Digital Signal Processing

Chapter 3. Digital filters

Zhenyu Huang

bighuang@sjtu.edu.cn

Study Points

- Difference equations and Z-transform (2H)
- FIR filter and design (2H)
- IIR filter and design (2H)
- Pole-zero design (2H)
- Filter realization (2H)
- Lab 2

Z transform and DTFT

The Bilateral Z transform is defined as

$$x[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = Z\{x[n]\}$$

Inverse Z transform is formally represented as

$$x[n] = \frac{1}{2\pi j} \int_C X(z) z^{n-1} dz$$

where C represents a closed contour that circles the origin by running in a counterclockwise direction through the region of convergence. This integral is not generally easy to compute.

This equation can be useful to prove theorems.

Z transform and DTFT

The Bilateral Z transform is defined as

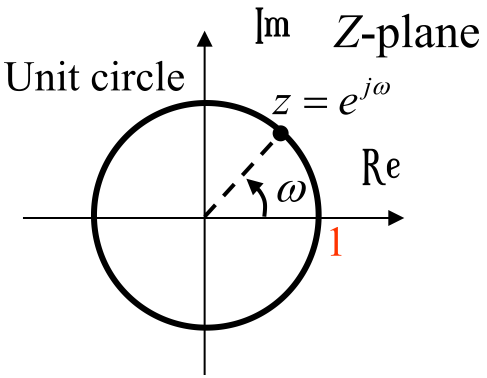
$$x[n] \leftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = Z\{x[n]\}$$

The relationship between Z transform and DTFT: $\boxed{z = re^{j\omega}}$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] r^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \left(x[n] r^{-n} \right) e^{-j\omega n} = F\left\{ x[n] r^{-n} \right\}$$

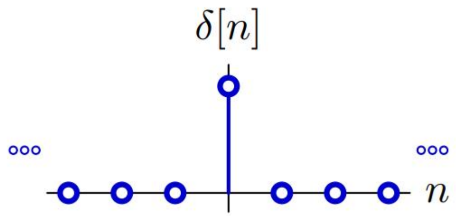
If $\sum_{n=-\infty}^{\infty} \left| x[n] \right| r^{-n} < \infty$, then DTFT exists.

So ROC of Z transform relays on $\left| z \right| = r$



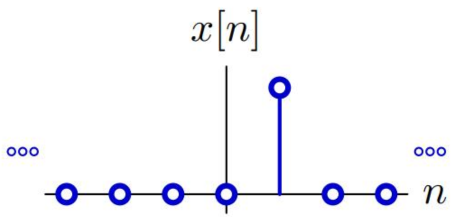
If ROC includes the unit circle, the DTFT of the signal exists.

Z transform of an impulse function



$$x[n] = \delta[n]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0} = 1$$

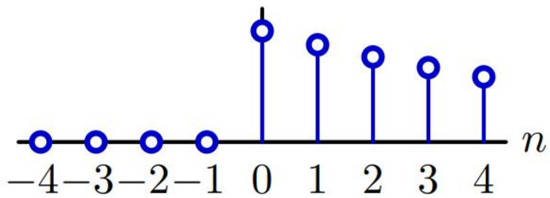


$$x[n] = \delta[n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[1]z^{-1} = z^{-1}$$

Z transform of an exponential function

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} =$$

$$\boxed{\left|\frac{7}{8} z^{-1}\right| < 1, \text{ i.e., } |z| > \frac{7}{8}.} \Rightarrow \boxed{\frac{1}{1 - \frac{7}{8} z^{-1}}}$$

The Z transform $X(z)$ is a function of z defined for all z inside a **Region of Convergence (ROC)**.

$$x[n] = \left(\frac{7}{8}\right)^n u[n] \quad \leftrightarrow \quad X(z) = \frac{1}{1 - \frac{7}{8} z^{-1}}; \quad |z| > \frac{7}{8}$$

ROC: $|z| > \frac{7}{8}$

Example 1

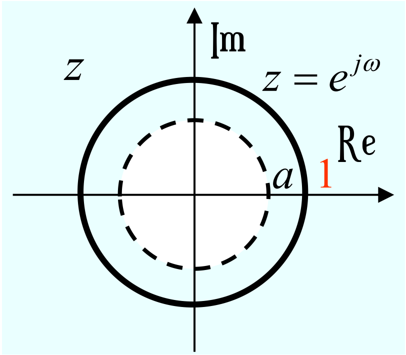
$x[n] = a^n u[n]$ is a right-sided signal.

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$\text{ROC}: X(z) < \infty \quad \Rightarrow \quad \left|az^{-1}\right| < 1 \quad \Rightarrow \quad |z| > |a|$$

$$\therefore \; X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

If $|a| < 1$, DTFT exists.



If $a = 1$, $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}, \quad |z| > 1$$

$$X(z) = \frac{z}{z-a}, \quad |z| > |a|$$

Zeros: 0, Poles: a

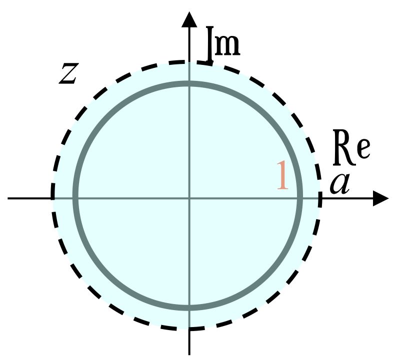
Example 2

$x[n] = -a^n u[-n-1]$ is a left-sided signal.

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$\text{ROC}: X(z) < \infty \quad \Rightarrow \quad |a^{-1}z| < 1 \quad \Rightarrow \quad |z| < |a|$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}}, \quad |z| < |a|$$



If $|a| > 1$, DTFT exists

If $a = 1$, $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}, \quad |z| > 1$$

$$X(z) = \frac{z}{z-a}, \quad |z| < |a|$$

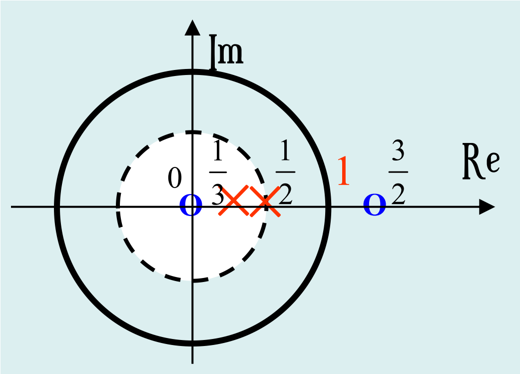
Zeros: 0, Poles: a

Example 3

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$$

$$X(z) = \sum_{n=0}^{\infty}\left\{7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right\} z^{-n} = 7\sum_{n=0}^{\infty}\left(\frac{1}{3}z^{-1}\right)^n - 6\sum_{n=0}^{\infty}\left(\frac{1}{2}z^{-1}\right)^n$$

ROC $\left|\frac{1}{3z}\right| < 1 \cap \left|\frac{1}{2z}\right| < 1 \rightarrow |z| > \frac{1}{2}$



$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}, \quad |z| > \frac{1}{2}$$

Inverse Z transform —— Rational form

Example1: $x[n] = a^n u[n]$ is a right-sided signal.

$$\therefore \; X(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

For the rational form of the z transform: $\dfrac{X(z)}{z} = \dfrac{N(z)}{D(z)}$

Single poles: $\dfrac{X(z)}{z} = \sum_{i=1}^{n} \dfrac{K_i}{z - p_i}$ $\qquad K_i = (z - p_i)\dfrac{X(z)}{z}\Big|_{z=p_i}$

$$\therefore \quad x(n) = \sum_i K_i (p_i)^n u(n)$$

Inverse Z transform —— Rational form

Multi-poles:

$$\frac{X(z)}{z}=\frac{N(z)}{(z-a)^3}=\frac{K_1}{(z-a)^3}+\frac{K_2}{(z-a)^2}+\frac{K_3}{z-a}$$

$$\frac{z}{(z-a)^3}\Leftrightarrow\frac{1}{2}n(n-1)\,a^{n-2}u(n-2)$$

$$K_1=(z-a)^3\frac{X(z)}{z}\Big|_{z=a}$$

$$\frac{z}{(z-a)^2}\Leftrightarrow n\,a^{n-1}u(n-1)$$

$$K_2=\frac{d}{dz}[(z-a)^3\frac{X(z)}{z}]\Big|_{z=a}$$

$$\frac{z}{z-a}\Leftrightarrow a^n u(n)$$

$$K_3=\frac{1}{2!}\frac{d^2}{dz^2}[(z-a)^3\frac{X(z)}{z}]\Big|_{z=a}$$

$$\therefore\quad x(n)=\frac{K_1}{2}n(n-1)\,a^{n-2}u(n-2)+K_2na^{n-1}u(n-1)+K_3a^nu(u)$$

Inverse Z transform —— Rational form

Conjugated complex poles

$$\frac{X(z)}{z} = \frac{K_1}{z - z_1} + \frac{K_2}{z - z_2} \qquad z_{1,2} = c \pm \mathrm{j}d = ae^{\pm \mathrm{j}\beta}$$

$$K_1 = (z - c - jd)\frac{X(z)}{z}\bigg|_{z=c+jd} = |K_1| \angle\theta_1 = A + jB$$

If X(z) is rational form with all real coefficients, there is

$$K_2 = K_1^* = |K_1| \angle -\theta_1 = A - jB$$

$$\begin{aligned} x(n) &= K_1 a^n e^{j\beta n} + K_2 a^n e^{-j\beta n} \\ &= |K_1| a^n e^{j(\beta n + \theta_1)} + |K_1| a^n e^{-j(\beta n + \theta_1)} \\ &= 2|K_1| a^n \cos(\beta n + \theta_1) \end{aligned}$$

Example 4

$$X(z)=\frac{z^3+6}{(z+1)(z^2+4)} \qquad |z|>2$$

Solution: $$\frac{X(z)}{z}=\frac{z^3+6}{z(z+1)(z^2+4)}=\frac{K_1}{z}+\frac{K_2}{z+1}+\frac{K_3}{z-j2}+\frac{K_3^*}{z+j2}$$

$$p_1=0, \quad p_2=-1, \qquad p_{3,4}=\pm j2=2e^{\pm j\frac{\pi}{2}}$$

$$K_1=\frac{6}{4}=\frac{3}{2} \qquad K_2=\frac{z^3+6}{z(z^2+4)}\bigg|_{z=-1}=\frac{5}{-5}=-1$$

$$K_3=\frac{z^3+6}{z(z+1)(z+j2)}\bigg|_{z=j2}=\frac{-j8+6}{j2(1+j2)\cdot j4}=\frac{-3+j4}{4(1+j2)}=\frac{\sqrt{5}}{4}\angle 63.5^\circ$$

$$\therefore \qquad x(n)=\frac{3}{2}\delta(n)-(-1)^n+\frac{\sqrt{5}}{2}(2)^n\cos\left(\frac{\pi}{2}n+63.5^\circ\right) \qquad n\ge 0$$

Properties of Z transform: convolution

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = x[n] * h[n]$$

$Y(z) = H(z)X(z)$, ROC at least the intersection of the ROCs of $H(z)$ and $X(z)$, can be bigger if there is pole/zero cancellation. *e.g.*

$$\begin{aligned} H(z) &= \frac{1}{z-a}, \quad |z| > a \\ X(z) &= z-a, \quad z \neq \infty \\ Y(z) &= 1 \quad \text{ROC all } z \end{aligned}$$

$$\boxed{H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}}$$ — **The System Function**

H(z) + ROC tells us everything about system

Properties of Z transform: delay

(1) Time Shifting $x[n - n_0] \longleftrightarrow z^{-n_0} X(z)$

The rationality of $X(z)$ unchanged, *different* from LT. ROC unchanged except for the possible addition or deletion of the origin or infinity

$n_o > 0 \Rightarrow$ ROC $z \neq 0$ (maybe)

$n_o < 0 \Rightarrow$ ROC $z \neq \infty$ (maybe)

Rational z-transforms

$x[n]$ = linear combination of exponentials for $n > 0$ and for $n < 0$

$X(z)$ is rational

$$X(z) = \frac{N(z)}{D(z)}$$ Polynomials in z

— characterized (except for a gain) by its poles and zeros

Properties of Z transform

Example:

$$\boxed{H(z)=\frac{2z+3}{z-0.5}}$$

$$H(z)=\frac{2+3z^{-1}}{1-0.5z^{-1}}$$

$$\Rightarrow Y(z)=\frac{2+3z^{-1}}{1-0.5z^{-1}}X(z)$$

$$\Rightarrow Y(z)-0.5z^{-1}Y(z)=2X(z)+3z^{-1}X(z)$$

$$\Rightarrow y[n]-0.5y[n-1]=2x[n]+3x[n-1]$$

$$y[n]=0.5y[n-1]+2x[n]+3x[n-1]$$

$$\boxed{H(z)=\frac{2z^2+3z}{z-0.5}}$$

$$H(z)=\frac{2z+3}{1-0.5z^{-1}}$$

$$\Rightarrow Y(z)=\frac{2z+3}{1-0.5z^{-1}}X(z)$$

$$\Rightarrow Y(z)-0.5z^{-1}Y(z)=2zX(z)+3X(z)$$

$$\Rightarrow y[n]-0.5y[n-1]=2x[n+1]+2x[n]$$

$$y[n]=0.5y[n-1]+2x[n+1]+2x[n]$$

Properties of Z transform

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$\Downarrow \text{ No poles at } \infty, \text{ if } M \leq N$$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then degree $N(z) \leq$ degree $D(z)$

Properties of Z transform

(8) Stability

LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$\Rightarrow$ Frequency Response $H(e^{j\omega})$ (DTFT of $h[n]$) exists.

$\Leftrightarrow$ ROC of $H(z)$ includes
the unit circle $|z| = 1$

A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

Z transform of basic signals and relations

Z definition

$$\delta[n] \xrightarrow{Z} 1 \quad \{z\}: all$$

time-domain shift

$$\delta[n-m] \xrightarrow{Z} z^{-m} \quad \{z\}: all, \notin 0 \ or \ \infty$$

summation

$$u[n] \xrightarrow{Z} \frac{1}{1-z^{-1}} \quad |z| > 1$$

reverse

$$\boxed{x[-n] \xleftrightarrow{Z} X(z^{-1}),\ \mathrm{ROC} = R^{-1}}$$

$$u[-n-1] = u[-n] - \delta[n] \xrightarrow{Z} \frac{1}{1-z} - 1 = \frac{-1}{1-z^{-1}} \quad |z| < 1$$

scaling

$$\boxed{z_0^n x[n] \xleftrightarrow{Z} X\left(\frac{z}{z_0}\right),\ \mathrm{ROC} = |z_0| R}$$

scaling

$$a^n u[n] \xrightarrow{Z} \frac{1}{1-(z/a)^{-1}} = \frac{1}{1-az^{-1}} \quad |z| > |a|$$

$$a^n u[-n-1] \xrightarrow{Z} \frac{1}{1-(z/a)^{-1}} = \frac{-1}{1-az^{-1}} \quad |z| < |a|$$

Differential

$$\boxed{nx[n] \longleftrightarrow -z\frac{dX(z)}{dz}, \quad \text{same ROC}}$$

Differential

$$na^n u[n] \xrightarrow{Z} z\frac{az^{-2}}{\left(1-az^{-1}\right)^2} = \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad |z| > |a|$$

$$n\,a^n u[-n-1] \xrightarrow{Z} \frac{az^{-1}}{\left(1-az^{-1}\right)^2} \quad |z| < |a|$$