Ex. 1 (6 points)

We take at random three cards (without replacement) from an ordinary deck of cards (of 52 cards).

1) calculate the measure of information (in bits) when all three are hearts. (2 points)
2) calculate the measure of information (in bits) when none of the cards is hearts. (2 points)
3) calculate the measure of information (in bits) when all three are aces (i.e. A). (2 points)

Solution

1) $I = -\log_2 p = -\log_2\left(\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{11}{50}\right) = 6.27$ bits.
2) $I = -\log_2 p = -\log_2\left(\frac{39}{52}\cdot\frac{38}{51}\cdot\frac{37}{50}\right) = 1.27$ bits.
3) $I = -\log_2 p = -\log_2\left(\frac{4}{52}\cdot\frac{3}{51}\cdot\frac{2}{50}\right) = 12.43$ bits.

Ex. 2 (8 points)

Y and Z are two continuous random variables. Y has an exponential probability density distribution $p(y)$ over $y \in [0,\infty]$: $p(y) = e^{-y}$. Z has a uniform probability density distribution: $p(z) = 1/\alpha$ for $z \in]0,\alpha]$, else $p(z) = 0$.

1) Calculate the differential entropies $H(Y)$ and $H(Z)$ for these two continuous random variables. (6 points)
2) Find the value of α for which these differential entropies are the same. (2 points)

Solution

1) Differential entropy H for a probability density distribution $p(x)$ is $-\int_x p\,(x)\log p(x)dx$, so:

$$H(Z) = -\int_0^\alpha \frac{1}{\alpha}\log\left(\frac{1}{\alpha}\right)dz = \frac{\alpha}{\alpha}\log(\alpha) = \log(\alpha)$$

and

$$H(Y) = -\int_0^\infty e^{-y}\log(e^{-y})dy = \log(e)\int_0^\infty y\,e^{-y}dy$$

an integral which we can evaluate using integration-by-parts: $\int u dv = uv - \int v du$, with: $u = y$, $dv = e^{-y}dy$, $v = -e^{-y}$, $du = dy$, so:

$$\int_0^\infty y\, e^{-y} dy = [-ye^{-y}]_0^\infty + \int_0^\infty e^{-y}\, dy = 1$$

Thus, $H(Y) = \log(e)$.

2) We see that $H(Z) = H(Y)$ when $\log(\alpha) = \log(e)$, or in other words, $\alpha = e = 2.718$.

Ex. 3 (5 points)

Show that for independent discrete random variables X and Y, and $Z = X + Y$, $I(X;Z) - I(Y;Z) = H(X) - H(Y)$.

Solution

Using the definition of mutual information for discrete random variables, $I(X;Y) = H(Y) - H(Y|X)$, we can write

$$\begin{aligned} I(X;Z) - I(Y;Z) &= H(Z) - H(Z|X) - H(Z) + H(Z|Y) \\ &= H(X+Y|Y) - H(X+Y|X) \\ &= H(X|Y) - H(Y|X) \\ &= H(X) - H(Y) \end{aligned}$$

The first step follows from the fact that modifying the mean doesn't change the entropie. For the second step, we used the fact that the conditional entropy $H(X|Y)$ is equal to $H(X)$ for independent random variables X and Y.

Ex. 4 (14 points)

There are two random variables X and Y. Their joint probability is

$Y \setminus X$	$x_1 = 0$	$x_2 = 1$
$y_1 = 0$	$1/8$	$3/8$
$y_2 = 1$	$3/8$	$1/8$

Define another random variable $Z = XY$. Calculate in bit/symbol:

1) $H(X)$, $H(Z)$, $H(XZ)$ (6 points)
2) $H(X|Y)$, $H(X|Z)$, $H(Z|X)$ (6 points)
3) $I(X;Y)$ (2 points)

Solution

1) $p(x_1) = p(x_1y_1) + p(x_1y_2) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

   $p(x_2) = p(x_2y_1) + p(x_2y_2) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

   $H(X) = -\sum_i p\,(x_i)\log_2 p(x_i) = 1$ bit/symbol (2 points)

   $$\begin{bmatrix} Z \\ P(Z) \end{bmatrix} = \begin{bmatrix} z_1 = 0 & z_2 = 1 \\ \frac{7}{8} & \frac{1}{8} \end{bmatrix}$$

$H(Z) = -\sum_k^2 p\,(z_k)\log_2 p(z_k) = -\left(\frac{7}{8}\log_2\frac{7}{8} + \frac{1}{8}\log_2\frac{1}{8}\right) = 0.544$ bit/symbol (2 points)

$p(x_1) = p(x_1z_1) + p(x_1z_2),\ p(x_1z_2) = 0,\ p(x_1z_1) = p(x_1) = 0.5$

$p(x_2z_1) = p(z_1) - p(x_1z_1) = \frac{7}{8} - 0.5 = \frac{3}{8}$

$p(z_2) = p(x_1z_2) + p(x_2z_2),\ p(x_2z_2) = p(z_2) = \frac{1}{8}$

$H(XZ) = -\sum_i \sum_k p\,(x_iz_k)\log_2 p(x_iz_k) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{3}{8}\log_2\frac{3}{8} + \frac{1}{8}\log_2\frac{1}{8}\right) =$ 1.406 bit/symbol (2 points)

2) $H(XY) = -\sum_i \sum_j p\,(x_iy_j)\log_2 p(x_iy_j) = -2\left(\frac{1}{8}\log_2\frac{1}{8} + \frac{3}{8}\log_2\frac{3}{8}\right) = 1.811$ bit/symbol

$p(y_1) = p(x_1y_1) + p(x_2y_1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$

$p(y_2) = p(x_1y_2) + p(x_2y_2) = \frac{3}{8} + \frac{1}{8} = \frac{1}{2}$

$H(Y) = -\sum_j p\,(y_j)\log_2 p(y_j) = 1$ bit/symbol

$H(X|Y) = H(XY) - H(Y) = 1.811 - 1 = 0.811$ bit/symbol (2 points)

$H(X|Z) = H(XZ) - H(Z) = 1.406 - 0.544 = 0.862$ bit/symbol (2 points)

$H(Z|X) = H(XZ) - H(X) = 1.406 - 1 = 0.406$ bit/symbol (2 points)

3) $I(X;Y) = H(X) - H(X|Y) = 1 - 0.811 = 0.189$ bit/symbol (2 points)

Ex. 5 (6 points)

1) A television frame can be considered to consist of 3×10^5 pixels, where all pixels vary independently. Each pixel has 128 possible brightness levels, and the brightness levels are equally probable. How much information is contained in one frame of the image? (2 points)
2) If a broadcaster narrates the television image using 1,000 Chinese characters chosen from a set of 10,000 characters, how much information is transmitted by the broadcaster during the narration? Assume that the vocabulary of Chinese characters is equally probable and that the characters are independent. (2 points)
3) What is the minimum number of Chinese characters the broadcaster needs to describe the image accurately? (2 points)

Solution

1)

$$H(X) = \log_2 n = \log_2 128 = 7 \text{ bit/symbol}$$
$$H(X^N) = NH(X) = 3 * 10^5 * 7 = 2.1 * 10^6 \text{ bit/figure}$$

2)

$$H(Y) = \log_2 m = \log_2 10000 = 13.288 \text{ bit/symbol}$$
$$H(Y^M) = MH(Y) = 1000 * 13.288 = 13288 \text{ bit/phrases}$$

3)

$$K = \frac{H(X^N)}{H(Y)} = \frac{2.1 * 10^6}{13.288} = 158037$$

Ex. 6 (10 points)

Suppose a random variable X with $a_1 \le x \le a_2$ and a random variable Y with $b_1 \le y \le b_2$. The joint probability density of the two random variables is:

$$P_{XY}(x, y) = 4 \cdot \frac{x - a_1}{(a_2 - a_1)^2} \cdot \frac{y - b_1}{(b_2 - b_1)^2}$$

1) Compute $H(X)$ and $H(Y)$ (6 points).

$$P_X(x) = \int_{b_1}^{b_2} P_{XY}(x, y)\, dy = 2 \cdot \frac{x - a_1}{(a_2 - a_1)^2},$$

$$H(X) = -\int_{a_1}^{a_2} P_X(x)\, log_2\, P_X\,(x)\, dx = -\int_{a_1}^{a_2} 2 \cdot \frac{x - a_1}{(a_2 - a_1)^2} log_2\, 2 \cdot \frac{x - a_1}{(a_2 - a_1)^2} dx.$$

Let $t = 2 \cdot \frac{x - a_1}{(a_2 - a_1)^2}$, then

$$H(X) = -\frac{(a_2 - a_1)^2}{2ln2} \int_0^{\frac{2}{a_2 - a_1}} t\, ln\, t\, dt$$
$$= -\frac{(a_2 - a_1)^2}{2ln2} \cdot \frac{1}{(a_2 - a_1)^2} \left(2\, ln \left(\frac{2}{a_2 - a_1}\right) - 1\right)$$
$$H(X) = log_2(a_2 - a_1) - ln\, 4e.$$

Similarly, we can obtain

$$H(Y) = log_2(b_2 - b_1) - ln\, 4e.$$

2) Then compute $H(XY)$ and $I(X; Y)$ (4 points).

From 1) we can know that $X$and Y are independent, therefore

$$H(XY) = H(X) + H(Y) = log_2(a_2 - a_1)(b_2 - b_1) - 2\, ln\, 4e,$$
$$I(X; Y) = 0.$$

Ex. 7 (8 points)

There is a first-order stationary Markov chain $X_1, X_2, \cdots, X_r, \cdots, X_N$. Each X_r takes a value from the set $\{a_1, a_2, a_3\}$. The initial probability $P(X_1)$ is

$P(X_1=a_1)=1/2$, $P(X_1=a_2)=P(X_1=a_3)=1/4$. The transition probability matrix $P(a_j|a_i)$ is

$i \setminus j$	$j=1$	$j=2$	$j=3$
$i=1$	$1/2$	$1/4$	$1/4$
$i=2$	$2/3$	0	$1/3$
$i=3$	$2/3$	$1/3$	0

Calculate in bit/symbol $H(X_1X_2X_3)$, $H_3(X_1X_2X_3)$.

Solution

$H(X_1X_2X_3)=H(X_1)+H(X_2|X_1)+H(X_3|X_2X_1)=H(X_1)+H(X_2|X_1)+H(X_3|X_2)$

$H(X_1)=-\frac{1}{2}\log_2\frac{1}{2}-\frac{1}{4}\log_2\frac{1}{4}-\frac{1}{4}\log_2\frac{1}{4}=1.5$ bit/symbol

The joint probability of X_1 and X_2 is

$p(X_1X_2)$	$X_2=a_1$	$X_2=a_2$	$X_2=a_3$
$X_1=a_1$	$1/4$	$1/8$	$1/8$
$X_1=a_2$	$1/6$	0	$1/12$
$X_1=a_3$	$1/6$	$1/12$	0

So, $H(X_2|X_1)=-\sum_i^3\sum_j^3 p\left(X_1=a_i,X_2=a_j\right)\log_2 p\left(X_2=a_j|X_1=a_i\right)=\frac{1}{4}\log_2 2+\frac{1}{8}\log_2 4+\frac{1}{8}\log_2 4+\frac{1}{6}\log_2\frac{3}{2}+\frac{1}{12}\log_2 3+\frac{1}{6}\log_2\frac{3}{2}+\frac{1}{12}\log_2 3=1.209$ bit/symbol

The probability of X_2 is $P(X_2=a_1)=14/24$, $P(X_2=a_2)=5/24$, $P(X_2=a_3)=5/24$.

The joint probability of X_2 and X_3 is

$p(X_2X_3)$	$X_3=a_1$	$X_3=a_2$	$X_3=a_3$
$X_2=a_1$	$7/24$	$7/48$	$7/48$
$X_2=a_2$	$5/36$	0	$5/72$
$X_2=a_3$	$5/36$	$5/72$	0

We have, $H(X_3|X_2)=\frac{7}{24}\log_2 2+\frac{7}{48}\log_2 4+\frac{7}{48}\log_2 4+\frac{5}{36}\log_2\frac{3}{2}+\frac{5}{72}\log_2 3+\frac{5}{36}\log_2\frac{3}{2}+\frac{5}{72}\log_2 3=1.258$ bit/symbol

So $H(X_1X_2X_3)=1.5+1.209+1.258=3.967$ bit/symbol (6 points)

$H_3(X_1X_2X_3)=\frac{3.967}{3}=1.322$ bit/symbol. (2 points)

Ex. 8 (6 points)

Compute the channel capacity of the following channels.

1) $\begin{bmatrix} p_1-\epsilon & p_2-\epsilon & 2\epsilon \\ p_2-\epsilon & p_1-\epsilon & 2\epsilon \end{bmatrix}$ (3 points)

2) $\begin{bmatrix} p_1-\epsilon & p_2-\epsilon & 2\epsilon & 0 \\ p_2-\epsilon & p_1-\epsilon & 0 & 2\epsilon \end{bmatrix}$ (3 points)

Solution

1) It is a quasi symmetrical channel.

   It can be divided into two submatrices: $\begin{bmatrix} p_1-\epsilon & p_2-\epsilon \\ p_2-\epsilon & p_1-\epsilon \end{bmatrix}$, $\begin{bmatrix} 2\epsilon \\ 2\epsilon \end{bmatrix}$.

   Since $N_1 = M_1 = 1-2\epsilon$, $N_2 = 2\epsilon$, $M_2 = 4\epsilon$,

   We have,

$$\begin{aligned} C = {} & \log_2 N - H(p_{i_1}, p_{i_2}, \cdots, p_{i_m}) - \sum_{k=1}^{K} N_k \log_2 M_k \\ = {} & \log_2 2 - H(p_1-\epsilon, p_2-\epsilon, 2\epsilon) - (1-2\epsilon)\log_2(1-2\epsilon) - 2\epsilon\log_2(4\epsilon) \\ = {} & 1 + (p_1-\epsilon)\log_2(p_1-\epsilon) + (p_2-\epsilon)\log_2(p_2-\epsilon) + 2\epsilon\log_2(2\epsilon) - (1-2\epsilon)\log_2(1-2\epsilon) - 2\epsilon\log_2(4\epsilon) \\ = {} & (1-2\epsilon)\log_2\frac{2}{1-2\epsilon} + (p_1-\epsilon)\log_2(p_1-\epsilon) + (p_2-\epsilon)\log_2(p_2-\epsilon) \text{ bit/symbol} \quad \text{(3 points)} \end{aligned}$$

2) It is a quasi symmetrical channel.

   It can be divided into two submatrices: $\begin{bmatrix} p_1-\epsilon & p_2-\epsilon \\ p_2-\epsilon & p_1-\epsilon \end{bmatrix}$, $\begin{bmatrix} 2\epsilon & 0 \\ 0 & 2\epsilon \end{bmatrix}$.

   Since $N_1 = M_1 = 1-2\epsilon$, $N_2 = M_2 = 2\epsilon$,

   We have,

$$\begin{aligned} C = {} & \log_2 N - H(p_{i_1}, p_{i_2}, \cdots, p_{i_m}) - \sum_{k=1}^{K} N_k \log_2 M_k \\ = {} & \log_{22} - H(p_1-\epsilon, p_2-\epsilon, 2\epsilon, 0) - (1-2\epsilon)\log_2(1-2\epsilon) - 2\epsilon\log_2(2\epsilon) \\ = {} & 1 + (p_1-\epsilon)\log_2(p_1-\epsilon) + (p_2-\epsilon)\log_2(p_2-\epsilon) + 2\epsilon\log_2(2\epsilon) - (1-2\epsilon)\log_2(1-2\epsilon) - 2\epsilon\log_2(2\epsilon) \\ = {} & 1 - (1-2\epsilon)\log_2(1-2\epsilon) + (p_1-\epsilon)\log_2(p_1-\epsilon) + (p_2-\epsilon)\log_2(p_2-\epsilon) \text{ bit/symbol} \quad \text{(3 points)} \end{aligned}$$

Ex. 9 (8 points)

The transition probability matrix of a discrete source is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

Calculate the channel capacity C and the corresponding probability distribution of the input of the channel if the average mutual information arrives at the channel capacity.

Solution

This is a general discrete source.

Since,

$$\begin{cases} \frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2} = \frac{1}{2}\beta_1 + \frac{1}{2}\beta_2 \\ \frac{1}{4}\log_2\frac{1}{4} + \frac{3}{4}\log_2\frac{3}{4} = \frac{1}{4}\beta_1 + \frac{3}{4}\beta_2 \end{cases}$$

we have

$$\begin{cases}\beta_1 = -1.3775 \\ \beta_2 = -0.6225\end{cases}$$

So, $C = \log_2\left(\sum_j 2^{\beta_j}\right) = \log_2(2^{-1.3775} + 2^{-0.6225}) = 0.049$ bit/symbol (4 points)

Then, $p(y_1) = 2^{\beta_1 - C} = 2^{-1.3775-0.049} = 0.372$
and $p(y_2) = 2^{\beta_2 - C} = 2^{-0.6225-0.049} = 0.638$
Since,

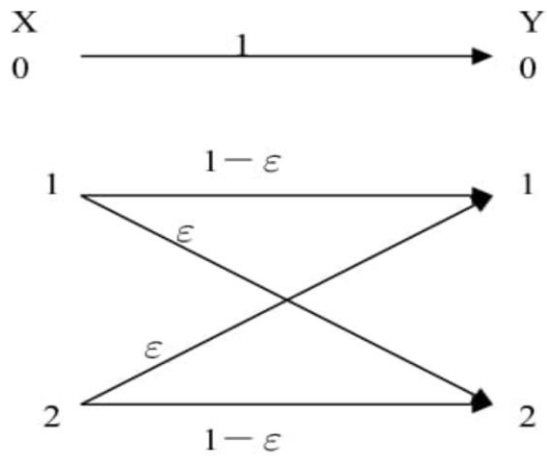
$$\begin{cases}0.372 = \dfrac{1}{2}p(x_1) + \dfrac{1}{4}p(x_2) \\ 0.628 = \dfrac{1}{2}p(x_1) + \dfrac{3}{4}p(x_2)\end{cases}$$

we have,

$$\begin{cases}p(x_1) = 0.488 \\ p(x_2) = 0.512\end{cases} \text{(4 points)}$$

Ex. 10 (12 points)

Calculate the channel capacity in bit/symbol of the following channel, and the corresponding probability distribution of the input of the channel if the average mutual information arrives at the channel capacity. Calculate the channel capacity in bit/symbol when $\epsilon = 0$ when $\epsilon = 1/2$.



Solution

The transition probability matrix of the channel is $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\epsilon & \epsilon \\ 0 & \epsilon & 1-\epsilon \end{bmatrix}$

Since $\sum_{j=1}^{3} p\left(y_j|x_i\right)\log p\left(y_j|x_i\right) = \sum_{j=1}^{3} p\left(y_j|x_i\right)\beta_j$ $(i = 1,2,3)$,

we have,

$$\begin{cases}(1-\epsilon)\log(1-\epsilon)+\epsilon\log\epsilon=(1-\epsilon)\beta_2+\epsilon\beta_3\\ \epsilon\log\epsilon+(1-\epsilon)\log(1-\epsilon)=\epsilon\beta_2+(1-\epsilon)\beta_3\\ \beta_1=0\end{cases}$$

We get $\beta_1=0$, $\beta_2=\beta_3=(1-\epsilon)\log(1-\epsilon)+\epsilon\log\epsilon$

So, $C=\log_2\sum_{j=1}^{32}{}^{\beta_j}=\log_2\left[1+2\times 2^{(1-\epsilon)\log_2(1-\epsilon)+\epsilon\log_2\epsilon}\right]=\log_2\left[1+2(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}\right]$ (5 points)

$$p(y_1)=2^{\beta_1-C}=2^{-C}=\frac{1}{1+2(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}$$

$$p(y_3)=p(y_2)=2^{\beta_2-C}=\frac{(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}{1+2(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}$$

Since $\sum_{i=1}^{3}p(x_i)p(y_j|x_i)=p(y_j)$ $(j=1,2,3)$,

We have,

$$\begin{cases}p(x_1)=p(y_1)\\ (1-\epsilon)p(x_2)+\epsilon p(x_3)=p(y_2)\\ \epsilon p(x_2)=(1-\epsilon)p(x_3)=p(y_3)\end{cases}$$

So,

$$\begin{cases}p(x_1)=p(y_1)=\dfrac{1}{1+2(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}\\ p(x_3)=p(x_2)=p(y_2)=\dfrac{(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}{1+2(1-\epsilon)^{(1-\epsilon)}\epsilon^{\epsilon}}\end{cases}\quad \text{(5 points)}$$

When $\epsilon=0$, $C=\log_2 3=1.585$ bit/symbol.

When $\epsilon=\frac{1}{2}$, $C=\log_2 2=1$ bit/symbol. (2 points)

Ex. 11 (9 points)

The mathematical model of a single-symbol discrete memoryless source X is $\begin{bmatrix}X\\P\end{bmatrix}=\begin{bmatrix}x_1 & x_2\\ a & 1-a\end{bmatrix}$. The sample space of the output Y is $\{y_1,y_2,y_3\}$. The transition probability matrix is

$$P=\begin{bmatrix}1/2 & 1/2 & 0\\ 1/2 & 1/4 & 1/4\end{bmatrix}$$

Calculate in bit/symbol

1) $H(Y)$ (3 points)
2) $H(Y|X)$ (2 points)
3) The channel capacity C (4 points)

Solution

1) The joint probability $p(x_i y_j)$ is

$X \setminus Y$	y_1	y_2	y_3
x_1	$a/2$	$a/2$	0
x_2	$(1-a)/2$	$(1-a)/4$	$(1-a)/4$

The probability distribution of Y is $p(y_1) = 1/2$, $p(y_2) = (1+a)/4$, $p(y_3) = (1-a)/4$

$$\begin{aligned} H(Y) = & \frac{1}{2}\log_2 2 + \frac{1+a}{4}\log_2\frac{4}{1+a} + \frac{1-a}{4}\log_2\frac{4}{1-a} \\ = & \frac{1}{2} + \frac{1}{4}\log_2\frac{16}{1-a^2} + \frac{a}{4}\log\frac{1-a}{1+a} \\ = & \frac{3}{2} + \frac{1}{4}\log_2\frac{1}{1-a^2} + \frac{a}{4}\log_2\frac{1-a}{1+a} \text{ bit/symbol} \end{aligned}$$

2) $H(Y|X) = -\left(\frac{a}{2}\log_2\frac{1}{2} + \frac{a}{2}\log_2\frac{1}{2} + \frac{1-a}{2}\log_2\frac{1}{2} + \frac{1-a}{4}\log_2\frac{1}{4} + \frac{1-a}{4}\log_2\frac{1}{4}\right) = \frac{3-a}{2}$ bit/symbol

3) We have

$$\begin{aligned} C = & \max_{p(x_i)} I(X;Y) = \max_{p(x_i)}[H(Y) - H(Y|X)] \\ = & \max_{p(x_i)}\left(\frac{a}{2} + \frac{1}{4}\log_2\frac{1}{1-a^2} + \frac{a}{4}\log_2\frac{1-a}{1+a}\right) \\ = & \frac{1}{\ln 2}\max_{p(x_i)}\left(\frac{a}{2}\ln 2 + \frac{1}{4}\ln\frac{1}{1-a^2} + \frac{a}{4}\ln\frac{1-a}{1+a}\right) \end{aligned}$$

$$\begin{aligned} & \frac{\partial\left(\frac{a}{2}\ln 2 + \frac{1}{4}\ln\frac{1}{1-a^2} + \frac{a}{4}\ln\frac{1-a}{1+a}\right)}{\partial a} \\ = & \frac{1}{2}\ln 2 + \frac{1}{4}\frac{2a}{1-a^2} + \frac{1}{4}\ln\frac{1-a}{1+a} + \frac{a}{4}\left(-\frac{1}{1-a} - \frac{1}{1+a}\right) \\ = & \frac{1}{2}\ln 2 + \frac{1}{2}\frac{a}{1-a^2} + \frac{1}{4}\ln\frac{1-a}{1+a} - \frac{a}{4}\frac{2}{1-a^2} \\ = & \frac{1}{2}\ln 2 + \frac{1}{4}\ln\frac{1-a}{1+a} \end{aligned}$$

In order to have $\frac{1}{2}\ln 2 + \frac{1}{4}\ln\frac{1-a}{1+a} = 0$, $\frac{1-a}{1+a} = \frac{1}{4}$

Therefore, $a = \frac{3}{5}$

$C = \frac{1}{2}\times\frac{3}{5} + \frac{1}{4}\log_2\frac{1}{1-\frac{9}{25}} + \frac{1}{4}\times\frac{3}{5}\log_2\frac{1-\frac{3}{5}}{1+\frac{3}{5}} = 0.161$ bit/symbol

Ex. 12 (5 points)

In image transmission, each frame has about 2.25×10^6 pixels. In order to reproduce the image well, each pixel has 16 brightness levels, and assumes equal probability distribution of brightness levels. If the signal-to-noise power ratio is 30 dB, calculate the channel bandwidth required to transmit one frame of picture per minute.

Solution

$I_{pixel} = \log_2 n = \log_2 16 = 4$ bit/pixel

$I_{frame} = NI_{pixel} = 2.25 \times 10^6 \times 4 = 9 \times 10^6$ bit/frame

$C = \frac{I_{frame}}{t} = \frac{9\times10^6}{60} = 1.5 \times 10^5$ bit/s

$W = \frac{C}{\log_2\left(1+\frac{S}{N}\right)} = \frac{1.5\times10^5}{\log_2(1+1000)} = 150499$ Hz

Ex. 13 (5 points)

The mathematical model of a discrete memoryless source is given below

$$\begin{bmatrix} X \\ P \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ \frac{1}{8} & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}$$

A binary lossless fixed-length coding for this source is conducted. If the required coding efficiency is 80% and the decoding error probability is smaller than 10^{-4}. Calculate the length of the source sequence N.

Solution

$H(X) = 2 \times \frac{1}{8}\log_2 8 + 2 \times \frac{3}{8}\log_2\frac{8}{3} = 1.811$ bit/symbol

$$\sigma^2(X) = E\{[I(x_i) - H(X)]^2\} = \sum_{i=1}^{4} p_i\,(\log_2 p_i)^2 - H(X)^2 = 0.471$$

$$\epsilon = \frac{H(X)}{\eta} - H(X) = 0.453$$

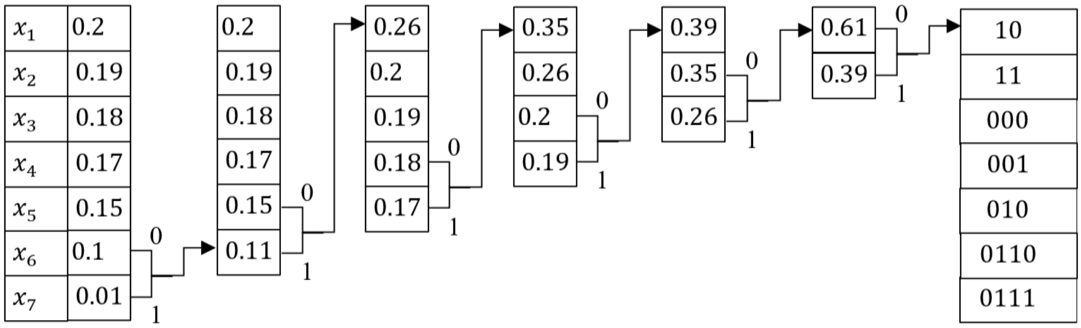
$$N \geq \frac{\sigma^2(X)}{\epsilon^2\delta} = 22972$$

Ex. 14 (6 points)

The mathematical model of a source is $\begin{bmatrix} X \\ P \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.2 & 0.19 & 0.18 & 0.17 & 0.15 & 0.1 & 0.01 \end{bmatrix}$. Construct the Huffman code, and calculate the average code length and coding information rate.

Solution

Huffman code: (2 points)



The average code length: $\overline{L} = 2 \times 0.2 + 2 \times 0.19 + 3 \times 0.18 + 3 \times 0.17 + 3 \times 0.15 + 4 \times 0.1 + 4 \times 0.01 = 2.72$ (2 points)

The entropy is $H(X) = -\sum_{i=1}^{7} p\,(x_i) log_2 p(x_i) = 2.609$ bit/symbol

The coding information rate is $R = \frac{H(X)}{\overline{L}} = \frac{2.609}{2.72} = 0.959$ bit/code symbol (2 points)

Ex. 15 (6 points)

The transition probability matrix of a discrete source is

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

1) When the probability distribution of the source is $p(a_1) = 2/3$, $p(a_2) = p(a_3) = 1/6$, construct the decoding function F according to the maximum a posteriori probability criterion, and calculate its average decoding error probability P_e. (3 points)
2) When the source is distributed according to equal probability, construct the decoding function F according to the maximum likelihood decoding criterion, and calculate its average decoding error probability P_e. (3 points)

Solution

1) The joint probability of X_1 and X_2 is

$p(a_1 b_2)$	b_1	b_2	b_3
a_1	$\frac{2}{6}$	$\frac{2}{12}$	$\frac{2}{12}$
a_2	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{24}$
a_3	$\frac{1}{24}$	$\frac{1}{24}$	$\frac{1}{12}$

According to the maximum a posteriori probability criterion, the decoding function is

$$\begin{cases} F(b_1) = a_1 \\ F(b_2) = a_1 \text{ (2 points)} \\ F(b_3) = a_1 \end{cases}$$

The average decoding error rate is $P_e = 1 - \frac{2}{6} - \frac{2}{12} - \frac{2}{12} = \frac{1}{3}$ (1 points)

2) According to the maximum likelihood decoding criterion, the decoding function is

$$\begin{cases} F(b_1) = a_1 \\ F(b_2) = a_2 \text{ (2 points)} \\ F(b_3) = a_3 \end{cases}$$

The average decoding error rate is $p_e = 1 - \frac{1}{3} * \left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}$ (1 points)

Ex. 16 (6 points)

In (8,4) binary linear block code, the check equation is

$$\begin{aligned} c_0 &= m_1 + m_2 + m_3 \\ c_1 &= m_0 + m_1 + m_2 \\ c_2 &= m_0 + m_1 + m_2 \\ c_3 &= m_0 + m_2 + m_3 \end{aligned}$$

Among them, m_0, m_1, m_2, m_3 are the information bits, c_0, c_1, c_2, c_3 are the parity bits, and the codeword $C = (m_3 m_2 m_1 m_0 c_3 c_2 c_1 c_0)$. Find the generation matrix G and parity check matrix H of this codeword.

Solution

The generation matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \text{ (3 points)}$$

The parity check matrix is

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ (3 points)}$$