

Signals

1.0 Introduction

Introducing mathematical description, representations and basic properties of signals and systems in general.

1.1 Signal classification

1.2 Transformations of the independent variable

1.3 Basic signals: exponential, sinusoidal, impulse, step, etc.

1.4 Continuous-time and Discrete-time systems

1.5 System properties: causality, stability, linearity, time-invariance

1.1 Signal classification

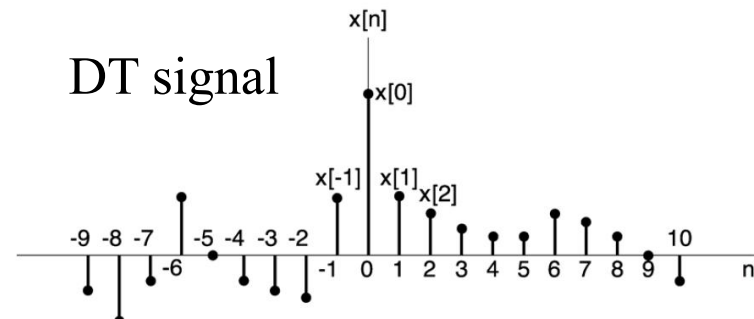
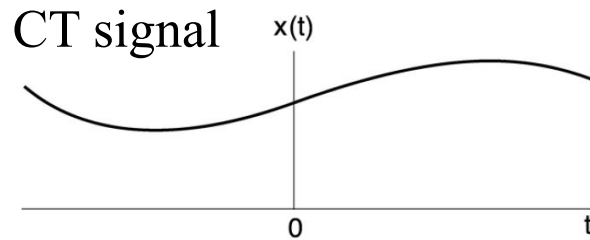
Signals are functions of independent variables that carry information. For example: voltages and currents in a circuit, Acoustic signals in speech, analog or digital Video signals, Biological signals ...

Independent variables can be continuous, e.g. many real signals in nature with physical meanings, can be discrete, e.g. digital image pixels, and can be 1-D, 2-D, ... dependent on the number of independent variables. In this course, we focus on 1-D signals with one independent variable of “time”.

Based on different features of signals, we can classify signals as

- **Continuous-time** and **discrete-time** signals
- **Even** and **odd** signals
- **Periodic** and **aperiodic** (non-periodic) signals
- **Deterministic** and **random** signals
- **Energy** and **power** signals

1.1.1 Continuous-time and discrete-time signals



Continuous-Time (**CT**) signals: $x(t)$, t is continuous

Discrete-Time (**DT**) signals: $x[n]$, n is integer values only

Weekly Dow-Jones
industrial average



Examples of DT signals

Digital Photo
with pixels



Courtesy of Jason Oppenheim.

A DT signal may represent a phenomenon for which the independent variable is inherently discrete. On the other side, Currently most DT signals are arisen from **sampling** of CT signals, i.e. human-made DT signals, which then can be represented and/or processed by modern digital hardware: computer and DSP chips.

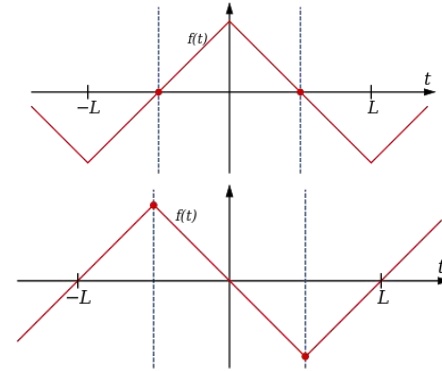
1.1.2 Even and odd signals

Even signal: $x(-t) = x(t)$ or $x[-n] = x[n]$.

Symmetric about the time origin.

Odd signal: $x(-t) = -x(t)$ or $x[-n] = -x[n]$.

Antisymmetric about the time origin.



Any signal can be decomposed into an even and an odd signals

$$x(t) = \frac{x(t) + x(-t)}{2} + \frac{x(t) - x(-t)}{2}, \quad \begin{matrix} \text{even} \\ x_e(t) = \frac{x(t) + x(-t)}{2} \end{matrix} \text{ and } \begin{matrix} \text{odd} \\ x_o(t) = \frac{x(t) - x(-t)}{2} \end{matrix}$$

Example: even or odd function of time? $x(t) = \begin{cases} \sin\left(\frac{\pi t}{T}\right), & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$

Solution:

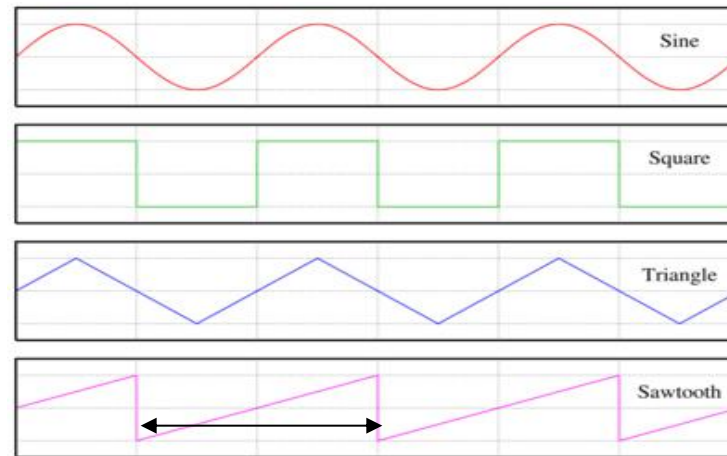
$$x(-t) = \begin{cases} \sin\left(-\frac{\pi t}{T}\right), & |t| \leq T \\ 0, & \text{otherwise} \end{cases} = \begin{cases} -\sin\left(\frac{\pi t}{T}\right), & |t| \leq T \\ 0, & \text{otherwise} \end{cases} = -x(t) \quad \text{So it is an Odd signal}$$

1.2.3 Periodic signals

Periodic signal is a function satisfy the following condition:

$$x(t) = x(t + T) \text{ or } x[n] = x[n + N]$$

Fundamental period: the *smallest* values of T for CT and n for DT to satisfy the periodicity conditions

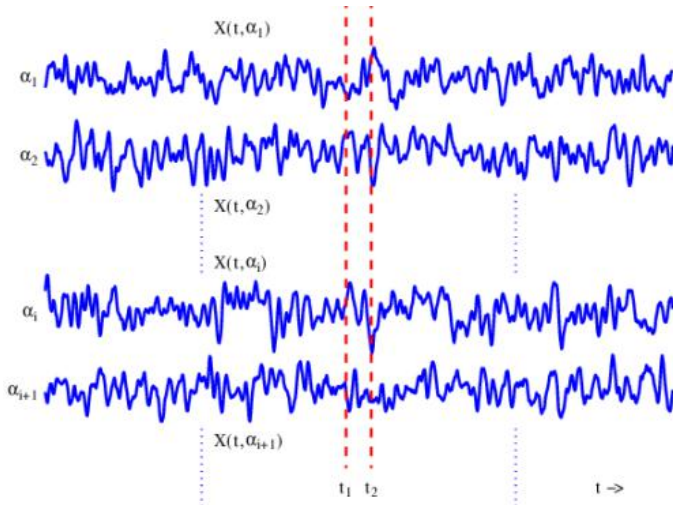


Fundamental
period

1.1.4 Deterministic and random signals

Deterministic: predictable, normally described by an explicit mathematical relationship with reasonable accuracy.

Radom: or stochastic, unpredictable, each observation of the phenomenon will be unique. A part of random processes, named **stationary** random, can be described by mathematical relationships in **statistics**.



Stationary random signals with constant statistical parameters: average, deviation ...

- 1) No physical data in practice can be truly deterministic because there is always a possibility that some unforeseen events in the future might influence the phenomenon to produce the data in a manner that was not originally considered.
- 2) On the other hand, it might be argued that no physical data are truly random, because an exact mathematical description might be possible if sufficient knowledge of the basic mechanisms of the phenomenon was available.

1.1.5 Power and energy signals

Instantaneous “**power**” (may not be related to the real power)

$$p(t) = |x(t)|^2 \quad \text{or} \quad p[n] = |x[n]|^2$$

Energy $E_{\infty} = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad \text{or} \quad E_{\infty} = \lim_{N \rightarrow \infty} \sum_{n=-N}^N |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$

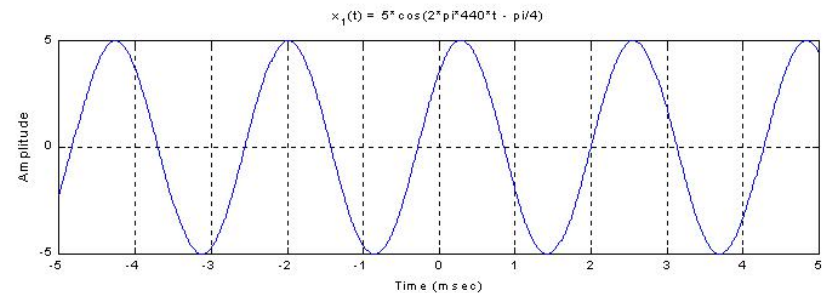
Average power $P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \quad \text{or} \quad P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^N |x[n]|^2$

(finite) Power signal $0 < P_{\infty} < \infty$

with infinite energy

$$x(t) = 5 \cos(2\pi \cdot 440t - \pi / 4)$$

$$E_{\infty} = \infty, \quad P_{\infty} = 12.5$$

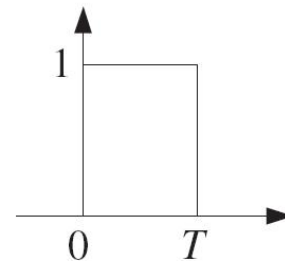


(finite) Energy signal $0 < E_{\infty} < \infty$

with zero average power

$$x(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

$$E_{\infty} = T, \quad P_{\infty} = 0$$



1.2 Transformations of the independent variable

Transformation of Signals is a central concept in signal and system analysis and has important physical meaning in the real world.

For example, in a high-fidelity audio system, an input signal representing music as recorded on a CD is modified to remove recording noise and enhance desirable characteristics (e.g. adding bass), and then amplified to drive the louder speak.

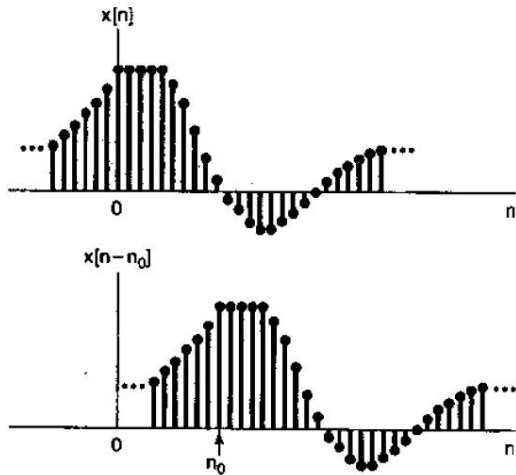
Here, we focus on basic signal transformations that involve simple modification of the independent variable, i.e., the time axis:

- Time shift: time delay and time advance
- Time reversal (reflection)
- Time scaling: time compression and time expansion
- Combination of time shift, reversal and scaling

1.2.1 Time shift and reversal

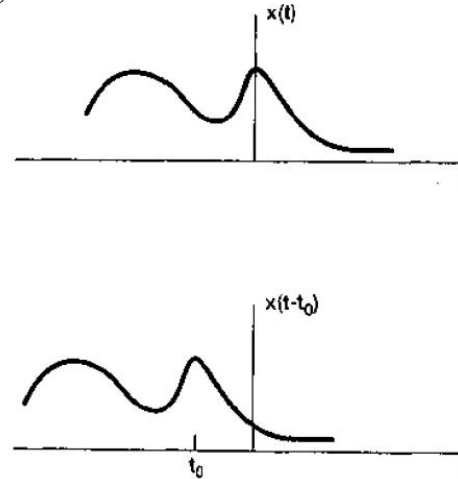
Time shift: $y(t) = x(t - t_0)$

$t_0, n_0 > 0$: delayed version



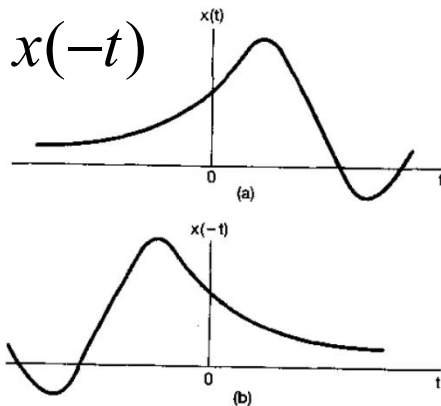
$y[n] = x[n - n_0]$, integer n_0

$t_0, n_0 < 0$: advanced version

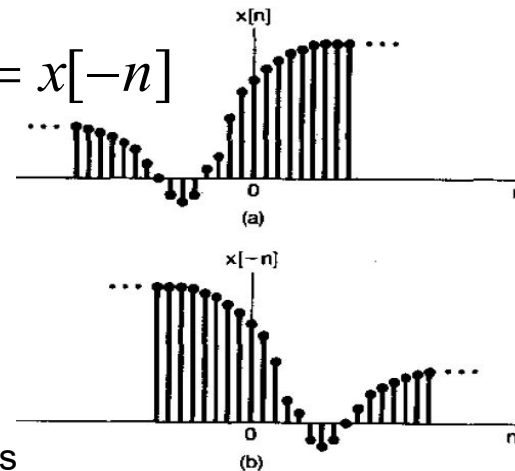


Time reversal about $t = 0$ or $n = 0$:

$y(t) = x(-t)$



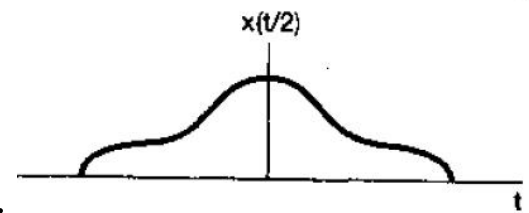
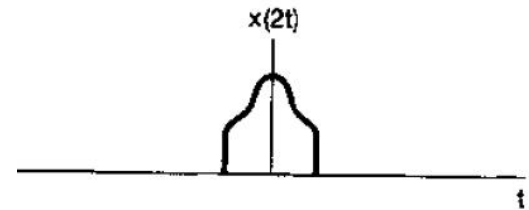
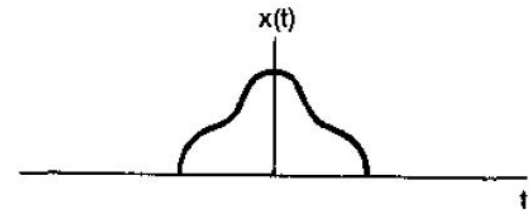
$y[n] = x[-n]$



1.2.2 Time scaling

CT signals $y(t) = x(at)$

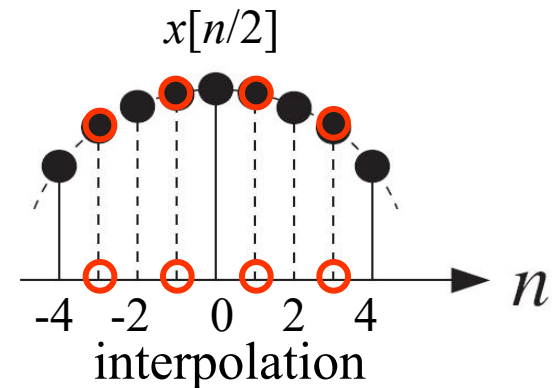
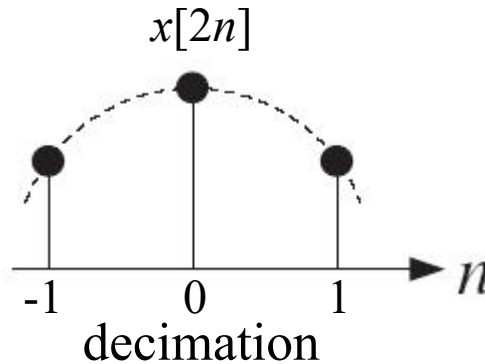
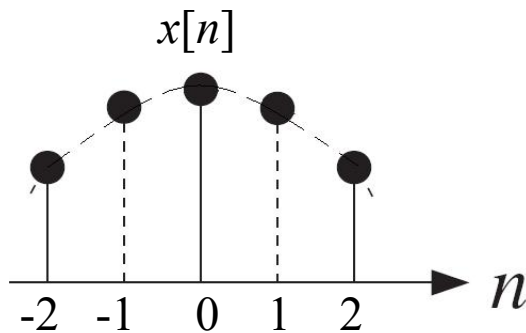
$|a| > 1$: linearly compressed by a factor of a
 $0 < |a| < 1$: linearly expanded by a factor of a
if $a < 0$, then reversed



DT signals $y[n] = x[kn]$, integer $k > 0$

$k > 1$: “**decimation**”, compressed? lost information

$0 < k < 1$: “**interpolation**”, expanded? add information



1.2.3 Combination transformation

Determine the effect of transforming $x(t)$ to obtain $y(t)$ of the form

$$y(t) = x(at - t_0)$$

Order of precedence:

$$y(t) = x(at - t_0)$$

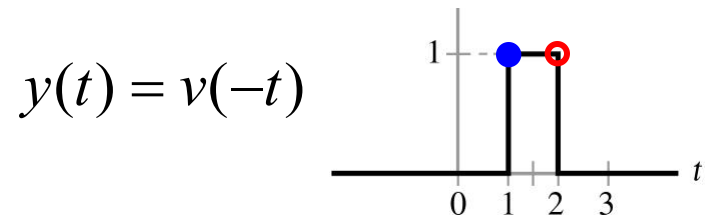
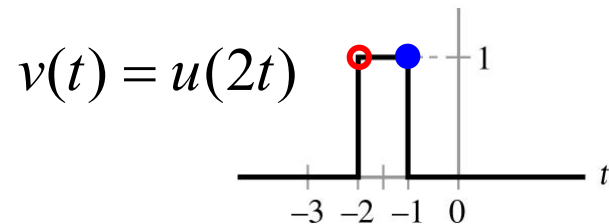
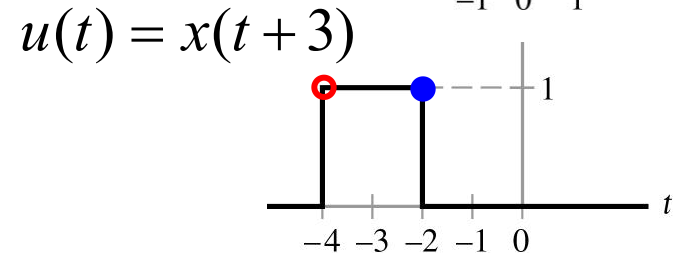
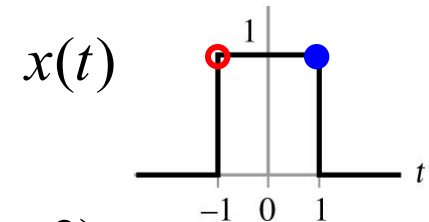
\Rightarrow

(1) Shift $u(t) = x(t - t_0)$

(2) Scale $v(t) = u(|a|t)$

(3) Reflection $y(t) = v(\text{sgn}(a) \cdot t)$

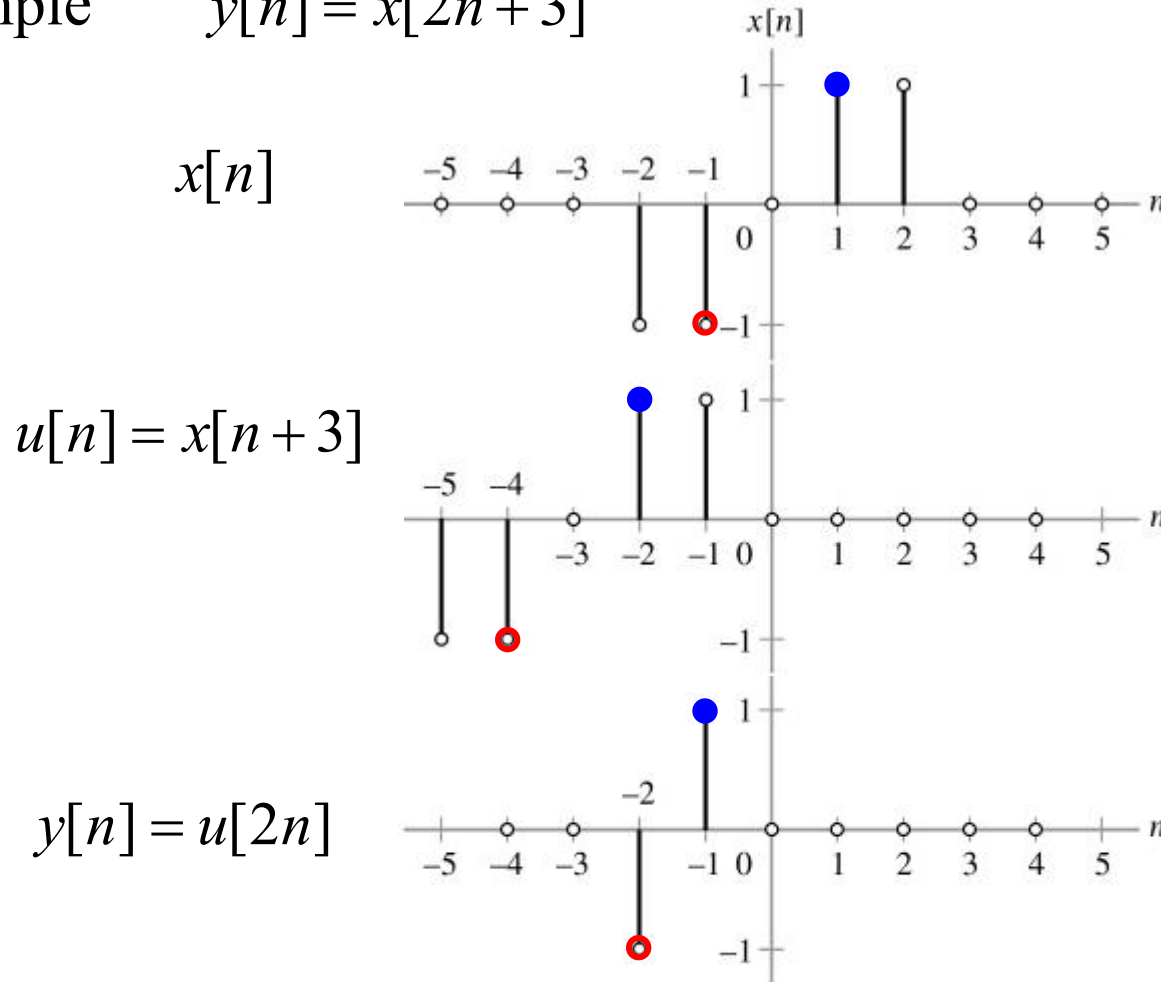
Example $y(t) = x(-2t + 3)$



check with selected points

1.2.3 Combination transformation

Example $y[n] = x[2n + 3]$



check with selected points

1.3 Basic signals

Signals introduced here are not only important themselves and but also frequently served as basic building blocks from which we can construct many other signals. These signals include

- Continuous-time exponential signals
- Discrete-time exponential signals
- Step function
- Impulse function
- Derivatives and integration of the impulse and step functions

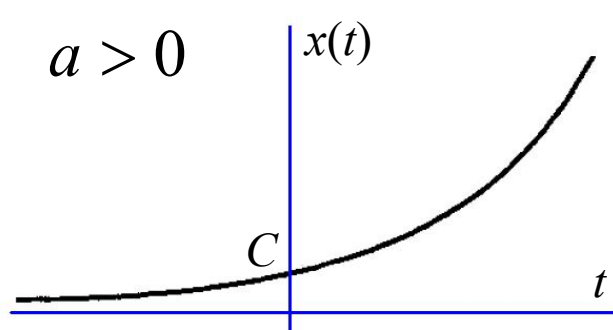
1.3.1 CT Exponential signals

The CT complex exponential signal is of the form

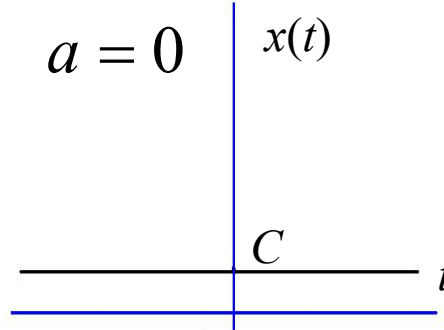
$$x(t) = Ce^{at} \quad C, a \text{ are generally complex numbers}$$

Depending on the values of these parameters, the complex exponential can exhibit several different characteristics.

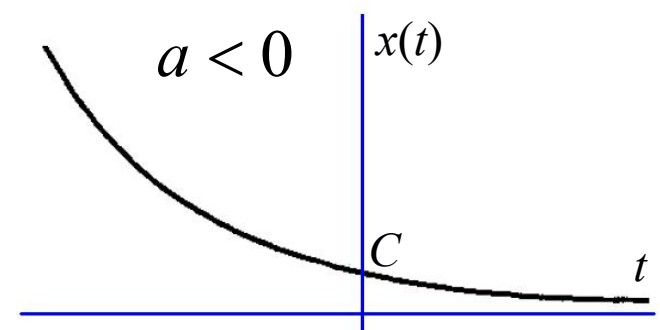
- Real CT exponential signals C, a are real numbers



Decaying exponential



Constant



Growing exponential

1.3.1 CT Exponential signals

- Complex CT exponential signals

$C = Ae^{j\phi}$ C is a complex number with modulus A and phase ϕ

$a = j\omega_0$ a is a purely imaginary number

So $Ce^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)}$ has periodic property, as

$$x(t) = Ce^{j\omega_0 t} = Ce^{j\omega_0 (t+T)} = Ce^{j\omega_0 t} e^{j\omega_0 T}$$

The Period T must satisfy: $e^{j\omega_0 T} = 1$

If $\omega_0 \neq 0$, then $T = \frac{2\pi k}{\omega_0}$ and the fundamental period is $T_0 = \frac{2\pi}{\omega_0}$

By using Euler's relation $e^{j\omega_0 t} = \cos(\omega_0 t) + j \sin(\omega_0 t)$

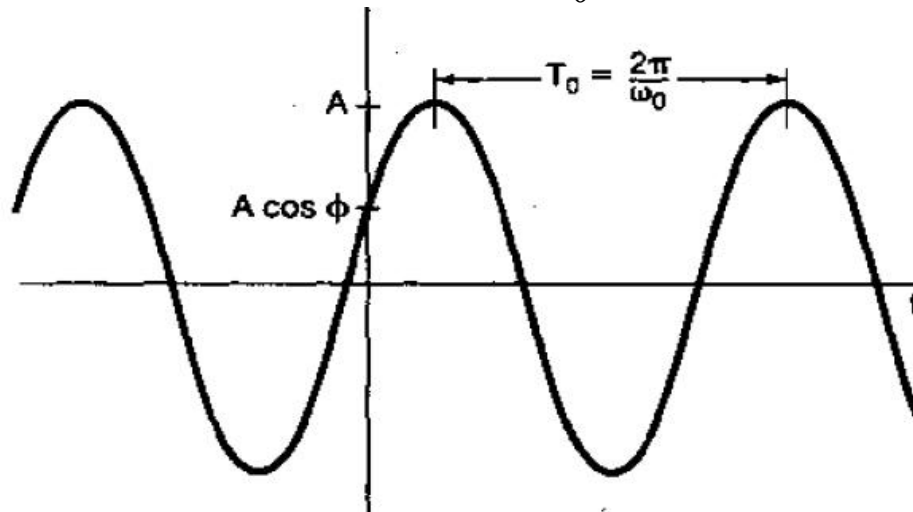
$$\cos(\omega_0 t) = 0.5e^{j\omega_0 t} + 0.5e^{-j\omega_0 t} = \text{Re}\{e^{j\omega_0 t}\}$$

$$\sin(\omega_0 t) = \text{Im}\{e^{j\omega_0 t}\}$$

1.3.1 CT Exponential signals

$$x(t) = Ce^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)} = A \cos(\omega_0 t + \phi) + jA \sin(\omega_0 t + \phi)$$

Real part $A \cos(\omega_0 t + \phi)$



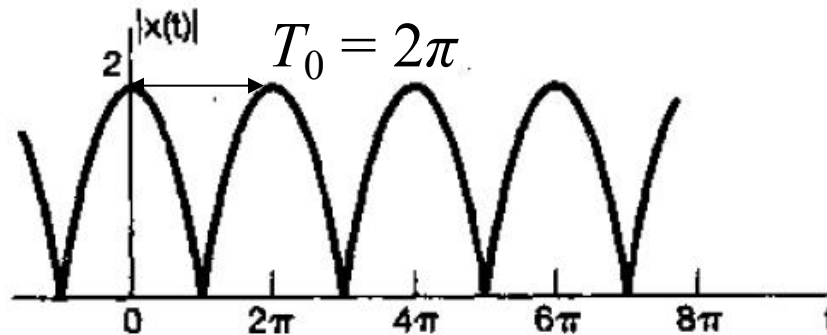
To increase or decrease the value of ω_0 will speed up or slow down the rate of oscillation. ω_0 is the fundamental radian frequency in (radian/s) and $f_0 = \omega_0 / 2\pi$ is the fundamental frequency in Hertz (1/s, Hz).

1.3.1 CT Exponential signals

Example: It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid and then we can obtain the magnitude of the signal.

$$\begin{aligned}x(t) &= e^{j2t} + e^{j3t} \\&= e^{j2.5t} (e^{-j0.5t} + e^{j0.5t}) \\&= e^{j2.5t} 2 \cos(0.5t)\end{aligned}$$

Magnitude is $|x(t)| = 2|\cos(0.5t)|$



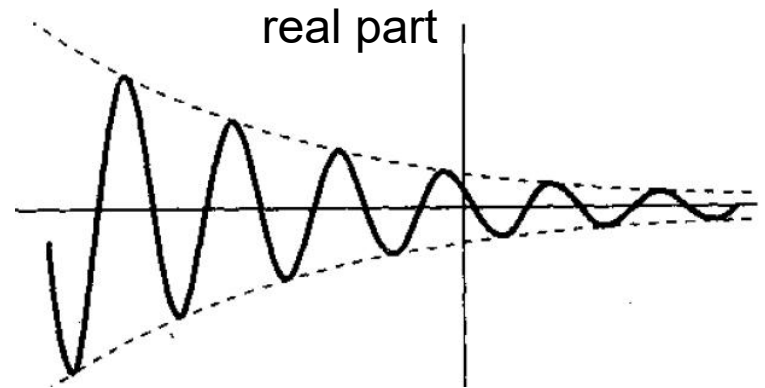
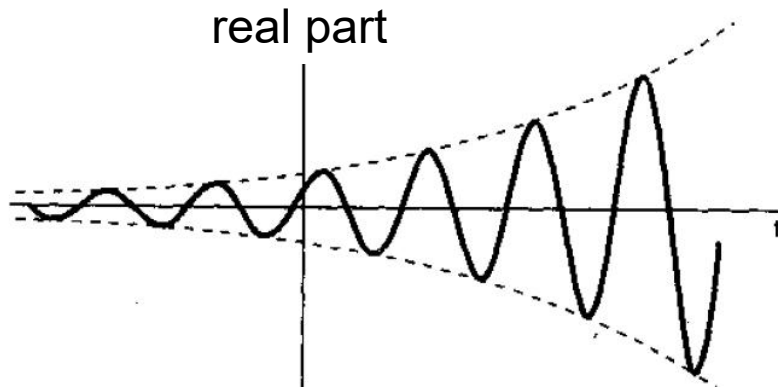
1.3.1 CT Exponential signals

- General complex CT exponential signals

$C = Ae^{j\phi}$ C is a complex number with modulus A and phase ϕ

$a = r + j\omega_0$ a is a general complex number with the real part r and the imaginary part ω_0

$$x(t) = Ce^{(r+j\omega_0)t} = Ae^{rt}e^{j(\omega_0 t + \phi)} = e^{rt}A\cos(\omega_0 t + \phi) + je^{rt}A\sin(\omega_0 t + \phi)$$



$r > 0$: Growing sinusoidal $r = 0$: Sinusoidal $r < 0$: Decaying sinusoidal

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as damped sinusoids, e.g. response of RLC circuits

1.3.2 DT Exponential signals

DT exponential signals $x[n] = C\alpha^n = Ce^{\beta n}$

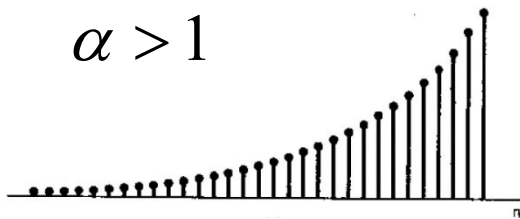
C, α are generally complex numbers

$C = Ae^{j\phi}$ is a complex number with modulus A and phase ϕ

$\alpha = e^{\beta}$ β is a general complex number

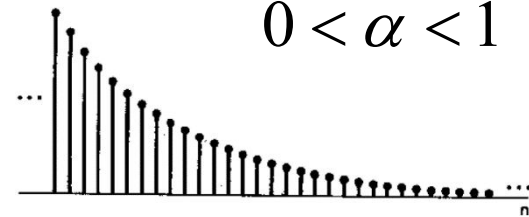
- Real DT exponential signals: C, α are real numbers

$\alpha > 1$

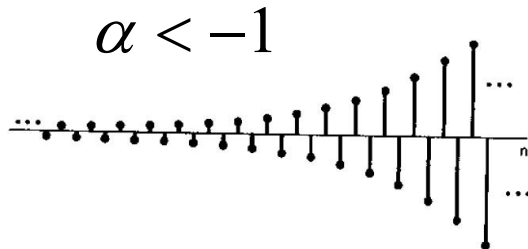


$\alpha = 1$

$0 < \alpha < 1$

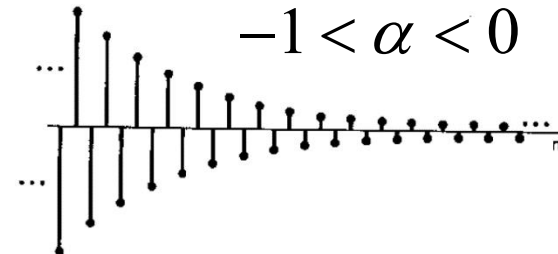


$\alpha < -1$



$\alpha = -1$

$-1 < \alpha < 0$

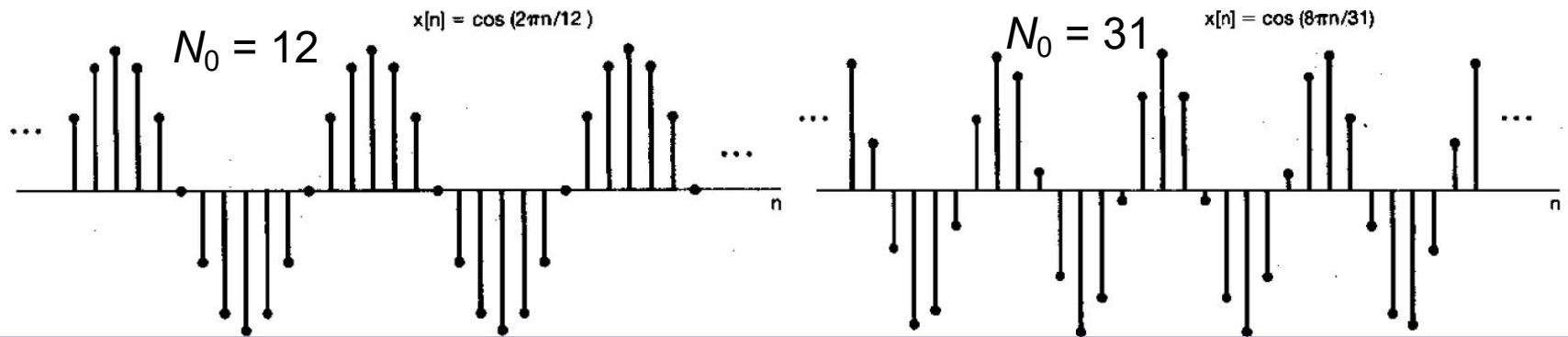


$|\alpha| > 1$: growing amplitude $|\alpha| < 1$: decaying amplitude

1.3.2 DT Exponential signals

- Complex DT exponential signals: $x[n] = C\alpha^n = Ce^{\beta n}$
 $C = Ae^{j\phi}$ is a complex number with modulus A and phase ϕ
 $\alpha = e^{\beta} = e^{j\omega_0}$ β is a pure imaginary value

So $x[n] = Ae^{j(\omega_0 n + \phi)} = A \cos(\omega_0 n + \phi) + jA \sin(\omega_0 n + \phi)$



$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1$$

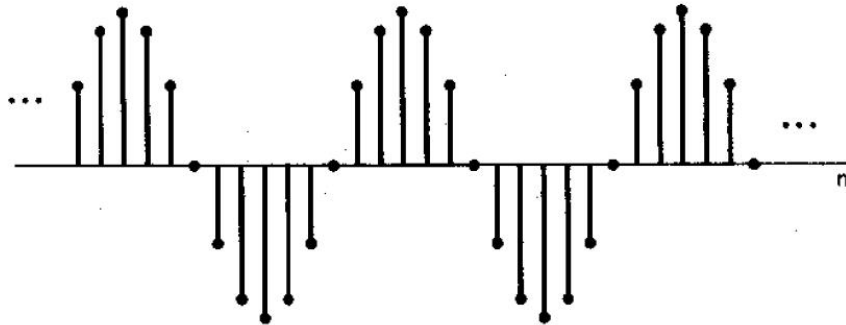
$$\Rightarrow \omega_0 N = 2\pi k \rightarrow \frac{\omega_0}{2\pi} = \frac{k}{N}, \quad k, N \text{ must be integer}$$

$x[n] = \cos(n/6)$ Periodic signal or not?

$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$, period?

Examples:

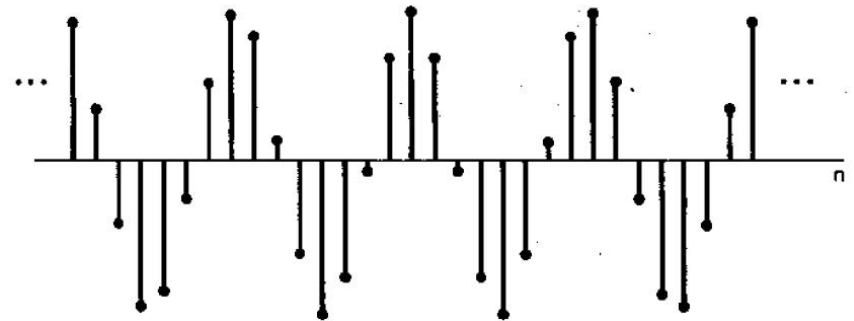
$$x[n] = \cos(2\pi n / 12)$$



$$N_0 = 12$$

$$x(t) = \cos(2\pi t / 12), \text{ period?}$$

$$x[n] = \cos(8\pi n / 31)$$



$$N_0 = 31$$

$$x(t) = \cos(8\pi t / 31), \text{ period?}$$

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \rightarrow e^{j\omega_0 N} = 1$$

$$\Rightarrow \omega_0 N = 2\pi k \rightarrow \frac{\omega_0}{2\pi} = \frac{k}{N}, \quad k, N \text{ must be integer}$$

$$x[n] = \cos(n / 6) \text{ Periodic signal or not?}$$

$$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}, \text{ period?}$$

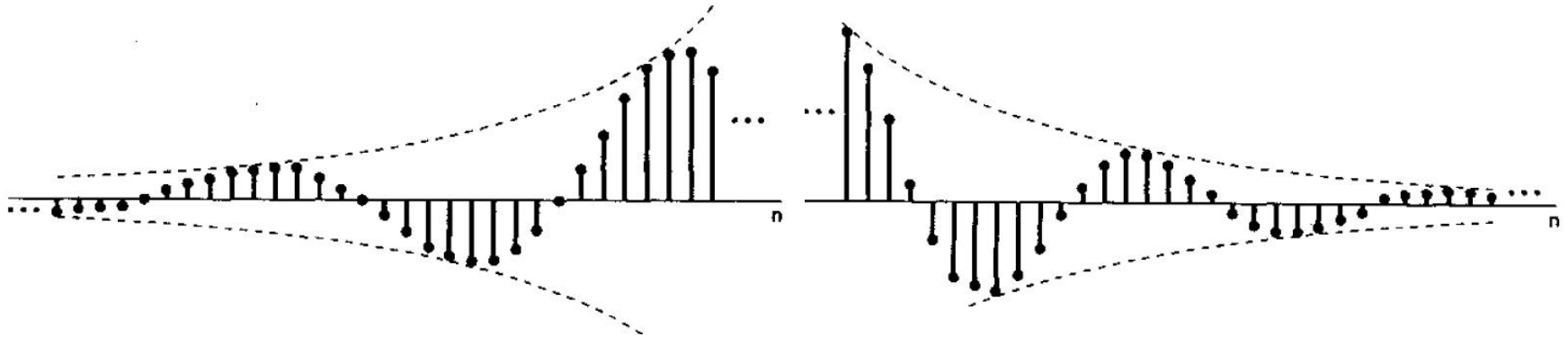
1.3.2 DT Exponential signals

- Complex DT exponential signals: $x[n] = C\alpha^n$

C, α are generally complex numbers

$$C = Ae^{j\phi} \quad \alpha = |\alpha|e^{j\omega_0}$$

So $x[n] = A|\alpha|^n e^{j(\omega_0 n + \phi)} = A|\alpha|^n \cos(\omega_0 n + \phi) + jA|\alpha|^n \sin(\omega_0 n + \phi)$



$|\alpha| > 1$: growing amplitude $|\alpha| < 1$: growing amplitude

$|\alpha| = 1$: Sinusoidal

1.3.2 DT Exponential signals

- Periodicity Properties of DT complex exponential:

While there are many similarities between CT and DT signals, there are differences in exponentials:

The oscillation rate increases with the increase of frequency

$$x(t) = e^{j\omega_0 t} \neq e^{j(\omega_0 + 2\pi k)t}$$

However, for DT, the oscillation rate does not continuously increase as the frequency increases. The period is 2π .

$$x[n] = e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n} = \dots = e^{j(\omega_0 + 2k\pi)n} = e^{j\omega_0 n}$$

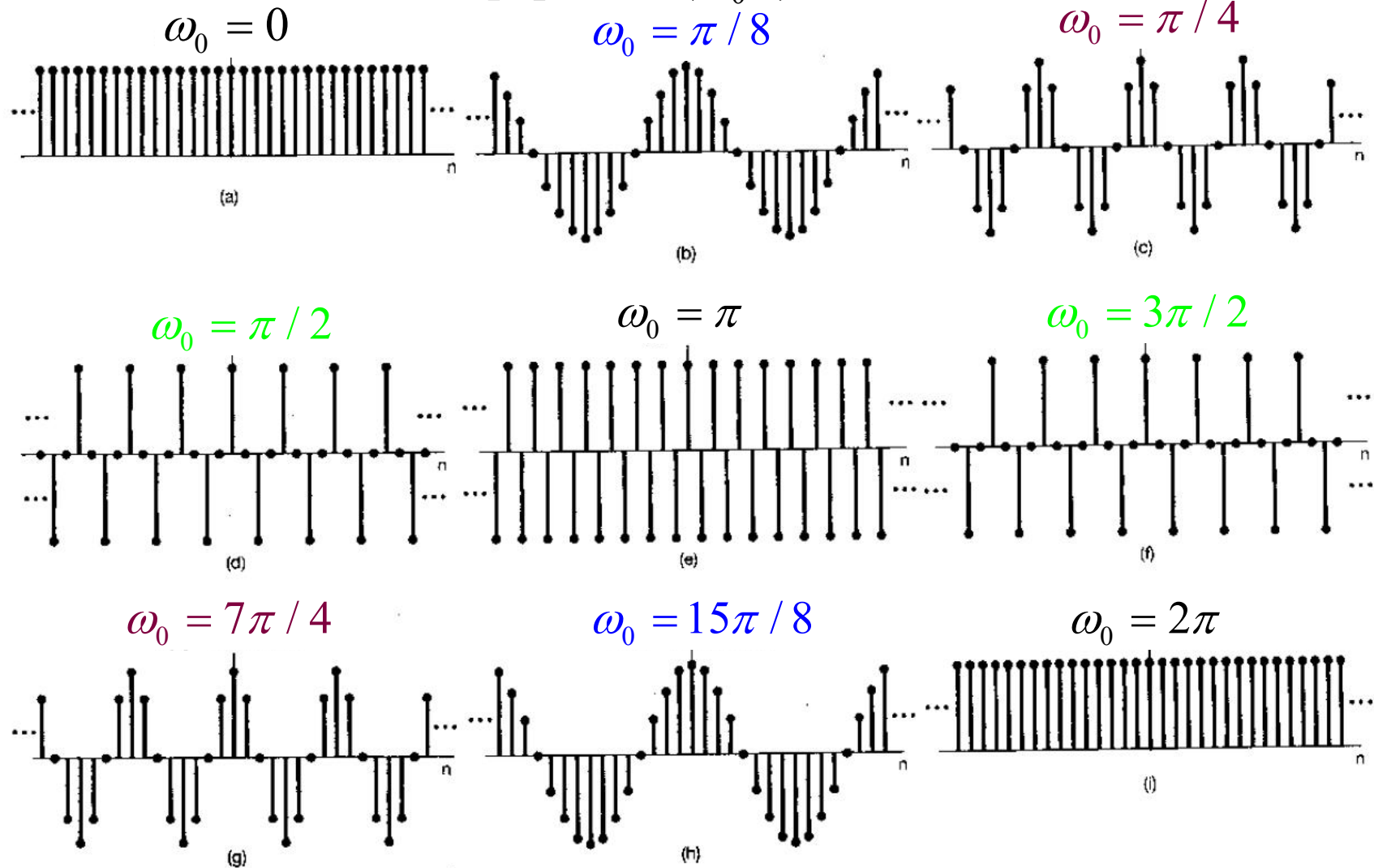
Therefore, we can observe the DT exponentials in any interval of 2π .

Normally

$$0 \leq \omega_0 < 2\pi \quad \text{or} \quad -\pi \leq \omega_0 < \pi$$

1.3.2 DT Exponential signals

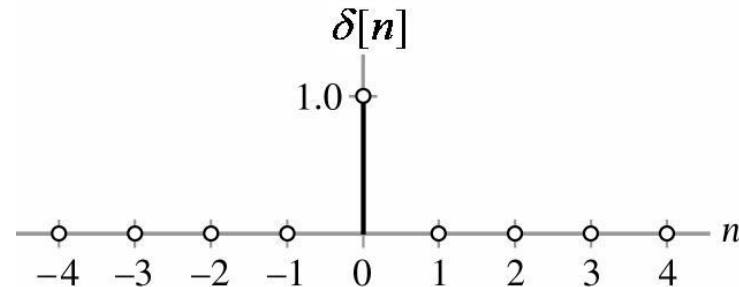
$$x[n] = \cos(\omega_0 n)$$



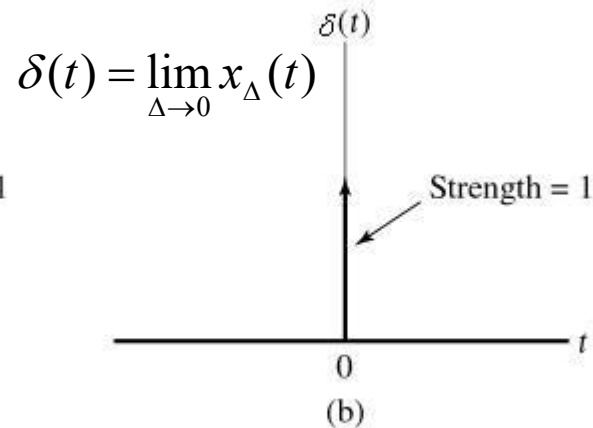
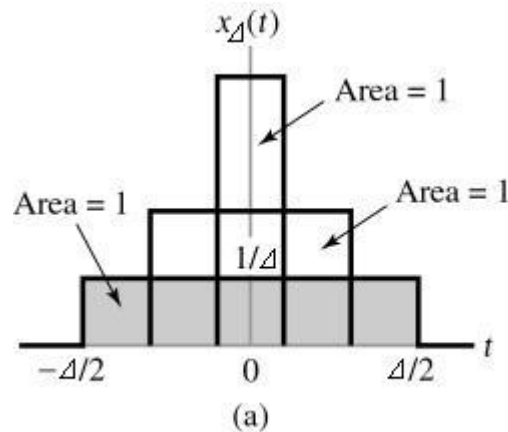
1.3.3 Unit impulse

Unit impulse

$$\text{DT} \quad \delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



$$\text{CT} \quad \begin{cases} \int_{-\infty}^{\infty} \delta(t) dt = 1 \\ \delta(t) = 0, & t \neq 0 \end{cases}$$

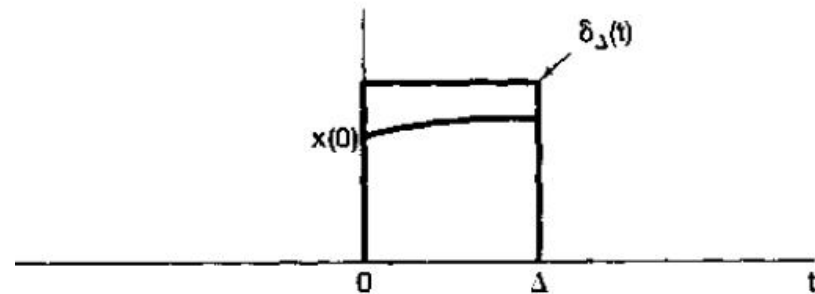
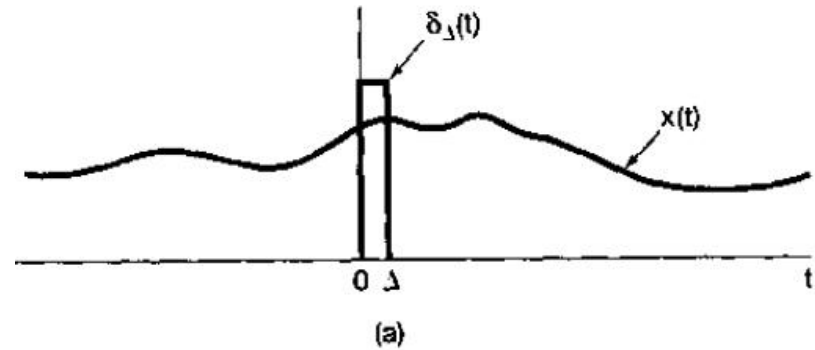
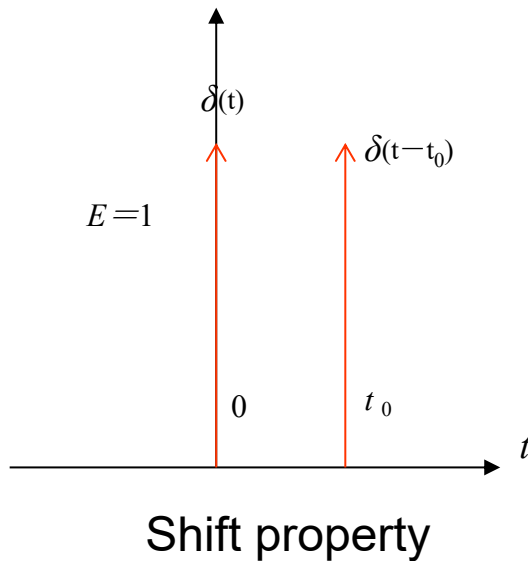


1.3.3 Unit impulse

Sampling property of Impulse

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0) \int_{-\infty}^{\infty} \delta(t-t_0)dt = x(t_0)$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$

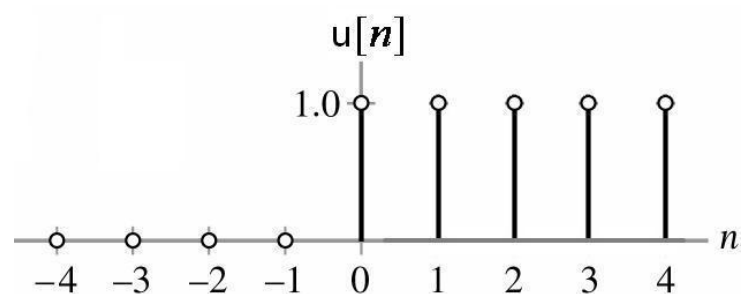


1.3.4 Unit step

Unit step

DT

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$
$$u[n] = \sum_{k=0}^{\infty} \delta[n - k]$$
$$\delta[n] = u[n] - u[n - 1]$$



CT

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$
$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau = \int_0^{\infty} \delta(t - \sigma) d\sigma$$
$$\delta(t) = \frac{du(t)}{dt}$$

