Regression Analysis: Inference

$$y = \beta_0 + \beta_1 x_1 + \ldots + \beta_k x_k + u$$

Statistics and Econometrics

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Roadmap

- Regression analysis with cross-sectional data
  - Basics: estimation, inference, analysis with dummy variables
  - More involved: model specification and data issues
- Advanced topics
  - Binary dependent variable models
  - Panel data analysis
  - Time series analysis

Outline (Wooldridge, Ch. 4.1 - 4.6)

- Sampling distribution of the OLS estimators
- Testing hypotheses about a single population parameter: t test
- Confidence intervals
- Testing hypotheses about a single linear combination of parameters
- Testing multiple linear restrictions: F test

Outline

- Sampling distribution of the OLS estimators
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Motivation

- The multiple regression model:

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + u$$

- Goal is to gain knowledge about the population parameters (β's) in the model
- Knowing the mean and variance of $\hat{\beta}_j$ is not enough. We need the sampling distribution of the OLS estimators to answer questions, such as
  - what we can say about the "true values"?
  - how to decide if a hypothesis is supported or not?

Sampling Distribution of OLS

Theorem (4.1, Normal Sampling Distribution)

With a "good" model,

$$\hat{\beta}_j \sim Normal\left(\beta_j, Var(\hat{\beta}_j)\right),$$

where the variance is given by

$$Var(\hat{\beta}_j) = \frac{\sigma^2}{SST_j(1 - R_j^2)}, \qquad j = 1, \ldots, k.$$

It implies:

$$\frac{\hat{\beta}_j - \beta_j}{sd(\hat{\beta}_j)} \sim \text{Normal}(0, 1), \qquad \text{where } sd(\hat{\beta}_j) = \sqrt{Var(\hat{\beta}_j)}$$

Sampling Distribution of OLS

In practice, σ^2 has to be estimated:

$$se(\hat{\beta}_j) = \frac{\hat{\sigma}}{\sqrt{SST_j(1 - R_j^2)}}, \qquad j = 1, \ldots, k,$$

which is called the standard error of $\hat{\beta}_j$.

Theorem (4.2, t-Distribution)

With a "good" model,

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1},$$

where $k + 1$ is the number of unknown parameters in the model, and $n - k - 1$ is the degrees of freedom (df).

Outline

- Sampling distribution of the OLS estimators
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Testing Simple Null Hypothesis

- Some questions of interest may be formulated as a simple null hypothesis about a population parameter,

$$H_0 : \beta_j = 0$$

- Eg. In the log wage model

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + \beta_3 tenure + u,$$

"$H_0 : \beta_1 = 0$", is economically interesting. If the null hypothesis is accepted, it implies that, holding *exper* and *tenure* fixed, a person's education level has no effect on wage.

Testing Simple Null Hypothesis

- To test a simple null hypothesis, the test statistic is usually called "the" *t* statistic or "the" *t* ratio

$$t_{\hat{\beta}_j} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \Rightarrow \text{evidence against Null}$$

if ≥ 0

- Sampling distribution of t statistic when H_0 is true
  - By Theorem 4.2, $t_{\hat{\beta}_j}$ has the t-distribution with $n - k - 1$ df
  - When df is large (> 30), the t distribution approaches the standard normal distribution

Testing Simple Null Hypothesis

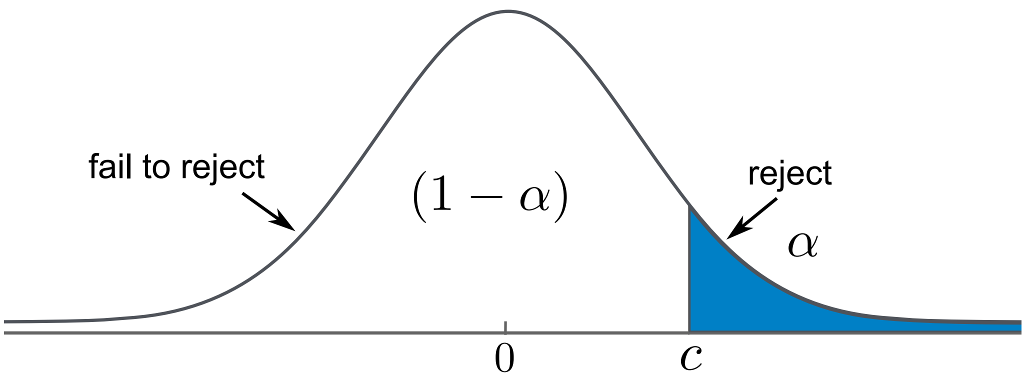
- t statistic along with a rejection rule (depends on alternative hypothesis and the chosen significance level) will be used to determine whether to accept the null hypothesis H_0
- Significance level
  - typical values: 1%, 5%, 10%
  - the probability of rejecting H_0 when it is true
- Alternative hypothesis
  - H_1 may be one-sided, or two-sided
  - $H_1 : \beta_j > 0$ or $H_1 : \beta_j < 0$ are one-sided
  - $H_1 : \beta_j \neq 0$ is a two-sided alternative

One-Sided Alternatives

- Testing against $H_1 : \beta_j > 0$
  - Pick a significance level, α
  - Look up the $(1-\alpha)^{th}$ percentile in a t distribution with $n-k-1$ df and call this c, the critical value (use normal critical values when df > 30)
  - Reject the null hypothesis if the t statistic is greater than c

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + u_i$$

$$H_0 : \beta_j = 0 \qquad\qquad H_1 : \beta_j > 0$$

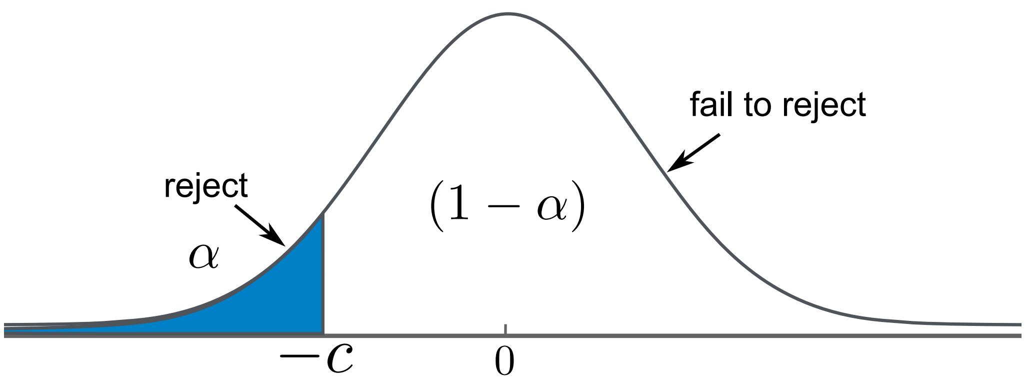


One-Sided Alternatives

- Testing against $H_1 : \beta_j < 0$
  - The critical value is just the negative of before because the t distribution is symmetric
  - Reject the null if $t_{\hat{\beta}_j} < -c$
  - If $t_{\hat{\beta}_j} \geq -c$ then we fail to reject the null

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + u_i$$

$$H_0 : \beta_j = 0 \qquad\qquad H_1 : \beta_j < 0$$

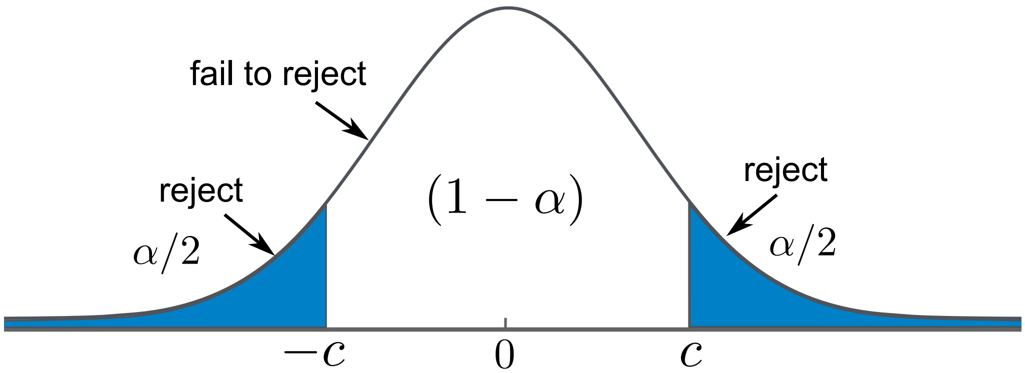


Two-Sided Alternatives

- For a two-sided test ($H_1 : \beta_j \neq 0$)
  - The critical value is based on $(1 - \alpha/2)$ percentile in a t distribution with $n - k - 1$ df
  - Reject $H_0 : \beta_j = 0$ if the absolute value of the t statistic is greater than c, i.e., $|t_{\hat{\beta}_j}| > c$

$$y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_k x_{ik} + u_i$$

$$H_0 : \beta_j = 0 \qquad\qquad H_1 : \beta_j \neq 0$$



Testing Simple Null Hypothesis

- Unless otherwise stated, the alternative is assumed to be two-sided
- In the case of $H_0 : \beta_j = 0$ and $H_1 : \beta_j \neq 0$,
  - if we reject the null, we typically say "x_j is statistically significant (or different from 0) at the α level" or
  - if we fail to reject the null, we typically say "x_j is statistically insignificant at the α level"

Testing Simple Null Hypothesis: An Example

Example 4.1. Log wage model (standard errors are in brackets):

$$\widehat{\log(wage)} = \underset{(.104)}{.284} + \underset{(.007)}{.092}\, educ + \underset{(.0017)}{.0041}\, exper + \underset{(.003)}{.022}\, tenure$$

$n = 526,\ R^2 = .316$

- Q. Is “returns to education” statistically significant at the 1% level, after controlling for experience and tenure?
- Hypotheses:

Testing Simple Null Hypothesis: An Example

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$n = 526,\ R^2 = .316$

- Q. Is "returns to education" statistically significant at the 1% level, after controlling for experience and tenure?
- Hypotheses: $H_0 : \beta_{educ} = 0$ vs $H_1 : \beta_{educ} \neq 0$
- Test statistic and decision rule:

Testing Simple Null Hypothesis: An Example

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$$\widehat{\log(wage)} = \underset{(.104)}{.284} + \underset{(.007)}{.092}\, educ + \underset{(.0017)}{.0041}\, exper + \underset{(.003)}{.022}\, tenure$$

$n = 526,\ R^2 = .316$

- Q. Is "returns to education" statistically significant at the 1% level, after controlling for experience and tenure?

  two side

  critical value = 99.5%

- Hypotheses: $H_0 : \beta_{educ} = 0$ vs $H_1 : \beta_{educ} \neq 0$
- Test statistic and decision rule: reject H_0 if $|t_{\hat{\beta}_{educ}}| > c$
- Critical value (large df, normal): R code: qnorm (0.995) ⇒ 2.5758

  qt (0.995, df=522) ⇒ 2.585

  n=526, larger enough

  (n-k-1 = 526-3-1 = 522)

  both work

Testing Simple Null Hypothesis: An Example

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$n = 526,\ R^2 = .316$

- Q. Is "returns to education" statistically significant at the 1% level, after controlling for experience and tenure?
- Hypotheses: $H_0 : \beta_{educ} = 0$ vs $H_1 : \beta_{educ} \neq 0$
- Test statistic and decision rule: reject H_0 if $|t_{\hat{\beta}_{educ}}| > c$
- Critical value (large df, normal): $c = 2.576$
- Conclusion:

Testing Simple Null Hypothesis: An Example

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$n = 526,\ R^2 = .316$

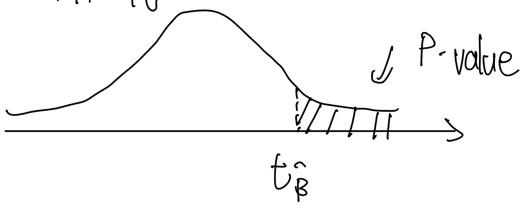
- Q. Is "returns to education" statistically significant at the 1% level, after controlling for experience and tenure?
- Hypotheses: $H_0 : \beta_{educ} = 0$ vs $H_1 : \beta_{educ} \neq 0$
- Test statistic and decision rule: reject H_0 if $|t_{\hat{\beta}_{educ}}| > c$
- Critical value (large df, normal): $c = 2.576$
- Conclusion: reject H_0 at the 1% level because

$$t_{\hat{\beta}_j} = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)}; \qquad |t_{\hat{\beta}_{educ}}| = .092/.007 = 13.149 > c$$

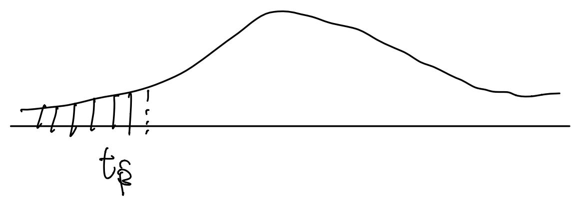
p-Values

- An alternative to the classical approach is to ask, "what is the smallest significance level at which the null would be rejected?"
  - Compute the t statistic
  - p-value is the probability that we'd observe a more extreme test statistic in the direction of the alternative hypothesis than we did, if the null is true
  - Smaller the p-value, stronger the evidence against H_0

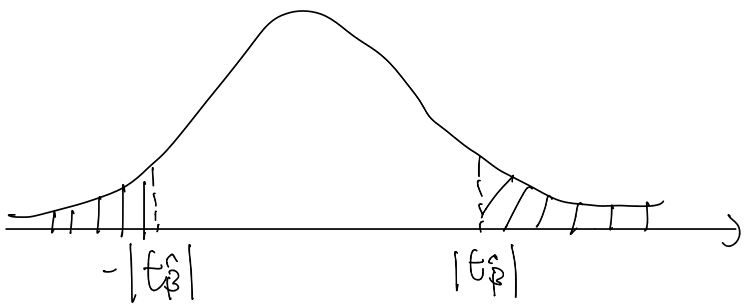
$H_1: \beta_j > 0$



$H_1: \beta_j < 0$



$H_1: \beta_j \neq 0$



if the region is small . the probability of seeing these $t_{\hat{\beta}}$ is small
P-value is small

↓

provide evidences against the null hypothesis

Question:

if P-value = 0.02. can we reject the null hypothesis at 1% level?

No. but we can reject at 5%

```
wage.m1 <- lm(log(wage) ~ educ + exper + tenure, data = data)
```

Estimate

Std. Error

at which significance level we can reject the null hypothesis

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.284360   0.104190   2.729  0.00656 **
educ        0.092029   0.007330  12.555  < 2e-16 ***
exper       0.004121   0.001723   2.391  0.01714 *
tenure      0.022067   0.003094   7.133 3.29e-12 ***
```

we can reject the null at 5% level
cannot reject the null at 1% level

small p-value provide strong evidence against the null hypothesis
(coefficient = 0)

we can reject the null at 0.1% level

linearHypothesis(wage.m1, "educ = 2")

p-Values and Testing Other Hypotheses

- p-values for t tests
  - R provides the t statistic, p-value (assuming a two-sided test) for $H_0 : \beta_j = 0$ in columns labeled "`t value`", and "`Pr(>|t|)`", respectively
  - If you want a one-sided alternative p-value, just divide the two-sided p-value by 2
- Testing other hypotheses
  - A more general form of the t statistic: $H_0 : \beta_j = a_j$
  - In this case, the appropriate t statistic is

$$t = \frac{\hat{\beta}_j - a_j}{se(\hat{\beta}_j)},$$

  where $a_j = 0$ for the standard test

Economical/Statistical Significance

- An independent variable is statistically significant when the size of the t-ratio $t_{\hat{\beta}_j}$ is sufficiently large (beyond the critical value c)
- An independent variable is economically (practically) significant when the size of the estimate $\hat{\beta}_j$ is sufficiently large (in comparison to the size of y)
- An important x should be both statistically and economically significant

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- Sampling distribution of the OLS estimators
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Confidence Intervals

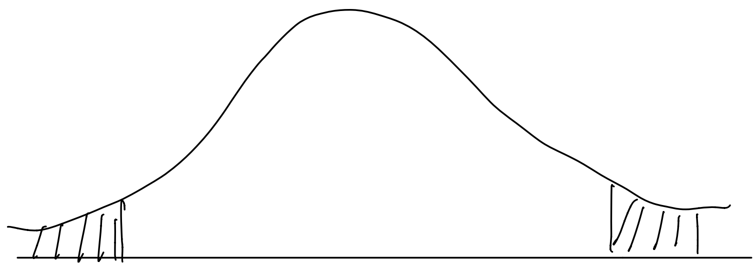
- The confidence interval (CI) for β_j is based on

$$\frac{\hat{\beta}_j - \beta_j}{se(\hat{\beta}_j)} \sim t_{n-k-1}$$

- A $(1-\alpha)$% CI is defined as

$$\hat{\beta}_j \pm c \cdot se(\hat{\beta}_j) = \left[\hat{\beta}_j - c \cdot se(\hat{\beta}_j), \hat{\beta}_j + c \cdot se(\hat{\beta}_j)\right],$$

where c is the $(1-\alpha/2)$ percentile in a t_{n-k-1} distribution



$$-c \le \frac{\hat{\beta}_j - \beta_j}{se(\beta_j)} \le c$$

$$\Longleftrightarrow \quad -c \cdot se(\hat{\beta}_j) - \hat{\beta}_j \le -\beta_j \le c \; se(\hat{\beta}_j) - \hat{\beta}_j$$

$$\Longleftrightarrow \quad \hat{\beta}_j + c \cdot se(\hat{\beta}_j) \ge \beta_j \ge \hat{\beta}_j - c \cdot se(\hat{\beta}_j)$$

Confidence Intervals and Two-Sided Tests

- When *df* is large (> 30), the t_{n-k-1} distribution is very close to the normal distribution and we use $N(0,1)$ critical values
  - eg. For large *df*, the 95% CI is about $\hat{\beta}_j \pm 1.96 \cdot se(\hat{\beta}_j)$
- The width of CI depends on the standard error $se(\hat{\beta}_j)$ and the critical value c
  - high confidence level $\rightarrow$ large c $\rightarrow$ wide CI
  - large standard error $\rightarrow$ wide CI
- CI and two-sided test (one-to-one relationship)
  - test "$H_0 : \beta_j = a_j$" against "$H_1 : \beta_j \neq a_j$"
  - reject H_0 at the α% significant level if (and only if) the $(1-\alpha)$% CI does not contain a_j

- reject H_0 at the $\alpha\%$ significant level if (and only if) the $(1-\alpha)\%$ CI does not contain a_j

If $(1-\alpha)\%$ CI does not contain a_j, we can reject H_0 at the $\alpha\%$ significant level

Proof.

$$\left.\begin{array}{ll} \text{either } a_j > \hat{\beta}_j + c \cdot se(\hat{\beta}_j) & \iff \dfrac{a_j - \hat{\beta}_j}{se(\hat{\beta}_j)} > c \\ \text{or } a_j < \hat{\beta}_j - c \cdot se(\hat{\beta}_j) & \iff \dfrac{a_j - \hat{\beta}_j}{se(\hat{\beta}_j)} < -c \end{array}\right\} \left|\dfrac{a_j - \hat{\beta}_j}{se(\hat{\beta}_j)}\right| > c$$

Confidence Intervals: An Example

- Example 4.1. Log wage model (standard errors are in brackets):

$$\widehat{\log(wage)} = \underset{(.104)}{.284} + \underset{(.007)}{.092}\, educ + \underset{(.0017)}{.0041}\, exper + \underset{(.003)}{.022}\, tenure,$$

$n = 526,\ R^2 = .316$

  - The 95% CI for β_{educ}: $n-k-1 = 522,\ c = 1.96$,

$$.092 \pm 1.96 \cdot (.007) = [.078, .106]$$

  - reject "$H_0 : \beta_{educ} = 0$" in favor of the two-sided H_1 at the 5% significant level

R code: confint(wage.ml,'educ',level=0.95)

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Testing A Linear Combination of Parameters

- In the log wage model,

$$\log(wage) = \beta_0 + \beta_1 educ + \beta_2 exper + u.$$

  Suppose we wish to see whether or not *educ* has the same effect on $\log(wage)$ as *exper*, i.e., to test

$$H_0 : \beta_1 - \beta_2 = 0 \quad \text{vs} \quad H_1 : \beta_1 - \beta_2 \neq 0,$$

  which involves a combination of 2 parameters

- R code: `linearHypothesis(wage.model, "educ - exper = 0")`

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Testing Multiple Linear Restrictions

- Everything we have done so far has involved testing a single linear restriction (eg, $\beta_1 = 0$ or $\beta_1 = \beta_2$)
- We may want to check whether or not a group of x variables has a joint effect on y (with the rest of x variables as controls)
  - i.e., testing exclusion restrictions - whether a group of parameters are all equal to zero

Testing Exclusion Restrictions

- The unrestricted model

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_k x_k + u$$

q number of constraints in the group

- q restrictions under the null hypothesis

$$H_0 : \beta_{k-q+1} = 0, \ldots, \beta_k = 0$$

- The alternative is just $H_1 : H_0$ is not true
- Under H_0, the restricted model

$$y = \beta_0 + \beta_1 x_1 + \cdots + \beta_{k-q} x_{k-q} + u_{(r)}$$

Testing Exclusion Restrictions

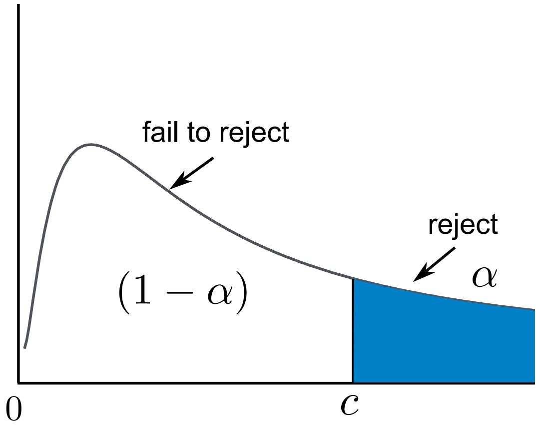
- To do the test, we need to estimate the restricted model without $x_{k-q+1}, \ldots, x_k$, as well as the unrestricted model with all x's included
- Intuitively, we want to know if the change in SSR is big enough to warrant inclusion of $x_{k-q+1}, \ldots, x_k$
- Test statistic

$$F = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)} \sim F_{q,n-k-1} \qquad \text{under } H_0$$

  - $q =$ number of restrictions, or $df_r - df_{ur}$
  - $n - k - 1 = df_{ur}$

F Statistic

- The F statistic is always positive, since the SSR from the restricted model cannot be less than the SSR from the unrestricted
- Reject H_0 if the increase in SSR when we move from the unrestricted to the restricted model is "big enough"
- Decision rule: reject H_0 if $F > c$ ($F_{q,n-k-1}$ critical value)



- F and t statistics
  - when $q = 1$, H_0 can be tested with either t stat or F stat

The R^2 Form of the F Statistic

- Using the fact that $SSR_r = SST(1 - R_r^2)$ and $SSR_{ur} = SST(1 - R_{ur}^2)$, we have

$$F = \frac{(R_{ur}^2 - R_r^2)/q}{(1 - R_{ur}^2)/(n - k - 1)},$$

  where r is restricted and ur is unrestricted

- This is called the R-squared form of the F statistic

Testing Exclusion Restrictions

- If H_0 is rejected, we say that $x_{k-q+1}, \ldots, x_k$ are jointly statistically significant At least one of the variable has impact on y
- If H_0 is not rejected, we say that $x_{k-q+1}, \ldots, x_k$ are jointly insignificant, which justifies dropping them from the model
- The p-value for F test is the probability of F distribution beyond observed F statistics

F Tests: An Example (Online Material Session 2.6)

- Example 4.9. Child birth weight and parents' education

$$\begin{aligned} bwght &= \beta_0 + \beta_1 cigs + \beta_2 parity + \beta_3 faminc \\ &\quad + \beta_4 motheduc + \beta_5 fatheduc + u \end{aligned}$$

  - *bwght*: birth weight
  - *cigs*: average cigarettes per day by mother
  - *parity*: birth order
  - *faminc*: family income
  - *motheduc*: years of education for mother
  - *fatheduc*: years of education for father

- Hypotheses: $H_0 : \beta_4 = 0$ and $\beta_5 = 0$ vs $H_1 : H_0$ is false

F Test: An Example (Online Material Session 2.6)

- Unrestricted model (ur)

$$\begin{aligned} bwght &= \beta_0 + \beta_1 cigs + \beta_2 parity + \beta_3 faminc \\ &+ \beta_4 motheduc + \beta_5 fatheduc + u \end{aligned}$$

$\rightarrow SSR_{ur}$

- Restricted model (r)

$$bwght = \beta_0 + \beta_1 cigs + \beta_2 parity + \beta_3 faminc + u_{(r)}$$

$\rightarrow SSR_r$

F Test: An Example (Online Material Session 2.6)

- The F statistic is the relative difference between SSR_r and SSR_{ur}

$$F = \frac{(SSR_r - SSR_{ur})/2}{SSR_{ur}/(n-6)}$$

- Under H_0, F follows the F-distribution

$$F = \frac{(SSR_r - SSR_{ur})/2}{SSR_{ur}/(n-6)} \sim F_{2,n-6},$$

  with $(2, n-6)$ degrees of freedom

- Decision rule: reject if $F > c$, where c is the $F_{2,n-6}$ critical value

F Test: An Example (Online Material Session 2.6)

- Use the data in bwght.RData: $n = 1191$, $R_r^2 = .0364$ and $R_{ur}^2 = .0387$.

$$F = \frac{(R_{ur}^2 - R_r^2)/2}{(1 - R_{ur}^2)/(n-6)} \approx 1.42$$

- The 5% $F_{2,n-6}$ critical value is $c = 3.00$
- According to the decision rule, H_0 is not rejected at the 5% level because $F < c$

F Test for Overall Significance of a Regression

- When $q = k$, the null "$H_0 : \beta_1 = 0, \ldots, \beta_k = 0$" is routinely tested by most regression packages, known as the F test for overall significance
- The null is that none of the independent variables has an effect on y. The restricted model is simply

$$y = \beta_0 + u$$

- The F stat under the null has an $F_{k,n-k-1}$ distribution. As the R-squared is zero under null, this F stat is

$$F = \frac{R^2/k}{(1-R^2)/(n-k-1)},$$

where R^2 is from the unrestricted model

Testing Exclusion Restrictions with t Statistic?

- We cannot test exclusion restrictions by checking each t statistic separately!
- A simple simulation

```
> x1 <- rnorm(100, mean = 1, sd = 2)
> x2 <- x1 + rnorm(100, mean = 1, sd = 1)
> y <- x1 + x2 + rnorm(100, mean = 1, sd = 8)
> m1 <- lm(y ~ x1 + x2)
> stargazer(m1, align = TRUE, no.space = TRUE)
```

Testing Exclusion Restrictions with t Statistic?

	Dependent variable:
	y
x1	-0.073
	(0.898)
x2	1.500^{*}
	(0.780)
Constant	0.563
	(1.197)
Observations	100
R^2	0.153
Adjusted R^2	0.136
Residual Std. Error	7.812 (df = 97)
F Statistic	8.759^{***} (df = 2; 97)
Note:	*p<0.1; **p<0.05; ***p<0.01

It is possible that a group of variables are jointly significant but individually insignificant. This is typically a symptom of multicollinearity.