Example 3.18. *Position control system*

In Example 3.8 (Section 3.4.1), we studied the locations of the closed-loop poles of the optimal position control system as a function of the parameter ρ. As we have seen, the closed-loop poles approach a Butterworth configuration of order two. This is in agreement with the results of this section. Since the open-loop transfer function

$$H(s) = \frac{\kappa}{s(s+\alpha)} \tag{3-507}$$

has no zeroes, both closed-loop poles go to infinity as $\rho \downarrow 0$.

Example 3.19. *Stirred tank*

As an example of a multiinput multioutput system consider the stirred tank regulator problem of Example 3.9 (Section 3.4.1). From Example 1.15 (Section 1.5.3), we know that the open-loop transfer matrix is given by

$$H(s) = \begin{pmatrix} \dfrac{0.01}{s+0.01} & \dfrac{0.01}{s+0.01} \\ \dfrac{-0.25}{s+0.02} & \dfrac{0.75}{s+0.02} \end{pmatrix}. \tag{3-508}$$

For this transfer matrix we have

$$\det\,[H(s)] = \frac{0.01}{(s+0.01)(s+0.02)}. \tag{3-509}$$

Apparently, the transfer matrix has no zeroes; all closed-loop poles are therefore expected to go to ∞ as $\rho \downarrow 0$. With the numerical values of Example 3.9 for R_3 and N, we find for the characteristic polynomial of the matrix Z

$$s^4 + s^2\left(-0.5 \times 10^{-3} - \frac{0.02416}{\rho}\right) + \left(0.4 \times 10^{-7} + \frac{0.7416 \times 10^{-5}}{\rho} + \frac{10^{-4}}{\rho^2}\right). \tag{3-510}$$

Figure 3.21 gives the behavior of the two closed-loop poles as ρ varies. Apparently, each pole traces a first-order Butterworth pattern. The asymptotic behavior of the roots for $\rho \downarrow 0$ can be found by solving the equation

$$s^4 - \frac{0.02416}{\rho}s^2 + \frac{10^{-4}}{\rho^2} = 0, \tag{3-511}$$

which yields for the asymptotic closed-loop pole locations

$$-\frac{0.1373}{\sqrt{\rho}} \quad \text{and} \quad -\frac{0.07280}{\sqrt{\rho}}. \tag{3-512}$$

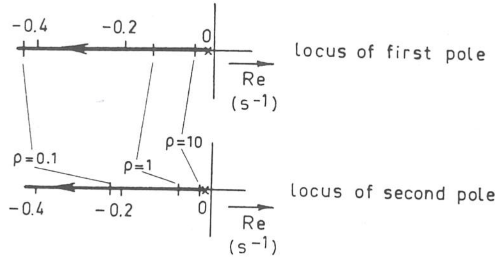

Fig. 3.21. Loci of the closed-loop roots for the stirred tank regulator. The locus on top originates from -0.02, the one below from -0.01.

The estimate **3-505** yields for the distance of the faraway poles to the origin

$$\frac{0.1}{\sqrt{\rho}}. \tag{3-513}$$

We see that this is precisely the geometric average of the values **3-512.**

Example 3.20. *Pitch control of an airplane*

As an example of a more complicated system, we consider the longitudinal motions of an airplane (see Fig. 3.22). These motions are characterized by the velocity u along the x-axis of the airplane, the velocity w along the z-axis of the airplane, the pitch θ, and the pitch rate $q = \dot{\theta}$. The x- and z-axes are rigidly connected to the airplane. The x-axis is chosen to coincide with the horizontal axis when the airplane performs a horizontal stationary flight.

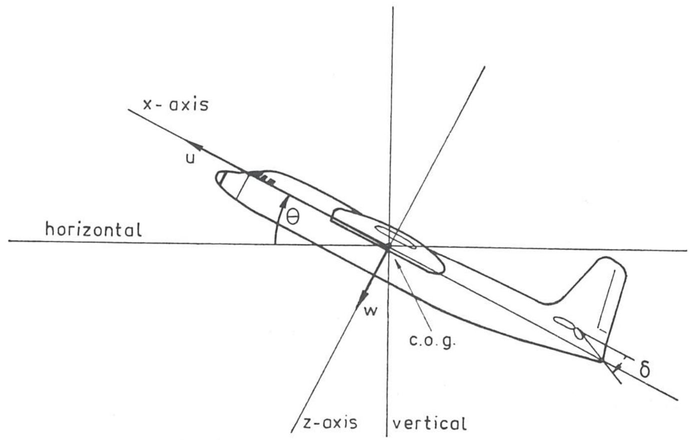

Fig. 3.22. The longitudinal motions of an airplane

The control variables for these motions are the engine thrust T and the elevator deflection δ. The equations of motion can be linearized around a nominal solution which consists of horizontal flight with constant speed. It can be shown (Blakelock, 1965) that the linearized longitudinal equations of motion are independent of the lateral motions of the plane.

We choose the components of the state as follows:

$$\begin{aligned} \xi_1(t) &= u(t), && \text{incremental speed along } x\text{-axis}, \\ \xi_2(t) &= w(t), && \text{speed along } z\text{-axis}, \\ \xi_3(t) &= \theta(t), && \text{pitch}, \\ \xi_4(t) &= q(t), && \text{pitch rate}. \end{aligned} \qquad \textbf{3-514}$$

The input variable, this time denoted by c, we define as

$$c(t) = \begin{pmatrix} T(t) \\ \delta(t) \end{pmatrix} \quad \begin{matrix} \text{incremental engine thrust}, \\ \text{elevator deflection}. \end{matrix} \qquad \textbf{3-515}$$

With these definitions the state differential equations can be found from the inertial and aerodynamical laws governing the motion of the airplane (Blakelock, 1965). For a particular medium-weight transport aircraft under cruising conditions, the following linearized state differential equation results:

$$\dot{x}(t) = \begin{pmatrix} -0.01580 & 0.02633 & -9.810 & 0 \\ -0.1571 & -1.030 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 0.0005274 & -0.01652 & 0 & -1.466 \end{pmatrix} x(t) + \begin{pmatrix} 0.0006056 & 0 \\ 0 & -9.496 \\ 0 & 0 \\ 0 & -5.565 \end{pmatrix} c(t). \qquad \textbf{3-516}$$

Here the following physical units are employed: u and w in m/s, θ in rad, q in rad/s, T in N, and δ in rad.

In this example we assume that the thrust is constant, so that the elevator deflection $\delta(t)$ is the only control variable. With this the system is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} -0.01580 & 0.02633 & -9.810 & 0 \\ -0.1571 & -1.030 & 0 & 120.5 \\ 0 & 0 & 0 & 1 \\ 0.0005274 & -0.01652 & 0 & -1.466 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ -9.496 \\ 0 \\ -5.565 \end{pmatrix} \delta(t). \qquad \textbf{3-517}$$

As the controlled variable we choose the pitch $\theta(t)$:

$$\theta(t) = (0, 0, 1, 0)x(t). \qquad \textbf{3-518}$$

It can be found that the transfer function from the elevator deflection $\delta(t)$ to the pitch $\theta(t)$ is given by

$$\frac{-5.565s^2 - 5.663s - 0.1112}{s^4 + 2.512s^3 - 3.544s^2 + 0.06487s + 0.03079}. \qquad \textbf{3-519}$$

The poles of the transfer function are

$$\begin{aligned} &-0.006123 \pm j0.09353, \\ &-1.250 \pm j1.394, \end{aligned} \qquad \textbf{3-520}$$

while the zeroes are given by

$$-0.02004 \quad \text{and} \quad -0.9976. \qquad \textbf{3-521}$$

The loci of the closed-loop poles can be found by machine computation. They are given in Fig. 3.23. As expected, the faraway poles group into a Butterworth pattern of order two and the nearby closed-loop poles approach the open-loop zeroes. The system is further discussed in Example 3.22.

Example 3.21. *The control of the longitudinal motions of an airplane*

In Example 3.20 we considered the control of the pitch of an airplane through the elevator deflection. In the present example we extend the system by controlling, in addition to the pitch, the speed along the x-axis. As an additional control variable, we use the incremental engine thrust $T(t)$. Thus we choose for the input variable

$$c(t) = \begin{pmatrix} T(t) \\ \delta(t) \end{pmatrix} \quad \begin{matrix} \text{incremental engine thrust}, \\ \text{elevator deflection}, \end{matrix} \qquad \textbf{3-522}$$