初等数论自测题二 参考答案与评分标准

一、名词解释

1. An algebraic number α is a complex number α that is a root of an integral coefficients polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ ($a_0, a_1, \cdots, a_n$ are integers).

……………………………………3 分

2. Given two integers a , b , if $m \mid (b-a)$, then we say a is congruent to b modulo m, and denote it by $a \equiv b (mod \quad m)$ ……………………………………3 分

3. Fundamental Theorem of Arithmetic:
Every integer greater than 1 can be written uniquely as a product of primes.

……………………………………4 分

二、应用题

4.
Sol:

a) Since the last digit of $504, 504^2, 504^3, \cdots$ are $4, 6, 4, \cdots$, it is quite easy to find the last digit of 504^{2022} is 6. ……………………………………5 分

b) The last three digits of the decimal representation of 504^{2022} is $504^{2022} (mod \quad 1000)$.

……………………………………3 分

Use repeated squaring method to find $504^{2022} (mod \quad 1000) = 416$.（参考课上例题写出过程）

……………………………………过程 10 分

So the last three digits of the decimal representation of 504^{2022} is 416..

……………………………………2 分

（注：求 $504^{2022} (mod \quad 1000)$ 的方法不唯一，我们之后还会介绍新的方法。）

三、计算题

5.
Sol:

a) $[\sqrt{1891}] = 43$,

$44^2-1891=45$, not a square.

$45^2-1891=134$, not a square.

$46^2-1891=225=15^2$, so $1891=46^2-15^2=31\times 61$

……………费马因子分解的过程 3 分、答案 1 分

（没有按题目要求用费马因子分解法的扣 3 分）

b) $\tau(1891)=4$, 1891 has 4 positive divisors. ……………………………………3 分

c) $\sigma(1891)=32\times 62=1984$, the sum of all positive divisors of 1891 is 1984.

……………………………………3 分

6.

Sol:

Let A_n be the number of nonnegative integral solutions of the Diophantine equation

$x+y+2z=n$.

The generating function of { A_n } is $f(t)=\dfrac{1}{(1-t)(1-t)(1-t^2)}$

……………………………………3 分

Partial fraction, we get $f(t)=-\frac{1}{2}\frac{1}{(t-1)^3}+\frac{1}{4}\frac{1}{(t-1)^2}-\frac{1}{8}\frac{1}{t-1}+\frac{1}{8}\frac{1}{t+1}$ ……………4 分

$$A_n=\frac{1}{n!}f^{(n)}(0)=\frac{2n^2+8n+7+(-1)^n}{8}$$

So the number of nonnegative integral solutions of the Diophantine equation

$x+y+2z=n$ is $\dfrac{2n^2+8n+7+(-1)^n}{8}$.

……………………………………3 分

7.

Sol:

$\dfrac{10}{81}$（参考课上例题写出求解过程和检验）

……………………………………过程 6 分、答案 2 分、按题设要求验算 2 分

四、证明题

8.

Proof：

$(10101)_b = b^4 + b^2 + 1.$ ··2 分

$b^4 + b^2 + 1 = (b^2 + b + 1)(b^2 - b + 1)$. ··2 分

Since $b > 1$， $b^2 + b + 1 > 1, b^2 - b + 1 > 1$, so $(10101)_b = b^4 + b^2 + 1$ is a composite number.

(一些同学作业里漏了这一步) ··2 分

9.

Proof：

We first prove that "every integer greater than or equal to 2 can be written as a product of primes."

Let n be an integer greater than or equal to 2, use induction on n.

When $n = 2$, 2 itself is a prime, the statement holds.

···3 分（注意本题用归纳法或其它等价方法（如递降法等）证明时归纳初始一定是从 2 开始）

Now, assume when $n \le k$ $(k \ge 2)$, the statement holds，i.e. an integer between 2 and k can be factorized into a product of primes. ··2 分

（注意是第二数学归纳法）

When $n = k + 1$,

If $k + 1$ is a prime, then the statement holds. ··1 分

If $k + 1$ is a composite, then we have:

$$k + 1 = m_1 m_2$$

where $2 \le m_1, m_2 \le k$ ··1 分

By induction assumption, m_1, m_2 can be written as a product of primes, we then get

$m_1 m_2$ can also be written as a product of primes. ···································2 分

（这里使用归纳假设时可看出为什么归纳初始是从 2 开始）

By induction, we prove the statement "every integer greater than or equal to 2 can be written as a product of primes." Clearly, it implies the statement "every integer greater than 2022 can be written as a product of primes." ···································1 分

10.

Proof:

a) If m is a square, say $m=n^2$ where n is an integer, then for any prime p,

$ord_p m = 2ord_p n$ is even. ……………………………………………………2 分

Conversely, if for any prime p, $ord_p m$ is even.

If m=1, then $m=1^2$ is a square.

If $m>1$, by the Fundamental Theorem of Arithmetic, $m = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$, where

$p_1, p_2, \cdots, p_l$ are distinct primes, $a_1, a_2, \cdots, a_l$ are positive integers.

Since for all i, $a_i = ord_{p_i} m$ is even, say $a_i = 2b_i$ where b_i is a positive integer.

Then $m = (p_1^{b_1} p_2^{b_2} \cdots p_l^{b_l})^2$ is a square. ……………………………………………………3 分

b) Assume not, then

$$\sqrt{m} = \frac{a}{b} \qquad (*)$$

where *a*,*b* are nonzero integers.

Squaring (*), we get：

$$mb^2 = a^2 \qquad (**)$$

By part a), since m is not a square, so there exists a prime p, s.t. $ord_p m$ is odd.

Apply ord_p to (**), we get：

$$ord_p m + 2ord_p b = 2ord_p a \qquad (***)$$

The LHS of (***) is odd, but the RHS of (***) is even, a contradiction.

Thus $\sqrt{m}$ is an irrational number.

……………………………………………………6 分

c) By part b), it suffices to show $2^{251}-1$ is not a square.

Clearly, $4 \mid 2^{251}$, so $2^{251}-1$ is a 4k+3 type integer.

Since the square of an odd is a 4k+1 type integer and the square of an even is a 4k type integer, $2^{251}-1$ is not a square.

By part b), $\sqrt{2^{251}-1}$ is an irrational number. ……………………………………3 分

11.

Proof:

a)Let $g(a_1,a_2,\cdots,a_n)$ be the Frobenius number of coprime positive integers $a_1,a_2,\cdots,a_n$, i.e. the largest integer that cannot be expressed as an non-negative integral combination of $a_1,a_2,\cdots,a_n$.

Since $gcd(8,10)=2$ and $g(a,b)=ab-a-b$ for coprime positive integers *a*,*b*, we have

$$g(8,9,10)=g(8,10,9)\le g(2,9)+2\times(g(\frac{8}{2},\frac{10}{2})+1)=(2\times9-2-9)+2\times((4\times5-4-5)+1)=31$$

Thus when $c>31$, the Diophantine equation $8x+9y+10z=c$ always has an nonnegative integer solution.

……………………………………4 分

b) If the Diophantine equation $8x+9y+10z=31$ has a nonnegative integer solution (x_0,y_0,z_0), then $8x_0+9y_0+10z_0=31$.

Since $x_0,y_0\ge0$, so $10z_0\le31$, hence $z_0=0$ or 1 or 2 or 3. …………………1 分

If $z_0=0$, then...参考课上例题补完检验

If $z_0=1$, then...参考课上例题补完检验

If $z_0=2$, then...参考课上例题补完检验

If $z_0=3$, then...参考课上例题补完检验

So when $c=31$, the Diophantine equation $8x+9y+10z=c$ has no nonnegative integer solution.

……………………………………5 分