第一章 向量与坐标

§1.10 三向量的双重向量积

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高等教育出版社
高等教育电子音像出版社

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教学重难点: 双重向量积几何关系的证明

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前面两个双重向量积的公式有共同的易于记忆的规律:

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$$\begin{aligned}\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= -(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}=(\boldsymbol{c}\times\boldsymbol{b})\times\boldsymbol{a}\\ &= (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c},\end{aligned}$$

所以 $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$.

比较以上两个公式可知, $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ 和 $(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$ 在一般情况下是两个不同的向量, 因此向量积不满足结合律.

前面两个双重向量积的公式有共同的易于记忆的规律: 三个向量的双重向量积等于中间的向量与其余向量的数量积的乘积减去括号中另一个向量与其余两向量的数量积的乘积.

利用双重向量积的公式可以证明拉格朗日(Lagrange)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}')=\begin{vmatrix}\boldsymbol{a}\cdot\boldsymbol{a}' & \boldsymbol{a}\cdot\boldsymbol{b}'\\ \boldsymbol{b}\cdot\boldsymbol{a}' & \boldsymbol{b}\cdot\boldsymbol{b}'\end{vmatrix}.$$

利用双重向量积的公式可以证明拉格朗日(Lagrange)恒等式

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a}' \times \boldsymbol{b}') = \begin{vmatrix} \boldsymbol{a} \cdot \boldsymbol{a}' & \boldsymbol{a} \cdot \boldsymbol{b}' \\ \boldsymbol{b} \cdot \boldsymbol{a}' & \boldsymbol{b} \cdot \boldsymbol{b}' \end{vmatrix}.$$

这是因为

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{a}' \times \boldsymbol{b}') = [(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a}'] \cdot \boldsymbol{b}'$$

利用双重向量积的公式可以证明拉格朗日(Lagrange)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}')=\begin{vmatrix}\boldsymbol{a}\cdot\boldsymbol{a}' & \boldsymbol{a}\cdot\boldsymbol{b}'\\ \boldsymbol{b}\cdot\boldsymbol{a}' & \boldsymbol{b}\cdot\boldsymbol{b}'\end{vmatrix}.$$

这是因为

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}') &= [(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}']\cdot\boldsymbol{b}'\\ &= [(\boldsymbol{a}\cdot\boldsymbol{a}')\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{a}')\boldsymbol{a}]\cdot\boldsymbol{b}'\end{aligned}$$

利用双重向量积的公式可以证明拉格朗日(Lagrange)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}')=\begin{vmatrix}\boldsymbol{a}\cdot\boldsymbol{a}' & \boldsymbol{a}\cdot\boldsymbol{b}'\\ \boldsymbol{b}\cdot\boldsymbol{a}' & \boldsymbol{b}\cdot\boldsymbol{b}'\end{vmatrix}.$$

这是因为

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}') &= [(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}']\cdot\boldsymbol{b}'\\ &= [(\boldsymbol{a}\cdot\boldsymbol{a}')\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{a}')\boldsymbol{a}]\cdot\boldsymbol{b}'\\ &= (\boldsymbol{a}\cdot\boldsymbol{a}')(\boldsymbol{b}\cdot\boldsymbol{b}')-(\boldsymbol{a}\cdot\boldsymbol{b}')(\boldsymbol{b}\cdot\boldsymbol{a}').\end{aligned}$$

利用双重向量积的公式可以证明拉格朗日(Lagrange)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}')=\begin{vmatrix}\boldsymbol{a}\cdot\boldsymbol{a}' & \boldsymbol{a}\cdot\boldsymbol{b}'\\ \boldsymbol{b}\cdot\boldsymbol{a}' & \boldsymbol{b}\cdot\boldsymbol{b}'\end{vmatrix}.$$

这是因为

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{a}'\times\boldsymbol{b}') &= [(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{a}']\cdot\boldsymbol{b}'\\ &= [(\boldsymbol{a}\cdot\boldsymbol{a}')\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{a}')\boldsymbol{a}]\cdot\boldsymbol{b}'\\ &= (\boldsymbol{a}\cdot\boldsymbol{a}')(\boldsymbol{b}\cdot\boldsymbol{b}')-(\boldsymbol{a}\cdot\boldsymbol{b}')(\boldsymbol{b}\cdot\boldsymbol{a}').\end{aligned}$$

拉格朗日恒等式的一个特殊情况是

$$(\boldsymbol{a}\times\boldsymbol{b})^2=\boldsymbol{a}^2\boldsymbol{b}^2-(\boldsymbol{a}\cdot\boldsymbol{b})^2.$$

例 1

试证明雅可比(Jacobi)恒等式

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} + (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} + (\boldsymbol{c} \times \boldsymbol{a}) \times \boldsymbol{b} = \mathbf{0}.$$

例 1

试证明雅可比(Jacobi)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{0}.$$

证 因为

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},$$

例 1

试证明雅可比(Jacobi)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{0}.$$

证 因为

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},$$

$$(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}=(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$$

例 1

试证明雅可比(Jacobi)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{0}.$$

证 因为

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},$$

$$(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}=(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$$

$$(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c},$$

例 1

试证明雅可比(Jacobi)恒等式

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}+(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}+(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=\boldsymbol{0}.$$

证 因为

$$(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}=(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a},$$

$$(\boldsymbol{b}\times\boldsymbol{c})\times\boldsymbol{a}=(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}-(\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b}$$

$$(\boldsymbol{c}\times\boldsymbol{a})\times\boldsymbol{b}=(\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}-(\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c},$$

三式相加即得结论.

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}' \\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}' \\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式.

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}' - (\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}' \\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b} - (\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') = \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}')$$

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}')\\ &= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}'\end{aligned}$$

例 2

证明

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}' - (\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}' \\
&= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b} - (\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.
\end{aligned}
$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}') \\
&= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}' - (\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}' \\
&= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}']\boldsymbol{a}' - [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}']\boldsymbol{b}'
\end{aligned}
$$

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}')\\ &= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}'\\ &= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}']\boldsymbol{a}'-[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}']\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'.\end{aligned}$$

例 2

证明

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.\end{aligned}$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}')\\ &= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}'\\ &= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}']\boldsymbol{a}'-[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}']\boldsymbol{b}'\\ &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'.\end{aligned}$$

(2) 第二个等式.

例 2

证明

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}' \\
&= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.
\end{aligned}
$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}') \\
&= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}' \\
&= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}']\boldsymbol{a}'-[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}']\boldsymbol{b}' \\
&= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'.
\end{aligned}
$$

(2) 第二个等式. 因为 $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}')=-(\boldsymbol{a}'\times\boldsymbol{b}')\times(\boldsymbol{a}\times\boldsymbol{b})$, 再由(1)即得结论.

例 2

证明

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'\\
&= (\boldsymbol{a}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{b}-(\boldsymbol{b}\boldsymbol{a}'\boldsymbol{b}')\boldsymbol{a}.
\end{aligned}
$$

证 (1) 第一个等式. 设 $\boldsymbol{a}\times\boldsymbol{b}=\boldsymbol{e}$, 得

$$
\begin{aligned}
(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}') &= \boldsymbol{e}\times(\boldsymbol{a}'\times\boldsymbol{b}')\\
&= (\boldsymbol{e}\cdot\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{e}\cdot\boldsymbol{a}')\boldsymbol{b}'\\
&= [(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{b}']\boldsymbol{a}'-[(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{a}']\boldsymbol{b}'\\
&= (\boldsymbol{a}\boldsymbol{b}\boldsymbol{b}')\boldsymbol{a}'-(\boldsymbol{a}\boldsymbol{b}\boldsymbol{a}')\boldsymbol{b}'.
\end{aligned}
$$

(2) 第二个等式. 因为 $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}'\times\boldsymbol{b}')=-(\boldsymbol{a}'\times\boldsymbol{b}')\times(\boldsymbol{a}\times\boldsymbol{b})$, 再由(1)即得结论.

习题课