Assignment 1

Group 3

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1. Forward Rates

1) If assuming annual compounding interest rate, we can get the expression:
$$(1+s_1)*(1+f_{1,2})=(1+s_2)^2$$
Therefore,
$$f_{1,2}=\frac{(1+s_2)^2}{(1+s_1)}-1=\frac{(1+6.9\%)^2}{(1+6.3\%)}-1=0.075$$

2) If assuming continuous compounding interest rate, we can get the expression:
$$e^{s_1}*e^{f_{1,2}}=e^{2*s_2}$$
We take the logarithm of the formula, in this case, we can obtain:
$$2*s_2=s_1+f_{1,2}$$
Therefore,
$$f_{1,2}=2*6.9\%-6.3\%=0.075$$

2. Pricing a Floating Rate Bond

For the floating rate bond, we have:

coupons: $c_i=r_{i-1}*F \quad (i=1,2,\dots,n)$
principal: F paid at t_n

Let d_i be the discount factor, we can get:
$$V(t=0)=\sum_{i=1}^{n}E(r_{i-1})*F*d_i+F*d_n$$
Because
$$\left(1+f_{0,i-1}\right)*\left(1+E(r_{i-1})\right)=\left(1+f_{0,i}\right)$$
$$d_i=1/(1+f_{0,i})$$
We have
$$E(r_{i-1})=\frac{\left(1+f_{0,i}\right)}{\left(1+f_{0,i-1}\right)}-1=\frac{d_{i-1}}{d_i}-1$$
Hence,
$$V(t=0)=\sum_{i=1}^{n}\left(\frac{d_{i-1}}{d_i}-1\right)*F*d_i+F*d_n$$
$$=\sum_{i=1}^{n}\left(\frac{d_{i-1}}{d_i}-1\right)*F*d_i+F*d_n$$
$$=\sum_{i=1}^{n}(d_{i-1}*F-d_i*F)+F*d_n$$

$$= d_0 * F = F$$

Based on the calculations, the value of the floating rate bond at t=0 is its face value.

3. Mortgage-Mathematics

a) We have the formula:

$$M_k = (1+c)M_{k-1} - B \quad (for\ k = 1, 2, \ldots, n)$$

Since the payment is fully amortised, which means $M_n = 0$, we have:

$$M_k = (1+c)^k M_0 - B[\frac{(1+c)^k - 1}{c}]$$

Because

$$P_k = B - cM_{k-1}$$

We have

$$P_k = B - c * \left((1+c)^{k-1}M_0 - B\left[\frac{(1+c)^{k-1} - 1}{c}\right]\right) = (B - cM_0) * (1+c)^{k-1}$$

Therefore,

$$V_0 = \sum_{k=1}^{n} \frac{P_k}{(1+r)^k} = \sum_{k=1}^{n} (B - cM_0) * \frac{(1+c)^{k-1}}{(1+r)^k}$$

$$= (B - cM_0) * \frac{(1+r)^n - (1+c)^n}{(r-c) * (1+r)^n}$$

Because

$$B = \frac{c(1+c)^n M_0}{(1+c)^n - 1}$$

We have

$$V_0 = \frac{cM_0}{(1+c)^n - 1} * \frac{(1+r)^n - (1+c)^n}{(r-c) * (1+r)^n}$$

b) From the above calculation of V_0, we can see that when n is close to ∞, $(1+r)^n$ and $(1+c)^n$ are all close to infinity. In this sense, the numerator of the formula to compute V_0 is equal to 0.

Therefore,

$$\lim_{n\to\infty} V_0 = 0$$

c) We know the fair mortgage value:

$$F_0 = \sum_{k=1}^{n} \frac{B}{(1+r)^k} = \frac{c(1+c)^n M_0}{(1+c)^n - 1} * \frac{(1+r)^n - 1}{r(1+r)^n}$$

So, we can get the W_0:

$$W_0 = F_0 - V_0 = \frac{c(1+c)^n M_0}{(1+c)^n - 1} * \frac{(1+r)^n - 1}{r(1+r)^n} - \frac{cM_0}{(1+c)^n - 1} * \frac{(1+r)^n - (1+c)^n}{(r-c) * (1+r)^n}$$

$$= \frac{cM_0}{[(1+c)^n - 1] * (1+r)^n} * [\frac{(1+c)^n * [(1+r)^n - 1]}{r} - \frac{(1+r)^n - (1+c)^n}{(r-c)}]$$

d) The duration of PO security is longer than the duration of IO security. Since we are dealing with the standard level-payment mortgage, the payment B of each period is the same and it is used to pay back both interest and principal. In the early periods, large M_k leads to large interests, therefore, the most of B is to pay the interest. As time goes, the interest payment becomes smaller and smaller. While the principal payments are in the opposite way, that is to say, the most of B is to pay the outstanding principal in the latter time periods, and the principal payment accounts for a small portion in the early stage when paying back the mortgage. Hence, we believe that the duration of PO security is longer than that of IO security.

4. Futures Hedging

1) To conduct the minimum-variance hedging, we want to minimise the variance of terminal cashflow, Y_T.

   Since we have:
   $$Y_T = Z_T + h(F_T - F_0)$$
   And
   $$Var(Y_T) = Var(Z_T) + h^2 Var(F_T) + 2hCov(Z_T, F_T)$$

   Therefore, to get the minimum of h by computing the first derivative, we have:
   $$h^* = -\frac{Cov(Z_T, F_T)}{Var(F_T)} = -150000 * \frac{\sigma_{S_T} * \sigma_{F_T} * \rho}{Var(F_T)}$$
   $$= -150000 * \frac{\frac{0.2 * 1.5}{\sqrt{4}} * \frac{0.2 * 1.2}{\sqrt{4}} * 0.7}{[(0.2 * 1.2)/\sqrt{4}]^2} = -131250$$
   Based on the calculations, the minimum variance hedging is -131250 pounds of orange juice futures.

2) After having h^*, we can obtain:
   $$Var(Y_T^*) = Var(Z_T) - \frac{Cov(Z_T, F_T)^2}{Var(F_T)} = Var(Z_T) * (1 - \rho^2) = 0.51 * Var(Z_T)$$
   And
   $$\sigma_{Y_T^*} = 0.714 * \sigma_{Z_T}$$

   Through comparison of $\sigma_{Y_T^*}$ and σ_{Z_T}, we can see that the minimum variance hedging is very effective, and the risk reduced to 71.4% compared to not conducting the hedge.

5. The Die Game

(a)

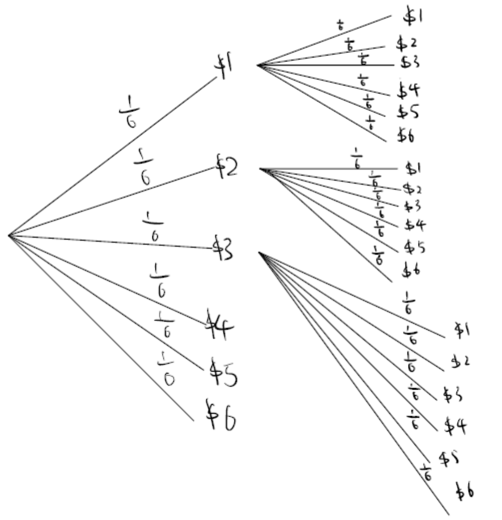
(1). if there is only one rolls, expected value is

$$\frac{1}{6}*1+\frac{1}{6}*2+\frac{1}{6}*3+\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6$$
$$=3.5$$

(2). If there are two throws, the expected value for each throw is 3.5. Therefore, only when the result of the first throw is 1,2, or 3, I will throw the second time, whose expected value is 3.5. The expected value is

$$\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6+$$
$$\frac{1}{6}*\left(\frac{1}{6}*1+\frac{1}{6}*2+\frac{1}{6}*3+\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6\right)*3$$
$$=4.25$$

The diagram below illustrates the calculation above.



(3). If there are three throws, the expected value of the second and third throw is 4.25. Therefore, only when the result of the first throw is 1,2,3 or 4, I will throw the second time. The expected value of the third throw is 3.5. Therefore, only when the result of the second throw is 1,2, or 3, I will throw the third time. Therefore, the expected value is

$$\frac{1}{6}*5+\frac{1}{6}*6+$$
$$\frac{1}{6}*\left(\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6\right)*4+$$
$$\frac{1}{6}*\frac{1}{6}*\left(\frac{1}{6}*1+\frac{1}{6}*2+\frac{1}{6}*3+\frac{1}{6}*4+\frac{1}{6}*5+\frac{1}{6}*6\right)*12$$
$$=\ 4.67$$

The diagram for this calculation is too complicate to be draw by hand. However, the idea is same as part (2) above.
Therefore, if I am risk-neutral, the fair value of this game to me is $4.67.

(b)
Since I am risk averse, I prefer outcomes with low uncertainty to those outcomes with high uncertainty. I will use log (base 10) utility to calculate the expected value of the fair game. First, calculating the log value of all possible payoff:

$$\log(1) = 0, \log(2) = 0.301, \log(3) = 0.477, \log(4) = 0.602, \log(5) = 0.698, \log(6) = 0.778$$

(1). if there is only one rolls, expected value is

$$\frac{1}{6} * \log(1) + \frac{1}{6} * \log(2) + \frac{1}{6} * \log(3) + \frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6) = 0.476$$

(2). If there are two throws, the log of expected value for each throw is 0.476. Therefore, only when the result of the first throw is 1 or 2, I will throw the second time. Therefore, the expected value is

$$\frac{1}{6} * \log(3) + \frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6) + \frac{1}{6} * \left(\frac{1}{6} * \log(1) + \frac{1}{6} * \log(2) + \frac{1}{6} * \log(3) + \frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6)\right) * 2 = 0.585$$

(3). If there are three throws, the expected value of the second and third throw is 0.585. Therefore, only when the result of the first throw is 1,2 or 3, I will throw the second time. The expected value of the third throw is 0.476. Therefore, only when the result of the second throw is 1 or 2, I will throw the third time. Therefore, the expected value is

$$\frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6) + \frac{1}{6} * \left(\frac{1}{6} * \log(3) + \frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6)\right) * 3 + \frac{1}{6} * \frac{1}{6} * \left(\frac{1}{6} * \log(1) + \frac{1}{6} * \log(2) + \frac{1}{6} * \log(3) + \frac{1}{6} * \log(4) + \frac{1}{6} * \log(5) + \frac{1}{6} * \log(6)\right) * 6$$

$$= 0.664$$

We could convert the 10 base log form to the original form:

$$10^{0.664} = 4.61$$

Therefore, if I am risk-averse, the fair value of this game to me is \$4.61.

6. Brownian Motion and Geometric Brownian Motion

(a) To calculate $E_0[B_{t+s}B_s]$, we write $B_{t+s} = (B_{t+s} - B_s + B_s)$

Then, we have

$$\begin{aligned} E_0[B_{t+s}B_s] &= E_0[(B_{t+s} - B_s + B_s) * B_s] \\ &= E_0[(B_{t+s} - B_s) * B_s + B_s * B_s] \\ &= E_0[(B_{t+s} - B_s) * B_s] + E_0[B_s^2] \\ &= E_0[(B_{t+s} - B_s)] * E_0[B_s] + E_0[B_s^2] \end{aligned}$$

Because we know

$$E_0[(B_{t+s} - B_s)] = 0$$
$$E_0[B_s^2] = Var(B_s) + E_0(B_s)^2 = s + 0^2 = s$$

Therefore,

$$E_0[B_{t+s}B_s] = 0 + s = s$$

(b) Because S_{t+s} follows the GBM (μ, σ), we have

$$E_t[S_{t+s}^2] = E_t\left[S_t^2 e^{2\left(\mu - \frac{\sigma^2}{2}\right)s + 2\sigma(B_{t+s} - B_t)}\right]$$

$$= E_t\left[S_t^2 e^{2\left(\mu-\frac{\sigma^2}{2}\right)s}\right] * E_t\left[e^{2\sigma(B_{t+s}-B_t)}\right]$$

According to the MGF of a normal random variable, we know

$$E_t\left[e^{2\sigma(B_{t+s}-B_t)}\right] = E_t\left[e^{s\sigma^2}\right]$$

Therefore,

$$E_t[S_{t+s}^2] = E_t\left[S_t^2 e^{2\left(\mu-\frac{\sigma^2}{2}\right)s}\right] * E_t\left[e^{s\sigma^2}\right] = S_t^2 e^{2\left(\mu-\frac{\sigma^2}{2}\right)s} * e^{s\sigma^2} = S_t^2 e^{2\mu s}$$

7. Tower Property of Conditional Expectations

There are 3 timepoints u, t, v. We know that u ≤ t but have no idea about the relationship between t and v.

1) If we assume that u ≤ t ≤ v:
   From the result of question 6, we know
   $$E_t[S_v^2] = S_t^2 e^{2\mu(v-t)}$$
   Therefore,
   $$E_u\left[E_t[S_v^2]\right] = E_u\left[S_t^2 e^{2\mu(v-t)}\right] = E_u[S_t^2] * E_u\left[e^{2\mu(v-t)}\right]$$

   Because we know
   $$E_u[S_t^2] = S_u^2 e^{2\mu(t-u)}$$
   $$E_u\left[e^{2\mu(v-t)}\right] = e^{2\mu(v-t)}$$
   Therefore,
   $$E_u\left[E_t[S_v^2]\right] = S_u^2 e^{2\mu(t-u)} * e^{2\mu(v-t)} = S_u^2 e^{2\mu(t-u)+2\mu(v-t)} = S_u^2 e^{2\mu(v-u)} = E_u[S_v^2]$$

   Because u ≤ t ≤ v, $E_u\left[E_t[S_v^2]\right] = E_u[S_v^2]$ is abide by the tower property of conditional expectations.

2) If we assume that u ≤ v ≤ t:
   When calculating $E_t[S_v^2]$, we are at time t. Since time t is the latest timepoint, $E_t[S_v^2]$ is a constant value. Then, we have
   $$E_t[S_v^2] = S_v^2$$
   Therefore,
   $$E_u\left[E_t[S_v^2]\right] = E_u[S_v^2]$$

Question 8

Code

For this question, we use Monte-Carlo simulation to estimate the price of an Asian call option. The first step is to estimate the price of underlying stock, and the second step is to price the option.

Estimating the payoff

First, we set all parameters according to the question.

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```
S0 <- 100  # Initial stock price
T <- 1  # time in years
Num_Periods = 11
dt <- T/Num_Periods # time
mu <- .05   # Annualized drift
sigma <- .25 # Annualized volatility
Num_Paths = 10000
```

Second, we estimate the price of underlying stock S_t that could be denoted as a gemetric Brownian motion. The price is stored in a matrix with 10000 rows and 12 columns.

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```
X <- matrix(rnorm(n=Num_Paths*Num_Periods, (mu - (sigma^2)/2)*dt, sigma*sqrt(dt)),Num_Pa
ths,Num_Periods)
X <- cbind(rep(0,Num_Paths),t(apply(X,1,cumsum)))
GBM <- S0*exp(X)
Time_Periods = seq(0,T,dt)
```

Pricing the option.

Now we have estimated the price of underlying stock at different time, we could price the Asian option now. We will price the option for K = 90, 100, 110 and 120.

First, we assume K = 90. The payoff of the Asian call option could be calculated using function: $h(X) = max(\frac{\sum_{i=1}^{m} S_{iT/m}}{m} - K)$.

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```
k=90
Mean_BM = rowMeans(GBM)
h=pmax(Mean_BM-k,0)
```

Second, we calculate the price of Asian call option in each option using funtion $E_0^Q[e^{-rt}h(X)]$.

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```
price=c()
for (i in 1:length(h)) price[i]=exp(-mu*T)*h[i]
price_mean_90=mean(price)
```

Third, we report the 95 CIs for the price of the Asian call option

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```
ci_95_90=t.test(price,conf.level = 0.95)[4]
```

We duplicate the calculation above to other value of K.

K = 100

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```
k=100
h=pmax(Mean_BM-k,0)
price=c()
for (i in 1:length(h)) price[i]=exp(-mu*T)*h[i]
price_mean_100=mean(price)
ci_95_100=t.test(price,conf.level = 0.95)[4]
```

K = 110

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```
k=110
h=pmax(Mean_BM-k,0)
price=c()
for (i in 1:length(h)) price[i]=exp(-mu*T)*h[i]
price_mean_110=mean(price)
ci_95_110=t.test(price,conf.level = 0.95)[4]
```

K = 120

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```
k=120
h=pmax(Mean_BM-k,0)
price=c()
for (i in 1:length(h)) price[i]=exp(-mu*T)*h[i]
price_mean_120=mean(price)
ci_95_120=t.test(price,conf.level = 0.95)[4]
```

The option prices and the corresponding CI for K = 90, 100, 100 and 120 are reported below.

```
When K = 90, the price of Asian call Option is 13.24004 , with CI(95%) [ 12.99503 13.485
05 ]
When K = 100, the price of Asian call Option is 6.804821 , with CI(95%) [ 6.610119 6.999
522 ]
When K = 110, the price of Asian call Option is 2.955695 , with CI(95%) [ 2.821018 3.090
372 ]
When K = 120, the price of Asian call Option is 1.120705 , with CI(95%) [ 1.036572 1.204
838 ]
```