Discrete source

Ex. 1 (18 points)

There are two binary random variables X and Y with the following **joint probability distribution**:

Y \ X	$x_1=0$	$x_2=1$
$y_1=0$	$1/8$	$3/8$
$y_2=1$	$3/8$	$1/8$

Additionally, define another random variable $Z = XY$ (the general product). Calculate the following:

(1) $H(X), H(Y), H(Z), H(X,Z), H(Y,Z), H(X,Y,Z)$; (6 points)

(2) $H(X/Y), H(Y/X), H(X/Z), H(Z/X), H(Y/Z), H(Z/Y), H(X/YZ), H(Y/XZ),$ $H(Z/XY)$; (9 points)

(3) $I(X;Y), I(X;Z), I(Y;Z)$(3 points)

Hint: $P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

Solution

(1)

$$p(x_1) = p(x_2) = \frac{1}{2} \quad H(X) = -\sum_i p(x_i)\, log_2\, p\,(x_i) = 1\ \ bit/symbol,$$

$$p(y_1) = p(y_2) = \frac{1}{2} \quad H(Y) = -\sum_j p(y_j)\, log_2\, p\,(y_j) = 1\ \ bit/symbol$$

$Z = XY$:

$$\begin{bmatrix} Z \\ P(Z) \end{bmatrix} = \begin{Bmatrix} z_1 = 0 & z_2 = 1 \\ \frac{7}{8} & \frac{1}{8} \end{Bmatrix}$$

$$H(Z) = -\sum_k^2 p(z_k) = -\left(\frac{7}{8}\log_2\frac{7}{8} + \frac{1}{8}\log_2\frac{1}{8}\right) = 0.544\ \ bit/symbol$$

$$p(x_1) = p(x_1z_1) + p(x_1z_2)$$
$$p(x_1z_2) = 0$$
$$p(x_1z_1) = p(x_1) = 0.5$$
$$p(z_1) = p(x_1z_1) + p(x_2z_1)$$
$$p(x_2z_1) = p(z_1) - p(x_1z_1) = \frac{7}{8} - 0.5 = \frac{3}{8}$$
$$p(z_2) = p(x_1z_2) + p(x_2z_2)$$
$$p(x_2z_2) = p(z_2) = \frac{1}{8}$$
$$H(XZ) = -\sum_i \sum_k p(x_iz_k)\log_2 p(x_iz_k) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{3}{8}\log_2\frac{3}{8} + \frac{1}{8}\log_2\frac{1}{8}) = 1.406\ bit/symbol$$
$$p(y_1) = p(y_1z_1) + p(y_1z_2)$$
$$p(y_1z_2) = 0$$
$$p(y_1z_1) = p(y_1) = 0.5$$
$$p(z_1) = p(y_1z_1) + p(y_2z_1)$$
$$p(y_2z_1) = p(z_1) - p(y_1z_1) = \frac{7}{8} - 0.5 = \frac{3}{8}$$
$$p(z_2) = p(y_1z_2) + p(y_2z_2)$$
$$p(y_2z_2) = p(z_2) = \frac{1}{8}$$
$$H(YZ) = -\sum_j \sum_k p(y_jz_k)\log_2 p(y_jz_k) = -(\frac{1}{2}\log_2\frac{1}{2} + \frac{3}{8}\log_2\frac{3}{8} + \frac{1}{8}\log_2\frac{1}{8}) = 1.406\ bit/symbol$$
$$p(x_1y_1z_2) = 0$$
$$p(x_1y_2z_2) = 0$$
$$p(x_2y_1z_2) = 0$$
$$p(x_1y_1z_1) + p(x_1y_1z_2) = p(x_1y_1)$$
$$p(x_1y_1z_1) = p(x_1y_1) = 1/8$$
$$p(x_1y_2z_1) + p(x_1y_1z_1) = p(x_1z_1)$$
$$p(x_1y_2z_1) = p(x_1z_1) - p(x_1y_1z_1) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$p(x_2y_1z_1)+p(x_2y_1z_2)=p(x_2y_1)$$
$$p(x_2y_1z_1)=p(x_2y_1)=\frac{3}{8}$$
$$p(x_2y_2z_1)=0$$
$$p(x_2y_2z_1)+p(x_2y_2z_2)=p(x_2y_2)$$
$$p(x_2y_2z_2)=p(x_2y_2)=\frac{1}{8}$$
$$H(XYZ)=-\sum_i\sum_j\sum_k p(x_iy_jz_k)\log_2 p(x_iy_jz_k)$$
$$=-\left(\frac{1}{8}\log_2\frac{1}{8}+\frac{3}{8}\log_2\frac{3}{8}+\frac{3}{8}\log_2\frac{3}{8}+\frac{1}{8}\log_2\frac{1}{8}\right)=1.811\ bit/symbol$$

(2)

$$H(XY)=-\sum_i\sum_j p(x_iy_j)\,log_2\,p\,(x_iy_j)=$$
$$=-\left(\frac{1}{8}log_2\frac{1}{8}+\frac{3}{8}log_2\frac{3}{8}+\frac{3}{8}log_2\frac{3}{8}+\frac{1}{8}log_2\frac{1}{8}\right)$$
$$=1.811\ \ bit/symbol$$
$$H(X/Y)=H(XY)-H(Y)=1.811-1=0.811\ \ bit/symbol$$
$$H(Y/X)=H(XY)-H(X)=1.811-1=0.811\ \ bit/symbol$$
$$H(X/Z)=H(XZ)-H(Z)=1.406-0.544=0.862\ \ bit/symbol$$
$$H(Z/X)=H(XZ)-H(X)=1.406-1=0.406\ \ bit/symbol$$
$$H(Y/Z)=H(YZ)-H(Z)=1.406-0.544=0.862\ \ bit/symbol$$
$$H(Z/Y)=H(YZ)-H(Y)=1.406-1=0.406\ \ bit/symbol$$
$$H(X/YZ)=H(XYZ)-H(YZ)=1.811-1.406=0.405\ \ bit/symbol$$
$$H(Y/XZ)=H(XYZ)-H(XZ)=1.811-1.406=0.405\ \ bit/symbol$$
$$H(Z/XY)=H(XYZ)-H(XY)=1.811-1.811=0\ \ bit/symbol$$

(3)

$$I(X;Y)=H(X)-H(X/Y)=1-0.811=0.189\ \ bit/symbol$$
$$I(X;Z)=H(X)-H(X/Z)=1-0.862=0.138\ \ bit/symbol$$
$$I(Y;Z)=H(Y)-H(Y/Z)=1-0.862=0.138\ \ bit/symbol$$

Multiple-symbol discrete stationary source with memory

Ex. 2 (10 points)

The messages in black and white meteorological facsimiles have only two types: black and white, i.e., the source $X=\{Black,\ White\}$.

(1) Let the probability of black appearing be $P(Black) = 0.3$ and the probability of white appearing be $P(White) = 0.7$ .Assume that the appearance of black and white messages on the map has **no correlation** before and after, find the entropy $H(X)$; (2 points)

(2) Now we consider a source that generates a very long sequence. Assume that the messages are correlated, with dependency relationships $P\left(X_{j+1} = White \mid X_j = White \right) = 0.9$, $P\left(X_{j+1} = Black \mid X_j = White \right) = 0.1$, $P\left(X_{j+1} = White \mid X_j = Black \right) = 0.2$, and $P\left(X_{j+1} = Black \mid X_j = Black \right) = 0.8$. Find the entropy $H_\infty(X)$ of this **stationary** first-order Markov source. (5 points)

(3) Calculate the redundancy of the two sources above, and explain their physical meanings. (3 points)

Solution

(1)

$$H(X) = -\sum_i p(x_i)\, log\, p\,(x_i) = -(0.3\, log\, 0.3 + 0.7\, log\, 0.7) = 0.881 \;\; bit/symbol$$

(2) We define the black message as e_1, and white as e_2.

$$\begin{cases} p(X_{j+1} = e_1) = p(X_j = e_1)p(X_{j+1} = e_1/X_j = e_1) + p(X_j = e_2)p(X_{j+1} = e_1/X_j = e_2) \\ p(X_{j+1} = e_2) = p(X_j = e_2)p(X_{j+1} = e_2/X_j = e_2) + p(X_j = e_1)p(X_{j+1} = e_2/X_j = e_1) \end{cases}$$

$$\begin{cases} p(X_{j+1} = e_1) = 0.8p(X_j = e_1) + 0.1p(X_j = e_2) \\ p(X_{j+1} = e_2) = 0.9p(X_j = e_2) + 0.2p(X_j = e_1) \end{cases}$$

Because it is stationary, we have:

$$p\left(X_j = e_1\right) = \mathrm{p}\left(X_{j+1} = e_1\right)$$
$$p\left(X_j = e_2\right) = \mathrm{p}\left(X_{j+1} = e_2\right)$$

Then we can get:

$$\begin{cases} p(X_{j+1} = e_2) = 2p(X_{j+1} = e_1) \\ p(X_{j+1} = e_1) + p(X_{j+1} = e_2) = 1 \end{cases}$$

$$\begin{cases} p(X_{j+1} = e_1) = 1/3 \\ p(X_{j+1} = e_2) = 2/3 \end{cases}$$

$$H_\infty(X) = \lim_{N\to\infty} H_N(X^N) = \lim_{N\to\infty} H(X_N | X^{N-1})$$

$$= -\sum_{\mathrm{i}} \sum_{\mathrm{j}} \mathrm{p}\left(X_j = \mathrm{e_i}\right)\mathrm{p}\left(X_{j+1} = \mathrm{e_j}/X_j = \mathrm{e_i}\right) \log \mathrm{p}\left(X_{j+1} = \mathrm{e_j}/X_j = \mathrm{e_i}\right)$$

$$= -(\frac{1}{3} \times 0.8 \log 0.8 + \frac{1}{3} \times 0.2 \log 0.2 + \frac{2}{3} \times 0.1 \log 0.1 + \frac{2}{3} \times 0.9 \log 0.9)$$

$$= 0.553 \;\; bit/symbol$$

(3)

$$\gamma_1 = \frac{H_0 - H(X)}{H_0} = \frac{log_2 2 - 0.881}{log_2 2} = 11.9\%$$

$$\gamma_2 = \frac{H_0 - H_\infty(X)}{H_0} = \frac{log_2 2 - 0.553}{log_2 2} = 44.7\%$$

The uncertainty of a memoryless source is greater than that of a source with memory. A source with memory contains more structured information, allowing for a greater degree of compression.

Continuous source

Ex. 3 (36 points)

Suppose that the joint probability distribution of random variables X and Y is

$$p(xy) = \frac{1}{2\pi\sqrt{SN}} \exp\left\{-\frac{1}{2N}\left[x^2\left(1+\frac{N}{S}\right) - 2xy + y^2\right]\right\}$$

with $N > 0$ and $S > 0$.

Noting that $\int_{-\infty}^{\infty} \exp(-at^2)dt = \sqrt{\frac{\pi}{a}}$ and $\int_{-\infty}^{\infty} t^2 \exp(-at^2)dt = \frac{1}{2}\sqrt{\frac{\pi}{a^3}}$, $a > 0$.

Compute:

1) $p(x)$ (9 points)
2) $H(X)$ (9 points)
3) $p(y)$(9 points)
4) $H(Y)$ (9 points)

Solution

1)

$$\begin{aligned}
p(x) &= \int_{-\infty}^{\infty} p\,(xy)dy \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{SN}} \exp\left\{-\frac{1}{2N}\left[x^2\left(1+\frac{N}{S}\right) - 2xy + y^2\right]\right\} dy \\
&= \frac{1}{2\pi\sqrt{SN}} \exp\left(-\frac{x^2}{2S}\right) \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2N}(y-x)^2\right] dy \\
&= \frac{1}{2\pi\sqrt{SN}} \exp\left(-\frac{x^2}{2S}\right) \sqrt{2N\pi} \\
&= \frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right)
\end{aligned}$$

2)

$$
\begin{aligned}
H(X) &= -\int_{-\infty}^{\infty} p\,(x)\log p(x)dx \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right) \log\left[\frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right)\right] dx \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right) \log\left(\frac{1}{\sqrt{2S\pi}}\right) dx - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right) \log\left[\exp\left(-\frac{x^2}{2S}\right)\right] dx \\
&= \frac{1}{\sqrt{2S\pi}} \log\left(\sqrt{2S\pi}\right)\sqrt{2S\pi} - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2S\pi}} \exp\left(-\frac{x^2}{2S}\right)\left(-\frac{x^2}{2S}\right) \log(e) dx \\
&= \log\left(\sqrt{2S\pi}\right) + \frac{\log(e)}{2} \\
&= \frac{1}{2}\log 2\pi e S
\end{aligned}
$$

3)

$$
\begin{aligned}
p(y) &= \int_{-\infty}^{\infty} p\,(xy)dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{SN}} \exp\left\{-\frac{1}{2N}\left[x^2\left(1+\frac{N}{S}\right) - 2xy + y^2\right]\right\} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{SN}} \exp\left\{-\frac{1}{2N}\left[(x-\frac{Sy}{S+N})^2\left(1+\frac{N}{S}\right) + \frac{N}{S+N}y^2\right]\right\} dx \\
&= \frac{1}{\sqrt{2\pi(S+N)}} \exp\left(-\frac{1}{2S+2N}y^2\right)
\end{aligned}
$$

4)

$$
\begin{aligned}
H(Y) &= -\int_{-\infty}^{\infty} p\,(y)\log p(y)dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(S+N)}} \exp\left(-\frac{1}{2S+2N}y^2\right) \log\left[\frac{1}{\sqrt{2\pi(S+N)}} \exp\left(-\frac{1}{2S+2N}y^2\right)\right] dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(S+N)}} \exp\left(-\frac{1}{2S+2N}y^2\right) \log\left(\frac{1}{\sqrt{2\pi(S+N)}}\right) dy \\
&\quad -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(S+N)}} \exp\left(-\frac{1}{2S+2N}y^2\right) \log\left[\exp\left(-\frac{1}{2S+2N}y^2\right)\right] dy \\
&= \log\left(\sqrt{2\pi(S+N)}\right) + \frac{\log(e)}{2} \\
&= \frac{1}{2}\log 2\pi e(S+N)
\end{aligned}
$$

Channel capacity

Ex. 4 (9 points)

Indicate the kinds of the following four channels (symmetric channel, strongly symmetric channel, quasi symmetrical channel, or general channel), and calculate the channel capacity in bit/symbol.

1) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$ (3 points)

2) $\begin{bmatrix} \frac{3}{11} & \frac{2}{11} & \frac{2}{11} & \frac{2}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{3}{11} & \frac{2}{11} & \frac{2}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{2}{11} & \frac{3}{11} & \frac{2}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{2}{11} & \frac{2}{11} & \frac{3}{11} & \frac{2}{11} \\ \frac{2}{11} & \frac{2}{11} & \frac{2}{11} & \frac{2}{11} & \frac{3}{11} \end{bmatrix}$ (3 points)

3) $\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$(3 points)

Solution

1) It is a symmetric channel.

$C = \log_2 m - H(p_{i_1}, p_{i_2}, \cdots, p_{i_m}) = \log_2 4 - 2 \times \frac{1}{2}\log_2 2 = 1$ bit/symbol

2) It is a strongly symmetric channel, with $m = 5$, $p = \frac{8}{11}$.

$C = \log_2 m - H(1-p, p) - p\log_2(m-1) = \log_2 5 + \frac{3}{11}\log_2\frac{3}{11} + \frac{8}{11}\log_2\frac{8}{11} - \frac{8}{11}\log_2 4 = 0.022$ bit/symbol

3) It is a quasi-symmetrical channel.

$C = \log_2 2 - H(0.5,0.2,0.3) - 0.8\log_2(0.8) - 0.2\log_2(0.4) =$ 0.0365 bit/symbol

Ex. 5 (20 points)

Calculate the channel capacity in bit/symbol of the following channel with the transition probability matrix of the channel is $P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0.1 & 0 & 0.2 & 0.6 & 0.1 \\ 0.4 & 0 & 0.3 & 0.2 & 0.1 \\ 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0.2 & 0 & 0.2 & 0.5 \end{bmatrix}$, and the corresponding probability distribution of the input of the channel if the average mutual information arrives at the channel capacity.

Solution

Since $\sum_{j=1}^{5} p\left(y_j|x_i\right)\log p\left(y_j|x_i\right) = \sum_{j=1}^{5} p\left(y_j|x_i\right)\beta_j$

we have,

$$\begin{cases} \log(1) = \beta_3 \\ 0.1log(0.1) + 0.2\log(0.2) + 0.6\log(0.6) + 0.1\log(0.1) = 0.1\beta_1 + 0.2\beta_3 + 0.6\beta_4 + 0.1\beta_5 \\ 0.4log(0.4) + 0.3log(0.3) + 0.2log(0.2) + 0.1log(0.1) = 0.4\beta_1 + 0.3\beta_3 + 0.2\beta_4 + 0.1\beta_5 \\ \log(1) = \beta_2 \\ 0.1log(0.1) + 0.2log(0.2) + 0.2log(0.2) + 0.5log(0.5) = 0.1\beta_1 + 0.2\beta_2 + 0.2\beta_4 + 0.5\beta_5 \end{cases}$$

We get $\beta_2 = \beta_3 = 0$, $\beta_1 = -3.209, \beta_4 = -1.718, \beta_5 = -2.193$

So,

$C = \log_2 \sum_{j=1}^{5} 2^{\beta_j} = \log_2[2 + 2^{-3.209} + 2^{-1.718} + 2^{-2.193}] = 1.396$ bits/symbol

(10 points)

$$p(y_1) = 2^{\beta_1 - C} = 0.041$$
$$p(y_2) = p(y_3) = 2^{\beta_2 - C} = 0.38$$
$$p(y_4) = 2^{\beta_4 - C} = 0.116$$
$$p(y_5) = 2^{\beta_5 - C} = 0.083$$

Since $\sum_{i=1}^{5} p\left(x_i\right)p\left(y_j|x_i\right) = p\left(y_j\right)$,

We have,

$$\begin{cases} 0.1p(x_2) + 0.4p(x_3) + 0.1p(x_5) = p(y_1) \\ p(x_4) + 0.2p(x_5) = p(y_2) \\ p(x_1) + 0.2p(x_2) + 0.3p(x_3) = p(y_3) \\ 0.6p(x_2) + 0.2p(x_3) + 0.2p(x_5) = p(y_4) \\ 0.1p(x_2) + 0.1p(x_3) + 0.5p(x_5) = p(y_5) \end{cases}$$

Finally,

$$p(x_1) = 0.342$$
$$p(x_2) = 0.138$$
$$p(x_3) = 0.035$$
$$p(x_4) = 0.354$$
$$p(x_5) = 0.131$$

(10 points)

Ex. 6 (7 points)

A telephone signal has an information rate of 5.6×10^4 bits/second. It is transmitted over a Gaussian channel with a noise **power spectral density** of $N_0 = 5 \times 10^{-6}$ $\boldsymbol{mW/Hz}$, a limited bandwidth W, and a limited power of signal S.

1) When W= 4 kHz, determine the minimum power of signal S (in watts) required for error-free transmission. (2 points)
2) When $W \to \infty$, determine the minimum power of signal S (in watts) required for error-free transmission. (5 points)

Hint:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

Solution

1)

$$C_t = W\, log\left(1 + \frac{S}{WN_0}\right)$$

$$S = WN_0\left(2^{\frac{C_t}{W}} - 1\right) = 4000 \times 5 \times 10^{-9} \times \left(2^{\frac{5.6\times10^4}{4000}} - 1\right) = 0.328 \quad W$$

2)

$W \to \infty$

$$C_t = \frac{S}{N_0} log_2\, e$$

$$S = \frac{C_t N_0}{log_2\, e} = \frac{5.6 \times 10^4 \times 5 \times 10^{-9}}{log_2\, 2.71828} = 1.94 \times 10^{-4} \quad W$$