Introduction to Discrete Mathematics

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Preface

If you are reading this book, you probably have never studied proofs before. So let me give you some advice, mathematical books are very different from fiction and even books in other sciences. Quite often you may see that some steps are missing and some steps are not really explained and just claimed as obvious. The main reason behind this is to make the ideas of the proof more visible and allow to grasp quickly the essence of proofs.

Since the steps are skipped you cannot just read the book and believe that you studied the topic; the best way to actually study the topic is to try to prove every statement before you read the actual proof in the book. In addition to this, I recommend trying to solve all the exercises in the book (you may find exercises in the middle and at the end of every chapter).

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Part I

Introduction to Mathematical Reasoning

Chapter 1

Proofs

1.1 Direct Proofs

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We start the discussion of the proofs in mathematics from an example of a proof in “everyday” life. Assume that we know that the following statements are true.

1. If a fish has fins and scales it is kosher,
2. if a fish has scales it has fins,
3. any salmon has scales.

Using these facts we may conclude that any salmon is kosher; indeed, any salmon has scales by the third statement, hence, by the second statement any salmon has fins, finally, by the first statement any salmon is kosher since it has fins and scales.

One may notice that this explanation is a sequence of conclusions such that each of them true because the previous one is true. Mathematical proof is also a sequence of statements such that every statement if the previous statement is true. If P and Q are some statement and Q is always true whether P is true, then we say that P implies Q. We denote the statement that P implies Q by $P \implies Q$.

In order to define the implication formally let us consider the following table.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Let P and Q be some statements. Then this table says that if P and Q both false, then $P \implies Q$ is true etc.

Exercise 1.1. *Let n be an integer.*

1. *Is it always true that "n^2 is positive" implies "n is not equal to 0"?*
2. *Is it always true that "$n^2 - n - 2$ is equal to 0" implies "n is equal to 2"?*

In the example we gave at the beginning of the section we used some *known* facts. But what it means to know something? In math we typically say that we know a statement if we can prove it. But in order to prove it we need to know something again, it is a problem! In order to solve it, mathematicians introduced the notion of an *axiom*. An axiom is a statement that believed to be true and when we prove a statement we prove it under the assumption that these axioms are true[1].

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For example, we may consider axioms of inequalities for real numbers.

1. Let $a, b \in \mathbb{R}$. Only one of the following is true:
    - $a < b$,
    - $b < a$, or
    - $a = b$.
2. Let $a, b, c \in \mathbb{R}$. Then $a < b$ iff $a + c < b + c$.
3. Let $a, b, c \in \mathbb{R}$. Then $a < b$ iff $ac < bc$ provided that $c > 0$ and $a < b$ iff $ac > bc$ if $c < 0$.
4. Let $a, b, c \in \mathbb{R}$. If $a < b$ and $b < c$, then $a < c$.

Let us now try to prove something using these axioms, we prove that if $a > 0$, then $a^2 > 0$. Note that $a > 0$, hence, by the third axiom $a^2 > 0$.

Similarly, we may prove that if $a < 0$, then $a^2 > 0$. And combining these two statements together we may prove that if $a \neq 0$, then $a^2 > 0$.

Such a way of constructing proof is called direct proofs.

Exercise 1.2. *Axiomatic system for a four-point geometry.*
Undefined terms: point, line, is on.
Axioms:

- *For every pair of distinct points x and y, there is a unique line ℓ such that x is on l and y is on l.*
- *Given a line ℓ and a point x that is not on ℓ, there is a unique line m such that x is on m and no point on ℓ is also on m.*
- *There are exactly four points.*
- *It is impossible for three points to be on the same line.*

Prove that there are at least two distinct lines.

[1]Note that in different parts of math axioms may be different

1.2 Constructing Proofs Backwards

However, sometimes it is not easy to find the proof. In this case one of the possible methods to deal with this problem is to try to prove starting from the end.

For example we may consider the statement $(a+b)^2 = a^2+2ba+b^2$. Imagine, for a second, that you have not learned about axioms. In this case you would write something like this:

$$(a+b)^2 = (a+b)\cdot(a+b) = \\ a(a+b)+b(a+b) = \\ a^2+ab+ba+b^2 = a^2+2ba+b^2.$$

Let us try to prove it completely formally using the following axioms.

1. Let $a, b, c \in \mathbb{R}$. If $a = b$ and $b = c$, then $a = c$.
2. Let $a, b, c \in \mathbb{R}$. If $a = b$, then $a + c = b + c$ and $c + a = c + a$.
3. Let $a, b, c \in \mathbb{R}$. Then $a(b + c) = ab + bc$.
4. Let $a, b, c \in \mathbb{R}$. Then $ab = ba$.
5. Let $a, b, c \in \mathbb{R}$. Then $a + b = b + a$.
6. Let $a, b, c \in \mathbb{R}$. Then $a \cdot a = a^2$.
7. Let $a, b, c \in \mathbb{R}$. Then $a + a = 2a$.

So the formal proof of the statement $(a+b)^2 = a^2+2ab+b^2$ is as follows. First note that $(a + b)^2 = (a + b)\cdot(a + b)$ (by axiom 5), hence, by axiom 1, it is enough to show that $(a+b)\cdot(a+b) = a+2ab+b^2$. By axiom 3 applied twice, $(a+b)\cdot(a+b) = a\cdot(a+b)+b\cdot(a+b) = a\cdot a+a\cdot b+b\cdot a+b\cdot b$; so by axiom 1, it is enough to show that $a\cdot a+a\cdot b+b\cdot a+b\cdot b = a^2+2ab+b^2$. Additionally, by axiom 6, $a\cdot a = a^2$ and $b\cdot b = b^2$. Hence, by axiom 2, it is enough to show that $a^2 + a\cdot b + b\cdot a + b^2 = a^2 + 2ab + b^2$. By axiom 4, $a\cdot b = b\cdot a$, hence, by axiom 2, $a\cdot b + b\cdot a = b\cdot a + b\cdot a$. Therefore by axiom 7, $a\cdot b + b\cdot a = 2b\cdot a$. Finally, by axiom 2, $a\cdot b + b\cdot a + a^2 + b^2 = 2b\cdot a + a^2 + b^2$ and by axiom 5, $a\cdot b+b\cdot a+a^2+b^2 = a^2+a\cdot b+b\cdot a+b^2$ and $2b\cdot a+a^2+b^2 = a^2+2b\cdot a+b^2$. Which finishes the proof by axiom 1.

1.3 Proofs in Real-life Mathematics

In this paragraph we explicitly used axioms to prove statement. However, it leads us to really long and hard to understand proofs (the last example in the previous section is a good example of this phenomenon). Because of this mathematicians tend to skip steps in the proofs when they believe that they are clear. This is the reason why it is arduous to read mathematical texts and it is very different from reading non-mathematical books. Add problem arising because of this tendency is that some mistakes may arise if we skip way to many steps. In the last two centuries there were several attempts to solve this issue, one approach to this we are going to discuss in the second part of this book.

End of The Chapter Exercises

1.3 Using the axioms of inequalities show that if a is a non-zero real number, then $a^2 > 0$.

1.4 Using the axioms of inequalities prove that for all real numbers a, b, and c,

$$bc + ac + ab \leq a^2 + b^2 + c^2.$$

1.5 Prove that for all integers a, b, and c, If a divides b and b divides c, then a divides c. Recall that an integer m divides an integer n if there is an integer k such that $mk = n$.

1.6 Show that square of an even integer is even.

1.7 Prove that 0 divides an integer a iff $a = 0$.

1.8 Using the axioms of inequalities, that if $a > 0$, b, and c are real numbers, then $b \geq c$ implies that $ab \geq ac$.

1.9 Using the axioms of inequalities, that if $a, b < 0$ are real numbers, then $a \leq b$ implies that $a^2 \geq b^2$.