Elementary Number Theory
Spring 2022   Problem Set I

2022 年 2 月 20 日

一、名词解释(2 points)

Write out the definitions(resp. contents) of following concepts(resp. theorems):

1. prime number
2. composite number
3. division algorithm
4. base-b representation of positive integers
5. a, b(resp. $a_1, a_2, \cdots, a_n$) coprime, where a, b(resp. $a_1, a_2, \cdots, a_n$) are integers
6. Fermat number
7. multiplicative function
8. Mersenne prime number
9. perfect number
10. Fundamental theorem of Arithmetic
11. Prime Number Theorem

二、计算题

1. (2 points)
   a)Convert 20220216 to hexadecimal and base-11 expansion.
   b)Convert $(20220216)_{16}$ to binary and octal.
   c)Convert $(20220216)_9$ to base-3 expansion.
   d)Add $(ABC)_{16}$ and $(123)_{16}$.
   e)Subtract $(2021)_{15}$ from $(ABCD)_{15}$.

2. (1 point)
   a)Find the quotient and remainder when 105 is divided by 11.
   b)Find the quotient and remainder when -105 is divided by 11.

3. (*You can skip this problem and just remember these consequences.*)
   * Let n be an integer, $(a_k a_{k-1} \cdots a_0)_{10}$ be the decimal representation of n. Prove the following:
   a) n can be divided by 2 iff(short for if and only if) the last digit of n can be divided by 2(i.e. a_0 can be divided by 2).
   b) n can be divided by 4 iff the last two digits of n can be divided by 4(i.e. $(a_1 a_0)_{10}$ can be divided by 4).
   c) n can be divided by 8 iff the last three digits of n can be divided by 8(i.e. $(a_2 a_1 a_0)_{10}$ can be divided by 8).
   d) n can be divided by 5 iff the last digit of n can be divided by 5(i.e. a_0 can be divided by 5).
   e) n can be divided by 25 iff the last two digits of n can be divided by 25(i.e. $(a_1 a_0)_{10}$ can be divided by 25).
   f) n can be divided by 3 iff the sum of digits of n can be divided by 3(i.e. $a_0 + a_1 + \cdots + a_k$ can be divided by 3).
   g) n can be divided by 9 iff the sum of digits of n can be divided by 9(i.e. $a_0 + a_1 + \cdots + a_k$ can be divided by 9).
   h) n can be divided by 11 iff the sum of digits in the odd positions of n minus the sum of digits in the even positions of n can be divided by 11(i.e. $(a_0 + a_2 + a_4 + \cdots) - (a_1 + a_3 + a_5 + \cdots)$ can be divided by 11).

4. (2 points)
   a)Is 122333444455555666666 a multiple of 2(resp. 3,4,5,9,11)? Give your reason.
   b)An integer(in decimal representation) $(2a2b2c)_{10}$ can be divided by 1320, find the values of a, b and c.

5. (1 point)
   a)Let $a_1, a_2, \cdots, a_n$ be integers, state the definitions of $a_1, a_2, \cdots, a_n$ being coprime and they being pairwise coprime, explain the difference.
   b)Give an example: three integers are coprime but not pairwise coprime.

6. (2 points)

   a)Is 1 a prime or a composite?

   b)Is 137 a prime or a composite? Give your reason. If 137 is a composite, factorize it into prime powers.

   c)Is 2021 a prime or a composite? Give your reason. If 2021 is a composite, factorize it into prime powers.

   d)Classify the primes and composites in $\{1, 11, 139, 2022, 2023, 10086\}$, give your reason.

7. (1 point)

   Let p be a prime, a be a non-negative integer, n be an integer,

   a)Write out the definition of $p^a||n$.

   b)Write out the definition of $ord_p n$.

   c)Calculate $ord_3 1080$.

8. (2 points)

   a)Find the integer s such that $11^s||2022!$.

   b)Find the integer t such that $8^t||224!$.

9. (2 points)

   a)Factorize 30! into the product of prime powers.

   b)The hexadecimal representation of 2022! ends in exactly m consecutive zeros, find the value of m.

   c)The duodecimal representation of $\binom{2022}{2101} = \frac{2022!}{2101!91!}$ ends in exactly n consecutive zeros, find the value of n.

10. (2 points)

    a)Calculate $gcd(1080, 1215)$ and $lcm(1080, 1215)$.

    b)Calculate $gcd(10^{2021}, 2021!)$ and $lcm(10^{2021}, 2021!)$.

    c)Calculate $gcd(10^{2021}, \binom{2022}{2101})$.

11. (3 points)

    Let $n = p_1^{a_1} p_2^{a_2} \cdots p_l^{a_l}$ be a positive integer, where $p_1, p_2, \cdots, p_l$ are distinct primes, $a_1, a_2, \cdots, a_l$ are positive integers.

a)let $\tau(n)$ be the number of positive divisors of n, write out the formula of $\tau(n)$(proof is not required).
b)let $\sigma(n)$ be the sum of the positive divisors of n, write out the formula of $\sigma(n)$(proof is not required).
c)Find the number of positive divisors of 2022.
d)Find the number of even positive divisors of 2022.
e)Find the sum of all positive divisors of 2022.
f)Find the sum of all even positive divisors of 2022.
g)Find the sum of all positive divisors which are multiples of 4 of 10000.
h)Use Fermat Factorization to factorize 6077 into the product of prime powers, then calculate $\nu(6077)$ and $\sigma(6077)$.

12. (2 points)
a) Estimate the number of primes between 10^{2022} and 10^{2023}(Take $\ln 10 \approx 2.302$).
b) Estimate the size of the 10^{2022}th prime(Take $\ln 10 \approx 2.302$).

三、应用题

1. (2 points)
A sunny afternoon, Sherlock Holmes and Doctor Watson are chatting while drinking black tea.
Watson: "Sherlock, Have you ever heard factorial? "
Holmes: "Sure. $n!$ *is the product of integers from* 1 *to* n*, e.g.* $3! = 1 \times 2 \times 3 = 6$, $4! = 1 \times 2 \times 3 \times 4 = 24$."
a) Calculate 5! and 10!.
Watson: "It seems that n! is a really huge number when n is large."
Holmes: "Yes, for example, no one can tell the exact value of 100!... *But I can tell you the number of consecutive 0s in the tail of* 100!."
b) Calculate the number of consecutive 0s in the tail of decimal representation of 100!
Watson: "Wonderful! This is amazing!"

Holmes smiles: "There are also other numbers correspond to the factorial, e.g. combinatorial numbers, it is also hard to tell the exact value of $\binom{100}{50} = \frac{100!}{50!50!}$*, but I can tell you the number of consecutive 0s in the tail of this number."*

c) Calculate the number of consecutive 0s in the tail of decimal representation of $\binom{100}{50} = \frac{100!}{50!50!}$.

Watson: "Marvelous! Sherlock, Is this the power of deduction?"

Holmes laughs: "No, Watson. It is the power of number theory."

2. (2 points)

   A dozen equals 12, and a gross equals 12^2*. Using duodecimal arithmetic, answer the following questions.*

   a) If 3 gross 7 dozen and 4 eggs are removed from a total of 11 gross and 3 dozen eggs, how many eggs are left?

   b) If 5 truckloads of 2 gross 3 dozen and 7 eggs each are delivered to the supermarket, how many eggs are delivered?

   c) If 11 gross 10 dozen and 6 eggs are divided in 3 groups of equal size, how many eggs are in each group?

四、证明题

1. (2 points)

   A recursive sequence $\{a_n\}$ is defined as following:

   $$a_1 = 1, \quad a_2 = 5, \quad a_{n+1} = a_n + 2a_{n-1} \quad (n \geq 2)$$

   Use induction to prove: for any positive integer n, $a_n = 2^n + (-1)^n$.

2. (2 points)

   Prove: Every integer greater than 2022 can be written as the product of primes(uniqueness is not required).

   (Here you are not allowed to use the consequence of the Fundamental Theorem of Arithmetic directly.)

3. (2 points)
   Let n be a natural number, prove that $n^4 + n^2 + 1$ is a prime if and only if $n = 1$.

4. (2 points)
   Let a, n be two positive integers greater than 1,
   a)Prove: $(a-1)|(a^n - 1)$.
   b)Prove: If $a^n - 1$ is a prime, then $a = 2$ and n is a prime.
   *(Thus $2^p - 1$ is a prime **only if** p is a prime.)*

5. (2 points)
   Let a, n be two positive integers greater than 1,
   a)Prove: If n is odd, then $(a+1)|(a^n + 1)$.
   b)Prove: If $a^n + 1$ is a prime, then n is a power of 2.

6. (2 points)
   a)Prove that there are infinitely many primes.
   b)Let p_i be the i-th prime, e.g. $p_1 = 2, p_2 = 3, \cdots$, prove that: for any positive integer n,
   $$p_{n+1} \leq p_1 p_2 \cdots p_n + 1$$
   c)Let p_n as above, use induction to prove: for any positive integer k, $p_k < 2^{2^k}$.
   (Here you are not allowed to use the consequence of Bertrand postulate directly.)

7. (2 points)
   Let $p_1, p_2 \cdots$ be the increasing sequence of all odd primes.
   a)Prove that for any integer N, there exist two adjacent primes p_k, p_{k+1}, such that $p_{k+1} - p_k > N$.
   b)Prove that the sum of two adjacent odd primes(i.e. $p_k + p_{k+1}$) is the product of three integers greater than 1.

8. (1 point)
   Two positive integers m and n are called an amicable pair if $\sigma(m) =$

$\sigma(n) = m + n$.

Prove that 220 and 284 is an amicable pair.

9. (2 points)

   a) Prove that $\tau(n)$ and $\sigma(n)$ are multiplicative functions.

   b) Prove that $\tau(n)$ is odd if and only if n is a square.

   c) Prove that $\sigma(n)$ is odd if and only if n is a square or twice a square.

   d) A number n is called "multiplicatively perfect" if the product of the positive divisors of n is n^2. Find all "multiplicatively perfect" numbers.

10. (2 points)

    a)Prove: $\sqrt{2022}$ is irrational.

    b)Prove: $\sqrt[3]{2}$ is irrational.

    c)Prove: $\sqrt{3} + \sqrt{5}$ is irrational.

    d)Prove: $\log_{30} 2022$ is irrational.

11. (3 points)

    a)Let m be a positive integer, prove that m is a square if and only if for any prime p, $ord_p m$ is even.

    b)Let m as above, prove that $\sqrt{m}$ is irrational if and only if m is not a square.

    c)Prove: $\sqrt{1000000000000000009}$ is irrational.

    d)Let m, n be positive integers, prove that $\sqrt{m} + \sqrt{n}$ is irrational if and only if at least one of m, n is not a square.

    e)Prove that for any positive integer x, $\sqrt{x} + \sqrt{x + 2022}$ is irrational.

12. (2 points)

    a) Let x, y be two real numbers, prove that

$$[x] + [y] \leq [x + y] \leq [x] + [y] + 1$$

$$[x] + [y] + 2[x + y] \leq [3x] + [3y]$$

    b) Prove that for any two positive integers m and n, $\frac{(3m)!(3n)!}{(m)!(n)!((m+n)!)^2}$ is an integer.

13. (3 points)

    a) Prove that almost all positive integers contain the digit "0", i.e. If we let $N(n)$ be the number of positive integers $\leq n$ whose decimal expansions do not contain the digit "0", then

    $$\lim_{n\to\infty} \frac{N(n)}{n} = 0$$

    b) Prove that almost all positive integers contain the chain "2022", i.e. If we let $N(n)$ be the number of positive integers $\leq n$ whose decimal expansions do not contain the chain "2022"(e.g. 20220228 contains the chain 2022, while 20200222 does not contain the chain 2022), then

    $$\lim_{n\to\infty} \frac{N(n)}{n} = 0$$

14. (3 points)

    a) Let $\pi(x)$ be the number of primes smaller than x, $a < b$ be two positive real numbers. Prove that:

    $$\lim_{x\to+\infty} (\pi(bx) - \pi(ax)) = +\infty$$

    b) Prove that the set of numbers of the form $\frac{p}{q}$(where p and q are primes) is dense in the set of positive real numbers, i.e. $\forall x \in \mathbb{R}^+, \forall \epsilon > 0$, there exist primes p, q, such that $|x - \frac{p}{q}| < \epsilon$.

15. (3 points)

    a) State Goldbach Conjecture(proof is not required).

    b) Prove that Goldbach Conjecture is equivalent to the statement "Every integer greater than 5 can be written as the sum of three primes".

注意事项

1. 以上是本课程第一部分（整数理论）讲解结束后的习题课内容，也是要求大家第一部分要掌握的内容的底限。我们不会直接提供习题答案给大家，大家需要根据上课笔记内容独力或和同学们合作讨论完成这些题目。

2. 这些题目不用交(作业题目是根据这些题目稍作变化后的题目),但是习题课上会让大家上台讲解这些题目。上台讲解会有讲解分,题号旁边的数字上台讲解习题正确的讲解分。

3. 平时成绩=签到考勤分(*10*分)+上课纪律分(*30*分)+作业分数(*20*分)+讲题分+期中考试成绩×40%,大于*100*按*100*计算。

4. 拒绝上台讲题或是上台后一言不发一字不写的会得到D的评分,第一次评分为D不扣平时分,从第二次评分为D开始,每次评分D扣*5*分平时分。

5. 总评成绩= 平时成绩×40%+ 期末考试卷面成绩×60%。

Suggest Reading

我们这学期的课只是教授一些初等数论中最基本的内容,我们推荐一些经典的初等数论教材(当然这方面的好书很多很多)供大家课外深入阅读:

1. G.H.Hardy and E.M.Wright著,《An introduction to the theory of numbers》(6th edition)
(国内出过一套图灵数学、统计学丛书,这本书也被翻译出版了。)

2. I.Niven, H.S.Zuckerman and H.L.Montgomery著,《An introduction to the theory of numbers》(5th edition)

3. A.Granville著,《Number Theory Revealed: An Introduction》和《Number Theory Revealed: A Masterclass》
(这是最近才写的书,所以里面介绍了很多新结果。后面那本书是前面那本书的扩展版。)

4. Kenneth Ireland,Michael Rosen,《A Classical introduction to Modern Number Theory》(第二版)
(这本书已经不算初等数论教材了,但这是一本好的入门书。)

5. 徐迟,《哥德巴赫猜想》
(*1978*年的春天,徐迟的报告文学《哥德巴赫猜想》使得数学家陈景润成了家喻户晓的人物,激起了无数人研究哥德巴赫猜想的热情,是值得一读的作品。)