Application of the Pontryagin's Minimum Principle

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Abstract

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Keywords Optimization · Control System · Pontryagin's Minimum Principle

1 Background Information

1.1 Finite-dimensional Optimization

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$, and $\|.\|$ is an Euclidean norm on $\mathbb{R}^n$. A point x^* is a **local minimum** of function f in it's definition domain, denoted as D_f, if there exists an constant $\delta > 0$ such that,

$$\forall x \in D, \quad |x - x^*| < \delta, \quad f(x^*) \leq f(x) \tag{1}$$

1.1.1 Unconstrained Optimization

We have the sufficient condition for optimality: if f is a twice continously differentiable function, and on the point $x^* \in D_f$ we have

$$\begin{cases} \nabla f(x^*) & = 0 \\ \nabla^2 f(x^*) = 0 & \text{(positively definite)} \end{cases} \tag{2}$$

Therefore, x^* is a local minimum of function f. Detailed demonstration of this theorem can be found in any *Optimization* course.

1.1.2 Constrained Optimization

Now we add constraints to D_f, which is defined by the equality constraints:

$$\forall x \in D_f, \quad h_1(x) = ... = h_m(x) = 0 \tag{3}$$

Suppose that $x^* \in D_f$ is a local minimum of f and a regular point of D. From the definition, the gradients at x^* ($\nabla h_{i \in [1,m]}$) are linearly independent.

Therefore,

$$\nabla f(x^*) \in \text{span}\{\nabla h_i(x^*), \quad i = 1, ..., m\} \tag{4}$$

It is equivlant to say that

$$\exists (\lambda_1^*, ..., \lambda_m^*) \in \mathbb{R}^m, \quad \nabla f(x^*) + \sum_{j=1}^{m} \lambda_j \nabla h_j(x^*) = 0 \tag{5}$$

We name $\lambda = (\lambda_1 \ \cdots \ \lambda_m)^T$ **Lagrange multipliers**.

The above equation can be decomposed into:

$$\forall i \in [1, m], \quad \left(\frac{\partial f(x)}{\partial x_i} + \sum_{j=1}^{m} \frac{\partial h_j(x)}{\partial x_i} \right) |_{x=x^*} = 0 \tag{6}$$

Now, consider the **augmented cost function** $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, defined by :

$$F(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) \tag{7}$$

If (x^*, λ^*) resp. the local constrained minimum and the coressponding vector of Lagrange multipliers, then the gradient of F at (x^*, λ^*) satisfies

$$\nabla F(x,\lambda)|_{x=x^*,\lambda=\lambda^*} = \begin{pmatrix} F_x(x,\lambda)|_{x=x^*,\lambda=\lambda^*} \\ F_\lambda(x,\lambda)|_{x=x^*,\lambda=\lambda^*} \end{pmatrix} = \begin{pmatrix} \nabla f(x^*) + \sum_{j=1}^m \lambda_j^* \nabla h_{j(x^*)} \\ h(x^*) \end{pmatrix} = 0 \tag{8}$$

The above condition in terms of Lagrange multipliers is necessary but not sufficient for constrained optimality.

1.2 Local minima of a functional

First we define a **functional**, which could be thought as a "function of a function". If there exist a one-to-one mapping between a variable quantity J and a function $y(t)$, then J is a functional denoted as $J[y(t)]$.

For example, in Section 2.1 we have

$$J = \int_{t_0}^{t_f} L(t,x,u)\ \mathrm{d}t \tag{9}$$

Here J is a functional of L denoted as $J[L(t,x,u)]$.

In this part, we are going to formally define **local minima of a functional**. Suppose V a function space with norm $\|.\|_p$. A *function* (not a variable !!) is a local minimum of J over a subset of V denoted as A, which is a real-valued functional defined on V, if

$$\exists \varepsilon > 0, \quad \forall y \in A, \quad \|y - y^*\|_p \le \varepsilon \quad \Longrightarrow \quad J(y^*) \le J(y) \tag{10}$$

1.3 Controled Dynamic

Consider an *ordinary differential equation (ODE)* as follows

$$\begin{cases} \dot{x}(t) = f(x(t)) \quad t > 0 \\ x(0) = x_0 \end{cases} \tag{11}$$

The unknown is the "curve" of the dynamical evolution of a certain "system" $x : \left[0, t_f\right] \to \mathbb{R}^n$, where t_f is the end of the time length.

Now, the function f also depends upon some control parameters belonging to a set $A \subset \mathbb{R}^m$, named as **collection of all admissible controls**. If these control parameters are time-variant, we consider the ODE

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \quad t > 0 \\ x(0) = x_0 \end{cases} \tag{12}$$

1.4 Euler-Lagrange Equation

2 Optimal Control

2.1 Optimal Control Problem

2.1.1 Formulation

Based on the ODE Equation on Section 1.3:

$$\dot{x}(t) = f(x, u, t), \quad x(t_0) = x_0 \tag{13}$$

We need another ingredient for the *Optimal Control Problem*: the **cost functional**, denoted as $J(u)$, which assigns a cost value to each admissible control. It can be written as:

$$J(u) \overset{\text{def}}{=} \int_{t_0}^{t_f} L(t, x(t), u(t)) \, \mathrm{d}t + \varphi(t_f, x_f(t_f)) \tag{14}$$

where the first term is the running cost, or the **Lagrangian**, depending on the time, and the second term is ther terminal cost, only taking account of the final state.

For example, if we want to minimize the time-consume of a certain system, like a marathon race, the cost functional could be expressed as

$$J(u) = \int_{t_0}^{t_f} 1 \, \mathrm{d}t \tag{15}$$

To formulate the optimal control problem, we follow the steps:
1. Establish the state equation $\dot{x} = f(x(t), u(t), t)$
2. Clarify the boundary conditions $x(t_0) = x_0, x(t_f) \in S$
3. Defining the performance index J
4. Confirming the admissible range $u(t) \in \Omega$, then add it to the state equations if needed (will be introduced in the later sections)

2.1.2 Analysis

In this section and the following section, we suppose that the control variable are free and is not limited by any equality or inequality constraint.

Our aim is to minimize the cost functional under the system expressed as

$$\dot{x}(t) = f(x(t), u(t), t) \quad \Longleftrightarrow \quad f(x(t), u(t), t) - \dot{x}(t) = 0 \tag{16}$$

It could be conceptualized as minimizing J under the constraint of the equation above. In this case, we can use the method introduced in Section 1.1.2.

Suppose that x is an n-dimensional state vector, in accordance, we introduce an n-dimensional *Lagrange vector* $\lambda(t) = (\lambda_1(t) \; \dots \; \lambda_n(t))^T$ to obtain the *augmented functional* J_{aug}:

$$J_{\text{aug}} = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} (L(x(t), u(t), t) + \lambda^T(t)[f(x(t), u(t), t) - \dot{x}(t)]) \, \mathrm{d}t \tag{17}$$

By introducing an auxiliary function which is also called **Hamilton function** defined as

$$H(x, u, \lambda, t) \overset{\text{def}}{=} L(x, u, t) + \lambda^T f(x, u, t) \tag{18}$$

The above formula could be written as

$$J_{\text{aug}} = \varphi\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} \left(H(x, u, \lambda, t) - \lambda^T \dot{x}\right) \,\mathrm{d}t \tag{19}$$

From the **Euler-Lagrange Equation** of the functional J_{aug}, deduced from Section 1.4, we can obtain the necessary conditions for the unconstraint optimal problem:

$$\begin{aligned} \lambda(t_f) &= \frac{\partial \varphi}{\partial x(t_f)} \\ \dot{\lambda} &= -\frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial u} &= 0 \end{aligned} \tag{20}$$

Meanwhile, the system state equation could be written a more compact form:

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{21}$$

2.1.3 Solution

The above analysis can be concluded as the **theorem**: If the control system

$$\dot{x}(t) = f(x, u, t), \quad x(t_0) = x_0 \tag{22}$$

can make the performance index functional

$$J = \varphi\big(x(t_f), t_f\big) + \int_{t_0}^{t_f} L(x(t), u(t), t) \,\mathrm{d}t \tag{23}$$

at a fixed-end-time t_f and a *free*-end-state $x\big(t_f\big)$. Then the following conditions and equations should hold simultaneously:

- The **boundary conditions**

$$\lambda(t_f) = \frac{\partial \varphi}{\partial x(t_f)}, \quad x(t_0) = x_0 \tag{24}$$

- The **control equation**

$$\frac{\partial H}{\partial u}\Big|_{u=u^*} = 0 \tag{25}$$

- The **canonical equations**

$$\dot{\lambda} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial \lambda} \tag{26}$$

Therefore, the procedure to solve the constraint functional extrema is divided into three steps:

1. From J, derive L and φ.
2. Give *Hamilton function*

$$H = L + \lambda^T f \tag{27}$$

3. Get the expression of the optimal control $u^* = u^*(x, \lambda)$ from the **control equation**
4. Substitute u^* into the **canonical equations**, use the **boundary conditions** to obtain the optimal state trajectory x^* and adjoint state trajectory λ^*
5. Substitute x^* and λ^* into u^* to obtain the optimal control.

3 The Pontryagin's aMaximum/Minimum Principle

4 Problem Formulation

4.1 State and Control

In this study, we analyze the dynamics of an electric vehicle (EV) navigating a real-world environment. Let $s(t) \in \mathbb{R}$ represent the cumulative distance traveled by the vehicle at time t, and $v(t) \in \mathbb{R}$ denote its velocity. Furthermore, the vehicle's total energy at any given moment is expressed as $w(t) \in \mathbb{R}$.

The operation of the EV is subject to several physical constraints:

- The speed of the vehicle $v(t)$ could not exceed a certain amount denoted as $v_{\max}$,
- The vehicle's total energy is bounded by an upper limit, thus constrained to the condition that $|w(t)| \leq E_{\max}$,
- The output power of the vhicle is also limited, thus, $|p(t)| \leq P_{\max}$

The control set, denoted as $\mathcal{U}$ is a compact and convex subset of $\mathbb{R}^3$.

Let's define the state and the control

$$\begin{aligned} \boldsymbol{x}(t) &= (s(t) \ \ v(t) \ \ w(t))^T \in \mathbb{R}^3 \\ \boldsymbol{u}(t) &= (v(t) \ \ w(t) \ \ p(t))^T \in \mathcal{U} \end{aligned} \tag{28}$$

4.2 Physical Analysis

Physical model implies the following consequences:

1) **The relation between the distance and the speed**, where t_0 and t_f are respectively the beginning time and the ending time:

$$v(t) = \frac{\mathrm{d}s(t)}{\mathrm{d}t}, \quad s(t) = s_0 + \int_{t_0}^{t_f} v(t) \ \mathrm{d}t \tag{29}$$

2) **The relation between the accerleration and the power also the vitesse of the vehicle**, where the first term derives from the motor force, the second term derives from the friction, θ denotes the inclination of the road and μ is the friction coefficient. The third force is the force of air resistence, ρ is the air density, C_d is the air resistence coefficient and A is the surface of the vehicle.

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{p(t)}{v(t)} - mg(\sin\theta + \mu\cos\theta) - \frac{1}{2}\rho C_d A v^2(t) \tag{30}$$

3) **The relation between the total energy (left) of the vehicle and its power output, in the meantime take in account of its power recupation.**

4.3 Dynamic of the System

For a admissible control $u \in \mathcal{U}$, the dynamics of the system is described by the following equation:

$$\forall t \geq 0, \quad \dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{31}$$