Chapter 2

Linear and Time-invariant Systems

2.0 Introduction

In this chapter, we introduce and discuss a number of methods for describing the relationship between the input and output signals of linear time-invariant (LTI) systems in the time domain.

Due to the superposition property of a linear system, we can represent the input in terms of a linear combination of a set basic signals, then we can use superposition to compute the output of the system in terms of its responses to these basic signals.

2.1 Convolution sum for discrete-time LTI systems

2.2 Convolution integral for continuous-time LTI systems

2.3 Properties of LTI systems

2.4 Linear constant-coefficient difference/differential equations

2.1 Convolution sum

An arbitrary discrete-time signal can be thought of as a weighted superposition of shifted (discrete-time) impulses.

$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots$$

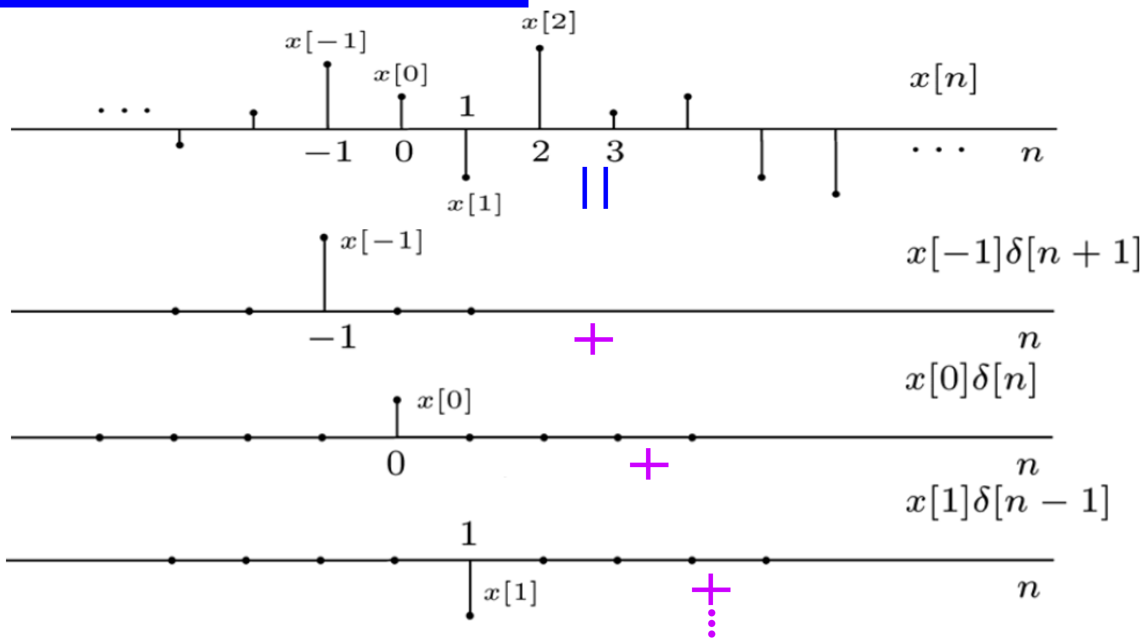
$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$x[n]$: the entire signal

$x[k]$: a specific value of the signal $x[n]$ at time k.

For any value of n, only one of the terms on the right-hand side of the equation is nonzero, and the scaling associated with that term is precisely $x[n]$

Weighted sum of scaled impulses



$$x[n] = \cdots + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + \cdots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$x[n]$: the entire signal, $x[k]$: a specific value of the signal $x[n]$ at time k.

2.1 Convolution sum

Impulse response $\delta[n] \rightarrow h[n]$

Shift impulse response $\delta[n-k] \rightarrow h_k[n]$

Time invariance

$$\boxed{\delta[n-k] \rightarrow h_k[n] = h[n-k]}$$

Linear-Scaling $x[k]\delta[n-k] \rightarrow x[k]h_k[n]$

Linear-Superposition:

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$

$$\qquad || \qquad\qquad\qquad\qquad ||$$

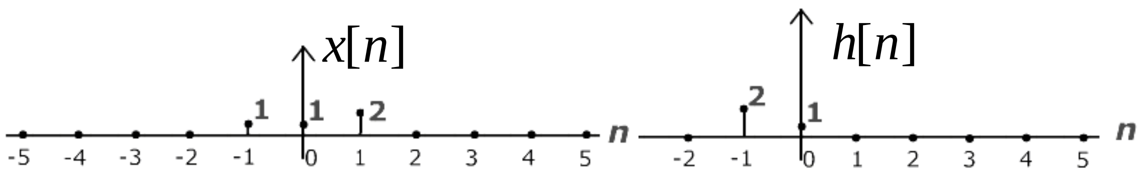
$$x[n] \qquad \rightarrow \qquad y[n]$$

Convolution sum

only for **Linear** and **Time-invariant** DT systems

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \triangleq x[n]*h[n]}$$

Convolution sum: LTI form

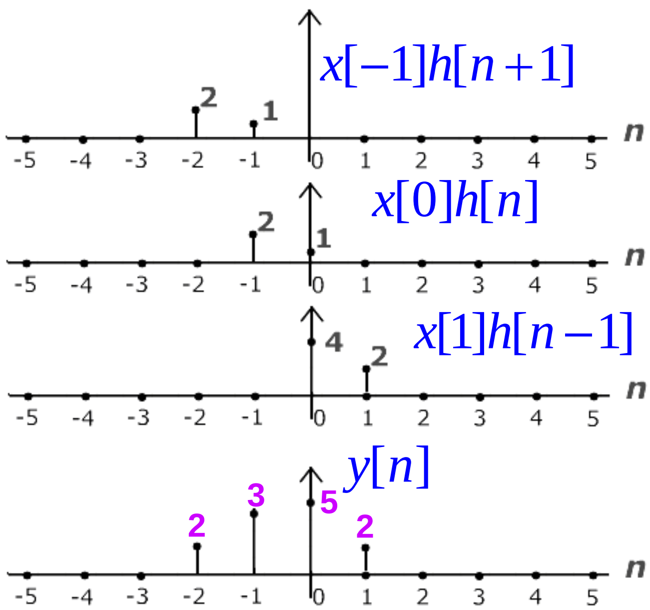


$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

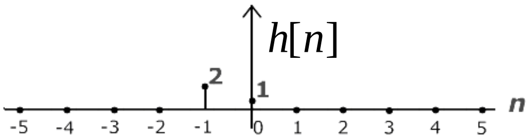
$$= x[-1]h[n+1]$$

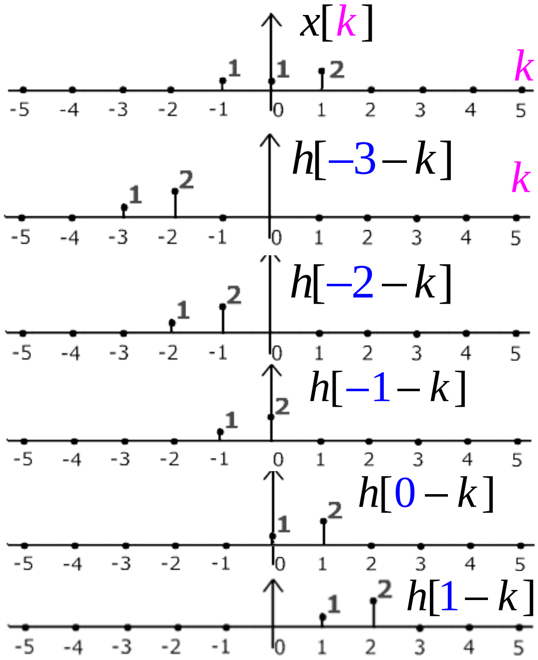
$$+ \ x[0]h[n]$$

$$+ \ x[+1]h[n-1]$$



Convolution sum: Time reversed and shifted form





$$y[-3]=\sum_{k=-\infty}^{\infty}x[k]h[-3-k]=0$$

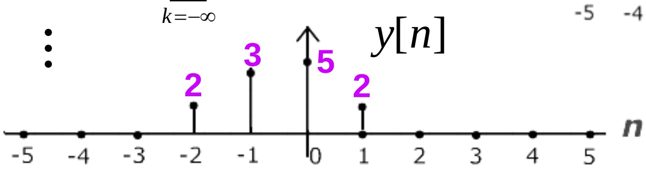
$$y[-2]=\sum_{k=-\infty}^{\infty}x[k]h[-2-k]=2$$

$$y[-1]=\sum_{k=-\infty}^{\infty}x[k]h[-1-k]=3$$

$$y[0]=\sum_{k=-\infty}^{\infty}x[k]h[-k]=5$$

$$y[1]=\sum_{k=-\infty}^{\infty}x[k]h[1-k]=2$$

$\vdots$

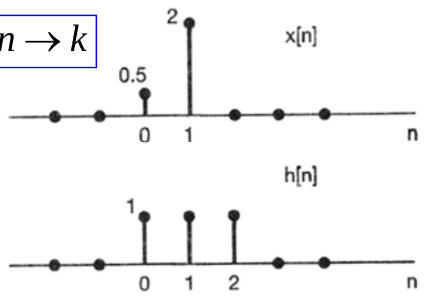


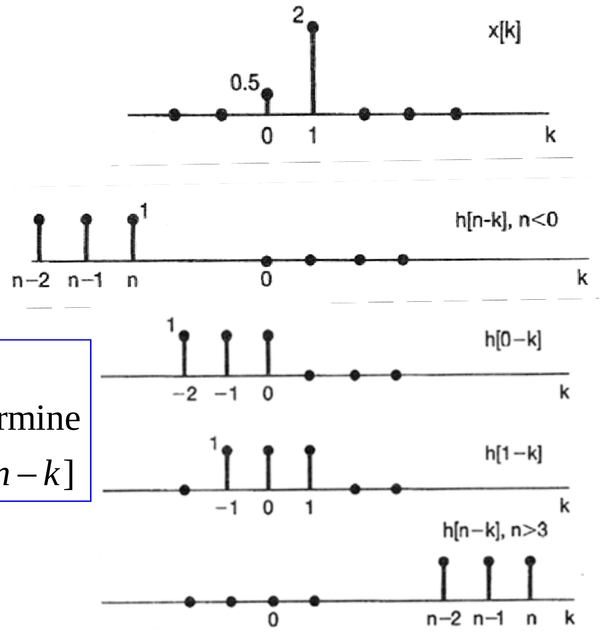
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

Example 1

$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=x[0]h[n-0]+x[1]h[n-1]=0.5h[n]+2h[n-1]$$

Step1: $n \rightarrow k$





Step2:
$n \rightarrow k$, draw $h[n-k]$ with n that determine the overlap area between $x[k]$ and $h[n-k]$

Step3:
response $y[n]$ is the overlap area between $x[k]$ and $h[n-k]$

2.2 Convolution integral

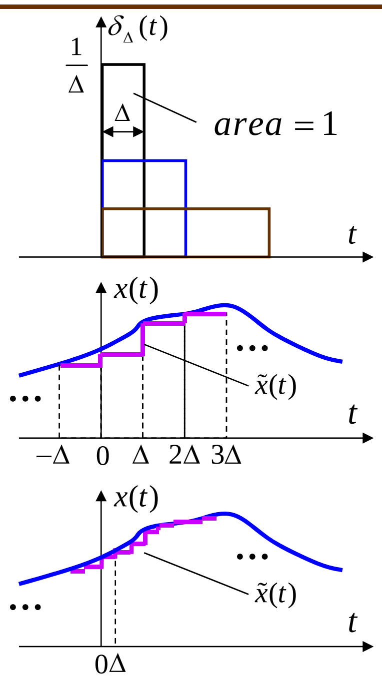
$$\delta_\Delta(t) \triangleq \begin{cases} \dfrac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases}$$

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t): \quad \begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 \\ \delta(t) = 0, \ t \ne 0 \end{cases}$$

$$\hat{x}(t) \triangleq \sum_{k=-\infty}^{\infty} x(k\Delta)[\delta_\Delta(t - k\Delta)\Delta]$$

$$\Delta \to 0: \ \ k\Delta \to \tau, \ \Delta \to \mathrm{d}\tau$$

$$\boxed{x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau}$$



An arbitrary CT signal can be represented by the integral of weighted and shifted unit impulse function $\delta(t)$

2.2 Convolution integral

Impulse response $\quad \delta(t) \to h(t)$

Shift impulse response $\quad \delta(t-\tau) \to h(t-\tau)$

Linear-Scaling $\quad x(\tau)\delta(t-\tau) \to x(\tau)h(t-\tau)$

Linear-Superposition:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau \to y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau$$

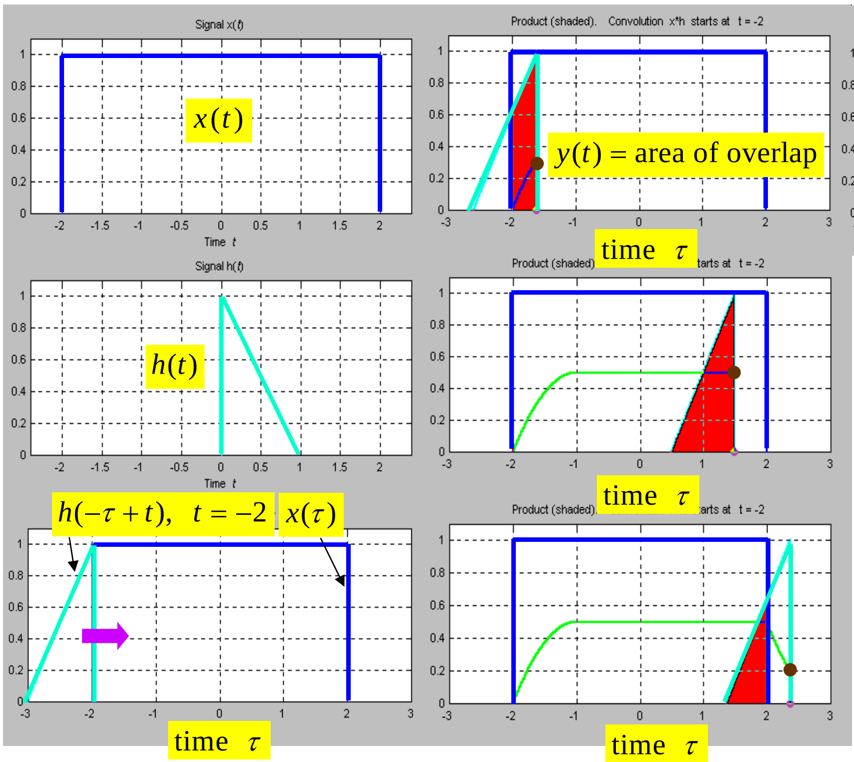
Convolution integral

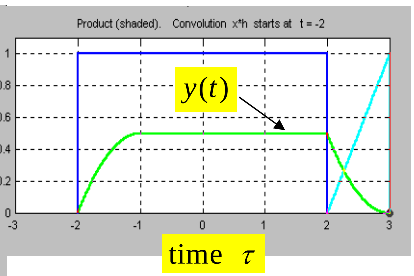
only for **Linear** and **Time-invariant** CT systems

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau \triangleq x(t) * h(t)}$$

Convolution integral: Time reversed and shifted form

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\mathrm{d}\tau \triangleq x(t)*h(t)$$





Steps:

1. reflection
2. shift
3. multiply & integral

2.3 Properties of LTI systems

The characteristics of an **LTI** system are **determined completely** by its unit impulse response, $h(t)$ or $h[n]$.

Following properties are effective for both CT and DT systems.

- Commutative $x(t)*h(t)=h(t)*x(t)$
- Distributive $x(t)*\left[h_1(t)+h_2(t)\right]=x(t)*h_1(t)+x(t)*h_2(t)$
- Associative $x(t)*\left[h_1(t)*h_2(t)\right]=\left[x(t)*h_1(t)\right]*h_2(t)$
- Time shifting $y(t-t_1-t_2)=x(t-t_1)*h(t-t_2)$
- LTI systems without memory
- Invertibility of LTI systems
- Causality for LTI systems
- Stability for LTI systems

2.3.1 Interconnection properties of LTI systems

- Commutative $x(t)*h(t)=h(t)*x(t)$

$$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

$$\overset{\lambda=t-\tau}{=}-\int_{\infty}^{-\infty}x(t-\lambda)h(\lambda)d\lambda=\int_{-\infty}^{\infty}h(\lambda)x(t-\lambda)d\lambda$$

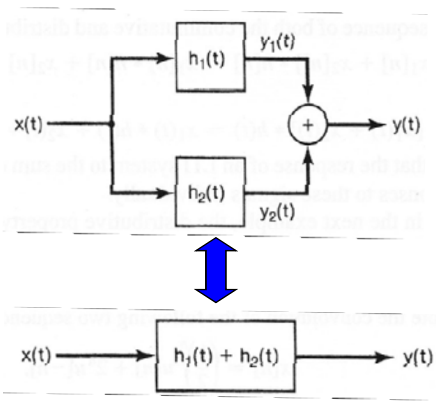
$$=h(t)*x(t)$$

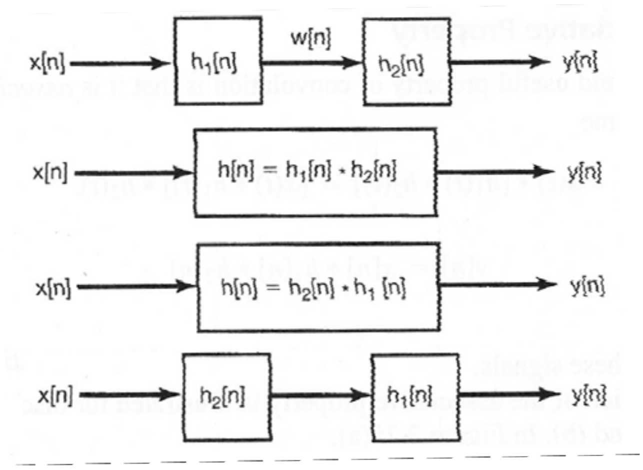
- Distributive $x(t)*\left[h_1(t)+h_2(t)\right]=x(t)*h_1(t)+x(t)*h_2(t)$

- Associative $x(t)*\left[h_1(t)*h_2(t)\right]=\left[x(t)*h_1(t)\right]*h_2(t)$

2.3.1 Properties of LTI systems

- Commutative $x(t)*h(t)=h(t)*x(t)$
- Distributive $x(t)*\left[h_1(t)+h_2(t)\right]=x(t)*h_1(t)+x(t)*h_2(t)$
- Associative $x(t)*\left[h_1(t)*h_2(t)\right]=\left[x(t)*h_1(t)\right]*h_2(t)$





2.3.2 LTI systems with and without memory

A system is memoryless if its output at any time depends only on the value of the input at that same time, i.e. $y[n]=Kx[n], y(t)=Kx(t)$

K is a constant.

$$\boxed{\begin{aligned}&y[n]=Kx[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k],\quad \gamma\triangleq n-k\\&\Rightarrow\begin{cases}k=n,\ \gamma=0: & h[0]=K\\ k\neq n,\ \gamma\neq 0: & h[\gamma]=0\end{cases}\Rightarrow h[\gamma]=K\delta[\gamma]\end{aligned}}$$

$$\boxed{\begin{aligned}&y(t)=Kx(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\mathrm{d}\tau,\quad \lambda\triangleq t-\tau\\&\Rightarrow\begin{cases}\tau=t,\lambda=0: & \int_{-\infty}^{\infty}h(\lambda)\mathrm{d}\lambda=K\\ \tau\neq t,\lambda\neq 0: & h(\lambda)=0\end{cases}\Rightarrow h(\lambda)=K\delta(\lambda)\end{aligned}}$$

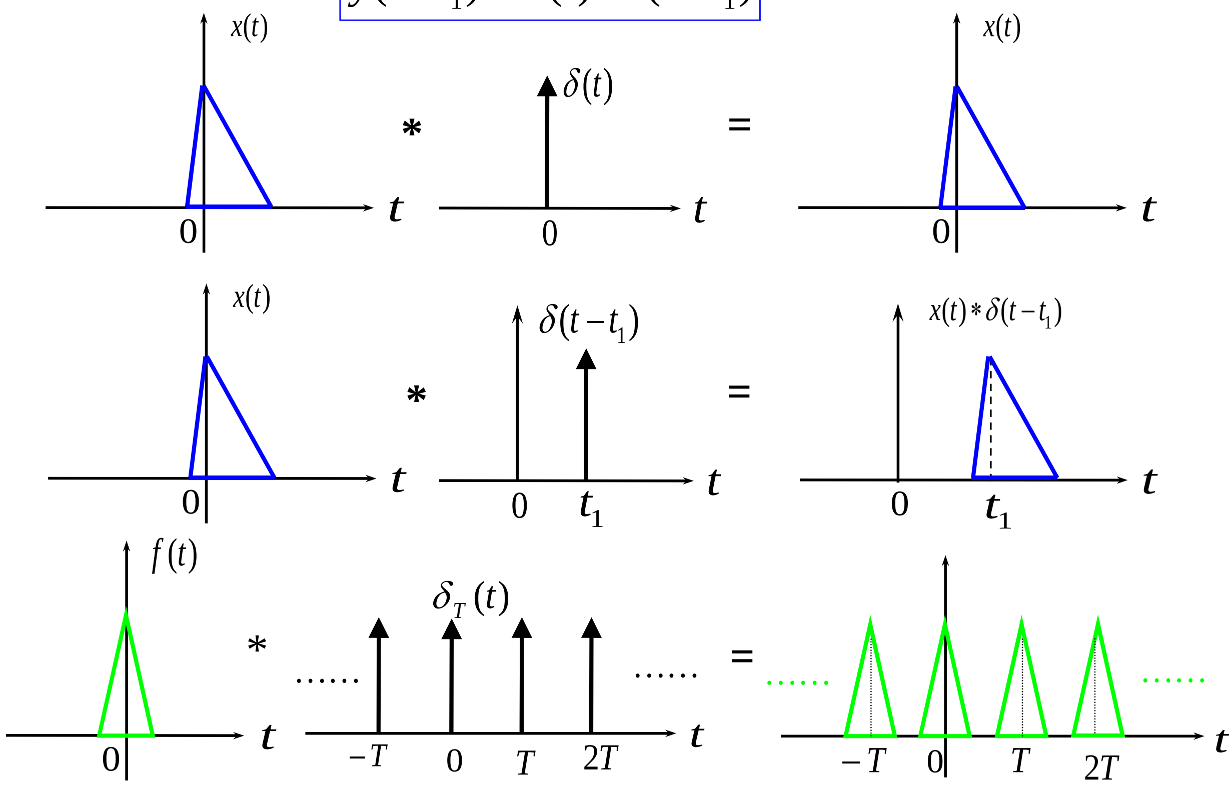
if $K=1$: $h[n]=\delta[t],\ h(t)=\delta(t)$

$y[n]=x[n]*h[n]=x[n]*\delta[n]=x[n]$

$y(t)=x(t)*h(t)=x(t)*\delta(t)=x(t)$

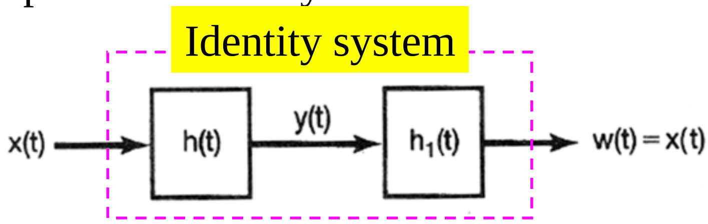
Example 2 Time shifting and convolution with δ function

$$\boxed{y(t-t_1)=x(t)*h(t-t_1)}$$



2.3.2 Invertibility of LTI systems

A system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system



$$y(t) = x(t) * h(t)$$

$$w(t) = y(t) * h_1(t) = x(t) * h(t) * h_1(t) = x(t)$$

$$\Rightarrow h(t) * h_1(t) = \delta(t)$$

Similarly $h[n] * h_1[n] = \delta[n]$

Example 3

Consider the LTI system consisting of a pure time shift $y(t) = x(t - t_0)$, find an inverse system which can recover the input $x(t)$ from the output $y(t)$.

Solution : $y(t) = x(t - t_0) = x(t) * \delta(t - t_0)$

$\Rightarrow h(t) = \delta(t - t_0)$

To recover the input $x(t)$ from the output $y(t)$,

$h_1(t) = \delta(t + t_0)$, then $h(t) * h_1(t) = \delta(t)$.

Thus, the inverse system is

$w(t) = y(t) * h_1(t) = y(t) * \delta(t + t_0) = y(t + t_0)$

Example 4 Consider the LTI system with impulse response

$$h[n] = u[n]$$

Using the convolution sum, the output to arbitray input is

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]u[n-k] = \sum_{k=-\infty}^{n} x[k].$$

find its inverse system.

Solution :

For an system $w[n] = y[n] - y[n-1] = \sum_{k=-\infty}^{n} x[k] - \sum_{k=-\infty}^{n-1} x[k] = x[n],$

its output is the input of the first system.

Thus, this sytem is the inverse system of the first system.

And its impulse response is $h_1[n] = \delta[n] - \delta[n-1]$

To check the identity property:

$$h[n] * h_1[n] = u[n] * (\delta[n] - \delta[n-1]) = u[n] * \delta[n] - u[n] * \delta[n-1]$$
$$= u[n] - u[n\text{-}1] = \delta[n]$$

2.3.3 Causality for LTI systems

The output of a causal system depends only on the present and past values of the input to the system.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Causal: $h[n\text{-}k] = 0,\ k > n$

$\Rightarrow \quad h[\gamma] = 0, \quad \gamma \triangleq n - k < 0$

$\therefore y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$

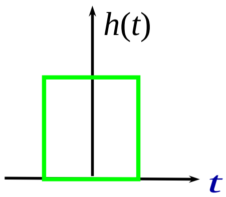
$$y[n] = x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$
$$= \cdots + h[-1]x[n+1] + h[0]x[n] + h[1]x[n-1] + \cdots$$

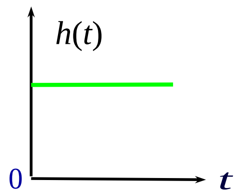
$$\overset{\text{causal}}{=} \quad h[0]x[n] + h[1]x[n-1] + \cdots \Rightarrow h[n] = 0 \text{ for } n < 0$$

$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau,$

Causal: $h(t-\tau) = 0, \ \tau > t$

$\Rightarrow \ h(\lambda) = 0, \ \lambda \triangleq t - \tau < 0$



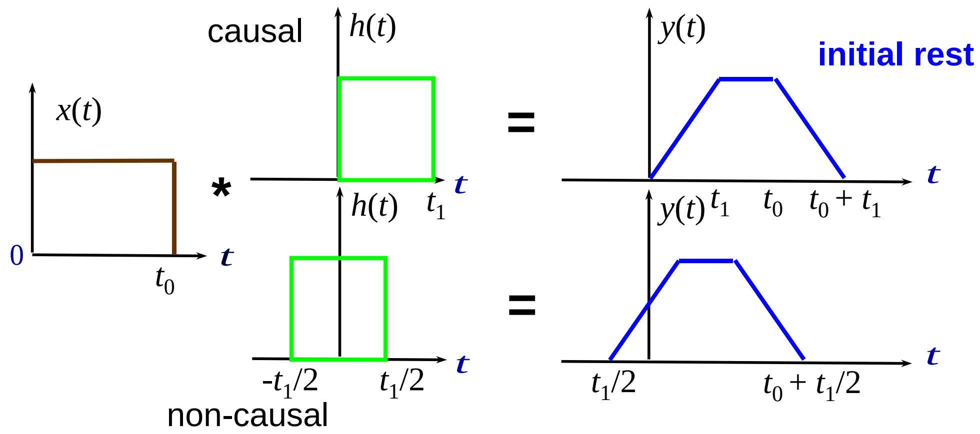


Example 5 Causal for a **linear** system is equivalent to the condition of **initial rest:** i.e. if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time.

$$\text{Causal: } y[n_0] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] = \sum_{k=-\infty}^{n_0} x[k]h_k[n]$$

$$y[n_0] = \ldots + x[n_0-2]h_{n_0-2}[n_0] + x[n_0-1]h_{n_0-1}[n_0] + x[n_0]h_{n_0}[n_0]$$

$$\text{If } x[n]=0, n \le n_0, \text{ then } y[n_0]=0$$



2.3.4 Stability for LTI systems

A system is stable if every bounded input produces a bounded output. An LTI system is stable **if and only if**

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty, \quad \int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

For any limited input $|x[n]| \le B$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$\le \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le \sum_{k=-\infty}^{\infty} |h[k]|B = B\sum_{k=-\infty}^{\infty} |h(k)|$$

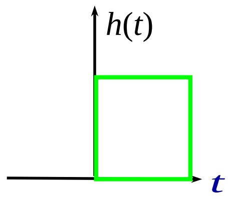
If $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$, then $|y[n]| < \infty$ for all n **Sufficient conditions**

2.3.5 Unit step response of an LTI system

The unit step response of an LTI system is the sum/integral of its unit impulse responses.

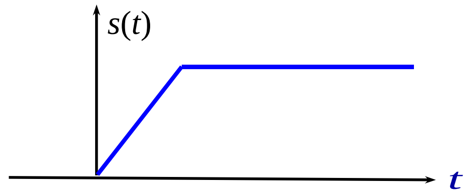
$$s[n] = h[n] * u[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$



$$s(t) = h(t) * u(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$



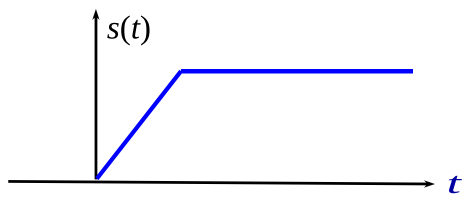
2.3.5 Unit step response of an LTI system

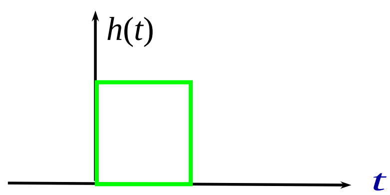
The unit impulse response of an LTI system is the first difference/derivative of its unit step response.

$$s[n] = \sum_{k=-\infty}^{n} h[k] \text{ and } s[n-1] = \sum_{k=-\infty}^{n-1} h[k]$$

$$\Rightarrow h[n] = s[n] - s[n-1]$$

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau \Rightarrow h(t) = \frac{\mathrm{d}s(t)}{\mathrm{d}t}$$





2.4 Causal LTI systems

An extremely important class of CT/DT systems are described by the relationships of inputs and outputs in terms of linear constant coefficient differential/difference equations.

Constant coefficient differential equations

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k} \qquad \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = x(t)$$

Constant coefficient difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad y[n] - 0.5y[n-1] = 2x[n] + 3x[n-1]$$

2.4.1 Linear differential equations

The order N refers to the highest derivative of the output $y(t)$ appearing in the equation.

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

In the case when N=0, $y(t)$ is an explicit function of $x(t)$ and its derivatives.

$$y(t) = b_0 x(t) + b_1 \frac{\mathrm{d}x(t)}{\mathrm{d}t} + b_2 \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2}$$

For N>0, the equation specifies the output implicitly in terms of the input. In order to obtain an explicit expression, we must solve the differential equation.

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = x(t)$$

2.4.1 Linear differential equations

Solve $$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t^k}$$

The complete solution consists of the sum of a homogeneous solution $y_h(t)$ and a particular solution $y_p(t)$: $y(t) = y_h(t) + y_p(t)$

A homogeneous solution y_h(t) is the solution to the differential equation with the input set to zero, referred to as the free response or natural response:

Hypothesize a solution of the form: $y_h(t) = Ae^{st} \Rightarrow \frac{\mathrm{d}^k Ae^{st}}{\mathrm{d}t^k} = s^k \left(Ae^{st}\right)$

Then, $\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y_h(t)}{\mathrm{d}t^k} = 0$ $\quad Ae^{st} \sum_{k=0}^{N} a_k s^k = 0 \quad \Rightarrow \quad \sum_{k=0}^{N} a_k s^k = 0$

Natural radian frequencies s_k $\quad y_h(t) = \sum_{k=1}^{N} A_k e^{s_k t}$

We need N initial conditions to determine A_k

2.4.1 Linear differential equations

Solve $$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{\mathrm{d}t^k}$$

A particular solution y_p(t) is the solution to the differential equation which has the form same as the input signal, referred to as the forced response:

Input	Particular Solution
1	c
t^m	$c_m t^m + c_{m-1} t^{m-1} + \cdots + c_1 t + c_0$
e^{at}	ce^{at}
$\cos(\omega t + \varphi)$	$c\cos(\omega t + \phi) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$

So the whole solution is $\quad y(t) = \sum_{k=1}^{N} A_k e^{s_k t} + y_p(t)$

N initial conditions to determine A_k 。

Example 6 $y''(t)+3y'(t)+2y(t)=x'(t)+2x(t)$

Input and initial conditions are $x(t)=t^2,\ y(0)=1,\ y'(0)=1$

Homogeneous solution $y_h(t)=Ae^{st}$ $Ae^{st}\sum_{k=0}^{N}a_k s^k=0 \Rightarrow \sum_{k=0}^{N}a_k s^k=0$

$s^2+3s+2=0 \ \therefore \ s_1=-1,\ s_2=-2$

$\therefore \ y_h(t)=A_1e^{-t}+A_2e^{-2t}$

Assume a particular solution $y_p(t)=P_2t^2+P_1t+P_0$

Substitute the particular solution into the differential equation with the input $x(t)=t^2$, gives $2P_2+3(2P_2t+P_1)+2(P_2t^2+P_1t+P_0)=2t+2t^2$

Compare the coefficients, determines $P_0=2,\ P_1=-2,\ P_2=1 \ \therefore y_p(t)=t^2-2t+2$

Whole solution: $y(t)=y_h(t)+y_p(t)=A_1e^{-t}+A_2e^{-2t}+t^2-2t+2$

According to the initial conditions: $A_1=1, A_2=-2$

So the whole solution: $y(t)=\underbrace{\left(e^{-t}-2e^{-2t}\right)}_{\text{free response}}+\underbrace{\left(t^2-2t+2\right)}_{\text{forced response}},t\geq 0$

2.4.1 Linear differential equations

The output of a system can be expressed as the sum of two components: one associated only with the initial conditions (the zero-input response), the other due only to the input signal (the zero-state response).

$$y(t)=y_{zp}(t)+y_{zs}(t)$$

Zero-input response $y_{zp}(t)$ has the form of the homogeneous solution.

Zero-state response $y_{zs}(t)$ has the form of the whole solution, i.e. the sum of the homogeneous solution and the input.

Example 7

$$y''(t)+3y'(t)+2y(t)=x'(t)+2x(t)$$

Input and initial conditions are $x(t)=t^2 \quad (t\ge 0), \quad y(0)=1,\ y'(0)=1$

Zero input solution: $y_{zi}(t)=Be^{st}$

$$s^2+3s+2=0 \qquad s_1=-1,\ s_2=-2 \qquad \therefore \quad y_{zi}(t)=B_1e^{-t}+B_2e^{-2t},\ t\ge 0$$

$$\begin{cases} B_1+B_2=1 \\ -B_1-2B_2=1 \end{cases} \quad \begin{cases} B_1=3 \\ B_2=-2 \end{cases} \qquad \therefore \quad y_{zi}(t)=3e^{-t}-2e^{-2t}, t\ge 0$$

Zero condition solution: $y_{zs}(t)=y_h(t)+y_p(t)=C_1e^{-t}+C_2e^{-2t}+t^2-2t+2$

According to the zero initial conditions: $y(0)=y'(0)=y''(0)=\ldots=0$

$$\begin{cases} C_1+C_2+2=0 \\ -C_1-2C_2-2=0 \end{cases} \Rightarrow \begin{cases} C_1=-2 \\ C_2=0 \end{cases} \qquad \therefore y_{zs}(t)=-2e^{-t}+t^2-2t+2$$

$$\boxed{\begin{aligned} y(t)&=y_{zi}(t)+y_{zs}(t)=\left(3e^{-t}-2e^{-2t}\right)+\left(-2e^{-t}+t^2-2t+2\right) \\ &=y_h(t)+y_p(t)=\left(e^{-t}-2e^{-2t}\right)+\left(t^2-2t+2\right) \qquad ,t\ge 0 \end{aligned}}$$

2.4.1 Linear differential equations

- The differential equation does not completely specify the output in terms of the input, and we need to identify auxiliary conditions to determine completely the input-output relationship for the system.
- The response of the system can be considered as the sum of the response only due to the input when the initial state is zero and the response only due to the stored energy at the initial state but not due to the input.
- Basically the time origin is defined as the time instant at which $x(t)$ is applied to the system, i.e., we assume that $x(t)=0$ for $t<0$.
- If the system is LTI and causal, before $x(t)$ is applied to the system, the response should be zero $y(t)=0$, $t<0$, and therefore the initial condition of the system is zero state, i.e. initial rest.
- Under the zero state condition, the response of the LTI system can be determined by the convolution of the impulse response and the input signal for $t>0$.

2.4.2 Linear difference equations

The discrete-time Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The difference equations can be solved in a manner exactly analogous to that for differential equations. The solution $y[n]$ can be written as the sum of a particular solution and a solution to the homogeneous equation.

Alternatively, to solve it, rewrite the difference equation into the form as

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \underbrace{\sum_{k=1}^{N} a_k y[n-k]}_{\text{previous outputs}}\right\}$$

It is a recursive equation for $N > 0$ because we need to know the previous outputs to calculate the current output. And the impulse response may be infinite, so it is an infinite impulse response (IIR) system.

$$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k] - \sum_{k=1}^{N} a_k h[n-k]\right\} = \begin{cases} \frac{b_n}{a_0} - \sum_{k=1}^{N} \frac{a_k}{a_0} h[n-k], 0 \le n \le M \\ -\sum_{k=1}^{N} \frac{a_k}{a_0} h[n-k], \quad \text{otherwise} \end{cases}$$

Example 8 $y[n] = 0.5y[n-1]+2x[n]+3x[n\text{-}1]$

Initial rest

$n \le -1, x[n] = 0$ and $y[n] = 0,$

$x = [1, 2, 3], n \ge 0,$

For each new sample $x[n]$, do

$y := 0.5w_1 + 2x + 3v_1$

$w_1 := y$

$v_1 := x$

$Step1:$

$y[0] := 0.5y[-1] + 2x[0] + 3x[-1] = 2x[0] = 2$

$w_1 := y[0] = 2$

$v_1 := x[0] = 1$

$Step2:$

$y[1] := 0.5w_1 + 2x[1] + 3v_1 = 0.5\times 2 + 2\times 2 + 3\times 1 = 8$

$w_1 := y[1] = 8$

$v_1 := x[1] = 2$

$Step3:$

$y[2] := 0.5w_1 + 2x[2] + 3v_1 = 0.5\times 8 + 2\times 3 + 3\times 2 = 16$

$w_1 := y[2] = 16$

$v_1 := x[2] = 3$

$Step4:$

$y[3] := 0.5\times 16 + 2\times 0 + 3\times 3 = 17$

$w_1 := y[3] = 17$

$v_1 := x[3] = 0$

$Step5:$

$y[4] := 0.5\times 17 + 2\times 0 + 3\times 0 = 8.5$

$w_1 := y[4] = 8.5$

$v_1 := x[4] = 0$

$\vdots$

Example 8 $y[n] = 0.5y[n-1] + 2x[n] + 3x[n\text{-}1]$

$$\textit{Initial rest}$$
$$n \le -1, x[n] = 0 \text{ and } y[n] = 0,$$
$$x = [1, 2, 3], n \ge 0,$$

$$h[n] = 0.5h[n-1] + 2\delta[n] + 3\delta[n\text{-}1]$$

$$h[0] = 0.5 \times 0 + 2 \times 1 + 3 \times 0 = 3$$
$$h[1] = 0.5 \times 3 + 2 \times 0 + 3 \times 1 = 4.5$$
$$h[2] = 0.5 \times 4.5 + 2 \times 0 + 3 \times 0 = 0.5 \times 4.5$$
$$\vdots$$
$$h[n] = 0.5^{n-1} \times 4.5, n \ge 1$$

2.4.2 Linear difference equations

If N = 0, it is non-recursive and it is an explicit function with inputs

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

The impulse function can be written as

$$h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k] = \begin{cases} \frac{b_n}{a_0}, 0 \le n \le M \\ 0, \quad \text{otherwise} \end{cases}$$

So the impulse response will last actually finite duration, and it is a *finite impulse response* (FIR) system.

Example 9

$$y[n] = 2x[n] + 3x[n\text{-}1]$$

Initial rest

$n \le -1, x[n] = 0$ and $y[n] = 0,$

$x = [1, 2, 3], n \ge 0,$

For each new sample $x[n]$, do

$y := 2x + 3v_1$

$v_1 := x$

$Step1:$

$y[0] := 2x[0] + 3x[-1] = 2x[0] = 2$

$v_1 := x[0] = 1$

$Step2:$

$y[1] := 2x[1] + 3v_1 = 2\times 2 + 3\times 1 = 7$

$v_1 := x[1] = 2$

$Step3:$

$y[2] := 2x[2] + 3v_1 = 2\times 3 + 3\times 2 = 12$

$v_1 := x[2] = 3$

$Step4:$

$y[3] := 2\times 0 + 3\times 3 = 9$

$v_1 := x[3] = 0$

$Step5:$

$y[4] := 2\times 0 + 3\times 0 = 0$

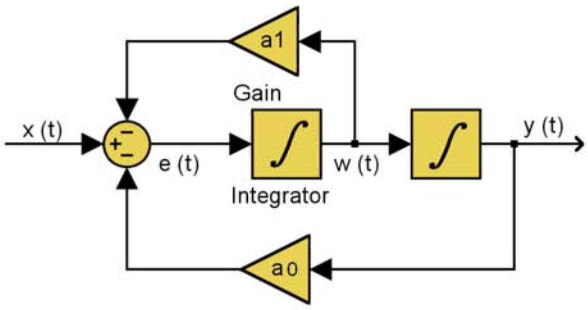
$v_1 := x[4] = 0$

$$\boxed{\begin{array}{l} h[n] = 2\delta[n] + 3\delta[n\text{-}1] \\ \Rightarrow \\ h[0] = 2, \\ h[1] = 3. \end{array}}$$

2.4.3 Block diagram representations of systems

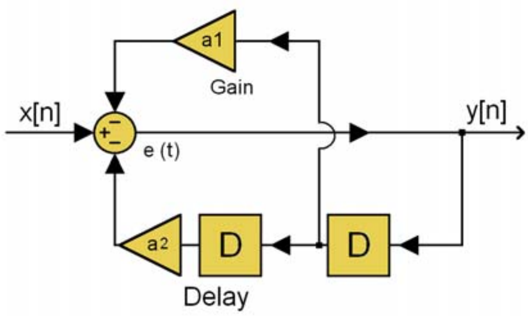
$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0 y(t) = x(t)$$

$$\Rightarrow \frac{d^2y(t)}{dt^2} = x(t) - a_0 y(t) - a_1\frac{dy(t)}{dt}$$



$$y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n]$$

$$\Rightarrow y[n] = x[n] - a_1 y[n-1] - a_2 y[n-2]$$



Assignment 2

- 2.1（b）（c）
- 2.6
- 2.14
- 2.21 (d)
- 2.24
- 2.25
- 2.28（a）（d)
- 2.29（c）（e）
- 2.61 (a)
- 2.62 (a)