Financial Analytics

Mean-Variance Optimization and the CAPM

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Outline

Mean-Variance: A Simple Motivating Example

Consider a one-period market with n securities satisfying:

$$\begin{aligned} \mathsf{E}[R_i] &= \mu, \quad i = 1, \ldots, n \\ \mathsf{Var}(R_i) &= \sigma^2, \quad i = 1, \ldots, n \\ \mathsf{Cov}(R_i, R_j) &= 0 \;\; \text{for all } i \neq j. \end{aligned}$$

Let w_i denote fraction of wealth invested in i^{th} security at time $t = 0$

- must have $\sum_{i=1}^{n} w_i = 1$ for any portfolio.

Consider now two portfolios:

Portfolio A: 100% invested in security 1 so that $w_1 = 1$ and $w_i = 0$ for $i > 1$.

Portfolio B: An equi-weighted portfolio so that $w_i = 1/n$ for $i = 1, \ldots, n$.

Then have

$$\begin{aligned} \mathsf{E}[R_A] &= \mathsf{E}[R_B] = \mu \\ \mathsf{Var}(R_A) &= \sigma^2 \\ \mathsf{Var}(R_B) &= \sigma^2/n. \end{aligned}$$

where R_A and R_B are random returns of portfolios A and B, respectively.

Mean-Variance: A Simple Motivating Example

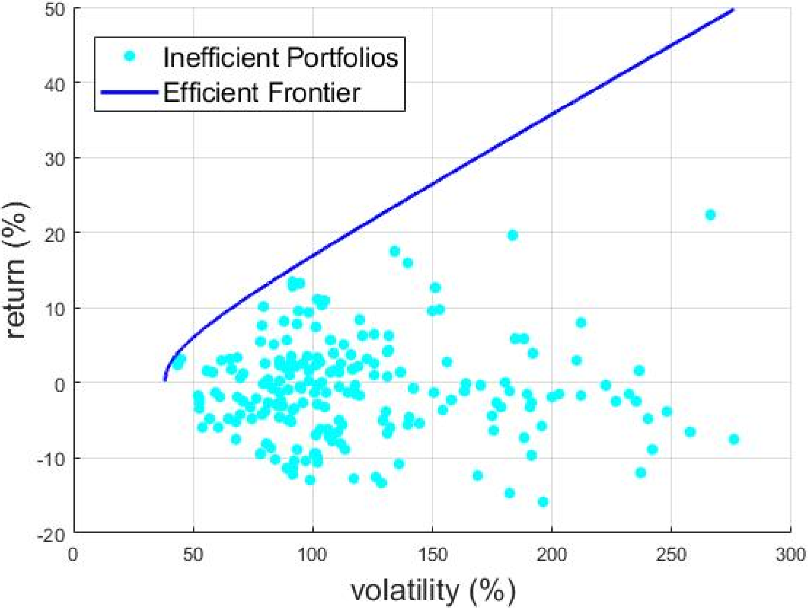
So both portfolios have same expected return but different return variances.

A risk-averse investor should clearly prefer portfolio B because this portfolio benefits from diversification without sacrificing any expected return.

- the central insight of Markowitz.

Consider figure on next slide:

- We simulated $m = 200$ random portfolios from universe of $n = 6$ securities.
- Expected return and volatility, i.e. standard deviation, plotted for each one
  - they are inefficient because each one can be improved.
- In particular, for same expected return it is possible to find an (efficient) portfolio with a smaller volatility.
- Alternatively, for same volatility it is possible to find an (efficient) portfolio with higher expected return.



Inefficient Portfolios
Efficient Frontier
return (%)
volatility (%)
50
40
30
20
10
0
-10
-20
0
50
100
150
200
250
300

Mean-Variance without a Riskfree Asset

- Have n risky securities with corresponding return vector $\mathbf{R}$ satisfying

$$\mathbf{R} \sim \mathsf{MVN}_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$

- Let $\mathbf{w} = [w_1 \cdots w_n]^\top$ where $w_i =$ fraction of wealth invested in i^{th} security.
- Mean-variance portfolio optimization problem is formulated as:

$$\min_{\mathbf{w}} \ \frac{1}{2}\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \tag{1}$$

$$\begin{aligned} \text{subject to} \qquad \mathbf{w}^\top \boldsymbol{\mu} &= p \\ \text{and} \qquad \mathbf{w}^\top \mathbf{1} &= 1. \end{aligned}$$

- (8) is a quadratic program (QP) and is also convex because $\boldsymbol{\Sigma} \succeq 0$
  - can therefore be solved via Lagrange multiplier methods.
- Note that specific value of p will depend on investor's level of risk aversion.

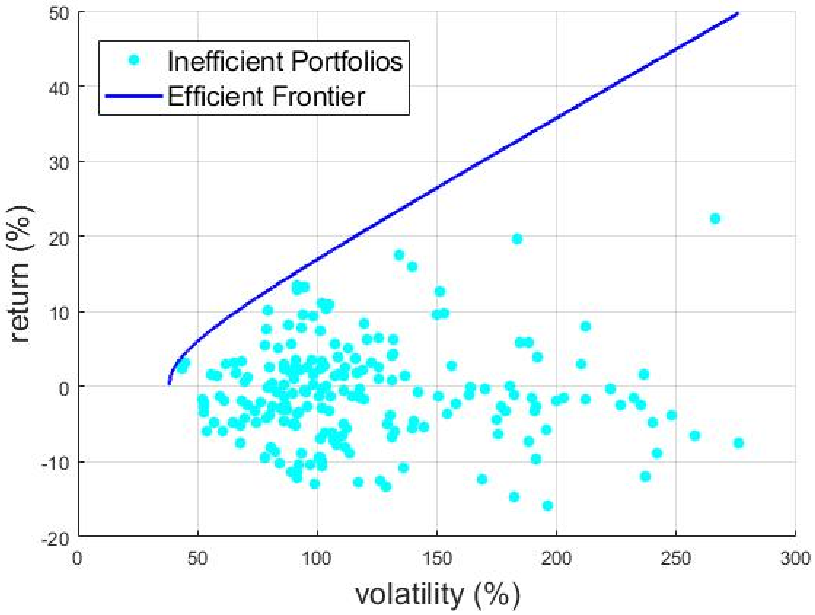
Mean-Variance without a Riskfree Asset

- When we plot the mean portfolio return p against the corresponding minimized portfolio volatility / standard deviation we obtain the so-called portfolio frontier.

- Can also identify the portfolio having minimal variance among all risky portfolios: the minimum variance portfolio.

  Let $\bar{R}_{mv}$ denote expected return of minimum variance portfolio.

- Points on portfolio frontier with expected returns greater than $\bar{R}_{mv}$ are said to lie on the efficient frontier.



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A 2-Fund Theorem

- Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be mean-variance efficient portfolios corresponding to expected returns p_1 and p_2, respectively, with $p_1 \neq p_2$.
- Can then be shown that all efficient portfolios can be obtained as linear combinations of $\mathbf{w}_1$ and $\mathbf{w}_2$
  - an example of a 2-fund theorem.

Mean-Variance with a Riskfree Asset

- Now assume there's a risk-free security with risk-free rate equal to r_f.
- Let $\mathbf{w} := [w_1 \cdots w_n]^\top$ be the vector of portfolio weights on the n risky assets
  - so $1 - \sum_{i=1}^n w_i$ is the weight on the risk-free security.
- Investor's portfolio optimization problem may then be formulated as

$$\min_{\mathbf{w}} \ \frac{1}{2}\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \tag{2}$$

$$\text{subject to} \qquad \left(1 - \sum_{i=1}^{n} w_i\right) r_f \ + \ \mathbf{w}^\top \boldsymbol{\mu} \ = \ p.$$

Mean-Variance with a Riskfree Asset

- Optimal solution to (2) given by

$$\mathbf{w} \;=\; \xi\, \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}) \tag{3}$$

  where $\xi := \sigma^2_{min}/(p - r_f)$ and

$$\sigma^2_{min} \;=\; \frac{(p - r_f)^2}{(\boldsymbol{\mu} - r_f \mathbf{1})^\top\, \boldsymbol{\Sigma}^{-1}\, (\boldsymbol{\mu} - r_f \mathbf{1})} \tag{4}$$

  is the minimized variance.

- While ξ (or p) depends on investor's level of risk aversion it is often inferred from the market portfolio.

Mean-Variance with a Riskfree Asset

- Taking square roots across (4) we obtain

$$\sigma_{min}(p) \;=\; \frac{(p - r_f)}{\sqrt{(\boldsymbol{\mu} - r_f\mathbf{1})^\top\, \boldsymbol{\Sigma}^{-1}\, (\boldsymbol{\mu} - r_f\mathbf{1})}} \tag{5}$$

  – so the efficient frontier $(\sigma_{min}(p),\, p)$ is linear when we have a risk-free security:

Does the Frontier of Risky Assets (Only) Play Any Role?

Can gain further insight as follows:

- Let R denote the (random) return of any portfolio of risky (only) securities.
- Now form a portfolio of the risk-free security with this risky portfolio.
- Return on this new portfolio is

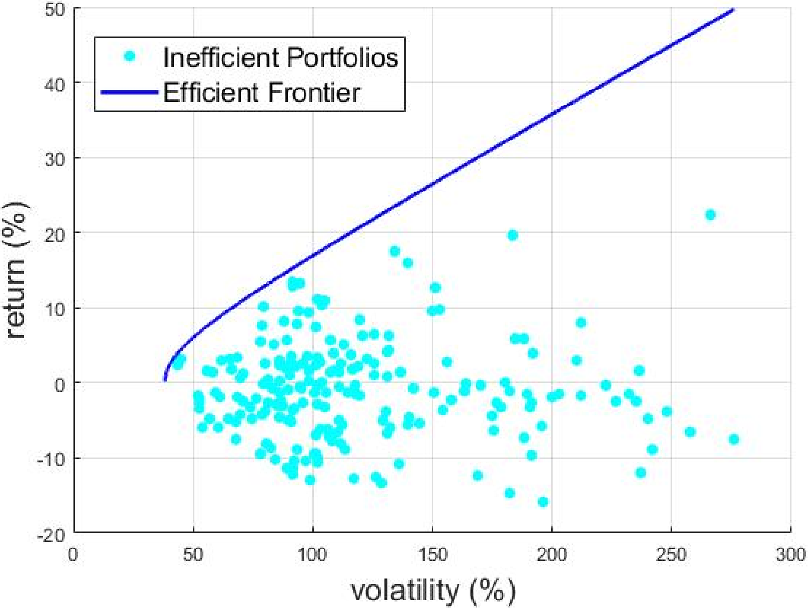
$$R_\alpha := \alpha r_f + (1-\alpha)R$$

- Also have

$$\begin{aligned}\bar{R}_\alpha &= \alpha r_f + (1-\alpha)\bar{R} \\ \sigma_\alpha &= (1-\alpha)\sigma_R\end{aligned}$$

So the mean and standard deviation of the portfolio varies linearly with α.

Question: What does this imply?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Efficient Frontier
return (%)
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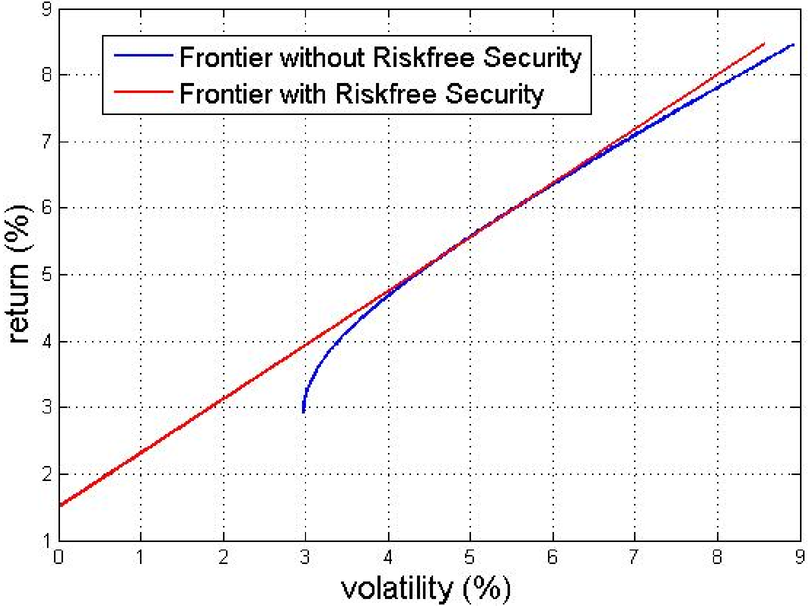

Frontier without Riskfree Security
Frontier with Riskfree Security
return (%)
volatility (%)
0
1
2
3
4
5
6
7
8
9

Mean-Variance with a Riskfree Asset

- In fact suppose $r_f < \bar{R}_{mv}$.
- Efficient frontier then becomes a straight line that is tangent to the risky efficient frontier and with a y-intercept equal to the risk-free rate.
- We also then have a 1-fund theorem:

  Every investor will optimally choose to invest in a combination of the risk-free security and the tangency portfolio.

Mean-Variance with a Riskfree Asset

- Recall the optimal solution to mean-variance problem given by:

$$\mathbf{w} \;=\; \xi\, \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f\mathbf{1}) \tag{6}$$

  where $\xi := \sigma^2_{min}/(p - r_f)$ and

$$\sigma^2_{min} \;=\; \frac{(p - r_f)^2}{(\boldsymbol{\mu} - r_f\mathbf{1})^\top\, \boldsymbol{\Sigma}^{-1}\, (\boldsymbol{\mu} - r_f\mathbf{1})} \tag{7}$$

  is the minimized variance.

- The tangency portfolio $\mathbf{w}^*$ is given by (6) except that it must be scaled so that its component sum to 1
  - this scaling removes the dependency on p.

Question: Describe the efficient frontier if no-borrowing is allowed.

Mean-Variance with a Riskfree Asset

Definition. The Sharpe ratio of a portfolio (or security) is the ratio of the expected excess return of the portfolio to the portfolio's volatility.

Definition. The Sharpe optimal portfolio is the portfolio with maximum Sharpe ratio.

Have already seen(!) that the tangency portfolio $\mathbf{w}^*$ is the Sharpe optimal portfolio of risky assets.

Weaknesses of Traditional Mean-Variance Analysis

- Traditional mean-variance analysis has many weaknesses when applied naively in practice.

  e.g. It often produces extreme portfolios combining extreme shorts with extreme longs
  - portfolio managers generally do not trust these extreme weights as a result.

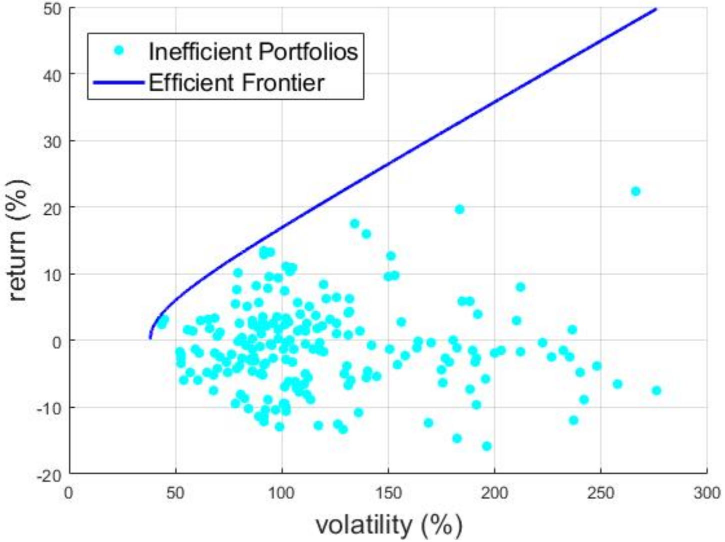
- This problem is typically caused by estimation errors in the mean return vector and covariance matrix.

- Consider again our original mean-variance portfolio optimization problem

$$\min_{\mathbf{w}} \ \frac{1}{2}\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$$

$$\begin{array}{rl} \text{subject to} & \mathbf{w}^\top \boldsymbol{\mu} = p \\ \text{and} & \mathbf{w}^\top \mathbf{1} = 1. \end{array}$$

which (as we vary p) leads to the efficient frontier of risky securities ...



- In practice, investors can't compute frontier since they don't know $\boldsymbol{\mu}$ or $\boldsymbol{\Sigma}$.
- The best we can do is approximate it. But how might we do this?

Weaknesses of Traditional Mean-Variance Analysis

One approach would be to simply estimate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ using historical data.

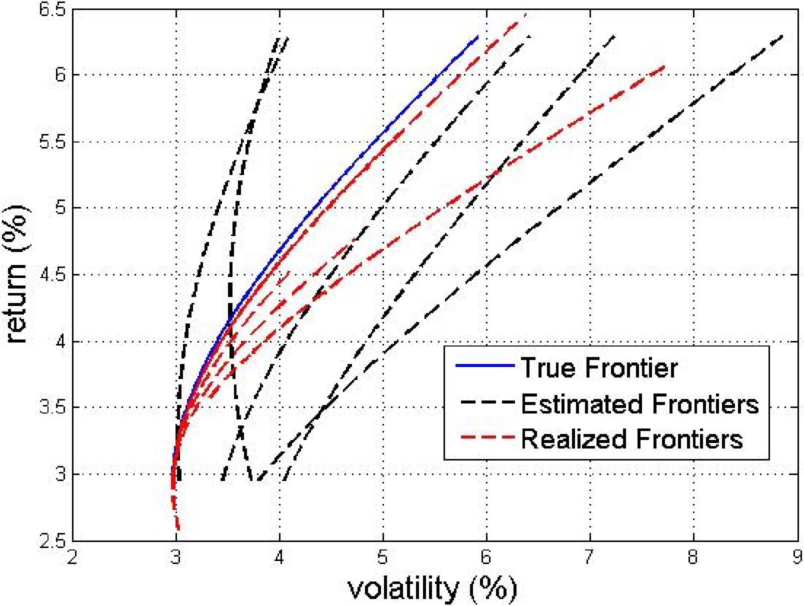
Each of the dashed black curves in next figure is an estimated frontier that we computed by:

1. Simulating $m = 24$ sample returns from the true distribution
    - which in this case was assumed to be multivariate normal.
2. Estimating $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ from this simulated data
3. Using these estimates ($\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$) to generate the (estimated) frontier.

The blue curve in the figure is the true frontier computed using $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

First observation is that the estimated frontiers are random and can differ greatly from the true frontier;

- an estimated frontier may lie below or above the true frontier or it may intersect it.



True Frontier
Estimated Frontiers
Realized Frontiers
return (%)
volatility (%)
6.5
6
5.5
5
4.5
4
3.5
3
2.5
2
3
4
5
6
7
8
9

Weaknesses of Traditional Mean-Variance Analysis

- An investor who uses such an estimated frontier to make investment decisions may end up choosing a poor portfolio.

  Question: But just how poor?

- The dashed red curves in the figure are the realized frontiers
  - the true mean-volatility tradeoff that results from making decisions based on the estimated frontiers.

- In contrast to the estimated frontiers, the realized frontiers must always (why?) lie below the true frontier.

- Some of the realized frontiers lie very close to the true frontier and so in these cases an investor might do very well.

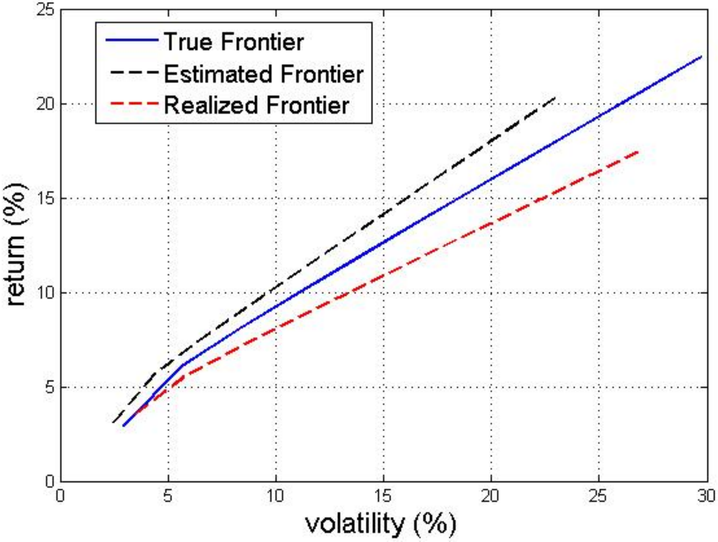
- But in other cases the realized frontier is far from the (unobtainable) true efficient frontier.

Overcoming These Weaknesses

As a result of these weaknesses, portfolio managers traditionally had little confidence in mean-variance analysis and therefore applied it rarely in practice.

Efforts to overcome these problems include:

1. The use of shrinkage estimators.
2. Imposing constraints, e.g. no short-sales and no borrowing, on the problem.
3. Bayesian techniques such as the Black-Litterman framework
   - also allows users to specify their own subjective views on the market in a consistent and tractable manner.
4. The use of robust optimization algorithms that explicitly account for uncertainty in parameter estimates.



- Figure displays estimated and realized frontiers obtained from a robust optimization algorithm.
- They lie much closer to the true frontier!

Portfolio Management Relative to a Benchmark

Quite common in practice for portfolio managers to manage and assess performance relative to a benchmark portfolio

- benchmark portfolio typically represents a particular asset class.

Within this asset class:

- A passive manager would aim to replicate the benchmark.
- An active manager would aim to outperform the benchmark.

Mean-variance framework can be easily adapted to the problem of outperforming a benchmark:

- Expected return replaced by the expected excess return $(\mathbf{w} - \mathbf{w}_B)^\top \mathbf{R}$.
- Return variance replaced by tracking error variance, i.e. $\mathrm{Var}(\mathbf{R}^\top (\mathbf{w} - \mathbf{w}_B))$.
- Still end up with a convex quadratic optimization problem!

Portfolio Management Relative to a Benchmark

e.g. A passive asset manager might solve

$$\min_{\mathbf{w}} \ \frac{1}{2}(\mathbf{w} - \mathbf{w}_B)^\top \boldsymbol{\Sigma}(\mathbf{w} - \mathbf{w}_B) \tag{8}$$

$$\text{subject to} \qquad \mathbf{w}^\top \mathbf{1} = 1$$

e.g. An active manager might solve

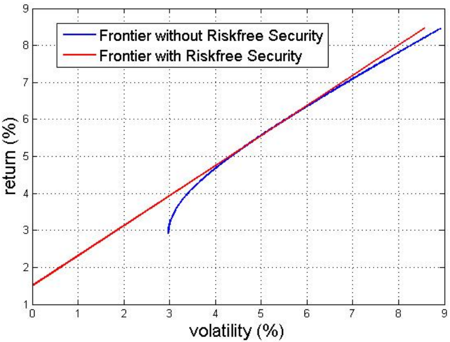
$$\max_{\mathbf{w}} \ (\mathbf{w} - \mathbf{w}_B)^\top \boldsymbol{\mu} \tag{9}$$

$$\text{subject to} \qquad \frac{1}{2}(\mathbf{w} - \mathbf{w}_B)^\top \boldsymbol{\Sigma}(\mathbf{w} - \mathbf{w}_B) \leq \sigma^2$$

$$\text{and} \qquad \mathbf{w}^\top \mathbf{1} = 1.$$

- Straightforward to also account for transactions costs & linear constraints.
- In fact, solution to (8) is $\mathbf{w} = \mathbf{w}_B$ unless we include transaction costs or some constraints.
- Note (8) is a much easier problem in practice as it does not involve $\boldsymbol{\mu}$.

The Capital Asset Pricing Model (CAPM)



- If every investor is a mean-variance optimizer then each of them will hold the same tangency portfolio of risky securities in conjunction with a position in the risk-free asset.
- Because the tangency portfolio is held by all investors and because markets must clear, we can identify this portfolio as the market portfolio.
- The efficient frontier is then termed the capital market line (CML).

The Capital Asset Pricing Model (CAPM)

- Now let R_m and $\bar{R}_m$ denote the return and expected return, respectively, of the market, i.e. tangency, portfolio.

- Central insight of the Capital Asset-Pricing Model is that in equilibrium the riskiness of an asset is not measured by the standard deviation of its return R but by its beta:

$$\beta := \frac{\mathsf{Cov}(R, R_m)}{\mathsf{Var}(R_m)}.$$

- In particular, there is a linear relationship between the expected return, $\bar{R} = \mathsf{E}[R]$, of any security (or portfolio) and the expected return of the market portfolio:

$$\bar{R} = r_f + \beta\,(\bar{R}_m - r_f). \tag{10}$$

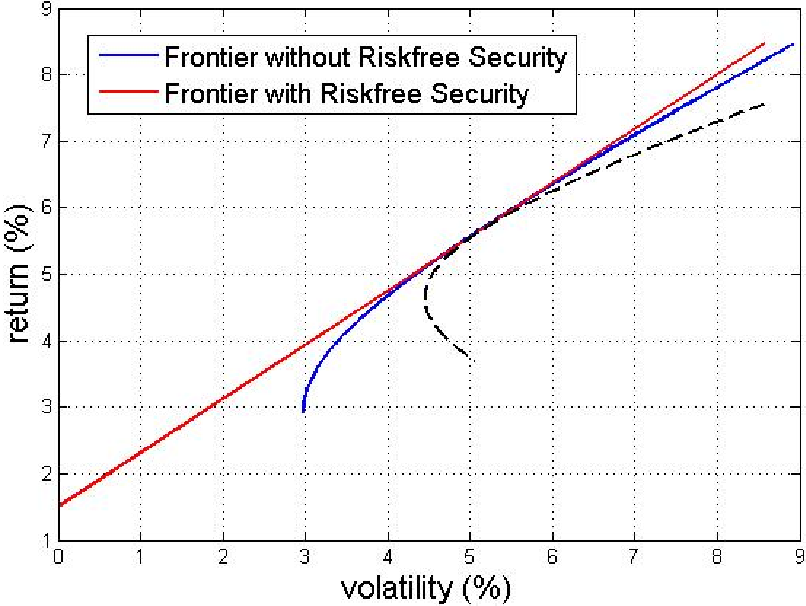
The Capital Asset Pricing Model (CAPM)

- In order to prove (10), consider a portfolio with weights α and weight $1-\alpha$ on the risky security and market portfolio, respectively.
- Let R_α denote the (random) return of this portfolio as a function of α.
- Then have

$$\begin{aligned} \mathsf{E}[R_\alpha] &= \alpha\bar{R} + (1-\alpha)\bar{R}_m \qquad (11) \\ \sigma^2_{R_\alpha} &= \alpha^2\sigma^2_R + (1-\alpha)^2\sigma^2_{R_m} + 2\alpha(1-\alpha)\sigma_{R,R_m}. \qquad (12) \end{aligned}$$

- As α varies, the mean and stand. dev. $(\mathsf{E}[R_\alpha], \sigma_{R_\alpha})$ trace out a curve.

Question: This curve cannot cross the efficient frontier. Why?


Frontier without Riskfree Security
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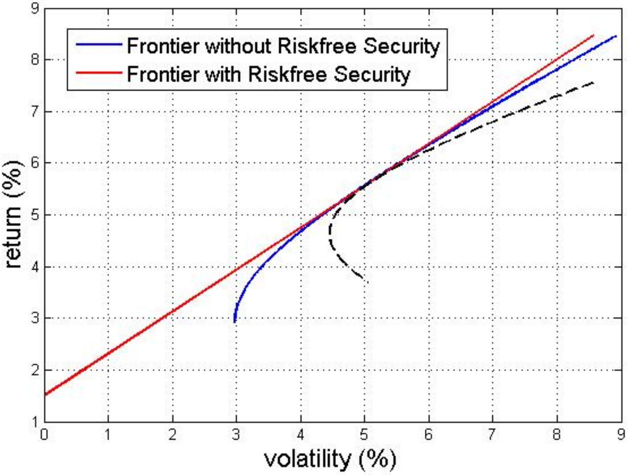
The Capital Asset Pricing Model (CAPM)

- Therefore at $\alpha = 0$ this curve must be tangent to the CML.
- So slope of the curve at $\alpha = 0$ must equal slope of the CML.
- Using (11) and (12) we see slope of CML is given by

$$
\begin{aligned}
\left.\frac{d\,\mathsf{E}[R_\alpha]}{d\,\sigma_{R_\alpha}}\right|_{\alpha=0} &= \left.\frac{d\,\mathsf{E}[R_\alpha]}{d\,\alpha}\Bigg/\frac{d\,\sigma_{R_\alpha}}{d\,\alpha}\right|_{\alpha=0} \\
&= \left.\frac{\sigma_{R_\alpha}\left(\bar{R}-\bar{R}_m\right)}{\alpha\sigma_R^2-(1-\alpha)\sigma_{R_m}^2+(1-2\alpha)\sigma_{R,R_m}}\right|_{\alpha=0} \\
&= \frac{\sigma_{R_m}\left(\bar{R}-\bar{R}_m\right)}{-\sigma_{R_m}^2+\sigma_{R,R_m}}.
\end{aligned}
$$



- Slope of CML is $\left(\bar{R}_m - r_f\right)/\sigma_{R_m}$ and equating the two therefore yields

$$\frac{\sigma_{R_m}\left(\bar{R} - \bar{R}_m\right)}{-\sigma_{R_m}^2 + \sigma_{R,R_m}} = \frac{\bar{R}_m - r_f}{\sigma_{R_m}} \tag{13}$$

which upon simplification gives (10).

The Capital Asset Pricing Model (CAPM)

- The CAPM is one of the most famous models in all of finance.
- Even though it arises from a simple one-period model, it provides considerable insight to the problem of asset-pricing.

  e.g. It's well-known that riskier securities should have higher expected returns in order to compensate investors for holding them. But how do we measure risk?

- According to the CAPM, security risk is measured by its beta which is proportional to its covariance with the market portfolio
  - a very important insight.
- This does not contradict the mean-variance formulation of Markowitz where investors use variance to measure risk
  - indeed we derived the CAPM from mean-variance analysis!

The CAPM Today

- Today it's understood that the CAPM is not an accurate model of reality
  - multi-factor models provide better explanations for returns.
- But the CAPM is still very influential.