Solving Kernel constraints with Wigner-Eckart

Contents

1 Harmonic Networks with Wigner-Eckart

1.1 Representation theory of the circle group over $\mathbb{C}$

See also here, chapter 5, for explanations regarding the representation theory of the circle group.

Let $G = SO(2) = \mathbb{R}/\mathbb{Z}$ be the group of rotations in dimension 2. We'll view it as a multiplicative or additive group depending on context. If $g \in G$ then we write g_+ for the additive counterpart in $\mathbb{R}/\mathbb{Z}$.

Let $L^2(S^1)$ be the Hilbert-space of square-integrable functions on S^1, with values in $\mathbb{C}$. It's scalar-product is given by:

$$\langle f, g \rangle = \int_{S^1} \overline{f(s)} g(s) ds.$$

Let the "elementary" functions or characters χ_m be given by

$$\chi_m(s) = e^{2\pi i m s}$$

for $m \in \mathbb{Z}$ and $s \in S^1 = \mathbb{R}/Z$. These functions form an orthonormal basis of $L^2(S^1)$.

Let $\rho : G \to L^2(S^1)$ be the linear representation given by

$$[\rho(g)(f)](s') = f(s' - g_+)$$

which just shifts functions.

Furthermore, the vector space spanned by χ_m over the complex numbers

$$\mathbb{C}\langle\chi_m\rangle = \{c \cdot \chi_m \mid c \in \mathbb{C}\}$$

is, as a representation, isomorphic to the m'th order irrep of G, denoted by V_m. Thus, we can write

$$L^2(S^1) \cong \widehat{\bigoplus_{m\in\mathbb{Z}}} V_m,$$

as an isomorphism of representations, where the hat means that we take the topological closure of the direct sum.

1.2 The Wigner-Eckart Theorem

If we write "V is a representation", without further clarification, then we mean that V is a vector space that comes equipped with a homomorphism $\rho_V : G \to \mathrm{Aut}(V)$.

In this section, we state and prove the Wigner-Eckart theorem, which we will use for solving kernel constraints. The treatment essentially follows the basis-independent form in [Agrawala, 1980].

The main ingredient for this theorem is Schur's Lemma, see [Jeevanjee, 2011]:

Proposition 1.1. *Let V and W be irreducible representations over a group G and let $f : V \to W$ be a linear equivariant map. Then either f is zero or an isomorphism.*

Furthermore, if the ground field is the complex numbers $\mathbb{C}$, then the set of endomorphisms, i.e. linear equivariant maps from V to V, is isomorphic to $\mathbb{C}$ itself:

$$\mathrm{End}(V) = \{c \cdot \mathrm{Id}_V \mid c \in \mathbb{C}\} \cong \mathbb{C}.$$

Furthermore, in order to state the theorem, we need the notion of representations on tensor products and spaces of linear functions between representations:

If T, U and V are representations, we can build the tensor product $T \otimes U$ and the space of linear functions $\mathrm{Lin}(U, V)$. Both carry a representation:

$$\rho(g)(t \otimes u) := \rho_t(t) \otimes \rho_U(u),$$

and

$$[\rho_{\mathrm{Hom}}(g)](f) := \rho_V(g) \circ f \circ \rho_U(g)^{-1}.$$

Definition 1.2. Let T, U and V be three representations. Then a representation operator is a linear equivariant map $\phi : T \to \mathrm{Lin}(U, V)$.

We have the following alternative description of representation operators, proven in [Agrawala, 1980]:

Proposition 1.3. *There is a 1–1 correspondence between representation operators $\phi : T \to \mathrm{Lin}(U, V)$ and linear equivariant maps $\phi' : T \otimes U \to V$. This correspondence is given by*

$$\phi'(t \otimes u) = \phi(t)(u).$$

Theorem 1.4. *Let T, U, V be G-representations, of which V is assumed to be irreducible. Let $\overline{K} : T \to \mathrm{Lin}(U, V)$ be a representation operator. Then $\overline{K}$ is constrained as follows:*

Assume that V appears n times as a direct summand in $T \otimes U$, i.e. there is an isomorphism of representations

$$T \otimes U \cong V^n \oplus W$$

for some other representation W that splits into irreducibles that are all non-isomorphic to V ($n = 0$ is possible and allowed). Let $p_i : T \otimes U \to V$ be the corresponding equivariant linear projections, $i = 1, \ldots, n$. Then $\overline{K}$ is given by

$$\overline{K}(t)(u) = \begin{pmatrix} c_1 & \ldots & c_n \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix} (t \otimes u) = \sum_{i=1}^{n} c_i \left(p_i(t \otimes u) \right)$$

for endomorphisms $c_i : V \to V$ independent of t and u. Furthermore, if the underlying field is the complex numbers $\mathbb{C}$, then the c_i are just complex numbers and called reduced matrix elements of the representation operator.

Sketch of proof. The idea is to use the correspondence Proposition 1.3 in order to get an equivalent description of the space of representation operators:

$$\begin{aligned} \mathrm{Hom}_G(T, \mathrm{Lin}(U, V)) &\cong \mathrm{Hom}_G(T \otimes U, V) \\ &\cong \mathrm{Hom}_G(V^n \oplus W, V) \\ &\cong \bigoplus_{i=1}^{n} \mathrm{Hom}_G(V, V) \oplus \mathrm{Hom}_G(W, V) \\ &= \bigoplus_{i=1}^{n} \mathrm{End}_G(V). \end{aligned}$$

In the second isomorphism, the iso $T \otimes U \cong V^n \oplus W$ was used. The third isomorphism just uses that linear equivariant maps can be described on each direct summand individually. The last equality uses that W does not contain V as a direct summand, and so by Schur's Lemma 1.1, there is no homomorphism $W \to V$. Now the result follows by taking the tuple $(c_1, \ldots, c_n) \in \bigoplus_{i=1}^{n} \mathrm{End}_G(V)$ corresponding to the representation operator $\overline{K}$ under the above isomorphism and explicitly tracing back the isomorphisms from bottom to top to find the form of $\overline{K}$.

The second statement about c_i being complex numbers in the case that the field is $\mathbb{C}$ follows from the second part of Schur's Lemma 1.1. □

Corollary 1.5. *Let $\{\varphi_i \mid i \in I\}$ be a basis of $\mathrm{End}_G(V)$. Then the compositions $\{\varphi_i \circ p_j \mid i \in I, j \in \{1, \ldots, n\}\}$ form a basis of $\mathrm{Hom}_G(T \otimes U, V) = \mathrm{Hom}_G(T, \mathrm{Lin}(U, V))$.*

Proof. Note that the elements $(0, \ldots, 0, \varphi_i, 0, \ldots, 0)$ form a basis of $\mathrm{End}_G(V)^n$. From the proof of Theorem 1.4 it follows that they get mapped by an isomorphism to $\varphi_i \circ p_j$, with j being the index containing φ_i. □

1.3 Solving for the kernel basis with Wigner-Eckart

Let $K : \mathbb{R}^2 \to \mathrm{Lin}(V_m, V_n)$ be a continuous equivariant kernel. By definition, this means that K is continuous (for that to make sense, view $\mathrm{Lin}(V_m, V_n)$ as $\mathbb{C}^{\dim V_m \cdot \dim V_n}$) and it fulfils the equivariance property

$$K(g \cdot x) = \rho_n(g) \circ K(x) \circ \rho_m(g)^{-1}, \tag{1}$$

where $G = SO(2)$ acts as rotations on $\mathbb{R}^2$. However, K is not assumed to be linear in any straightforward way. Since $V_m \cong V_n \cong \mathbb{C}$ when viewed as $\mathbb{C}$-vectorspaces, we can identify $\mathrm{Lin}(V_m, V_n)$ with $\mathbb{C}$. Under this identification, we have $K : \mathbb{R}^{\to}\mathbb{C}$ and are wondering how this function looks like. Note that we will freely move back and forth between these identifications. We will show the following kernel constraint, used, but not proven, for the first time in [Worrall et al., 2016]:

Proposition 1.6. *There is a continuous function $c : \mathbb{R}_{\geq 0} \to \mathbb{C}$ such that K is given, under the identifications from above, by*

$$K(s) = c(|s|) \cdot \chi_{n-m}(s/|s|).$$

Here, $s/|s| \in \mathbb{R}^2$ has norm 1 and is thus viewed as an element in S^1. If $n \neq m$ then $c(0) = 0$.

We will prove this statement using the Wigner-Eckart Theorem 1.4 in its basis-independent form. The idea is to use K to construct a representation operator $\overline{K} : L^2(S^1) \to \mathrm{Lin}(V_m, V_n)$ for each restriction of K to a ring of constant radius in $\mathbb{R}^2$. Since $L^2(S^1) \otimes V_m$ contains V_n exactly once as a direct summand and since we are over $\mathbb{C}$, the theorem tells us that this representation operator is described by exactly one complex number. If we trace back what this means, we end up with the description from above.

Proof of Proposition 1.6. The strategy is to consider each circle-restriction $K_r : S^1 \to \mathrm{Lin}(V_m, V_n), s \mapsto K(rs)$ separately, where $r \in \mathbb{R}_{>0}$ acts as a scalar. Clearly, such a K_r still fulfils the equivariant constraint Equation 1. Thus, consider a fixed r and write, by abuse of notation, $K = K_r$.

We define an "extension" $\overline{K} : L^2(S^1) \to \mathrm{Lin}(V_m, V_n)$ to which we will apply Wigner-Eckart. It is given by:

$$\overline{K}(f)(v) = \int_{S^1} f(s)K(s)(v)ds.$$

Clearly, $\overline{K}$ is linear in f and v. Furthermore, it is equivariant (in the following, G is written additively when acting on f and multiplicatively when acting on K), since:

$$\begin{aligned}
\overline{K}(g_+ \cdot f)(v) &= \int_{S^1} (g_+ \cdot f)(s)K(s)(v)ds \\
&= \int_{S^1} f(s - g_+)K(s)(v)ds \\
&= \int_{S^1} f(s)K(g \cdot s)(v)ds \\
&= \int_{S^1} f(s)\left[\rho_n(g) \circ K(s) \circ \rho_m(g)^{-1}\right](v)ds \\
&= \rho_n(g)\left(\int_{S^1} f(s)K(s)\left(\rho_m(g)^{-1}(v)\right)ds\right) \\
&= \rho_n(g)\left(\overline{K}(f)\left(\rho_m(g)^{-1}(v)\right)\right) \\
&= \left(\rho_{\mathrm{Hom}}(g)\left(\overline{K}(f)\right)\right)(v).
\end{aligned}$$

Consequently, we can apply Wigner-Eckart Theorem 1.4 to find the structure of $\overline{K}$. For doing so, we need to define a projection operator $p : L^2(S^1) \otimes V_m \to V_n$. We do it as follows:

$$p(f \otimes \chi_m) = \langle f \cdot \chi_m, \chi_n \rangle \chi_n,$$

where V_m and V_n are viewed as generated from χ_m and χ_n, respectively. $f \cdot \chi_m$ means the element-wise product. Setting $f = \chi_{n-m}$ we indeed see that this map is surjective. From Wigner-Eckart, we obtain that

$$\overline{K}(f)(\chi_m) = c \cdot \langle f \cdot \chi_m, \chi_n \rangle \chi_n$$

for some constant $c \in \mathbb{C}$. With the identifications from before, $\overline{K}(f) : \mathbb{C} \to \mathbb{C}$ is given by multiplication with $c \cdot \langle f \cdot \chi_m, \chi_n \rangle$.

What's missing is now how to trace this back to a statement about the appearance of the original circle map $K : S^1 \to \mathrm{Lin}(V_m, V_n)$. We do this by viewing elements of S^1 as functions in $L^2(S^1)$ by the corresponding Dirac delta-functions. Therefore, let δ_s be the Dirac-delta function at $s \in S^1$. Then we get:

$$\overline{K}(\delta_s)(v) = \int_{S^1} \delta_s(s')K(s')(v)ds = K(s)(v),$$

by general behaviour of the Dirac-delta. Thus, we can compute K as follows:

$$\begin{aligned} K(s)(\chi_m) &= \overline{K}(\delta_s)(\chi_m) \\ &= c \cdot \langle \delta_s \cdot \chi_m, \chi_n \rangle \\ &= c \cdot \int_{S^1} \overline{\delta_s(s') \cdot \chi_m(s')} \cdot \chi_n(s')ds' \\ &= c \cdot \int_{S^1} \delta_s(s') \cdot \chi_{n-m}(s')ds' \\ &= c \cdot \chi_{n-m}(s). \end{aligned}$$

Now, remember that we did abuse of notation, i.e. we have just computed that $K_r : S^1 \to \mathbb{C}$ is given by $K_r(s) = c(r) \cdot \chi_{n-m}(s)$ with $c(r)$ depending on r. This means that $K(s) = K_{|s|}(s/|s|) = c(|s|) \cdot \chi_{n-m}(s/|s|)$. Since the kernel K is continuous, the map c needs to be continuous. Furthermore, $c(0) = 0$ also due to continuity, unless $m = n$ in which case χ_0 is constant and so $c(0)$ can in principle take any complex number. $\square$

2 General Steerable CNNs with Wigner-Eckart

2.1 Strategy for finding steerable kernel-bases for arbitrary groups

The example of harmonic networks in the last section highlights how the Wigner-Eckart theorem can be applied in order to find a basis for steerable kernels. However, this treatment was rather simple in four ways:

1. As $\mathbb{C}$ is algebraically closed, we obtained from Schur's Lemma that the endomorphisms of V_n are just given by multiplication with a complex number. Over the real numbers, the theory requires us to classify the endomorphisms of the irreps in a more careful manner.

2. Since V_n appeared exactly once as a direct summand in $L^2(S^1) \otimes V_m$, there was only the need to define one projection operator and not several.

3. The only projection-operator was extremely simple, just mapping $f \otimes \chi_m$ to $\langle f \cdot \chi_m, \chi_n \rangle \chi_n$. Later, we will see that this in general involves very non-trivial Clebsch-Gordan coefficients.

4. The bases of the irreps V_m were just given by one character χ_m, which is considerably simpler than in the general theory.

In order to deal with these complications, we will therefore in the next subsection prove a Wigner-Eckart theorem for steerable kernel bases that works for all groups that are usually considered. The end-result will be an explicit description of the steerable kernel-bases using exactly three ingredients, already hinted at with the complications described above. These ingredients are:

1. A basis for the G-endomorphisms of all irreps.

2. All Clebsch-Gordan coefficients underlying the decompositions of all tensor products $V_l \otimes V_m$ into irreps.

3. Orthonormal bases of V_l when viewed as a direct summand in $L^2(S)$, where $S = S^1$ or $S = S^2$ in applications.

2.2 Version of the Wigner-Eckart theorem for general steerable kernels

In this section, we provide a version of the Wigner-Eckart theorem specifically for steerable kernels. We formulate it as general as possible, so that we can apply it in many settings. The definitions are deliberately a little vague and the proofs therefore only meant as a "hint" for how to prove the statements in generality.

In this section, let G be a compact topological group, for example $SO(2)$, $O(2)$, C_n, D_n, $SO(3)$, $O(3)$, $SU(1)$, $SU(2)$, $SU(3)$ etc.

Furthermore, assume G acts on a space S like S^1 or S^2 from the left. Also assume that S carries a space of square-integrable functions $L^2(S)$ with values in a field $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. G acts on this space by $(g \cdot f)(s) = f(g^{-1} \cdot s)$. Furthermore, $L^2(S)$ is assumed to be a Hilbert-space by means of the scalar product

$$\langle f, g \rangle = \int_S \overline{f(s)} g(s) ds.$$

In that formula, the complex conjugation does not do anything if $\mathbb{K} = \mathbb{R}$.

We also assume that the irreps are given by $(V_m)_{m \in \mathbb{Z}}$ (or indexed by $m \in \mathbb{N}$) and that every irrep appears exactly once in $L^2(S)$ as a direct summand.

For m, let $[m]$ denote the $\mathbb{K}$-dimension of V_m. Let $\{Y_i^m \mid i \in \{1, \ldots, [m]\}\}$ be an orthonormal standard basis of $V_m \subseteq L^2(S)$, such that the union of all these functions is an orthonormal basis of $L^2(S)$. For example, if $G = SO(2)$, $S = S^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $\mathbb{K} = \mathbb{C}$, then these functions are just the characters $Y_1^m = \chi_m$. For $\mathbb{K} = \mathbb{R}$, these function are given by $Y_1^m = \cos_m$ and $Y_2^m = \sin_m$ (probably up to some scalar!). For $G = SO(3)$ and $S = S^2$, we obtain the spherical harmonics.

For lack of a better notation, let $\overline{m}$ denote the $\mathbb{K}$-dimension of $\mathrm{End}_G(V_m)$. Denote by $\{\varphi_r \mid r \in \{1, \ldots, \overline{m}\}$ a basis of this space.

Now, with somewhat inconvenient notation, we need to define the Clebsch-Gordan coefficients. Thus, assume that we have given two irreps V_l, V_m. For a third irrep V_n, let $[n, (l, m)]$ denote the number of times V_n appears in a direct sum decomposition of $V_l \otimes V_m$ (This number can be larger than 1! For example, it turns out that V_0 is twice a direct summand of $V_m \otimes V_m$ for $m \geq 1$ and $\mathbb{K} = \mathbb{R}$). Thus, for $n \in \mathbb{Z}$ and

$s \in \{1, \dots, [n, (l, m)]\}$ there are copies $V_n^s \subseteq V_l \otimes V_m$ of V_n in the tensor product such that we get an inner direct sum decomposition

$$V_l \otimes V_m = \bigoplus_{n \in \mathbb{Z}} \bigoplus_{s=1}^{[n,(l,m)]} V_n^s.$$

Let $Y_k^{s;n}$ be basis elements in V_n^s corresponding to the standard basis elements Y_k^n of V_n. Then, we can write the standard basis elements $Y_i^l \otimes Y_j^m$ of $V_l \otimes V_m$ by means of these basis elements as follows:

$$Y_i^l \otimes Y_j^m = \sum_{n \in \mathbb{Z}} \sum_{s=1}^{[n,(l,m)]} \sum_{k=1}^{[n]} q_{k,(i,j)}^{s;n,(l,m)} Y_k^{s;n}.$$

The indices $q_{k,(i,j)}^{s;n,(l,m)}$ are called the *Clebsch-Gordan coefficients* corresponding to an explicit decomposition of $V_l \otimes V_m$ into irreps.

Note that the Clebsch-Gordan coefficients immediately induce equivariant projections $q^{s;n,(l,m)} : V_l \otimes V_m \to V_n$, given on the basis by

$$q^{s;n,(l,m)}(Y_i^l \otimes Y_j^m) = \sum_{k=1}^{[n]} q_{k,(i,j)}^{s;n,(l,m)} Y_k^n.$$

Thus, for fixed s, n, l and m, $q^{s;n,(l,m)}$ can be viewed as a matrix of shape $[n] \times ([l] \cdot [m])$. If $[n, (l, m)] = 1$, then for convenience we drop the index s and just write $q^{n,(l,m)}$.

Our final ingredient is the following: for $l \in \mathbb{Z}$, let $p_l : L^2(S) \to V_l$ be the canonical projection, given explicitly by

$$p_l(f) = \sum_{i=1}^{[l]} \langle f, Y_i^l \rangle Y_i^l.$$

To reduce clutter, we denote by p_l also the projection $p_l = p_l \otimes \mathrm{Id} : L^2(S) \otimes V_m \to V_l \otimes V_m$. By means of the correspondence from Proposition 1.3, we also view p_l as a homomorphism $p_l : \mathrm{Hom}_G(L^2(S), \mathrm{Hom}_{\mathbb{K}}(V_m, V_l \otimes V_m))$ when need arises.

Proposition 2.1. *There is an isomorphism*

$$\begin{tikzcd} \mathrm{Hom}_G(S, \mathrm{Hom}_{\mathbb{K}}(U, V)) \arrow[r, bend left, "\overline{(\cdot)}"] & \mathrm{Hom}_G(L^2(S), \mathrm{Hom}_{\mathbb{K}}(U, V)) \arrow[l, bend left, "(\cdot)|_S"] \end{tikzcd}$$

between the space of continuous steerable kernels on the left and the space of continuous representation operators on the right, given by $\overline{K}(f)(u) = \int_S f(s)K(s)(u)ds$ *and* $K'|_S(s)(u) = K'(\delta_s)(u)$*, where* δ_s *is the Dirac Delta function.*

Note that the Wigner-Eckart theorem 1.6 is formulated for three G-representations T, U and V and in terms of V being a direct summand of $T \otimes U$. However, as we also saw (or will see), we actually use it slightly different. Namely, we have a space $L^2(S) \otimes V_m$ and first project to all the spaces $V_l \otimes V_m$ such that V_n is (possibly several times) a direct summand. Furthermore, we are not interested in the specific form of *one representation operator* but want to parameterize the space of all such operators. This means that we are well-advised to make the basis of representation operators more explisit. This is the content of the following theorem:

Theorem 2.2. *Let V_m and V_n be irreps. Let there be the following sets of projections and endomorphisms, as defined above:*

1. $\left\{p_l = p_l \otimes \mathrm{Id} : L^2(S) \otimes V_m \to V_l \otimes V_m \mid l \in \mathbb{Z}\right\}$*, is the set of canonical projections.*
2. $\left\{q^{s;n,(l,m)} : V_l \otimes V_m \to V_n \mid s \in \{1, \dots, [n, (l, m)]\}\right\}$ *is the set of projections from $V_l \otimes V_m$ to V_n that emerge from the Clebsch-Gordan coefficients.*
3. $\{\varphi_r \mid r = 1, \dots, \overline{n}\}$ *is a basis of* $\mathrm{End}_G(V_n)$.

Then the set of all possible compositions

$$\left\{\varphi_r \circ q^{s;n,(l,m)} \circ p_l\right\}$$

form a basis of $\mathrm{Hom}_G(L^1(S), \mathrm{Lin}(V_m, V_n))$ *by setting, with abuse of notation,*

$$(\varphi_r \circ q^{s;n,(l,m)} \circ p_l)(f)(v) = (\varphi_r \circ q^{s;n,(l,m)} \circ p_l)(f \otimes v).$$

Furthermore, the set of compositions $\varphi_r \circ q^{s;n,(l,m)} \circ p_l|_S$ forms a basis of steerable kernels $S \to \mathrm{Lin}(V_m, V_n)$.

Proof. The projections $q^{s;n,(l,m)} \circ p_l$ correspond to the projections in the Wigner-Eckart theorem 1.4 and Corollary 1.5. Then it follows directly from that Corollary that the compositions $\varphi_r \circ (q^{s;n,(l,m)} \circ p_l) = \varphi_r \circ q^{s;n,(l,m)} \circ p_l$ form a basis of the space of representation operators. Consequently we see from Proposition 2.1 that we get a basis of steerable kernels by restricting to S. □

The final goal is to get more insight into how the Steerable kernel-basis, given by the compositions $\varphi_r \circ q^{s;n,(l,m)} \circ p_l|_S$, looks like more explicitly. Fortunately, the projections q are already given in matrix-form. We now do the same for p:

Corollary 2.3. *For $s \in S$, let $p_l|_S(s) : V_m \to V_l \otimes V_m$. Then, with respect to the standard bases of $V_l \otimes V_m$ and V_m, the matrix-elements of this are given by:*

$$[p_l|_S(s)]_{(i,j),h} = \begin{cases} Y_i^l(s), & j = h \\ 0, & j \neq h \end{cases}$$

Proof. This follows directly from the computation

$$\begin{aligned}
[p_l|_S(s)]\,(Y_h^m) &= p_l(\delta_s \otimes Y_h^m) \\
&= p_l(\delta_s) \otimes Y_h^m \\
&= \sum_{i=1}^{[l]} \left\langle \delta_s, Y_i^l \right\rangle \left(Y_i^l \otimes Y_h^m\right) \\
&= \sum_{i=1}^{[l]} Y_i^l(s) \left(Y_i^l \otimes Y_h^m\right).
\end{aligned}$$

In the last step, the following computation was used, which follows from the properties of the Dirac-delta:

$$\langle \delta_s, f \rangle = \int_S \delta_s(s') f(s') ds' = f(s).$$

□

We note the following intuitive interpretation of this Corollary: Let $p_l|_S(s) \in \mathrm{Hom}_{\mathbb{K}}(V_m, V_l \otimes V_m)$ be given by a $([l] \times [m]) \times [m]$-matrix. Let $Y^l(s)$ be the column-vector with entries $Y_i^l(s)$. Then this matrix is given by:

$$p_l|_S(s) = \left(\begin{bmatrix} Y^l(s) & 0 & \dots & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & Y^l(s) & 0 & \dots & 0 \end{bmatrix} \quad \dots \quad \begin{bmatrix} 0 & \dots & 0 & Y^l(s) \end{bmatrix}\right)$$

Thus far, we have written all projections explicitly in matrix-form. We furthermore assume that also the basis-endomorphisms $\varphi_r : V_n \to V_n$ are given by their corresponding bases, i.e. we have

$$\varphi_r(Y_k^n) = \sum_{k'=1}^{[n]} (\varphi_r)_{k'k} \, Y_{k'}^n.$$

Finally, we get the explicit matrix-form of the basis-kernels:

Theorem 2.4 (Basis-Kernels, matrix-form). *The basis-kernels are given by*

$$\varphi_r \circ q^{s;n,(l,m)} \circ p_l|_S(s) = \varphi_r \cdot \begin{pmatrix} Y^l(s)^T \cdot q_1^{s;n,(l,m)} \\ \vdots \\ Y^l(s)^T \cdot q_{[n]}^{s;n,(l,m)} \end{pmatrix},$$

Where each "dot" denotes just conventional matrix multiplication.

Before we prove this, note that the shapes of the involved matrices makes sense:

1. $Y^l(s)^T$ is of shape $1\times[l]$ and $q^{s;n,(l,m)}$ of shape $[l]\times[m]$, so they can be multiplied. The result is of size $1 \times [m]$. When stacking all these results, the outcome is of shape $[n] \times [m]$.
2. φ_r is of shape $[n] \times [n]$ and thus can be multiplied with the right outcome to something of shape $[n] \times [m]$ again.
3. Shape $[n] \times [m]$ is exactly what we expect for a linear map $V_m \to V_n$, so the outcome is of the required form.

Proof. Since φ does not change in the computation, we only need to engage with the second and third factor. Denote by $\odot$ the elementwise product of two matrices, followed by a sum over all resulting entries. Then we obtain:

$$\begin{aligned} q^{s;n,(l,m)} \circ p_l|_S(s) &= \begin{pmatrix} q_1^{s;n,(l,m)} \\ \vdots \\ q_{[n]}^{s;n,(l,m)} \end{pmatrix} \cdot \left(\begin{bmatrix} Y^l(s) & 0 & \dots & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & Y^l(s) & 0 & \dots & 0 \end{bmatrix} \quad \dots \quad \begin{bmatrix} 0 & \dots & 0 & Y^l(s) \end{bmatrix}\right) \\ &= \left(q_k^{s;n,(l,m)} \odot \begin{bmatrix} 0 & \dots & 0 & Y^l(s) & 0 & \dots & 0 \end{bmatrix}\right)_{k,j=1}^{[n],[m]} \\ &= \left(Y^l(s)^T \cdot q_{k,(-,j)}^{s;n,(l,m)}\right)_{k,j=1}^{[n],[m]} \\ &= \left(Y^l(s)^T \cdot q_k^{s;n,(l,m)}\right)_{k=1}^{[n]}, \end{aligned}$$

which is exactly what we wanted to show. □

2.3 Harmonic Networks Revisited

As we saw in Section 1.3 and proved with quite some effort, a basis of steerable kernels $S^1 \longrightarrow \mathrm{Lin}(V_m, V_n)$ for $SO(2)$ over $\mathbb{C}$ is given by the character χ_{n-m}. How can we see this again, using what we learned in the last section? We just determine the three "ingredients" that the theory requires:

1. The endomorphisms $\mathrm{End}_{SO(2)}(V_m)$ have Id_{V_m} as a basis, due to Schur's Lemma. Thus, we can just ignore these endomorphisms altogether, since postcomposition with the identity doesn't change anything.

2. $V_l \otimes V_m \cong V_{l+m}$, and the basis element $\chi_l \otimes \chi_m$ maps directly to the basis element χ_{l+m} by this isomorphism. Thus, the only existing Clebsch-Gordan coefficient is just 1.

3. The orthonormal bases of V_m are just given by the characters χ_m.

Finally, note that $V_{n-m} \otimes V_m \cong V_n$. Thus, $l = n - m$ is the only index such that V_n appears within $V_l \otimes V_m$. Plugging everything into Theorem 2.4, we see that χ_{n-m} is a basis for steerable kernels.

2.4 Representation Theory of the circle group over $\mathbb{R}$

In this section, we do the representation-theoretic preparation, determining the "ingredients" that we need in order to apply Theorem 2.4 to determine bases for steerable kernels of $SO(2)$ over $\mathbb{R}$.

We view $S^1 = \mathbb{R}/2\pi\mathbb{Z}$, that is we take the interval $[0, 2\pi]$ as the space where our functions are defined. Consequently, we have to put the fraction $\frac{1}{2\pi}$ before all of our integrals, different from what we did in our treatment of $SO(2)$ over $\mathbb{C}$.

The irreps of $SO(2)$ over $\mathbb{R}$ are given by V_m, $m \in \mathbb{N}_{\geq 0}$, where for $m \geq 1$, $V_m = \mathbb{R}^2$ as a vector space. The action is given by

$$\rho_m(\phi)(v) = \begin{pmatrix} \cos(m\phi) & -\sin(m\phi) \\ \sin(m\phi) & \cos(m\phi) \end{pmatrix} \cdot v$$

for $\phi \in SO(2) \cong \mathbb{R}/2\pi\mathbb{Z}$. The trivial representation is given by $V_0 = \mathbb{R}$ together with the trivial action.

Now look at square-integrable functions $L^2(S^1)$ that we now assume to take *real values.* As before, $SO(2)$ acts on this space by $(g \cdot f)(s) = f(s - g_+)$. This space again forms a Hilbert space using the scalar product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{S^1} f(s)g(s)ds.$$

For notational simplicity, we write $\cos_m$ for the function that maps s to $\cos(ms)$, and analogously for $\sin_m$. One then can show the following, which we take as a given:

Proposition 2.5. *The functions $\cos_m$, $\sin_m$, $m \geq 1$ span an irreducible invariant subspace of $L^2(S^1)$ of dimension 2, explicitly given by*

$$\mathbb{R}\langle \cos_m, \sin_m \rangle = \{\alpha \cos_m + \beta \sin_m \mid \alpha, \beta \in \mathbb{R}\}$$

which is equivariantly isomorphic to V_m *by* $\cos_m \mapsto \begin{pmatrix}1\\0\end{pmatrix}$ *and* $\sin_m \mapsto \begin{pmatrix}0\\1\end{pmatrix}$. *Furthermore,* $\sin_0 = 0$ *and* $\cos_0 = 1$ *are constant functions and their span is* 1*-dimensional and equivariantly isomorphic to* V_0 *by* $\cos_0 \mapsto 1$.

Finally, the functions $\cos_m, \sin_m$ *form an orthogonal basis of* $L^2(S^1)$*, i.e. every function can be written uniq über Lucas?uely as a (possibly infinite) linear combination of these basis functions (This is a standard result about Fourier series')*[1]. *Overall, these results mean that there is a direct sum decompostion*

$$L^2(S^1) = \widehat{\bigoplus_{m\geq 0}} V_m.$$

Note that in the following we will often write V_m for $R\langle \cos_m, \sin_m\rangle$ etc. when from the context it is clear that the space lies in $L^2(S^1)$. Also note that $\cos_m = Y_1^m$ and $\sin_m = Y_2^m$ in the notation of Section 2.2. This already determines one of our ingredients for applying Theorem 2.4!

We now describe the endomorphisms of the irreps, our second ingredient:

Proposition 2.6. *We have* $\mathrm{End}(V_0) \cong \mathbb{R}$*, i.e. multiplications with all real numbers are valid endomorphisms of* V_0. *For* $m \geq 1$*, we get*

$$\mathrm{End}(V_m) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\},$$

which is the set of all scaled rotations of $\mathbb{R}^2$. *When identifying* $\mathbb{R}^2 \cong \mathbb{C}$*, we can also view these transformations as arbitrary multiplications with a complex number.*

As a consequence, $\mathrm{Id}_{\mathbb{R}}$ *is a basis for* $\mathrm{End}(V_0)$ *and* $\left\{\mathrm{Id}_{\mathbb{R}^2}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$ *a basis for* $\mathrm{End}(V_m)$ *for* $m \geq 1$.

Proof Sketch. For an arbitrary matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that commutes with all rotation matrices $\rho_m(\phi)$, i.e. $M \circ \rho_m(\phi) = \rho_m(\phi) \circ M$, one can easily show the constraints $a = d$ and $b = -c$, from which the result follows. $\square$

We now do the explicit decomposition of $V_k \otimes V_m$ into irreps, which will give us the Clebsch-Gordan coefficients that we need. This will be the final ingredient needed for applying Theorem 2.4. For doing so, we first need some trigonometric formulas in our disposal:

Proposition 2.7. *There are the following laws underlying sinus and cosinus. The first two are well-known and the last three follow directly from the first two using* $\sin_{-l} = -\sin_l$ *and* $\cos_{-l} = \cos_l$ *where needed.*

1. $\sin_{l+m} = \sin_l \cos_m + \cos_l \sin_m$.
2. $\cos_{l+m} = \cos_l \cos_m - \sin_l \sin_m$.
3. $\cos_l \cos_m = \frac{1}{2}\left[\cos_{l+m} + \cos_{l-m}\right]$.
4. $\sin_l \cos_m = \frac{1}{2}\left[\sin_{l+m} + \sin_{l-m}\right] = \frac{1}{2}\left[\sin_{l+m} - \sin_{m-l}\right]$.

[1] They are *not quite* orthonormal I believe, since $\langle \cos_m, \cos_m\rangle = \langle \sin_m, \sin_m\rangle = \frac{1}{2}$ for $m \geq 1$ (is that correct? My brother says this is wrong)

5. $\sin_l \sin_m = \frac{1}{2}[\cos_{l-m} - \cos_{l+m}]$

The following Lemma can easily be proven, so we take it as a given:

Lemma 2.8. *Let $f : T \longrightarrow U$ be a linear equivariant map. Then $\ker(f) = \{t \in T \mid f(t) = 0\}$ is an invariant linear subspace of T.*

Proposition 2.9. *We have the following decomposition results:*

1. *For $l = m = 0$ we have $V_0 \otimes V_0 \cong V_0$ by $q^{0,(0,0)} = \left(\begin{bmatrix}1\end{bmatrix}\right)$.*
2. *For $l = 0$, $m > 0$ we have $V_0 \otimes V_m \cong V_m$ by $q^{m,(0,m)} = \left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$.*
3. *For $l > 0$, $m = 0$, we get $V_l \otimes V_0 \cong V_l$ by $q^{l,(l,0)} = \left(\begin{bmatrix}1\\0\end{bmatrix} \atop \begin{bmatrix}0\\1\end{bmatrix}\right)$.*
4. *For $l, m \geq 0$ and $l \neq m$ we get $V_l \otimes V_m \cong V_{|l-m|} \oplus V_{l+m}$. The Clebsch-Gordan coefficients are given by $q^{|l-m|,(l,m)} = \left(\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix} \atop \begin{bmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}\right)$ and $q^{l+m,(l,m)} = \left(\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix} \atop \begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}\right)$*
5. *For $l = m > 0$, we get an isomorphism $V_m \otimes V_m \cong V_{2m} \oplus V_0^2$. We obtain the Clebsch-Gordan coefficients $q^{1;0,(m,m)} = \left(\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\right)$, $q^{2;0,(m,m)} = \left(\begin{bmatrix}0 & -\frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}\right)$ and $q^{2m,(m,m)} = \left(\begin{bmatrix}\frac{1}{2} & 0\\ 0 & -\frac{1}{2}\end{bmatrix} \atop \begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{2} & 0\end{bmatrix}\right)$, the last one corresponding to the Clebsch-Gordan coefficients $q^{l+m,(l,m)}$ from above.*

Proof. The proof of 1, 2 and 3 is clear.

For 4, consider the standard basis $\{b_{cc}, b_{cs}, b_{sc}, b_{ss}\}$ of $V_l \otimes V_m$, where for example $b_{cc} = \cos_l \otimes \cos_m$ etc. Our goal is to express these basis-elements with respect to basis-elements of invariant subspaces. We do this by explicitly constructing an isomorphism to a decomposition of irreps. To that end, let $p : V_l \otimes V_m \longrightarrow L^2(S^1)$ be given by $f \otimes g \mapsto f \cdot g$, which is clearly equivariant and linear. Let $b'_{cc} = p(b_{cc})$ etc. We get as image of p the set

$$\mathbb{R}\langle b'_{cc}, b'_{cs}, b'_{sc}, b'_{ss}\rangle = \mathbb{R}\langle \cos_l \cdot \cos_m, \cos_l \cdot \sin_m, \sin_l \cdot \cos_m \sin_l \cdot \sin_m\rangle,$$

From rules 1, 3, 4 and 5 of Proposition 2.7 we obtain:

$$b'_{cc} - b'_{ss} = \cos_{l+m},\ b'_{cs} + b'_{sc} = \sin_{l+m},\ b'_{cc} + b'_{ss} = \cos_{l-m},\ b'_{sc} - b'_{cs} = \sin_{l-m}.$$

Since these are linearly independent generators, we obtain:

$$\operatorname{Im}(p) = \mathbb{R}\langle \cos_{l+m}, \sin_{l+m}, \cos_{l-m}, \sin_{l-m}\rangle = V_{|l-m|} \oplus V_{m+l}.$$

Note for the last step that due to symmetry, $\cos_{l-m} = \cos_{m-l}$ and $\sin_{l-m} = -\sin_{m-l}$. Now define the following second set of basis-elements in $V_l \otimes V_m$, corresponding to the basis-elements $V_{|l-m|} \oplus V_{m+l}$ by means of the isomorphism p:

$$c_1 = b_{cc} - b_{ss},\ c_2 = b_{cs} + b_{sc},\ c_3 = b_{cc} + b_{ss},\ c_4 = b_{sc} - b_{cs}.$$

We obtain $V_l \otimes V_m = \mathbb{R}\langle c_1, c_2\rangle \oplus \mathbb{R}\langle c_3, c_4\rangle \cong V_{m+l} \oplus V_{|l-m|}$. From the following equations we can read off the Clebsch-Gordan coefficients:

$$b_{cc} = \frac{1}{2}\left[c_1 + c_3\right],\ b_{cs} = \frac{1}{2}\left[c_2 - c_4\right],\ b_{sc} = \frac{1}{2}\left[c_2 + c_4\right],\ b_{ss} = \frac{1}{2}\left[c_3 - c_1\right]$$

The result follows.

Now, we prove 5. We have $l = m$ and still consider the same map p and overall notation. Note that $b'_{sc} - b'_{cs} = 0$ and $b'_{cc} + b'_{ss} = 1$ is the constant function. Then by what was proven above, $\{c_1, c_2\}$ spans a space isomorphic to V_{2m}, c_3 a space isomorphic to the span of $\cos_0$, i.e. V_0, and c_4 spans the kernel, which is one-dimensional and also an invariant subspace due to Lemma 2.8, and therefore it spans a space isomorphic to V_0 as well. Overall, we obtain $V_m \otimes V_m = \mathbb{R}\langle c_1, c_2\rangle \oplus \mathbb{R}\langle c_3\rangle \oplus \mathbb{R}\langle c_4\rangle \cong V_{2m} \oplus V_0^2$. From this, we can as before read off the Clebsch-Gordan coefficients and obtain the claimed result. □

2.5 Solving the kernel constraint of $SO(2)$ over $\mathbb{R}$ with Wigner-Eckart

Now we have done all needed preparation and can solve the kernel constraint explicitly, using the matrix-form of Wigner-Eckart for steerable kernels, Theorem 2.4.

Proposition 2.10. *Let $K : S^1 \to \mathrm{Lin}(V_m, V_n)$ be an equivariant kernel, where V_m and V_n are irreps. Then the following holds:*

1. *For $m = n = 0$, we get $K(s) = a \cdot \left(1\right)$ for an arbitrary real number $a \in \mathbb{R}$.*
2. *For $m = 0$, $n > 0$, a basis for equivariant kernels is given by $\begin{pmatrix}\cos_n \\ \sin_n\end{pmatrix}$ and $\begin{pmatrix}-\sin_n \\ \cos_n\end{pmatrix}$.*
3. *For $m > 0$ and $n = 0$, a basis for equivariant kernels is given by $\begin{pmatrix}\cos_m & \sin_m\end{pmatrix}$, $\begin{pmatrix}\sin_m & -\cos_m\end{pmatrix}$.*
4. *For $m, n > 0$, a basis for equivariant kernels is given by the results stated in the proof.*

Proof. The proof of 1 is clear.

For 2, note that V_n can only appear in $V_l \otimes V_0$ if $l = n$. The relevant Clebsch-Gordan coefficients are therefore $q^{n,(n,0)} = \left(\begin{bmatrix}1 \\ 0\end{bmatrix} \begin{bmatrix}0 \\ 1\end{bmatrix}\right)$. Furthermore, the orthonormal basis of $V_l = V_n$ is given by $\cos_n$, $\sin_n$, which we have to write as a row-vector. Our final ingredient is the endomorphism basis of V_n, which is given by $\varphi_1 = \mathrm{Id}_{\mathbb{R}^2}$ and $\varphi_2 = \begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix}$. Overall, the basis-kernels are given by

$$\varphi_i \cdot \begin{pmatrix} \begin{bmatrix}\cos_n(s) & \sin_n(s)\end{bmatrix} \cdot \begin{bmatrix}1 \\ 0\end{bmatrix} \\ \begin{bmatrix}\cos_n(s) & \sin_n(s)\end{bmatrix} \cdot \begin{bmatrix}0 \\ 1\end{bmatrix} \end{pmatrix} = \varphi_i \cdot \begin{pmatrix}\cos_n(s) \\ \sin_n(s)\end{pmatrix}.$$

The result follows.

For 3, we find V_0 only in $V_l \otimes V_m$ if $l = m$, and even twice so. The relevant Clebsch-Gordan coefficients are therefore given by $q^{1;0,(m,m)} = \left(\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right)$ and $q^{2;0,(m,m)} = \left(\begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right)$. The basis-functions in $V_l = V_m$ are $\cos_m$ and $\sin_m$, again written as a row-vector. Finally, $V_n = V_0$ has only $\mathrm{Id}_{\mathbb{R}}$ as a basis-endomorphism, so this can be ignored. We obtain the following basis for steerable kernels:

$$\left(\begin{bmatrix} \cos_m & \sin_m \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right) = \begin{pmatrix} \frac{1}{2}\cos_m & \frac{1}{2}\sin_m \end{pmatrix}$$
$$\left(\begin{bmatrix} \cos_m & \sin_m \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right) = \begin{pmatrix} \frac{1}{2}\sin_m & -\frac{1}{2}\cos_m \end{pmatrix}.$$

For 4, we consider only the case $m > n$. The case $m = n$ and $m < 0$ can probably be considered analogously and leads by trigonometric formulae to the same result (to check). We have

$$V_{m-n} \otimes V_m \cong V_n \oplus V_{2m-n}, \;\; V_{m+n} \otimes V_m \cong V_n \oplus V_{2m+n},$$

i.e. $l = m-n$ and $l = m+n$ leads to a tensor product decomposition containing V_n, but no other l. Thus, the relevant Clebsch-Gordan coefficients are $q^{n,(m-n,m)}$ and $q^{n,(m+n,m)}$, which are both equal and given by $\begin{pmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \end{pmatrix}$. For $l = m-n$ and $l = m+n$, the basis functions of V_{m-n} and V_{m+n} are furthermore given by $\cos_{n-m}$, $\sin_{n-m}$ and $\cos_{m+n}$, $\sin_{m+n}$ respectively. Finally, V_n has again the two basis endomorphisms $\varphi_1 = \mathrm{Id}_{\mathbb{R}^1}$ and φ_2. Now, we do the computation for $l = m-n$, since for $l = m+n$ it is exactly the same and obtain the following two basis-kernels:

$$\varphi_i \cdot \begin{pmatrix} \begin{bmatrix} \cos_{m-n} & \sin_{m-n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} \cos_{m-n} & \sin_{m-n} \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \end{pmatrix} = \varphi_i \cdot \begin{pmatrix} \frac{1}{2}\cos_{m-n} & \frac{1}{2}\sin_{m-n} \\ \frac{1}{2}\sin_{m-n} & -\frac{1}{2}\cos_{m-n} \end{pmatrix}.$$

The result follows. □

3 To Check/Do

1. Is it a problem for Wigner-Eckart that only the *closure* of the direct sum is the whole space?
2. Can the argument with Dirac deltas be made mathematically foolproof?
3. Should I just once explain that I mean "linear equivariant map" when I write "homomorphism"?
4. Is the footnote correct about the integral of cosinus with itself and so on?
5. In what sense, if at all, is Proposition 2.1 true? Since that's what I'm building everything on!

6. About the proposition that says that the steerable kernel space is the same as representation operators: If I want to prove this, I should probably generalize Lin(U,V) to being an arbitrary G-representation.

7. Do the Clebsch-Gordan coefficients really induce such equivariant projections? Didn't really check that.

8. About the matrix-form of the basis-kernels: Can this be viewed in terms of "vectorization", in order to make a connection to Maurice's proof in the 3d-steerable CNNs paper? Because I think this thing where Y^l appears several times could be written simpler by first putting q into "rows" in a sense. But maybe that's nonsense.

9. Is it better to write $\cos_n(s)$ or $\cos_n$ in the matrix-form of Wigner-Eckart?

10. In the last proof, I only consider the case $m > n$ so far. That's a bit ugly, does that need to be?

References

[Agrawala, 1980] Agrawala, V. (1980). Wigner-eckart theorem for an arbitrary group or lie algebra. *Journal of Mathematical Physics*, 21.

[Jeevanjee, 2011] Jeevanjee, N. (2011). *An introduction to tensors and group theory for physicists.* Birkhäuser, New York, NY.

[Worrall et al., 2016] Worrall, D. E., Garbin, S. J., Turmukhambetov, D., and Brostow, G. J. (2016). Harmonic networks: Deep translation and rotation equivariance. *CoRR*, abs/1612.04642.