Digital Signal Processing

Introduction of the course

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2023

1 Introduction: course aim

- Signals carry information, which can be represented in digital form and treated by digital Systems.

- The aim of the course is to understand fundamental concepts in digital signal processing (DSP) and know how to solve typical DSP problems.

- Help everyone grasp DSP knowledge Systematically!

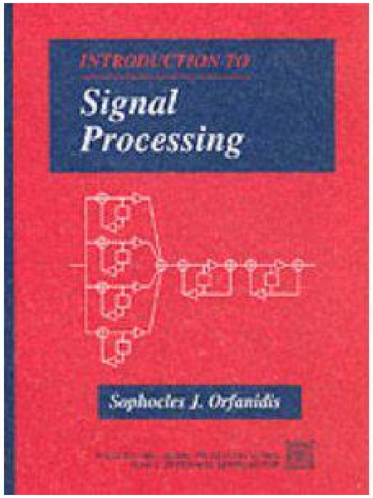
2 Introduction: course arrangement

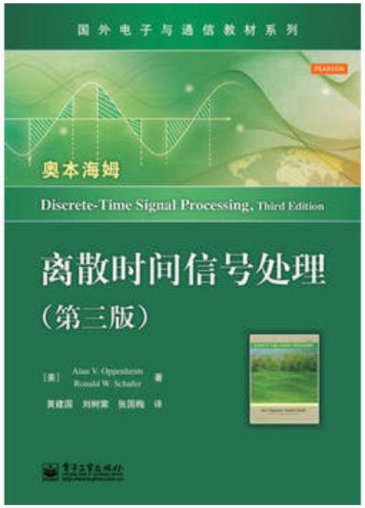
- Introduction
- Ch.1: Data sampling and reconstruction (6h)
- Ch.2: Fourier analysis (12h)
- Ch.3: Digital filters (14h)
- Ch.4: Random signal analysis (14h)

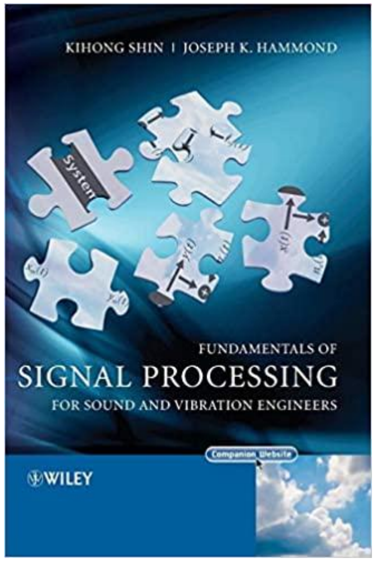
- Assignments: 3 reports (15%, 15%, 15%)
- Final exam (50%)
- Average score (2022)   : 88%

3 Introduction: learning materials for the course

- Course notes (Canvas)
- Sophocles Orfanidis, Introduction to Signal Processing, Prentice Hall, 1995. (Digital version is available)
- Discrete-time signal processing (3^{rd} Edition), Oppenheim, 2015
- Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, Shin, 2008







4 Contact Us

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Chapter 1

Signals and Systems

1.0 Introduction

Introducing mathematical description, representations and basic properties of signals and systems in general.

1.1 Signal classification

1.2 Transformations of the independent variable

1.3 Basic signals: exponential, sinusoidal, impulse, step, etc.

1.4 Continuous-time and Discrete-time systems

1.5 System properties: causality, stability, linearity, time-invariance

1.1 Signal classification

Signals are functions of independent variables that carry information. For example: voltages and currents in a circuit, Acoustic signals in speech, analog or digital Video signals, Biological signals …

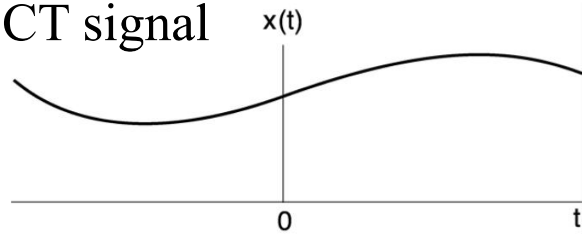
Independent variables can be continuous, e.g. many real signals in natural with physical meanings, can be discrete, e.g. digital image pixels, and can be 1-D, 2-D, … dependent on the number of independent variables. In this course, we focus on 1-D signals with one independent variable of "time".

Based on different features of signals, we can classify signals as

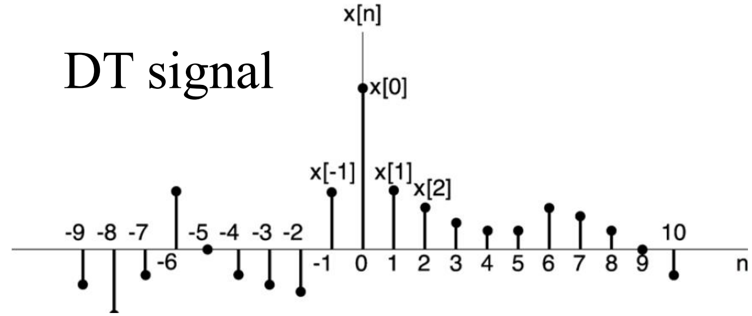
- **Continuous-time** and **discrete-time** signals
- **Even** and **odd** signals
- **Periodic** and **aperiodic** (non-periodic) signals
- **Deterministic** and **random** signals
- **Energy** and **power** signals

1.1.1 Continuous-time and discrete-time signals

CT signal



DT signal

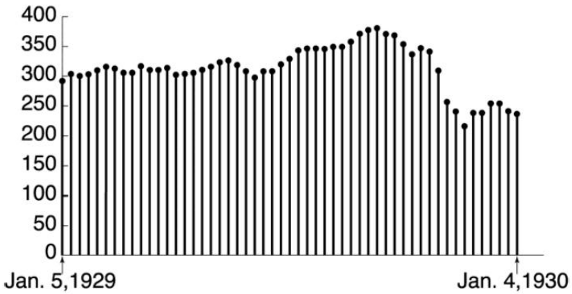


Continuous-Time (**CT**) signals: $x(t)$, t is continuous

Discrete-Time (**DT**) signals: $x[n]$, n is integer values only

Examples of DT signals

Weekly Dow-Jones industrial average



Digital Photo with pixels

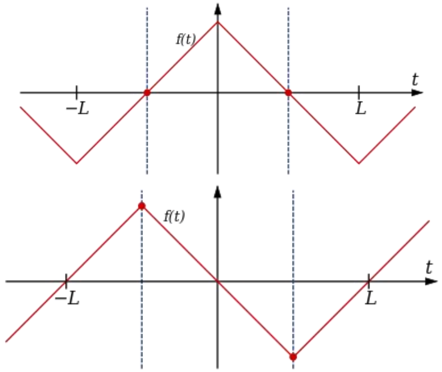
Courtesy of Jason Oppenheim.

A DT signal may represent a phenomenon for which the independent variable is inherently discrete. On the other side, Currently most DT signals are arisen from **sampling** of CT signals, i.e. human-made DT signals, which then can be represented and/or processed by modern digital hardware: computer and DSP chips.

1.1.2 Even and odd signals

Even signal: x(-t) = x(t) or x[-n] = x[n].
Symmetric about the time origin.

Odd signal: x(-t) = -x(t) or x[-n] = -x[n].
Antisymmetric about the time origin.



Any signal can be decomposed into an even and an odd signals

$$x(t)=\frac{x(t)+x(-t)}{2}+\frac{x(t)-x(-t)}{2},\quad \overset{\text{even}}{x_e(t)}=\frac{x(t)+x(-t)}{2}\ \text{and}\ \overset{\text{odd}}{x_o(t)}=\frac{x(t)-x(-t)}{2}$$

Example: even or odd function of time? $x(t)=\begin{cases}\sin\left(\dfrac{\pi t}{T}\right), & -T\le t\le T\\ 0, & otherwise\end{cases}$

Solution:

$$x(-t)=\begin{cases}\sin(-\dfrac{\pi t}{T}), & |t|\le T\\ 0, \text{otherwise}\end{cases}=\begin{cases}-\sin(\dfrac{\pi t}{T}), & |t|\le T\\ 0, \text{otherwise}\end{cases}=-x(t)$$

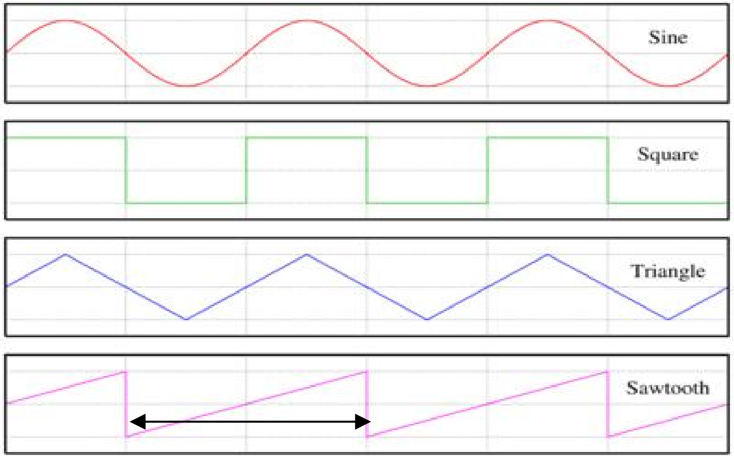
So it is an Odd signal

1.2.3 Periodic signals

Periodic signal is a function satisfy the following condition:

$$x(t) = x(t+T) \quad or \quad x[n] = x[n+N]$$

Fundamental period: the *smallest* values of T for CT and n for DT to satisfy the periodicity conditions

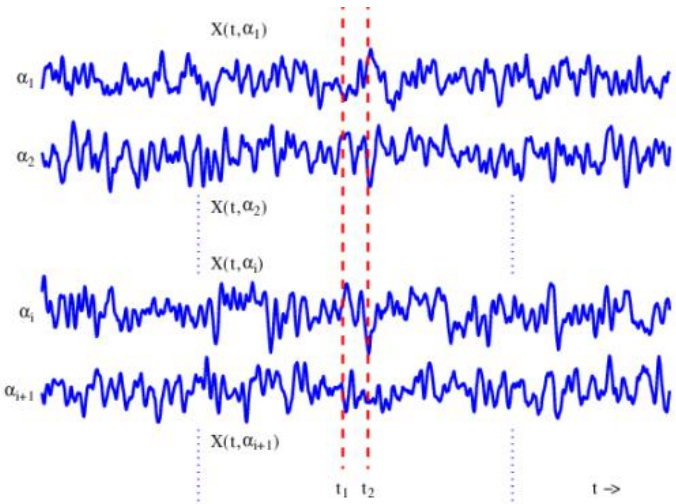


Fundamental period

1.1.4 Deterministic and random signals

Deterministic: predictable, normally described by an explicit mathematical relationship with reasonable accuracy.

Radom: or stochastic, unpredictable, each observation of the phenomenon will be unique. A part of random processes, named **stationary** random, can be described by mathematical relationships in **statistics**.



Stationary random signals with constant statistical parameters: average, deviation …

1) No physical data in practice can be truly deterministic because there is always a possibility that some unforeseen events in the future might influence the phenomenon to produce the data in a manner that was not originally considered.

2) On the other hand, it might be argued that no physical data are truly random, because an exact mathematical description might be possible if sufficient knowledge of the basic mechanisms of the phenomenon was available.

1.1.5 Power and energy signals

Instantaneous "**power**" (may not be related to the real power)

$$p(t) = |x(t)|^2 \quad \text{or} \quad p[n] = |x[n]|^2$$

Energy $$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \quad \text{or} \quad E_\infty = \lim_{N\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

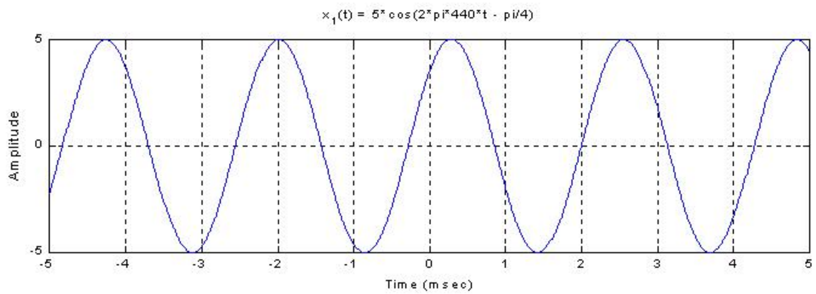
Average power $$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \quad \text{or} \quad P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

(finite) Power signal $0 < P_\infty < \infty$

with infinite energy

$$x(t) = 5\cos(2\pi \cdot 440t - \pi/4)$$

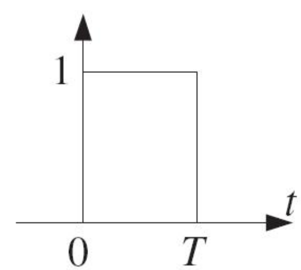
$$E_\infty = \infty, \quad P_\infty = 12.5$$



(finite) Energy signal $0 < E_\infty < \infty$

with zero average power

$$x(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & otherwaise \end{cases}$$



$$E_\infty = T, \quad P_\infty = 0$$

1.2 Transformations of the independent variable

Transformation of Signals is a central concept in signal and system analysis and has important physical meaning in the real world.

For example, in a high-fidelity audio system, an input signal representing music as recorded on a CD is modified to remove recording noise and enhance desirable characteristics (e.g. adding bass), and then amplified to drive the louder speak.

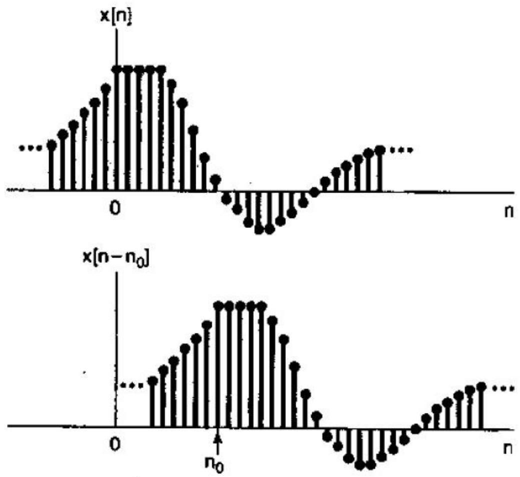
Here, we focus on basic signal transformations that involve simple modification of the independent variable, i.e., the time axis:

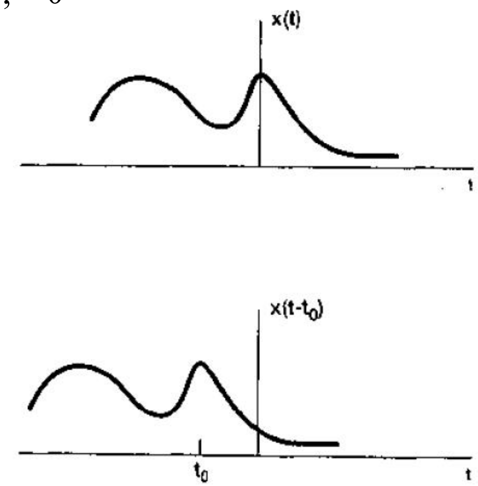
- Time shift: time delay and time advance
- Time reversal (reflection)
- Time scaling: time compression and time expansion
- Combination of time shift, reversal and scaling

1.2.1 Time shift and reversal

Time shift: $y(t) = x(t - t_0)$ $\qquad$ $y[n] = x[n - n_0]$, integer n_0

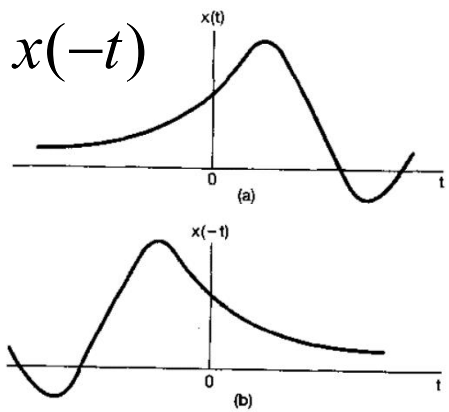
$t_0, n_0 > 0$: delayed version $\qquad$ $t_0, n_0 < 0$: advanced version



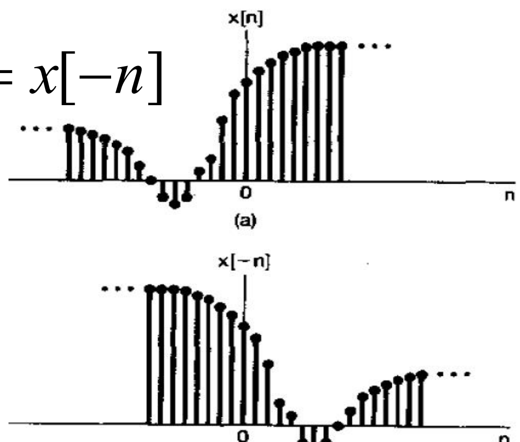


Time reversal about $t = 0$ or $n = 0$:

$y(t) = x(-t)$

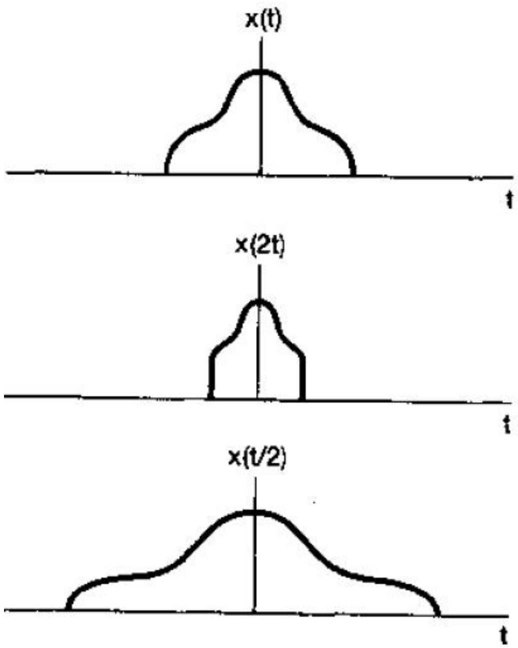


$y[n] = x[-n]$



1.2.2 Time scaling

CT signals $y(t) = x(at)$

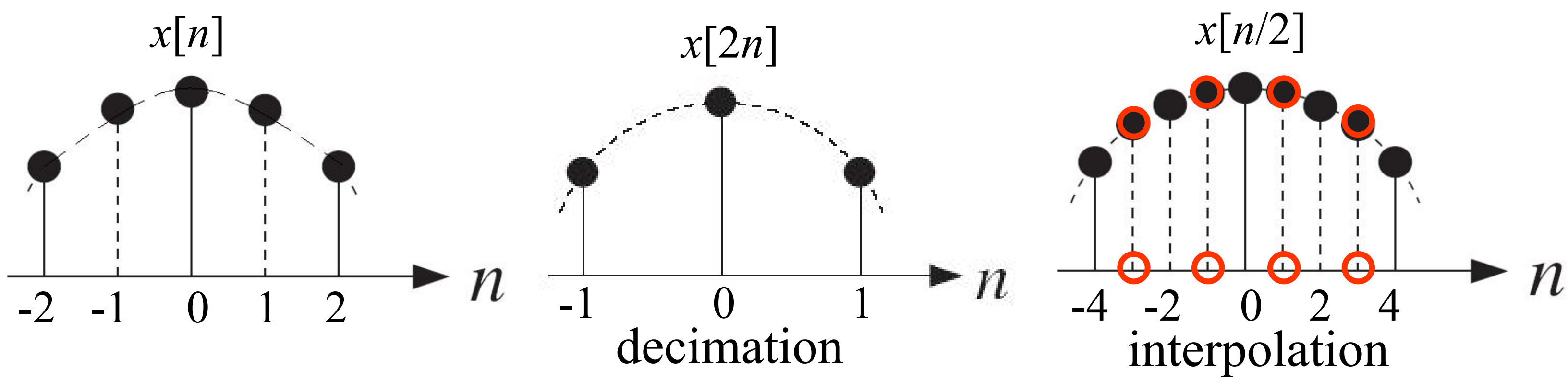


$\lvert a \rvert > 1$:	linearly compressed by a factor of a
$0 < \lvert a \rvert < 1$:	linearly expanded by a factor of a
if a < 0, then reversed	

DT signals $y[n] = x[kn]$, integer $k > 0$

$k > 1$: "**decimation**", compressed? lost information

$0 < k < 1$: "**interpolation**", expanded? add information



1.2.3 Combination transformation

Determine the effect of transforming $x(t)$ to obtain $y(t)$ of the form

$$y(t) = x(at - t_0)$$

Order of precedence:

$$y(t) = x(at - t_0)$$

$\Rightarrow$

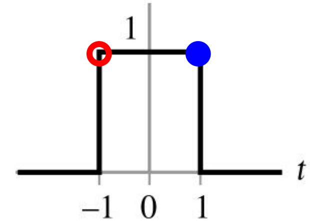
(1) Shift $\quad u(t) = x(t - t_0)$

(2) Scale $\quad v(t) = u(|a|t)$

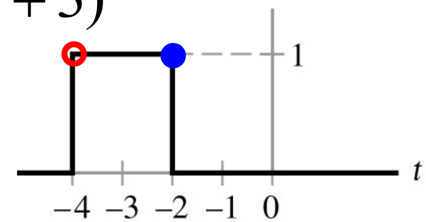
(3) Reflection $\quad y(t) = v(\mathrm{sgn}(a) \cdot t)$

Example $\quad y(t) = x(-2t + 3)$

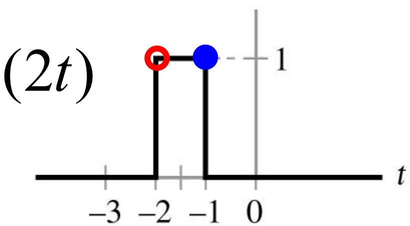
$x(t)$



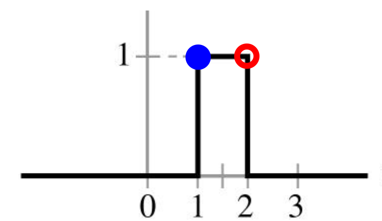
$u(t) = x(t + 3)$



$v(t) = u(2t)$



$y(t) = v(-t)$



check with selected points

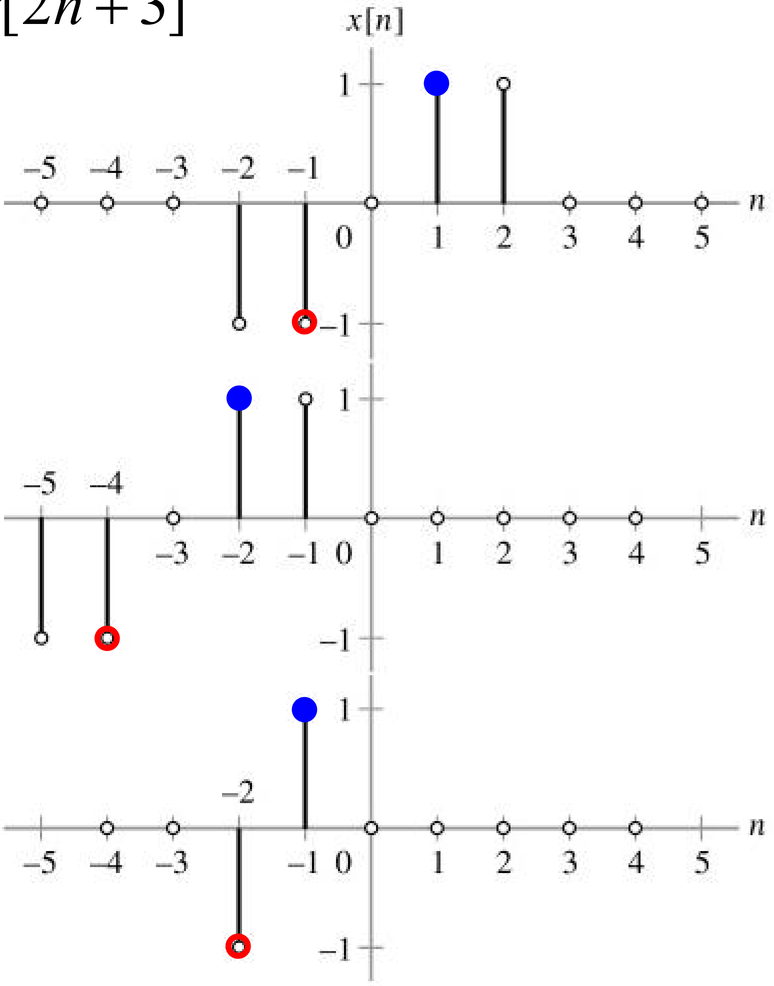
1.2.3 Combination transformation

Example $y[n] = x[2n+3]$

$x[n]$

$u[n] = x[n+3]$

$y[n] = u[2n]$



check with selected points

1.3 Basic signals

Signals introduced here are not only important themselves and but also frequently served as basic building blocks from which we can construct many other signals. These signals include

– Continuous-time exponential signals
– Discrete-time exponential signals
– Step function
– Impulse function
– Derivatives and integration of the impulse and step functions

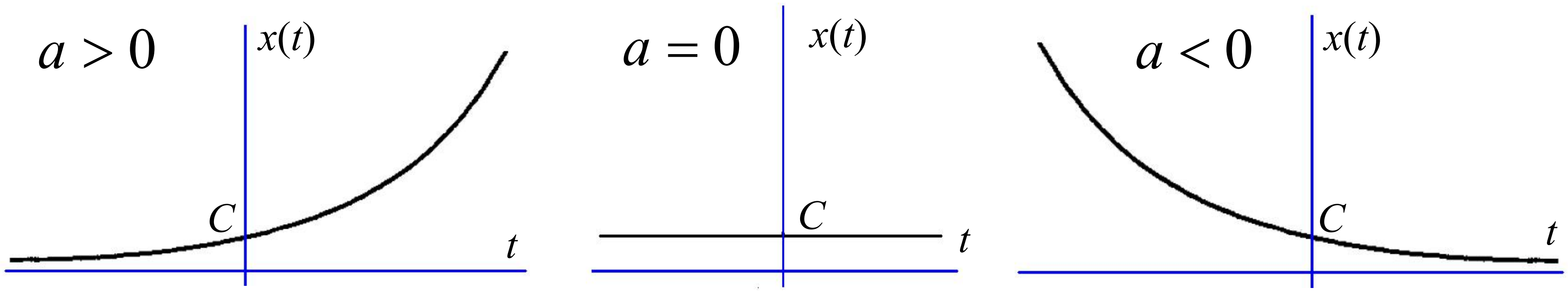
1.3.1 CT Exponential signals

The CT complex exponential signal is of the form

$$x(t) = Ce^{at} \qquad C, a \text{ are generally complex numbers}$$

Depending on the values of these parameters, the complex exponential can exhibit several different characteristics.

- Real CT exponential signals C, a are real numbers



Decaying exponential | Constant | Growing exponential

1.3.1 CT Exponential signals

- Complex CT exponential signals

$C = Ae^{j\phi}$ C is a complex number with modulus A and phase φ

$a = j\omega_0$ a is a purely imaginary number

So $\boxed{Ce^{j\omega_0 t} = Ae^{j(\omega_0 t+\phi)}}$ has periodic property, as

$$x(t) = Ce^{j\omega_0 t} = Ce^{j\omega_0 (t+T)} = Ce^{j\omega_0 t}e^{j\omega_0 T}$$

The Period T must satisfy: $e^{j\omega_0 T} = 1$

If $\omega_0 \neq 0$, then $T = \dfrac{2\pi k}{\omega_0}$ and the fundamental period is $T_0 = \dfrac{2\pi}{\omega_0}$

By using Euler's relation $e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$

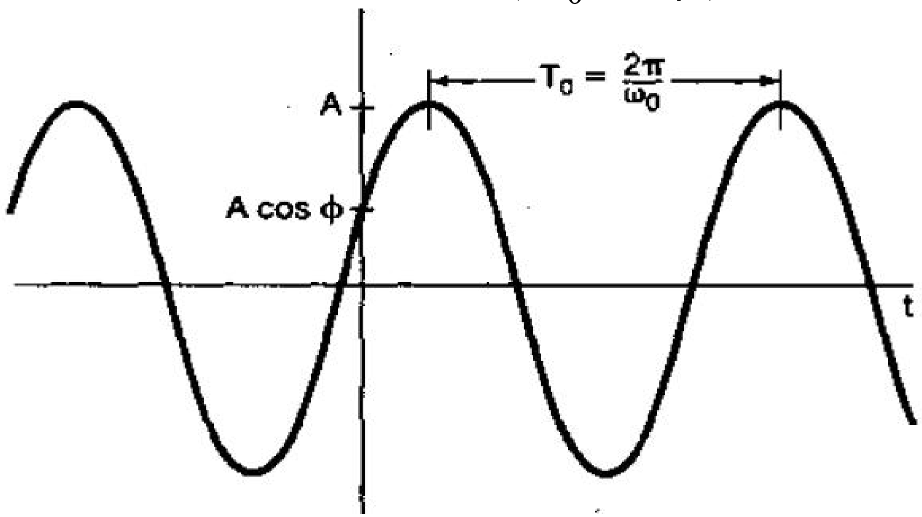
$$\cos(\omega_0 t) = 0.5e^{j\omega_0 t} + 0.5e^{-j\omega_0 t} = \mathrm{Re}\{e^{j\omega_0 t}\}$$

$$\sin(\omega_0 t) = \mathrm{Im}\{e^{j\omega_0 t}\}$$

1.3.1 CT Exponential signals

$$x(t) = Ce^{j\omega_0 t} = Ae^{j(\omega_0 t + \phi)} = A\cos(\omega_0 t + \phi) + jA\sin(\omega_0 t + \phi)$$

Real part $A\cos(\omega_0 t + \phi)$



To increase or decrease the value of ω_0 will speed up or slow down the rate of oscillation. ω_0 is the fundamental radian frequency in (radian/s) and $f_0 = \omega_0 / 2\pi$ is the fundamental frequency in Hertz (1/s, Hz).

1.3.1 CT Exponential signals

Example: It is sometimes desirable to express the sum of two complex exponentials as the product of a single complex exponential and a single sinusoid and then we can obtain the magnitude of the signal.

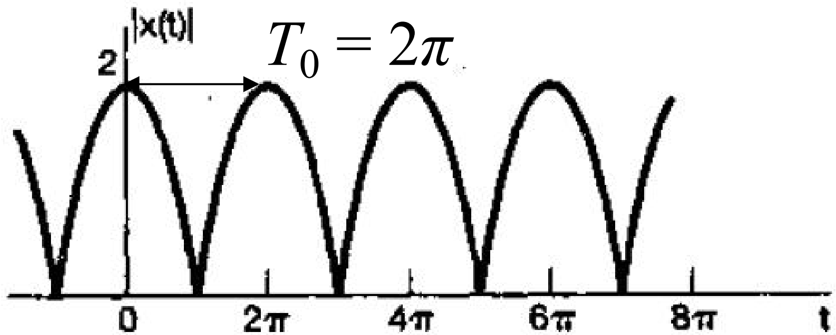
$$x(t) = e^{j2t} + e^{j3t}$$

$$= e^{j2.5t}(e^{-j0.5t} + e^{j0.5t})$$

$$= e^{j2.5t} 2\cos(0.5t)$$

Magnitude is $|x(t)| = 2|\cos(0.5t)|$

$T_0 = 2\pi$



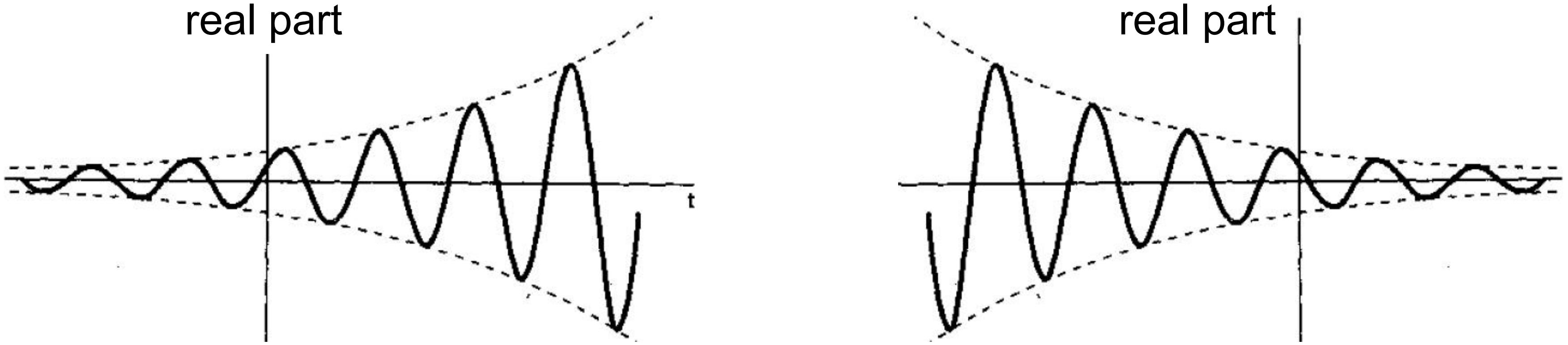
1.3.1 CT Exponential signals

- General complex CT exponential signals

$C = Ae^{j\phi}$ C is a complex number with modulus A and phase φ

$a = r + j\omega_0$ a is a general complex number with the real part r and the imaginary part ω_0

$$x(t) = Ce^{(r+j\omega_0)t} = Ae^{rt}e^{j(\omega_0 t+\phi)} = e^{rt}A\cos(\omega_0 t+\phi) + je^{rt}A\sin(\omega_0 t+\phi)$$



$r > 0$: Growing sinusoidal $\boxed{r = 0\text{: Sinusoidal}}$ $r < 0$: Decaying sinusoidal

Sinusoidal signals multiplied by decaying exponentials are commonly referred to as damped sinusoids, e.g. response of RLC circuits

1.3.2 DT Exponential signals

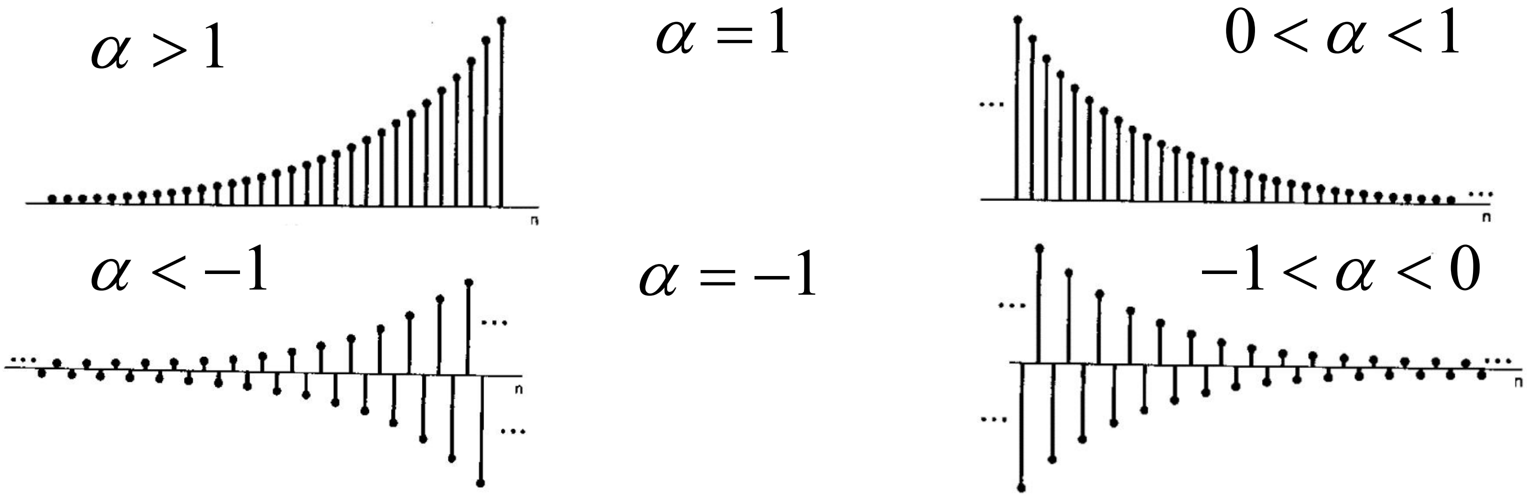
DT exponential signals $x[n] = \boxed{C\alpha^n} = Ce^{\beta n}$

C, α are generally complex numbers

$C = Ae^{j\phi}$ is a complex number with modulus A and phase φ

$\alpha = e^{\beta}$ β is a general complex number

- Real DT exponential signals： C, α are real numbers



$|\alpha| > 1$: growing amplitude $|\alpha| < 1$: decaying amplitude

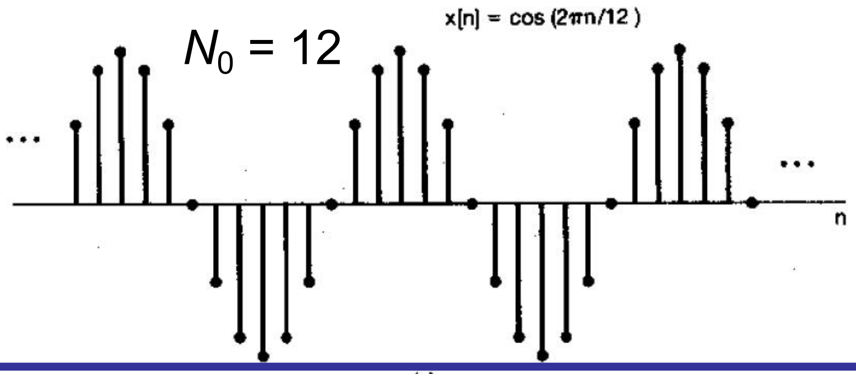
1.3.2 DT Exponential signals

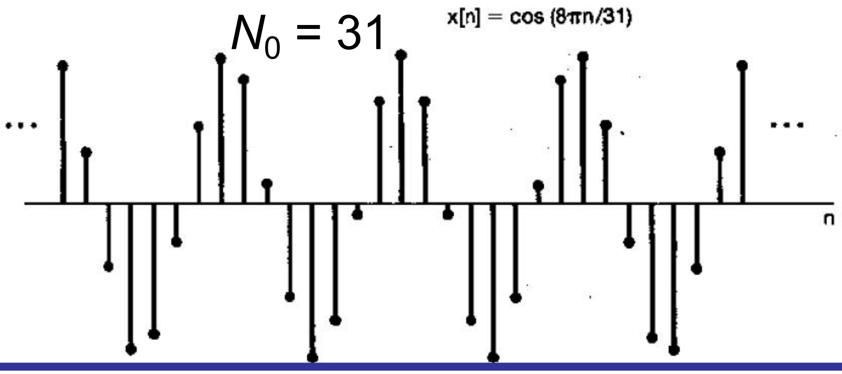
- Complex DT exponential signals: $x[n] = C\alpha^n = Ce^{\beta n}$

  $C = Ae^{j\phi}$ is a complex number with modulus A and phase φ

  $\alpha = e^{\beta} = e^{j\omega_0}$ β is a pure imaginary value

  So $x[n] = Ae^{j(\omega_0 n+\phi)} = A\cos(\omega_0 n+\phi) + jA\sin(\omega_0 n+\phi)$





$$e^{j\omega_0 n} = e^{j\omega_0(n+N)} \rightarrow e^{j\omega_0 N} = 1$$

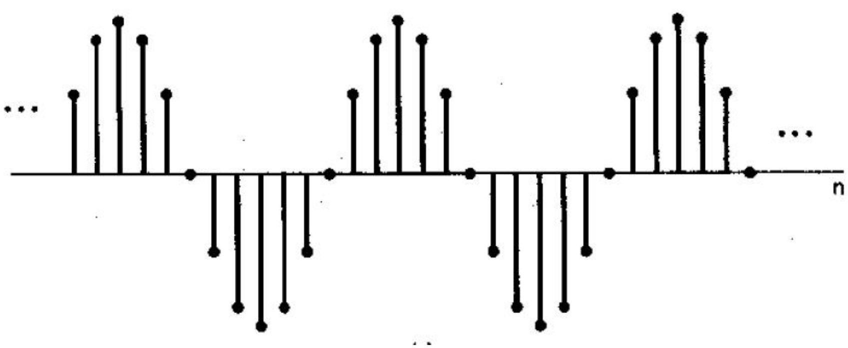
$$\Rightarrow \omega_0 N = 2\pi k \rightarrow \frac{\omega_0}{2\pi} = \frac{k}{N}, \quad k, N \text{ must be integer}$$

$x[n] = \cos(n/6)$ Periodic signal or not?

$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$, period?

Examples:

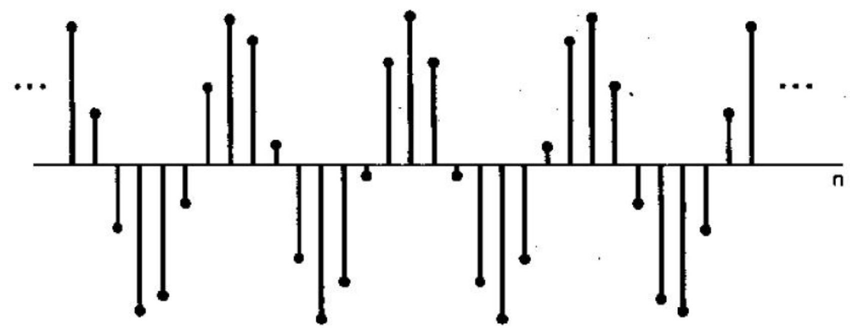
$x[n] = \cos(2\pi n / 12)$



$N_0 = 12$

$\boxed{x(t) = \cos(2\pi t / 12), \text{period}?}$

$x[n] = \cos(8\pi n / 31)$



$N_0 = 31$

$\boxed{x(t) = \cos(8\pi t / 31), \text{period}?}$

$$e^{j\omega_0 n} = e^{j\omega_0 (n+N)} \quad \rightarrow \quad e^{j\omega_0 N} = 1$$

$$\Rightarrow \omega_0 N = 2\pi k \quad \rightarrow \quad \frac{\omega_0}{2\pi} = \frac{k}{N}, \quad k, N \text{ must be integer}$$

$\boxed{x[n] = \cos(n/6) \text{ Periodic signal or not?}}$

$\boxed{x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}, \quad \text{period?}}$

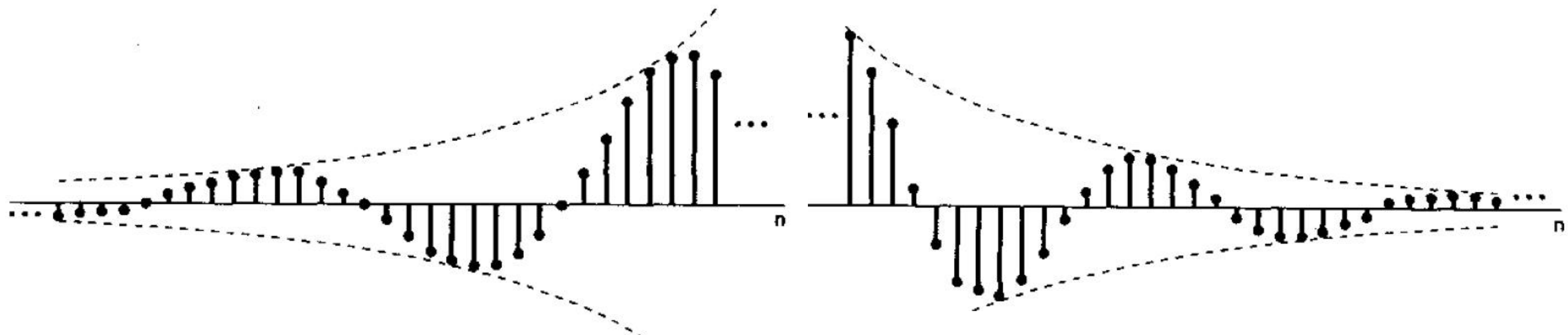
1.3.2 DT Exponential signals

- Complex DT exponential signals: $x[n] = C\alpha^n$

  C, α are generally complex numbers

  $C = Ae^{j\phi} \qquad \alpha = |\alpha| e^{j\omega_0}$

So $x[n] = A|\alpha|^n e^{j(\omega_0 n + \phi)} = A|\alpha|^n \cos(\omega_0 n + \phi) + jA|\alpha|^n \sin(\omega_0 n + \phi)$



$|\alpha| > 1$: growing amplitude $\qquad |\alpha| < 1$: growing amplitude

$|\alpha| = 1$: Sinusodal

1.3.2 DT Exponential signals

- Periodicity Properties of DT complex exponential：

While there are many similarities between CT and DT signals, there are differences in exponentials:

The oscillation rate increases with the increase of frequency

$$x(t) = e^{j\omega_0 t} \neq e^{j(\omega_0 + 2\pi k)t}$$

However, for DT, the oscillation rate does not continuously increase as the frequency increases. The period is 2π.

$$x[n] = e^{j\omega_0 n} = e^{j(\omega_0 + 2\pi)n} = \cdots = e^{j(\omega_0 + 2k\pi)n} = e^{j\omega_0 n}$$

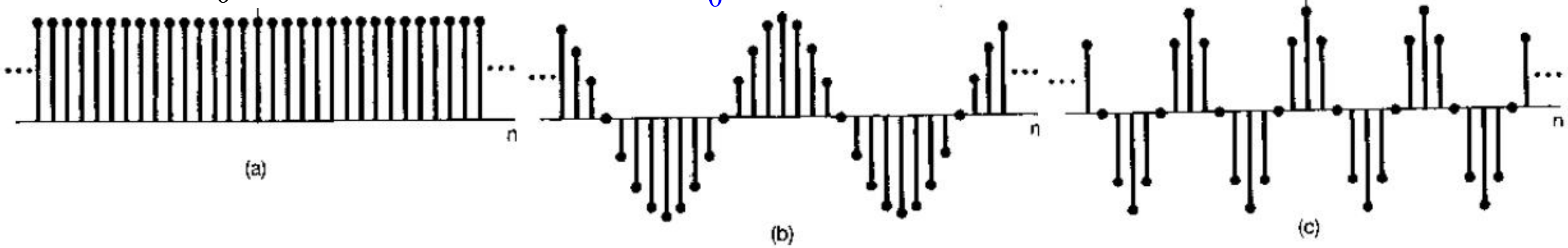
Therefore, we can observe the DT exponentials in any interval of 2π. Normally

$$0 \leq \omega_0 < 2\pi \quad \text{or} \quad -\pi \leq \omega_0 < \pi$$

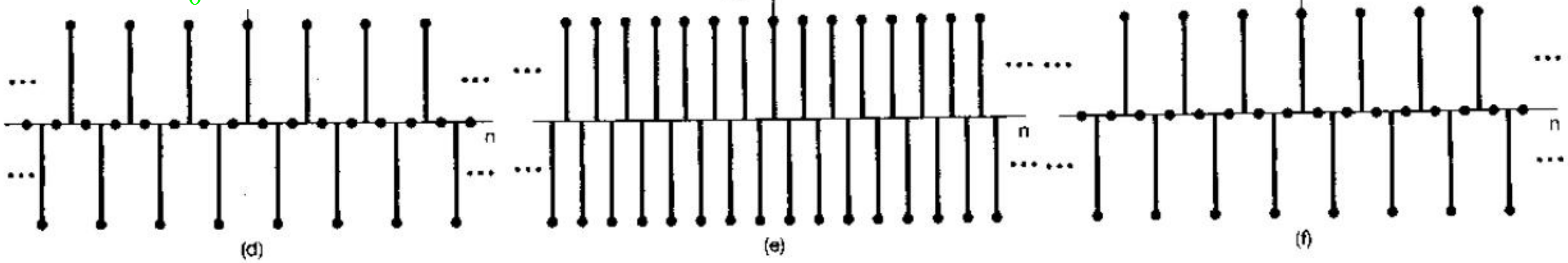
1.3.2 DT Exponential signals

$$x[n] = \cos(\omega_0 n)$$

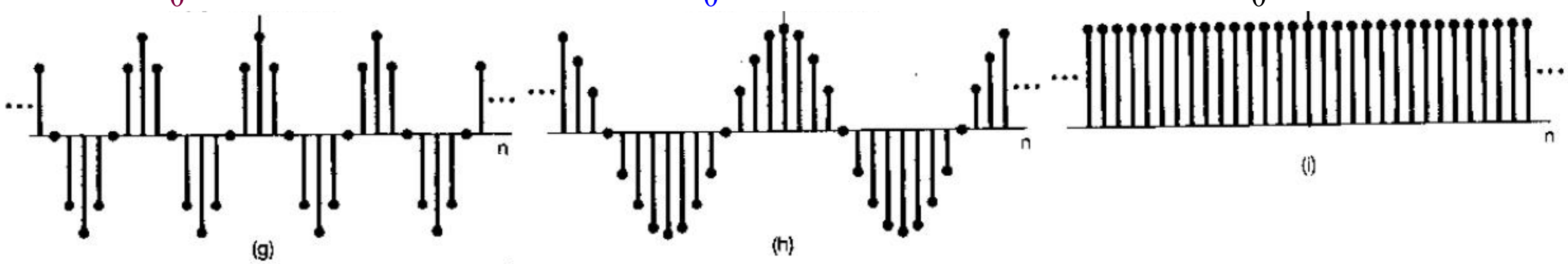
$\omega_0 = 0$ $\quad$ $\omega_0 = \pi / 8$ $\quad$ $\omega_0 = \pi / 4$



$\omega_0 = \pi / 2$ $\quad$ $\omega_0 = \pi$ $\quad$ $\omega_0 = 3\pi / 2$

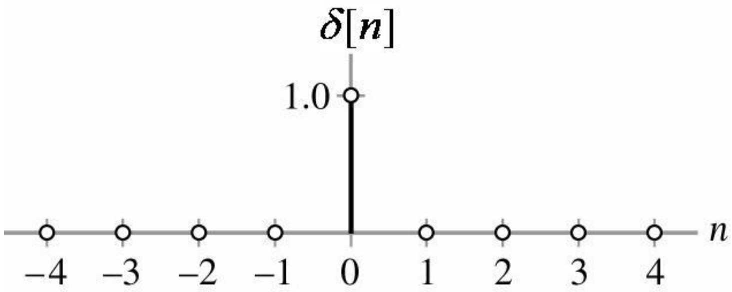


$\omega_0 = 7\pi / 4$ $\quad$ $\omega_0 = 15\pi / 8$ $\quad$ $\omega_0 = 2\pi$



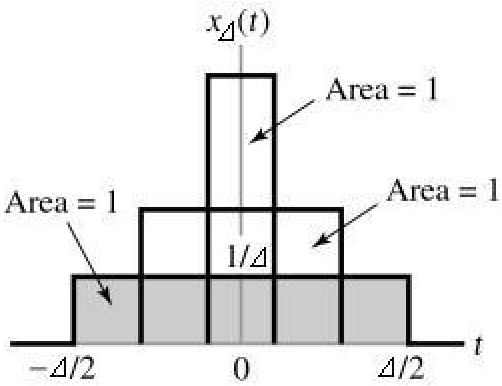
1.3.3 Unit impulse

Unit impulse



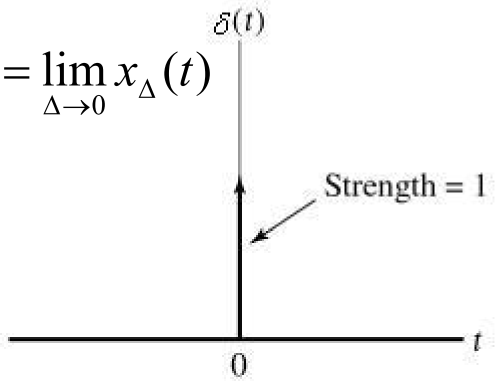
DT $$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

CT $$\begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 \\ \delta(t) = 0, \quad t \neq 0 \end{cases}$$



(a)

$$\delta(t) = \lim_{\Delta \to 0} x_\Delta(t)$$



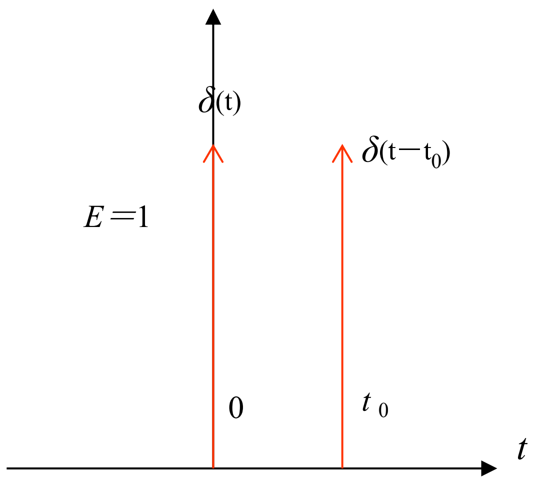
(b)

1.3.3 Unit impulse

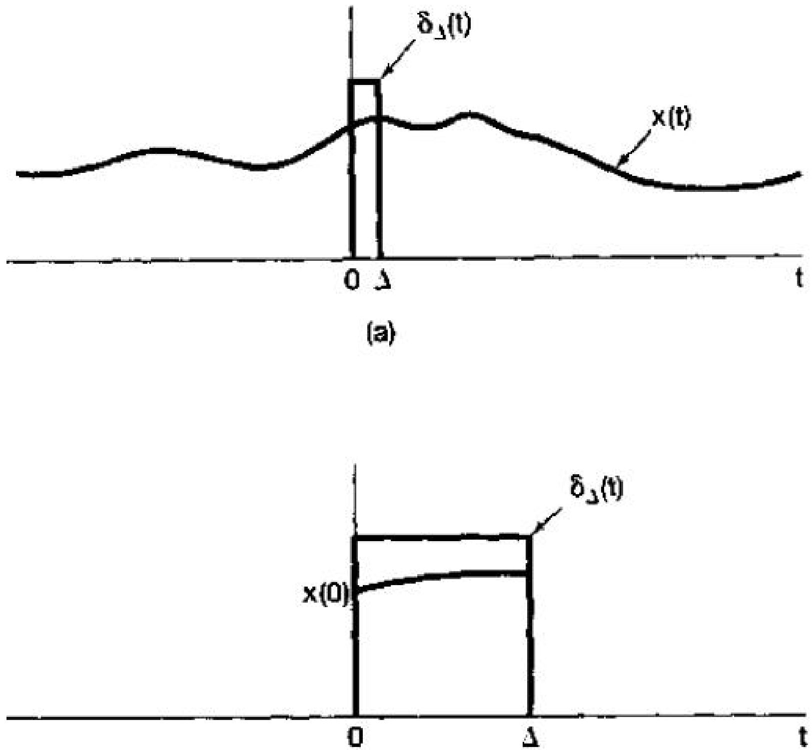
Sampling property of Impulse

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = \int_{-\infty}^{\infty} x(t_0)\delta(t-t_0)dt = x(t_0)\int_{-\infty}^{\infty}\delta(t-t_0)dt = x(t_0)$$

$$x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]$$



Shift property



1.3.4 Unit step

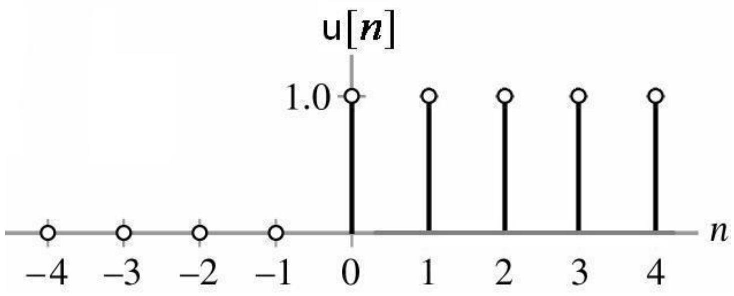
Unit step

DT

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$$\delta[n] = u[n] - u[n-1]$$

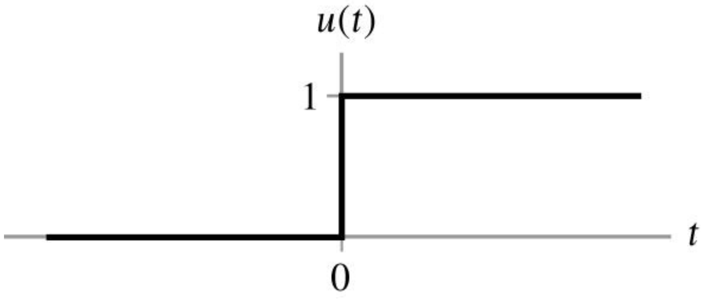


CT

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \int_{0}^{\infty} \delta(t-\sigma)\,d\sigma$$

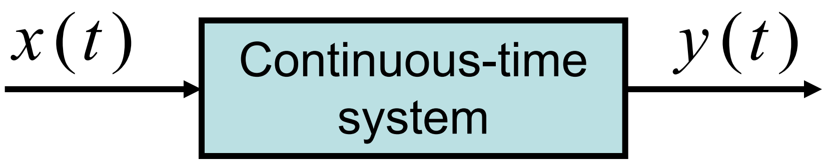
$$\delta(t) = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$$



1.5 Continuous- and discrete-time systems

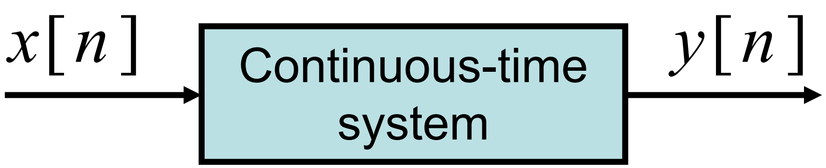
Systems are an interconnection of components or subsystems in the real world. In the signal processing, a system can be viewed as a process in which input signals are transformed by the system or cause the system to respond in some way, resulting in other signals as outputs.

Continuous-time system



$$x(t) \rightarrow y(t)$$

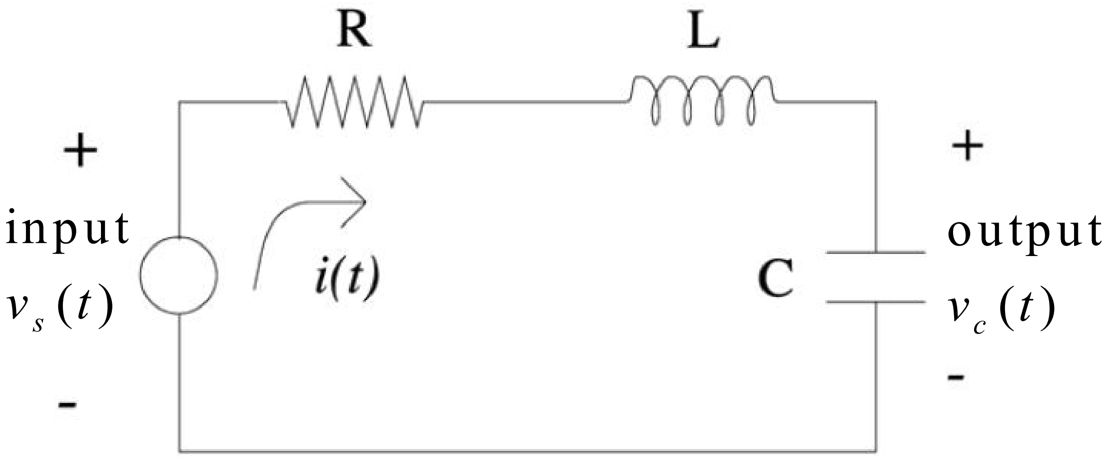
Discrete-time system



$$x[n] \rightarrow y[n]$$

1.5.1 Simple examples of systems

RLC circuit system



$$\begin{cases} R\,i(t) + L\dfrac{di(t)}{dt} + v_c(t) = v_s(t) \\ i(t) = C\dfrac{dv_c(t)}{dt} \end{cases}$$

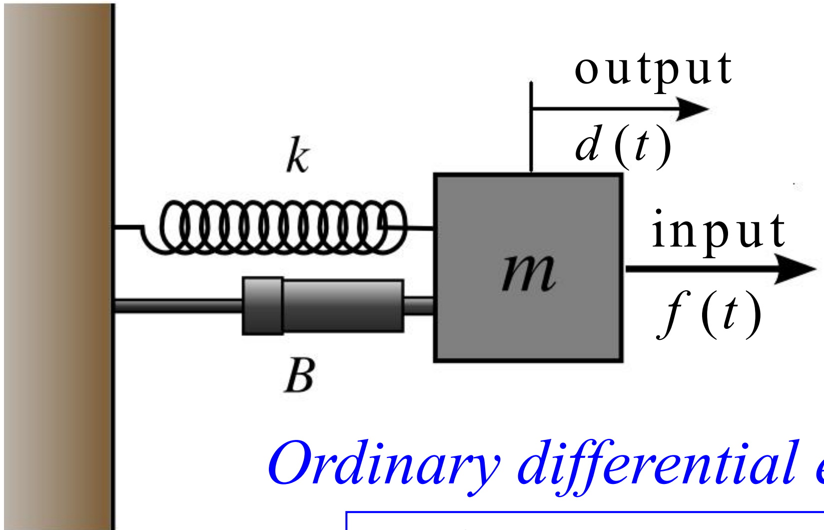
$$v_s(t) \triangleq x(t)$$
$$v_c(t) \triangleq y(t)$$

$$\boxed{LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)}$$

Determining the output $y(t)$ with respect to $x(t)$ requires initial conditions.

1.5.1 Simple examples of systems

Mass-damper-spring dynamic system



$$m\ddot{d} = f(t) - kd - B\dot{d}$$

$$f(t) \triangleq x(t)$$

$$d(t) \triangleq y(t)$$

Ordinary differential equation (ODE)

$$\boxed{m\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + B\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ky(t) = x(t)}$$

Comparing with $$\boxed{\mathrm{LC}\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \mathrm{RC}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = x(t)}$$

Very different physical systems may be modeled mathematically in very similar ways. So typical system properties are of interests.

1.5.1 Simple examples of systems

Month balance in a bank

Consider a simple model for the balance in a bank account from month to month. Specifically, let $y[n]$ denote the balance at the end of the nth month, and suppose that $y[n]$ evolves from month to month according to the *difference equation*

$$y[n] = 1.01\,y[n-1] + x[n]$$

where $x[n]$ represents the net deposit (i.e., deposits minus withdrawals) during the nth month and the term $1.01y[n\text{-}1]$ models the fact that we accrue 1% interest each month.

With some initial conditions: e.g. $y[0] = 1000$, we can calculate $y[n]$ according to the input $x[n]$.

1.5.1 Simple examples of systems

1) A very rich class of systems (but by no means all systems of interest to us) are described by differential and difference equations.

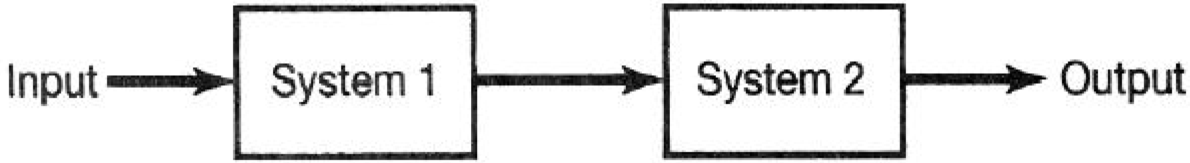
2) Such an equation, by itself, does not completely describe the input-output behavior of a system: we need auxiliary conditions (initial conditions, boundary conditions).

3) In some cases the system of interest has time as the natural independent variable and is causal. However, that is not always the case.

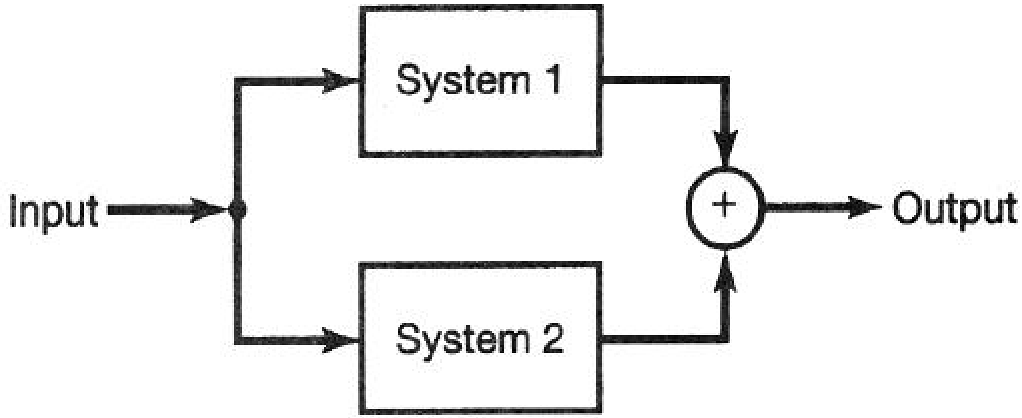
4) Very different physical systems may have very similar mathematical descriptions.

1.5.2 Interconnections of systems

Series (cascade) interconnection

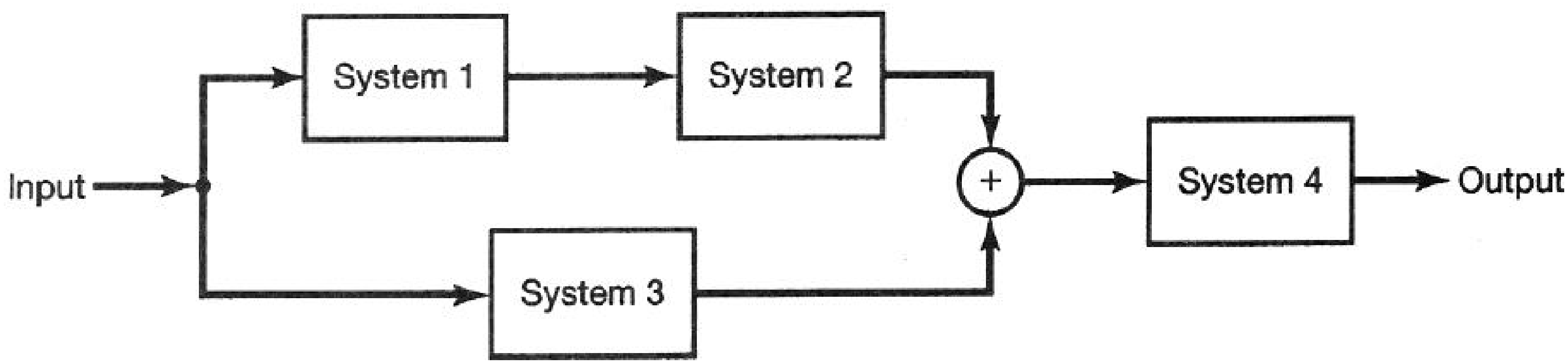


Parallel interconnection

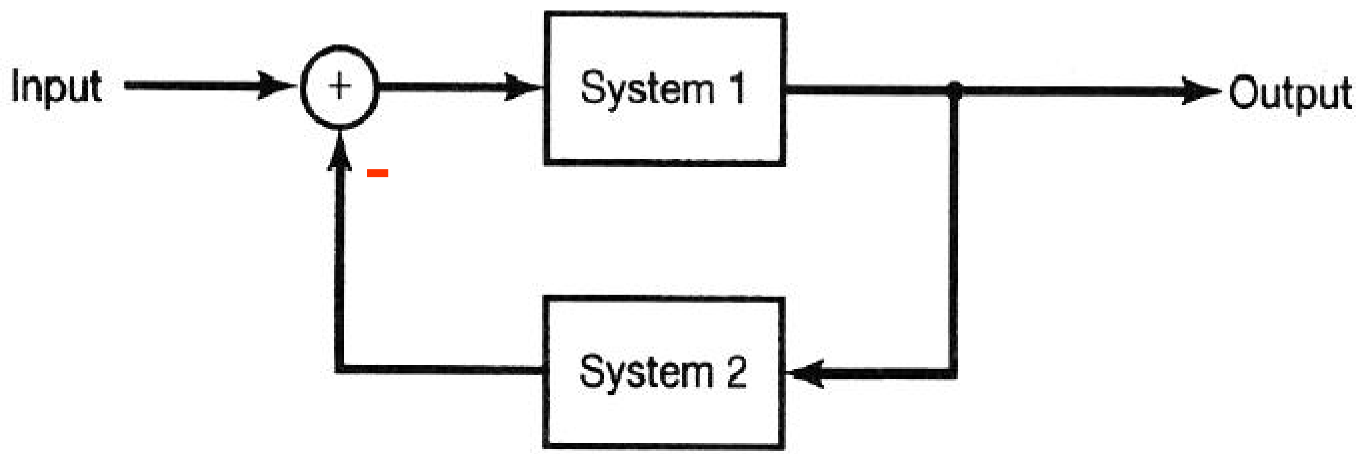


1.5.2 Interconnections of systems

Series-parallel interconnection



Feedback interconnection



1.6 Basic system properties

System properties have important physical interpretations and relatively simple mathematical descriptions. These properties are

- Memory
- Invertibility
- Causality
- Stability
- Time invariance
- Linearity

1.6.1 Systems with and without memory

A system is said to possess memory if its output signal depends on past or future values of the input and output signal.

A system is said to be memoryless if its output signal depends only on the present value of the input and output signal.

Memoryless systems

(1) $v(t) = Ri(t)$ (resistor)

(2) $y(t) = (2x(t) - x^2(t))^2$

(3) $y(t) = \dfrac{d}{dt}x(t)$ (differentiator)

Memory systems

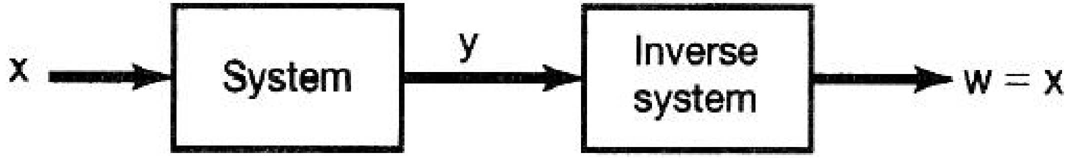
(4) $i(t) = \dfrac{1}{L}\displaystyle\int_{-\infty}^{t} v(\tau)d\tau$ (inductor)

(5) $y[n] = \dfrac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$ (averager)

(6) $y[n] = \displaystyle\sum_{k=-\infty}^{n} x[k] = y[n-1] + x[n]$ (accumulator)

1.6.2 Invertibility and inverse systems

A system is said to be invertible if distinct inputs lead to distinct outputs, which means the inputs of the system can be recovered from the outputs. This type system is important in communication and encryption.



$$y(t) = H\{x(t)\} \quad x(t) = H^{inv}\{y(t)\} = H^{inv}\{H\{x(t)\}\}$$

$$\Rightarrow H^{inv}H = I$$

Example

$y(t) = 2x(t)$ whose inverse system is $w(t) = \frac{1}{2}y(t)$

Noninvertible systems: $y[n] = 0,\ y(t) = x^2(t)$

1.6.3 Causality

A system is *causal* if the output at **any** time depends only on values of the input at the present time and in the past. Such a system is often referred to as being nonanticipative, as the system output does not anticipate future values of the input. So all memoryless systems are causal.

Example $$y[n] = 2x[n] + x[n+1]$$

Causal or noncausal? $$y[n] = x[-n]$$

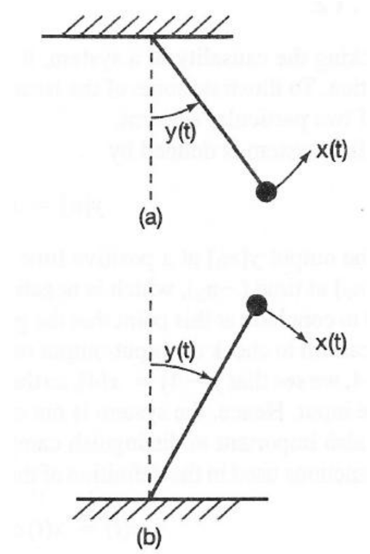
$$y(t) = x(t)\cos(t+1)$$

Causal systems are of great importance for real processing systems with the variable of time, but they are not essential constraint in applications in which the independent variable is not time, e.g. image processing and those recorded data.

1.6.4 Stability

A system is *stability* if the input to a stable system is bounded, then the output must also be bounded and therefore cannot diverge.

Bounded input and bounded output: BIBO principle



Example $$y[n] = \sum_{k=-\infty}^{n} u[k]$$

If we suspect that a system is unstable, then a useful way to verify this is to find a specific bounded input that leads to an unbounded output. However, if we want to verify a system stable, we must check for stability by using a way that does not utilize specific examples of input signals.

Example $$y(t) = e^{x(t)}$$

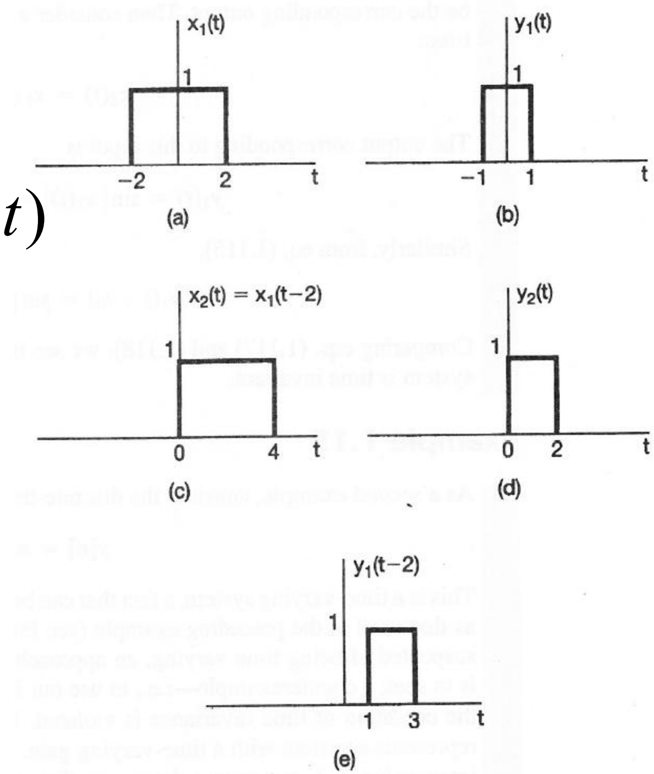
1.6.5 Time invariance

A system is *time invariant* if the behavior and characteristics of the system are fixed over time. In the signals and systems language, a system is time invariant is a time shift in the input signal results in an identical time shift in the output signal for any input and any time shift.

Examples

(1) $y(t) = \sin[x(t)]$ (3) $y(t) = x(2t)$

(2) $y[n] = nx[n]$



1.6.6 Linearity

A system is *linear* if it satisfies superposition (or additivity) and scaling (or homogeneity) properties.

$$\text{Superposition: } x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

$$\text{Scaling: } ax_1(t) \rightarrow ay_1(t), \text{ where } a \text{ is any complex constant}$$

Combing two properties to give

$$\boxed{\text{Linearity: } ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)}$$

Examples

$$(1)\ y(t) = tx(t)$$

$$(2)\ y(t) = 2x(t) + 3$$

$$(3)\ y(t) = x^2(t)$$

Incrementally linear system
with zero-input response