Digital signal processing

Chapter 1. Data sampling and reconstruction

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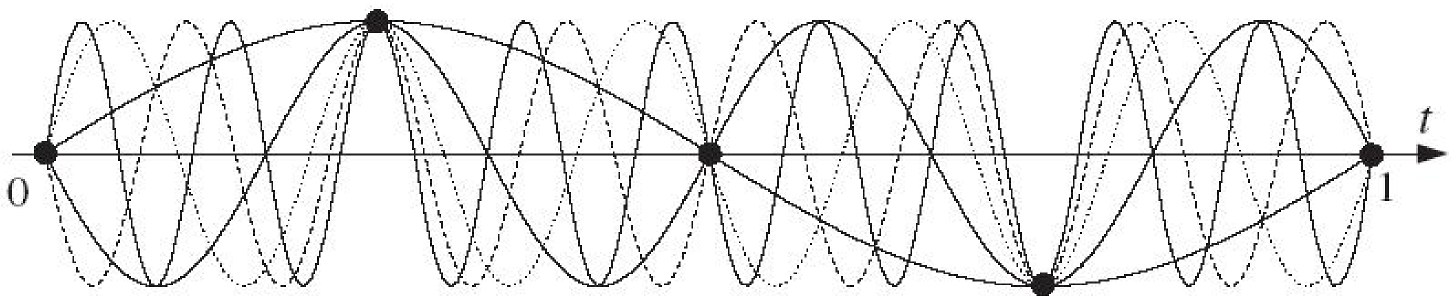
Study Points

- Sampling Theorem
- Antialiasing
- Reconstruction
- Lab 1

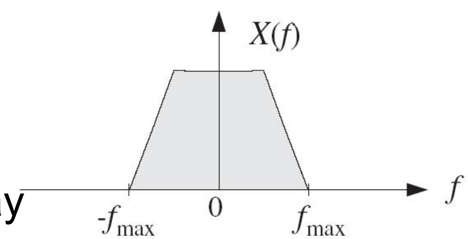
Sampling theorem: Physical meanings

- Without additional conditions, a signal cannot be uniquely represented by a series of samplings:



Sampling theorem states that for accurate representation of a signal x(t) by its time samples x(nT), two conditions:

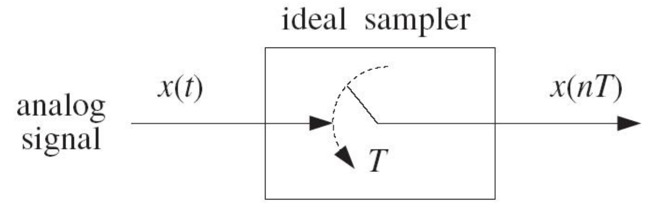
1. x(t) must be bandlimited, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency, say f_{max}, and no frequencies beyond that.



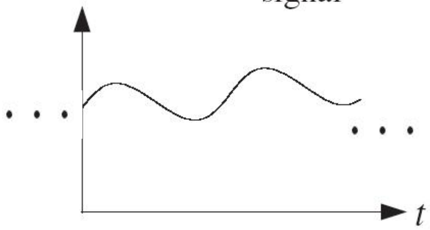
2. The sampling rate f_s must be chosen to be at least twice the maximum frequency f_{max}, that is, $\boxed{f_s > 2 f_{max}}$

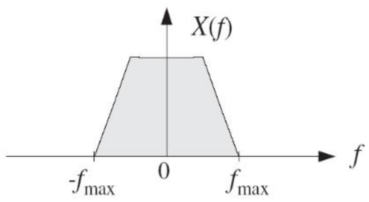
Sampling theorem: illustration

The analog signal $x(t)$ is periodically measured every T seconds.



$\boxed{f_s = 1/T}$



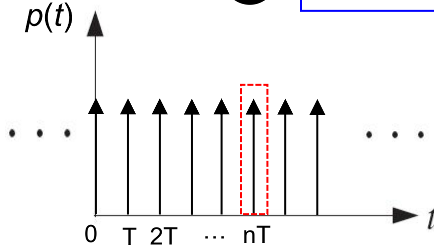


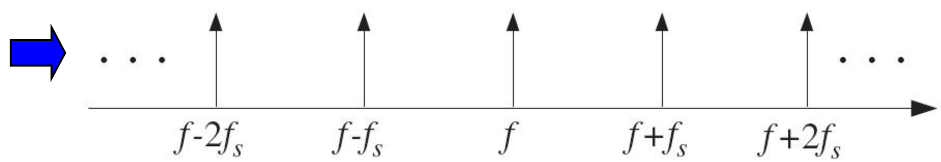
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$\otimes$ multiply

$*$ Convolution

$$\frac{1}{T} \sum_{k=-\infty}^{+\infty} \delta(f - kf_s)$$





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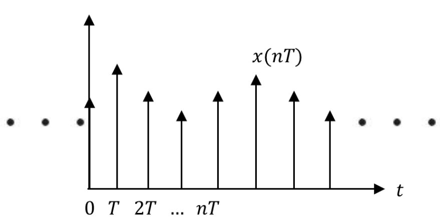
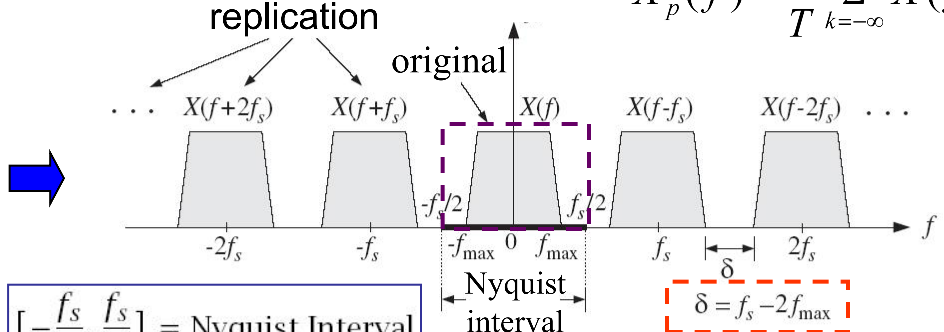
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(t)\delta(t - nT)$$

$$X_p(f) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X(f - kf_s)$$



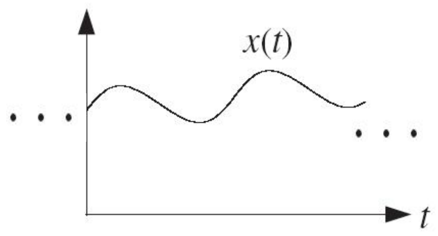


$\boxed{[-\frac{f_s}{2}, \frac{f_s}{2}] = \text{Nyquist Interval}}$

Sampling theorem: illustration detail

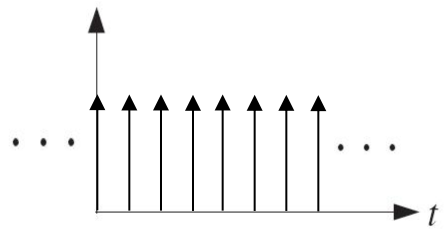
A series of pulses $p(t)$ with a period T are given to sample an arbitrary signal $x(t)$

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$



Multiply $p(t)$ and $x(t)$

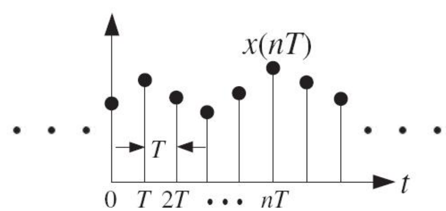
$$x_p(t)=x(t)\,p(t)$$



$$x_p(t)=\sum_{n=-\infty}^{\infty}x(nT)\,\delta(t-nT)$$

The property of Fourier transform is given as

$$\mathcal{F}[x(t)y(t)]=\frac{1}{2\pi}[X(j\omega)*Y(j\omega)]=X(jf)*Y(jf)$$



The Fourier transform of $x_p(t)$ is given as:

$$X_p(j\omega)=\frac{1}{2\pi}X(j\omega)*P(j\omega)$$

Sampling theorem: illustration detail

Define $\omega_s = 2\pi/T$ as the sampling (angle) frequency, the Fourier transform of $p(t)$ can be given as:

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - k\omega_s)$$

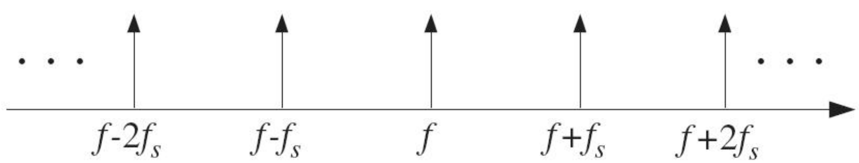
Obtain the Fourier coefficients with inverse transform:

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}$$

Substituting a_k into $P(j\omega)$

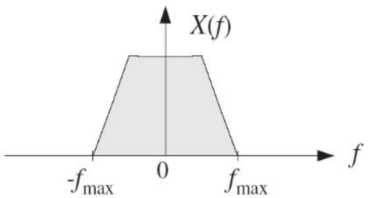
$$P(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - k\omega_s) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

The results of Fourier transform are shown in the figure



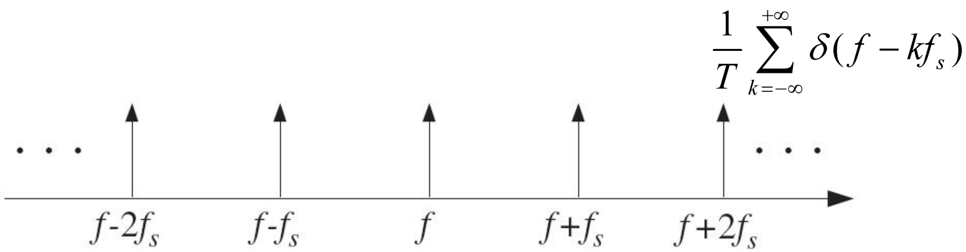
Sampling theorem: illustration detail

Assume the spectrum of $x(t)$ is as shown in the schematic.



$$X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * \left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)\right]\right]$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(j\left(\omega - k\omega_s\right)\right)$$

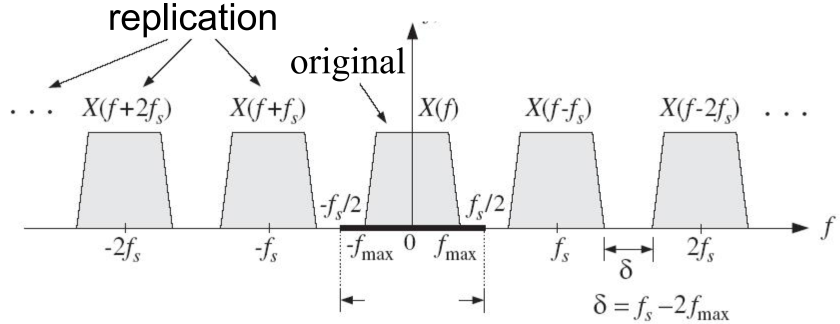


Express the Fourier transform with f_s

$$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f - kf_s)$$

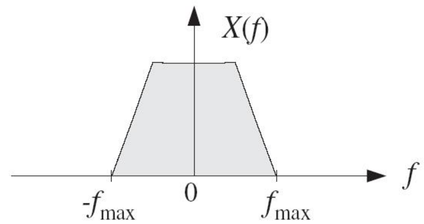
$$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f - kf_s)$$

The result of the convolution is:

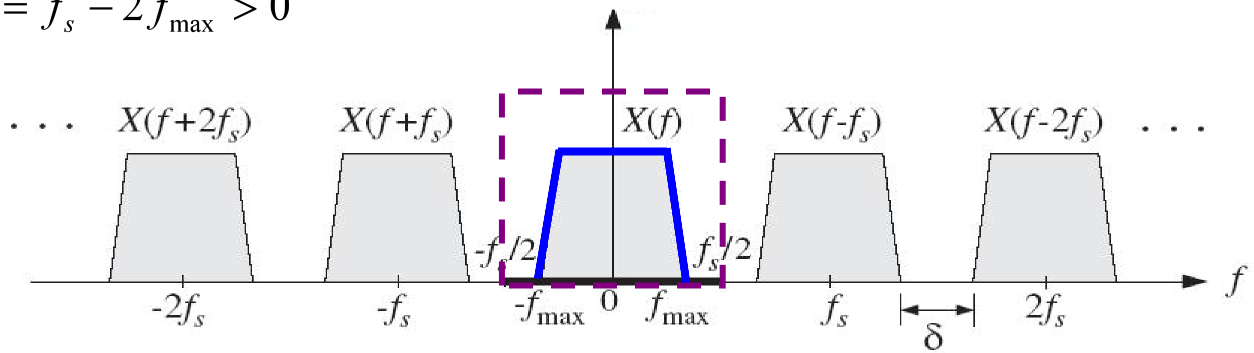


Sampling theorem: Aliasing phenomenon

Original spectrum

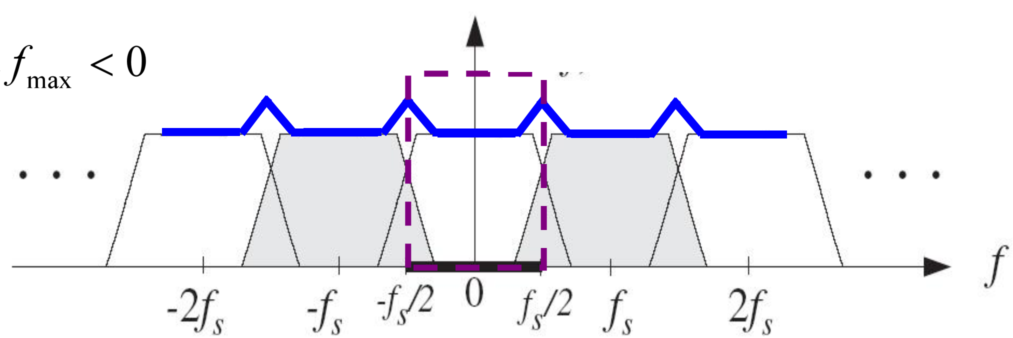


$\delta = f_s - 2f_{\max} > 0$



Spectrum of the sampling signal satisfying Sampling theorem

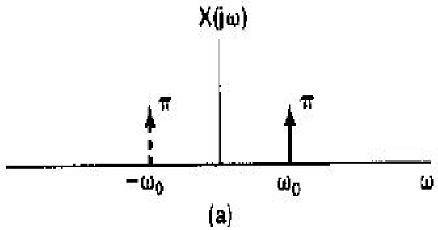
$\delta = f_s - 2f_{\max} < 0$



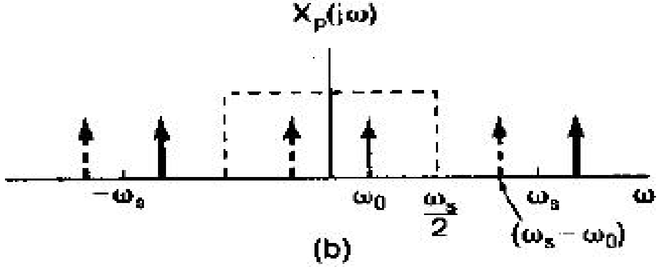
Aliasing phenomenon

Sampling theorem: Aliasing phenomenon

Original signal



Sampled signal with



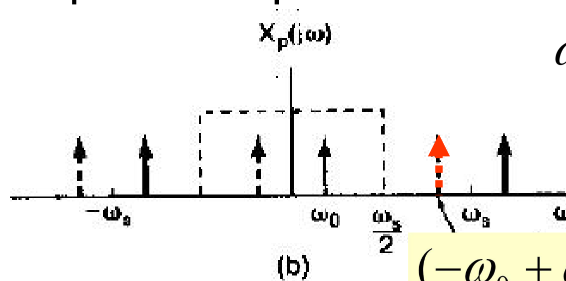
Replicated frequencies

$$\omega_a = (\pm\omega_0 + k\omega_s) \quad k \neq 0$$

$$\boxed{\omega_0 < \frac{\omega_s}{2}\,?}$$

Replicated spectra

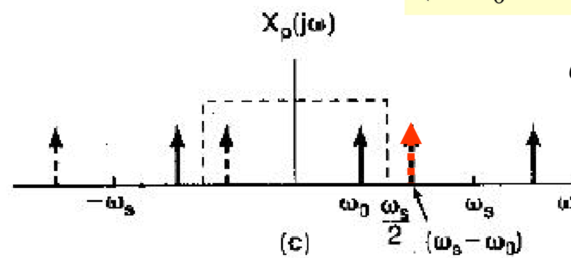
$$\omega_0 = \frac{\omega_s}{6}$$



$$\omega_a = \frac{5}{6}\omega_s$$

$$(-\omega_0 + \omega_s)$$

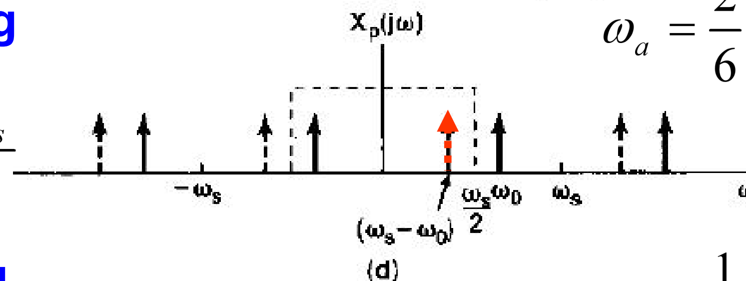
$$\omega_0 = \frac{2\omega_s}{6}$$



$$\omega_a = \frac{4}{6}\omega_s$$

Aliasing

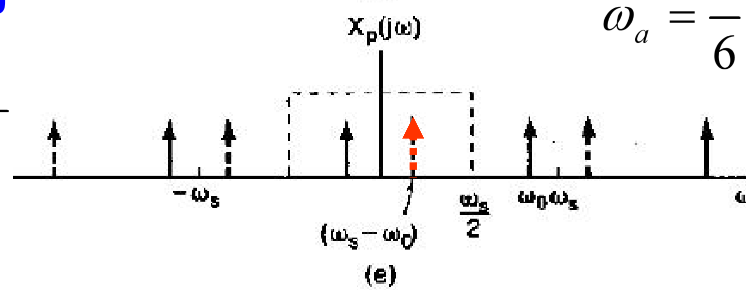
$$\omega_0 = \frac{4\omega_s}{6}$$



$$\omega_a = \frac{2}{6}\omega_s < \frac{1}{2}\omega_s$$

Aliasing

$$\omega_0 = \frac{5\omega_s}{6}$$



$$\omega_a = \frac{1}{6}\omega_s < \frac{1}{2}\omega_s$$

Sampling theorem: Aliasing phenomenon

The signal

$$x(t) = \sin(\pi t) + 4\sin(3\pi t)\cos(2\pi t)$$

where t is in msec, is sampled at a rate of 3 kHz. Determine the signal $x_a(t)$ aliased with $x(t)$. Then, determine two other signals $x_1(t)$ and $x_2(t)$ that are aliased with the same $x_a(t)$, that is, such that $x_1(nT) = x_2(nT) = x_a(nT)$.

Solution: To determine the frequency content of $x(t)$, we must express it as a sum of sinusoids. Using the trigonometric identity $2\sin a\cos b = \sin(a+b) + \sin(a-b)$, we find:

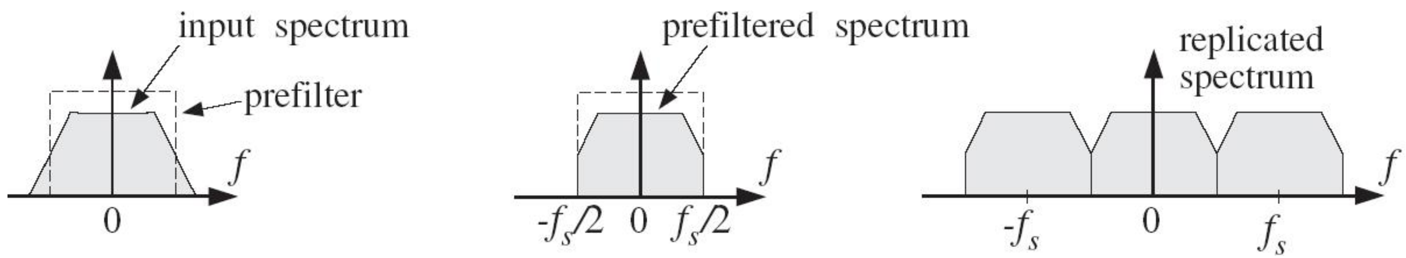
$$x(t) = \sin(\pi t) + 2\left[\sin(3\pi t + 2\pi t) + \sin(3\pi t - 2\pi t)\right] = 3\sin(\pi t) + 2\sin(5\pi t)$$

$$\boxed{f_a = f \bmod (f_s)}$$

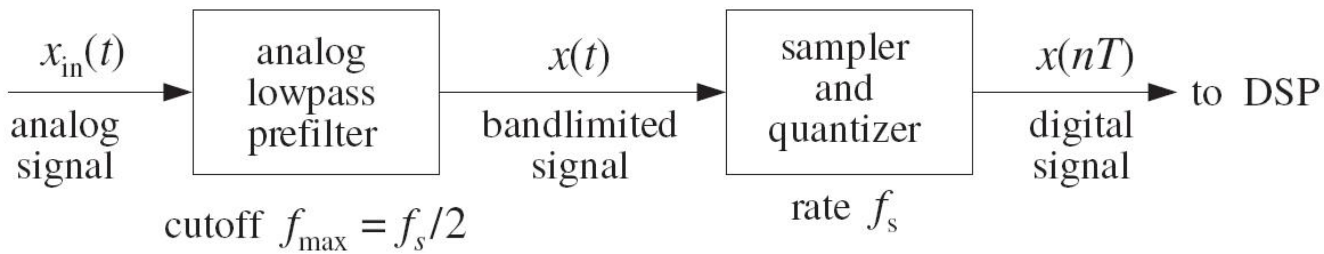
Thus, the frequencies present in $x(t)$ are $f_1 = 0.5$ kHz and $f_2 = 2.5$ kHz. The first already lies in the Nyquist interval $[-1.5, 1, 5]$ kHz so that $f_{1a} = f_1$. The second lies outside and can be reduced mod f_s to give $f_{2a} = f_2 \bmod (f_s) = 2.5 \bmod (3) = 2.5 - 3 = -0.5$. Thus, the given signal will "appear" as:

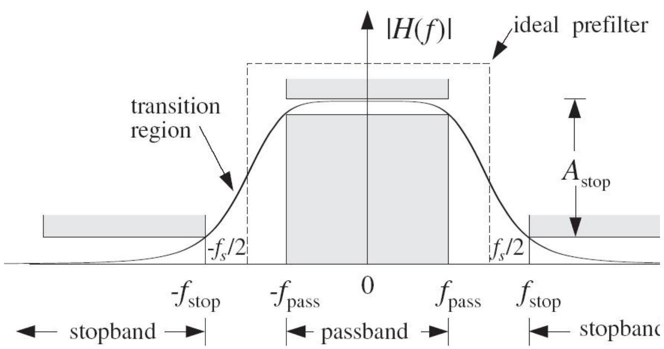
$$\begin{aligned} x_a(t) &= 3\sin(2\pi f_{1a}t) + 2\sin(2\pi f_{2a}t) \\ &= 3\sin(\pi t) + 2\sin(-\pi t) = 3\sin(\pi t) - 2\sin(\pi t) \\ &= \sin(\pi t) \end{aligned}$$

Sampling theorem: Antialiasing prefilter

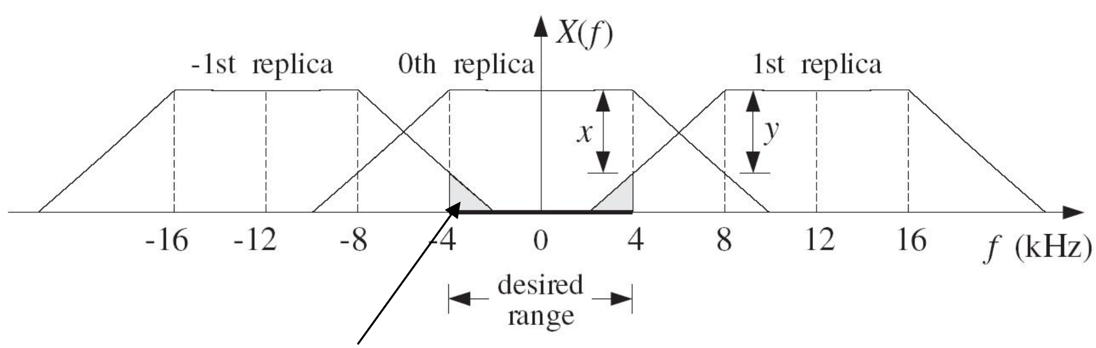


It should be emphasized that the rate f_s must be chosen to be high enough so that, after the prefiltering operation, the surviving signal spectrum within the Nyquist interval $[-f_s/2, f_s/2]$ contains all the significant frequency components for the application at hand.





Practical antialiasing prefilter



Remainders due to practical prefilter

Sampling theorem: Hardware limits

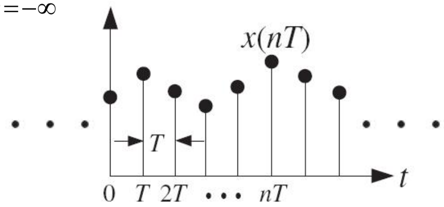
The sampling theorem provides a lower bound on the allowed values of f_s. The hardware used in the application imposes an upper bound.

In any case, there is a total processing or computation time for acquire sampling, data process and reconstruction, say totally $\boldsymbol{T_{proc}}$ seconds required for each sample. Thus, $\boldsymbol{T \geq T_{proc}}$ or, $f_{proc} = 1/T_{proc}$, $\boldsymbol{f_s \leq f_{proc}}$,

$$2f_{max} \leq f_s \leq f_{proc}$$

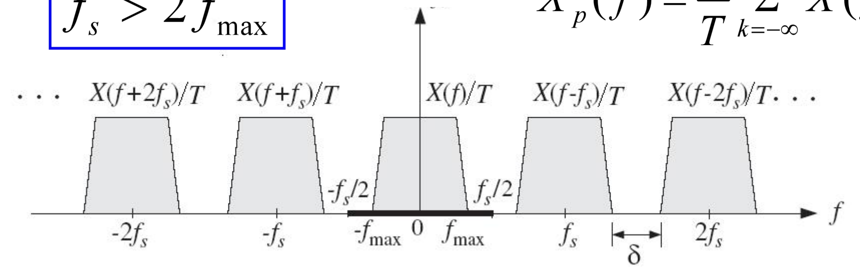
Reconstruction with an ideal low-pass filter

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$$



$$\boxed{f_s > 2f_{\max}}$$

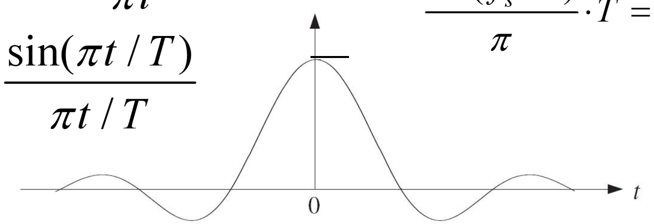
$$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f-kf_s)$$



$$h(t) = T\frac{\sin(\pi t/T)}{\pi t} = \frac{\sin(\pi t/T)}{\pi t/T}$$

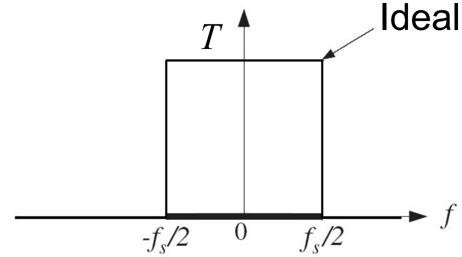
$*$

$$\frac{2\pi(f_s/2)}{\pi}\cdot T = 1$$



$\otimes$

$\mathcal{F}^{-1}$



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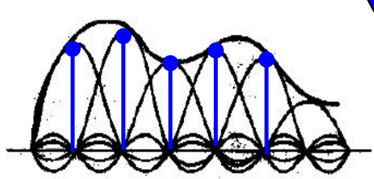
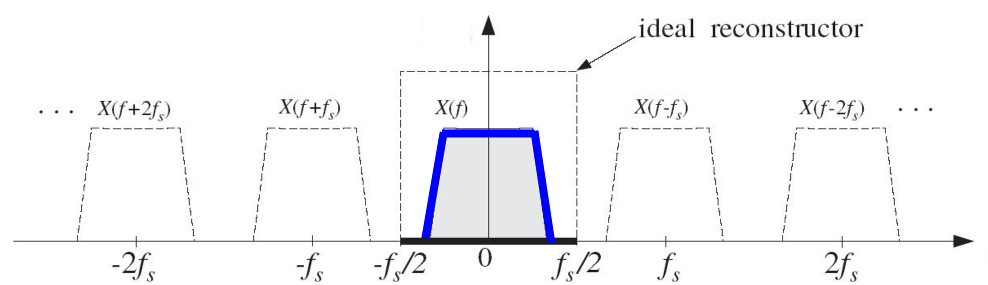
$$y(t) = x_p(t) * h(t)$$
$$= x(t)$$



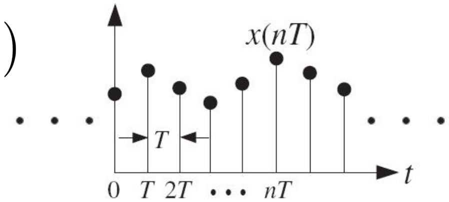
$\mathcal{F}$



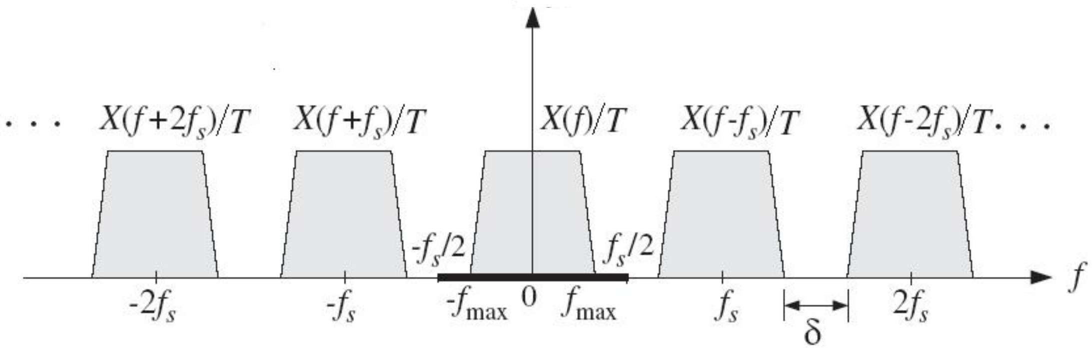
Ideal low-pass filter is uncausal

Reconstruction with an ideal low-pass filter

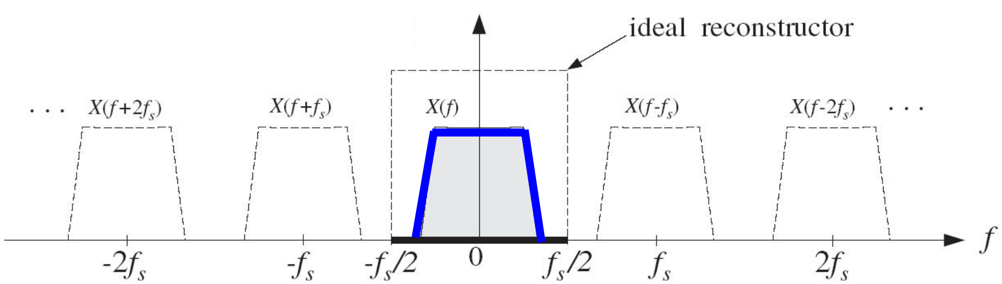
The sampled signal is given as: $$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$



The spectrum of the sampled signal : $$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f-kf_s)$$



In order to restore the signal, a window with the range ($-f_s/2$, $f_s/2$) is given:



Reconstruction with an ideal low-pass filter

The spectrum of an ideal rectangular window in frequency domain is:

$$H(f) = H(j\omega) = \begin{cases} T & ,|\omega| < \omega_c \\ 0 & ,|\omega| > \omega_c \end{cases}$$

The spectrum of *Y* (*t*) can be calculate by :

The property of Fourier transform is given as :

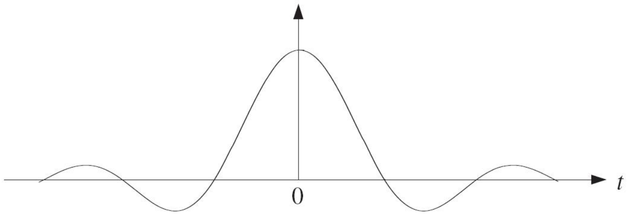
$$\mathcal{F}\left[x(t) * y(t)\right] = X(j\omega)Y(j\omega)$$

y (*t*) can be calculated by :

$$y(t) = x_p(t) * h(t)$$

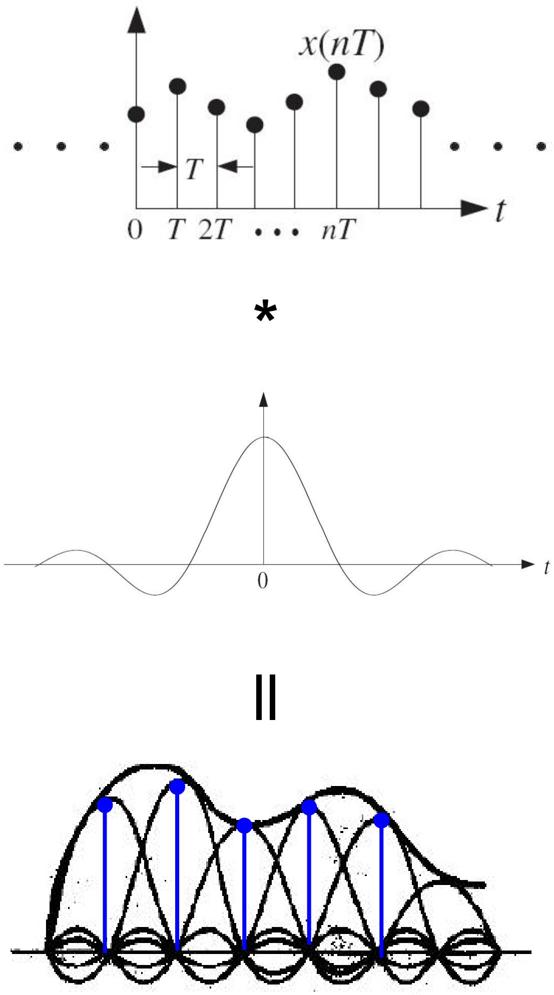
The Inverse Fourier transform of *H* (*f*) is:

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(j\omega)e^{j\omega t}d\omega = \frac{T}{2\pi}\int_{-\frac{\omega_s}{2}}^{+\frac{\omega_s}{2}} e^{j\omega t}d\omega = \frac{\sin(\pi t / T)}{\pi t / T}$$



Reconstruction with an ideal low-pass filter

Then the original signal *y* (*t*) can be calculate by: $y(t) = x_p(t) * h(t)$

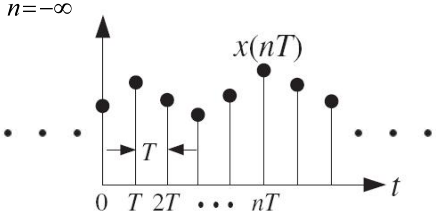


From the Figure given left :

Ideal low-pass filter is non causal

Reconstruction with an non-ideal low-pass filter

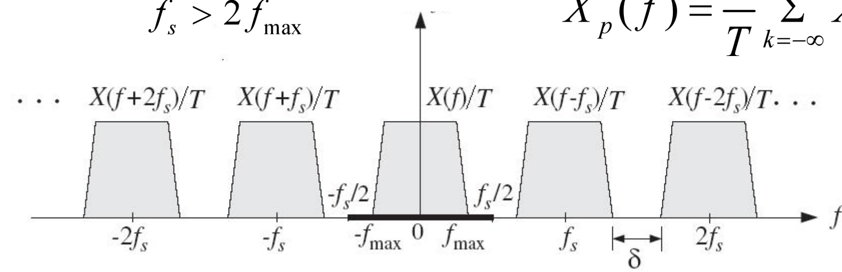
$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$



$\mathcal{F}$

$f_s > 2f_{\max}$

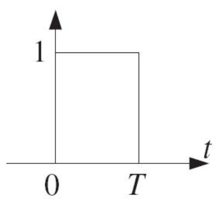
$$X_p(f) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X(f - kf_s)$$



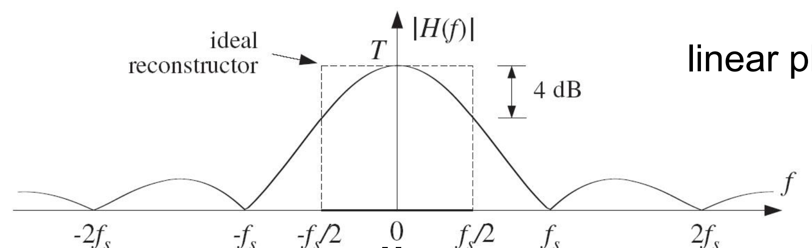
$$\otimes$$

$$h(t) = u(t) - u(t - T)\ *$$

Causal



$\mathcal{F}$



linear phase

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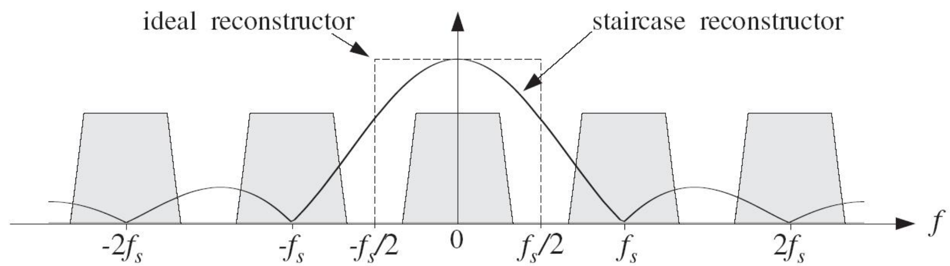
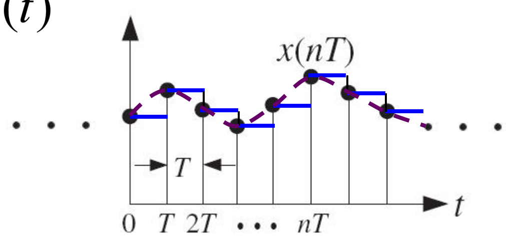
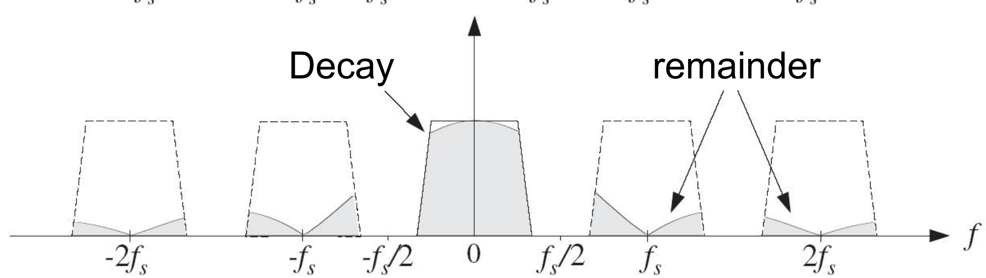
$$y(t) = x_p(t) * h(t)$$

$$\approx x(t)$$



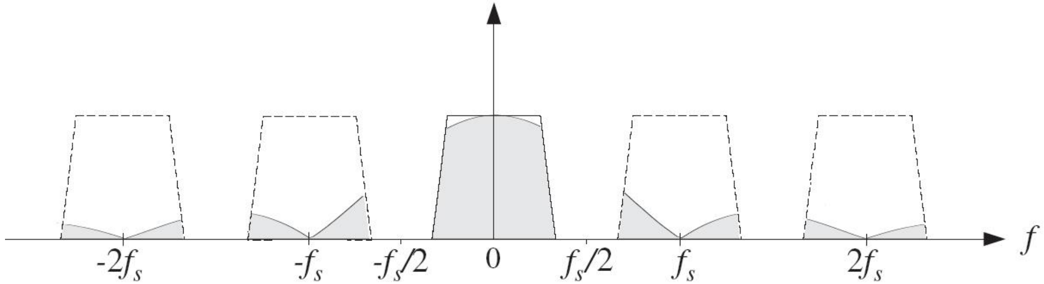




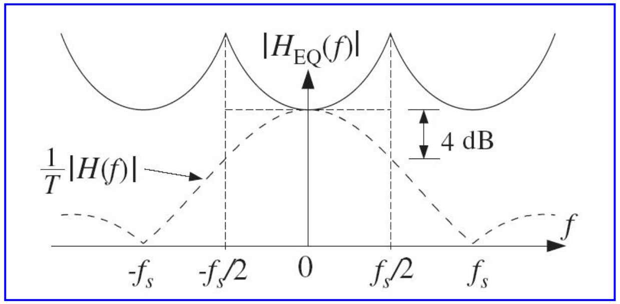
Zero-order holder

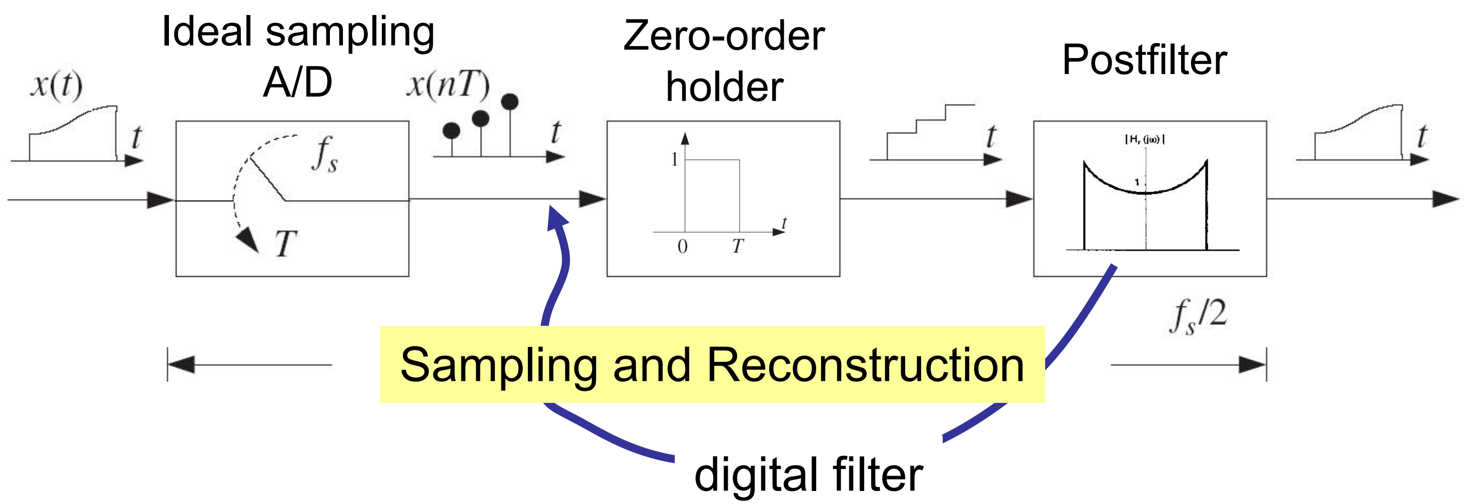
Sampling and Reconstruction without distortion

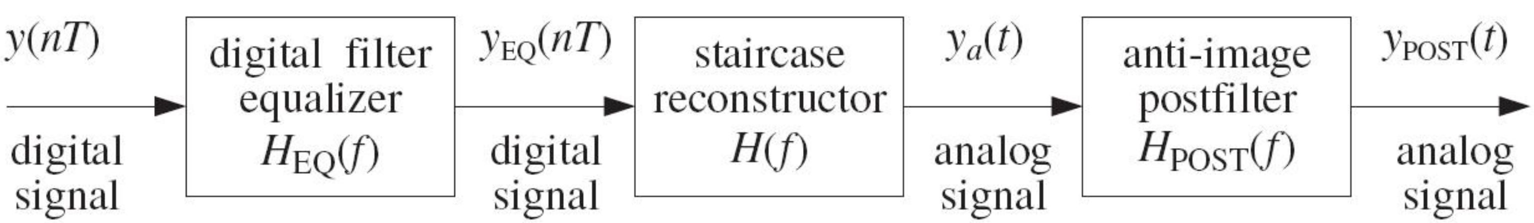
Non-ideal reconstructor



Anti-image postfilter and Equalizer filter

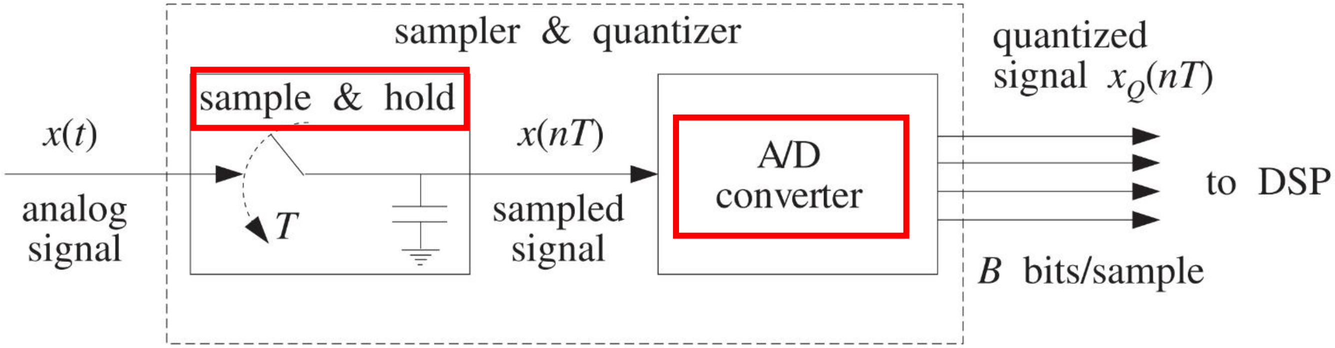






Quantization Process

Analog to Digital



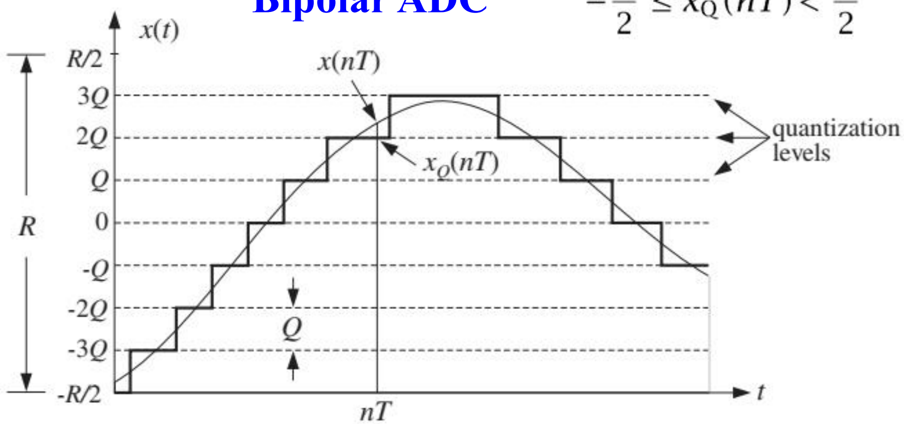
Quantization width or Quantizer resolution Q is defined as

$$Q = \frac{R}{2^B}$$

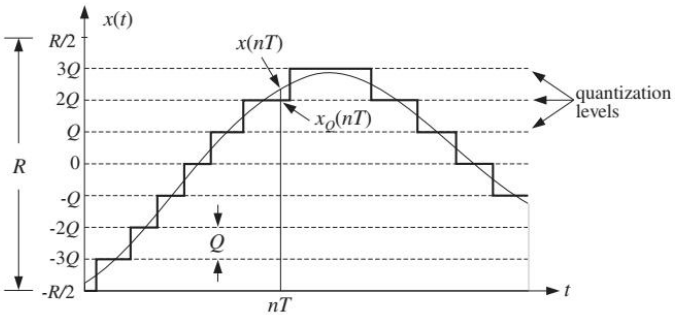
where R is the full-scale range

Bipolar ADC

$$-\frac{R}{2} \leq x_Q(nT) < \frac{R}{2}$$



Quantization Process



Quantization error (rounding)

$$e(nT) = x_Q(nT) - x(nT)$$

$$e = x_Q - x$$

$$-\frac{Q}{2} \le e < \frac{Q}{2}$$

$$\bar{e} = \frac{1}{Q}\int_{-Q/2}^{Q/2} e\, de = 0$$

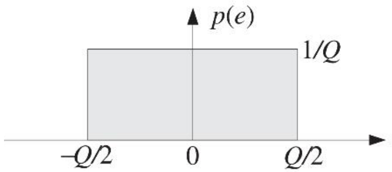
$$\overline{e^2} = \frac{1}{Q}\int_{-Q/2}^{Q/2} e^2\, de = \frac{Q^2}{12}$$

root-mean-square (rms)

$$\boxed{e_{\text{rms}} = \sqrt{\overline{e^2}} = \frac{Q}{\sqrt{12}}}$$

Quantization noise

$$p(e) = \begin{cases} \dfrac{1}{Q} & \text{if } -\dfrac{Q}{2} \le e \le \dfrac{Q}{2} \\ 0 & \text{otherwise} \end{cases}$$



$$\int_{-Q/2}^{Q/2} p(e)\, de = 1$$

$$E[e] = \int_{-Q/2}^{Q/2} e p(e)\, de \quad \text{and} \quad E[e^2] = \int_{-Q/2}^{Q/2} e^2 p(e)\, de$$

Signal to Noise Ratio (SNR)

Dynamic range (6 dB per bit rule)

$$20\log_{10}\left(\frac{R}{Q}\right) = 20\log_{10}(2^B) = B \cdot 20\log_{10} 2,$$

$$\boxed{SNR = 20\log_{10}\left(\frac{R}{Q}\right) = 6B \ \text{ dB}}$$

ADC Device Price

Device	Bits	Frequency	Price
ADC1175-50CIMTX/NOPB	8bit	50M	19.92
ADC081S021CIMF/NOPB	8bit	200k	4.72
TLC1549IP	10bit	38k	28.9
MCP3021A5T-E/OT	10bit	22.3k	4.42
MCP3208	12bit	100k	20.01
TLC2543IDWR	12bit	66k	37.23
MCP3302-CI/ST	13bit	100k	27.4
AD9240ASZRL	14bit	10M	81.54
AD7616BSTZ	16bit	1M	94.96
ADS1115IDGSR	16bit	860k	23.58
PCAP01AD	17bit	500k	45.94
AD7608BSTZ	18bit	200k	251.46
MCP3422A0-E/MS	18bit	37.5k	19.19
AD7789BRMZ	19bit	16.6k	22.89
AD7781BRUZ	20bit	16.7k	96.08

https://www.szlcsc.com/news/13434.html?c=Z(立创商城）