics}}$$

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Learnable Parameters

Example Applications

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[1] Daniel E. Worrall et al. (2016). “Harmonic Networks: Deep Rotation and Translation Equivariance” In: Conference on Computer Vision and Pattern Recognition (CVPR)

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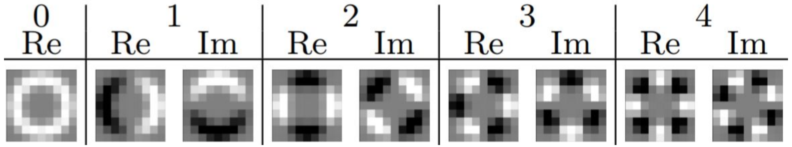
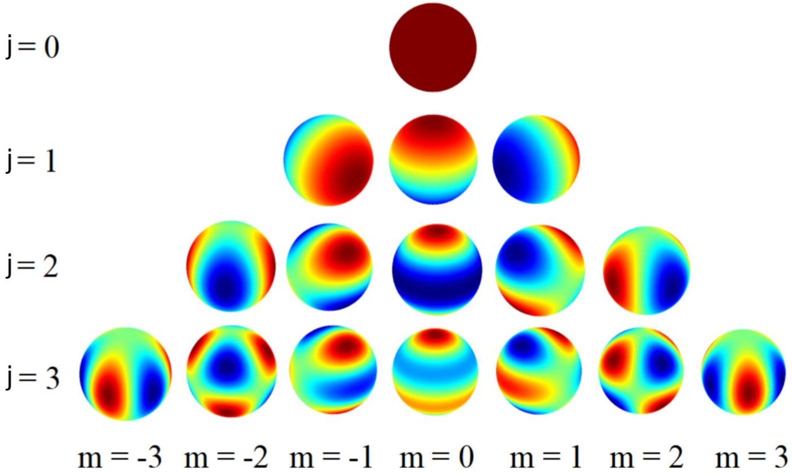
Circular harmonics



Spherical harmonics



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