# BS1820: Maths and Statistics Foundations for Analytics

## Statistics 2

**Zhe Liu**

Imperial College Business School

Email: zhe.liu@imperial.ac.uk

# Outline

# 2.1 Motivation

A statistical hypothesis is an assertion or conjecture about the population.

Often we cannot prove whether a hypothesis is right or wrong with absolute certainty because we do not have absolute knowledge of the entire population.

So we resort to a random sample to judge the hypothesis at some confidence/significance level – this is what hypothesis testing concerns about.

**Example:** A computer screen manufacturer advertises a new screen that uses 82W on average. It can be assumed that the usage among screen products is normally distributed with a known variance $\sigma^2 = 4^2(\mathsf{W}^2)$.

As a consumer, you want to test the manufacturer's claim and plan to take some measurements of power usage for this type of screens.

What should you do?

# 2.2 Steps in Hypothesis Testing

1. State the null $(H_0)$ and alternative $(H_1)$ **hypotheses**
2. Specify **significance level** $\alpha$
3. Choose and calculate the **test statistic**
4. Equivalent approaches to **draw a conclusion** (accept/reject):
   - Calculate critical value and compare with test statistic
   - Calculate $p$-value and compare with significance level $\alpha$

# 2.3 Hypotheses

The first step is to state the null hypothesis vs alternative hypothesis.

$$H_0 : \ \mu = \mu_0$$
$$H_1 : \ \mu \neq \mu_0$$

where $\mu_0 = 82$ is the manufacturer's claimed average power usage.

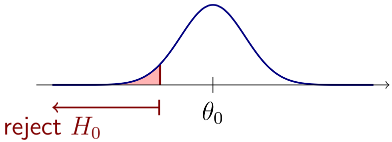
In the end, we either **reject** $H_0$ or **cannot reject** $H_0$.

**Remark:** Note that failing to reject $H_0$ does not mean $H_0$ is definitely true! It only means that we do not have sufficient evidence to support $H_1$.

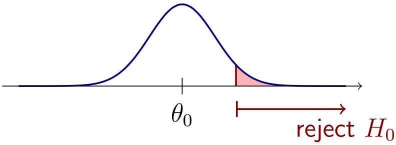
# 2.4 One-Tailed vs Two-Tailed Tests

Choose **one-tailed** or **two-tailed** test based on the hypotheses.

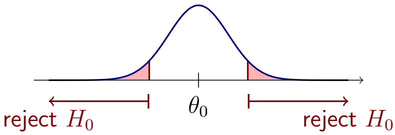
$$H_0: \ \theta \geq \theta_0$$
$$H_1: \ \theta < \theta_0$$




$$H_0: \ \theta \leq \theta_0$$
$$H_1: \ \theta > \theta_0$$




$$H_0: \ \theta = \theta_0$$
$$H_1: \ \theta \neq \theta_0$$



# 2.5 Sample Data

Having set up the null and alternative hypotheses, we collect sample data and calculate the sample statistics we are interested in.

**Example (continued):**
12 measurements of the power usage are recorded:

$$82, 86, 84, 84, 92, 83, 93, 80, 83, 84, 82, 86$$

We can then compute:

$$\bar{x} = 84.92$$

Is this **different enough** from the null hypothesis $(H_0 : \mu_0 = 82)$?

If so, we can reject $H_0$ but otherwise we may not.

# 2.6 Test Statistics

We want to "**standardize**" the sample statistics (**e.g.** $\bar{X}, \widehat{p}$) to standard test statistics (**e.g.** $z$-score or $t$-score) that follow certain distribution (**e.g.** standard normal, student-$t$).

– Normal population mean ($\sigma$ **known**): $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$

– Normal population mean ($\sigma$ **unknown**): $T = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$

– Large population mean: $Z = \dfrac{\bar{X} - \mu_0}{S/\sqrt{n}} \approx \mathcal{N}(0, 1)$

– Population proportion: $Z = \dfrac{\widehat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \approx \mathcal{N}(0, 1)$

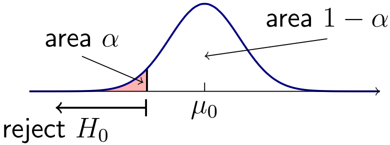
**Example (continued):** Since we have a normal population with known $\sigma$, we use the $z$-score:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{84.92 - 82}{4/\sqrt{12}} = 2.53$$

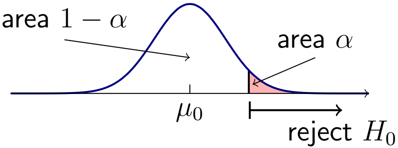
# 2.7 Critical Values

Compare test statistics with proper **critical values** to draw a conclusion.

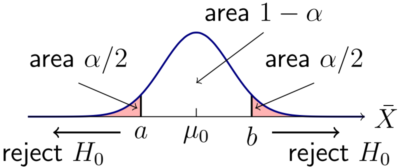
Reject $H_0$ if $Z < -z_\alpha$ or $T < -t_{n-1}^{\alpha}$




Reject $H_0$ if $Z > z_\alpha$ or $T > t_{n-1}^{\alpha}$




Reject $H_0$ if $|Z| > z_{\alpha/2}$ or $|T| > t_{n-1}^{\alpha/2}$




**Remark:** The probability that the test statistic lies outside the critical value is $\alpha$.

## 2.7 Critical Values

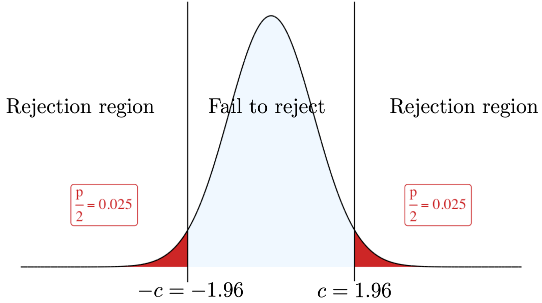
**Example (continued):** Since we have a normal population with known $\sigma$, we use the $z$-score:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{84.92 - 82}{4/\sqrt{12}} = 2.53$$

For this two-tailed test, if we choose the significance level $\alpha = 0.05$, we have

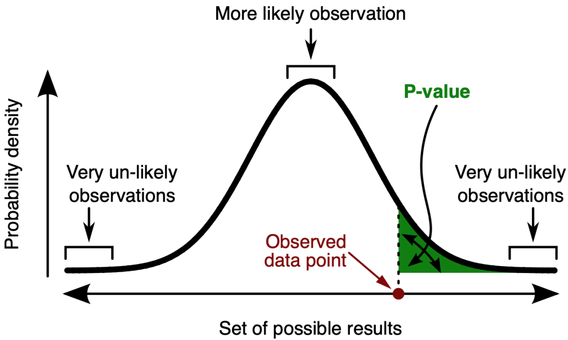
$$z > z_{\alpha/2} = 1.96$$

and hence we can reject the null hypothesis.



# 2.8 $p$-Value

Instead of comparing test statistic with calculated critical value (based on $\alpha$), we can also compute "something" based on test statistic and compare it with $\alpha$.




A **p-value** (shaded green area) is the probability of an observed (or more extreme) result assuming that the null hypothesis is true.

**Remark:** $p$-value is advantageous since it's *associated with* the test statistic (data). Once calculated, it can be compared with any significance level $\alpha$.

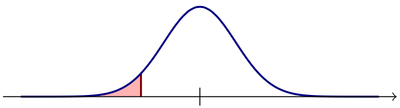
Figure source: *Wikipedia*

# 2.8 *p*-Value

$$p\text{-value} = \mathsf{P}(\text{data } \textbf{as or more extreme} \text{ than } \textbf{observed data} \mid H_0)$$

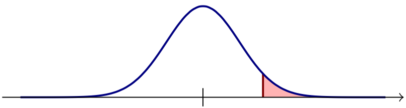
$H_0:\ \theta \geq \theta_0$ vs $H_1:\ \theta < \theta_0$

$p$-value $= \mathsf{P}(z \leq Z)$ or $\mathsf{P}(t_{n-1} \leq T)$



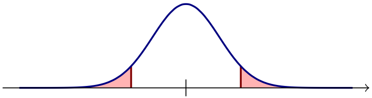
$H_0:\ \theta \leq \theta_0$ vs $H_1:\ \theta > \theta_0$

$p$-value $= \mathsf{P}(z \geq Z)$ or $\mathsf{P}(t_{n-1} \geq T)$



$H_0:\ \theta = \theta_0$ vs $H_1:\ \theta \neq \theta_0$

$p$-value $= 2 \times \mathsf{P}(z \geq |Z|)$ or $2 \times \mathsf{P}(t_{n-1} \geq |T|)$



**Remark: Reject $H_0$ if $p$-value $< \alpha$.** (Why?)

# 2.8 *p*-Value

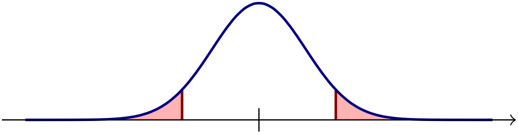
**Example (continued):** Since we have a normal population with known $\sigma$, we use the $z$-score:

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{84.92 - 82}{4/\sqrt{12}} = 2.53$$

For this two-tailed test, the $p$-value is

$$p = 2 \times \mathsf{P}(z \geq |2.53|) = 0.011$$

and hence we can reject $H_0$ at $\alpha = 0.05$ or $0.1$, but we cannot at $\alpha = 0.01$.



**Remark:** We can repeat the "critical value" approach at various $\alpha$ and reach the same conclusion. Clearly the "$p$-value" approach saves effort!

# 2.9 Steps in Hypothesis Testing (Recap)

1. State the null $(H_0)$ and alternative $(H_1)$ **hypotheses**

   $\Rightarrow$ this determines one-tailed or two-tailed test

2. Specify **significance level** $\alpha$

3. Choose and calculate the **test statistic**

   $\Rightarrow$ this determines $t$-test or $z$-test

4. Equivalent approaches to **draw a conclusion** (accept/reject):

   – Calculate critical value and compare with test statistic:

     $\Rightarrow$ Reject $H_0$ if test statistic more "extreme" critical value

   – Calculate $p$-value and compare with significance level $\alpha$:

     $\Rightarrow$ Reject $H_0$ if $p$-value $< \alpha$