A brief guide to the numerical scheme implemented in Basilisk

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Chapter 1

Introduction

1.1 Equations

The solved equations are the Navier-Stokes incompressible one given by,

$$\nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\rho} \left(-\nabla p + \nabla \cdot (2\mu \mathcal{D}) \right) + \mathbf{a} \tag{1.2}$$

where $\mathcal{D}$ is the deformation tensor and $\mathbf{a}$ stands for extra acceleration than can be originated by several causes, as for example the gravitational forces, surface tension forces, viscoelastic forces, etc.

Simultaneously it can be computed any tracer, c_i or VOF tracer T_i,

$$\partial_t T_i + \mathbf{u} \cdot \nabla T_i = 0 \tag{1.3}$$

$$\partial_t c_i + \nabla \cdot (\mathbf{u} c_i) = 0 \tag{1.4}$$

Observe that these tracer can be passively advected with the fluid or, in a more common (and interesting situation), can affect actively the fluid motion as for example by altering the distribution of the fluid properties

$$\rho = \rho(T_i, c_i) \quad \text{and} \quad \mu = \mu(T_i, c_i). \tag{1.5}$$

1.2 Momentum equation

The advection part of momentum $(\mathbf{u} \cdot \nabla)\mathbf{u}$ can be more easily computed profiting the incompressibility of the fluid. In effect, the divergence of the dyadic product $\mathbf{uu}$ (also written as $\mathbf{u} \bigotimes \mathbf{u}$) writes

$$\nabla \cdot (\mathbf{uu}) = \mathbf{u}(\nabla \cdot \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{u} = (\mathbf{u} \cdot \nabla)\mathbf{u}.$$

Observe that the Gauss's theorem allows its computation by evaluating the fluxes through the cell's faces,

$$\int_\Omega \nabla \cdot (\mathbf{uu})\, d\Omega = \int_\Sigma \mathbf{u}(\mathbf{u} \cdot \mathbf{n})\, d\Sigma$$

This conservative approach of the Gauss's theorem, i.e. to calculate temporal evolution at the cell center by computing fluxes across cell faces is thoroughly used in the numerical scheme.

Flowchart of the generic time step

Starting point
Velocities: (a)centered: $\mathbf{u}^n$
and (b) normal face: $u^{f,n}$
Centered pressure: p^n
Centered pressure gradient + extra-accelerations (no viscosity):

$$\mathbf{g}^n = -\nabla p^n + \mathbf{a}^n$$

Centered tracers: $c_i^{n-1/2}$
VOF tracers: $T_i^{n-1/2}$

event dtmax
Set the maximum imposed step size (if any)

event Stability
Reduces time step if necessary according to CFL or other criteria.

event VOF
VoF tracers are advected to time $n+1/2$:

$$T_i^{n+1/2} = T_i^{n-1/2} + \Delta t \nabla \cdot (T_i^n \mathbf{u}^n)$$

event tracer_advection
Using the Bell-Colella-Glaz scheme are advanced the center values of the tracers to the instant $n+1/2$. This method is essentially an extrapolation from center values to face value and from time $n-1/2$ to n, $c_i^{f,n} = \mathbf{f}(c_i^{n-1/2}, u^{f,n})$, followed by an explicit update,

$$\begin{aligned} c_i^{n+1/2} &= c_i^{n-1/2} + \Delta t \nabla \cdot (c_i^n \mathbf{u}^n) \\ &= c_i^{n-1/2} + \frac{\Delta t}{h^2} \int_{\partial \mathcal{C}} c_i^{f,n} u^{f,n} \, d\Sigma \end{aligned}$$

Not present at the moment but I wonder it not would be convenient to add a: **event tracer_diffusion** here

event properties
The relevant properties like viscosity and density are advanced to the step $n+1/2$. Typically depend on the tracers,

$$\rho^{n+1/2}, \mu^{n+1/2} = \mathbf{f}(T_i^{n+1/2}, c_i^{n+1/2})$$

event advection_term
With a variant of the BCG scheme are calculated normal face velocities at instant $n+1/2$,

$$u_p^{f,n+1/2} = \mathbf{f}(\mathbf{u}^n, \mathbf{g}^n).$$

Note that the extrapolation takes into account either the pressure gradient and extra-accelerations by means of $\mathbf{g}^n$. The label p denotes that it is a PREDICTION. i.e, its solenoidal character is not guaranteed ($\nabla \cdot \mathbf{u}_p \neq 0$).

PROJECTION STEP: Consist in correct the predicted face velocity $u_p^{f,n+1/2}$ to make it solenoidal. To this end we derive a correction for the face velocities,

$$\begin{aligned} u^{f,n+1/2} &= u_p^{f,n+1/2} + \Delta u^f \\ &= u_p^{f,n+1/2} - \nabla p_f(\Delta t)/\rho \end{aligned}$$

Attention, p_f is a correction NOT the real pressure field.

Advection term is computed using again BCG[a],

$$\mathbf{u}^{f,n+1/2} = \mathbf{f}(\mathbf{u}^n, \mathbf{g}^n, u^{f,n+1/2})$$
$$\frac{\mathbf{u}_a^{n+1} - \mathbf{u}^n}{\Delta t} = \frac{1}{h^2} \int_{\partial \mathcal{C}} u^{f,n+1/2} \mathbf{u}^{f,n+1/2} \, d\Sigma.$$

[a] Note that $\mathbf{u}^{f,n+1/2}$ is the velocity vector at faces while $u^{f,n+1/2}$ is the component *normal* to the face, relevant for calculation of fluxes.

event viscosity
It solves implicitly the equation

$$\frac{\mathbf{u}_v^{n+1} - \mathbf{u}_a^{n+1}}{\Delta t} = \nabla \cdot [2\mu \mathcal{D}_v^{n+1}] - \frac{\nabla p^n}{\rho} + \mathbf{a}^n$$

$\mathbf{u}_a^{n+1}$ is the velocity after considering the advection term.
Due to *correction(-Δt)*, the center velocity at the end of this event is

$$\mathbf{u}_*^{n+1} = \mathbf{u}_v^{n+1} - \Delta t(-\frac{\nabla p^n}{\rho} + \mathbf{a}^n)$$

Normal face velocity at instant $n+1$, $u_*^{f,n+1}$, are constructed averaging $\mathbf{u}_*^{n+1}$.

event acceleration
Once the accelerations at faces are calculated at the instant $n+1$, $a^{f,n+1}$, the face velocity is consequently augmented

$$u_{**}^{f,n+1} = u_*^{f,n+1} + \Delta t a^{f,n+1}$$

event projection
A projection step with $u_{**}^{f,n+1}$ is done. As a result a normal face velocities are divergence free

$$u^{f,n+1} = u_{**}^{f,n+1} - \nabla p^{n+1}(\Delta t)/\rho$$

The correction pressure p^{n+1} is considered also the pressure field at instant $n+1$.
Face values of the enriched normal pressure gradient $g^{f,n+1}$ are calculated from the pressure field p^{n+1} and acceleration $a^{f,n+1}$. Centered value $\mathbf{g}^{n+1}$ are calculated simply by averaging. Finally, centered velocity at $n+1$ is simply,

$$\mathbf{u}^{n+1} = \mathbf{u}_*^{n+1} + \Delta t[-(\nabla p^{n+1})/\rho + \mathbf{a}^{n+1}]$$

End timestep

1.3 Additional considerations

The basilisk uses a fractional step method consisting in:

$$\begin{array}{lrcl}
\textbf{event advection} & \mathbf{u}_a^{n+1} & = & \mathbf{u}^{n-1} - \Delta t(\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2}) \\
\textbf{event viscosity}, \textit{correction}(\Delta t) & \mathbf{u}_r^{n+1} & = & \mathbf{u}_a^{n+1} + \Delta t(-\nabla p^n/\rho + \mathbf{a}^n) \\
\textbf{event viscosity}, \textit{viscous}() & \mathbf{u}_v^{n+1} & = & \mathbf{u}_r^{n+1} + \Delta t(\nabla\cdot[2\mu\mathcal{D}_v^{n+1}]) \\
\textbf{event viscosity}, \textit{correction}(\text{-}\Delta t) & \mathbf{u}_*^{n+1} & = & \mathbf{u}_v^{n+1} - \Delta t(-\nabla p^n/\rho + \mathbf{a}^n) \\
\textbf{event projection} & \mathbf{u}^{n+1} & = & \mathbf{u}_*^{n+1} + \Delta t(-\nabla p^{n+1}/\rho + \mathbf{a}^{n+1})
\end{array} \tag{1.6}$$

Summing up all this fractional steps, the general scheme is,

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t(-\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2} - \nabla p^{n+1}/\rho + \nabla\cdot[2\mu\mathcal{D}_v^{n+1}] + \mathbf{a}^{n+1}) \tag{1.7}$$

being the deformation tensor $\mathcal{D}$ calculated with the velocity, $\mathbf{u}_v^{n+1}$, resulting of the implicit approximate equation,

$$\mathbf{u}_v^{n+1} = \mathbf{u}^n + \Delta t(-\mathbf{u}^{n+1/2}\cdot\nabla\mathbf{u}^{n+1/2} - \nabla p^n/\rho + \nabla\cdot[2\mu\mathcal{D}_v^{n+1}] + \mathbf{a}^n) \tag{1.8}$$

Since the acceleration event is located after the viscosity event, in case that the acceleration depend upon the velocity it would be,

$$\mathbf{a}^{n+1} = \mathbf{a}^{n+1}(\mathbf{u}_*^{n+1})$$

Should not be the acceleration being calculated with $\mathbf{u}_v^{n+1}$ instead of $\mathbf{u}_*^{n+1}$?