Elementary Number Theory
Spring 2022 Problem Set II

2022 年 3 月 30 日

一、名词解释(2 points)

Write out the definitions(resp. contents) of following concepts(resp. theorems):

1. algebraic number
2. algebraic integer
3. Frobenius number of $a_1, a_2, \cdots, a_n$, where $a_1, a_2, \cdots, a_n$ are coprime integers.

二、计算题

1. (2 points)
   Use Euclidean algorithm to find:
   a) $gcd(123456789, 987654321)$
   b) $lcm(1541, 943)$

2. (3 points)
   a)Find all integer roots of the polynomial $5x^5 - x^4 + 10106x^3 - 2017x^2 - 8089x - 4$.
   b)Find all rational roots of the polynomial $5x^5 - x^4 + 10106x^3 - 2017x^2 - 8089x - 4$.

3. (2 points)
   a)Is the Diophantine equation $72792x + 268926y = 22242$ solvable?Give your reason(You don't have to find out an integer solution even if your answer is "yes").
   b)Is there any nonnegative integer solution of the Diophantine equation $72792x + 268926y = 22242$?Give your reason(You don't have to find out a nonnegative integer solution even if your answer is "yes").

4. (2 points)
   Find all nonnegative integer solutions of the Diophantine equation $2052x + 1273y = 45486$.

5. (2 points)

   Find a fractional $\frac{p}{q}$(where p, q are positive integers less than 1000) satisfy the following conditions:

$$|\frac{p}{q} - \pi| < 10^{-6}$$

   where π is a math constant and $e \approx 3.1415926$.

   Check your results.

6. (2 points)

   Find all integer solutions of the Diophantine equation $70x_1 + 42x_2 + 30x_3 = 1526$.

7. (2 points)

   Find all nonnegative integer solutions of the following linear system of Diophantine equations

$$209x_1 + 25x_2 + 385x_3 = 2418$$

$$528x_1 + 35x_2 + 275x_3 = 2536$$

8. (2 points)

   a)Let n be a positive integer, find the number of nonnegative integral solutions of the Diophantine equation $x + y + 2z = n$.

   b)Find the number of nonnegative integral solutions of the Diophantine equation $x + y + 2z = 2022$.

   c)Determine the positive integer n such that the Diophantine equation $x + y + 2z = n$ has exactly 1000000 nonnegative integral solutions.

9. (3 points)

   Find all integer solutions of the Pythagorean equation(亦称勾股数方程)

$$x^2 + y^2 = z^2$$

10. (2 points)

   a) Find all positive integers x, such that $x^2 + 81$ is a square.

b) Find all positive integers y, such that $y^2 - 100$ is a square.

Fraction of the form $\frac{1}{n}$ is called unit fraction, which was used in the times of the pharaohs. The following is a problem on unit fractions.

11. (3 points)

a) Let $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$ be a positive integer, where $p_1, p_2, \cdots, p_l$ are distinct primes, $a_1, a_2, \cdots, a_l$ are positive integers. Find the number of integral solutions of the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

b) Find the number of integral solutions of the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{1001}$$

三、应用题

1. (2 points)

A sunny afternoon, Sherlock Holmes and Doctor Watson are chatting while drinking black tea.

Watson: *Sherlock, Have you ever heard Diophantus?*

Holmes: *Of course. Diophantus is the "father of algebra", he was the first to employ symbols in Greek algebra, and used symbols for an unknown quantity, as well as for algebraic operations and for powers, also, the Diophantine equation is named after him...*

Watson: *Sounds like a really great people... Anything known for his life, e.g. years in which he lived?*

Holmes: *It's a pity that essentially nothing is known of his life, and there has been much debate regarding the precise years in which he lived. Historians believed that Diophantus lived in the Silver Age, but it is hard to pinpoint the exact years in which he lived. The greatest amount of information about Diophantus's life comes from the possibly*

fictitious collection of riddles written by Metrodorus around 5th Century, the most famous is as follows:
His boyhood lasted 1/6 of his life;
His beard grew after 1/12 more;
He married after 1/7 more, and his son was born five years later;
The son lived to half his father's age, and the father died four years after the son.

a) Find the age when Diophantus leave the world.

Watson: *Sounds like a legend...Sherlock, Are there any other ancient people also focus on the Diophantine equation?*

Holmes smiles: *In fact, Chinese did some deep research in this area, e.g. an ancient Chinese mathematician Zhang Qiujian, who lived in the sixth century, posted the "hundred fowls" problem, asks:*
"If a cock is worth 5 coins, a hen worth 3 coins, and three chicken together worth 1 coin. How many cocks, hens, and chickens, totaling 100, can be bought for 100 coins?"

b) Solve the above "hundred fowls" problem.

Watson: *Amazing! I'll try some equations myself!*

c) Find all nonnegative integer solutions of the following linear system of Diophantine equations

$$209x_1 + 25x_2 + 385x_3 = 2418$$

$$528x_1 + 35x_2 + 275x_3 = 2536$$

2. (2 points)
A businessman returning home from conferences in Los Angles and Frankfurt, changes his dollars and euros into RMB, we know that $1dollar = 7.09yuan$ and $1euro = 7.67yuan$. If both the amount of dollars and euros are greater than 1500, and he receives 26164.14yuan in total, how much of each currency did he exchange?(Note that the answer may not unique)

3. (2 points)
   The China Railway Station offers three types of rail tickets from Beijing to Shanghai. First-Class tickets are 933 yuan, Second-Class tickets are 553 yuan and the Business Class tickets are 1748 yuan. If 1368 passengers pay a total of 1076769 yuan on a trip, how many of each type of the tickets were sold?(Note that the answer may not unique)

4. (2 points)
   The Indian astronomer and mathematician Mahavira, who lived in the ninth century, posted this puzzle:
   A band of 23 weary travelers entered a lush forest where they found 63 piles of bananas each pile contains the same number of bananas and another pile of 7 bananas. They divided the bananas equally. How many bananas at least in each pile? Solve this puzzle.

5. (3 points)
   a)A post office just have stamps of two values: 3fen and 8fen. Which postage amount can be made up exactly just by using these stamps(We assume that there are infinite stamps of each value), which cannot be? In particular, If the postage amount is 35fen, how to combine the stamps to get the amount?(Note that the answer may not unique)
   b)Another post office is left with stamps of only two values. The staffs found that there are exactly 24 postage amount that cannot be made up using these stamps, including 30fen. What are the values of stamps?(Note that the answer may not unique)
   c)Still another post office just have stamps of three values: 6fen, 10fen and 15fen. Which postage amount can be made up exactly just by using these stamps(We assume that there are infinite stamps of each value), which cannot be?
   In particular, If the postage amount is 48fen, how to combine the stamps to get the amount?
   d)The fourth post office is left with stamps of only three values. The

staffs found that there are exactly 6 postage amount that cannot be made up using these stamps, including 4fen and 9fen. What are the values of stamps? (Note that the answer may not unique)

四、证明题

1. (2 points)

   a) Prove that for all natural numbers n, $gcd(12n+1, 30n+2) = 1$.

   b)(1959, IMO) Prove that for all natural numbers n, the fraction

   $$\frac{21n+4}{14n+3}$$

   is irreducible(i.e. the numerator and denominator of the fraction are coprime).

2. (2 points)

   Let a, b be two positive integers, q_k, r_k be the quotients and remainders occurred in the repeated division algorithms to find $g.c.d.(a, b)$, i.e.

   $$a = bq_1 + r_1, \quad 0 < r_1 < b$$

   $$b = r_1 q_2 + r_2, \quad 0 < r_2 < r_1$$

   $$\cdots$$

   $$r_i = r_{i+1} q_{i+2} + r_{i+2}, \quad 0 < r_{i+2} < r_{i+1}$$

   $$\cdots$$

   P_k, Q_k are the numbers in the Extended Euclidean Algorithm, i.e.

   $$P_0 = 1, \quad P_1 = q_1, \quad P_k = q_k P_{k-1} + P_{k-2} \quad (k \geq 2)$$

   $$Q_0 = 0, \quad Q_1 = 1, \quad Q_k = q_k Q_{k-1} + Q_{k-2} \quad (k \geq 2)$$

   Use induction to prove: for all positive integer k,

   $$Q_k a - P_k b = (-1)^{k+1} r_k$$

3. (2 points)

   Let a, b, c be integers(a, b nonzero), consider the Diophantine equation

$$ax + by = c \quad (*)$$

   a)Prove that (*) has integer solutions iff $gcd(a, b)|c$.

   b)Prove that the equation (*) has infinitely many integer solutions if $gcd(a, b)|c$.

4. (3 points)

   a)Let $x_0 = \frac{p}{q}(p, q$ are coprime integers) be a rational root of an integral coefficients polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 (a_0, a_1, \cdots, a_n$ are integers), prove that

$$q|a_n \quad and \quad p|a_0$$

   b)Let x_0 be a rational root of a monic integral coefficients polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 (a_0, a_1, \cdots, a_{n-1}$ are integers), prove that x_0 is an integer.

   c)Let m be a positive integer, Use b) to prove that $\sqrt{m}$ is irrational iff m is not a square.

   d)Use b) to prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

   e)Let $d \neq 0, 2$ be an integer, prove that $x^3 - dx + 1$ is an irreducible polynomial in $\mathbb{Q}[X]$.

   f)By calculus, we know that the cubic polynomial $x^3 + px + q$ always has a real root, and we actually have Cardano's formula to get a root:

$$x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

   Prove that $\sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2022^3}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{2022^3}{27}}}$ is an irrational number.

5. (2 points)

   Let α be an algebraic number. Prove that there is an integer n such that $n\alpha$ is an algebraic integer.

6. (2 points)

   Let $a_1, a_2, \cdots, a_n, c$ be integers($a_1, a_2, \cdots, a_n$ nonzero), consider the Diophantine equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = c \quad (*)$$

   a)Prove that (*) has integer solutions iff $gcd(a_1, a_2, \cdots, a_n)|c$.

   b)Prove that the equation (*) has infinitely many integer solutions if $gcd(a_1, a_2, \cdots, a_n)|c$.

7. (2 points)

   a)Prove that for all positive integer n, the Diophantine equation

$$x^2 + y^2 = 100^n$$

   has positive integer solutions.

   b)Prove that for all positive integer n, the Diophantine equation

$$x^2 + y^2 + z^2 = 101^n$$

   has positive integer solutions.

   c)Prove that for all positive integer n, the Diophantine equation

$$x^2 + y^2 + z^2 + w^2 = 102^n$$

   has positive integer solutions.

8. (2 points)

   Prove that the equation

$$x^2 = y^3 + z^{2022}$$

   has infinitely many positive positive integer solutions.

Given coprime positive integers $a_1, a_2, \cdots, a_n$, the largest integer that cannot be expressed as an nonnegative integral combination of these numbers is called the ***Frobenius number*** *of $a_1, a_2, \cdots, a_n$, and denoted by $g(a_1, a_2, \cdots, a_n)$.*

9. (2 points)

Let a, b be coprime integers, c be an integer, consider the Diophantine equation

$$ax + by = c \quad (*)$$

a)Prove that if $c > ab - a - b$, then (*) has an nonnegative integer solution.

b)Prove that if $c = ab - a - b$, then (*) has no nonnegative integer solution.

10. (3 points)

a)Let $a_1, a_2, \cdots, a_n$ be coprime positive integers, prove that the Frobenius number of $a_1, a_2, \cdots, a_n$ exists.

b)Prove that if $c > 9$, then the Diophantine equation $5x + 6y + 8z = c$ has an nonnegative integer solution.

c)Prove that if $c = 9$, then the Diophantine equation $5x + 6y + 8z = c$ has no nonnegative integer solution.

d)Prove that if $c > 29$, then the Diophantine equation $6x + 10y + 15z = c$ has an nonnegative integer solution.

e)Prove that if $c = 29$, then the Diophantine equation $6x + 10y + 15z = c$ has no nonnegative integer solution.

11. (3 points)

Let a, b, c be pairwise coprime positive integers, prove the following statements:

a)If $n > 2abc - ab - bc - ca$, then the Diophantine equation $bcx + cay + abz = n$ has an nonnegative integer solution.

b)If $n = 2abc - ab - bc - ca$, then the Diophantine equation $bcx + cay + abz = n$ has no nonnegative integer solution.

注意事项

1. 以上是本课程第二部分（丢番图方程理论）讲解结束后的习题课内容，也是要求大家第二部分要掌握的内容的底限。我们不会直接提供习题答案给大家，大家需要根据上课笔记内容独力或和同学们合作讨论完成这些题目。

2. 这些题目不用交（作业题目是根据这些题目稍作变化后的题目），但是习题课上会让大家上台讲解这些题目。上台讲解会有讲解分，题号旁边的数字上台讲解习题正确的讲解分。

3. 平时成绩＝签到考勤分（*10*分）+上课纪律分（*30*分）+作业分数（*20*分）+讲题分+期中考试成绩×40%，大于*100*按*100*计算。

4. 拒绝上台讲题或是上台后一言不发一字不写的会得到D的评分，第一次评分为D不扣平时分，从第二次评分为D开始，每次评分D扣*5*分平时分。

5. 总评成绩＝ 平时成绩×40%+ 期末考试卷面成绩×60%。

6. **新：疫情原因，我们期中时不一定能按原计划举行期中考试，如果不能举行期中考试的话，平时成绩和总评成绩计算公式将变更、作业评价会影响平时成绩。**

Suggest Reading

“将一个立方数分成两个立方数之和，或一个四次幂分成两个四次幂之和，或者一般地将一个高于二次的幂分成两个同次幂之和，这是不可能的。关于此，我确信已发现了一种美妙的证法，可惜这里空白的地方太小，写不下。”

《费马大定理，一个困惑了世间智者358 年的谜》（西蒙.辛格著，薛密译），广西师范大学出版社，2013