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THE COMPLEXITY OF THE OPTIMAL SEARCHER PATH PROBLEM

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In this note we show that the problem of finding an optimal searcher path that maximizes the probability of detecting a stationary target by the end of a fixed time is NP-complete. We also demonstrate that the problem of finding a path that minimizes mean time to detection is NP-hard.

This note discusses the complexity of optimally searching for a stationary target in discrete time and space. In this problem, a target's position is fixed within a gridwork of cells. A searcher moves through this gridwork in an attempt to locate the target. At each time step, the searcher must either remain at its current location or move to a neighboring cell. For the discussion of complexity in this note, it is convenient to view the search problem in terms of a finite, connected graph. In this (equivalent) formulation, cells correspond to vertices of a graph and edges of the graph connect neighboring cells.

We consider two versions of the optimal searcher path problem. In the first version (denoted by PD), the time horizon is finite and the measure of search effectiveness is probability of detection. In the second version (denoted by ET), the time horizon is infinite and the measure of search effectiveness is expected time to detection. Our main results are that PD is NP-complete and that ET is NP-hard.

The problem of searching for a target in discrete time and space has received considerable attention. Stone (1975) outlines a straightforward algorithm for solving the search problem whenever the target is stationary and the searcher effort is infinitely divisible. Further, Stone extends this analysis to the situation in which the searcher effort must be applied in discrete units. When the detection function is exponential, Brown (1980) introduces a powerful technique that generalizes the results for infinitely divisible effort to moving targets. Washburn (1980) applies this technique to moving target problems involving discrete effort. Stewart (1980) adapts this technique to the construction of optimal searcher paths by using branch and bound methods. Richardson (1981) further explores these methods. Eagle (1984) gives an alternate approach: he formulates the problem as a partially observable Markov decision process (POMDP) and suggests a method for reducing the extensive calculations associated with standard POMDP solution procedures. Stewart, Richardson and Eagle all assume a finite time horizon and take probability of detection as their measure of search effectiveness. Lössner and Wegener (1982) consider a problem similar to the optimal searcher path problem in which switching costs constrain the searcher motion. In this problem, the time horizon is infinite and expected cost to detection is the measure of effectiveness. Wegener (1982), whose analysis motivated this note, shows that the switching costs problem is NP-hard.

Throughout our discussion, we use Garey and Johnson (1979) as the standard reference for results on the theory of complexity.

Definitions

Both optimal searcher path problems we will consider use the following general framework.

The search space is a finite, connected graph $G = (V, E)$. V denotes the set $\{v_1, \ldots, v_n\}$ of vertices of G, and E denotes the set of edges. Each edge in E can be viewed as an unordered pair of vertices $\{v_i, v_j\}$. A single stationary target is located at one of the vertices of G. Further, a prior target location distribution is given, with p_i being the probability that the target is located at vertex v_i. (All probabilities in this note are assumed to be rational numbers.)

The search for the target will be conducted in discrete time. During each time step, the searcher can look at one vertex. One of the search problems to be considered has a finite time horizon T; in the other problem, the time horizon is infinite. If there is a finite horizon, let S denote the set $\{1, 2, \ldots, T\}$. Otherwise, let S be the set of positive integers.

The searcher's motion is constrained by the structure of the graph G. More precisely, let Ψ be the set of functions $\psi : S \rightarrow V$ with the property that for any

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two consecutive integers k and $k+1$ in S, either $\psi(k) = \psi(k+1)$ or $\{\psi(k), \psi(k+1)\} \in E$. Then Ψ defines the set of available search plans. Under plan ψ, vertex $\psi(k)$ is searched during step k. The conditions on the set Ψ guarantee that at each time step the searcher will either remain at the current vertex or move to a neighboring vertex.

If the searcher looks at the vertex occupied by the target, it is not certain that a detection will occur. Instead, the detection capability of the searcher is described by a glimpse probability function g. If at time step k the searcher looks at vertex v, then $g(v, k)$ is the probability that a detection will occur, given that the target is at vertex v. When the time horizon is infinite, we drop the dependence on time and assume that g is a function only of vertex v. Further, in this case, we assume that $g(v) > 0$ for all vertices v.) No false detections occur and detection at one time step is independent of detection at any other time step.

Problem Statements

In terms of the preceding definitions, the two problems can be stated as follows.

First, suppose that there is a finite time horizon T. In this case, the measure of search effectiveness is probability of detection. The probability of detection, $P_T(\psi)$, provided by a plan $\psi \in \Psi$, is computed as follows. For each pair (i, k) with $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant T$, define $f(i, k, \psi)$ by

$$f(i, k, \psi) = \begin{cases} 1 - g(v_i, k) & \text{if } \psi(k) = v_i \\ 1 & \text{otherwise.} \end{cases}$$

Then $P_T(\psi)$ is given by

$$P_T(\psi) = \sum_{i=1}^{n} p_i\left(1 - \prod_{k=1}^{T} f(i, k, \psi)\right). \tag{1}$$

Now, using the format of Garey and Johnson, we can state the first optimal searcher path problem in terms of a generic instance and a question.

Probability of Detection (PD)

Instance: A finite connected graph $G = (V, E)$ with $n = |V|$ vertices, probability distribution $(p_1, \ldots, p_n)$ on V, a time horizon T, glimpse probabilities $g(v, k)$ for $v \in V$ and $1 \leqslant k \leqslant T$, and a bound B with $0 \leqslant B \leqslant 1$.

Question: Is there a search plan $\psi \in \Psi$ with $P_T(\psi) \geqslant B$?

For the second problem, suppose that the time horizon is infinite. Now the measure of search effectiveness is the expected time to detection. For each $\psi \in \Psi$, this time is computed as follows. If k is a positive integer, then by restricting ψ to integers between 1 and k, we can view ψ as a search plan for time horizon k. Thus, $P_k(\psi)$ can be defined as in (1). The expected time to detection, $E(\psi)$, for the plan ψ is given by

$$E(\psi) = \sum_{k=1}^{\infty} k(P_k(\psi) - P_{k-1}(\psi)). \tag{2}$$

By definition, $P_0(\psi) = 0$.

Note that since the graph G is finite and connected and the glimpse probabilities $g(v)$ are positive for all vertices v, the expected time to detection for an optimal plan is finite. In fact, suppose that it is possible to start at some vertex v, visit each vertex at least once, and return to v in t time steps. Further, let u denote the minimum value of $g(v)$ over all vertices v. Then the expected time to detection for an optimal plan ψ^* is bounded by

$$E(\psi^*) \leqslant \sum_{m=1}^{\infty} mtu(1-u)^{m-1} = t/u.$$

Now we can state the second optimal searcher path problem.

Expected Time (ET)

Instance: A finite connected graph $G = (V, E)$ with $n = |V|$ vertices, a probability distribution $(p_1, \ldots, p_n)$ on V, glimpse probabilities $g(v) > 0$ for $v \in V$, and a bound $B > 0$.

Question: Is there a search plan $\psi \in \Psi$ with $E(\psi) \leqslant B$?

A Known NP-Complete Problem

The problem of finding a Hamiltonian path in a graph is NP-complete. (See problem GT39 in Garey and Johnson.) This decision problem can be stated more precisely as follows.

Hamiltonian Path (HP)

Instance: A finite, connected graph $G = (V, E)$ with $n = |V|$ vertices.

Question: Is there an ordering $\langle v_1, \ldots, v_n \rangle$ of the vertices of G with $\{v_i, v_{i+1}\} \in E$ for $1 \leqslant i < n$?

In the following proofs, we show that HP transforms to an instance of both PD and ET.

PD IS NP-Complete

The problem PD is in NP since a polynomial time nondeterministic algorithm need only guess a search plan ψ and then evaluate $P_T(\psi)$ in Equation 1. The search plan that the algorithm must guess is just a sequence of T vertices with either equal or neighboring consecutive vertices. This requirement on consecutive vertices can be checked by scanning the set E. Further, evaluating the formula in Equation 1 involves no more than $2n$ additions and nT multiplications. Taken together, these observations indicate that the operations can be performed in a time that can be bounded by a polynomial in $n + T$. However, our definition of a generic instance of PD includes the function g, which depends on both n and T. Thus, the operations can be performed by a polynomial time nondeterministic algorithm.

Next we define a function that maps instances of HP to instances of PD. An instance of HP consists of a finite, connected graph $G = (V, E)$ with $n = |V|$ vertices. The corresponding instance of PD includes the same graph G, the uniform prior distribution on vertices given by $p_i = 1/n$ for $i = 1, \ldots, n$, the time horizon $T = n$, the glimpse probabilities $g(v, k) = 1/2$ for $v \in V$ and $1 \leq k \leq T$, and the bound $B = 1/2$. In general, the length of an instance of PD depends on both n and T, while the length of an instance of HP depends on n but not T. However, in the specified transformation, $T = n$. Thus, this transformation, which amounts to simply augmenting the instance of HP with some straightforward values, can be performed in a time that is polynomial in the length of an instance of HP.

Now note that the graph G in the instance of HP has a Hamiltonian path if and only if the transformed problem is a yes-instance of PD. If G has a Hamiltonian path, then construct a search plan that follows this path. This plan yields a probability of detection of $1/2$. If the transformed problem is a yes-instance, then some search plan ψ of length n yields a probability of detection of at least $1/2$. However, it is easy to see that the maximum value of Equation 1 for a plan of length n is $1/2$ and that this value occurs when the plan visits each vertex exactly once. Thus, search plan ψ must trace a Hamiltonian path.

Since PD is in NP and the problem HP transforms to PD, the decision problem PD is NP-complete.

ET Is NP-Hard

Define a function that maps instances of HP to instances of ET as follows. The instance of HP consists of a finite, connected graph $G = (V, E)$ with $n = |V|$ vertices. The corresponding instance of ET includes the same graph G, the uniform prior distribution $p_i = 1/n$ for $i = 1, \ldots, n$, glimpse probabilities $g(v) = 1/2$ for $v \in V$, and a bound $B = (3n + 1)/2$. As in the case of PD, this transformation simply augments the instance of HP and can be performed in polynomial time.

Now note that the graph G in the instance of HP has a Hamiltonian path if and only if the transformed problem is a yes-instance of ET. If G has a Hamiltonian path, then construct the search plan ψ that oscillates back and forth along this path while repeating the endpoint on each turn. The expected time to detection for this plan is

$$\begin{aligned} E(\psi) &= \sum_{l=1}^{\infty} \sum_{k=1}^{n} [(l-1)n + k]/(2^l n) \\ &= \sum_{l=1}^{\infty} [2nl - n + 1]/2^{l+1} \\ &= (3n + 1)/2. \end{aligned}$$

If the transformed problem is a yes-instance, then some search plan ψ yields an expected time to detection that is less than or equal to $(3n + 1)/2$. However, if no Hamiltonian path exists, then the expected time to detection must be greater than $(3n + 1)/2$. So a yes-instance must yield a Hamiltonian path.

The fact that problem HP does transform to ET demonstrates that the decision problem ET is NP-hard. Whether ET is NP-complete remains unresolved. We note that one approach to encoding a search plan is to list all vertices in the order in which the searcher visits them. In the decision problem ET, a search plan consists of an infinite sequence of vertices. Thus, with this encoding of search plans, it would not be possible to construct a polynomial time nondeterministic algorithm that guesses search plans and evaluates their expected time to detection.

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