初等数论期中自测题参考答案与评分标准

一、名词解释

1. An integer x greater than 1 is called a prime number if it has no positive divisors other than 1 and x itself. ………………………………3 分

2. $\sigma(n)=(\frac{p_1^{a_1+1}-1}{p_1-1})(\frac{p_2^{a_2+1}-1}{p_2-1})\cdots(\frac{p_l^{a_l+1}-1}{p_l-1})$ ………………………………3 分

3. Prime Number Theorem:

Let $\pi(n)$ be the number of primes smaller than or equal to n, then

$$\lim_{n\to\infty}\frac{\pi(n)}{n/\ln n}=1$$

………………………………4 分

二、应用题

4.

Sol:

Assume the values of two stamps are a,b,where a,b are two positive integers, and $a<b$.

Clearly, a,b are coprime, otherwise every integer which is not a multiple of $gcd(a,b)$ cannot be represented as a nonnegative integral combination of a and b. ………3 分

Since the Frobenius number $g(a,b)=ab-a-b$, every integer greater than $g(a,b)$ can be represented as a nonnegative integral combination of a and b, and for positive integers between 0 and $g(a,b)$, there are exactly $\frac{(a-1)(b-1)}{2}$ integers that cannot be represented as a nonnegative integral combination of a and b, we have

$\frac{(a-1)(b-1)}{2}=24$ ………………………………4 分

Factorize 24, we get all possible integer solutions of the above equality:

$a=2,b=49$, $a=3,b=25$, $a=4,b=17$, $a=5,b=13$, $a=7,b=9$ ………4 分

Since $ax+by=30$ has no nonnegative integer solutions, the only possible solution is

$a=4,b=17$ ………………………………3 分

Now, solve the Diophantine equation $4x+17y=48$, we find the general solutions are:

$x=12+17t, y=0-4t$, $t\in\mathbb{Z}$.（注：通解形式不唯一） ································2 分

Nonnegative integer solution requires $t=0$, so all nonnegative integral solution of

$4x+17y=48$ is $x=12, y=0$. ···1 分

Thus it is possible to combine the stamps to get a postage amount of 48fen, we can combine 12 stamps of 4fen to get the postage amount, and that is the only way to get the postage amount. ···3 分

三、计算题

5.

Sol:

a) 1 is neither a prime, nor a composite. ···1 分

b) $[\sqrt{137}]=11$, primes less than or equal to 11 are 2,3,5,7,11, all of them are not divisors of 137(Check!). ···3 分

Thus 137 is a prime. ···2 分

（直接写 137 没有除了自身和 1 以外的正因子的扣 2 分）

c) $100=2^2\times 5^2$ ···2 分

100 is a composite. ···2 分

6.

Sol:

All nonnegative integral solution of $666x+468y=38448$ are $x=50, y=11$ and

$x=24, y=48$. ·····················解方程的过程 6 分、答案 4 分（每个解 2 分）

7.

Sol:

a) $(2227)_8=1175=(979)_{11}$. ··················计算过程 3 分、答案 2 分

b) $(5316)_8=(101011001110)_2=(ACE)_{16}$. ························过程 3 分、答案 2 分

8.

Sol:

All nonnegative integral solution of the original linear system of Diophantine equations is $x=3, y=1, z=4$. ……………………解方程组的过程 7 分、答案 3 分

四、证明题

9.

Proof:

a) It suffices to prove the consequence when $0 \le x, y < 1$. ……………………………2 分

In this case, $[x]+[y]+[x+y]=[x+y]$, without loss of generality, we can assume $x \le y$,

then $[x+y] \le [2y] \le [2x]+[2y]$, the consequence holds. ……………………………5 分

b) It suffices to prove that: for every prime p,

$$ord_p((2m)!(2n)!) \ge ord_p((m)!(n)!((m+n)!))$$ ……………………………3 分

Now,

$$ord_p((2m)!(2n)!) = \sum_{r=1}^{\infty}([\frac{2m}{p^r}]+[\frac{2n}{p^r}])$$ ……………………………4 分

$$ord_p((m)!(n)!((m+n)!)) = \sum_{r=1}^{\infty}([\frac{m}{p^r}]+[\frac{n}{p^r}]+[\frac{m+n}{p^r}])$$ ……………………………4 分

By a), we have $[\frac{2m}{p^r}]+[\frac{2n}{p^r}] \ge [\frac{m}{p^r}]+[\frac{n}{p^r}]+[\frac{m+n}{p^r}]$ for each r. ………………………2 分

So $ord_p((2m)!(2n)!) \ge ord_p((m)!(n)!((m+n)!))$, we get the consequence.

10.

Proof:

Use Induction on n.

When $n=2$, we know that the Frobenius number of two coprime positive integers x,y exists and $g(x,y)=xy-x-y$. ……………………………1 分

Assume when $n \ge k (k \ge 2)$, the Frobenius number of k coprime positive integers exists.

……………………………2 分

Now $n=k+1$, we have $k+1$ coprime positive integers $a_1, \cdots, a_{k+1}$.

It suffices to prove that there exists an integer N, such that

$$a_1x_1 + a_2x_2 + \cdots + a_{k+1}x_{k+1} = c \qquad (1)$$

has non-negative integral solutions for all integers $c > N$ (then the exact lower-bound of these N is just the Frobenius number of $a_1, \cdots, a_{k+1}$). ······························3 分

Let $d_1 = (a_1, a_2)$, then $d_1, a_3, \cdots, a_{k+1}$ are k coprime integers, we know that the equation (1) equivalents to the following system of equations:

$$\begin{cases} a_1x_1 + a_2x_2 = d_1y_1 & (2) \\ d_1y_1 + a_3x_3 + \cdots + a_{k+1}x_{k+1} = c & (3) \end{cases}$$

By induction assumption, (3) always has non-negative integral solutions when $c \geq g(d_1, a_3, \cdots, a_{k+1})$.

When $c \geq g(d_1, a_3, \cdots, a_{k+1}) + d_1(g(\frac{a_1}{d_1}, \frac{a_2}{d_1}) + 1)$, we can further get a non-negative integral solution of (3) such that $y_1 \geq g(\frac{a_1}{d_1}, \frac{a_2}{d_1}) + 1$, which makes (2) has a non-negative integral solution simultaneously, this equivalents to (1) has non-negative integral solutions when $c \geq g(d_1, a_3, \cdots, a_{k+1}) + d_1(g(\frac{a_1}{d_1}, \frac{a_2}{d_1}) + 1)$.

Now, $g(a_1, a_2, \cdots, a_{k+1}) = inf\{N \in \mathbb{Z} : a_1x_1 + a_2x_2 + \cdots + a_{k+1}x_{k+1} = c$ has non-negative integral solutions for all integers $c > N\}$ ($g(a_1, a_2, \cdots, a_{k+1})$ exists since this set is nonempty and has a lower bound -1).

···4 分