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THE THEORY OF SEARCH

II. TARGET DETECTION†

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"KINEMATIC BASES," the first paper of this series, discussed the geometric and kinematic factors involved in search—the positions, motions, and contacts of observers and targets. Probability was introduced only in assuming specific relative positions for the observer and target.

The present paper discusses the uncertainties inherent in the act of detection under various specific conditions of contact. In the course of the discussion a body of methods for applying probability to problems of detection is developed. It must be emphasized, however, that these methods are conditioned by the particular situation in the case of visual detection because the different elementary acts of looking or 'glimpses' are essentially independent trials. The reason for the distinction follows.

A situation in visual search can be determined at a glance: Clouds or hills between the searcher and the position searched, for example, are readily discernible and it is also relatively simple to determine whether there is enough light to see by and whether there is an absence of haze. In contrast, it is becoming more apparent that the conditions of sonar and radar 'visibility' vary greatly for different places and times, and that the quality of these conditions cannot be ascertained by the simple methods of direct observation. This leads to the possibility that different 'glimpses' are dependent on each other. Thus, if the first 'glimpse' fails to show the target, it suggests that the prevailing conditions are poorer than previously supposed. From this is then inferred that the next 'glimpse' will be

† This is Part II of a series of three papers on *The Theory of Search*, by B. O. Koopman. Part I, *Kinematic Bases*, appeared on page 324 of the June 1956 issue. Part III, *The Optimum Distribution of Searching Effort*, will appear in a later issue.

even more likely to fail. This problem is discussed, although only briefly, in the last section of the paper.

Two basic facts underlie every type of detection:

(*a*) There is a certain set of physical requirements which have to be met if detection is to be possible, and which, if met, will in fact make detection possible, though not necessarily inevitable: Targets must obviously not be too far away; their view from the observer must not be completely obstructed; to be seen there must be some illumination; the radar will not reveal them if the atmospheric conditions or background echoes are too bad; sonar detection requires that the sound path be not completely bent away from the observer by water refraction; etc.

(*b*) Even when the physical conditions make detection possible, it will by no means inevitably occur: Detection is an event which under definite conditions has a definite probability, the numerical value of which may be zero or unity or anything in between. Thus when the target just barely fulfills the physical conditions for possible detection, the probability of detection will be close to zero (at least when the time for observation is very limited). As the conditions improve, the chance of detection increases, and it may become close to or equal to unity; detection becomes practically certain. Experience in everyday life shows that we may be looking for an object in plain sight and yet sometimes fail to find it. Cases are known where observational aircraft flying on clear sunny days on observational missions have passed close over large ships and yet failed to detect them. And a host of operational statistics give further confirmation of this point. It must be constantly realized that every instrumentality of detection is based in last analysis on a human being, and its success is accordingly influenced by his attention, alertness, and fatigue, and the whole chain of events which occur between the impact of the message on his sense organs and his mental response thereto. Furthermore, even under physical conditions that are as fixed and constant as it is practicable to make them, innumerable rapid fluctuations in them are still apt to occur (a radar target changes its aspect from moment to moment with the continual rocking of the ship, sonar ranges experience short-term oscillations about their mean, etc.); and as a result, a target which may not be detected at one instant may be detected if sought a moment later.

In view of (*a*), one part of the study of detection requires the physical conditions for detection to be explored; in view of (*b*), the other part requires the probabilities of detection, when the former conditions are given, to be obtained. In the last section of this paper, the effect of statistically combining observations made under operational conditions in which the physical situation is not constant is considered.

INSTANTANEOUS PROBABILITIES OF DETECTION

SUPPOSE THAT the physical conditions (distances, etc.) remain fixed and that the observer is looking for the target ('looking' will mean trying to detect with the means considered, visual, radar, sonar, etc.).

There are two possibilities: *First*, the observer may be making a succession of brief 'glimpses'; a typical case of this is in the echo-ranging procedure in which each sweep or scan affords one opportunity for detection (glimpse), successive ones occurring two or three minutes apart. *Second*, the observer may be looking continuously; a typical case is the observer fixing his eyes steadily on the position where he is trying to detect the target. The case of radar is intermediate; on account of the scanning it would belong to the first case, but if the scanning is very fast, and especially when there is persistency of the image on the scope, it may be treated as belonging to the second. Likewise, visual detection by a slow scan through a large angle belongs to the first rather than the second case. Very often the decision to regard a method of detection in the first or in the second way depends simply on which affords the closest or most convenient approximation.

In the case of separated glimpses the important quantity is the instantaneous probability, g, of detection by one glimpse. When n glimpses are made under unchanging conditions the probability of detection, p_n, is given by the formula

$$p_n = 1-(1-g)^n. \tag{1}$$

This is because $1-p_n$ is the probability of failing to detect with n glimpses, and for this to occur the target must fail to be detected at every single one of the n glimpses; each such failure having the probability $1-g$ and the n failures being independent events, we conclude that $1-p_n=(1-g)^n$; hence (1). When $g=0$, obviously $p_n=0$, but if $g>0$ and even if g is very small, p_n can be made as close to 1 as we please by increasing n sufficiently; in other words, once the physical conditions give some chance, however small, of detecting on one glimpse, enough glimpses under the same conditions will lead, with practical certainty, to eventual detection.

To find the mean or expected number $\bar{n}$ of glimpses for detection we must first find the probability, P_n, that detection will occur precisely at the nth glimpse (and not before). This is the product of the probability that it will not occur during the first $n-1$ glimpses, $(1-g)^{n-1}$, times the probability that a detection will occur on a single glimpse (the nth), g; it is accordingly $P_n=(1-g)^{n-1}g$. The required mean number $\bar{n}$ is, according to the theory of probability, $1P_1+2P_2+3P_3+\cdots$, and thus

$$\begin{aligned}\bar{n} &= \sum_{n=1}^{\infty} n\,(1-g)^{n-1}\,g = g+2\,(1-g)\,g+3\,(1-g)^2\,g+\cdots \\ &= -g\,\frac{d}{dg}\left[1+(1-g)+(1-g)^2+\cdots\right] \\ &= -g\,\frac{d}{dg}\,\frac{1}{1-(1-g)} = -g\,\frac{d}{dg}\left(\frac{1}{g}\right) = \frac{1}{g}.\end{aligned} \tag{2}$$

Turning to the case of continuous looking, the important quantity is the probability $\gamma\, dt$ of detecting in a short time interval of length dt. The quantity γ is called the instantaneous probability density (of detection). When the looking is done continuously during a time, t, under unchanging conditions, the probability, $p(t)$, of detection is given by

$$p(t) = 1 - e^{-\gamma t}. \tag{3}$$

To prove this, consider $q(t) = 1 - p(t)$, the probability of failure of detection during the time t. For detection to fail during the time $t+dt$ [probability $= q\,(t+dt)$], detection must fail both during t [probability $= q(t)$] and during dt (probability $= 1 - \gamma\, dt$), and multiplying these probabilities of independent events we obtain $q(t+dt) = q(t)\,(1-\gamma\, dt)$, which is equivalent to the differential equation

$$dq(t)/dt = -\gamma\, q(t).$$

The solution of this equation on the assumption that $q(0) = 1$ (no detection when no time is given to looking) is $q(t) = e^{-\gamma t}$: whence (3). Again it is true that if there is the least chance of detection in time dt (i.e., if $\gamma > 0$) the chance of detection increases to virtual certainty as the looking time t becomes sufficiently large.

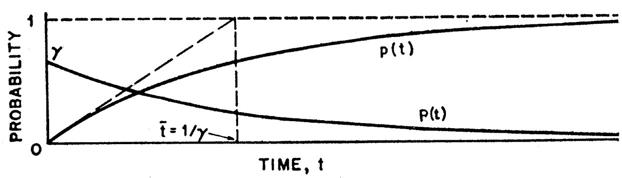

Fig. 1. Probabilities of detection under fixed conditions.

To find the mean or expected time $\bar{t}$ at which detection occurs, observe that the probability $P(t)\, dt$ of detection between t and $t+dt$ (when looking has been continuing from the initial time 0) is the product of probability of no detection before t, times probability of a detection during dt, i.e., $P(t)\, dt = e^{-\gamma t}\, \gamma\, dt$. $\bar{t}$ is found by integration:

$$\bar{t} = \int_0^\infty t\, e^{-\gamma t}\, \gamma\, dt = \frac{1}{\gamma}. \tag{4}$$

Figure 1 shows the graphs of the probability, $p(t)$, of detection *during* the time t, and $P(t)$ of detection *at* the time t, and gives the construction of $\bar{t}$ as the abscissa of the intercept with the horizontal line of unit ordinate of the tangent to $p(t)$ at the origin.

Since equation (3) reduces to equation (1) when γ is taken as $\gamma = -\log(1-g)$ and $t = n$ (glimpses one unit of time apart), Fig. 1 serves to show the quantitative behavior of p_n and P_n: the difference is that only discrete points ($t=1, 2, 3, \cdots$) on the curve are used, and $\bar{n}$ is no longer given by the tangent intercept but rather by a secant intercept.†

When, as usually occurs in actual search, the distances, and hence the probability quantities g or γ, change as time goes on, (1) must be replaced by

$$p_n = 1 - \prod_{i=1}^{n}(1-g_i) = 1-(1-g_i)\ (1-g_2)\ (1-g_3)\ \cdots, \tag{5}$$

which takes into account the fact that g will change from glimpse to glimpse: g_i is the probability of detection for the ith glimpse. And (3) must be replaced by

$$p(t) = 1 - \exp\left(-\int_0^t \gamma_t\, dt\right), \tag{6}$$

where the possible change in the probability density of detection as time goes on is put into evidence by the subscript in γ_t. The reasoning leading to these equations is precisely similar to that in the earlier case. But the probabilities p_n, $p(t)$ do not necessarily approach unity as n or t increases; thus when $\int_0^\infty \gamma_t\, dt$ is finite, the chance of detection $p(t)$ never exceeds $1-\exp(-\int_0^\infty \gamma_t\, dt) < 1$.

The instantaneous probability quantities g and γ depend, as we have said, on the sum total of physical conditions. For example, in visual detection γ depends on the range r from target to observer, on the meteorological state (illumination and haze), on the size and brightness of target against the background, on the observer's facilities, altitude, etc. And corresponding lists can be made out for radar and sonar detection. Throughout the remainder of the present paper, only the dependence on range will be explicitly considered, i.e., we shall write

$$g = g(r), \qquad \gamma = \gamma(r). \tag{7}$$

It will be legitimate to apply the results either when all the other conditions remain practically unchanged during the operation considered, or when the other conditions have been shown not to influence the results to the degree of approximation that is accepted.

Since the instantaneous probability quantities tend to decrease to zero as the range r increases and to be large when the range is small, their graph against r will be of the character shown in Fig. 2. In case A, the instantaneous probability density reaches a finite maximum at zero range (probability of detecting target when flying over target is less than unity). In case B, this maximum is infinite (probability of detection when flying over

† Throughout this paper, log is used to denote *natural logarithm*, and $\log_{10}$ to denote *common logarithm*.

target is unity). In case C, the effect of sea return on radar diminishes the probability of detection when over target. In case D, the instantaneous probability is infinite when $r<R$: detection is sure to occur as soon as the target gets within this critical range R.

The last case, while not altogether realistic, is often not very far from the truth. A very useful rough approximation is to assume further that the instantaneous probability is zero for $r>R$. Then detection is sure and immediate within the range R and is impossible beyond R. This assumption will be called the *definite-range law of detection.*

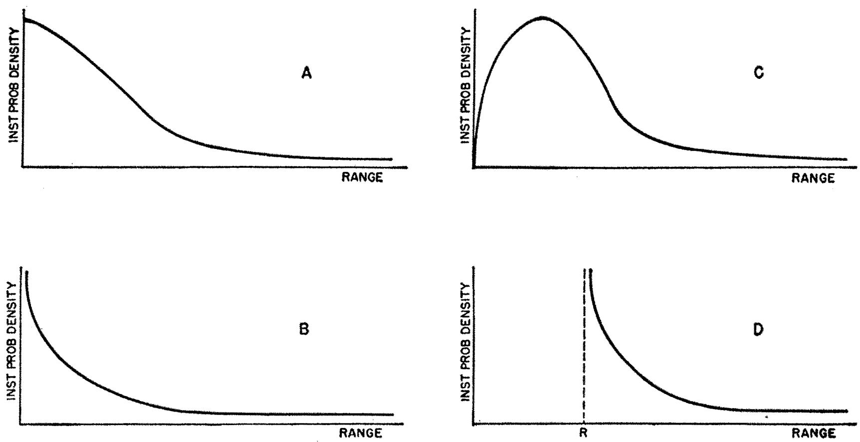

Fig. 2. Instantaneous probability at various distances.

An important example showing the evaluation of the function $\gamma(r)$ is in the case when the following assumptions are made:

1. The observer is at height h above the ocean, on which the target is cruising.
2. The observer detects the target by seeing its wake.
3. The instantaneous probability of detection γ is proportional to the solid angle subtended at the point of observation by the wake.

The calculation of the solid angle is shown in Fig. 3 for an area of ocean which is a rectangle of length a toward the observer and width b perpendicular to the direction of observation (perpendicular to the page in Fig. 3-A). The infinitesimal solid angle is the product of the angle α subtended by a, and the angle β subtended by b. The radian measure of α is c/s. By similar triangles, $c/a=h/s$ and hence $\alpha=ah/s^2$. And the radian measure of β is obviously b/s. Hence solid angle $=\alpha\beta=abh/s^3=$ area of rectangle times h/s^3. The actual area A of the target's wake is not rectangular, but can be regarded as made up of a large number of rectangles like the above, the solid angle being the sum of the corresponding solid angles. Hence, when the dimensions of A are small in comparison with h, r, and s, we have the formula

$$\text{solid angle} = Ah/s^3 = Ah/(h^2+r^2)^{3/2}. \tag{8}$$

Since γ is assumed to be proportional to the solid angle, we obtain

$$\gamma = kh/s^3 = kh/(h^2+r^2)^{3/2}, \tag{9}$$

where the constant k depends on all the factors that we a reregarding as fixed and not introducing explicitly, such as contrast of wake against

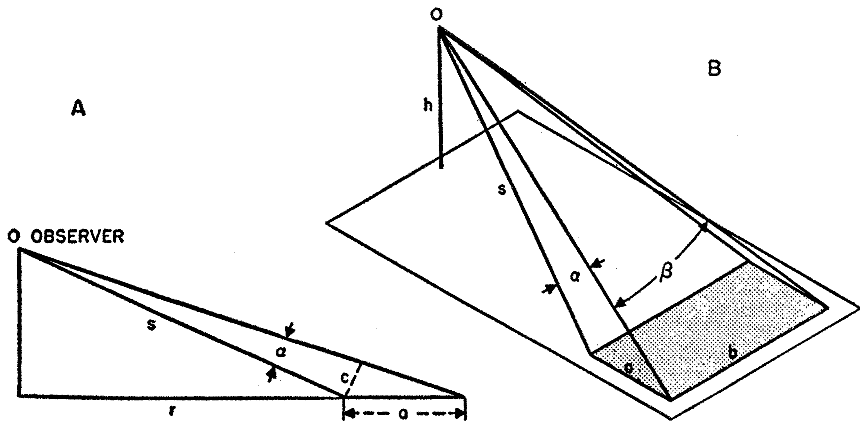

Fig. 3. Solid angle subtended by wake.

ocean, observer's ability (number of lookouts and their facilities), meteorological conditions, etc.; and of course k contains A as a factor. Dimensionally, $k = [L^2\ T^{-1}]$. In the majority of cases r is much larger than h, and (9) can be replaced by the satisfactory approximation

$$\gamma = kh/r^3. \tag{10}$$

Formulas (9) and (10) lead to cases A and B respectively of Fig. 2; the property of detection which they express shall be called the inverse-cube law of sighting. When the subject of vision is studied, it will be found that many changes in this law have to be made to obtain a high degree of approximation under the various conditions of practice. Nevertheless the inverse-cube law gives a remarkably useful approximation. Its use in the present paper is chiefly as an illustration of the general principles.

DEPENDENCE OF DETECTION ON TRACK

WHEN THE OBSERVER AND TARGET are moving over the ocean in their respective paths, which may be straight or curved and at constant or changing speeds, the continuous change in their relative positions constantly changes the instantaneous probability of detection; we have to deal with the functions g_t and γ_t and calculate probabilities of detection by means of formulas (5) and (6). It is convenient to draw the target's track C (Fig. 4) relative to the observer. The latter need not be moving in fixed course and speed over the ocean, although that is very often the case. The coordinates used have been described in the preceding paper, "Kinematic Bases" (see Fig. 4 and equation (2) of that paper).

In Fig. 4, the target is at (ξ,η) at the time t, so that the equations of the target's relative motion are

$$\xi=\xi(t), \qquad \eta=\eta(t), \tag{11}$$

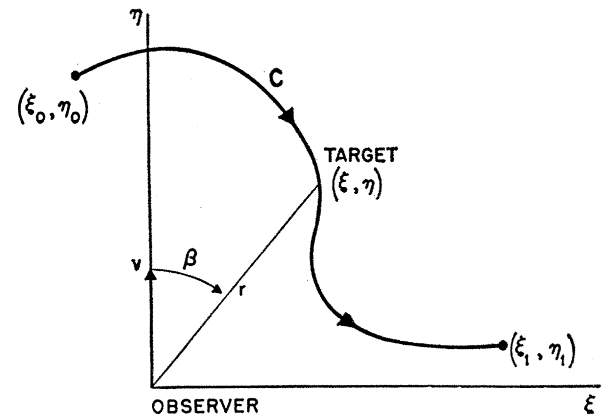

Fig. 4. Target's relative track.

where initially, at $t=t'$, $\xi_0=\xi(t')$, $\eta_0=\eta(t')$, and finally, at $t=t''$, $\xi_1=\xi(t'')$, $\eta_1=\eta(t'')$. The target describes the relative track C. Accordingly, (7) becomes [writing $\xi^2(t)$ for $\{\xi(t)\}^2$]:

$$g=g(\sqrt{\xi^2(t)+\eta^2(t)})=g_t, \qquad \gamma=\gamma(\sqrt{\xi^2(t)+\eta^2(t)})=\gamma_t. \tag{12}$$

Hence according to equations (5) and (6) the probabilities p_C of detection are given by either of the following

$$p_C=1-\prod_{i=1}^{n}[1-g(\sqrt{\xi^2(t_i)+\eta^2(t_i)})], \tag{13}$$

$$p_C=1-\exp\left[-\int_{t'}^{t''}\gamma(\sqrt{\xi^2(t)+\eta^2(t)})\,dt\right]. \tag{14}$$

In (13), t_i is the time (epoch) of the ith 'glimpse' or scan, and n is the number of glimpses between t' and t'': $t'\leqq t_1<t_2<\cdots<t_n\leqq t''$. In (14),

the integral is actually a line integral along C; if w is the relative speed (not necessarily constant), we may write [with s = arc length of C from (ξ_0, η_0)]:

$$p_C = 1 - \exp[-\int_C \gamma(r)\, ds/w]. \qquad (15)$$

Formulas (13) and (14) may be united into

$$p_C = 1 - e^{-F[C]}, \qquad (16)$$

where for the case of separate glimpses

$$F[C] = -\sum_{i=1}^{n} \log[1 - g(\sqrt{\xi^2(t_i) + \eta^2(t_i)})], \qquad (17)$$

and in the case of continuous looking

$$F[C] = \int_C \gamma(r)\, ds/w. \qquad (18)$$

This quantity $F[C]$ will be called the *sighting potential.* It has the important property of additivity: If C_1 and C_2 are two tracks and $C = C_1 + C_2$ is their combination or sum, and if $p_C = p_{C_1+C_2}$ is the probability of sighting on at least one track, $p_{C_1+C_2}$ is still obtained by formula (16) and

$$F[C_1 + C_2] = F[C_1] + F[C_2]. \qquad (19)$$

This is an immediate consequence of the usual equation for combining probabilities of events which may not be mutually exclusive:

$$p_C = 1 - (1 - p_{C_1})\,(1 - p_{C_2}) = p_{C_1} + p_{C_2} - p_{C_1} p_{C_2}.$$

The additivity applies, of course, to the sum of any number of paths. One application is to the calculation of p_C when C is complicated, but made up out of a sum of simple pieces such as straight lines. Another application is in the case of two or more inter-communicating observers: C_1 can be the path of the target relative to the first and C_2 that relative to the second, etc.

A most important case, and one which will chiefly concern us in this paper, is when both observer and target are moving at constant speed and course. The results of "Kinematic Bases" become applicable. Track C is a straight line, and the speed w is a constant (as long as C is not turned). It is convenient to make the calculations with the aid of the coordinates (x,y) of Fig. 5, where x is the lateral range. The equations of motion which take the place of (11) are x = constant, $y = wt$, where t is measured from the epoch of closest approach, and where, furthermore, the positive direction of the y-axis is that of the target's relative motion; this convention is used throughout this chapter. The potential $F[C]$ is given by the appropriate one of the formulas

$$F[C]=-\sum_{i=1}^{n}\log[1-g(\sqrt{x^2+w^2t_i^2})]=-\sum_{i=1}^{n}\log[1-g(\sqrt{x^2+y_i^2})]; \quad (20)$$

$$F[C]=\int_{t'}^{t''}\gamma(\sqrt{x^2+w^2t^2})\,dt=\frac{1}{w}\int_{y'}^{y''}\gamma(\sqrt{x^2+y^2})\,dy, \quad (21)$$

where y_i is the distance of target at the ith glimpse to its closest position, and (x',y') and (x'',y'') are the extremities of C: $x'=x''=x=$ constant, $y'=wt'$, $y''=wt''$.

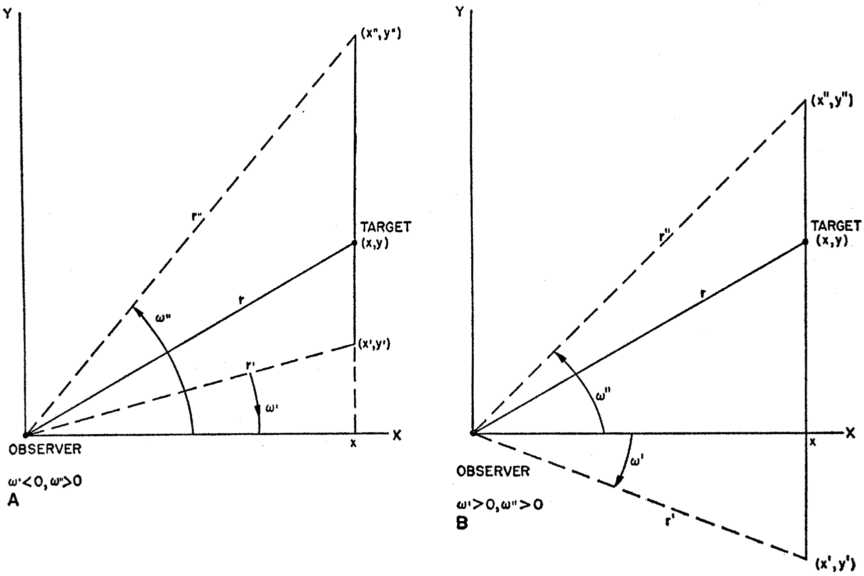

Fig. 5. Detection at fixed speed and course.

In the case of the inverse-cube law (9),

$$F[C]=\frac{kh}{w}\int_{y'}^{y''}\frac{dy}{(h^2+x^2+y^2)^{3/2}}$$

$$=\frac{m}{h^2+x^2}\left(\frac{y'}{\sqrt{h^2+x^2+(y'')^2}}-\frac{y'}{\sqrt{h^2+x^2+(y')^2}}\right), \quad (22)$$

and for (10),

$$F[C]=\frac{kh}{w}\int_{y'}^{y''}\frac{dy}{(x^2+y^2)^{3/2}}=\frac{m}{x^2}\left(\frac{y''}{r''}-\frac{y'}{r'}\right)=\frac{m}{x^2}\,(\sin w'+\sin w''), \quad (23)$$

where in each case $\qquad m=kh/w, \qquad (24)$

and where r' and r'' are the ranges of the extremities of C, and w' and w'' the angles they subtend with the normal to C.

THE LATERAL-RANGE DISTRIBUTION

WHEN THE OBSERVER AND TARGET are on their straight courses at constant speeds for a long time before and after their closest approach, the probability of detection, $p(x)$, is a function of the lateral range x. The graph of $p(x)$ against x is called the lateral-range curve and expresses the distribution in lateral range. In consequence of (16), $p(x)$ is given by

$$p(x)=1-e^{-F(x)}, \tag{25}$$

where $F(x)$ is the value of $F[C_x]$, C_x being an infinite straight line at the perpendicular distance x from the observer. The value of $F(x)$ is found by applying equation (20), summing over all integral values of i, or equa-

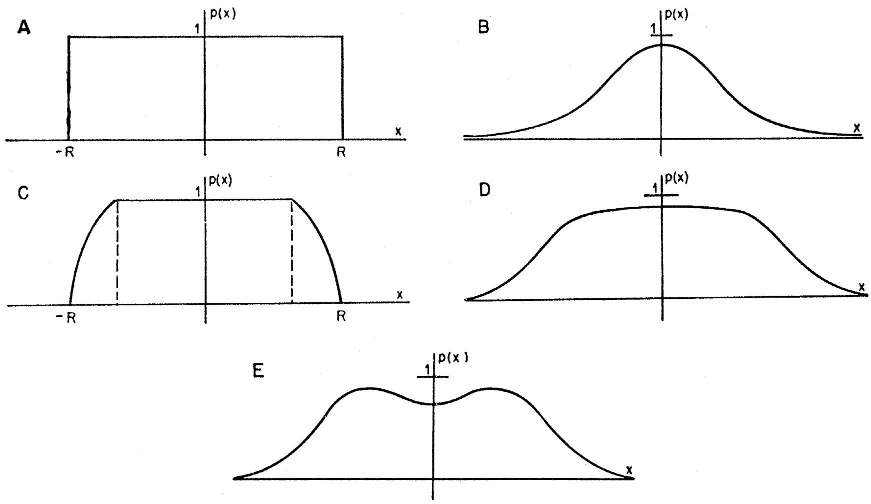

Fig. 6. Lateral-range curves.

tion (21) with $y'=-\infty$ and $y''=\infty$ in the glimpse or the continuous looking cases, respectively.

With continuous looking, (21) applies. For the definite-range law, $p(x)=1$ or 0 according as $-R<x<R$ or not, and the lateral-range curve is Fig. 6-A. For the inverse-cube law,

$$p(x)=1-e^{-2m/(h^2+x^2)} \quad \text{or} \quad p(x)=1-e^{-2m/x^2}, \tag{26}$$

according to whether (22) or (23) is used; the curve is shown in Fig. 6-B in the former case.

With intermittent glimpses taking place T units of time apart, equation (20) applies. For the definite-range law, $p(x)=0$ when $x>R$ or $x<-R$, and $p(x)=1$ when the length $2\sqrt{R^2-x^2}$ of relative track during which the target is within range R of the observer is greater than wT, i.e., when

$$-\sqrt{R^2-\tfrac{1}{4}w^2T^2} \leqq x \leqq \sqrt{R^2-\tfrac{1}{4}w^2T^2};$$

but

$$p(x)=2\sqrt{R^2-x^2}/wT$$

in intermediate cases, this being the probability that the target be glimpsed while within range R. The lateral-range curve is shown in Fig. 6-C.

Other typical lateral-range curves are those of Fig. 6-D and -E. The dip at $x=0$ in Fig. 6-E shows the effect of sea return (radar) or pinging over the target (sonar).

The area W under the lateral-range curve is called the *effective search* (*or sweep*) *width*:

$$W=\int_{-\infty}^{+\infty} p(x)\, dx. \tag{27}$$

It has the following interpretation: If the observer moves through a swarm of targets uniformly distributed over the surface of the ocean (N per unit area on the average) and either all at rest or all moving with the same vector velocity $\boldsymbol{u}$, the average number N_0 detected per unit time is

$$N_0=NwW. \tag{28}$$

For suppose that t is such a long period of time that the length of time during which a target is within range of possible detection is small in comparison with t. Then the number of targets passing during the period t through detection range (i.e., exposing themselves to detection) and having the lateral range between x and $x+dx$ is $Nwt\,dx$ (since such targets are in an area of $wt\,dx$ square miles). On the average $p(x)\,Nwt\,dx$ of these will be detected. Hence the average total number detected is

$$\int_{-\infty}^{+\infty} p(x)\, Nwt\, dx.$$

Dividing this by t and applying equation (27), equation (28) is obtained.

Since for continuous looking with a definite-range law, $W=2R$, we may describe W as follows: The effective search width is twice the range of a definite-range law of detection, which is equivalent to the given law of detection in the sense that each of the two laws detects the same number of uniformly distributed targets in identical velocity.

The product wW is called the *effective search* (*or sweep*) *rate*.

When the distribution of targets is uniform and when their speed is given but their course is not, w has to be replaced by its average $\bar{w}$ (taken as uniformly distributed in track angle ϕ) i.e., we must write

$$\bar{w}=(1/2\pi)\int_0^{2\pi} w\,d\phi,$$

so that (28), $N_0=\bar{N}\bar{w}W$, may hold (here $W=2R$). [See "Kinematic Bases," equations (1) and (4).]

In the case of the simplified inverse-cube law, equations (26) and (27) give, by carrying out the integration (see below):

$$W=2\sqrt{2\pi m}=2\sqrt{2\pi kh/w}, \tag{29}$$

so that the search width is proportional to the square root of the altitude and inversely proportional to the square root of the target's relative speed. Furthermore, if there are n aircraft flying the same path without mutual interference (or if there are n observers having the same facilities operating independently of one another in the same aircraft), W is replaced by $W\sqrt{n}$.

This results from the additivity of the potentials which has the effect that k is replaced by nk in equations (22) and (23), and thus that m is replaced by nm [see (24)]. Thus the statement that W is replaced by $W\sqrt{n}$ is a consequence of equation (29).

The integration leading to (29) is performed by introducing equation (26) into (27) and changing to the new variable of integration:

$$z=\sqrt{2m}/x;$$

and then integrating by parts. Use is made of the well-known equation $\int_0^\infty e^{-z^2}\,dz=\tfrac{1}{2}\sqrt{\pi}$.

By its definition, $p(x)$ is the probability (not probability density) that a target, known to have the lateral range x, be detected. On the other hand, $p(x)\,dx/W$ is the probability that a target, known to have been detected, have a lateral range between x and $x+dx$ (in this case $p(x)/W$ is a probability density). This fact (actually a consequence of Bayes' theorem in probability) is easily seen, as follows: The detected target may be thought of as chosen at random from the set of all detected targets; the chance that its lateral range be between x and $x+dx$ is equal to the proportion of targets in this set which have such a lateral range; from the previous calculations, this proportion is seen to be

$$Nw\,p(x)\,dx/NwW=p(x)\,dx/W.$$

THE DISTRIBUTION IN TRUE RANGE

AGAIN WE SUPPOSE that the observer makes constant speed and course and that the targets do likewise and are distributed uniformly over the surface of the ocean with the density N (average number per unit area). Relative to the observer, the targets all move parallel to the y-axis in the direction of increasing y. How many targets are detected on the average per unit time in the small region of area $dx\,dy$ of Fig. 7? The number will be proportional to N and to $dx\,dy$, and may accordingly be represented by $N\,p(x,y)\,dx\,dy$, where $p(x,y)$, which may be described as the rate of first contacts at the point (x,y) per unit area and per unit density of targets, is obtained by the argument which follows.

The number of targets entering $dx\,dy$ in unit time is $Nw\,dx$. A given target's probability of being detected therein is the product of the probability that it fail to be detected before entering this region times the probability that, when not previously detected, it be detected while crossing $dx\,dy$, i.e., during the time $dt = dy/w$. The former probability

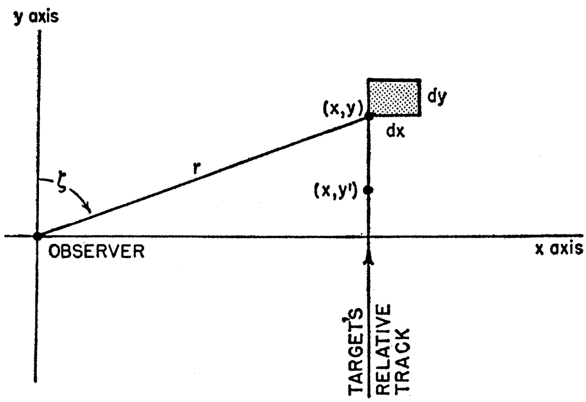

Fig. 7. Target detection at given relative position.

is $e^{-F(x,y)}$ in virtue of equation (16), where $F(x,y)$ is given in the case of glimpses by equation (20), with i summed over all values for which $y_i < y$ (with sufficient accuracy we may write $y_i = y - iwT$ and sum for $i = 1, 2, \cdots, \infty$); and in the case of continuous looking, by equation (21) with $y' = -\infty$ and $y'' = y$. The latter probability is given by $g(r)\,dy/wT$ (intermittent glimpses, one every T units of time, dy/wT being the probability that a glimpse occur while target is in $dx\,dy$), or by $\gamma(r)\,dy/w$ (continuous looking). Thus the probability is

$$e^{-F(x,y)}\,g(r)\,dy/wT \quad \text{or} \quad e^{-F(x,y)}\,\gamma(r)\,dy/w,$$

according to whether glimpsing or continuous looking is used. To obtain the mean number of detections per unit time in $dx\,dy$, these expressions are multiplied by the number of targets exposed to such detection, $Nw\,dx$.

Hence the answer to the question in italics above is supplied by the following expressions for $p(x,y)$:

$$\begin{aligned} p(x,y) &= e^{-F(x,y)}\, g(r)/T \\ F(x,y) &= -\sum_{i=1}^{\infty} \log[-g(\sqrt{x^2+(y-iwT)^2})] \end{aligned} \tag{30}$$

for intermittent glimpsing, and

$$\begin{aligned} p(x,y) &= e^{-F(x,y)}\, \gamma(r) \\ F(x,y) &= \frac{1}{w} \int_{-\infty}^{y} \gamma(\sqrt{x^2+y^2})\, dy \end{aligned} \tag{31}$$

for continuous looking.

It is seen by carrying out the differentiation that in the case of equation (31),

$$p(x,y) = w\ (\partial/\partial y)[1-e^{-F(x,y)}]. \tag{32}$$

In the case of (30), the corresponding formula is

$$p(x,y) = w\ \Delta_y[1-e^{-F(x,y)}], \tag{33}$$

where the operation Δ_y applied to a function denotes the result of the following process: first, replace y in the function by $y+\Delta y$, where $\Delta y = wT$; second, subtract the original value of the function from the new; third, divide by Δy.

If A is a plane region moving with the observer over the ocean, the average number Q_A of targets detected per unit time within A is (by addition of averages)

$$Q_A = N \iint_A p(x,y)\, dx\, dy. \tag{34}$$

In particular, when A embraces the whole plane, equations (32) and (33) lead from equation (34) to the previously obtained expression $N_0 = NwW$ of (28) by straightforward calculation. When $A = A_R$ is a circle of radius R centered on the observer, (34) expressed in polar coordinates (r,ζ) ($\zeta=$ angle from positive y-axis to vector r drawn from observer to target) becomes

$$Q(R) = N \iint_A p(x,y)\, dx\, dy = N \int_0^R dr \int_0^{2\pi} r\, p(r \sin\zeta, r\cos\zeta)\, d\zeta.$$

Now the number of targets detected per unit time at a distance (true range) from the observer between r and $r+dr$ is of the form $N\, \rho(r)\, dr$ (being proportional to both N and dr), and since its integral from 0 to R must, for every value of R, be equal to $Q(R)$, it follows (by equating the two integral expressions for $Q(R)$ and differentiating through with respect to R, etc.) that

$$\rho(r)=\int_0^{2\pi} r\, p(r\sin\zeta, r\cos\zeta)\, d\zeta. \tag{35}$$

$\rho(r)\,dr$ may be described as the rate of detection in the range interval $(r, r+dr)$ at unit target density.

If, now, a target is known to be detected but at unknown range, the probability that the range of detection has been between r and $r+dr$ is $\rho(r)\,dr/wW$. For this target may be thought of as chosen at random from the set of all the NwW detected targets, of which there are $N\,\rho(r)\,dr$ detected at range between r and $r+dr$. Hence the probability that the target be detected at such a range is the quotient

$$N\,\rho(r)\,dr/NwW=\rho(r)\,dr/wW.$$

The function $\rho(r)$ [or the equivalent functions $N\,\rho(r)$ or $\rho(r)/wW$] expresses the distribution in (true) range, and the graphs of these functions

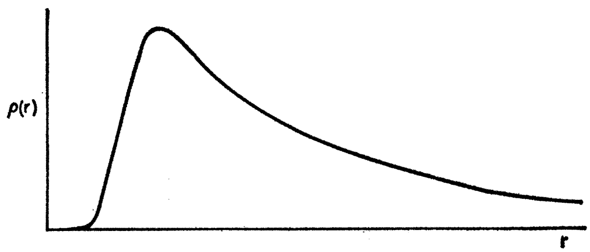

Fig. 8. True-range curve.

against r are called range curves. They fall considerably for small values of r, since relatively few targets come close to the observer by chance, and of these a still smaller number are apt to survive undetected up to a close proximity of the observer. Figure 8 shows a typical range curve (actually, for the inverse-cube law); as the situation approaches the definite range law, the curve humps up indefinitely about the value $r=R$ of the definite range, and falls to the axis of abscissas elsewhere.

The mean value of the range of detection is given by

$$\bar{r}=\frac{1}{wW}\int_0^{\infty} r\,\rho(r)\,dr=\frac{1}{wW}\int_0^{2\pi} r^2\, p(r\sin\zeta, r\cos\zeta)\, d\zeta. \tag{36}$$

In the case of the simplified inverse-cube law, we have equation (23), in which we set $w'=\frac{1}{2}\pi-\zeta$ and $w''=\frac{1}{2}\pi$ to obtain

$$F[C_y]=\frac{m}{x^2}(1+\cos\zeta)=\frac{m}{r^2}\,\frac{1+\cos\zeta}{\sin^2\zeta}=\frac{m}{2r^2}\csc^2(\tfrac{1}{2}\zeta).$$

Thus
$$\rho(x,y) = w \exp\left(-\frac{m}{2r^2}\csc^2 \tfrac{1}{2}\zeta\right)\frac{m}{r^3}.$$

Hence†
$$\rho(r) = \frac{wm}{r^2}\int_0^{2\pi} \exp\left(-\frac{m}{2r^2}\csc^2 \tfrac{1}{2}\zeta\right) d\zeta = \frac{2\pi wm}{r^2}\left[1-\operatorname{erf}\left(\frac{\sqrt{\tfrac{1}{2}m}}{r}\right)\right], \quad (37)$$

where $\operatorname{erf}X$ (the 'error function' or 'probability integral') is defined as
$$\operatorname{erf}X = \frac{2}{\sqrt{\pi}}\int_0^X e^{-x^2}\,dx.$$

By expressing m in terms of the search width W by means of equation (29), equation (37) is reduced to
$$\rho(r) = \frac{wW^2}{4r^2}\left[1-\operatorname{erf}\left(\frac{W}{4\sqrt{\pi}r}\right)\right]. \quad (39)$$

This is the function actually graphed in Fig. 8. For values of r not over about 15 miles, it is in reasonable, rough agreement with operational data (visual), but farther out it has too high an ordinate.

RANDOM SEARCH

In the last two sections, both observer and target were on straight courses at constant speeds; this represents the extreme of simplicity of paths. At the other extreme is the case where both are moving in complicated paths over the ocean and at speeds which may vary in the course of time, a case which is called that of random search. If the position of the target is in the area of interest A (which may be many hundred square miles) in which the observer is moving, and if the observer is without pre-knowledge indicating that the target is more likely to be in one part of A

† This depends on the evaluation of the integral
$$\phi(\lambda) = \int_0^{\frac{1}{2}\pi} e^{-\lambda\csc^2\theta}\,d\theta.$$

This is done by the following device: Differentiating with respect to λ,
$$\phi'(\lambda) = -\int_0^{\frac{1}{2}\pi} e^{-\lambda\csc^2\theta}\csc^2\theta\,d\theta = e^{-\lambda}\int_0^{\infty} e^{-\lambda\cot^2\theta}\,d\cot\theta = -\tfrac{1}{2}\sqrt{\frac{\pi}{\lambda}}\,e^{-\lambda},$$

as is seen on changing to the variable of integration $x=\sqrt{\lambda}\cot\theta$, and the use of the formula $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. Integrating $\phi'(\lambda)$ with respect to λ, observing that $\phi(0)=\frac{1}{2}\pi$, we obtain
$$\phi(\lambda) = \tfrac{1}{2}\pi - \sqrt{\pi}\int_0^{\lambda} e^{-\lambda}\frac{d\lambda}{2\sqrt{\lambda}} = \tfrac{1}{2}\pi\,[1-\operatorname{erf}\sqrt{\lambda}], \quad (38)$$

as it appears on changing to the variable of integration $\mu=\sqrt{\lambda}$.

than in another, a good approximation to the probability p that the observer make a contact is given on the basis of the following three assumptions:

1. The target's position is uniformly distributed in A.

2. The observer's path is random in A in the sense that it can be thought of as having its different (not too near) portions placed independently of one another in A.

3. On any portion of the path which is small relatively to the total length of path but decidedly larger than the range of possible detection, the observer always detects the target within the lateral range $\frac{1}{2}W$ on either side of the path and never beyond.

These assumptions lead to the formula of random search

$$p = 1 - e^{-WL/A}, \tag{40}$$

where A = area in square miles, W = effective search width in miles, L = total length of observer's path in A in miles.

To prove this, suppose that the observer's path L is divided into n equal portions of length L/n. If n is large enough so that most of the pieces are randomly related to any particular one, the chance of failing to detect during the whole path L is the product of the chances that detection fail during motion along each piece. If, further, L/n is such that

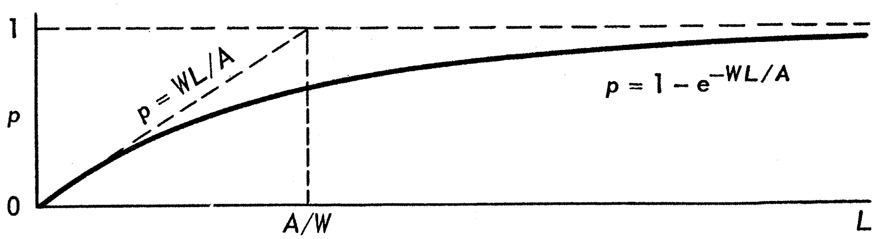

Fig. 9. Detection with random search.

most of the pieces of this length are practically straight and considerably longer than the range of detection, then in virtue of (3) the latter chance of detection is the probability that the target be in the area swept (whose value is WL/n square miles), and this probability is WL/nA, assumption (1). Hence the chance that along all of L there be no detection is $(1 - WL/nA)^n$, and hence

$$p = 1 - \left(1 - \frac{WL}{nA}\right)^n; \qquad p = 1 - e^{-WL/A} \quad \text{for large } n.$$

This reasoning assumes, of course, that a large n having these properties exists. This is essentially assumption (2).

If the exponential in equation (40) is replaced by the first two terms in its power-series expansion, the equation is replaced by $p = WL/A$; this corresponds to the probability in the case that L consists of a single straight line, or a path so little bent that there is practically no overlapping of swept regions: The total area swept is WL and the chance of the target being in it is WL/A. The departure from this simple value represents the effect of random overlapping of swept areas.

Figure 9 shows the way in which the probability increases with the length of observer's path L. For smaller values of L, it is closely approximated by its tangent $p=WL/A$. For much larger values, it approaches unity, exhibiting a 'saturation' or 'diminishing-returns' effect.

PARALLEL SWEEPS

SEARCH BY PARALLEL SWEEPS is a method frequently employed, and many apparently more complicated schemes turn out to be equivalent to parallel sweeps, either exactly or with sufficient approximation for practical purposes. A target is at rest on the ocean in an unknown position, all equal areas having the same chance of containing it; it is decided to search along a large ('infinite') number of parallel lines on the ocean, their common distance apart, or sweep spacing, being S miles; what is the probability $P(S)$ of detection? Or again, the target's speed and direction are known, but the position is uniformly distributed as above; it is possible to search in equally spaced parallel paths relative to the target, i.e., in the plane mov-

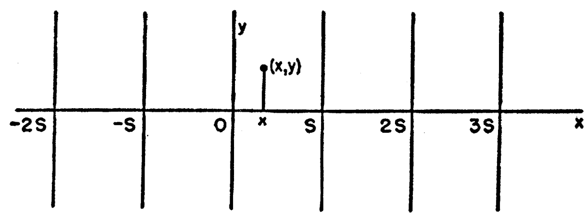

Fig. 10. Parallel sweeps.

ing with the target's motion and in which it appears to be a fixed point, as in the first case. It is immaterial whether all the parallel paths are traversed by the same observer or by different observers having similar observing characteristics.

In Fig. 10 the parallel paths are shown referred to a system of rectangular coordinates; the axis of ordinates is along one of the paths and the target's (unknown) position is at (x,y), and $0 \leqq x < S$. It is observed that this inequality, expressing the fact that the target is in the strip immediately to the right of the axis of ordinates, is a consequence of the method of choice of the axes, and implies no restriction in the position of the target.

The first step in calculating $P(S)$ is to write down the lateral ranges of the target from the various observer paths. For paths at or to the left of the axis of ordinates, the lateral ranges are x, $x+S$, $x+2S$, $x+3S$, $\cdots$.

For those to the right, the ranges are $S-x$, $2S-x$, $3S-x$, $\cdots$. All these cases may be combined into the absolute-value formula

$$\text{lateral range} = |x-nS|, \tag{41}$$

where $0 \leqq x < S$ and $n=0, \pm 1, \pm 2, \pm 3, \cdots$. Equation (25) is now applied to find the probability $p_n = p(n\text{th lateral range})$ of detection by the nth sweep when the target's position is given as (x,y):

$$p_n = 1 - e^{-F(|x-nS|)},$$

where the potential F is given by the appropriate formula. The probability of no detection by the nth sweep is $1-p_n$; that of no detection by any sweep is the (infinite) product $\prod (1-p_n)$ for all values of n ($n<0$, $n=0$, $n>0$); and the probability that at least one sweep detect a target given at (x,y) is

$$P(x,S) = 1 - \prod_{n=-\infty}^{+\infty} e^{-F(|x-nS|)},$$

or, finally,

$$P(x,S) = 1 - e^{-\Phi(x,S)}, \tag{42}$$

with

$$\Phi(x,S) = \sum_{n=-\infty}^{+\infty} F(|x-nS|). \qquad (0 \leqq x < S),$$

This is essentially a repetition of the argument proving the additivity of the potentials.

It remains to find the probability of detection $P(S)$ when the target's position (the value of x) is not given, but has a uniform distribution between 0 and S. An easy probability argument shows that $P(S)$ is the average of $P(x,S)$ over all values of x in this interval:

$$P(S) = \frac{1}{S}\int_0^S (1-e^{-\Phi(x,S)})\,dx, \quad \text{with} \quad \Phi(x,S) = \sum_{n=-\infty}^{+\infty} F(|x-nS|). \tag{43}$$

This gives the general solution of the problem.

The effective visibility E is defined as half that sweep spacing for which the probability of detection by parallel sweeps is one half. In other words, E is determined as the solution of the equation $P(2E) = \frac{1}{2}$.

Three cases are of particular interest. The first is that of continuous looking on the assumption of a definite-range law. Detection will surely occur if, and only if, the target happens to be within the definite range R of either of the two adjacent sweeps. The chance for this is $2R/S = W/S$ when $S > 2R = W$, and unity when $S \leqq W$. It is easy to see that the effective visibility $E = W$.

The second case is that of the inverse-cube law (which will be taken here in its simplest form). We obtain $\Phi(x,S)$ with the aid of equation (26),

$$\Phi(x,S) = 2m \sum_{n=-\infty}^{+\infty} \frac{1}{(x-nS)^2} = 2m \frac{\pi^2}{S^2} \csc^2\left(\frac{\pi x}{S}\right), \tag{44}$$

the latter equality resulting from a well-known formula of analysis (obtained, e.g., from the expansion of the sine in an infinite product by taking logarithms and then differentiating twice). Inserting this expression into equation (43), we must find

$$P(S) = \frac{1}{S} \int_0^S \left[1 - \exp\left(-\frac{2m\pi^2}{S^2} \csc^2 \frac{\pi x}{S}\right)\right] dx.$$

This is found by means of equation (38), on setting $\theta = \pi x/S$, $\lambda = 2m\pi^2/S^2$. The result, which can be transformed by means of equation (29), is

$$P(S) = \mathrm{erf}\left(\frac{\pi\sqrt{2m}}{S}\right) = \mathrm{erf}\left(\frac{\sqrt{\pi}W}{2S}\right). \tag{45}$$

We are now in a position to express m and W in terms of the effective visibility E. To find E we solve

$$P(2E) = \mathrm{erf}\left(\frac{\pi\sqrt{2m}}{2E}\right) = \mathrm{erf}\left(\frac{\sqrt{\pi}\,W}{4E}\right) = \tfrac{1}{2}.$$

The tables of the probability integral show that $\mathrm{erf}0.477 = 0.5$; hence

$$\pi\sqrt{2m}/2E = \sqrt{\pi}\,W/4E = 0.477,$$

i.e.,

$$m = 0.046\,E^2, \qquad W = 1.076\,E. \tag{46}$$

These values substituted into equation (45) give

$$P(S) = \mathrm{erf}(0.954\,E/S). \tag{47}$$

A third case is useful to consider, although strictly speaking it is not one of parallel sweeps but of uniform random search. It may be described as the situation which arises when the searcher attempts to cover the whole area uniformly by a path or paths which place about the same length of track in each strip but which operate within a given strip in the manner of the searcher making a random search. Let all the strips be cut by two horizontal lines a distance of b miles apart and suppose that the search is for a target inside the large rectangle bounded by these lines and two vertical lines NS miles apart, as shown in Fig. 11. The area is $A = NSb$ square miles. Assume that the total length of track is equal to that of all included parallel sweeps, $L = Nb$, then apply equation (40); we obtain

$$P(S) = 1 - e^{-W/S}, \tag{48}$$

independent of N and b.

These three cases may be represented by means of a common diagram (Fig. 12) by plotting $P = P(1/n)$ where $n = 1/S$ is the sweep density, or number of sweeps per mile. At one extreme is the case of the definite-range law, at the other the case of random search. All actual situations can be regarded as leading to intermediate curves, i.e., lying in the shaded region.

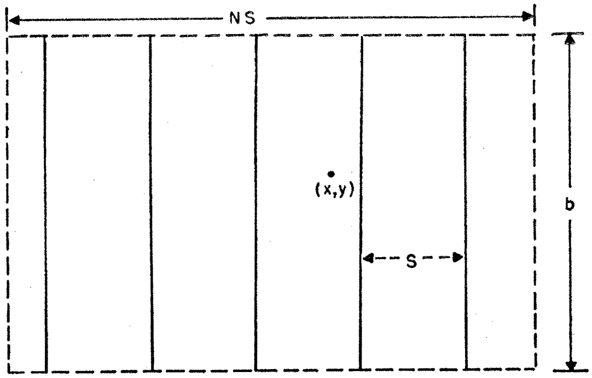

Fig. 11. A rectangle of random sweeps.

The inverse-cube law is close to a middle case, a circumstance that indicates its frequent empirical use, even in cases where the special assumptions upon which its derivation was based are largely rejected.

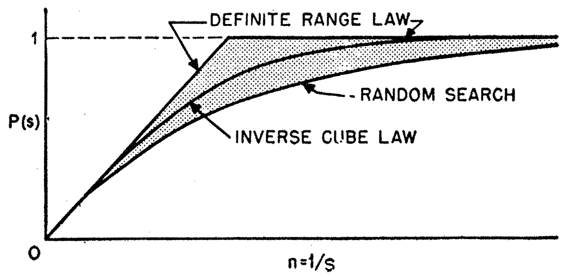

Fig. 12. Probabilities with parallel sweeps.

FORESTALLING

When the observer is using two different means of detection simultaneously and independently (i.e., when neither interferes with or aids the other), it is sometimes necessary to know the probability of making a first detection by a particular one of the two means. Since the second means of detection can deprive the first of a chance of detecting (by detecting the target first) this probability may be lower than if the second means had not been present: we say that the second can forestall the first. For example, when both radar and visual detection are possible, in gathering data bearing on the effectiveness of the radar, the possibility of visual forestalling of the radar must be taken into account.

Again, when the target is itself capable of detecting the observer, and if it is important to detect the target before it can detect the observer (as when the former is a surfaced submarine that can submerge if it detects the observer first and so deprive it of its chance of detecting), it is important to find the probability that the observer detect the target first, before it has been forestalled by the target's detection.

Just as the chance of detection is mathematically equivalent to that of hitting a target continuously exposed to our fire (intensity varying in general with the time), so the question of forestalling is mathematically identical with that of hitting the target before it hits us, in the case where it is an enemy continuously firing back.

It will be sufficient to consider the case of continuous looking with the instantaneous probability $\gamma_t\, dt$ for the first means of detection without forestalling, the probability $p(t)$ of detection when there is no forestalling being given by equation (6). Let $\gamma_t'\, dt$ and $p'(t)$ be the corresponding quantities for the second means of detection, or for the target's detection of the observer in the second example above.

If $P(t)$ is the required probability of first detection during the interval of time from 0 to t by the first means, we consider the value of $P(t+dt)$. It is the probability of an event which can succeed in either of the following mutually exclusive ways: either by having the required detection between 0 and t, or by having neither means detect during this period, and having a detection by the first means between t and $t+dt$. This leads to the equation

$$P(t+dt)=P(t)+[1-p(t)]\,[1-p'(t)]\,\gamma_t\, dt,$$

whence a differential equation is obtained, the solution of which is

$$P(t)=\int_0^t \gamma_t \exp\left[-\int_0^t (\gamma_t+\gamma_t')\, dt\right] dt. \tag{49}$$

Precisely the same reasoning leads to the expression

$$P'(t)=\int_0^t \gamma_t' \exp\left[-\int_0^t (\gamma_t+\gamma_t')\, dt\right] dt$$

for the probability of a first detection by the second means in the time interval $0,t$.

Note that the sum $P(t)+P'(t)$ is the probability of a first detection either by the first means or by the second, in other words, the probability of a detection by some means between 0 and t. The expression obtained by adding the above equations and carrying out one integration is

$$P(t)+P'(t)=1-\exp\left[-\int_0^t (\gamma_t+\gamma_t')\, dt\right],$$

which is simply the expression (6) with γ_t replaced by $\gamma_t+\gamma_t'$, the latter being the instantaneous probability when both means of detection act in conjunction (additivity of potentials).

When a large number of independent trials of the detection experiment are made under identical conditions and all cases that have resulted in a first detection by the first means are sorted out and the precise epochs of this detection are averaged, the result will (statistically) be equal to

$$\bar{t}=\frac{\int_0^\infty t\,\gamma_t \exp[-\int_0^t (\gamma_t+\gamma_t')\,dt]\,dt}{\int_0^\infty \gamma_t \exp[-\int_0^t (\gamma_t+\gamma_t')\,dt]\,dt}. \tag{50}$$

The denominator is proportional to the total number of first detections by first means, and the result of dividing $\gamma_t \exp[-\int_0^t (\gamma_t+\gamma_t')\,dt]\,dt$ by the denominator is the proportion of such detections between t and $t+dt$. Thus the expression in equation (50) represents the expected value $\bar{t}$ of t, the epoch of detection by the first means.

As a first application we consider the case of constant instantaneous probabilities of detection, $\gamma_t=\gamma$, $\gamma_t'=\gamma'$. Equations (49) and (50) reduce to

$$P(t)=\frac{\gamma}{\gamma+\gamma'}\,[1-e^{-(\gamma+\gamma')t}], \qquad \bar{t}=\frac{1}{\gamma+\gamma'}.$$

It is thus seen that the proportion of the total number of first contacts by the first means (as $t\to\infty$) is $\gamma/(\gamma+\gamma')$, and, correspondingly, by the second, $\gamma'/(\gamma+\gamma')$. And the mean time elapsed to the former is the same as for the latter, i.e., $1/(\gamma+\gamma')$; this is different from the mean time $1/\gamma$ when no forestalling had been possible.

As a second application we consider the straight-track case, and assume that the observer is an aircraft and the target a surfaced submarine. If the observer sights the wake of the submarine, his ability to detect may reasonably be taken as the inverse-cube law of equation (10), and if the submarine sights the horizontal surfaces of the aircraft's wing, the same law (with k replaced by a different constant k') can reasonably be assumed for the submarine's detection of the aircraft. If it is assumed that the submarine dives as soon as it detects the aircraft, what is the probability that the aircraft detect the submarine, as a function of lateral range x? By how much is its effective search width decreased by this new possibility?

Equation (10) under the circumstance of the lateral-range distribution leads to

$$\gamma_t=\frac{kh}{(x^2+w^2t^2)^{3/2}}, \qquad \gamma_t'=\frac{k'h}{(x^2+w^2t^2)^{3/2}};$$

only in the present case the time interval is from $-\infty$ to t instead of from 0 to t. With these changes, equation (49) leads to the following expression for the probability of sighting the submarine before the time t:

$$P(x,t)=m\int_{-\infty}^{wt} \exp\left[-\frac{m+m'}{x^2}\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)\right]\frac{dy}{(x^2+y^2)^{3/2}},$$

where $m=kh/w$ and $m'=k'h/w$. The integral can be evaluated explicitly when it is noted that the integrand is proportional to the derivative of the exponential expression, i.e.,

$$\frac{1}{(x^2+y^2)^{3/2}}\exp\left[-\frac{m+m'}{x^2}\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)\right]$$
$$=-\frac{1}{m+m'}\frac{d}{dy}\exp\left[-\frac{m+m'}{x^2}\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)\right].$$

The result, on setting $t=+\infty$, gives the following probability of sighting the submarine some time on its whole straight course.

$$P(x,\infty)=[m/(m+m')]\,[1-e^{-2(m+m')/x^2}].$$

To find the value of the search width (which will be denoted by W^*), this expression must be used in the place of (26) in equation (27). The answer is obtained from (29) by replacing m by $m+m'$ and then multiplying the result by $m/(m+m')$; it is

$$W^*=2\sqrt{2\pi m}\sqrt{\frac{m}{m+m'}}=W\sqrt{\frac{m}{m+m'}}.$$

Thus the effect of forestalling is to multiply W by a factor $\sqrt{m/(m+m')}$ that is less than unity. And the probability of detection even when the target is flown over ($x=0$) is $P(0,\infty)=m/(m+m')$ instead of unity, as it would have been in the absence of forestalling.

For a definite-range law, that means of detection which has the greater range will always forestall the other. (Of course, this is strictly true only in the case of continuous looking.)

CONCLUSION—OPERATIONAL DISTRIBUTIONS

RETURNING TO FIRST PRINCIPLES, as set forth at the beginning of this paper, it has been laid down as basic that detection, even when possible, is an uncertain event; and the whole subsequent course of development of this chapter has been toward the calculation of probabilities of detection. But an essential restriction has been imposed in all these calculations: The one source of uncertainty that has been considered is the human fallibility of the observer, and the sudden uncontrollable fluctuations in the physical state of affairs, but not in the random element introduced by unknown, long-term variations in the underlying physical conditions (conditions which are expressible as parameters). Thus, as we have said earlier, under given meteorological conditions of visibility V, the observer will have a definite chance $\gamma(r)\,dt$ of sighting a target of given size A and background contrast C; and subsequent deductions have been made on the assumption that while the range r may vary in a given manner in the course of time, the parameters V, A, and C all remain fixed. The distributions calculated on this assumption can be expected to agree with the distributions found empirically when the results of a large number of experiments are obtained, all of which are performed under the same conditions of visibility and size and contrast of the target, geometrical quantities like r alone being allowed to vary. But as soon as operational results are compiled that refer to cases in which V, A, and C vary from incident to incident, an altogether different situation is present: The cause of the uncertainty of the event of detection is two-fold, being dependent not only on the human fallibility of the observer and short-term fluctuations, but on the more or less unknown and heterogeneous nature of the underlying physical conditions. And it is important to realize that in many cases this second factor may outweigh the first. When this is judged to be the case, it may well be expedient to employ a highly simplified law of detection, such as the definite-range law, and then seek to explain the distributions found in the operational data simply by averaging the calculated results of such laws over different possible values of the parameters. Thus if the definite-range law is assumed, mathematical expressions deduced from it will involve this range R; then it may be considered that in the operational incidents different values of R are present; by choosing appropriate frequencies for the different values of R and combining or averaging the theoretical results over such a distribution of R, a good agreement may often be found with the observations.

It must be emphasized that equations such as (1), (3), (5), and (6) are true only when the first cause of uncertainty alone is present, and when the underlying physical conditions remain constant (and are known to be of constant, though not necessarily of known, values) throughout the course of the looking. Thus in proving (1), the probability of detection for one glimpse was g, of not detecting $1-g$; now precisely at the point where it was asserted that the probability of failure to detect at each and every one of the first n glimpses is $(1-g)^n$, the assumption that the n different events are independent was made. This is justified only in two cases: *first*, when the only uncertainty is in the observer's chance performance so that his different opportunities (glimpses) are regarded as repeated independent trials (as in successive tosses of a coin); *second*, when there are indeed changes in physical conditions, but of such a rapidly fluctuating character that if no detection is known to occur at one glimpse, no inference can be drawn regarding the physical conditions pertaining to any other glimpse. But if, for example, the visibility V is not fully known, the fact that earlier glimpses have failed to detect may lead to the presumption that V is less than might otherwise have been supposed, and hence that later chances of detection are less: the expression $(1-g)^n$ is false.

The method of procedure is clear. The first step is to carry out the calculations as described in the previous sections of this paper, assuming fixed conditions (such as V, A, C). The second step is to average the results obtained for the probabilities (e.g., over the possible values of V, A, C, with appropriate weighting). Only the final result can reasonably be expected to furnish the probabilities which accord with the operational data. What is true of probabilities is also true of mean or expected values defined by them.

This can be illustrated by many practical examples and three simple cases are mentioned here. Firstly, suppose that the lateral-range curve involves a parameter λ referring to an unknown factor in the underlying physical conditions. Its equation is $p=p(x,\lambda)$. Once the distribution of frequencies with which the different values of λ occur in an operational situation has been estimated, the operational lateral-range curve $p=p_{\text{op}}(x)$ (i.e., the one furnished by a histogram of the observed data) is found by

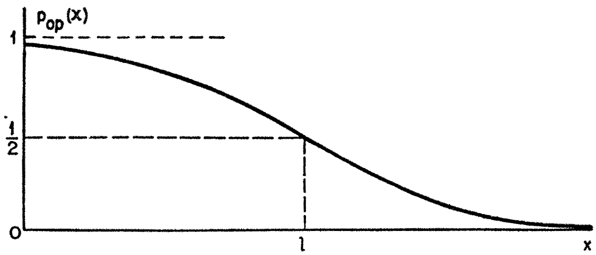

Fig. 13. Lateral-range curve based on a normal distribution of definite ranges.

averaging $p(x,\lambda)$ over the values of λ on the basis of this frequency. Thus it might be reasonable in some cases to assume that the values of λ are normally distributed about a known mean l with a known standard deviation σ. Accordingly,

$$p_{\text{op}}(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{+\infty} p(x,\lambda)\, e^{-(\lambda-l)^2/2\sigma^2}\, d\lambda.$$

Thus if $p(x,\lambda)$ results from a definite-range law of range $R=\lambda$, so that

$$p(x,\lambda)=1, \quad \text{when } x<\lambda; \qquad p(x,\lambda)=0, \quad \text{when } x>\lambda,$$

the equation becomes

$$p_{\text{op}}(x)=\frac{1}{\sigma\sqrt{2\pi}}\int_{x}^{+\infty} e^{-(\lambda-l)^2/2\sigma^2}\, d\lambda=½\left(1-\operatorname{erf}\frac{x-l}{\sigma\sqrt{2}}\right),$$

the graph of which is shown in Fig. 13. It is noted that when $x=0$, $p_{\text{op}}(0)$ is slightly less than unity, whereas it should exactly equal unity.

This is because the normal distribution of definite ranges allows a (slight) chance of negative ranges, a physical absurdity. It would have been more realistic to have assumed a skew nonnegative distribution (e.g., Pearson's Type III distribution, $k\,\lambda^{m-1}\,e^{-\beta\lambda}$).

As a second example, consider the search for a fixed target by two parallel sweeps at distance S apart. If the underlying conditions are the same during the two sweeps, and if $p(x)$ is the lateral-range probability, the chance of detection of a target between the paths and x miles from one of them is shown by the usual reasoning to be

$$P(x,S)=1-[1-p(x)]\,[1-p(S-x)]=p(x)+p(S-x)-p(x)\,p(S-x).$$

If $p(x)=p(x,\lambda)$ a weighted averaging process must be performed in order to get the operational probability $P_{\text{op}}(x,S)$ from $P(x,S,\lambda)$ given by the above equation. And of course if x is determined at random between 0 and S, a second averaging must be done to get $P_{\text{op}}(S)$, the operational probability of detecting the target given only to be somewhere between the sweeps and with a given distribution of physical conditions. The order in which these two averagings are done is immaterial. A corresponding treatment is given in the case of infinitely many parallel sweeps. It may be remarked that the operational effective visibility E_{op}, which is defined by the equation (see 'Parallel Sweeps'):

$$P_{\text{op}}(2E_{\text{op}})=\int P(2E_{\text{op}},\lambda)\,f(\lambda)\,d\lambda=\tfrac{1}{2}$$

is quite different from the average $\bar{E}$:

$$\bar{E}=\int E(\lambda)\,f(\lambda)\,d\lambda,$$

of the effective visibility defined under fixed conditions corresponding to a particular value of λ. Here $f(\lambda)$ is the assumed frequency with which the values of λ are taken to be distributed under the operational conditions in question.

As a third example, suppose that a radar set is chosen at random from a lot, only the fraction ϵ of which are in good adjustment, the remaining $1-\epsilon$ not in a condition to make any detections possible. The radar lateral-range curve $p(x)$ for a radar set in good adjustment and, e.g., mounted on an aircraft, must be multiplied by ϵ to obtain the operational curve that will be obtained when many observations are made with the aid of many sets chosen in this way. When a set or similar observing instrumentality or setup is not giving the results that could be expected of it, it is often said to be working at an efficiency less than 100 per cent. In the above case, a natural definition of efficiency is 100 ϵ. In more complicated cases, the concept, while useful as a general concept, may not be convenient to define in all precision.

In conclusion, the following principles are laid down: (*a*) If the object of the calculation of probabilities, averages, and similar statistical detection quantities is to coordinate and explain the data of the operations of the past, then the heterogeneity of conditions (dispersion of slowly varying parameters) is placed at the apex of the discussion, the influence of 'subjective' probabilities and short-term fluctuations (the main subject of this chapter) usually playing a secondary role. (*b*) If on the other hand the object of the calculation is to obtain contemplated performance data for the design of search plans to be used in the future, and, as is generally the case, when the conditions (slowly varying parameters) are known, then the probabilities originating from subjective and rapidly fluctuating sources occupy the center of the stage; any study of the heterogeneity of conditions is made only in order to check the sensitivity of the search plan to accidental imperfections in the knowledge of the conditions.