Digital Signal Processing

Chapter 3. Digital filters

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Study Points

- Difference equations and Z-transform (1H)
- FIR filter and design (3H)
- IIR filter and design (1H)
- Pole-zero design (1H)
- Filter realization (1H)
- Lab 2

Review： Z transform

According to eigenfunction of LTI systems, for DT system, there is

$$x[n]=z^n \longrightarrow \boxed{H(z)} \longrightarrow H(z)z^n$$

$H(z)=\sum_{n=-\infty}^{\infty} h(n)z^{-n}$, assume the summation is convergent

where z^n is an eigenfunction of LTI systems and $z=re^{j\omega}$ is a complex number.

The Bilateral Z transform is defined as

$$x[n] \leftrightarrow X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}=Z\{x[n]\}$$

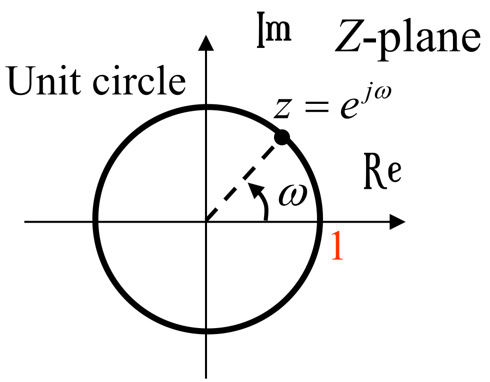
Z transform and DTFT

The relationship between Z transform and DTFT:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]r^{-n}e^{-j\omega n}=\sum_{n=-\infty}^{\infty}\left(x[n]r^{-n}\right)e^{-j\omega n}=F\left\{x[n]r^{-n}\right\}$$

If $\sum_{n=-\infty}^{\infty}\left|x[n]\right|r^{-n}<\infty$, then DTFT exists.

So ROC of Z transform relays on $\left|z\right|=r$



If ROC includes the unit circle, the DTFT of the signal exists.

Rational z-transforms

$x[n]$ = linear combination of exponentials for $n > 0$ and for $n < 0$

$X(z)$ is rational

$$X(z) = \frac{N(z)}{D(z)}$$ Polynomials in z

— characterized (except for a gain) by its poles and zeros

Z transform and L transform

The relationship between Z transform and Laplace transform:

Sampling $x_s(t) = x(t)\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty}x(nT)\delta(t-nT)$

$\because \quad L[\delta(t-nT)] = e^{-nTs}$

$\therefore \quad X_s(s) = \sum_{n=-\infty}^{\infty}x(nT)\,e^{-nTs}$

According to Z transform: $X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty}x[n]\,z^{-n}$

If $z = e^{sT}$ $\quad X(e^{sT}) = \sum_{n=-\infty}^{\infty}x[n]e^{-nTs} = X_s(s)$

$$\boxed{X_s(s) = X(z)\Big|_{z=e^{sT}}}$$

Example 1

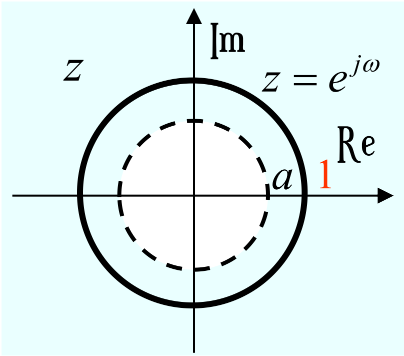
$x[n] = a^n u[n]$ is a right-sided signal.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

ROC: $X(z) < \infty \;\Rightarrow\; |az^{-1}| < 1 \;\Rightarrow\; |z| > |a|$

$$\therefore \; X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

If $|a| < 1$, DTFT exists.



If $a = 1$, $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}, \quad |z| > 1$$

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

Zeros: 0, Poles: a

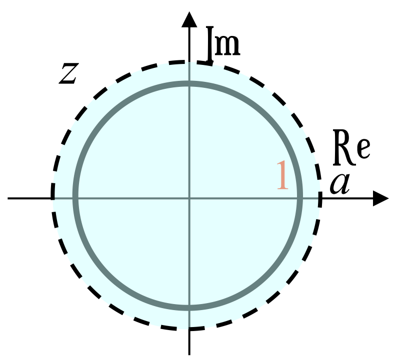
Example 2

$x[n] = -a^n u[-n-1]$ is a left-sided signal.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$\text{ROC: } X(z) < \infty \quad \Rightarrow \quad |a^{-1}z| < 1 \quad \Rightarrow \quad |z| < |a|$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$



If $|a| > 1$, DTFT exists

If $a = 1$, $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}, \quad |z| > 1$$

$$X(z) = \frac{z}{z-a}, \quad |z| < |a|$$

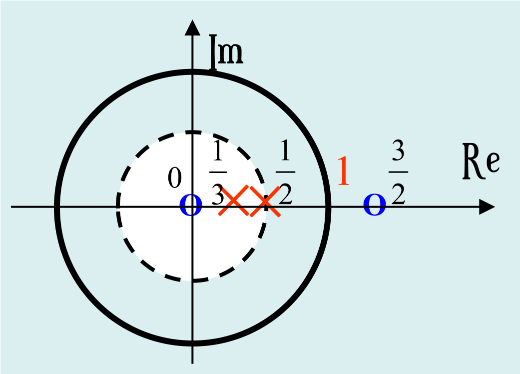
Zeros: 0, Poles: a

Example 3

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty}\left\{7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right\}z^{-n} = 7\sum_{n=0}^{\infty}(\frac{1}{3}z^{-1})^n - 6\sum_{n=0}^{\infty}(\frac{1}{2}z^{-1})^n$$

ROC $\quad \left|\frac{1}{3z}\right| < 1 \cap \left|\frac{1}{2z}\right| < 1 \rightarrow |z| > \frac{1}{2}$



$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}, \quad |z| > \frac{1}{2}$$

Inverse Z transform —— Rational form

Example1: $x[n] = a^n u[n]$ is a right-sided signal.

$$\therefore \ X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

For the rational form of the z transform: $\dfrac{X(z)}{z} = \dfrac{N(z)}{D(z)}$

Single poles: $\dfrac{X(z)}{z} = \displaystyle\sum_{i=1}^{n} \frac{K_i}{z-p_i}$ $\qquad K_i = (z-p_i)\dfrac{X(z)}{z}\Big|_{z=p_i}$

$$\therefore \quad x(n) = \sum_i K_i (p_i)^n u(n)$$

Inverse Z transform —— Rational form

Multi-poles:

$$\frac{X(z)}{z}=\frac{N(z)}{(z-a)^3}=\frac{K_1}{(z-a)^3}+\frac{K_2}{(z-a)^2}+\frac{K_3}{z-a}$$

$$\frac{z}{(z-a)^3}\Leftrightarrow\frac{1}{2}n(n-1)\,a^{n-2}u(n-2)$$

$$K_1=(z-a)^3\frac{X(z)}{z}\Big|_{z=a}$$

$$\frac{z}{(z-a)^2}\Leftrightarrow n\,a^{n-1}u(n-1)$$

$$K_2=\frac{d}{dz}[(z-a)^3\frac{X(z)}{z}]\Big|_{z=a}$$

$$\frac{z}{z-a}\Leftrightarrow a^n u(n)$$

$$K_3=\frac{1}{2!}\frac{d^2}{dz^2}[(z-a)^3\frac{X(z)}{z}]\Big|_{z=a}$$

$$\therefore\quad x(n)=\frac{K_1}{2}n(n-1)\,a^{n-2}u(n-2)+K_2na^{n-1}u(n-1)+K_3a^nu(u)$$

Inverse Z transform —— Rational form

Conjugated complex poles

$$\frac{X(z)}{z} = \frac{K_1}{z - z_1} + \frac{K_2}{z - z_2} \qquad z_{1,2} = c \pm \mathrm{j}d = ae^{\pm \mathrm{j}\beta}$$

$$K_1 = (z - c - jd)\frac{X(z)}{z}\Big|_{z=c+jd} = |K_1| \angle\theta_1 = A + jB$$

If $X(z)$ is rational form with all real coefficients, there is

$$K_2 = K_1^* = |K_1| \angle -\theta_1 = A - jB$$

$$\begin{aligned} x(n) &= K_1 a^n e^{j\beta n} + K_2 a^n e^{-j\beta n} \\ &= |K_1| a^n e^{j(\beta n + \theta_1)} + |K_1| a^n e^{-j(\beta n + \theta_1)} \\ &= 2|K_1| a^n \cos(\beta n + \theta_1) \end{aligned}$$

Example 4

$$X(z)=\frac{z^3+6}{(z+1)(z^2+4)} \qquad |z|>2$$

Solution:
$$\frac{X(z)}{z}=\frac{z^3+6}{z(z+1)(z^2+4)}=\frac{K_1}{z}+\frac{K_2}{z+1}+\frac{K_3}{z-j2}+\frac{K_3^*}{z+j2}$$

$$p_1=0, \quad p_2=-1, \qquad p_{3,4}=\pm j2=2e^{\pm j\frac{\pi}{2}}$$

$$K_1=\frac{6}{4}=\frac{3}{2} \qquad K_2=\frac{z^3+6}{z(z^2+4)}\bigg|_{z=-1}=\frac{5}{-5}=-1$$

$$K_3=\frac{z^3+6}{z(z+1)(z+j2)}\bigg|_{z=j2}=\frac{-j8+6}{j2(1+j2)\cdot j4}=\frac{-3+j4}{4(1+j2)}=\frac{\sqrt{5}}{4}\angle 63.5^\circ$$

$$\therefore \qquad x(n)=\frac{3}{2}\delta(n)-(-1)^n+\frac{\sqrt{5}}{2}(2)^n\cos(\frac{\pi}{2}n+63.5^\circ) \qquad n\ge 0$$

Properties of Z transform

(1) Time Shifting $\quad x[n-n_0] \longleftrightarrow z^{-n_0}X(z)$

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow$ ROC $z \neq 0$ (maybe)

$n_o < 0 \Rightarrow$ ROC $z \neq \infty$ (maybe)

Properties of Z transform

(6)

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$, can be bigger if there is pole/zero cancellation. *e.g.*

$$\begin{aligned} H(z) &= \frac{1}{z-a}, \quad |z| > a \\ X(z) &= z - a, \quad z \neq \infty \\ Y(z) &= 1 \quad \text{ROC all } z \end{aligned}$$

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}}$$

— The System Function

H(z) + ROC tells us everything about system

Properties of Z transform

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$\Downarrow$ No poles at ∞, if $M \leq N$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then degree $N(z) \leq$ degree $D(z)$

Properties of Z transform

Example:

$$\boxed{H(z)=\frac{2z+3}{z-0.5}}$$

$$H(z)=\frac{2+3z^{-1}}{1-0.5z^{-1}}$$

$$\Rightarrow Y(z)=\frac{2+3z^{-1}}{1-0.5z^{-1}}X(z)$$

$$\Rightarrow Y(z)-0.5z^{-1}Y(z)=2X(z)+3z^{-1}X(z)$$

$$\Rightarrow y[n]-0.5y[n-1]=2x[n]+3x[n-1]$$

$$y[n]=0.5y[n-1]+2x[n]+3x[n\text{-}1]$$

$$\boxed{H(z)=\frac{2z^2+3z}{z-0.5}}$$

$$H(z)=\frac{2z+3}{1-0.5z^{-1}}$$

$$\Rightarrow Y(z)=\frac{2z+3}{1-0.5z^{-1}}X(z)$$

$$\Rightarrow Y(z)-0.5z^{-1}Y(z)=2zX(z)+3X(z)$$

$$\Rightarrow y[n]-0.5y[n-1]=2x[n+1]+2x[n]$$

$$y[n]=0.5y[n-1]+2x[n+1]+2x[n]$$

Properties of Z transform

(8) Stability

LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

$\Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

Z transform of basic signals and relations

Z definition

$$\delta[n] \xrightarrow{Z} 1 \quad \{z\}: all$$

time-domain shift

$$\delta[n-m] \xrightarrow{Z} z^{-m} \quad \{z\}: all, \notin 0 \ or \ \infty$$

summation

$$u[n] \xrightarrow{Z} \frac{1}{1-z^{-1}} \quad |z| > 1$$

reverse

$$\boxed{x[-n] \xleftrightarrow{Z} X(z^{-1}),\ \mathrm{ROC} = R^{-1}}$$

$$u[-n-1] = u[-n] - \delta[n] \xrightarrow{Z} \frac{1}{1-z} - 1 = \frac{-1}{1-z^{-1}} \quad |z| < 1$$

scaling

$$\boxed{z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right),\ \mathrm{ROC} = |z_0| R}$$

scaling

$$a^n u[n] \xrightarrow{Z} \frac{1}{1-(z/a)^{-1}} = \frac{1}{1-az^{-1}} \quad |z| > |a|$$

$$a^n u[-n-1] \xrightarrow{Z} \frac{1}{1-(z/a)^{-1}} = \frac{-1}{1-az^{-1}} \quad |z| < |a|$$

Differential

$$\boxed{nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}, \quad \text{same ROC}}$$

Differential

$$na^n u[n] \xrightarrow{Z} z\frac{az^{-2}}{\left(1-az^{-1}\right)^2} = \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad |z| > |a|$$

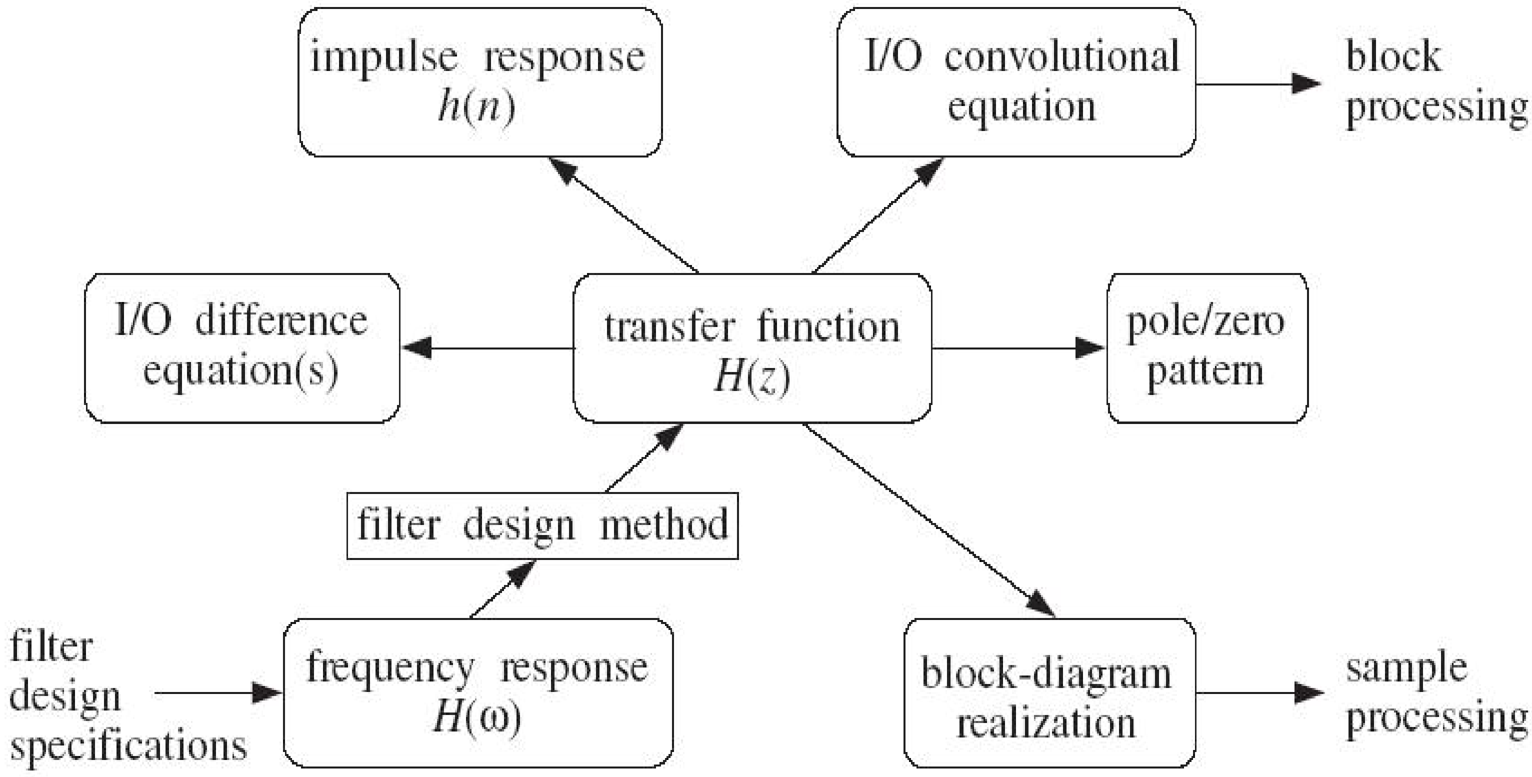
$$n\,a^n u[-n-1] \xrightarrow{Z} \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad |z| < |a|$$

3.1 Filter system representation

The subject of digital filter design is very extensive. Here, we present only a small available design methods — our objective being to give some flavor and practical methods in digital filter designs.

Filter design —— construct the transfer function of a filter that meets prescribed frequency response specifications

3.1 Filter system representation



- Frequency response $H(\omega)$
- Transfer function $H(z)$
- I/O difference equation

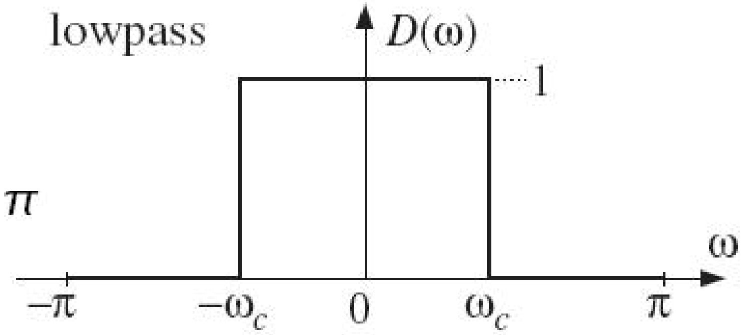
- Impulse response $h(n)$
- I/O convolutional equation
- Pole/zero pattern

3.1 Ideal filter

Frequency response $D(\omega)$ and Impulse response $d(k)$

$$D(\omega)=\sum_{k=-\infty}^{\infty} d(k)e^{-j\omega k} \underset{\text{Inverse DTFT}}{\overset{\text{DTFT}}{\Leftrightarrow}} d(k)=\int_{-\pi}^{\pi} D(\omega)e^{j\omega k}\frac{d\omega}{2\pi}$$

For an ideal lowpass filter



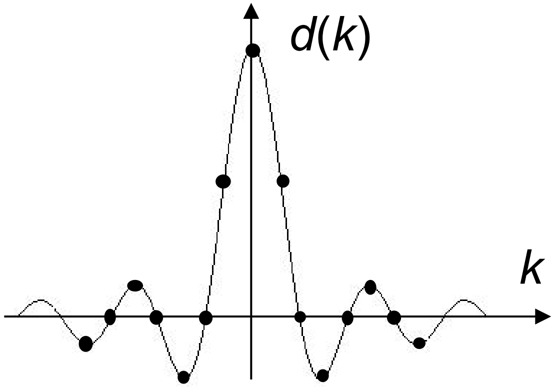
$$D(\omega)=\begin{cases} 1, & \text{if } -\omega_c \le \omega \le \omega_c \\ 0, & \text{if } -\pi \le \omega < -\omega_c, \text{ or } \omega_c < \omega \le \pi \end{cases}$$

$$d(k) = \int_{-\pi}^{\pi} D(\omega)e^{j\omega k}\frac{d\omega}{2\pi} = \int_{-\omega_c}^{\omega_c} 1\cdot e^{j\omega k}\frac{d\omega}{2\pi}$$

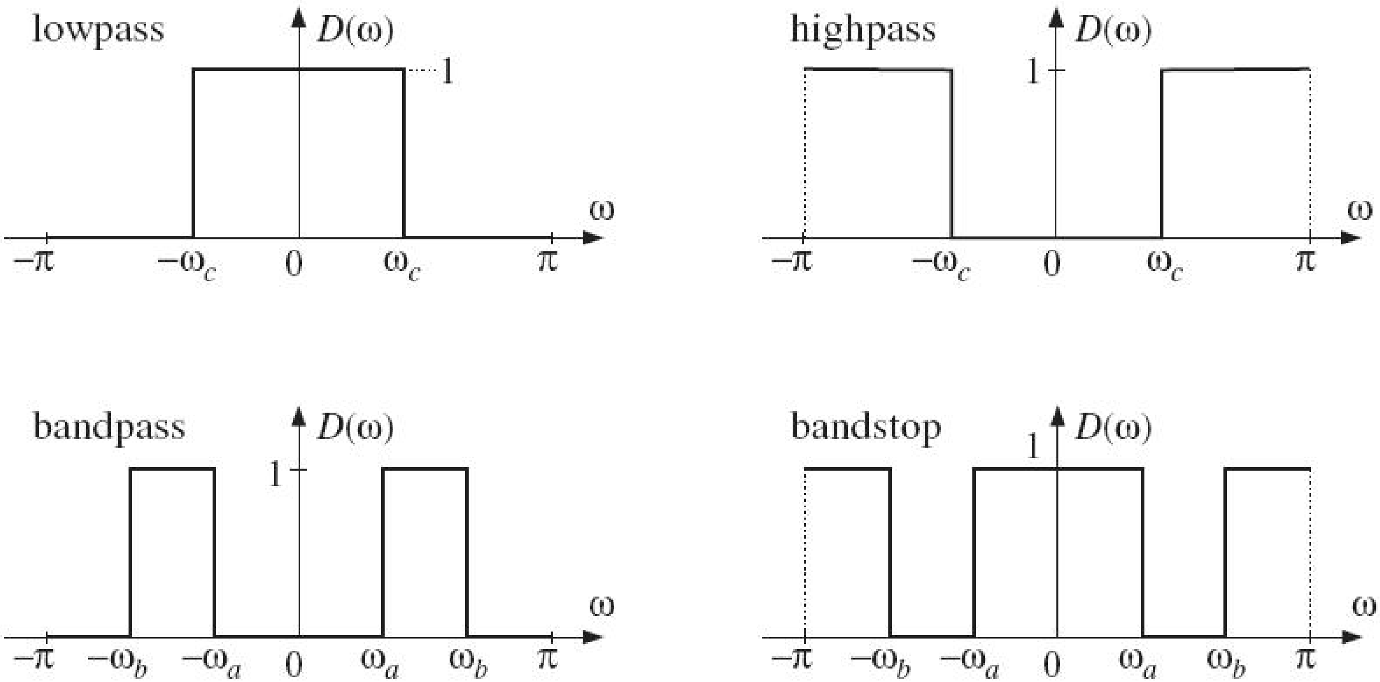
$$= \left[\frac{e^{j\omega k}}{2\pi jk}\right]_{-\omega_c}^{\omega_c} = \frac{e^{j\omega_c k}-e^{-j\omega_c k}}{2\pi jk}$$

$$= \frac{\sin(\omega_c k)}{\pi k} \qquad -\infty < k < \infty$$

$$d(0)=\frac{\omega_c}{\pi} \qquad k=0$$



3.1 Ideal filter

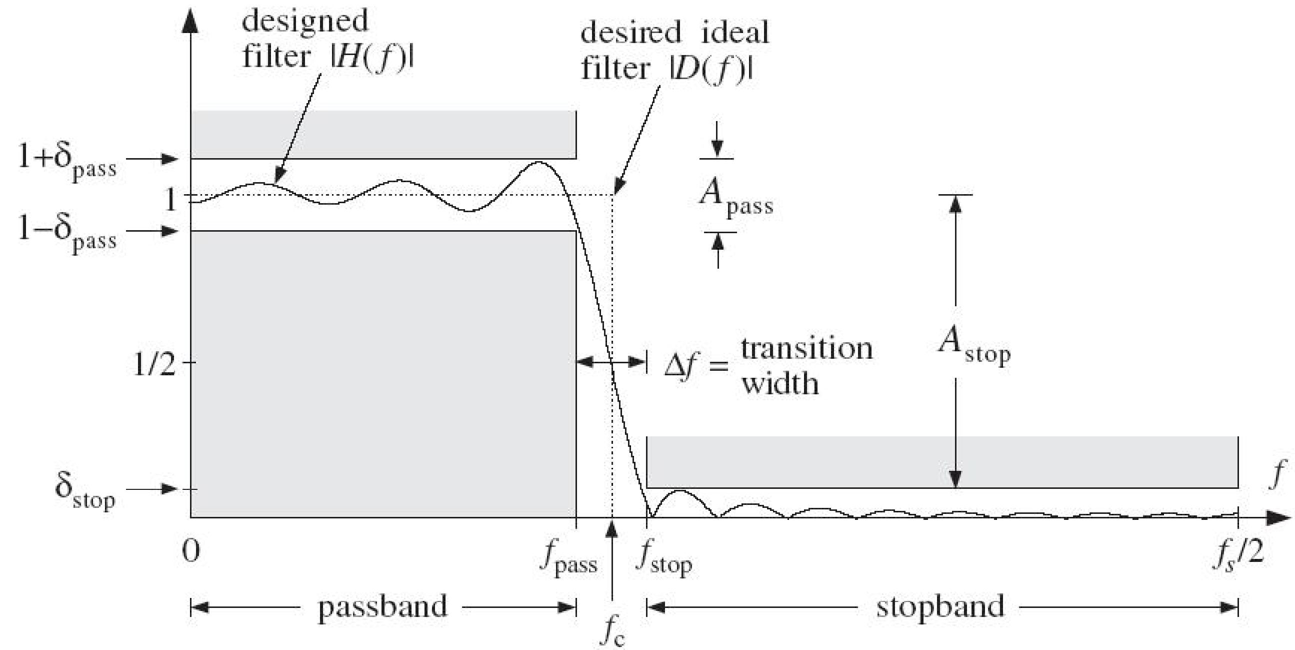


(highpass filter) $$d(k) = \delta(k) - \frac{\sin(\omega_c k)}{\pi k}$$

(bandpass filter) $$d(k) = \frac{\sin(\omega_b k) - \sin(\omega_a k)}{\pi k}$$

(bandstop filter) $$d(k) = \delta(k) - \frac{\sin(\omega_b k) - \sin(\omega_a k)}{\pi k}$$

3.1 Practical filter



Normalized digital frequencies

$$\omega_{pass} = \frac{2\pi f_{pass}}{f_s}, \quad \omega_{stop} = \frac{2\pi f_{stop}}{f_s}, \quad \omega_c = \frac{2\pi f_c}{f_s}, \quad \Delta\omega = \frac{2\pi \Delta f}{f_s}$$

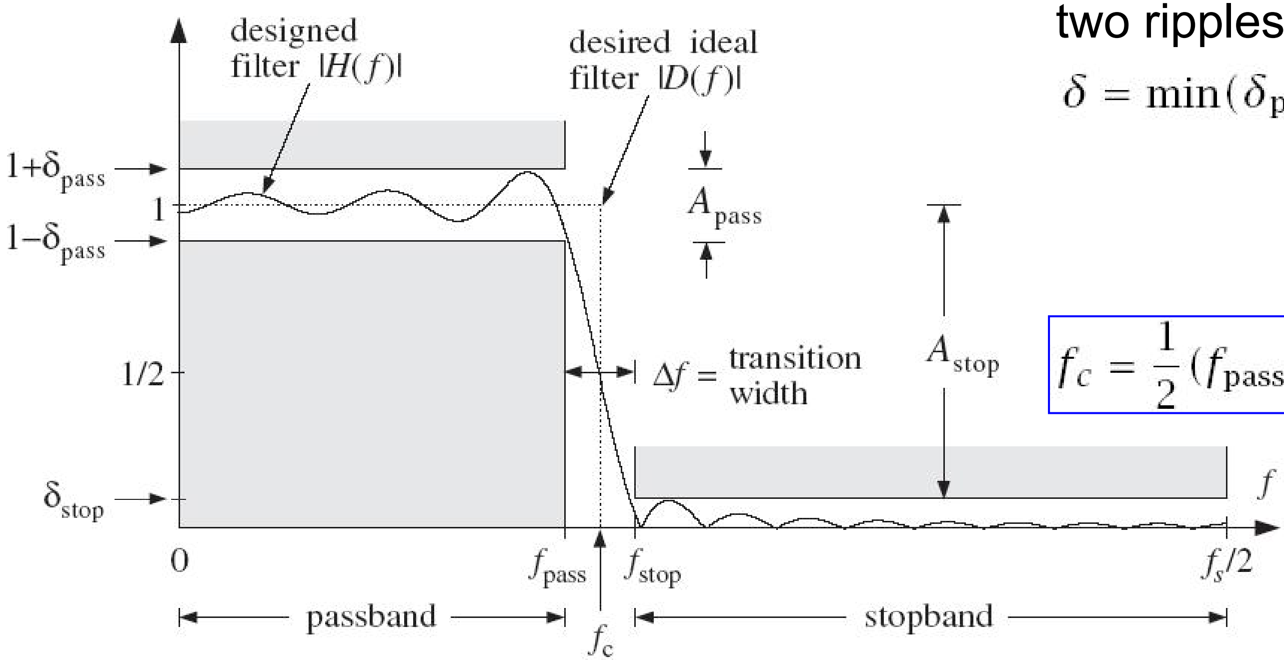
3.1 Practical filter

The designer can arbitrarily specify the amount of passband and stopband overshoot δ_{pass}, δ_{stop}, as well as the transition width Δf. The overshoots are expressed in dB.

$$A_{\text{pass}} = 20\log_{10}\left(\frac{1+\delta_{\text{pass}}}{1-\delta_{\text{pass}}}\right), \quad A_{\text{stop}} = -20\log_{10}\delta_{\text{stop}}$$

Normally design filter with equal passband and stopband ripples. Therefore, specify the smaller of the two ripples, e.g. stopband ripple

$$\delta = \min(\delta_{\text{pass}}, \delta_{\text{stop}}) \quad \boxed{\delta = \delta_{\text{stop}}}$$



$$\boxed{f_c = \frac{1}{2}(f_{\text{pass}} + f_{\text{stop}}), \quad \Delta f = f_{\text{stop}} - f_{\text{pass}}}$$

Normalized digital frequencies

$$\omega_{\text{pass}} = \frac{2\pi f_{\text{pass}}}{f_s}, \quad \omega_{\text{stop}} = \frac{2\pi f_{\text{stop}}}{f_s}, \quad \omega_c = \frac{2\pi f_c}{f_s}, \quad \Delta\omega = \frac{2\pi\Delta f}{f_s}$$

3.1 Linear difference equations

The discrete-time Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

The difference equations can be solved in a manner exactly analogous to that for differential equations. The solution $y[n]$ can be written as the sum of a particular solution and a solution to the homogeneous equation.

Alternatively, to solve it, rewrite the difference equation into the form as

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \underbrace{\sum_{k=1}^{N} a_k y[n-k]}_{\text{previous outputs}}\right\}$$

It is a recursive equation for $N > 0$ because we need to know the previous outputs to calculate the current output. And the impulse response may be infinite, so it is an infinite impulse response (IIR) system.

$$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k] - \sum_{k=1}^{N} a_k h[n-k]\right\} = \begin{cases} \dfrac{b_n}{a_0} - \displaystyle\sum_{k=1}^{N} \frac{a_k}{a_0} h[n-k], 0 \le n \le M \\ -\displaystyle\sum_{k=1}^{N} \frac{a_k}{a_0} h[n-k], \quad \text{otherwise} \end{cases}$$

3.1 DT LTI system analysis with Z-transform

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Z transforms on both sides to obtain the rational expression:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example 1

$$y[n] = 0.5y[n-1] + 2x[n] + 3x[n\text{-}1]$$

Initial rest

$n \le -1, x[n] = 0$ and $y[n] = 0,$

$x = [1, 2, 3], n \ge 0,$

For each new sample $x[n]$, do

$y := 0.5w_1 + 2x + 3v_1$

$w_1 := y$

$v_1 := x$

$Step 1:$

$y[0] := 0.5y[-1] + 2x[0] + 3x[-1] = 2x[0] = 2$

$w_1 := y[0] = 2$

$v_1 := x[0] = 1$

$Step 2:$

$y[1] := 0.5w_1 + 2x[1] + 3v_1 = 0.5 \times 2 + 2 \times 2 + 3 \times 1 = 8$

$w_1 := y[1] = 8$

$v_1 := x[1] = 2$

$Step 3:$

$y[2] := 0.5w_1 + 2x[2] + 3v_1 = 0.5 \times 8 + 2 \times 3 + 3 \times 2 = 16$

$w_1 := y[2] = 16$

$v_1 := x[2] = 3$

$Step 4:$

$y[3] := 0.5 \times 16 + 2 \times 0 + 3 \times 3 = 17$

$w_1 := y[3] = 17$

$v_1 := x[3] = 0$

$Step 5:$

$y[4] := 0.5 \times 17 + 2 \times 0 + 3 \times 0 = 8.5$

$w_1 := y[4] = 8.5$

$v_1 := x[4] = 0$

$\vdots$

Example 2

$$y[n] = 0.5y[n-1] + 2x[n] + 3x[n\text{-}1]$$

$$\textit{Initial rest}$$

$$n \le -1, x[n] = 0 \text{ and } y[n] = 0,$$

$$x = [1, 2, 3], n \ge 0,$$

$$h[n] = 0.5h[n-1] + 2\delta[n] + 3\delta[n\text{-}1]$$

$$h[0] = 0.5 \times 0 + 2 \times 1 + 3 \times 0 = 2$$

$$h[1] = 0.5 \times 2 + 2 \times 0 + 3 \times 1 = 4$$

$$h[2] = 0.5 \times 4 + 2 \times 0 + 3 \times 0 = 2$$

$$\vdots$$

$$h[n] = 0.5^{n-1} \times 4.5, n \ge 1$$

3.1 Linear difference equations

If $N = 0$, it is non-recursive and it is an explicit function with inputs

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} x[n-k]$$

The impulse function can be written as

$$h[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \delta[n-k] = \begin{cases} \frac{b_n}{a_0}, 0 \le n \le M \\ 0, \quad \text{otherwise} \end{cases}$$

So the impulse response will last actually finite duration, and it is a *finite impulse response* (FIR) system.

Example 3

$$y[n] = 2x[n]+3x[n\text{-}1]$$

Initial rest

$n \le -1, x[n] = 0$ and $y[n] = 0,$

$x = [1, 2, 3], n \ge 0,$

For each new sample $x[n]$, do

$y := 2x + 3v_1$

$v_1 := x$

$Step 1:$

$y[0] := 2x[0] + 3x[-1] = 2x[0] = 2$

$v_1 := x[0] = 1$

$Step 2:$

$y[1] := 2x[1] + 3v_1 = 2\times 2 + 3\times 1 = 7$

$v_1 := x[1] = 2$

$Step 3:$

$y[2] := 2x[2] + 3v_1 = 2\times 3 + 3\times 2 = 12$

$v_1 := x[2] = 3$

$Step 4:$

$y[3] := 2\times 0 + 3\times 3 = 9$

$v_1 := x[3] = 0$

$Step 5:$

$y[4] := 2\times 0 + 3\times 0 = 0$

$v_1 := x[4] = 0$

$$\frac{Y(z)}{X(z)} = \frac{2+3z^{-1}}{1} = \frac{2z+3}{z}$$

$$\boxed{\begin{aligned} &h[n] = 2\delta[n]+3\delta[n\text{-}1] \\ &\Rightarrow \\ &h[0] = 2, \\ &h[1] = 3. \end{aligned}}$$

3.2 FIR filter properties

FIR filter design —— construct the finite impulse response coefficient vector $\mathbf{h} = [h_0, h_1, \ldots, h_{N-1}]$ to meet a set of desired specifications.

I/O equation $y_n = b_0 x_n + b_1 x_{n-1} + \cdots + b_L x_{n-L}$

Transfer function $H(z) = N(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_L z^{-L}$

Impulse response $\mathbf{h} = [b_0, b_1, \ldots, b_L]$

Advantages: linear phase property and their guaranteed stability because of the absence of poles.

Disadvantage: requirement of sharp filter specifications can lead to long filter lengths, consequently increasing their computational cost.

3.2 FIR Window Method

The window method consists of truncating, or rectangularly windowing, the doublesided *d*(*k*) to a finite length. For example, we may keep only the coefficients:

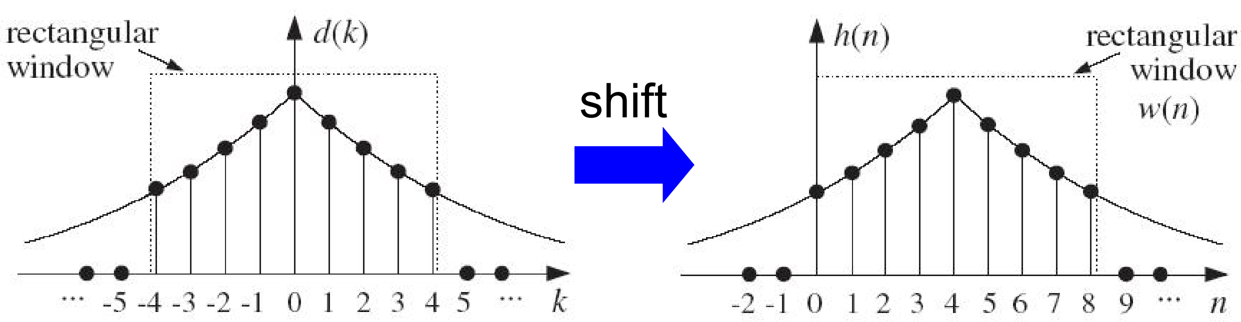
$$d(k) = \int_{-\pi}^{\pi} D(\omega)\, e^{j\omega k} \frac{d\omega}{2\pi}, \qquad -M \le k \le M$$

FIR impulse response approximating the infinite ideal response:

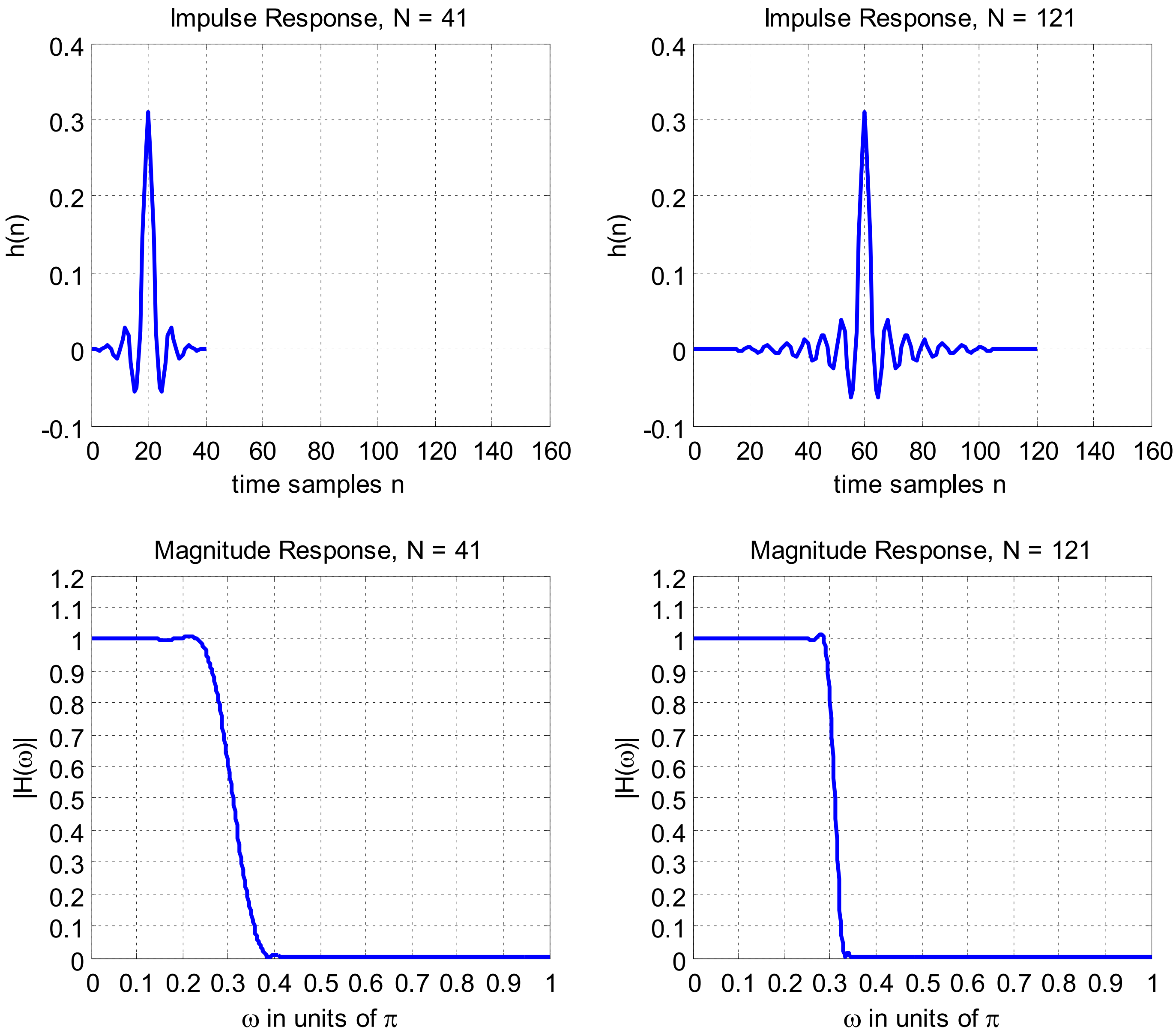
$$\mathbf{d} = [d_{-M}, \ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots, d_M]$$

To make the filter **causal** we may shift the time origin to the left of the vector and re-index the entries accordingly:

$$h(n) = d(n-M), \qquad n = 0, 1, \ldots, N-1$$



3.2 FIR Window Method



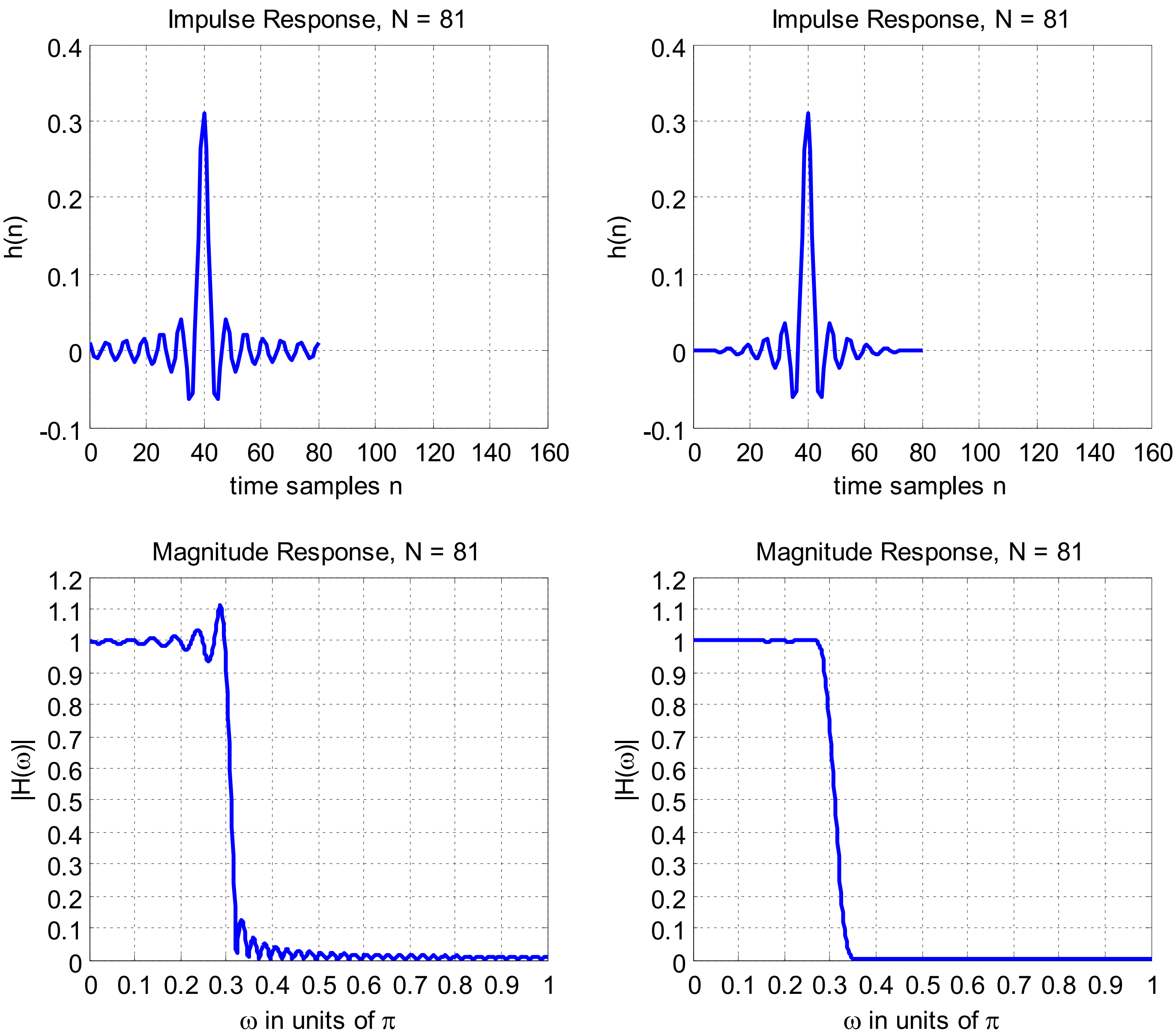
***N* increases**

1. The ripple size decreases, resulting in flatter passband and stopband.

2. The transition width decreases.

3. The largest ripples tend to cluster near the passband-to-stopband discontinuity and do not get smaller (overshoot 8.9%- Gibbs phenomenon).

3.2 FIR Window Method



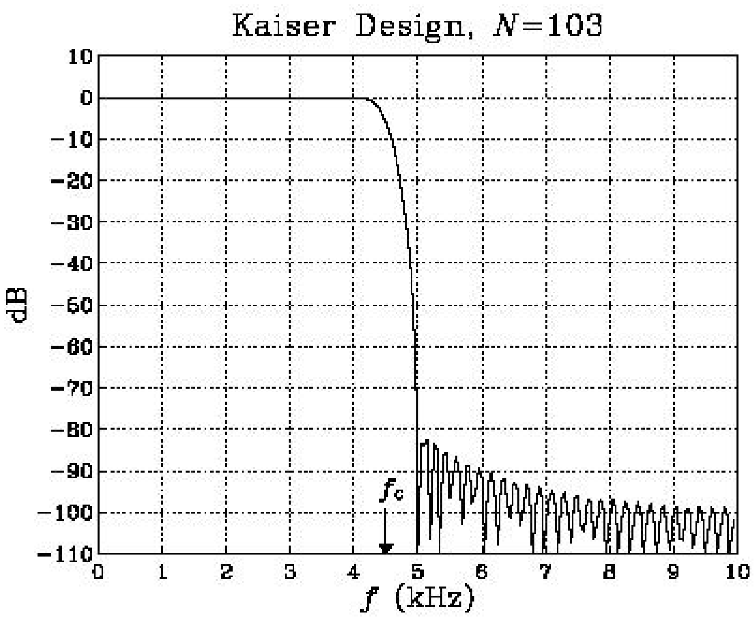
Rect vs. Hamming

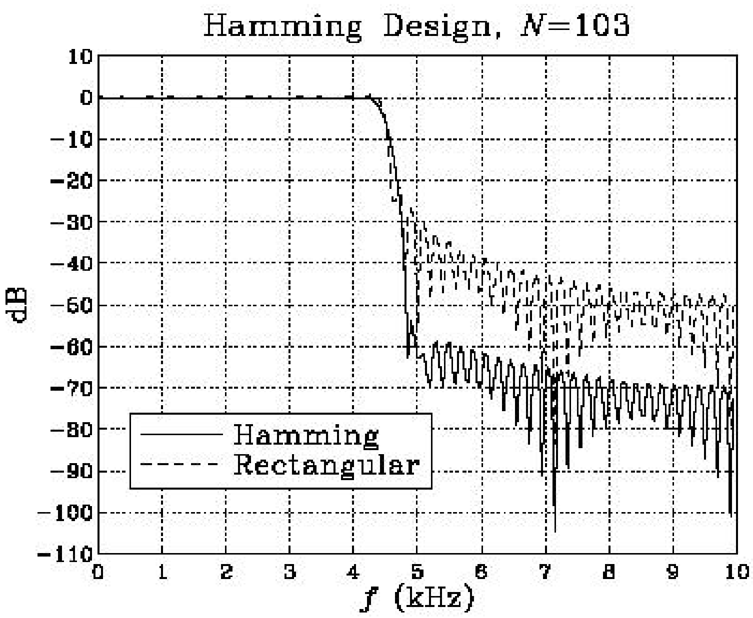
1. The ripples of the rectangular window are virtually eliminated from the Hamming window (overshoot 0.2%)

2. The price for eliminating the ripples is loss of resolution, which is reflected into a wider transition width.

3.2 FIR Window Method

Specifications for rectangular, Hamming, and Kaiser windows.





3.2 FIR filtering with the convolution method

Consider a causal FIR filter of order M with impulse response $h(n)$, $n = 0, 1, \ldots, M$.

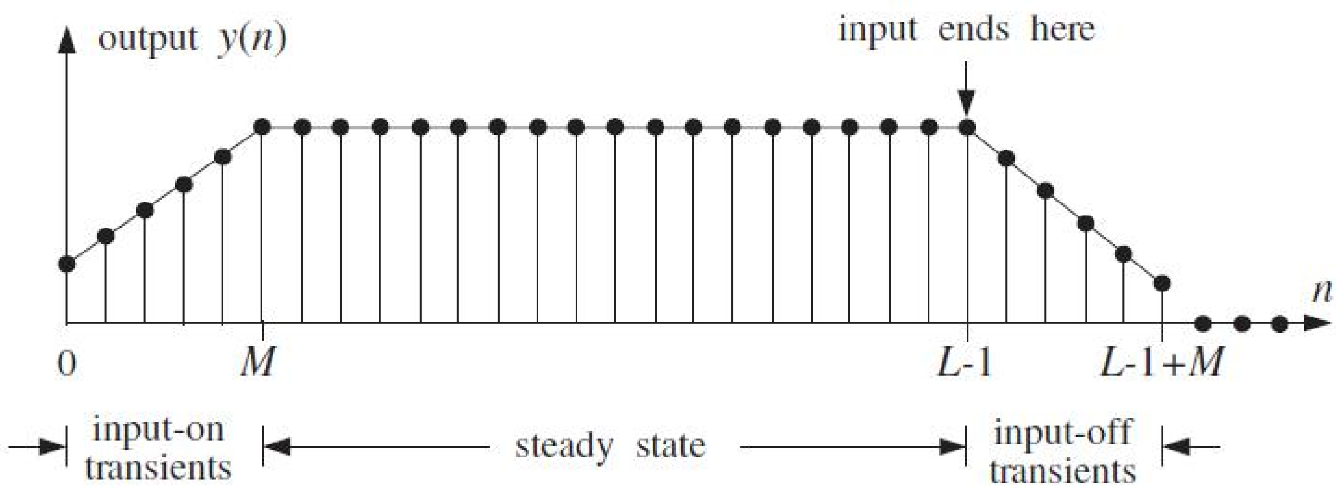
$$\mathbf{h} = [h_0, h_1, \ldots, h_M] \qquad \text{Its length} \quad \boxed{L_h = M + 1}$$

The convolution of the length-L input $\mathbf{x}$ $\quad y(n) = \sum_m h(m)x(n-m)$

$$\mathbf{h} = \boxed{M+1}$$

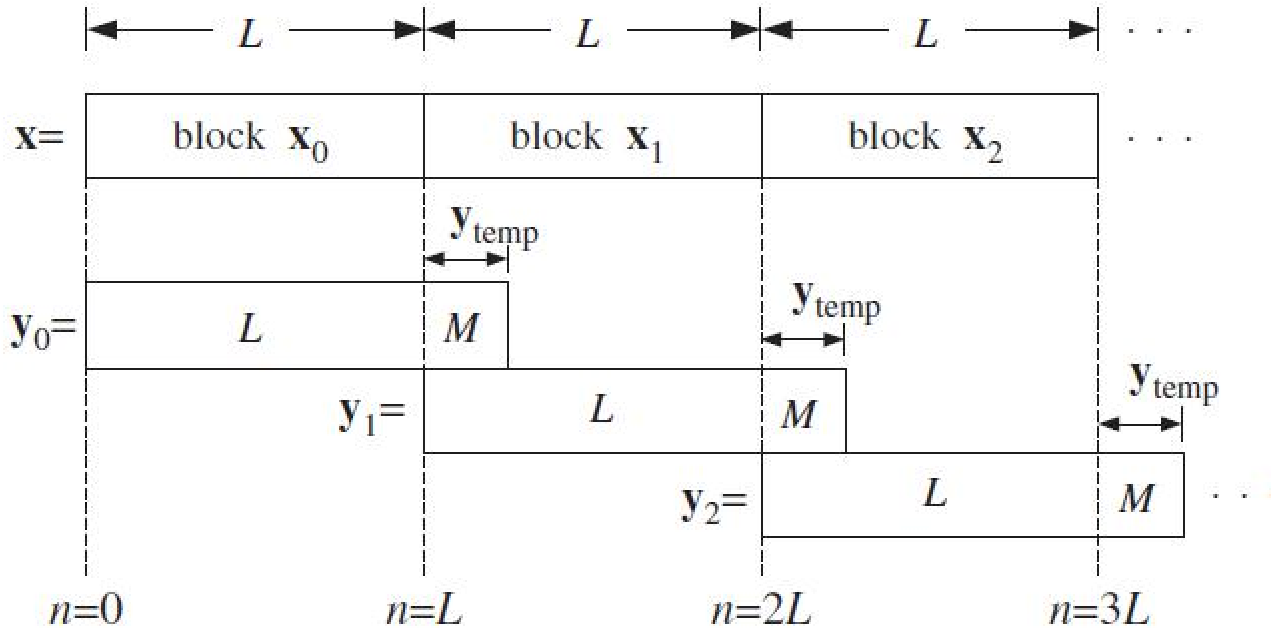
$$\mathbf{x} = \boxed{\quad L \quad}$$

$$\mathbf{y} = \mathbf{h} * \mathbf{x} = \boxed{\quad L \quad \vdots \quad M \quad}$$



3.2 Overlap-add block convolution method

A practical approach is to divide the long input into *contiguous* non-overlapping blocks of manageable length, say L samples, then filter each block and piece the output blocks together to obtain the overall output, as shown in Fig.   Thus, processing is carried out block by block.



$$\mathbf{y}_0 = \mathbf{h} * \mathbf{x}_0$$
$$\mathbf{y}_1 = \mathbf{h} * \mathbf{x}_1$$
$$\mathbf{y}_2 = \mathbf{h} * \mathbf{x}_2$$

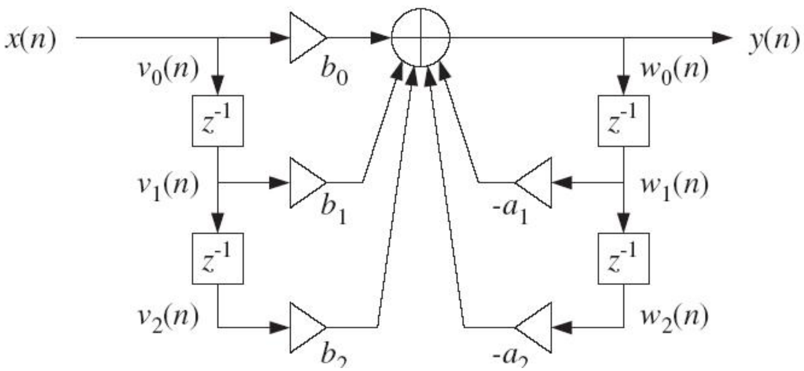
3.3 IIR filter properties

IIR filter design —— construct the numerator and denominator coefficient vectors **b** = [b_0, b_1, . . . , h_M] and **a** = [1, a_1, . . . , a_M] to meet a set of desired specifications.

Second-order IIR filter

$$H(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$y_n = -a_1 y_{n-1} - a_2 y_{n-2} + b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}$$



for each input sample x do:

$$v_0 = x$$
$$w_0 = -a_1 w_1 - a_2 w_2 + b_0 v_0 + b_1 v_1 + b_2 v_2$$
$$y = w_0$$
$$v_2 = v_1, \quad w_2 = w_1$$
$$v_1 = v_0, \quad w_1 = w_0$$

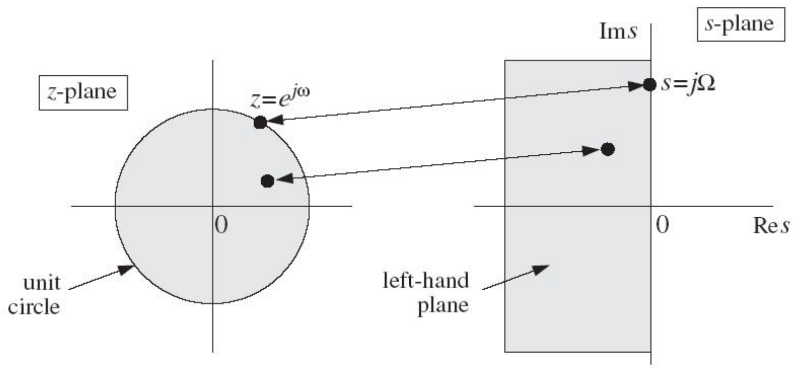
Advantages: low computational cost and their efficient implementation in cascade of second-order sections.

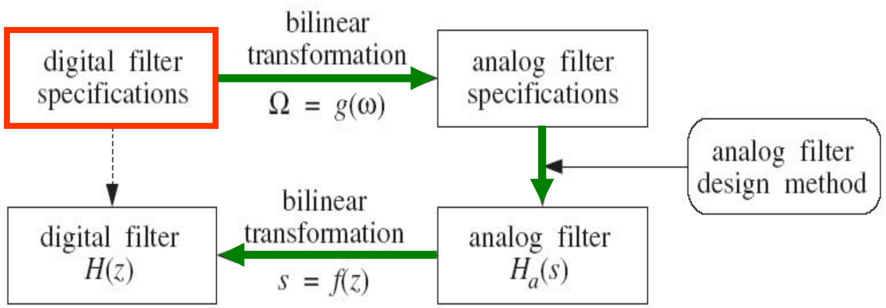
Disadvantage: potential for instabilities introduced when the quantization of the coefficients pushes the poles outside the unit circle.

For IIR filters, linear phase cannot be achieved exactly over the entire Nyquist interval, but approximately over the relevant passband of the filter.

3.3 IIR Bilinear Transformation

Simplest and effective method of designing IIR filters with prescribed magnitude response specifications is the **bilinear transformation** method.
The method maps the digital filter into an equivalent analog filter according to well-developed analog filter design methods, such as **Butterworth**, **Chebyshev**, or **elliptic** filter designs. The designed analog filter is then mapped back into the desired digital filter.





$$z = e^{sT}$$

$$z = e^{sT} = \frac{e^{sT/2}}{e^{-sT/2}} = \frac{1 + (\frac{s}{2}T) + \frac{1}{2}(\frac{s}{2}T)^2 + \ldots}{1 + (-\frac{s}{2}T) + \frac{1}{2}(-\frac{s}{2}T)^2 + \ldots} \approx \frac{1 + \frac{s}{2}T}{1 - \frac{s}{2}T}$$

$$z - z\frac{s}{2}T \approx 1 + \frac{s}{2}T$$

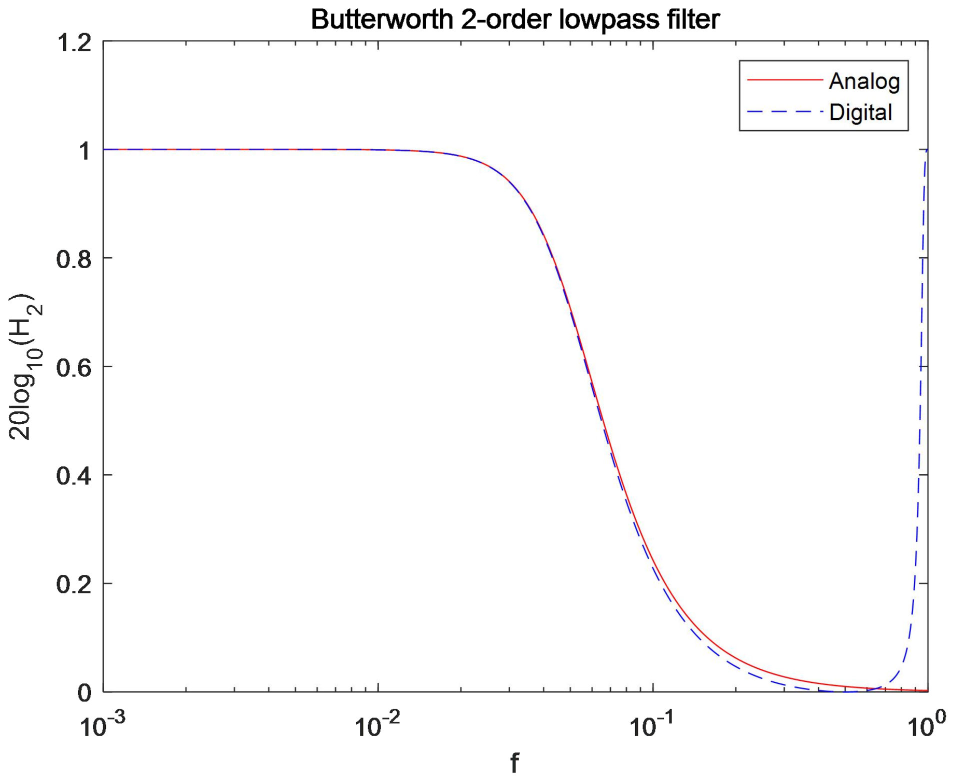
$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

3.3 IIR Bilinear Transformation

Butterworth lowpass filter (2-order)

$$H(s)=\frac{1}{(\frac{s}{w_c})^2+\sqrt{2}(\frac{s}{w_c})+1}$$

$$s=\frac{2}{T}\left(\frac{z-1}{z+1}\right) \qquad T=1/f_s$$



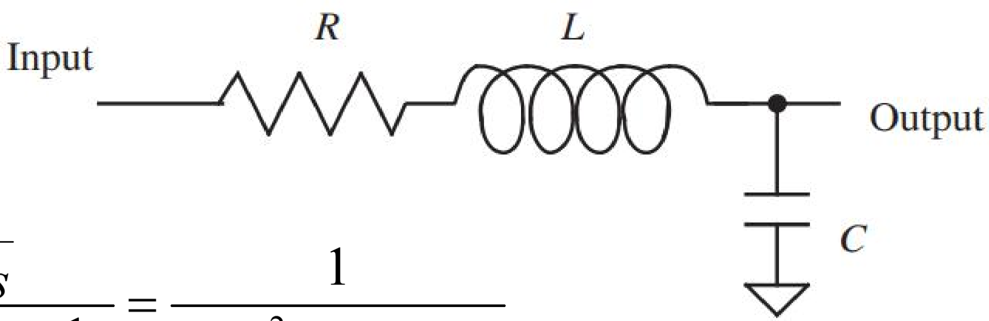
Butterworth lowpass filter (even-order)[1]

$$H(s)=\prod_{i=1}^{N/2}(\frac{\omega_c{}^2}{s^2+2\cos(\theta_i)\omega_c s+\omega_c{}^2}),\ \ \theta_i=(i-0.5)\times 180/N$$

[1]Ellis G. Filters in control systems[J]. Control System Design Guide, 2012, 9: 165.

3.3 Digital filter design —— IIR Examples

Example:



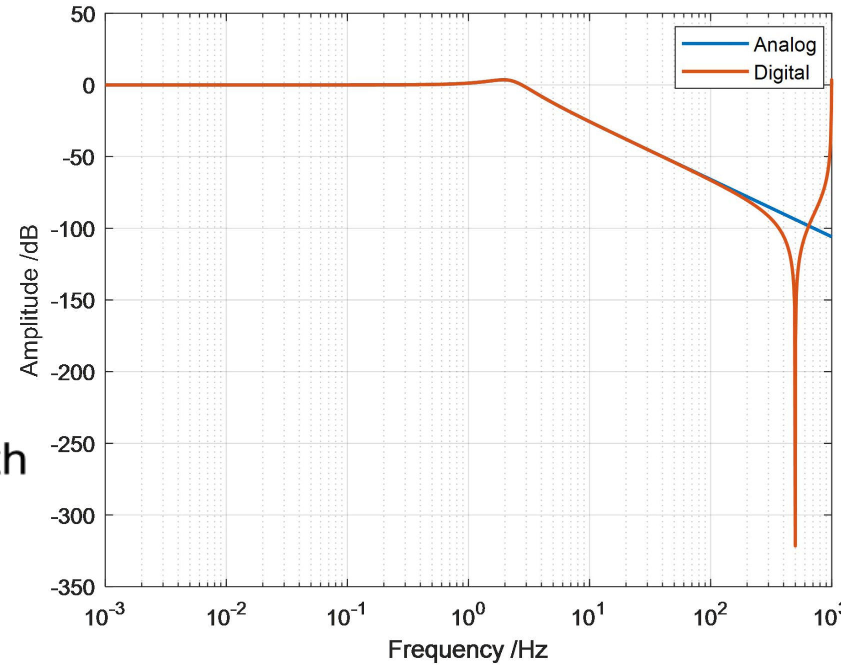
$$H(s) = \frac{Output}{Input} = \frac{\frac{1}{Cs}}{R + Ls + \frac{1}{Cs}} = \frac{1}{LCs^2 + RCs + 1}$$

$s = j\omega$

$L = 1\ \mathrm{H},\ C = 0.5\ \mathrm{F},\ R = 3\ \Omega$

$f_s = 1000\ \mathrm{Hz}$

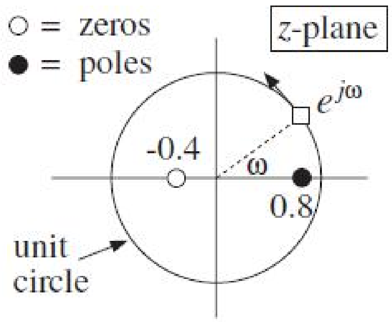
Find the transfer function $H(s)$ of the circuit shown in the figure. Calculate its digital filter using the bilinear method with the circuit parameters above.

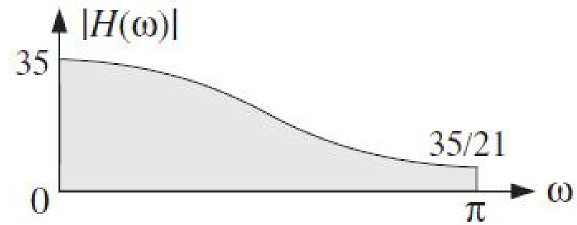


3.4 Pole/zero filter design

Example 1

$$H(z) = \frac{5(1+0.4z^{-1})}{1-0.8z^{-1}} \quad \Rightarrow \quad \boxed{H(\omega) = \frac{5(1+0.4e^{-j\omega})}{1-0.8e^{-j\omega}}}$$



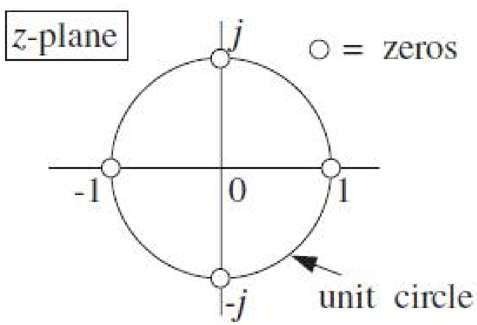


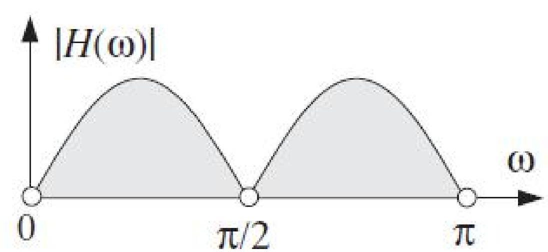
Example 2

An FIR filter is described by the I/O equation:

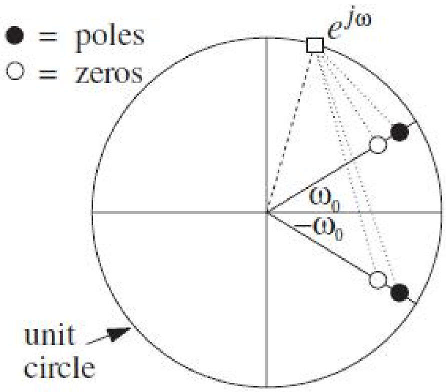
$$y(n) = x(n) - x(n-4)$$

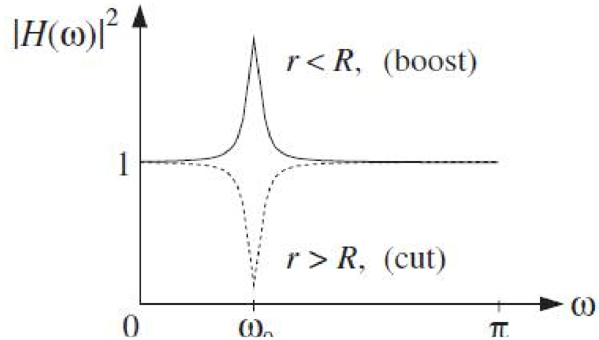
$$Y(z) = X(z) - z^{-4}X(z) \quad \Rightarrow \quad H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-4} \qquad z = 1, j, -1, -j$$





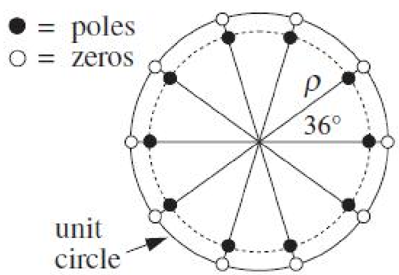
3.4 Pole/zero filter design

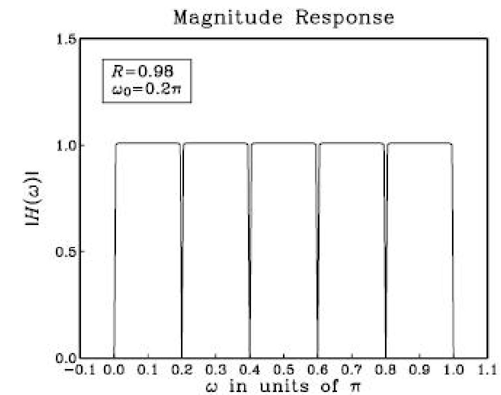


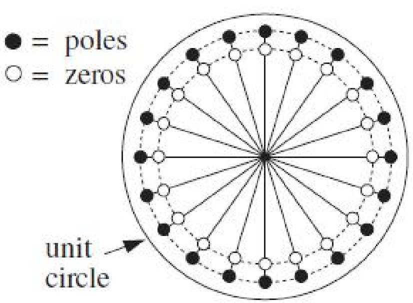


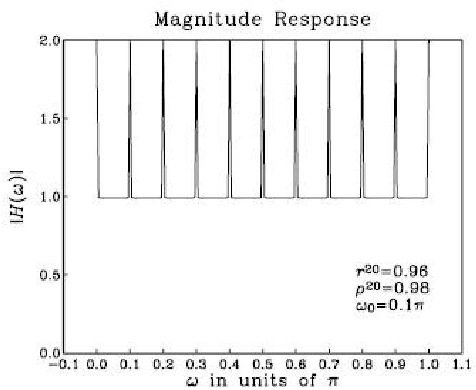
$$p = Re^{j\omega_0}, z = re^{j\omega_0}$$

Notch and comb filters









3.5 Digital filter Realization

After a filter is designed, it must be *realized* by developing a signal flow diagram that describes the filter in terms of operations on sample sequences.

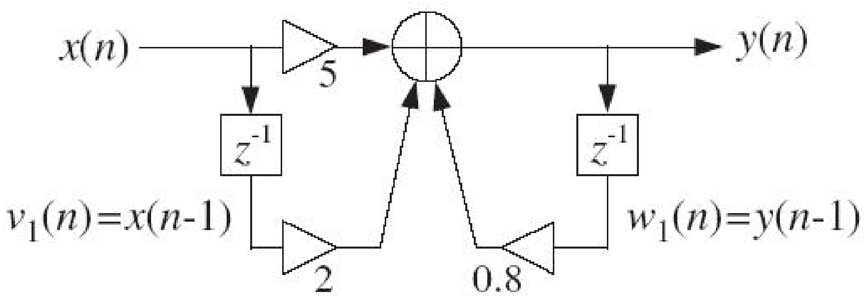
A given transfer function may be realized in many ways. All realizations may be seen as "factorizations" of the same transfer function, but different realizations will have different numerical properties.

- Efficient in terms of the number of operations or storage elements
- Numerical stability and reduced round-off error
- Better for fixed-point algorithm

3.5 Digital filter Realization

A straightforward approach for IIR filter realization **Direct form I** where the difference equation is evaluated directly.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5 + 2z^{-1}}{1 - 0.8z^{-1}}$$



for each input sample x do:

$y = 0.8w_1 + 5x + 2v_1$

$v_1 = x$

$w_1 = y$

This form is practical for small filters, but may be inefficient and impractical (numerically unstable) for complex designs

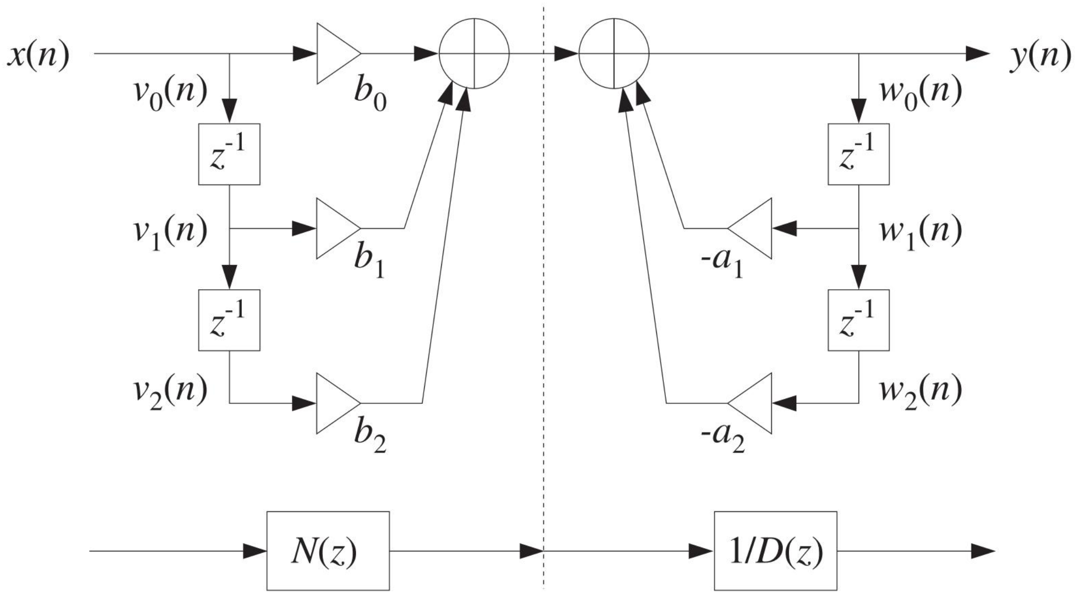
This form requires 2N delay elements (for both input and output signals) for a filter of order N

3.5 Digital filter Realization

A standard second-order sections, SOS

$$y_n = (b_0x_n + b_1x_{n-1} + b_2x_{n-2}) + (-a_1y_{n-1} - a_2y_{n-2})$$

This regrouping corresponds to splitting the big adder of the direct form realization of Fig. 7.1.1 into two parts, as shown in Fig. 7.2.1.



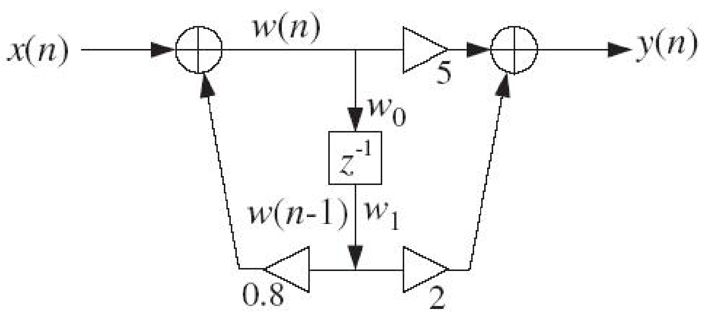
3.5 Digital filter Realization

Direct form II only needs N delay units, where N is the order of the filter, half as much as Direct Form I.

$$H(z)=\frac{Y(z)}{X(z)}=\frac{5+2z^{-1}}{1-0.8z^{-1}}$$

$$W(z)=\frac{1}{1-0.8z^{-1}}X(z)$$

$$Y(z)=(5+2z^{-1})W(z)$$



for each input sample x do:

$w_0 = 0.8w_1 + x$

$y = 5w_0 + 2w_1$

$w_1 = w_0$

The disadvantage is that Direct Form II increases the possibility of arithmetic overflow for filters of resonance.

3.5 Digital filter Realization

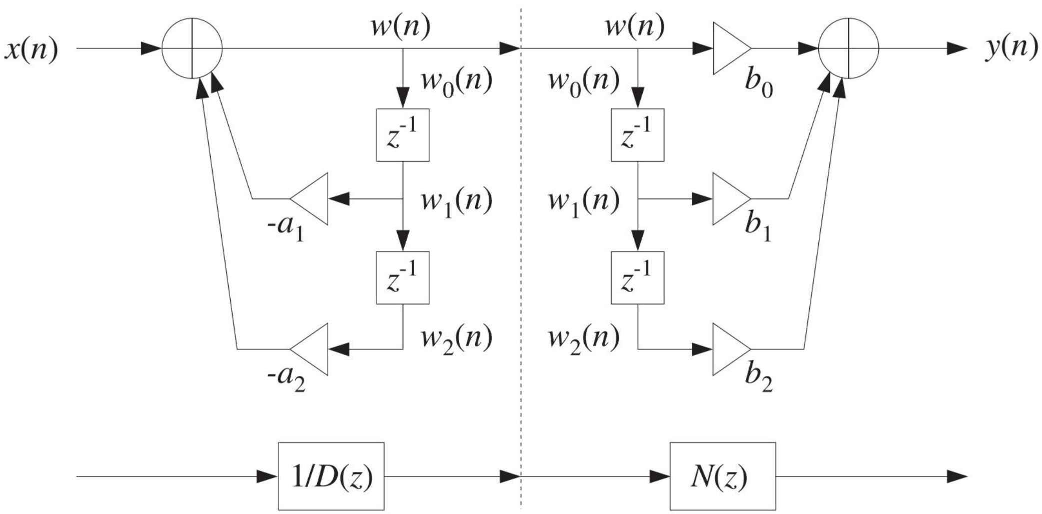


Fig. 7.2.2 Interchanging $N(z)$ and $1/D(z)$.

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z)$$

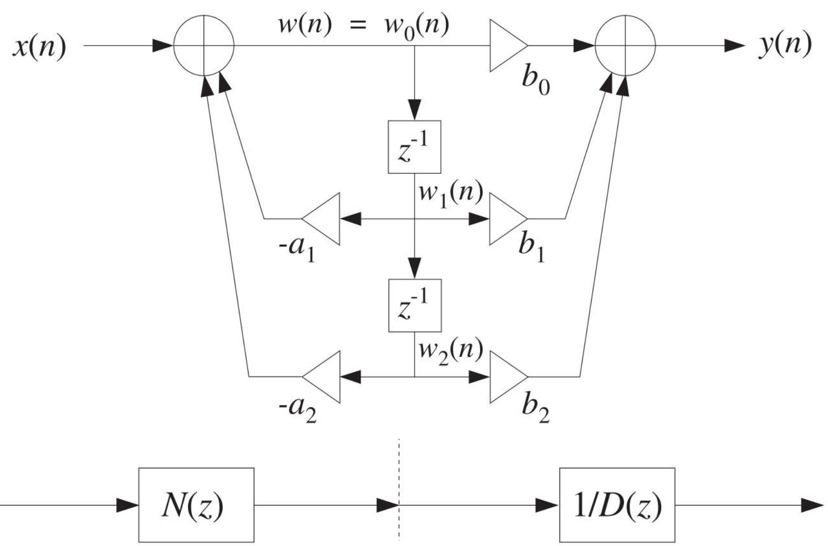
$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z)$$

$$W(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} X(z) = \frac{1}{D(z)} X(z)$$

$$Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2}) W(z) = N(z) W(z)$$

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2)$$

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2)$$



for each input sample x do:

$$w_0 = x - a_1 w_1 - a_2 w_2$$

$$y = b_0 w_0 + b_1 w_1 + b_2 w_2$$

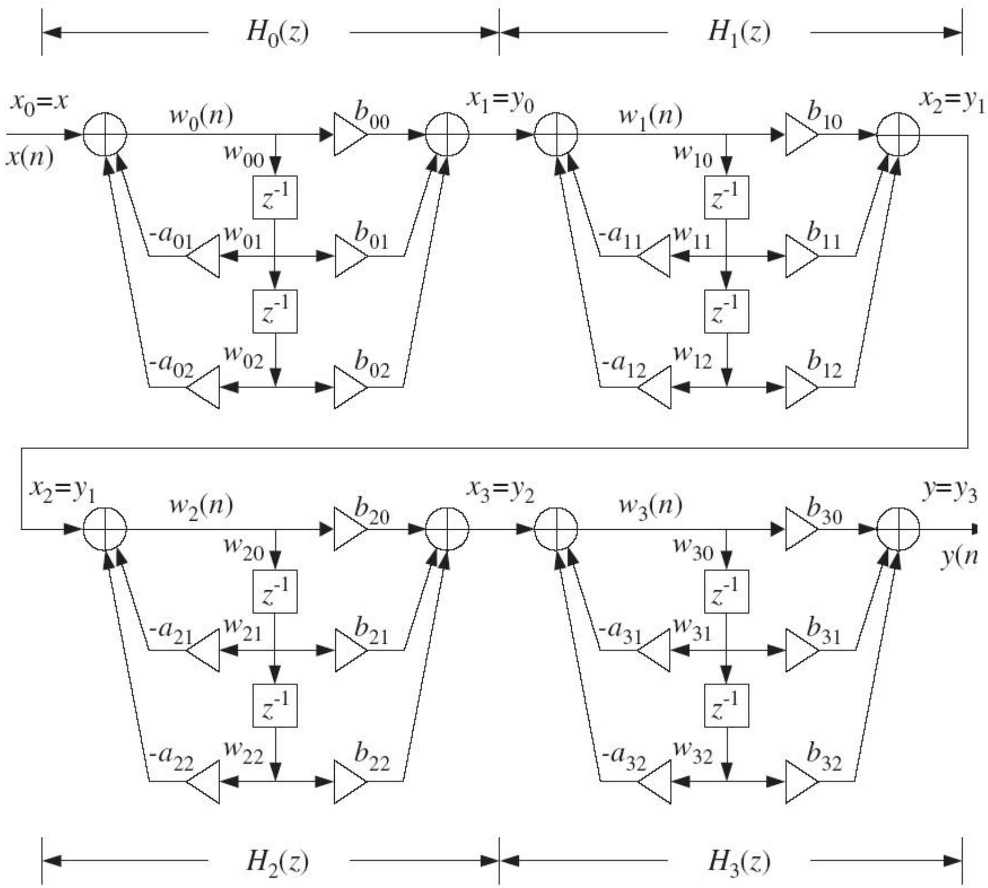
$$w_2 = w_1$$

$$w_1 = w_0$$

3.5 Digital filter Realization

Realize a higher-order (N > 2) digital filter as a cascaded series of second-order biquadratric sections

$$H(z) = \prod_{i=0}^{K-1} H_i(z) = \prod_{i=0}^{K-1} \frac{b_{i0} + b_{i1}z^{-1} + b_{i2}z^{-2}}{1 + a_{i1}z^{-1} + a_{i2}z^{-2}}$$



$x_0(n) = x(n)$

for $i = 0, 1, \ldots, K-1$ *do:*

$w_i(n) = x_i(n) - a_{i1}w_i(n-1) - a_{i2}w_i(n-2)$

$y_i(n) = b_{i0}w_i(n) + b_{i1}w_i(n-1) + b_{i2}w_i(n-2)$

$x_{i+1}(n) = y_i(n)$

$y(n) = y_{K-1}(n)$

3.5 Digital filter Realization

Once the denominator and numerator polynomials have been factored into their quadratic factors, each quadratic factor from the numerator may be paired with a quadratic factor from the denominator to form a second-order section.

This pairing of numerator and denominator factors and the ordering of the resulting SOSs is *not unique*, but the overall transfer function will be the same. In practice, however, the particular pairing/ordering may make a difference.

In a hardware realization, the internal multiplications in each SOS will generate a certain amount of roundoff error which is then propagated into the next SOS. The net roundoff error at the overall output will depend on the particular pairing/ordering of the quadratic factors. The optimal ordering is the one that generates the *minimum* net roundoff error. Finding this optimal ordering is a difficult problem and is beyond the scope of this book.