Digital Signal Processing

Chapter 2. Fourier analysis

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Study Points

- Fourier Series
- Fourier Transform
- Discrete-Time Fourier Transform (DTFT) 2h

- Window Effects 2h
- Discrete Fourier Transform and FFT 2h
- Power spectral density 2h
- Lab 2 (20%)

2.0 Fourier analysis in digitals

Through Fourier series (FS) and Fourier transform (FT), spectral representations can be obtained for a given periodic or nonperiodic continuous or discrete time signal in terms of a frequency spectrum, which is composed of the amplitude and phase spectrums.

The discrete-time Fourier transform (DTFT) is applied to time-domain samplings and therefore obtains *continuous* frequency spectrum. Although the discrete Fourier transform is also applied to time-domain samplings, the frequency spectrum is *discrete*.

The DTF and its fast implementation, the fast Fourier transform (FFT), have three major uses in DSP:

(a) numerical computation of the frequency spectrum;
(b) efficient implementation of convolution by the FFT;
(c) coding of waveforms.

2.1 Fourier series

A periodic continuous-time signal $x(t)$ that satisfies ***Dirichlet conditions*** can be represented by the Fourier series.

The Fourier series can be expressed in terms of a sum of sines and cosines, or just sines or just cosines with clear physical meaning. The form is given by

$$x(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left(a_k \cos k\omega_0 t + b_k \sin k\omega_0 t\right)$$

$$= \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k \cos(k\omega_0 t + \phi_k)$$

where

$$a_0 = \frac{2}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)\,dt$$

τ_0 a fundamental period

$$a_k = \frac{2}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)\cos k\omega_0 t\,dt$$

$$b_k = \frac{2}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t)\sin k\omega_0 t\,dt$$

$$c_k = \sqrt{a_k^2 + b_k^2} \qquad \phi_k = \arctan(b_k / a_k)$$

2.1 Fourier series

The most general form (exponential Fourier series) of this important representation is given by

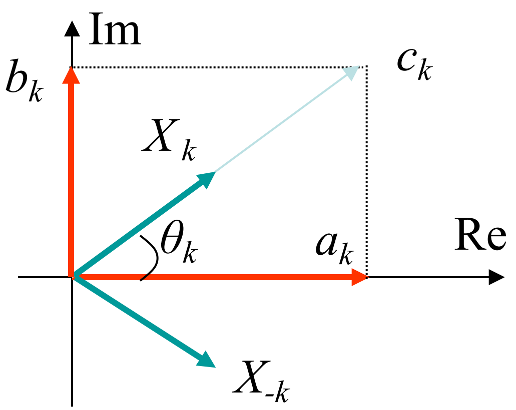
$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t} \quad for \quad -\tau_0/2 \le t \le \tau_0/2$$

where

$$\omega_0 = 2\pi/\tau_0$$

$$X_k = \frac{1}{\tau_0}\int_{-\tau_0/2}^{\tau_0/2} x(t) e^{-jk\omega_0 t} dt$$

$$|X_k| = |X_{-k}|$$



$$a_0 = 2X_0$$

$$a_k = X_k + X_{-k}$$

$$b_k = j(X_k - X_{-k})$$

,

2.1 Fourier series

Parseval's Formula for Periodic Signals

The mean of the product $x(t)x^*(t)$,

where $x^*(t)$ is the complex conjugate of $\tilde{x}(t)$

can be expressed in terms of the Fourier-series coefficients

$$\overline{x(t)x^*(t)} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)x^*(t)\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \left|x(t)\right|^2 dt$$

$$= \sum_{k=-\infty}^{\infty} X_k X_k^* = \sum_{k=-\infty}^{\infty} \left|X_k\right|^2$$

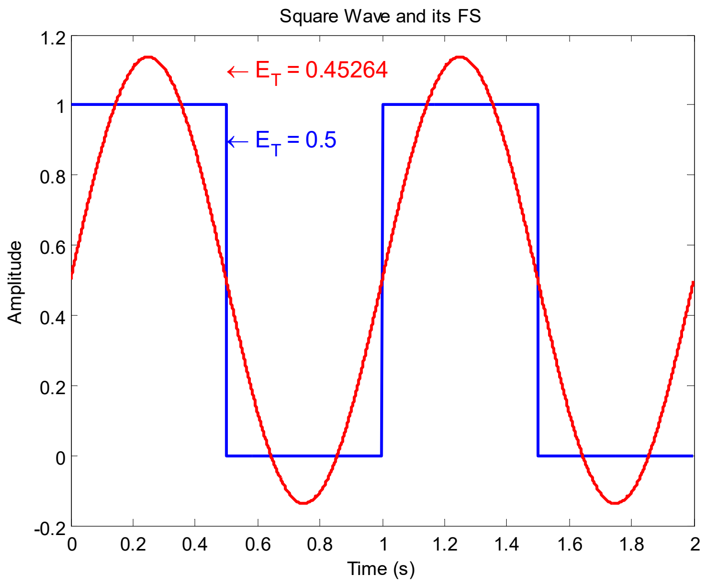
Parseval's theorem provides a formula that can be used to calculate the average power by using the Fourier-series coefficients.

2.1 Fourier series - Simulation

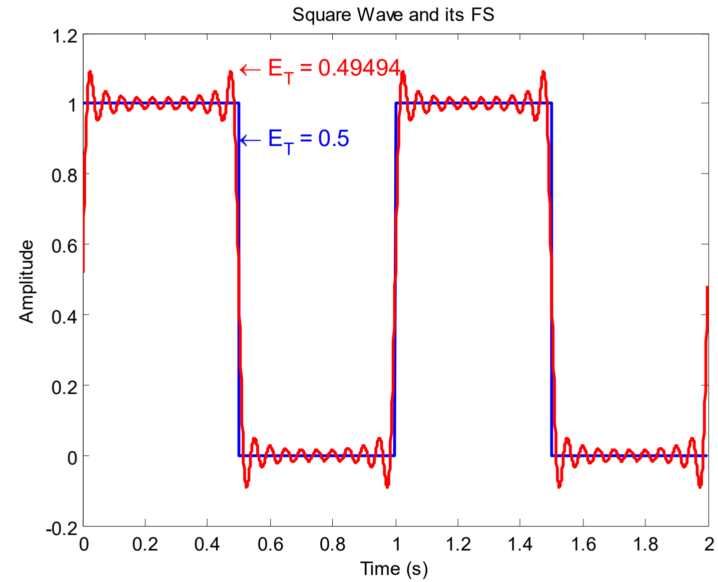
```
clear,clc          % close all
fs = 1000;         % sample frequency
T = 1;             % cycle
n = 2;             % number of cycles
t = 0:1/fs:T-1/fs;          % one cycle of signal
tt = 0:1/fs:n*T-1/fs;       % Time of signal
Amp = 1;           % amplitude of square wave
Duty = 0.5;        % duty cycle
% Signal
xp = zeros(1,length(t));
xp(1:T*Duty*fs) = Amp;
x = zeros(1,length(tt));
for jj = 1:n
    aa = (jj-1)*T*fs;
    x(aa+1:aa+T*fs) = xp;
end
E1 = 1/T*sum(abs(xp).^2)/fs;      % average power
str1 = num2str(E1);
% FS
m = 20;          % highest order of Fourier series
X = 0;           % sum of series
E2 = 0;          % power in frequency domain
for jj = -m:m
    an = 1/T*sum(xp.*exp(-1j*jj*2*pi/T*t))/fs;
    X = X+an*exp(1j*jj*2*pi/T*tt);
    E2 = E2+abs(an)^2;
end
str2 = num2str(E2);
```

```
figure
fig1 = plot(tt,x,'b',tt,real(X),'r-');
set(fig1,'linewidth',2)
xlabel('Time (s)')
ylabel('Amplitude')
title('Square Wave and its FS')
text(Duty*T,0.9*Amp,['\fontsize{12}\color{blue}\leftarrow
E_T = ' str1])
text(Duty*T,1.1*Amp,['\fontsize{12}\color{red}\leftarrow
E_T = ' str2])
```

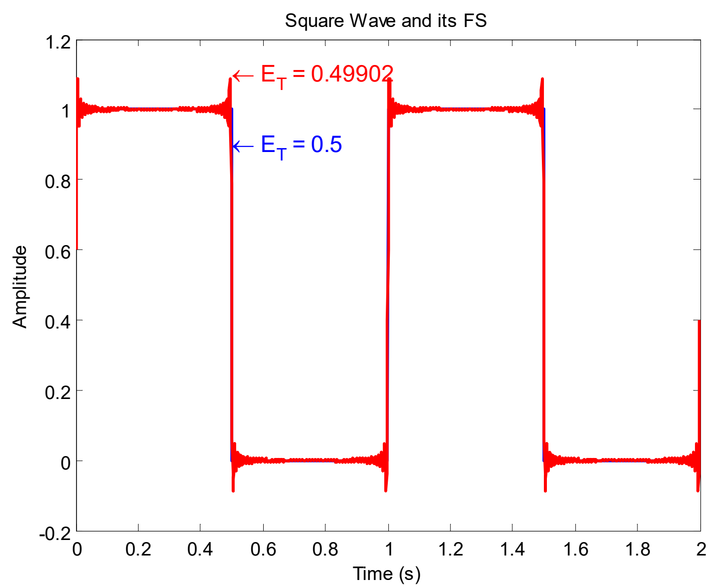
2.1 Fourier series - Simulation



1 order of FS

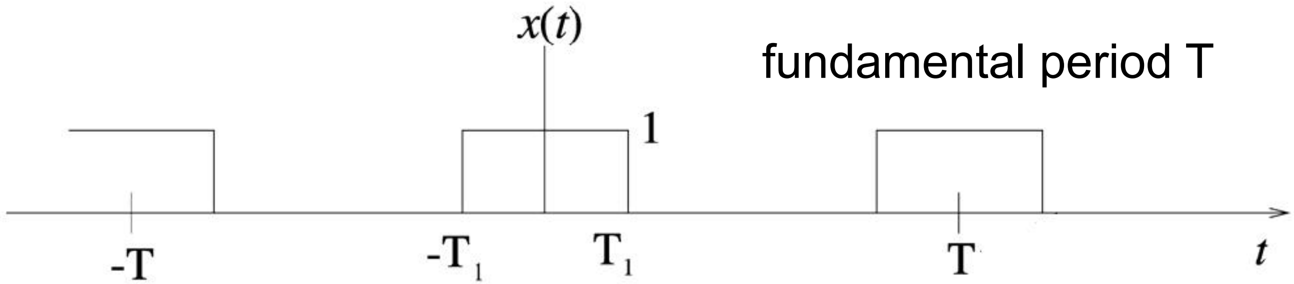


20 orders of FS



100 order of FS

2.2 Fourier transform

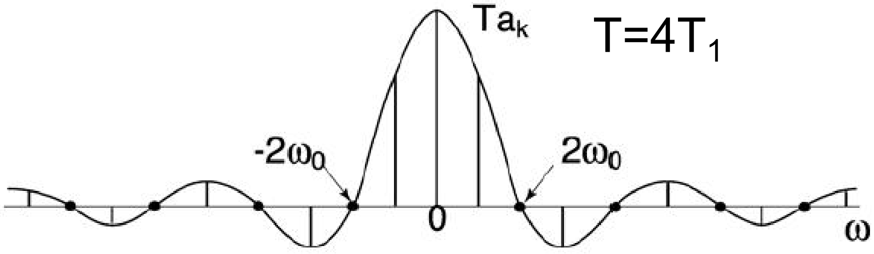


fundamental period T

FS coefficients $a_k = \dfrac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$

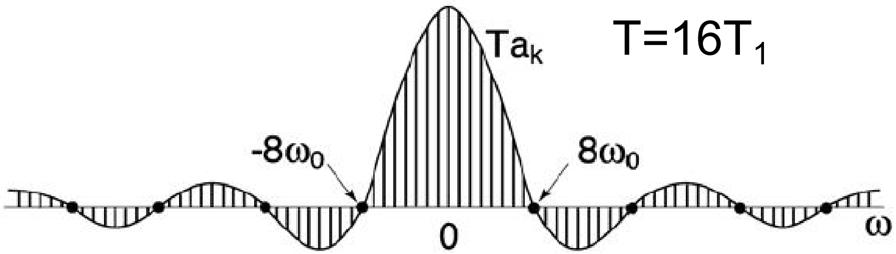
$$\boxed{a_k T = \frac{2\sin(k\omega_0 T_1)}{k\omega_0},\ \omega = k\omega_0}$$

T_1 is kept fixed and T is increased.



$T=4T_1$

$$a_k T = \frac{2\sin(2\pi k/4)}{k\omega_0}$$

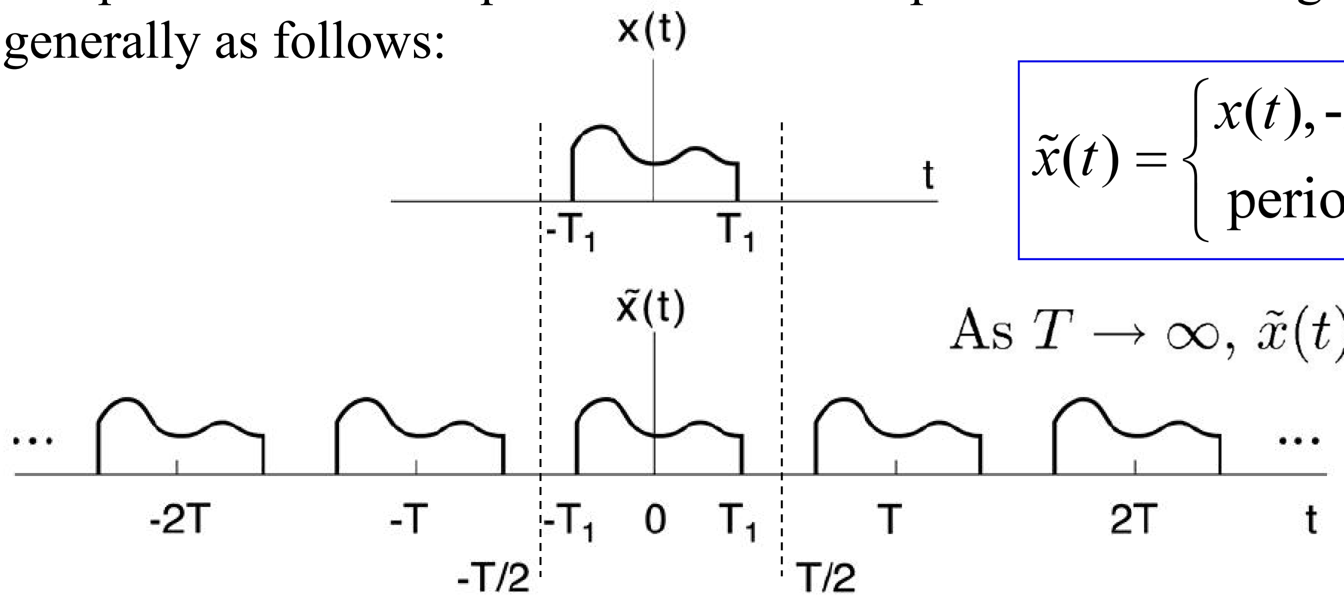


$T=16T_1$

$$a_k T = \frac{2\sin(2\pi k/16)}{k\omega_0}$$

2.2 Fourier transform

The phenomenon of square wave with the period T increasing can be represented generally as follows:



$$\tilde{x}(t) = \begin{cases} x(t), -T/2 < t < T/2 \\ \text{periodic}, \quad |t| > T/2 \end{cases}$$

As $T \to \infty$, $\tilde{x}(t) = x(t)$ for all t

For a periodic signal:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = 2\pi / T$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} \mathrm{d}t = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} \mathrm{d}t$$

2.2 Fourier transform

Define Fourier coefficient $X(jk\omega_0) = Ta_k$

$$Ta_k = \int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,\mathrm{d}t$$

$T \to \infty \;\Rightarrow\; k\omega_0 \to \omega$. Therefore, $X(j\omega) = Ta_k = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t$

$$\boxed{\text{Analysis FT equation: } X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,\mathrm{d}t}$$

As we know, for $-T/2 < t < T/2$, there is

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} Ta_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$T \to \infty \;\Rightarrow\; \omega_0 \to \mathrm{d}\omega, k\omega_0 \to \omega.$$

$$\boxed{\text{Synthesis FT equation: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,\mathrm{d}\omega}$$

2.2 Fourier transform

The signals should satisfy

1. From engineering view, just the finite energy $\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t < \infty$

2. More strickly to say, *Dirichlet* conditions

Condition 1. $x(t)$ is absolutely integrable over one period, i.e.

$$\int_T |x(t)| dt < \infty$$

Condition 2. In a finite time interval, $x(t)$ has a finite number of maxima and minima.

Condition 3. In a finite time interval, $x(t)$ has only a finite number of discontinuities.

2.2 Fourier transform

The Fourier transform of a nonperiodic continuous-time signal $x(t)$ is

$$X(j\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt$$

Assume that signal $x(t)$ is real, then

$$\mathrm{Re}\,X(j\omega)=\int_{-\infty}^{\infty}x(t)\cos\omega t dt$$

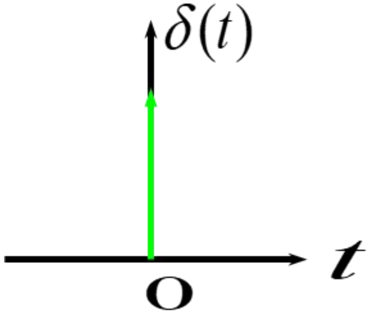
$$\mathrm{Im}\,X(j\omega)=-\int_{-\infty}^{\infty}x(t)\sin\omega t dt$$

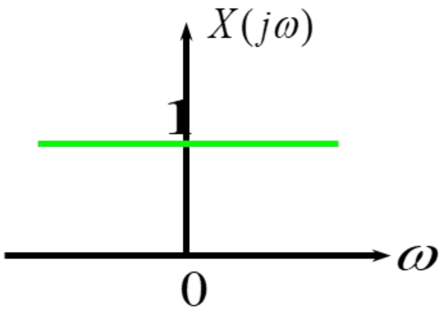
The inverse Fourier transform of $X(j\omega)$ is $x(t)=\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}\frac{d\omega}{2\pi}$

Impulse function

$$X(j\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=1$$

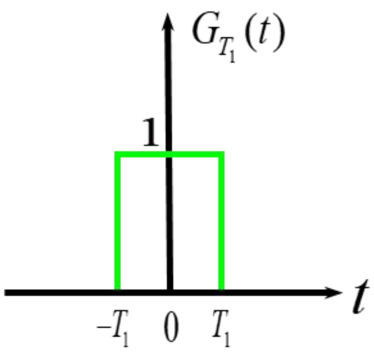
$$\therefore\ \delta(t)\Leftrightarrow 1 \qquad \delta(t-t_0)\Leftrightarrow e^{-j\omega t_0}$$

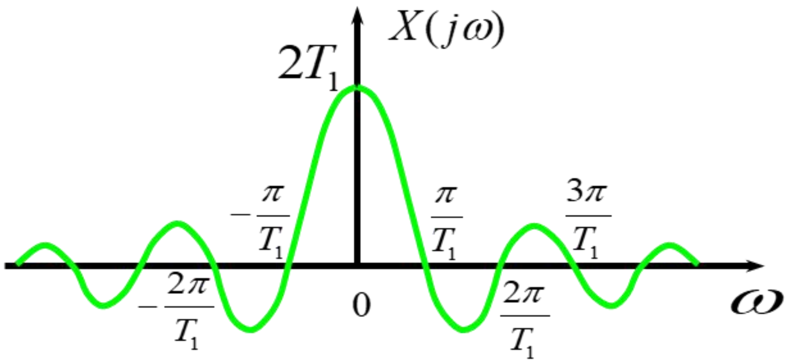




Gate function

$$X(j\omega)=\int_{-T_1}^{T_1}e^{-j\omega t}dt=\frac{2\sin(\omega T_1)}{\omega}=2T_1\,Sa(\omega T_1)$$





2.2 Fourier transform

By introducing impulses, FT can represent periodic signals which have infinite energy.

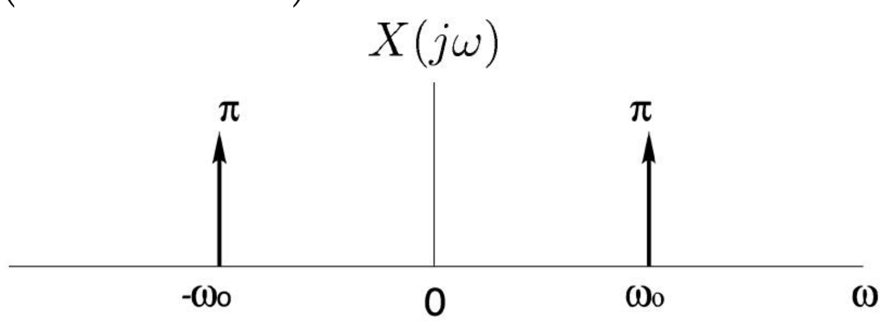
For $X(j\omega)=\delta(\omega-\omega_0)$, there is $x(t)=\dfrac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}\mathrm{d}\omega=\dfrac{1}{2\pi}e^{j\omega_0 t}$

So for a periodic signal $e^{j\omega_0 t}\Leftrightarrow 2\pi\delta(\omega-\omega_0)$

Generally, periodic signals

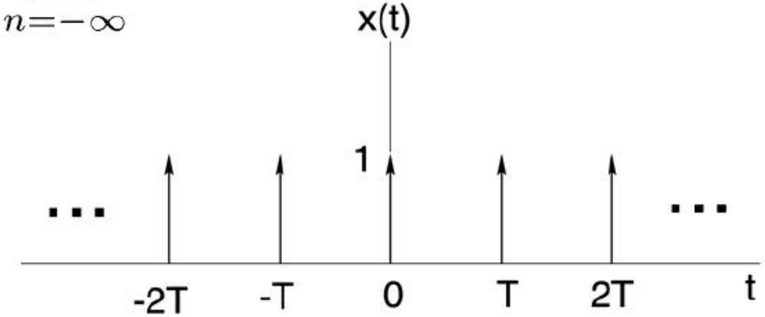
$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}\Leftrightarrow X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0)$$

$$\cos(\omega_0 t)=\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)/2\Leftrightarrow \pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$



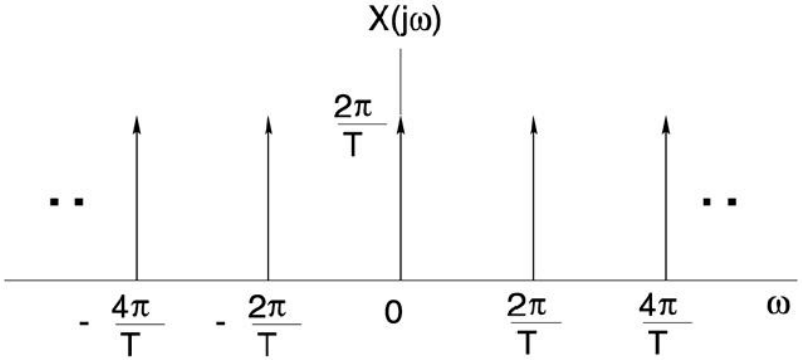
2.2 Fourier transform

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \text{ — Sampling function}$$



FS $\quad x(t) \leftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\delta(\omega - k\frac{2\pi}{T})$$



2.2 Fourier transform

1) Linearity $ax(t)+by(t) \leftrightarrow aX(j\omega)+bY(j\omega)$

2) Time shifting $x(t-t_0) \leftrightarrow e^{-j\omega t_0}X(j\omega)$

Unchanged magnitude

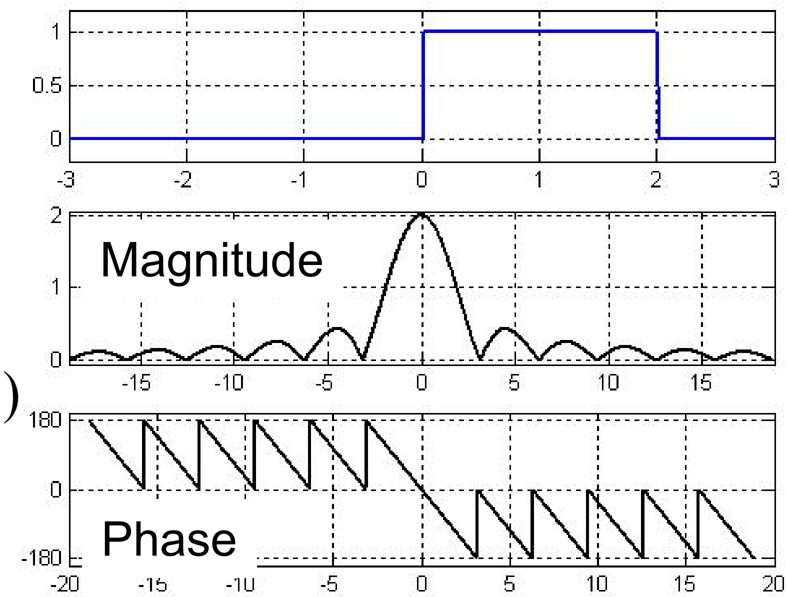
$$\left|e^{-j\omega t_0}X(j\omega)\right| = \left|e^{-j\omega t_0}\right|\left|X(j\omega)\right| = \left|X(j\omega)\right|$$

Linear change in phase

$$\arg\left(e^{-j\omega t_0}X(j\omega)\right) = \arg\left(X(j\omega)\right) + \arg(e^{-j\omega t_0})$$

$$= \arg\left(X(j\omega)\right) - \omega t_0$$

A shifted gate function

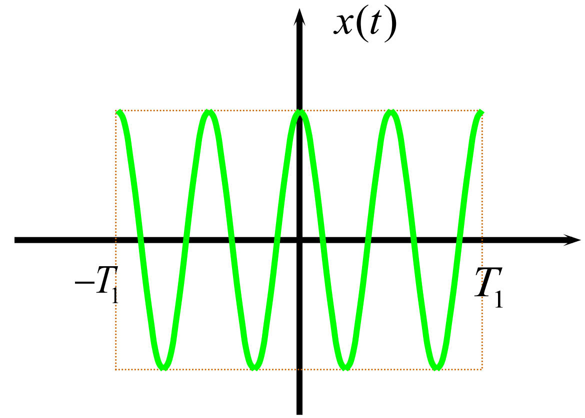


2.2 Fourier transform

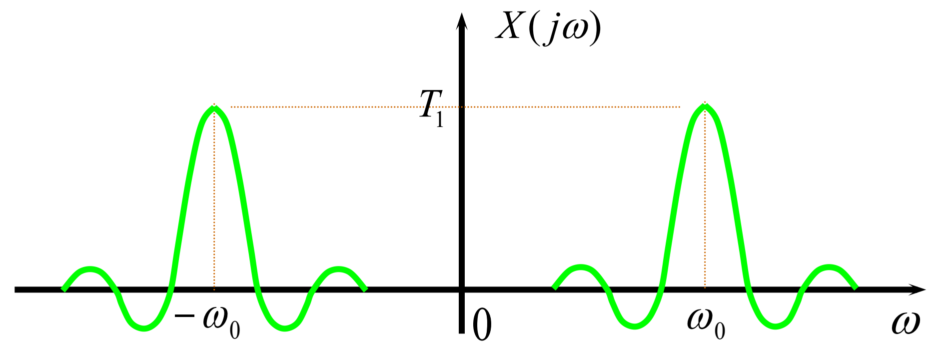
3) Frequency shifting $x(t)e^{\pm j\omega_0 t} \Leftrightarrow X[j(\omega \mp \omega_0)]$

Modulation

$$G_{T_1}(t)\cos(\omega_0 t) \Leftrightarrow \frac{1}{2}X[j(\omega \mp \omega_0)]$$



$$\cos(\omega_0 t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$



2.2 Fourier transform

4) Conjugation and conjugate symmetry $x^*(t) \leftrightarrow X^*(-j\omega)$

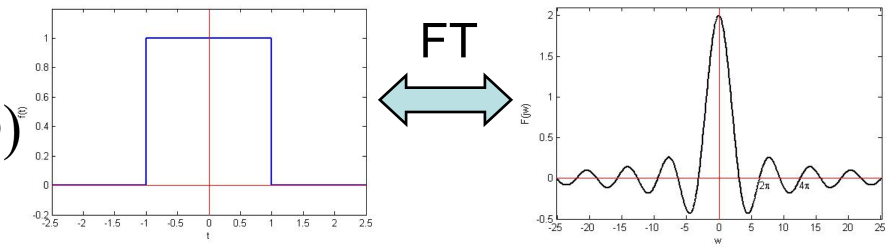
For real signals: $X(j\omega) = X^*(-j\omega)$

$\lvert X(-j\omega)\rvert = \lvert X(j\omega)\rvert$	*Even*
$\angle X(-j\omega) = -\angle X(j\omega)$	*Odd*
$Re\{X(-j\omega)\} = Re\{X(j\omega)\}$	*Even*
$Im\{X(-j\omega)\} = -Im\{X(j\omega)\}$	*Odd*

5) Differentiation and Integration

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leftrightarrow j\omega X(j\omega)$$

$$\int_{-\infty}^{t} x(\tau)\mathrm{d}\tau \leftrightarrow \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$



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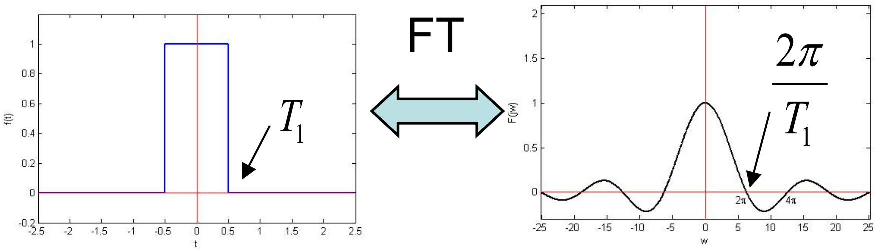


6) Time and frequency Scaling

$$x(at) \leftrightarrow \frac{1}{|a|}X(\frac{j\omega}{a})$$

A signal is compressed in one field and it is extended in another field.

$$\text{Let } a = -1,\ x(-t) \leftrightarrow X(-j\omega)$$

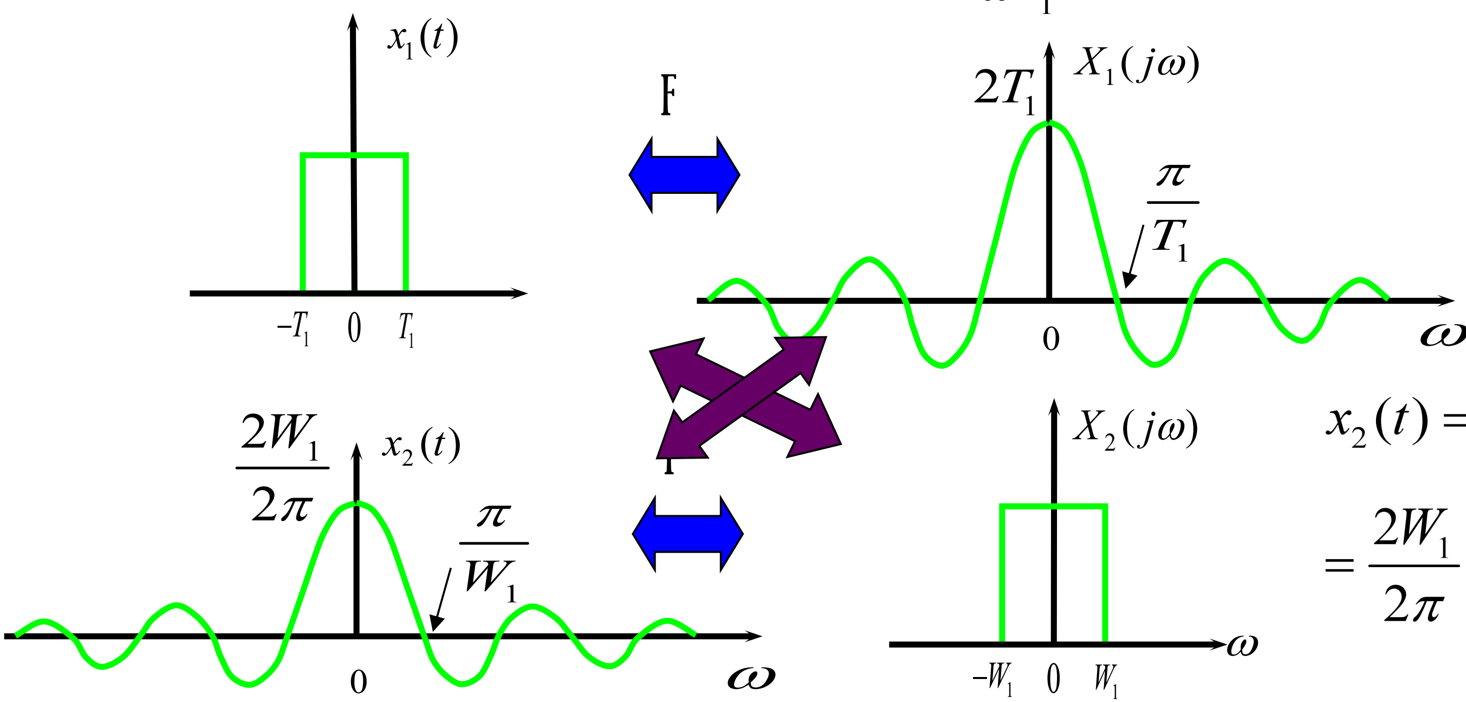
Reversing a signal in time also reverses its Fourier transform

2.2 Fourier transform

7) Duality $X(j\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\mathrm{d}t,\quad x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\mathrm{d}\omega$

The Fourier transform and inverse transform relations are similar but not quite identical in form. This leads to the *duality* of the Fourier transform.

$$X_1(j\omega)=\int_{-T_1}^{T_1} e^{-j\omega t}dt=\frac{2T_1\sin(\omega T_1)}{\omega T_1}$$



$$x_2(t)=\frac{1}{2\pi}\int_{-W_1}^{W_1} e^{j\omega t}d\omega$$

$$=\frac{2W_1}{2\pi}\frac{\sin(tW_1)}{tW_1}$$

8) Parseval's relation Proof can be found in Textbook

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \mathrm{d}\omega = \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f$$

total energy in the time domain

total energy in the frequency domain

$|X(j\omega)|^2$ is often referred to as the energy-density spectrum, with units of energy per Hz

Compare the FT Parseval's relation with that for periodic signal:

$$\frac{1}{T} \int_T |x(t)|^2 \, \mathrm{d}t = \sum_k |a_k|^2$$

average power in the time domain

average power of all harmonic components

9) Convolution property   Proof can be found in Textbook

$$y(t) = h(t) * x(t) \xleftrightarrow{F} Y(j\omega) = H(j\omega)X(j\omega)$$

$H(j\omega)$ is the Fourier transform of the impulse response $h(t)$, referred as the frequency response of the system.

Since $h(t)$ completely characterizes an LTI system, $H(j\omega)$ plays important roles in system analysis in the frequency domain.

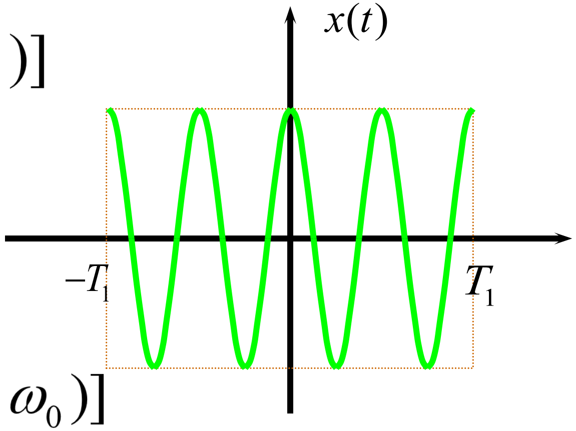
In using Fourier analsysis to study LTI systems, we restrict the systems whose impulse response possess Fourier transforms, i.e. $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$, in other words, the stable LTI system can be analyzed by FT.

10) Multiplication property

$$y(t)=h(t)x(t)\Leftrightarrow Y(f)=H(f)*X(f),Y(j\omega)=\frac{1}{2\pi}H(j\omega)*X(j\omega)$$

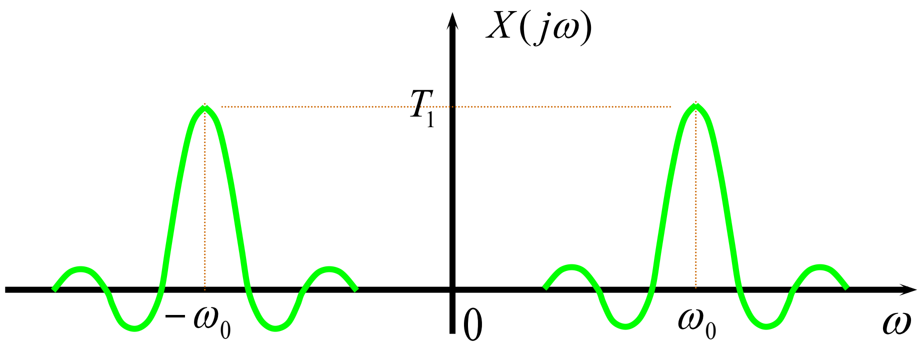
Explain Frequency shifting with multiplication property

Frequency shifting $x(t)e^{\pm j\omega_0 t}\Leftrightarrow X[j(\omega\mp\omega_0)]$



Modulation

$$G_{T_1}(t)\cos(\omega_0 t)\Leftrightarrow\frac{1}{2}X[j(\omega-\omega_0)]+\frac{1}{2}X[j(\omega+\omega_0)]$$



Properties of Fourier transform

Time convolution

$$x_1(t) \otimes x_2(t) \leftrightarrow X_1(j\omega) X_2(j\omega)$$

Frequency convolution

$$x_1(t) x_2(t) \leftrightarrow \frac{1}{2\pi} X_1(j\omega) \otimes X_2(j\omega)$$

Parseval's Formula for Nonperiodic Signals

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

In effect, the quantity $|X(j\omega)|^2$ represents the energy density per unit bandwidth of the signal at frequency ω and is often referred to as the *energy spectral density*.

2.2 Fourier transform

Comparison between FS and FT

	FS	FT
Analyzed Object	A periodic signal	A nonperiodic signal
Frequency Domain	Discrete Harmonic frequency	Continuous Whole frequency axis
Definition of Function Value	Frequency component value	Frequency component density

2.3 Discrete-time Fourier transform —— DTFT

The *original analog signal* $x(t)$ is measured by an ideal sampler at sampling instants $t = nT$. The *sampled signal* is $\hat{x}(t) = \sum_{n=-\infty}^{n=\infty} x(nT)\delta(t-nT)$

The Fourier transform: $$\hat{X}(\omega) = \int_{-\infty}^{\infty} \hat{x}(t)e^{-j\omega t}dt$$

$$\hat{X}(\omega) = \int_{-\infty}^{\infty} \left\{ \sum_{n=-\infty}^{n=\infty} x(nT)\delta(t-nT) \right\} e^{-j\omega t}dt$$

$$= \sum_{n=-\infty}^{n=\infty} \left\{ \int_{-\infty}^{\infty} x(nT)\delta(t-nT)e^{-j\omega t}dt \right\}$$

$$= \sum_{n=-\infty}^{n=\infty} \left\{ x(nT)\int_{-\infty}^{\infty} \delta(t-nT)e^{-j\omega t}dt \right\}$$

$$= \sum_{n=-\infty}^{n=\infty} x(nT)e^{-j\omega(nT)}$$

$T = 1$ $\Rightarrow$

$$\boxed{\begin{aligned} X[e^{j\omega}] &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi}\int_{2\pi} X[e^{j\omega}]e^{j\omega n}d\omega \end{aligned}}$$

2.3 DTFT —— Properties

Periodicity

$$\hat{X}(f+f_s)=\sum_{n=-\infty}^{n=\infty} x(nT)e^{-2\pi j(f+f_s)(nT)}=\sum_{n=-\infty}^{n=\infty} x(nT)\left\{e^{-2\pi jf(nT)}e^{-2\pi jn(f_sT)}\right\}=\sum_{n=-\infty}^{n=\infty} x(nT)e^{-2\pi jf(nT)}=\hat{X}(f)$$

$$X[e^{j(\omega+2\pi)}]=\sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2\pi)n}=\sum_{n=-\infty}^{\infty} x[n]\left\{e^{-j\omega n}e^{-j2\pi n}\right\}=X[e^{j\omega}]$$

Because of this periodicity, one may restrict the frequency interval to just one period, namely, the *Nyquist interval*. The periodicity in *f* implies that $\hat{X}(f)$ will extend over the entire frequency axis, in accordance with our expectation that the sampling process introduces high frequencies into the original spectrum.

2.3 DTFT —— Properties

Recover from spectrum

$x(nT)$ may be recovered from $\hat{X}(f)$:

$$x(nT) = \frac{1}{f_s}\int_{-f_s/2}^{f_s/2} \hat{X}(f)e^{2\pi jfnT}df = \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\frac{\omega}{2\pi}$$

where $\omega = 2\pi f / f_s$

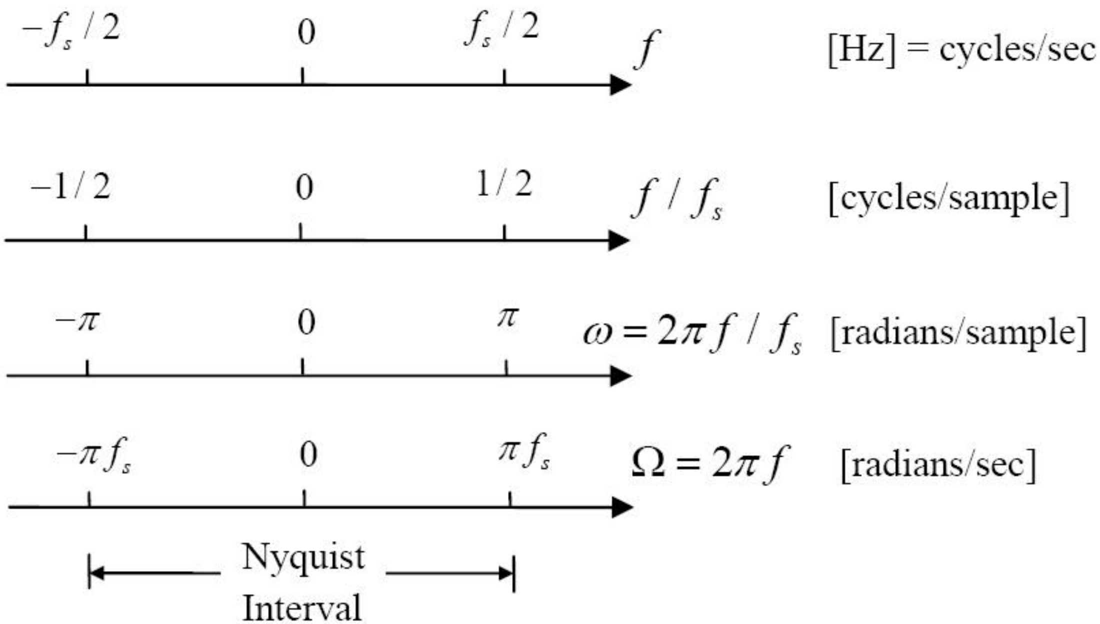
Numerical approximation

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-2\pi jft}dt \simeq \sum_{n=-\infty}^{n=\infty} x(nT)e^{-2\pi jf(nT)} \cdot T = T\hat{X}(f)$$

$$or \;\; X(f) = \lim_{1/T\to\infty} T\hat{X}(f)$$

2.3 DSP frequency units

DSP frequency units

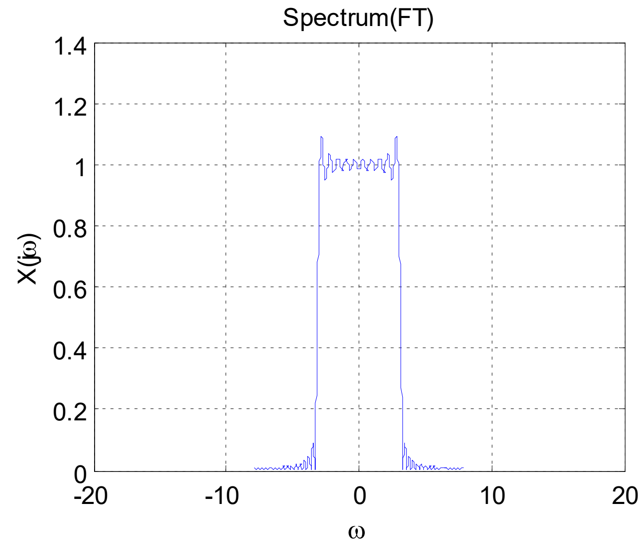


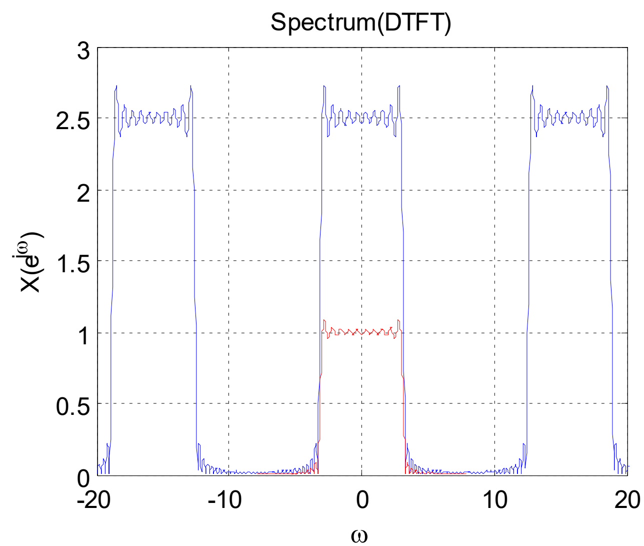
Compare various frequency scales that are commonly used in DSP.

In the DSP applications, where the sampling rates are determined in conditions, the most convenient set of units is simply in terms of *f*. In fixed-rate applications, the units of $\omega = 2\pi f / f_s$ and *f/fs* are the most convenient.

2.3 DTFT —— Simulation

```
fs = 2.5;                          % sampling frequency
…
x = Amp*sinc(t);
…
% DTFT
w_DTFT = -4*pi:dw:4*pi;      % 2pi*(f/fs)
n_DTFT = length(w_DTFT);
k = -T*fs:T*fs;                    % sampling sequence ID
X_DTFT = zeros(1,n_DTFT);
for ii = 1:length(k)
    tmp = x(ii)*exp(-1j*w_DTFT*ii);
    X_DTFT = X_DTFT + tmp;
end
subplot(222),plot(w_DTFT*fs, abs(X_DTFT))
xlabel('\omega')
ylabel('X(e^{j\omega})')
xlim([-20, 20])
title('Spectrum(DTFT)');
```

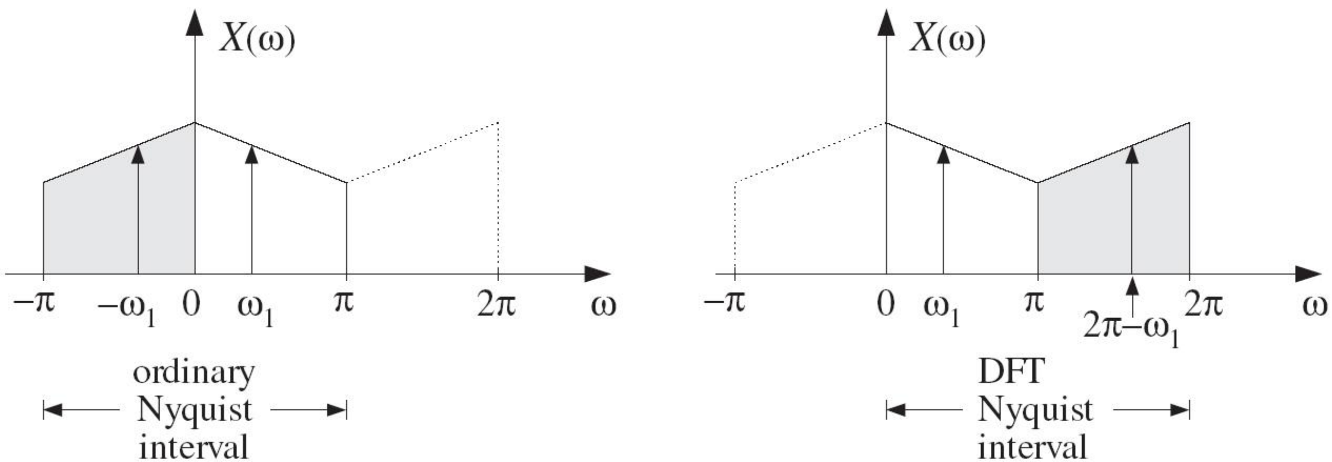




2.3 DTFT —— Nyquist interval

$$X[e^{j\omega}] = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

This expression may be computed at any desired value of ω in the Nyquist interval $-\pi \le \omega \le \pi$. It is customary in developing computational algorithms to take advantage of the periodicity of $X(\omega)$ and map the conventional symmetric Nyquist interval $-\pi \le \omega \le \pi$ onto the right-sided one $0 \le \omega \le 2\pi$.



Equivalent Nyquist intervals: $\{-f_s/2, f_s/2\} \Leftrightarrow \{0, f_s\}$

2.3 DTFT —— Practical considerations

In actual spectrum computations, two additional approximations must be made:

(a) Finite number of samples $x(nT)$ in calculations. This approximation leads to the concept of a ***time window*** and the related effects of ***smearing*** and ***leakage*** of the spectrum.

(b) A finite set of frequencies f at which to evaluate $\hat{X}(f)$.

Proper choice of this set allows the development of various efficient computational algorithms for the **DFT**.

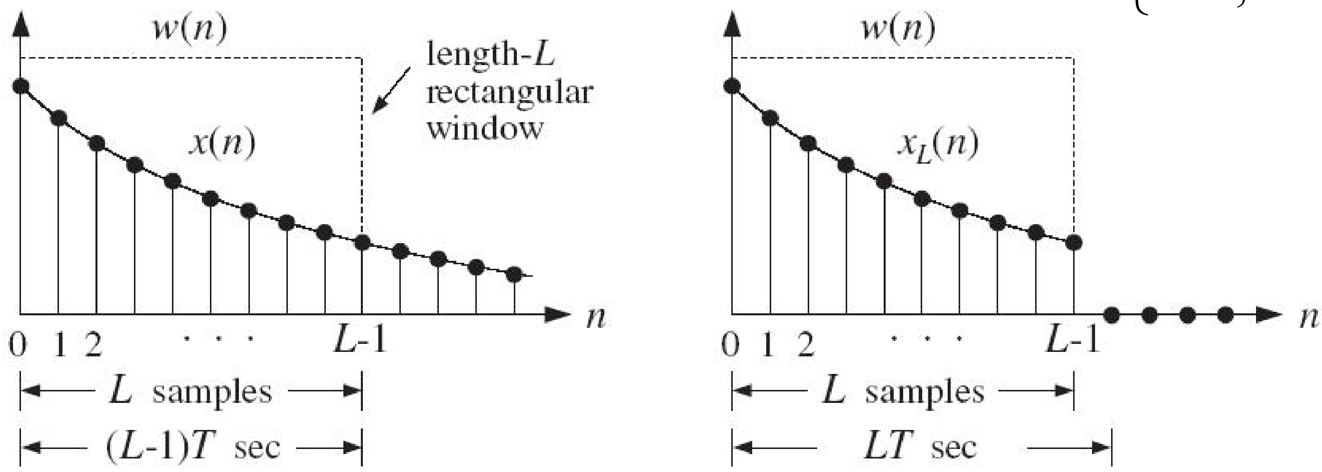
2.4 Time-windowing

Keeping only a finite number of samples, say $x(nT)$, $0 \le n \le N-1$ requires a so called ***time-windowing*** process. The time duration of the record samples is $T_N = NT$, where *T* is sample interval.

The windowed signal may be though of as an infinite signal which is zero outside the range of the window and agrees with the original one within the window. To express this mathematically, we define the rectangular window of length *N*:

$$w(n) = \begin{cases} 1, & \text{if } 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Define the windowed signal as follows $x_N(n) = x(n)w(n) = \begin{cases} x(n), & \text{if } 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$



2.4 Resolution and Leakage

The windowing process has two major effects:

(1) reduce the ***frequency resolution*** of the computed spectrum

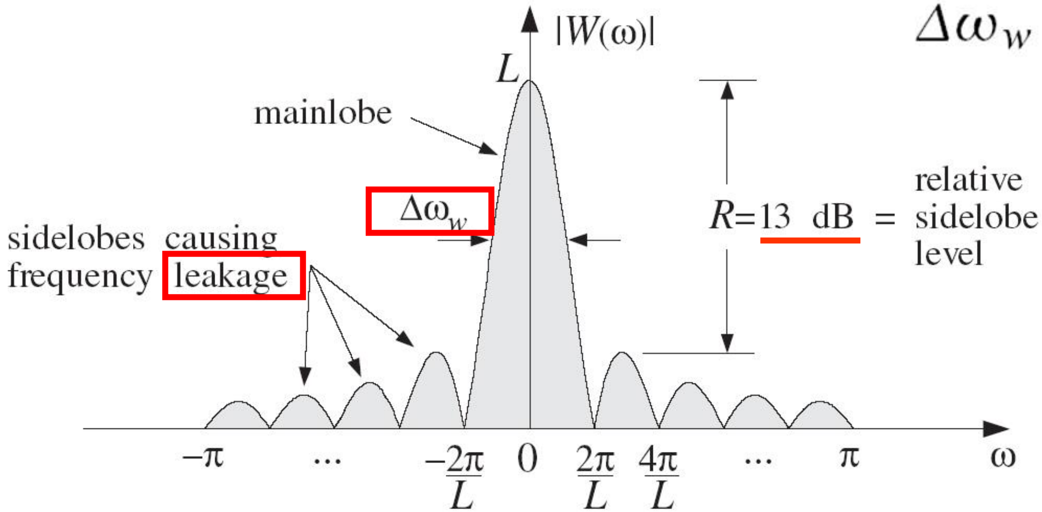
(2) introduce spurious high-frequency components into the spectrum, referred to as "***frequency leakage***".

$$x_L(n) = x(n)w(n)$$

FT $\downarrow$

$$X_L(\omega) = \int_{-\pi}^{\pi} X(\omega')W(\omega-\omega')\frac{d\omega'}{2\pi}$$

$$\Delta\omega_w = \frac{2\pi}{L}$$



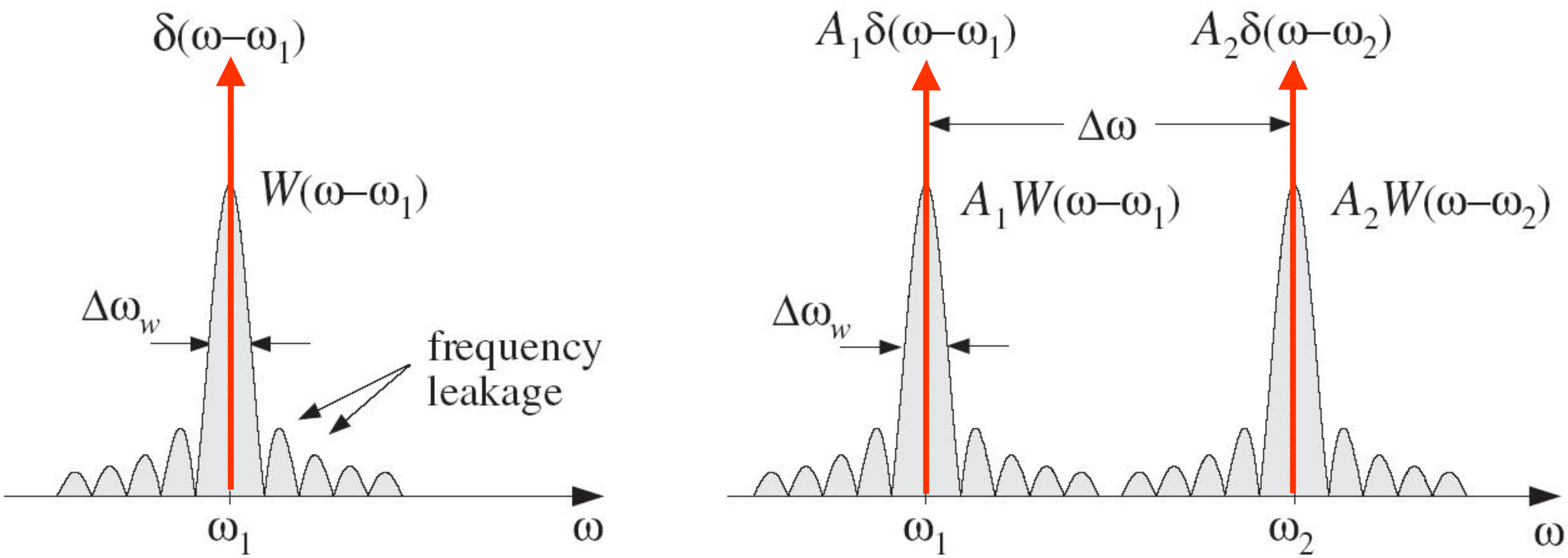
Magnitude spectrum of rectangular window.

2.4 Resolution and Leakage

$$x_L(n) = x(n)w(n)$$

FT $\downarrow$

$$X_L(\omega) = \int_{-\pi}^{\pi} X(\omega')W(\omega - \omega')\frac{d\omega'}{2\pi}$$



Spectra of windowed single and double sinusoids.

2.4 Resolution and Leakage

The ***resolvability*** condition that the two sinusoids appear as two distinct ones is that their frequency separation *Δf* be greater than the ***mainlobe width***:

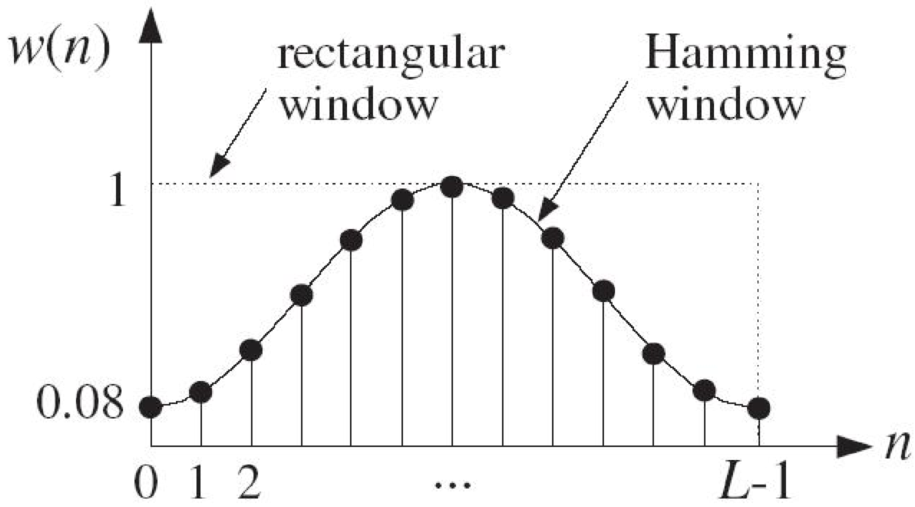
$$\boxed{\Delta f \geq \Delta f_w = \frac{f_s}{L}} \qquad \text{(frequency resolution)}$$

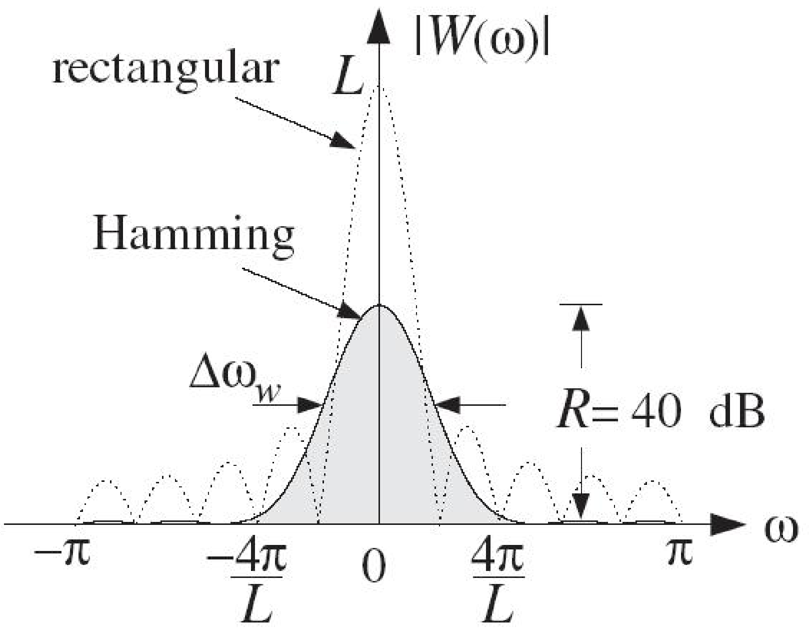
The minimum number of samples required to achieve a desired frequency resolution *Δf* can be determined. The smaller the desired separation, the longer the data record.

The ***sidelobes***, on the other hand, determine the amount of ***frequency leakage*** and are undesirable artifacts of the windowing process. They must be suppressed as much as possible because they may be confused with the mainlobes of weaker sinusoids that might be present.

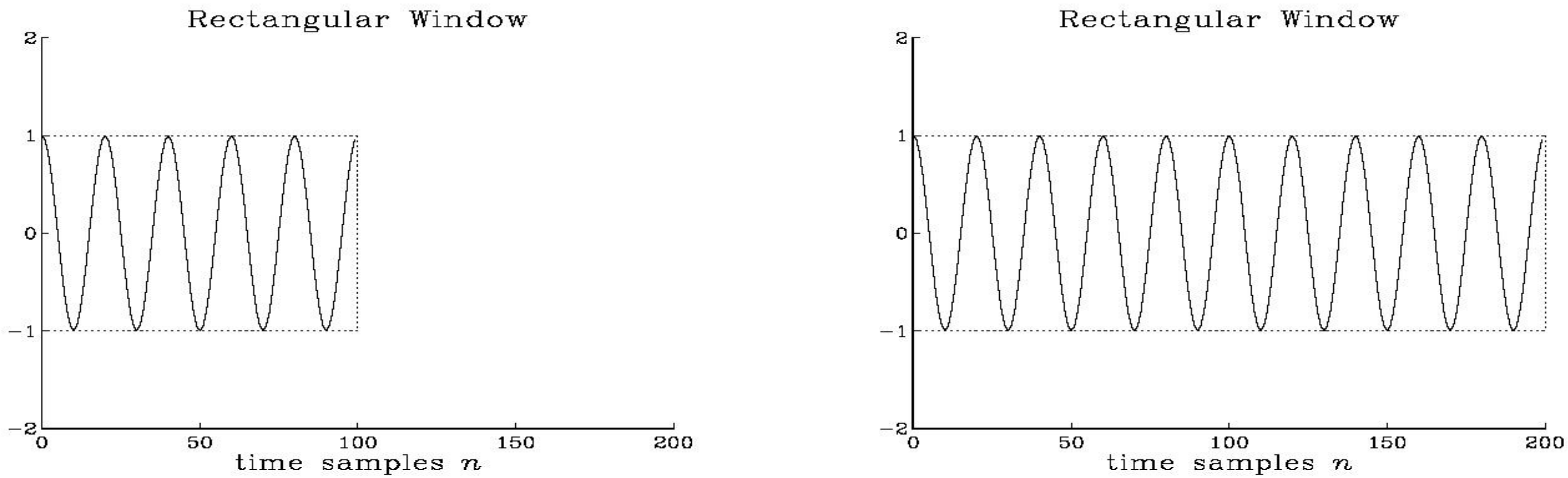
The standard technique for suppressing the sidelobes is to use a non-rectangular window — a window that cuts off to zero ***less sharply and more gradually*** than the rectangular one.

2.4 Window type

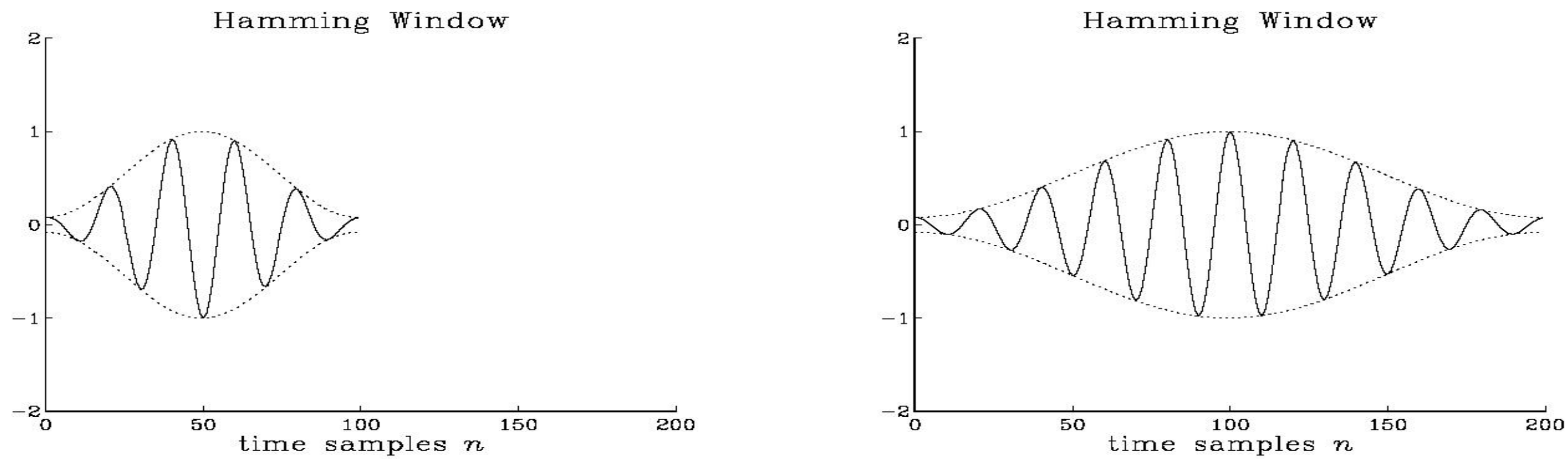




2.4 Window type

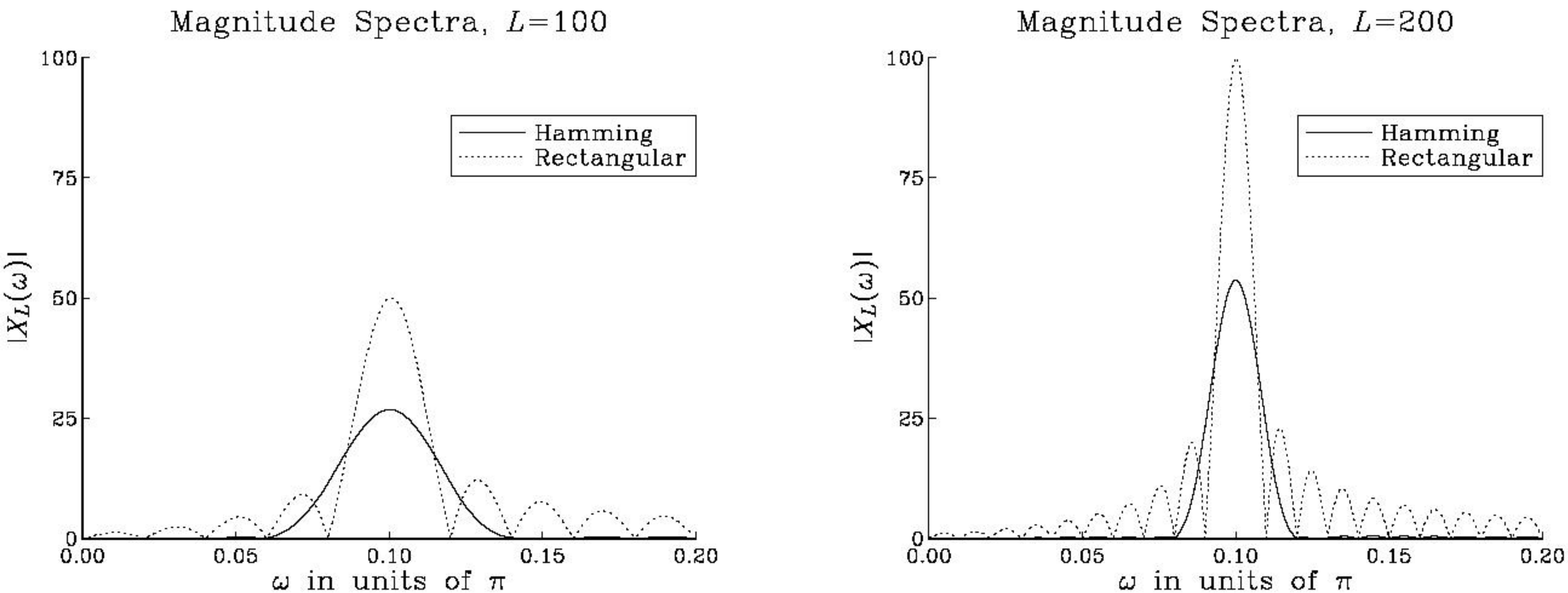


Rectangularly windowed sinusoids of lengths $L = 100$ and $L = 200$.



Hamming windowed sinusoids of lengths $L = 100$ and $L = 200$.

2.4 Window type



Rectangular and Hamming spectra for $L = 100$ and $L = 200$.

2.5 Discrete Fourier transform —— DFT

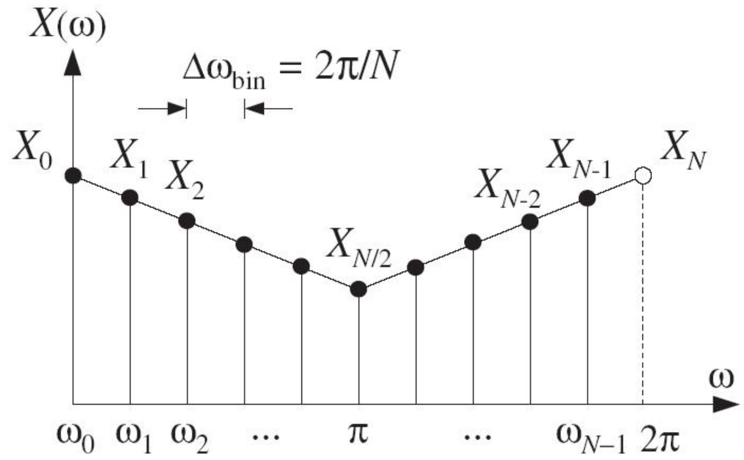
The ***N-point DFT*** of a *length-L* signal is defined to be the DTFT evaluated at N equally spaced frequencies over the full Nyquist interval, $0 \le \omega \le 2\pi$.

"DFT frequencies" are defined in radians per sample

$$\omega_k = \frac{2\pi k}{N}, \qquad k = 0, 1, \ldots, N-1$$

N-point DFT will be, for $k = 0, 1, \ldots, N-1$:

$$X(\omega_k) = \sum_{n=0}^{L-1} x(n) e^{-j\omega_k n} \qquad \text{(}N\text{-point DFT of length-}L\text{ signal)}$$



2.5 DFT —— Physical vs. computation resolution

Bin width in DFT, in rads/sample: $\Delta\omega_{\text{bin}} = \dfrac{2\pi}{N}$

represent the spacing between the DFT frequencies at which the DTFT is computed, noted as the ***Computational*** frequency resolution

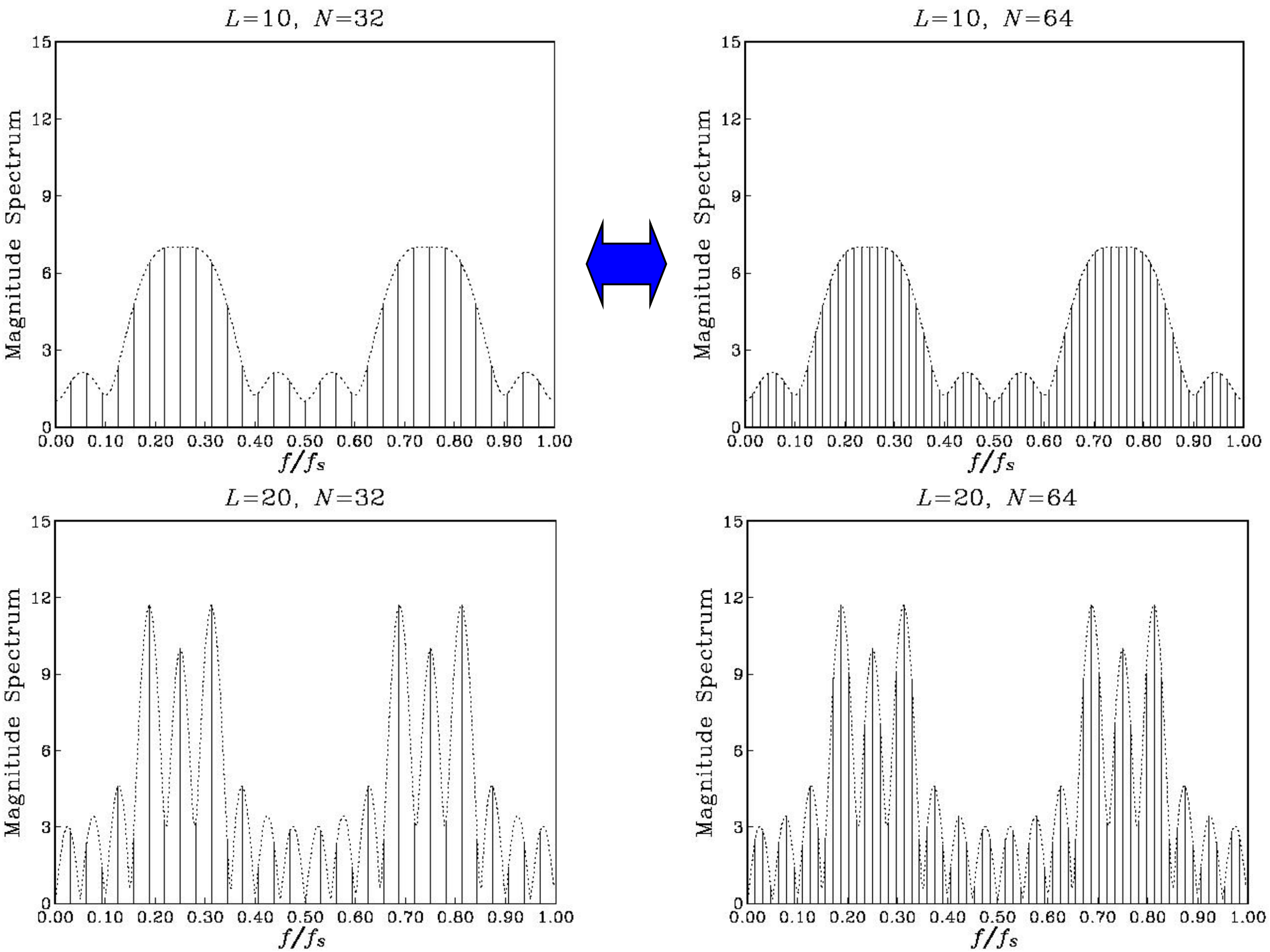
Avoid to confuse:

Frequency resolution width: $\Delta f = f_s / L$

Refer to the minimum resolvable frequency separation between two sinusoidal components, noted as the ***Physical*** frequency resolution

2.5 DFT —— Physical vs. computation resolution

An analog signal consisting of three sinusoids of frequencies f_1 = 2 kHz, f_2 = 2.5 kHz, and f_3 = 3 kHz: $x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$, sampled at the rate of 10 kHz.

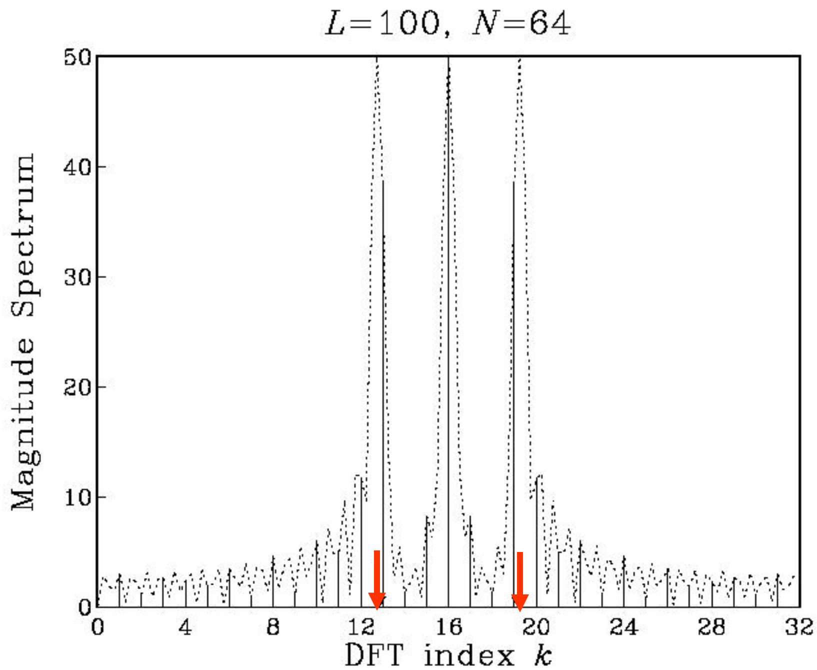


If the length *L* of the signal is not large enough to provide sufficient physical resolution, then increasing the length *N* of the DFT would only put more points on the wrong curve.

2.5 DFT —— Accuracy of peaks in spectrum

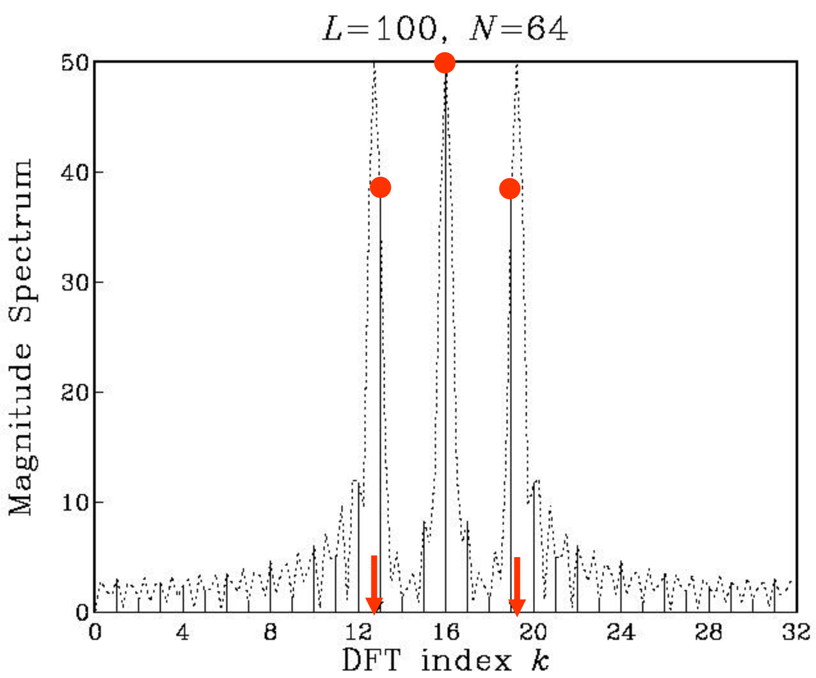
$$x(t) = \cos(2\pi f_1 t) + \cos(2\pi f_2 t) + \cos(2\pi f_3 t)$$

Another issue related to physical and computational resolution is the question of how accurately the DFT represents the peaks in the spectrum.



2.5 DFT —— Accuracy of peaks in spectrum

$f_1/f_s = 0.2$, $f_2/f_s = 0.25$, and $f_3/f_s = 0.3$



For each sinusoid present in the signal, say, at frequency f_0, the DTFT will exhibit a mainlobe peak arising from the shifted window $W(f - f_0)$. When we evaluate the N-point DFT, we would like the peak at f_0 to **coincide** with one of the N DFT frequencies.

This will happen if there is an integer $0 \le k_0 \le N - 1$, such that

$$f_0 = f_{k_0} = \frac{k_0 f_s}{N} \quad \Rightarrow \quad \boxed{k_0 = N\frac{f_0}{f_s}}$$

2.5 DFT —— Zero-padding

In principle, the two lengths L and N can be specified independently of each other: L is the number of time samples in the data record and can even be infinite; N is the number of frequencies at which we choose to evaluate the DTFT.

Most discussions of the DFT assume that $L = N$.
If $L < N$, pad $N-L$ zeros at the end of the data record to make it of length N.
If $L > N$, reduce the data record to length N by wrapping it modulo-N

$$\mathbf{x} = [x_0, x_1, \ldots, x_{L-1}]$$

$$\mathbf{x}_D = [x_0, x_1, \ldots, x_{L-1}, \underbrace{0, 0, \ldots, 0}_{D \text{ zeros}}]$$

$$X_D(\omega) = \sum_{n=0}^{L+D-1} x_D(n) e^{-j\omega n} = \sum_{n=0}^{L-1} x_D(n) e^{-j\omega n} + \sum_{n=L}^{L+D-1} x_D(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{L-1} x(n) e^{-j\omega n} = X(\omega)$$

2.5 DFT —— Zero-padding

Padding the D zeros to the front of the signal will be equivalent to a delay by D samples, which in the z-domain corresponds to multiplication by z^{-D} and in the frequency domain by $e^{-j\omega D}$.

$$\mathbf{x} = [x_0, x_1, \ldots, x_{L-1}]$$

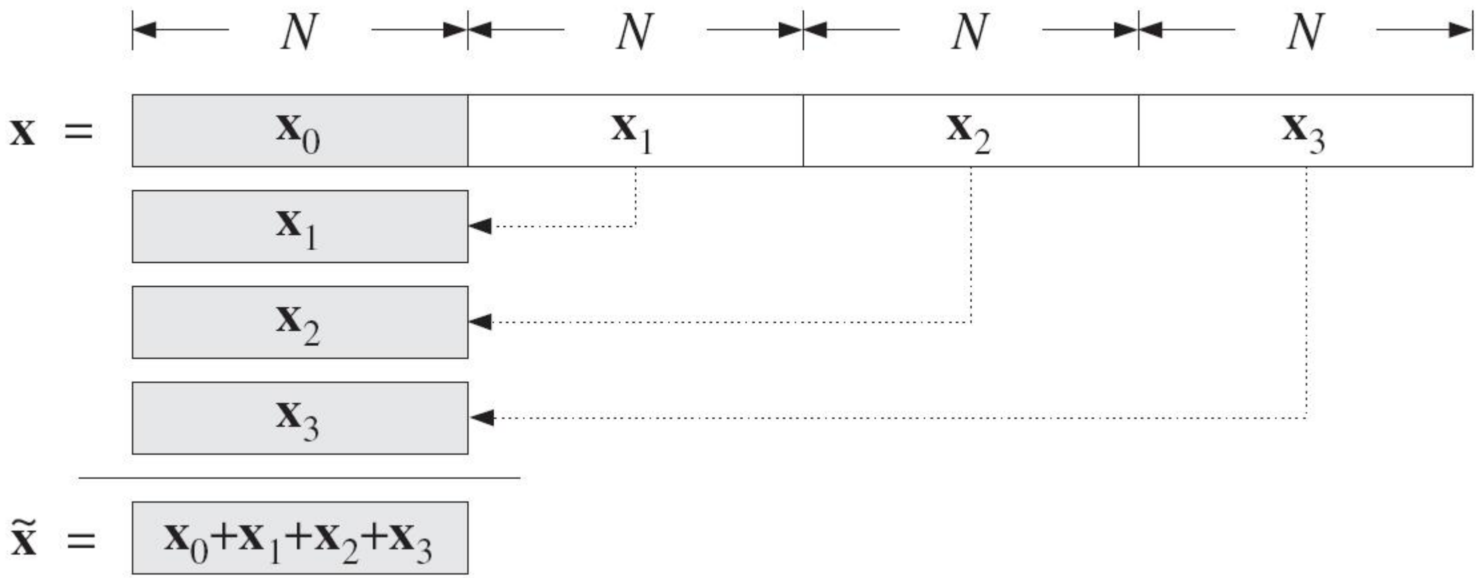
$$\mathbf{x}_D = [\underbrace{0, 0, \ldots, 0}_{D \text{ zeros}}, x_0, x_1, \ldots, x_{L-1}]$$

will have DTFTs and DFTs:

$$X_D(\omega_k) = e^{-j\omega_k D} X(\omega_k), \quad k = 0, 1, \ldots, N-1$$

2.5 DFT —— Wrapping

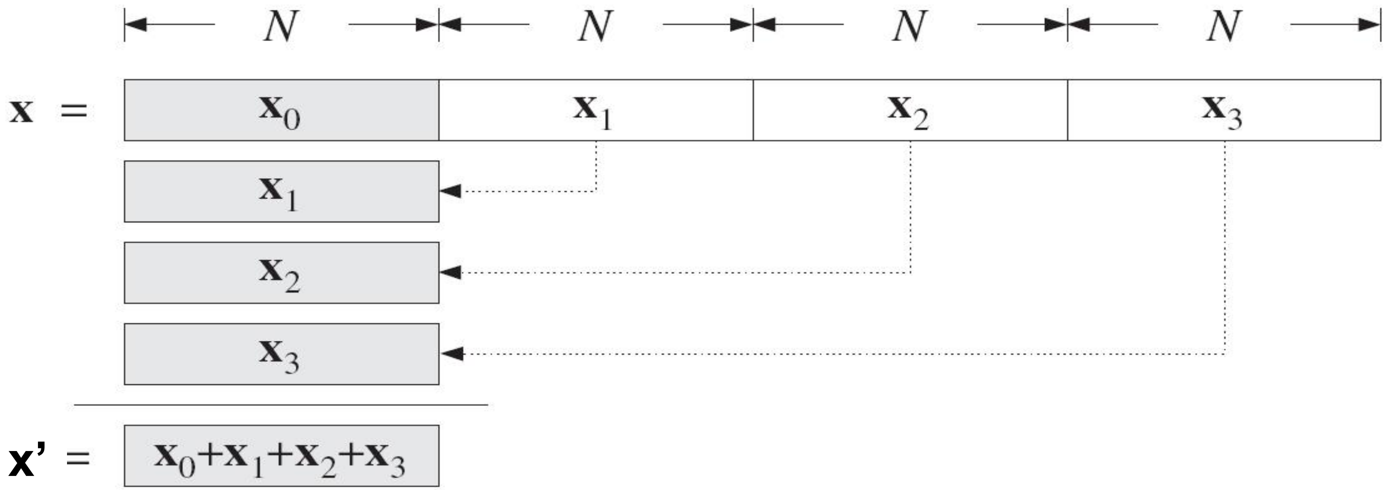
$L > N$



The modulo-N reduction or wrapping of a signal is defined by dividing the signal **x** into contiguous non-overlapping blocks of length *N*, wrapping the blocks around to be time-aligned with the first block, and adding them up. If *L* is not an integral multiple of *N*, then the last sub-block will have length less than *N*; in this case, we may pad enough zeros at the end of the last block to increase its length to *N*.

2.5 DFT —— Wrapping

$L > N$



The wrapping of a signal is defined by dividing the signal **x** into contiguous non-overlapping blocks of length *N*, wrapping the blocks around to be time-aligned with the first block, and adding them up.

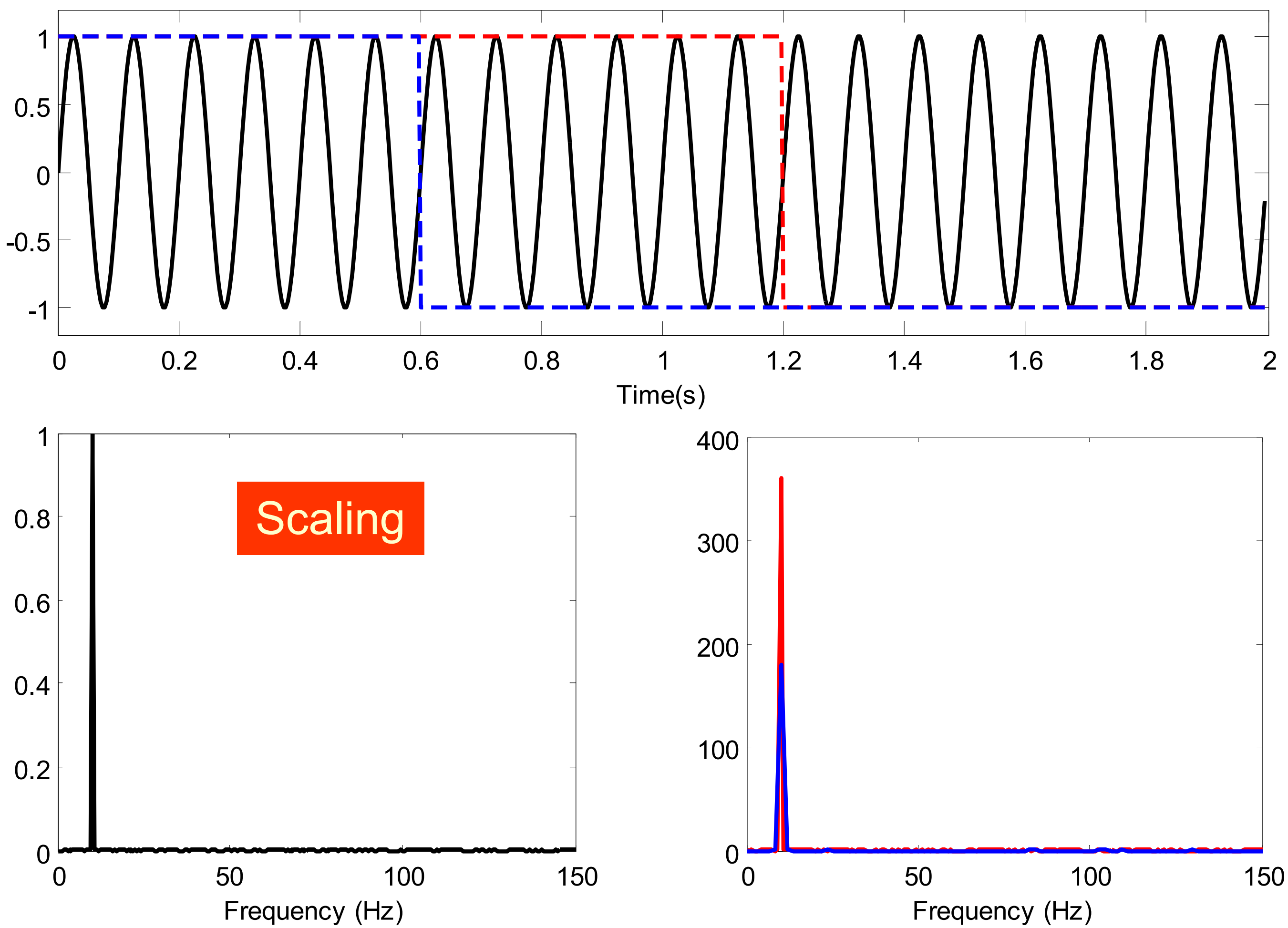
If *L* is not an integral multiple of *N*, then the last sub-block will have length less than *N*; in this case, we may pad enough zeros at the end of the last block to increase its length to *N*.

The length-*N* wrapped signal **x'** has the same *N*-point DFT as the original unwrapped signal **x**

$$\boxed{\tilde{X}_k = X_k \quad \text{or,} \quad \tilde{X}(\omega_k) = X(\omega_k)}\,, \qquad k = 0, 1, \ldots, N-1$$

2.5 DFT —— Periodic signal

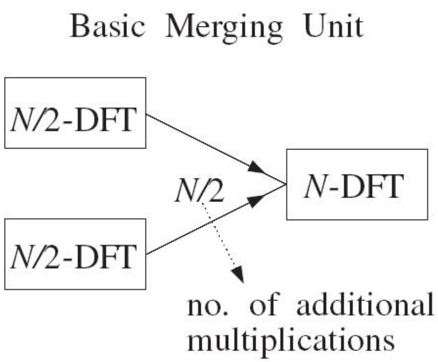
Frequency components of a periodic signal represented by DFT

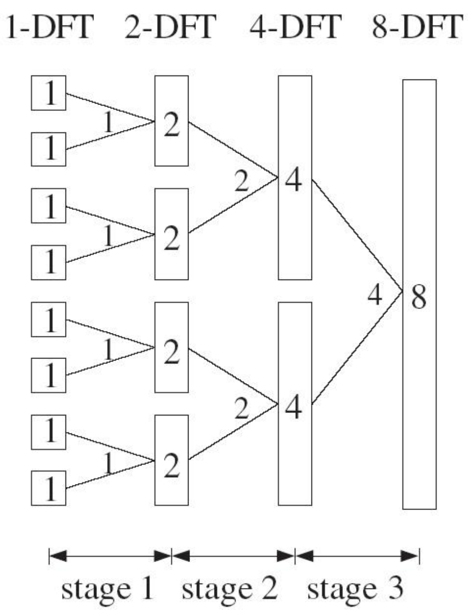


2.6 FFT

The fast Fourier transform is a fast implementation of the DFT. It is based on a divideand-conquer approach in which the DFT computation is divided into smaller, simpler, problems and the final DFT is rebuilt from the simpler DFTs.
In the simplest Cooley-Tukey version of the FFT, the dimension of the DFT is successively divided in half until it becomes unity. This requires the initial dimension *N* to be a power of two:

$$\boxed{N = 2^B} \quad \Rightarrow \quad B = \log_2(N)$$





FFT for arbitrary length signal：https://zjuturtle.com/2017/12/26/fft/