PHYS330 Advanced Lab: Coupled Torsional Oscillators and Fourier Methods

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I. INTRODUCTION

In PHYS220 (Oscillations & Waves), you should have learned about the Coupled Oscillator system and in PHYS230 (Intermediate Lab) there is an experiment on the torsional oscillator. In this lab, the system is mechanical, and the two oscillators are coupled to each other — and to the outside world — via magnetism. But the general concept of coupled oscillators applies to any system that has interacting 'normal modes', whether it be mechanical, electronic, electromagnetic, acoustical, or even quantum-mechanical in character. You'll use Fourier methods (a frequency analysis technique) to learn concepts and terminology which are portable across all these domains.

First, what is a 'normal mode'? For a very simple mechanical oscillator, like a spring-and-mass system, there's only one coordinate, and its natural motion is sinusoidal, and at a particular frequency. As soon as a system has more than one moving part, it needs multiple coordinates to describe it, and these coordinates can undergo motions much more complicated than a single sinusoid. But there are special kinds of system-wide motion in which each coordinate does oscillate as a pure sinusoid, and for which one single frequency prevails across the whole system. Such a pattern in space is called a normal mode of the system.

Clearly a strung and tuned violin is a multi-particle system, and very complicated motions of its parts can be imagined. But if one string is carefully excited in its fundamental mode, then each part of that string, and of the bridge, and also of the body of the violin, will oscillate sinusoidally, all at one common frequency. For this Coupled Oscillator experiment, we've provided a simpler mechanical system than a violin, chiefly characterized by only two coordinates, and therefore possessing two normal modes of interest. But as is the case in general, you'll learn that a given normal mode involves motions of all the parts of a system.

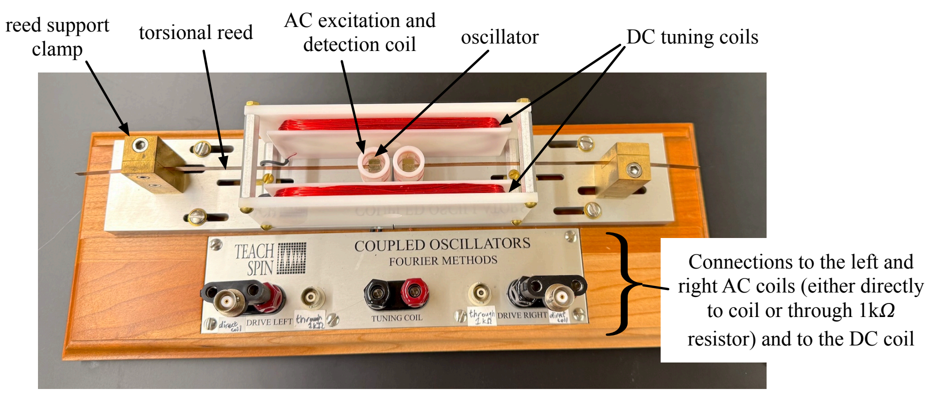

FIG. 1. Labeled photo of the Teach Spin coupled oscillator apparatus. The oscillator-related components are left-right symmetric.

II. UNDERSTANDING THE COUPLED-OSCILLATORS SYSTEM

A. Torsional-Reed Oscillators

The mechanical oscillator in this system is torsional in character, and is based on the elastic twisting of a thin ribbon, or reed, made of phosphor-bronze. One end of the reed is clamped, and the other end is free, but has attached to it a mass providing rotational inertia. As isometric view of such a system in Fig. 2 illustrates the coordinate system used in our description. The reed is clamped at $x = -L$, but near $x = 0$ the reed, and the mass it bears, is free to move. There's a 'wiggling mode' in which the end of the reed moves sideways (e.g., like a diving springboard), with the mass moving mostly in the $\pm y$-direction. While that motion is easily visible, it occurs at a rather low natural frequency (of order 6 Hz), and it is *not* the mode of interest to your studies.

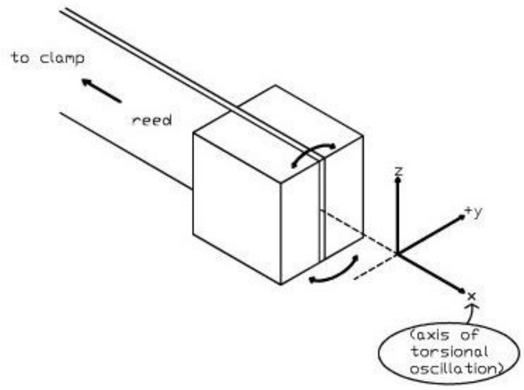

FIG. 2. An isometric view of (one) torsional-reed oscillator, and an xyz-coordinate system to describe it.

Instead, you'll study a different mode, in which the center-of-the-mass stays fixed in space, and the free end of the reed twists instead of wiggling. In such a motion, the 'central fiber' of the reed does not move at all, and each part of the mass moves with $x =$ constant, but with y and z changing due to rotation about the x-axis. This mode is described by an angle of rotation (of the mass) about the x-axis, which we will call θ. This mode has a natural frequency near 150 Hz, and at the low amplitudes you'll be using, its motion is nearly invisible to the eye.

Since the mass undergoes rotation about the x-axis in this mode, it certainly has a rotational inertia. If we model the mass as a cube of side l (and $l = 1/4\,\text{inch} = 6.35\,\text{mm}$ for your mass), and of density ρ (and $\rho = 7.51\,\text{g/cm}^3$ for your material), then it has mass $m = \rho l^3 = 1.92\,\text{g}$ in your oscillator. The rotational inertia of this object, for rotation about its center of mass and around the x-axis, is given by $I = (1/6)\rho l^5 = (1/6)ml^2$, with an estimated value of $I \approx 0.129\,\text{g cm}^2 = 1.29 \times 10^{-8}\,\text{kg m}^2$.

That gives the 'inertia term' of this oscillator. The 'spring-constant' part of it comes from the torsional elasticity of the phosphor-bronze strip. We describe this by a torsional constant κ, which gives the torque arising per unit rotation (measured in radians) of the reed's free end. The torsional reed has width, or height-in-z, of $w = 1/4\,\text{inch} = 6.35\,\text{mm}$, and thickness-in-$y$ of $t = 0.010\,\text{inch} = 0.254\,\text{mm}$. Its length-in-$x$ is $L \approx 4\,\text{inch}$ or 100 mm. Elasticity theory then predicts a torsion constant of approximately

$$\kappa = G\frac{wt^3}{3L} \tag{1}$$

where G is the shear modulus of the material of about $41\,\text{GPa} = 41 \times 10^9\,\text{N/m}^2$ for phosphor-bronze. This predicts $\kappa \approx 0.014\,\text{N m}$. I.e., the reed will act back with a torque of $0.014\,\text{N m}$ per radian of twist applied to its end.

The I and κ values computed above are only estimates based on material properties, and neither is easy to measure individually. But torsional oscillations provide a good check on a combination, since the (angular) frequency of torsional oscillations is predicted to be

$$\omega = \frac{\kappa}{I}\ . \tag{2}$$

The estimates above give the predicted angular frequency $\omega = (1.4 \times 10^{-2}\,\text{N m}/1.3 \times 10^{-8}\,\text{kg m}^2)^{1/2} \approx 1040\,\text{s}^{-1}$, which predicts an (ordinary) frequency of oscillation of $f = \omega/(2\pi) \approx 160\,\text{Hz}$, close to frequencies that you'll observe.

B. How the reed's torsional motion is excited and detected

The torsional mode near 150 Hz would be hard to observe or detect, except that in your oscillator the mass is actually a permanent magnet. Its magnetic moment is perpendicular to the square faces of the two slabs attached to the reed, and the magnetic-moment vector $\vec{\mu}$ points along the y-axis when the reed is untwisted. (The size of the magnetic moment is $\mu = MV$, where $M \approx 1.0 \times 10^6\,\text{A/m}$ is the magnetization of the NdFeB material used, and $V = l^3$ is the volume of the magnet. This gives $\mu \approx 0.26\,\text{A m}^2$.)

Now, a magnetic moment $\vec{\mu}$ interacts with a magnetic field $\vec{B}$ to give a torque

$$\vec{\tau} = \vec{\mu} \times \vec{B}\ , \tag{3}$$

and an interaction energy $U_{\text{mag}} = -\vec{\mu} \cdot \vec{B}$. Now look at one of your (two) torsional-reed oscillators, to see that the oscillator's reed is holding the magnet-block inside a coil wound around a white cylindrical form having an axis pointing along z. When a current i runs through this coil, it generates a field in the z-direction at the magnet's location, of magnitude $k_z i$ (where $k_z \approx 6.6\,\text{mT/A}$ can be predicted from the coil geometry and number of turns). With $\vec{\mu}$ in the y-direction, and $\vec{B}$ in the z-direction, the cross-product gives a torque in the x-direction as desired; this is the direction of torque needed to twist the reed. It has the predicted size $|\tau| = \mu B \sin 90°$. For a current of $i = 0.1\,\text{A}$, which gives $B = 0.66\,\text{mT}$, the torque is $172 \times 10^{-6}\,\text{N m}$. Applied to a reed of torsion-constant κ, we expect (from a dc current and a static B) a static angular deflection

$$\Delta\theta = \frac{\tau}{\kappa} \approx \frac{172 \times 10^{-6}\,\text{N m}}{1.4 \times 10^{-2}\,\text{N m}} \approx 12 \times 10^{-3}\,\text{rad}\ , \tag{4}$$

which is only about 2/3rds of a degree. While this static deflection is small, a torque of this magnitude generated by an alternating current $i(t)$ at one of the reed-mass system's resonant frequency can have a large effect. We will be able to 'pump up' the oscillator to a larger oscillation amplitude.

This coil system, driven by a current, allows the oscillation to be excited; but in reverse, this magnet-in-coil combination also allows the oscillation to be detected, electronically. That's due to Faraday's Law: the magnet moment $\vec{\mu}$ in rotational motion about the x axis will create a changing magnetic flux through any turn of the coil. That flux is approximately zero when $\theta = 0$, but varies above and below zero as magnet rotates. The emf generated in the coil is given by the time rate of change of the flux, and (by a reciprocity theorem) it turns out the emf per unit angular velocity of rotation is equal to the previously computed torque per unit current, $\tau/i = \mu k_z \approx 1.7 \times 10^{-3}\,\text{N m/A} = 1.7 \times 10^{-3}\,\text{V/(rad/s)}$.

Now if the reed develops a sinusodal oscillation in θ given by $\theta(t) = A\cos(\omega t)$, where $\omega \approx 1040\,/\text{s}$ gives the angular frequency of oscillation at resonance, and A is the amplitude of the angular motion, then the angular velocity is

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\omega A \sin(\omega t)\ , \tag{5}$$

and this has a peak value of ωA. Even for an amplitude of oscillation of only $1° = 0.017\,\text{rad}$, this has a magnitude of $18\,\text{rad/s}$. Then the emf in the coil is an ac voltage, at the resonant frequency, with a voltage amplitude

$$\text{emf} = (\mu k_z)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \left(1.7 \times 10^{-3}\,\frac{\text{V}}{\text{rad/s}}\right)(18\,\text{rad/s}) = 31\,\text{mV} \tag{6}$$

which is detectable with an oscilloscope, and especially using more sensitive devices like the SRS770 Frequency Analyzer combined with Fourier methods!

C. Tuning the oscillator

Thus far the natural frequency of the oscillator is set by the rotational inertia of the magnet's mass, and the torsional elasticity of the reed, but it is not otherwise adjustable. To give the system another independent variable, there is a provision to change the effective torsion constant. That's provided by an external and static magnetic field, which immerses the whole oscillator in a field $\vec{B}_y$ in the y-direction.

The effect of that field is to supplement the system's elastic potential energy $\kappa\theta^2/2$ with another term, $U_{\text{mag}} = -\vec{\mu} \cdot \vec{B}_y = -\mu B_y \cos\theta \approx -\mu B_y(1 - \theta^2/2) = \text{const} + \mu B_y \theta^2/2$. So the total potential energy becomes $(\kappa + \mu B_y)\theta^2/2$, and the effective torsion constant becomes $\kappa + \mu B_y$. So now the predicted torsional-oscillation frequency becomes

$$\omega = \sqrt{\frac{\kappa + \mu B_y}{I}}\ . \tag{7}$$

This predicts that $\omega^2 = (2\pi f)^2 = (\kappa + \mu B_y)/I$.

This B_y is generated by another set of coils, called the Tuning Coils, wound on two rectangular white plastic frames, and B_y will be proportional to the current i_T in these coils: $B_y = k_y i_T$. This new coil constant can be easily computed in the limiting case that the tuning coils are long in the x-direction, in which case their wires form a '4-wire field'. Along the center of such a structure (where, in our case, the magnet-blocks lie), the field is easily calculated, and the geometry of the structure and the number of turns give $k_y \approx 3.4\,\text{mT/A}$. So now we have the prediction that the experimentally-accessible quantity ω^2 ought to be a linear function of the coil current i_T , with intercept κ/I and slope $\mu k_y/I$. If a dc current of $i_T = \pm 2\,\text{A}$ is sent through these coils, we get $(\kappa + \mu B_y) \approx (0.014 \pm 0.0018)\,\text{N m}$, showing that the effective torsion constant can be changed by order $\pm 13\%$, and the frequency of the resonance changed by order $\pm 7\%$. That represents the ability to 'fine tune' the frequency of torsional oscillation, using a non-contact and real-time external parameter to do so.

D. Coupling two oscillators

If you look at your Coupled-Oscillator unit, you'll see two of these reed-based torsional oscillators in place, with their magnet-masses in each other's proximity. If you check using a little compass, you'll find that the two oscillators have their magnetic moments pointing in opposite directions (one along the $+y$, the other along the $-y$, direction). But the oscillators are otherwise nominally identical. To a good approximation, each one is excited by, and detected by, its own vertical-axis coil; but the two oscillators live in a single common B_y field generated by the Tuning Coil. Now if we name these oscillators #1 and #2, and if we assume they have the same rotational inertia I, but perhaps slightly different torsion constants κ_1 and κ_2 , we can imagine that the two oscillators have two separate oscillation frequencies, given by

$$\omega_1^2 = \frac{\kappa_1 + \mu B_y}{I} \ \text{ and } \ \omega_2^2 = \frac{\kappa_2 - \mu B_y}{I} \ . \tag{8}$$

The signs differ precisely because one magnet has its $\vec{\mu}$ lying along $\vec{B}_y$, while the other has its $\vec{\mu}$ lying opposed to $\vec{B}_y$. But B_y is a field magnitude common to both oscillators, and it is controlled by a single external current i_T in the tuning coil. So a plot of ω_1^2 and of ω_2^2 , both as functions of i_T, ought to display two straight lines, of similar y-intercept, but of opposite slopes. In particular, there is a value of i_T at which these two lines ought to cross – that is the say, there's a current i_T which produces a field B_y which 'tunes' the two oscillators to have a single, common, oscillation frequency (despite, for example, trifling differences in their construction).

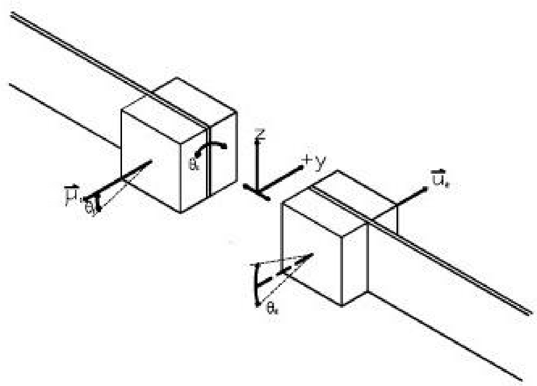

FIG. 3. Two magnetic moments, each on its own reed oscillator, each with its own angular coordinate for torsional oscillations.

That would be called a 'mode crossing' because of the crossing of two normal-mode frequency lines in a plot, but the crossing does not, in fact, occur. Instead, you'll see an 'avoided crossing', which is the single most general feature that can be found in coupled-oscillator systems across the breadth of physics. If your two oscillators each interacted with the field B_y , but not with each other, you'd see a crossing – but they are deliberately situated, with their magnets separated by distance r along the x-direction, so that they do interact. The interaction is the direct dipole-dipole coupling of two magnets, which gives another interaction energy U_{int}, which we'll now compute.

Let μ_1 (oscillator #1) be a point magnetic dipole regarded as the source of a magnetic field. Then located at vector displacement $\vec{r}$ from this source, a second magnetic dipole μ_2 (oscillator #2) finds itself immersed in the dipole field from μ_1:

$$\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\left(3\frac{\vec{\mu}_1\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{\mu}_1}{r^3}\right) \ , \tag{9}$$

and therefore experiences a magnetic interaction energy

$$U_{\text{int}} = -\vec{\mu}_2\cdot\vec{B}_1(\vec{r}) = \frac{\mu_0}{4\pi}\left(-3\frac{(\vec{\mu}_1\cdot\vec{r})(\vec{\mu}_2\cdot\vec{r})}{r^5} + \frac{\vec{\mu}_1\cdot\vec{\mu}_2}{r^3}\right) \ . \tag{10}$$

For the case at hand, even during oscillations of each dipole, the magnetic moments μ lie in the $y-z$ plane, while the inter-dipole displacement $\vec{r}$ is along the x-direction, so two of the dot-products vanish, and this interaction energy reduces to

$$U_{\text{int}} = \frac{\mu_0}{4\pi}\left(\frac{\vec{\mu}_1\cdot\vec{\mu}_2}{r^3}\right) \ . \tag{11}$$

Now if we adopt the angular coordinates shown in Fig. 3, the dot product gives

$$U_{\text{int}} = -\frac{\mu_0}{4\pi}\frac{\mu_1\mu_2}{r^3}\cos(\theta_1-\theta_2) \approx -\frac{\mu_0}{4\pi}\frac{\mu_1\mu_2}{r^3}\left[1-\frac{(\theta_1-\theta_2)^2}{2}\right]. \tag{12}$$

where we've used the small-argument form of the cosine function to simplify the expression.

This expression, when expanding the square brackets, includes a leading term which depends on r, but not on the angles (it gets more negative as r is decreased, corresponding to the attractive forces that these oppositely-directed moments exert on each other). Here, we particularly care the angle-dependent term containing $(\theta_1-\theta_2)^2$, which is lowest when the angles are equal, corresponding to the magnetic torques that these moments exert on each other, which tend to co-align the two angles. Most importantly, expanding this interaction term (i.e., the quadratic), we'll find a term with the product $\theta_1\theta_2$, which will couple these two oscillators together and thereby totally change their behavior.

Now finally we can write the total angle-dependent potential energy of this system, including the elastic terms for each reed, each magnet's interaction with the field B_y, and the two magnets' interaction with each other:

$$U_{\text{net}} = \frac{1}{2}\kappa_1\theta_1^2 + \frac{1}{2}\kappa_2\theta_2^2 + \frac{1}{2}(\mu B_y)\theta_1^2 + \frac{1}{2}(-\mu B_y)\theta_2^2 + \frac{1}{2}\left[\frac{\mu_0}{4\pi}\frac{\mu^2}{r^3}\right](\theta_1-\theta_2)^2\ . \tag{13}$$

Here, constant offsets in U_{net} are ignored, and we've assumed that the two magnetic moments are approximately the same size ($\mu_1=\mu_2\equiv\mu$). This expression also contains the small-angle approximation used earlier. Recall the numerical magnitudes of the values: the torsion constants κ are about 0.0140 N m, the coupling to the B_y field gives μB_y of about 0.0018 N m at a tuning-coil current of $i_T = 2$ A. The coupling constant

$$\lambda \equiv \left[\frac{\mu_0}{4\pi}\frac{\mu^2}{r^3}\right] \tag{14}$$

turns out to have magnitude about 0.0008 N m when the center-to-center separation of the two dipoles is taken to be $r = 20$ mm. So the coupling of the two oscillators to each other seems to be really weak relative to the reed's torsional constant, with $\lambda/\kappa \approx 0.06$, but nevertheless it turns out to have crucial consequences. The reason is that we can 'tune to the crossing', i.e., put the system into a condition in which the two oscillators would otherwise have matching frequency. In such a case, motion of one oscillator will drive the other oscillator resonantly, so even this weak coupling can have large consequences.

III. USING THE COUPLED-OSCILLATOR SYSTEM: SINUSOIDAL DRIVE

A. Initial adjustment

It's time to set theory temporarily aside, and actually excite this coupled-oscillator system. For now, we'll ignore the Tuning Coil, and we'll start with ordinary electronic excitation and detection methods. You'll want an external sine-function generator which you can hand-tune over the range 100-200 Hz, and you'll want a dual-trace oscilloscope for detection.

Before you do any electrical measurements on your Coupled Oscillator, you want to ensure it's properly adjusted. Use a fingertip or thin plastic probe to deflect one of the reeds sideway (i.e. in the y-direction) and then 'let it go', which will excite the 'wobble motion' of that reed. (Recall: this is *not* the motion we're studying later on. This step is used to check the system only.) The attractive force of interaction between the two magnets ensures that both reeds will participate in this wobble. What you want is for both reeds to be free to move; in particular, check that their top and bottom edges are not rubbing against the vertical-cylinder coil forms which enclose both magnets. If the wobble motion does not go on for many seconds, there is some friction involved. To eliminate it, let the instructor know of this problem. Once you have confirmed that the 'wobble motions' are properly centered and mechanically free, you can hand-damp out any remaining wobble motion. But now you can be quite sure that the torsional motion you'll be investigating is also free and very nearly undamped.

B. Excitation of one oscillator with function generator

The circuit to connect is shown in Fig. 4, where you are directly exciting the left oscillator. Set the function generator to about 1-Volt amplitude, and apply it to one drive coil's BNC connector so that current flows through the built-in 1 kΩ resistor as shown. This turns the generator into a 'current source' of about 1-mA amplitude. The connection shown to the 'scope's ch. 1 is then a voltage surrogate for the drive current being applied to the coil. Meanwhile ch. 2, connected via a BNC adapter to the banana plugs wired directly to this coil, will show the sum of two voltages: one is the $i(t) \cdot R$ resistive drop across the drive coil, and the other is the Faraday's-Law emf generated in the coil by the torsional motion of the oscillator. At resonance, this 'back emf' will dominate over the resistive part of the signal.

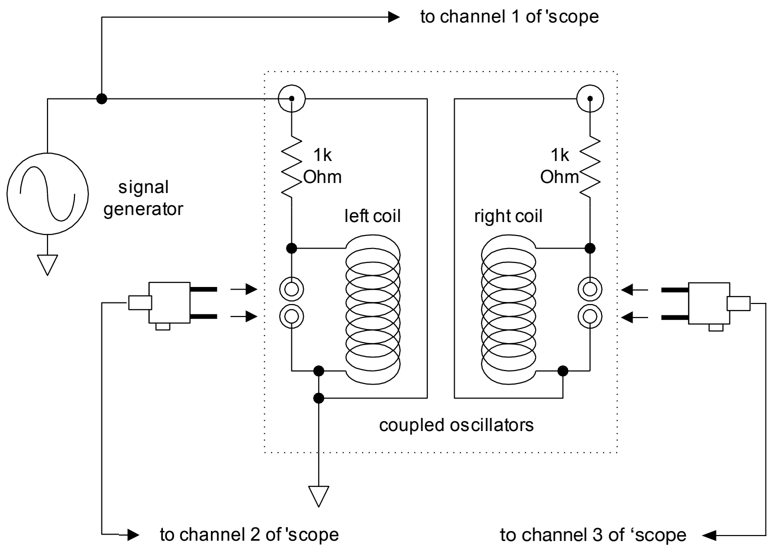

FIG. 4. The first setup to excite one oscillator using the function generator. The excitation voltage and the induced emf in the two small coils are all measured simultaneously using three channels on an oscilloscope. The unfilled triangles indicate the common ground.

Now set your 'scope so the ch. 1 signal triggers the 'scope, and displays a few cycles across the screen. Set ch. 2 to display as well, and set it to a sensitivity of about 10 mV/div. (For now, you don't need to display ch. 3.) For a generic value of the drive frequency, you expect on ch. 2 to see the iR drop across the coil, which might have an amplitude of about 10 mV, and will certainly be in phase with the drive being displayed on ch. 1. What you want to do is to vary the drive frequency, quite slowly, over the 100-200 Hz range, and look for a departure from this mere iR-drop behavior. That departure will show up with several characteristics:

- it will be of larger amplitude, perhaps up to 40 mV;
- it will show a departure in phase compared to the drive waveform; and
- it will take some time to develop, since you're driving a high-Q resonant system.

See if you can close in on the resonant frequency, at which the ch. 2 signal should be maximal in amplitude, and back in phase with the drive. You will have to tune to a fraction of 1 Hz (say 0.1 Hz) to locate the center of the resonance.

After you find 'the resonance', go looking for another – it might be about 10 Hz away in frequency. It will have similar characteristics as the one you found first, though it might differ somewhat in size.

C. Using XY mode to check driving of system (try skipping this to move faster)

Configure your 'scope to the XY-mode, in which the ch. 1 signal drives the display spot left-right, and the ch. 2 signal drives it up-down, on the display. Away from resonance, you'll see a nearly 'flat line'. Near, but not at, resonance, you'll see an open curve, a titled ellipse, which indicates phase shifts between drive current and emf response. At resonance, the ellipse will collapse back to a line, but a tilted line.

This titled-line plot is also very useful for checking to see if you are over-driving the system. If you're on resonance, and reduce the generator's amplitude by a factor of two, the whole plot you're seeing should shrink by a factor of 2. If instead it causes the line to open up to an ellipse, this says the resonance frequency for larger drive differs from the resonance frequency for smaller drive. That sort of anharmonicity is a sign of overdriving the system. You might use the new and smaller amplitude to repeat the search, and find more precise values for the two resonant frequencies of the two normal modes of the system.

D. Measuring the excitation voltage, left coil and right coil voltages simultaneously

How is it that there are two resonant frequencies? The answer is that, even though you're exciting and detecting only one of your two oscillators, the magnetic coupling with the second oscillator forms a coupled system, and therefore the system has two normal modes. Either modes can be excited by resonant drive at any point in the system.

Here is an alternative, and even more informative, way to interrogate the system. Go back to the time-based oscilloscope mode. Connect the right coil to the ch. 3 of the 'scope using the BNC cable and get it to display. Choose a similar voltage per division as for ch. 2. Now go back and excite the two resonant modes you discovered previously. Observe, sketch, and describe the phase and amplitude of the three signals on the 'scope screen for these modes.

For the connection to the right oscillator, there will be no iR-drop at all, and the 'scope will display, on ch. 3, purely an emf signal, directly related to the angular velocity of the reed/magnet you're not exciting directly. Now only the coupling of the two oscillators provides a pathway for the ch. 1 drive to communicate to the ch. 3 response. But you should see very similar indications of two resonances, with the advantage that there is no iR-background under your ch. 2 signal. There will also be, in this mode, this distinction between the two resonances: at the center of the second resonance, the phase of the response will be 180° different, ie. flipped in sign, compared to the case of the first resonance you saw.

Record the excitation amplitude, excitation frequency, first oscillator amplitude voltage, second oscillator amplitude voltage, and phase difference (between the two oscillator voltages) in a table. Scan the frequency over the $\sim 30\,\text{Hz}$ range of interest. Do this at intervals of $\sim 1\,\text{Hz}$ when not at the resonances, but stepping down to $\sim$ 0.1–0.2 Hz when close to the resonaces.

You will soon tire of the fine-tuning required on your signal generator, so instead of tediously moving the frequency by 1 Hz or even 0.1 Hz at a time, over a range of frequencies 10 or 100 Hz wide, it's time to try all the frequencies at once. That is to say, it's time to use Fourier methods.

IV. EXCITING THE SYSTEM BY NOISE, AND BY CHIRP, WAVEFORMS

The method you're about to apply to the Coupled Oscillator system is the same as you might have used on an acoustic system. The idea is to send white noise – a superposition of sinusoids at all frequencies – into the system, and then to Fourier analyze the results, to see which frequencies are preferentially transmitted by the system.

In the present case, you can use the Source Out of the 770, configured with the SOURCE button to be white noise of 1000 mV rms measure, as your source of noise. You can send that, through the $1\,\text{k}\Omega$ resistor, to one of the drive coils of your Coupled Oscillator. Then the emf generated in the other vertical-axis coil, serving as pick-up coil for the other oscillator, is the signal you'll want to Fourier analyze. Connect it to the A-input of the 770, and set up as usual to measure a spectrum. Remember to use the AutoRange function to optimize the 770's input for the (rather low) level of signal you'll be picking up.

You know that the resonances you care about lie in the 100-200 Hz range, so you might pick a frequency span of 390 Hz (and a start frequency of 0 Hz). This will entail an acquisition time of 1.024 s. You will see a Coupled-Oscillator's full spectrum in a second, and upon repeated acquisitions you'll see the statistical fluctuations to be expected from noise excitation. (Use the Continuous mode of Triggering, under the INPUT button, to get ongoing updates of the spectrum.) You can use the Average function to reduce these fluctuations, though at a cost in update time.

The MEASure button will let you use the Measure soft-key to choose to measure spectrum, or power spectral density. The Display soft-key will let you pick Log Magnitude, which will give you a fine view of the large dynamic range you can cover. The Scale button will let you pick Top Reference and dB/div settings for your vertical scale. Recall that 80 dB of vertical scale stands for a voltage range of 10^4 which you can cover. With so large a range of sensitivity, you're sure to see other modes than the two you previously discovered by 'scope and scanning. Here are some clues about 'other peaks':

- Look for signals at 60, 120, 180 Hz (or integer multiples of your local line frequency) – these represent interference (for example, direct inductive or capacitive coupling to your pick-up coil) and might persist even if the noise drive were to be removed.
- Look for signals about 6 (or 12) Hz, or for sidebands 6 (or 12) Hz away from the main peaks – these may be due to the modulation of the torsional modes by the 'wobble' motion of the reeds. If you damp away that wobble using temporary by-hand intervention, you might see these diminish.
- Look for the non-zero width-in-frequency of the main modes – this is due either to finite acquisition time, or to the finite lifetime of the torsional modes due to their damping.
- Look for unexplained or unassigned modes – this is where Fourier methods allow unexpected (or undesired) discoveries.

For the best measurement of the frequencies of the main torsional modes, you'll want to decrease the 770's frequency span even more, perhaps by three more factors of 2. You can also change the Start Frequency of the span, so as to center its coverage where you want it. You'll see two separate kinds of disadvantages come with this zoomed-in view:

- The acquisition time goes up by that many factors of 2; for a 49-Hz span, it will have risen to 8.192 s. Averaging will further slow the response of the system to any changes.
- The spectra you see will look (and be) 'noisier', ie. subject to larger fluctuations. That's because you are spreading a fixed amount of noise power across more bins in frequency, so there's less noise power (and thus greater fluctuation) per bin.

You can cure this latter problem, and very dramatically, by changing the Source configuration from Noise to Chirp. In the Noise mode, an rms output of 1000 mV has to contain frequency components over the full 0-100 kHz range, no matter what span you actually choose to analyze. But in the Chirp mode, the Source synthesizes a noise-like equal superposition of only those frequencies you are configured to analyze. Since it can now (for example) devote 1000 mV of output range to a span of 49 Hz rather than 100 kHz, the spectral power per bin analyzed can go up by $2^{11} = 2048$ (!). You may have to use the AutoRange button to deal with the much larger signals that result. But when using the Chirp waveform, you will also want to change the Window mode to Uniform.

You can also deal with part of former problem. Chirp source or not, a span of 49 Hz still requires an acquisition time of > 8 seconds. But it does not require Averaging to reduce the fluctuations, since the Chirp waveform is synthetic and deterministic, it is in fact periodic with the period given by the acquisition time. So what you get on one acquisition is what you'll get on any other.

Clearly with your fine views of spectral peaks, you're now in position to use the Marker function to estimate the resonant frequencies to about 0.1-Hz precision and accuracy.

V. TUNING THE NORMAL MODES

There is no special interest in the exact numbers you get for the resonant frequencies you've measured, except if they can be understood and changed systematically. Clearly the use of different length, thickness, or width of the torsional reeds would have put the resonances at different locations in frequency.

But the fact that there are two main resonances is no accident; it's due to the fact that there are two oscillators, and that they're coupled together to give two normal modes. Your next goal is to 'tune' the individual oscillators' frequencies, to see the systematic changes this creates in the two normal-mode frequencies.

The Tuning Coil previously mentioned is your way to achieve this. You can use up to ± 2 Amp currents continuously, or ± 3 Amp briefly, of dc current in these coils. There is a self-resetting fuse in series with the coil, which will interrupt the current if it gets too hot. If you find that the current suddenly drops to zero, turn the supply voltage down to zero, wait about a minute, and then watch the current as you dial up the supply voltage again.

The current you send through the Tuning Coils will create a y-directed field (of about $3.4\,\mathrm{mT/A}$) in the region of the reed-mounted magnets. But before taking spectroscopic data on their effect, you ought first to check the centering of the tuning coils relative to the oscillators. Here's what to do, and what to look for:

- Have the tuning coils connected to a dc power supply, and be ready to hand-dial that from $i_T = 0$ to 2 Amperes and back. Now look down from above at your two oscillators' reeds (no need to have either drive coil connected at this point), and watch to see if the magnets move (in the y-direction) when you dial up the current.
- If you see motion, it's due to the imperfect centering (in the y-direction) of the coils relative to the magnets. On the centerline of the coils, the B_y-field is gradient-free, so that field exerts the desired torques, but not any forces, on the magnets. But if the magnets find themselves not on centerline, then gradients exist in B_y, creating forces on the permanent magnets. You might see one magnet pulled in the $+y$, the other in the $-y$ direction (because the same field gradient is acting on the oppositely-directed magnetic moments of the two magnets).
- Once you see this effect, you want to eliminate or minimize it. To do so, you want to change the y-position of the tuning coil's centerline. You can do so most easily by loosening two brass round-head clamping screws, and then sliding sideways, or tilting the coil forms slightly (shimming underneath one or the other of the plastic forms) to get the centering you want.
- When you can see no y-motion of the magnets upon changing i_T from 0 to 2 Amperes, you are adequately aligned. In particular, you can now be sure that tuning the coils will not move the magnets and reeds to the point of touching the drive-coil forms (and thereby spoiling their high Q of torsional oscillations).

Given proper alignment of the tuning coils, you can now restore Noise or Chirp drive of one oscillator, and the Fourier-analysis of the pick-up from the other, to do the spectroscopy of your resonant modes. But now you have an independent variable, the tuning-coil current i_T , to affect the two dependent-variable frequencies you're seeing. You should first do an exploration over the full -2-A to +2-A range of i_T, to confirm that the two frequencies do indeed change. You will find a range of current over which the frequencies scarcely change at all; that's the range over which the two frequencies attain a minimum separation. Outside of this range of i_T, expect the two frequencies to move apart from each other. Note well that the two frequencies do not 'coalesce' or cross as functions of i_T.

Gather data for a series of choices of i_T, of the higher (f_+) and lower (f_-) normal-mode frequencies. When you've learned how to do this over the (-2, +2)-A range, get a few points outside this range. Limit the tuning-coil current to the ±3-A range, and spend less than a minute at ±3-A levels before allowing the coils to cool.

VI. THEORY OF THE NORMAL-MODE FREQUENCIES, AND THE AVOIDED CROSSING

The 'Fourier spectroscopy' you've now done has given you a data table – for each of a number of values of tuning current i_T, you have measured f_+ and f_- , the higher- and lower-frequency normal-mode resonant frequencies. What can be done with that data?

Theory (below) suggests forming some new plots. First form the values $\omega_+ = 2\pi f_+$, and $\omega_- = 2\pi f_-$, and then the combinations $(\omega_+^2 + \omega_-^2)$ and $(\omega_+^2 - \omega_-^2)^2$, and plot those as functions of i_T. These are the two main plots for studying the avoided crossing. The theory predicts the emergence of a constant for the first, and a parabola for the second. On these new plots, especially with the fits added, any discrepant data points will stand out.

Recall that for *un*-coupled oscillators you expected two separate frequencies ω_1 and ω_2 for oscillators #1 and #2, and you expected the quantities ω_1^2 and ω_2^2 each to display a straight-line, but differently-sloping variation as a function of i_T. But that would give rise to a crossing of the two straight lines, and as a result, you'd have gotten for $(\omega_+^2 - \omega_-^2)^2$ a parabola, but one having a minimum value of zero. The data for coupled oscillators says otherwise – the quantities ω_+^2 and ω_-^2 reach a minimum, but non-zero, separation. Why?

The answer comes from the finding the system's equations of motion, and solving them. Using the complete potential-energy expression for U_{int}, we can get the response to the torques on the two oscillators via

$$I\ddot{\theta}_1 = \Sigma\tau = -\frac{\partial U_{\text{net}}}{\partial\theta_1} + (\mu k_z)i_1(t) \;\text{ and}$$
$$I\ddot{\theta}_2 = \Sigma\tau = -\frac{\partial U_{\text{net}}}{\partial\theta_2} + (\mu k_z)i_2(t)\;. \tag{15}$$

Here i_1 and i_2 are the currents send into the two vertical-axis drive coils. These become

$$I\ddot{\theta}_1 + (\kappa_1 + \mu B_y)\theta_1 + \lambda(\theta_1 - \theta_2) = (\mu k_z)i_1(t) \;\text{ and}$$
$$I\ddot{\theta}_2 + (\kappa_2 - \mu B_y)\theta_2 + \lambda(\theta_2 - \theta_1) = (\mu k_z)i_2(t)\;, \tag{16}$$

which are two coupled 2nd-order differential equations.

Now we look for a normal mode of the *un*driven system by setting $i_1 = 0 = i_2$, and asserting the existence of oscillations at a single *common* frequency ω:

$$\theta_1(t) = A_1\exp(-i\omega t) \;\text{ and }\; \theta_2(t) = A_2\exp(-i\omega t)\;. \tag{17}$$

Notice that we claim purely sinusoidal motion of both θ-coordinates, and at one single system-wide frequency.

Under this assumption, the coupled differential equations for $\theta_1(t)$ and $\theta_2(t)$ turn into coupled *linear* equation for amplitudes A_1 and A_2. They can be written in matrix form as

$$\begin{bmatrix} -I\omega^2 + (\kappa_1 + \mu B_y + \lambda) & -\lambda \\ -\lambda & -I\omega^2 + (\kappa_2 - \mu B_y + \lambda) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\;. \tag{18}$$

This homogeneous set of equations does not give A_1 or A_2, but instead gives (two versions of) the ratio A_1/A_2. Those two versions are inconsistent, unless the determinant of the matrix above vanishes. Writing out that determinant gives a quadratic equation for the unknown ω^2, which will give two solutions for ω^2. To simplify the form of that solution, consider two frequencies the system could have: (1) let Ω_1 be the frequency at which θ_1 would oscillate, if θ_2 were held fixed at 0; and (2) let Ω_2 be the frequency at which θ_2 would oscillate, if θ_1 were held fixed at 0. From the equations of motion, we find that these imaginable frequencies are given by

$$I\Omega_1^2 = \kappa_1 + \mu B_y + \lambda \;\text{ and }\; I\Omega_2^2 = \kappa_2 - \mu B_y + \lambda\;, \tag{19}$$

and this gives a simpler form to the vanishing of the determinant, which becomes

$$I(\Omega_1^2 - \omega^2)\cdot I(\Omega_2^2 - \omega^2) - \lambda^2 = 0\;. \tag{20}$$

The quadratic equation to solve is

$$\omega^4 - (\omega_1^2 + \Omega_2^2)\omega^2 + \Omega_1^2\Omega_2^2 - \frac{\lambda^2}{I^2} = 0 \,. \tag{21}$$

There are two solutions for ω^2 and we'll call the larger of them ω_+^2, and the smaller ω_-^2; the two are given by

$$\omega_\pm^2 = \frac{1}{2}\left[\Omega_1^2 + \Omega_2^2 \pm \sqrt{(\Omega_1^2 + \Omega_2^2)^2 - 4\left(\Omega_1^2\,\Omega_2^2 - \frac{\lambda^2}{I^2}\right)}\,\right] = \frac{1}{2}\left[\Omega_1^2 + \Omega_2^2 \pm \sqrt{(\Omega_1^2 - \Omega_2^2)^2 + \left(\frac{2\lambda}{I}\right)^2}\,\right] \tag{22}$$

This is the form which motivates plotting the two linear combinations mentioned above, since it predicts

$$\omega_+^2 + \omega_-^2 = \Omega_1^2 + \Omega_2^2 = \frac{1}{I}\big(\kappa_1 + \mu B_y + \lambda + \kappa_2 - \mu B_y + \lambda\big) = \frac{1}{I}\big(\kappa_1 + \kappa_2 + 2\lambda\big), \tag{23}$$

and since B_y drops out of this prediction, it gives a flat-line plot as a function of I_T.

The predictions for $\omega_\pm^2$ can also be used to give the *squared* difference of squares,

$$(\omega_+^2 - \omega_-^2)^2 = (\Omega_1^2 - \Omega_2^2)^2 + \left(\frac{2\lambda}{I}\right)^2 , \tag{24}$$

and since Ω_1^2 and Ω_2^2 are both linear in B_y, and hence in I_T, this is predicted to display a *parabolic* dependence on the tuning current. That parabola, in turn, is predicted to reach down to a minimum value of $\big(2\lambda/I\big)^2$, directly related to the coupling λ between the two oscillators. The x-axis location of the parabola's minimum is the condition

$$\Omega_1^2 = \Omega_2^2 \;, \quad \text{i.e.,} \quad (\kappa_1 + \mu B_y + \lambda) = (\kappa_2 - \mu B_y + \lambda), \tag{25}$$

or at $2\mu B_y = \kappa_2 - \kappa_1$; this just gives the value of the B_y-field at which the two oscillators have been "brought into tune".

At this minimum-separation point, we have $(\omega_+^2 - \omega_-^2)^2 = 0^2 + (2\lambda/I)^2$, so we get $\omega_+^2 - \omega_-^2 = +2\lambda/I$. Meanwhile, we can put $\kappa_{\rm avg} \equiv (\kappa_1 + \kappa_2)/2$ and write $\omega_+^2 + \omega_-^2 = (2\kappa_{\rm avg} + 2\lambda)/I$ so now it is easy to solve for the values of ω_+^2 and ω_-^2 at the minimum separation:

$$\omega_+^2 = \frac{\kappa_{\rm avg} + 2\lambda}{I} \quad \text{and} \quad \omega_-^2 = \frac{\kappa_{\rm avg}}{I} \quad \text{(at minimum separation).} \tag{26}$$

So quite directly from the data, you can read off the values of $\kappa_{\rm avg}/I$ and λ/I, and using the computed value of rotational inertia I will then give deduced values for $\kappa_{\rm avg}$ and for λ, to be compared to earlier estimates.

Away from minimum-separation, we can define $B_y^{\rm min} = (\kappa_2 - \kappa_1)/(2\mu)$ as that B_y-value needed to attain the minimum-separation condition, and then it's easy to show that

$$(\omega_+^2 - \omega_-^2)^2 = \left(\frac{2\mu}{I}\right)^2 (B_y - B_y^{\rm min})^2 + \left(\frac{2\lambda}{I}\right)^2 . \tag{27}$$

This form displays the parabola most clearly, and it shows that a best-fit of the data to a parabola (against B_y) will give parameter values $(2\mu/I)^2$, $B_y^{\rm min}$, and $(2\lambda/I)^2$. Again using the computed value of I, these will give deduced values for μ and for λ. With values for those parameters in hand, the entire theoretical model is specified, and plots of ω_+^2 and ω_-^2, or of ω_+ and ω_-, or of f_+ and f_-, can be generated and laid atop the data. For a canonical view of the data, plot measured ω_+^2 and ω_-^2 data-points, overlay the theory's predicted curves for ω_+^2 and ω_-^2, and add in the plots of the Ω_1^2 and Ω_2^2 functions, to see the asymptotes toward which the theoretical curves tend. You'll be getting the best view of the system's avoided crossing.

A. Additional predictions of the theory (can skip)

One motivation for making such plots is that it's then easy to model how they would change if the coupling λ were to vary. (Notice that in the hardware, it is feasible—by loosening three screws—to adjust the separation r between the two magnets, and notice also that modest changes in r will make substantial changes in λ, due to the r^{-3} dependence.) You should find that a smaller λ-value entails that the two curves for ω_+^2 and ω_-^2 will approach each other more closely—they "more narrowly avoid" a crossing. But in principle, for any degree of coupling, however small, the curves do not cross.

Another prediction of the theory can be worked out, and that is for the ratio A_1/A_2 of amplitudes of the two separate coordinates θ_1 and θ_2. This is algebraically easy only at the minimum separation point $B_y = B_y^{\text{min}}$, and there you will find

$$\begin{aligned} \text{in the } \omega_- \text{ mode, } A_1 = +A_2, &\quad \theta_1(t) = +\theta_2(t), \\ \text{but in the } \omega_+ \text{ mode, } A_1 = -A_2, &\quad \theta_1(t) = -\theta_2(t). \end{aligned} \tag{28}$$

In fact this makes it clear why (at minimum separation) we found $\omega_-^2 = \kappa_{\text{avg}}/I$, independent of the coupling λ; since the two magnets oscillate in unison in the ω_- mode, the coupling energy $\lambda(\theta_1 - \theta_2)^2/2$ is, and remains, zero. The surprise is that the two oscillators can and do oscillate in unison, at equal amplitude, even though the drive is being applied to only one of them, and coupling seems to be "doing no work".

Away from the minimum-separation condition, $|A_1/A_2| \neq 1$, so each normal mode is characterized by having the motion "concentrated" in one of the two oscillators. Your spectroscopy has given you numerical values for all the constants you need to make a complete theoretical prediction for how this amplitude-ratio varies with the tuning.