FIR/IIR Filter Design

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April 4, 2024

Abstract

This article introduce the essence of FIR/IIR Filter Design.

Keywords LTI System · Fourier Transform · FIR · IIR

1 Introduction

In this section, we will introduce several fundamental concepts that will be applied in subsequent discussions.

1.1 Analysis of LTI Systems

A *Linear-Time Invariant (LTI) system* can be caracterized using a *Linear Difference Equation* as follows :

$$y(n) = \underbrace{-\sum_{k=1}^{N} a_k y(n-k)}_{\text{historical output values}} + \underbrace{\sum_{k=0}^{M} b_k x(n-k)}_{\text{input values}} \tag{1}$$

See Chapter 3.4.1, 2.5.1, 3.3.3 for more info.

Additionally, the system's response to any arbitrary input signal $x(n)$ can be understood through the process of convolution. This operation entails the input signal being convolved with the system's impulse response, denoted as $h(n)$, which is the response to a unit impulse input $\delta(n)$. Mathematically, this is represented as:

$$\begin{gathered} \delta(n) \xrightarrow{\mathcal{J}} h(n) \\ x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \xrightarrow{\mathcal{J}} y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \end{gathered} \tag{2}$$

2 From Ideal Filter to Practical Filter

2.1 Ideal Filter

When designing a Linear Time-Invariant (LTI) system for frequency selection purposes, the primary objective is to manipulate specific frequency components of the input signal. This manipulation is achieved through the system's frequency response, $H(\omega)$, which acts upon the input signal's frequency spectrum, $X(\omega)$, to produce the output signal's frequency spectrum, $Y(\omega) = H(\omega)X(\omega)$.

In essence, $H(\omega)$ serves as a filter that amplifies, attenuates, or entirely removes certain frequency components from $X(\omega)$ to construct $Y(\omega)$, thereby enabling the selective transmission or suppression of frequencies within the input signal.

For instance, consider the objective of implementing an ideal low-pass filter. Such a filter is characterized by its frequency response,

$$|H_{\text{lowpass}}(\omega)| = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \text{others} \end{cases} \tag{3}$$

where ω_c represents the cutoff frequency beyond which the filter attenuates the input signal.

Upon applying the Discrete Time Fourier Transform (DTFT), the filter's impulse response in the time domain :

$$h_{\text{lowpass}}(n) = \frac{\sin(\omega_c \pi n)}{\pi n}, \quad -\infty < n < \infty \tag{4}$$

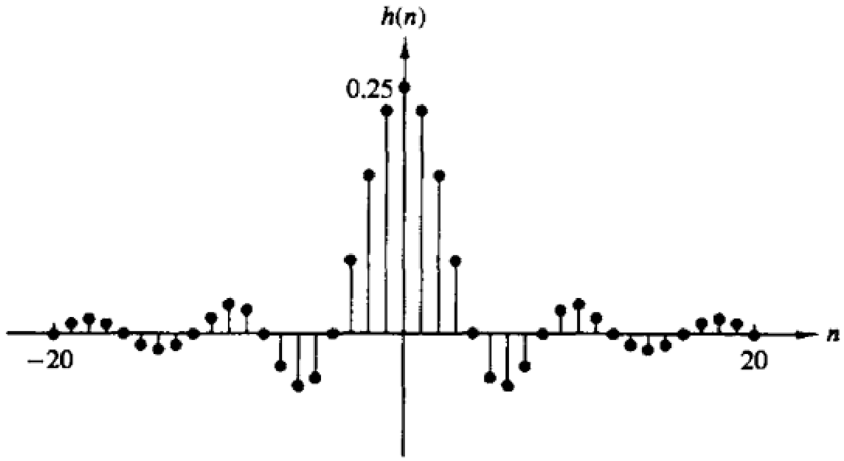

Figure 8.1 Unit sample response of an ideal lowpass filter.

This expression, however, is neither summable nor causal, rendering the filter unstable and, consequently, impracticable for real-world implementation. Therefore, while the concept of an ideal low-pass filter provides valuable theoretical insight, its physical realization is constrained by these limitations.

2.2 Practical Filter

It has been established that the concept of an ideal filter, while useful for theoretical analysis, is not feasible for practical implementation due to its non-causal and infinite-duration characteristics. However, by adopting a more pragmatic approach and allowing for slight deviations from the ideal model, it becomes possible to design filters that are practically implementable and effective.

Specifically, instead of insisting on a constant magnitude response, $|H(\omega)|$, across the entire passband, we can allow for minor variations in the response. This flexibility can lead to the development of filters that, although they do not strictly meet the ideal specifications, still perform satisfactorily for most practical applications. Such an approach acknowledges the trade-offs between the ideal theoretical models and the limitations imposed by physical and technological constraints.

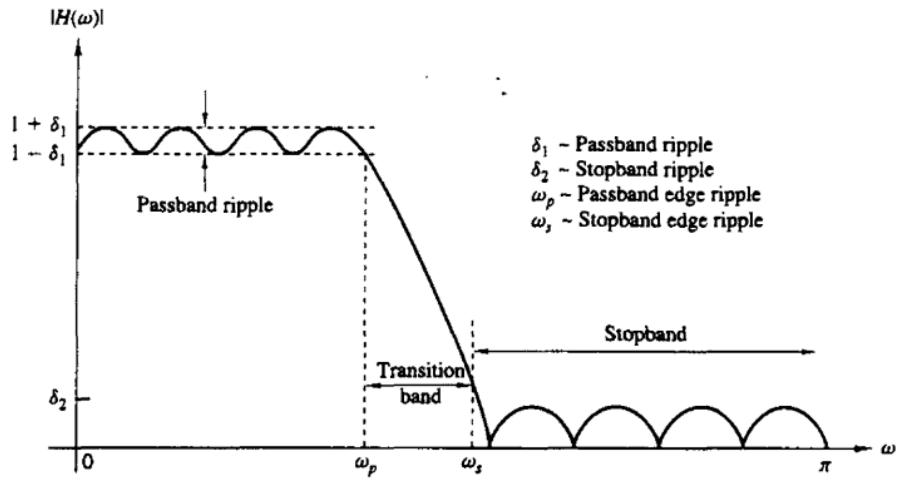

Figure 8.2 Magnitude characteristics of physically realizable filters.

3 Design of a FIR Filter

3.1 Caracterization of a FIR Filter

Consider a *FIR Filter* of length M, caracterized by the difference equation:

$$y(n) = b_0x(n) + b_1x(n-1) + ... + b_{M-1}x(n-M+1) = \sum_{k=0}^{M-1} b_kx(n-k) \tag{5}$$

where $x(n)$ is the input and $y(n)$ is the output.

Moreover, the output sequence of the FIR filter can also be expressed in the form of a convolution operation:

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k) \tag{6}$$

where the interval $[0, M-1]$ underscores the causal and finite-length attributes of the FIR filter.

3.2 FIR Filter Design Through Window Function Application

In the design process of digital filters, it is common to start with a specified desired frequency response, denoted as $H_d(\omega)$, and aim to derive a corresponding discrete-time impulse response, $h_d(n)$. Given that $h_d(n)$ represents a discrete signal, the Discrete Time Fourier Transform (DTFT) establishes the relationship as follows:

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n} \overset{\mathcal{F}}{\longleftrightarrow} h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)e^{j\omega n}\,\mathrm{d}\omega \tag{7}$$

To apply $h_d(n)$ in practical scenarios, it must be truncated to a finite length, typically at $n = M-1$. This truncation is accomplished by employing a rectangular window function defined as:

$$w(n) = \begin{cases} 1 & n \in [0, M-1] \\ 0 & \text{others} \end{cases} \tag{8}$$

which modifies the unit sample response to $h(n) = h_d(n)w(n)$.

It is essential to reassess the impact of this truncation, or windowing, on the originally desired frequency response. The practical frequency response, $H(\omega)$, results from the convolution of $H_d(\omega)$ with the Fourier Transform of the window function, $W(\omega)$.

so that the unit sample response becomes

$$h(n) = h_d(n)w(n) \tag{9}$$

Now we should reexamine the effect of the window or truncating to the frequency response we expected. In this case,

$$H(\omega) = H_d(\omega) * W(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\nu)W(\omega-\nu)\,\mathrm{d}\nu \quad \text{where } W(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} \tag{10}$$

where $W(\omega)$ is determined through DTFT. The most common method to obtain $H(\omega)$, however, is to simply calculate the DTFT of $h(n)$.