

Some Topics in Credit Modeling

These notes introduce two other important topics in finance, namely securitization and credit modeling. Our introduction to securitization will be via an overview of mortgage mathematics from which we can motivate certain types of mortgage-backed securities. We will also discuss collateralized-debt-obligations (CDO) via a stylized one-period model. Credit modeling is an extremely important part of finance. Its origins pre-date modern finance and indeed go back to the very beginning of commerce. More recently the development of credit-default-swaps (CDS's) and other more complex credit derivatives (such as CDO's) are widely considered to have been one of the contributing factors to the global financial crisis of 2008 and beyond. We will also use our simple credit models to introduce some important incentive problems from corporate finance.

1 An Introduction to Securitization and CDO's

Securitization is the name given to the process of constructing new securities from the cash-flows generated by a pool of underlying securities. The main economic rationale behind securitization is that it enables the construction of new securities with a broad range of risk profiles. Different types of investors may therefore be interested in these new securities even if they had no interest in the underlying securities. If this is the case then there will be an increased demand for the underlying cash-flows and so the cost-of-capital is reduced for the issuers of the underlying securities.

1.1 Collateralized Debt Obligations

Collateralized debt obligations (CDOs) are a particular class of credit derivatives that are constructed from an underlying pool of fixed-income securities. They were first issued by banks in the mid-1990's. The original motivation for introducing CDO's was *regulatory arbitrage*. By keeping the equity tranche (see below) of the CDO a bank could effectively keep the entire economic risk of the underlying portfolio but because the notional principal of the tranch was much smaller, the bank faced must lighter capital requirements. Our goal here will be to find the expected losses in a simple 1-period CDO with the following[1] characteristics:

- The maturity is 1 year.
- There are $N = 125$ bonds in the reference portfolio.
- Each bond pays a coupon of one unit after 1 year if it has not defaulted.
- The recovery rate on each defaulted bond is zero.
- There are 3 **tranches** of interest: the **equity**, **mezzanine** and **senior** tranches with **attachment points** 0-3 defaults, 4-6 defaults and 7-125 defaults, respectively.

The equity tranche bears the first losses in the underlying portfolio up to a maximum of 3 default. The mezzanine tranche then bears the next losses up to an additional maximum of 3 defaults. Therefore the equity and mezzanine tranches bear the losses associated with the first size defaults on the underlying portfolio of 125 bonds. Only if there are more than six defaults does the senior tranche start incurring losses.

[1]This example is taken from "The Devil is in the Tails: Actuarial Mathematics and the Subprime Mortgage Crisis", by C. Donnelly and P. Embrechts in ASTIN Bulletin 40(1), 1-33.

Computing Expected Tranche Losses

We make the simple assumption that the probability q of defaulting within 1 year is identical across all bonds, X_i is the normalized asset value of the i^{th} credit, i.e. bond, and we assume

$$X_i = \sqrt{\rho} M + \sqrt{1-\rho}\, Z_i \tag{1}$$

where $M, Z_1, \ldots, Z_N$ are IID normal random variables. Note that the correlation between each pair of asset values is identical. We assume also that the i^{th} credit defaults if $X_i \leq \bar{x}_i$. Since the probability of default q is identical across all bonds we must therefore have

$$\bar{x}_1 = \cdots \bar{x}_N = \Phi^{-1}(q) \tag{2}$$

where Φ is the CDF of a standard normal random variable. It now follows from (1) and (2) that

$$\begin{aligned} \mathsf{P}(\text{Credit } i \text{ defaults} \,|\, M) &= \mathsf{P}(X_i \leq \bar{x}_i \,|\, M) \\ &= \mathsf{P}(\sqrt{\rho} M + \sqrt{1-\rho}\, Z_i \leq \Phi^{-1}(q) \,|\, M) \\ &= \mathsf{P}\left(Z_i \leq \frac{\Phi^{-1}(q) - \sqrt{\rho} M}{\sqrt{1-\rho}} \,\middle|\, M \right). \end{aligned}$$

Therefore conditional on M, the total number of defaults is Binomial(N, q_M) where

$$q_M := \Phi\left(\frac{\Phi^{-1}(q) - \sqrt{\rho} M}{\sqrt{1-\rho}} \right).$$

That is,

$$p(k \,|\, M) = \binom{N}{k} q_M^k (1-q_M)^{N-k}.$$

The unconditional probabilities can be computed by integrating numerically the binomial probabilities with respect to M so that

$$\mathsf{P}(k \text{ defaults}) = \int_{-\infty}^{\infty} p(k \,|\, M) \phi(M)\, dM$$

where $\phi(\cdot)$ is the standard normal PDF. We can now compute the expected (risk-neutral) loss on each of the three tranches according to

$$\begin{aligned} \mathrm{E}_0^Q\left[\text{Equity tranche loss}\right] &= 3 \times \mathsf{P}(3 \text{ or more defaults}) + \sum_{k=1}^{2} k\, \mathsf{P}(k \text{ defaults}) \\ \mathrm{E}_0^Q\left[\text{Mezzanine tranche loss}\right] &= 3 \times \mathsf{P}(6 \text{ or more defaults}) + \sum_{k=1}^{2} k\, \mathsf{P}(k+3 \text{ defaults}) \\ \mathrm{E}_0^Q\left[\text{Senior tranche loss}\right] &= \sum_{k=1}^{119} k\, \mathsf{P}(k+6 \text{ defaults}). \end{aligned}$$

Results for various values of ρ and q are displayed in the figure below. Regardless of the individual default probability q and correlation ρ we see

$$\mathrm{E}_0^Q\left[\% \text{ Equity tranche loss}\right] \geq \mathrm{E}_0^Q\left[\% \text{ Mezzanine tranche loss}\right] \geq \mathrm{E}_0^Q\left[\% \text{ Senior tranche loss}\right].$$

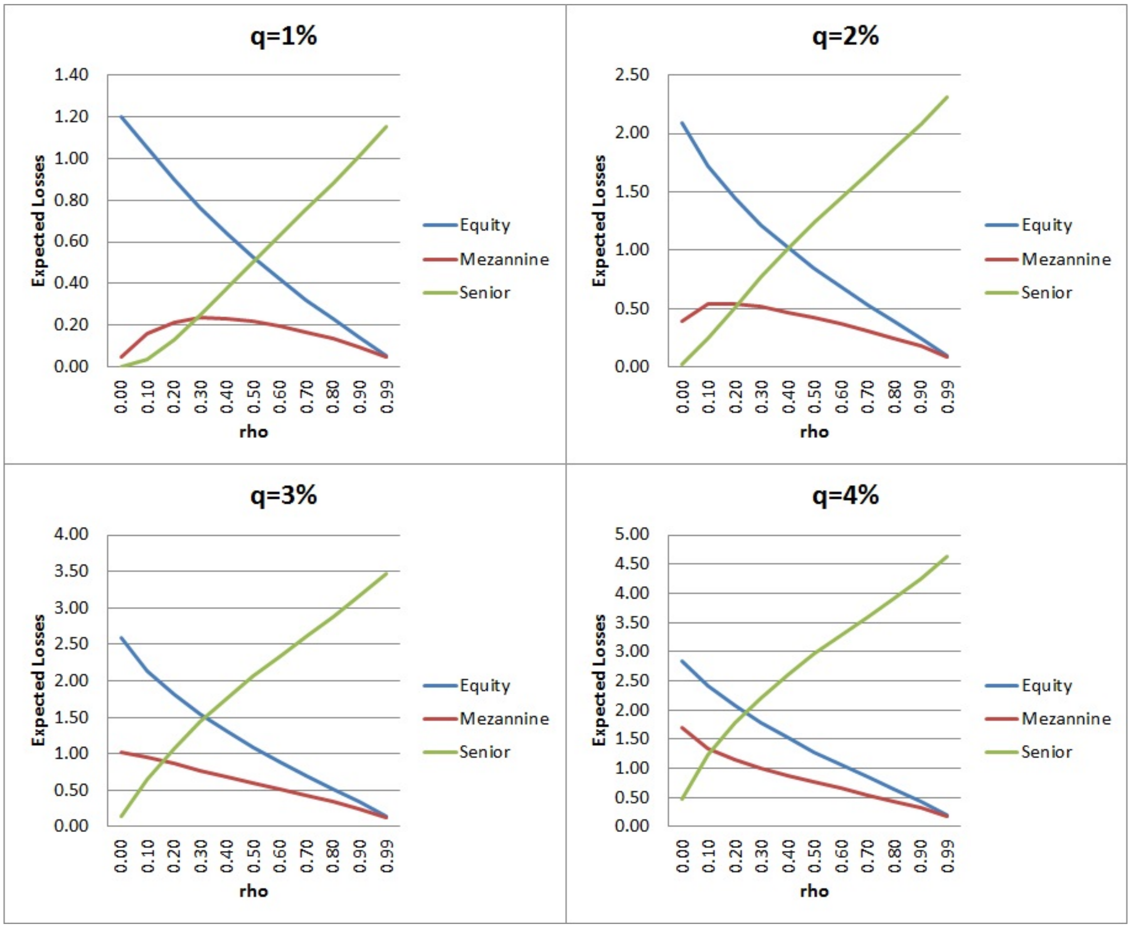
We also note that the expected losses in the equity tranche are always decreasing in ρ while mezzanine tranches are often relatively insensitive[2] to ρ. The expected losses in senior tranches (with upper attachment point of 100% or 125 units in our example) are always increasing in ρ.

Exercise 1 *How does the total expected loss in the portfolio vary with ρ?*

Remark 1 *The* dependence structure *we used in (1) to link the default events of the various bonds is the famous Gaussian-copula model.*

[2] This has important implications when it comes to model calibration, an issue we will not pursue further here.

Figure 1: Expected Tranche Losses As a Function of q and ρ



1.2 Other Asset-Backed Securities

Securities that are constructed by reassigning the cash-flows of some underlying group of securities are generally referred to as *asset-backed securities* (ABS). CDOs and MBSs are examples of these securities. Other common forms of ABS include collateralized loan obligations (CLOs) and securities constructed from pools of credit card loans, students loans or car loans. In fact it is possible to form more complex ABS from an underlying pool of other ABS. This leads, for example, to ABS CDOs where the CDO is constructed from an underlying pool of ABS, or CDO-squared's where a CDO is constructed from an underlying pool of CDO tranches. The structured credit / ABS market was at the heart of the 2008 financial crisis and came in for considerable criticism. As a result, many of the more complex ABS are no longer actively traded.

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2 Structural Credit Models and a Little Corporate Finance

The structural approach to credit modeling began with Merton in 1974 and was based on the *fundamental accounting equation*

$$\text{Assets} = \text{Debt} + \text{Equity} \tag{3}$$

which applies to all firms (in the absence of taxes). This equation simply states the obvious, namely that the asset-value of a firm must equal the value of the firm's debt and the firm's equity. This follows because all of the profits generated by a firm's assets will ultimately accrue to the debt- and equity-holders. While more complicated than presented here, the *capital structure* of a firm is such that debt-holders are more senior than equity holders. That means that in the event of bankruptcy debt-holders must be paid off in full before the equity-holders can receive anything. This insight allowed Merton[3] to write the time T value of the equity E_T as a call option on the value of the firm V_T with strike equal to the face value of the deb, D_T. Merton's model therefore implies

$$E_T = \max(0,\, V_T - D_T) \tag{4}$$

with default occurring if $V_T < D_T$. Note that (4) implicitly assumes that the firm is wound up at time T and that default can only occur at that time. These assumptions are not very realistic and have been relaxed in many directions since Merton's original work. Nonetheless, we can gain many insights from working with (4). First of all, we can take V_t to be the value of a traded asset (why?) so that risk-neutral pricing applies. If the firm does not pay dividends then we could assume, for example, that $V_t \sim \mathsf{GBM}(r, \sigma)$ so that E_t is the corresponding Black-Scholes price of a call option with maturity T, strike D_T and underlying security value V_t. From this we can compute other interesting quantities such as the (risk-neutral) probability of default.
While closed-form solutions can be obtained for the equity and debt values in Merton's model (as well as the Black-Cox extension we discuss below) it will be convenient to do much of our work in binomial lattice models as they will allow us to concentrate on the financial rather than mathematical aspects of the modeling.

2.1 The Merton Model

We assume the following parameters: $V_0 = 1,000$, $T = 7$ years, $\mu = 15\%$, $\sigma = 25\%$, $r = 5\%$ and the # of time periods = 7. The face value of the debt is 800 and the coupon on the debt is zero. The first task is to construct the lattice model for V_t and we do this following our usual approach to lattice construction. That is, we take $\nu = \left(\mu - \sigma^2/2\right)$, $\ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$, $d = 1/u$ and risk-neutral probability of an up-move $q = (e^{r\Delta t} - d)/(u - d)$. The resulting lattice of firm values is displayed below with values at time T corresponding to firm default marked with an asterisk.

```
Firm Value Lattice
                                                          6940.6
                                                  5262.6  3990.2
                                          3990.2  3025.5  2294.0
                                  3025.5  2294.0  1739.4  1318.9
                          2294.0  1739.4  1318.9  1000.0  758.2*
                  1739.4  1318.9  1000.0  758.2   574.9   435.9*
          1318.9  1000.0  758.2   574.9   435.9   330.5   250.6*
  1000.0  758.2   574.9   435.9   330.5   250.6   190.0   144.1*

  t = 0   t = 1   t = 2   t = 3   t = 4   t = 5   t = 6   t = 7
```

Now we are ready to price the equity and debt, i.e. corporate bonds, of the firm. We price the equity first by simply viewing it as a regular call option on V_T with strike $K = 800$ and using the usual risk-neutral backward evaluation approach. The bond or debt price can then be computed similarly or by simply observing that it must equal the difference between the firm-value and equity value at each time and state.

[3]But Black and Scholes also recognized the implications of option pricing for valuing corporate securities in their 1973 paper.

Equity Lattice

							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1127.8	687.1	358.4	128.3	0.0	0.0
	758.6	435.7	207.1	63.8	0.0	0.0	0.0
499.7	269.9	117.4	31.7	0.0	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Debt Lattice

							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		611.6	631.7	641.6	630.0	574.9	435.9*
	560.3	564.3	551.1	511.1	435.9	330.5	250.6*
500.3	488.3	457.5	404.2	330.5	250.6	190.0	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

We see the initial values of the equity and debt are 499.7 and 500.3, respectively. The **credit spread** can also be computed as follows. The **yield-to-maturity**, y, of the bond satisfies $500.3 = e^{-yT} \times 800$ which implies $y = 6.7\%$. The credit spread[4] is then given by $c = y - r = 1.7\%$ or 170 **basis points**.

Note that we could also easily compute the true or risk-neutral probability of default by constructing an appropriate lattice. Note that it is also easy to handle coupons. If the debt pays a coupon of C per period, then we write $E_T = \max(0,\, V_T - D_T - C)$ and then in any earlier period we have

$$E_t = \max\left(0,\ \left[qE^u + (1-q)E^d\right]/R - C\right)$$

where $R = e^{r\Delta t}$ and E^u and E^d are the two possible successor nodes in the lattice corresponding to up- and down-moves, respectively. As before, the debt value at a given node will be given by the difference between the firm and equity values at that node.

2.2 The Black-Cox Model

The Black-Cox model generalizes the Merton model by allowing default to also occur before time T. In our example we can assume default occurs the first time the firm value falls below the face value of the debt. In that case we can compute the value of the equity by placing 0 in those cells where default occurs and updating other cells using the usual backwards evaluation approach. As before the debt value at a given cell in the lattice is given by the difference between the firm and equity values in that cell. This results in the following equity and debt lattices:

[4]Note that in general the credit spread for a firm's bonds is usually a function of the time-to-maturity of the bond in question. It therefore makes more sense to talk about c_T rather than just c.

```
Equity Lattice (Black-Cox)
                                                         6140.6
                                                4501.6   3190.2
                                       3266.4   2264.5   1494.0
                              2336.9   1570.2    978.4    518.9
                     1640.8   1054.7    603.6    258.0      0.0
            1115.8    660.7    300.1      0.0      0.0      0.0
    703.9    328.5      0.0      0.0      0.0      0.0      0.0
350.0   0.0      0.0      0.0      0.0      0.0      0.0      0.0

t = 0   t = 1    t = 2    t = 3    t = 4    t = 5    t = 6    t = 7
```

```
Debt Lattice (Black-Cox)
                                                         800.0
                                                761.0    800.0
                                       723.9    761.0    800.0
                              688.6    723.9    761.0    800.0
                     653.2    684.7    715.3    742.0    758.2*
            623.6    658.2    699.9    758.2*   574.9*   435.9*
   614.9    671.5    758.2*   574.9*   435.9*   330.5*   250.6*
650.0   758.2*   574.9*   435.9*   330.5*   250.6*   190.0*   144.1*

t = 0   t = 1    t = 2    t = 3    t = 4    t = 5    t = 6    t = 7
```

We see that the debt-holders have benefitted from this new default regime with their value increasing from 500.3 to 650. Of course this increase has come at the expense of the equity holders whose value has fallen from 499.7 to 350. In this case the credit spread on the bond is -200 basis points! Negative credit spreads are generally not found in practice but have occurred in this case because the debt holders essentially own a down-and-in call option on the value of the firm with zero strike and barrier equal to the face value of the debt. The unreasonable value of the credit spread in this case is evidence against the realism of the specific default assumption made here. While it is true that a firm can default at any time, the barrier would generally be much lower than the face value of the long-term debt of 800. Note that we could easily use a different and time-dependent default barrier to obtain a more realistic value of the credit spread.

2.3 Some Incentive Problems in Corporate Finance

We now discuss some incentive problems that can arise between the equity-holders and debt-holders of a firm. These problems are important problems in corporate finance and very important in practice.

Taking on Good Investments

Recall Merton's structural lattice model of default from Section 2.1. Suppose now the firm is offered a great(!) investment opportunity. The fair value of the investment is $X = 100$ but the cost to the firm will only be 90 which is substantially less than X. The firm has no cash currently available, however, and would have to raise the cash, i.e. 90 dollars, from the current equity holders.

Question: Will the equity holders invest?

Answer: This is clearly an excellent deal since X is the *fair* risk-neutral value of the deal and yet it is available for only 90. We can model this situation by first adding X to the initial value of the firm and computing the resulting firm-value lattice. We obtain (with the time T default values again market with an asterisk):

```
Firm Value Lattice
                                                                   7634.7
                                                          5788.8   4389.3
                                                4389.3    3328.1   2523.4
                                      3328.1    2523.4    1913.3   1450.7
                            2523.4    1913.3    1450.7    1100.0    834.1
                  1913.3    1450.7    1100.0     834.1     632.4    479.5*
        1450.7    1100.0     834.1     632.4     479.5     363.6    275.7*
1100.0   834.1     632.4     479.5     363.6     275.7     209.0    158.5*

t = 0   t = 1     t = 2     t = 3     t = 4     t = 5     t = 6    t = 7
```

We can then compute the equity lattice in the usual manner:

```
Equity Lattice
                                                                   6834.7
                                                          5027.8   3589.3
                                                3665.4    2567.1   1723.4
                                      2639.5    1799.6    1152.4    650.7
                            1868.4    1224.8     726.9     339.0     34.1
                  1296.5     809.4     441.4     176.2      16.9      0.0
         881.1     520.9     261.0      91.5       8.4       0.0      0.0
586.9    327.7     151.3      47.4       4.2       0.0       0.0      0.0

t = 0   t = 1     t = 2     t = 3     t = 4     t = 5     t = 6     t = 7
```

Note that that equity value is now 586.9 which is an increase of $586.9 - 499.7 = 87.2$ dollars. Unfortunately, this increase is less than the 90 required to make the investment and so the equity-holders, if rational, would *not* make the investment! Of course what has happened is that some of the value has gone to the debt-holders (whose lattice we did not display). Their situation has improved since the injection of new equity has made default less likely than before.

This problem (of not making investments that make good economic sense from the overall firm's perspective) is known as the **debt-overhang** problem and can be very pronounced for firms that are relatively close to default.

Exercise 2 *(a) Suppose instead the current equity-holders raised the cash by issuing \$90 of new* non-voting *equity to* outside *investor. These new equity holders would own* $90/(90 + 499/7) = 15.26\%$ *of the firm. Would the original equity holders now vote to make the investment?*

(b) Is there anything wrong with the scenario in part (a)?

(c) What demands would rational outside investors make in order for them to inject \$90 of new equity into the firm so that the good investment could be made? Would the current equity holders agree to those demands?

(d) What would have to happen in order for the equity holders to make the new investment?

Taking on Bad Investments

In addition to the debt overhang problem, there are other incentive problems that can arise. Suppose for example, the fair value of the investment is again 100 but that it costs 110. Clearly from the perspective of the overall value of the firm this is a bad investment and should not be made. And yet from the point of view of the equity-holders it may actually be rational to do this if the investment increases the volatility of the firm. This makes sense if we recall that the equity-holders own a call option on the value of the firm and that the value of an option increases with volatility. It is possible that the increase in equity value due to the increase in volatility will exceed the decrease in equity value due to the poor quality of the investment. In that case it makes sense for the equity holders (who control the firm) to make the investment.

Exercise 3 *How might you model this situation using Merton's structural model?*

Other Incentive Problems in Corporate Finance

Incentive problems such as those outlined above can be and often are mitigated in part by the presence of *debt covenants*. Covenants are contracts that specify certain performance criteria that must be met. They often specify a maximum debt-to-asset ratio, for example, or restrict the size of dividend payments relative to working capital etc. Failure to satisfy a covenant will generally make the debt come due immediately. Nonetheless, incentive problems remain and can be found throughout corporate finance.

In the world of banking, for example, it is widely believed that governments will bail out any "too big-to-fail" institutions that are in financial distress. Governments, i.e. tax-payers, therefore provide an implicit subsidy to these institutions as it lowers their cost of capital. Moreover the size of this subsidy increases as the debt-to-equity ratio increases. This as well as the debt-tax-shield (see below) encourages[5] banks to have very high leverage ratios which of course makes them much more likely to fail.

Incentive problems can also arise between the management and the equity-holders of a firm. Corporate governance laws are such that it can be very difficult for the equity holders (who actually own the firm) to exert control over the management of the firm. This has often been blamed for the very high levels of executive compensation in the US where there is often very little relationship between compensation and performance.

From a modeling perspective, these incentive problems are typically handled using game theory / contract theory and other tools from micro-economics.

[5]Bankers have therefore lobbied very hard against regulations that require them to hold much higher levels of equity. See the recent book *"The Bankers' New Clothes"* by Admati and Hellwig for a discussion and compelling arguments as to why banks should be required to hold much higher levels of equity.