**Solutions to Practice Problems for *Probability Review***

Many of these problems are drawn from Arnold Barnett's excellent and recent textbook *Applied Probability: Models and Intuition* (Dynamic Ideas, LLC). (You do not need access to this book for the course and in fact, it's quite expensive to obtain it here in the U.K. as it needs to be shipped from the U.S.) Questions without "Challenging!" notation are within the exam scope.

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1. **Independence**

   If $A$ and $B$ are independent, show that $A$ and $B^c$ are also independent.

   **Solution**:

   $$\mathrm{P}(A \cap B^c) = \mathrm{P}(A) - \mathrm{P}(A \cap B) = \mathrm{P}(A) - \mathrm{P}(A)\mathrm{P}(B) = \mathrm{P}(A)(1 - \mathrm{P}(B)) = \mathrm{P}(A)\mathrm{P}(B^c).$$

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2. **Conditional Probability**

   An insurance company insures an equal number of male and female drivers. In any given year the probability that a male driver has an accident involving a claim is $\alpha$, independently of other years. The analogous probability for females is $\beta$. Assume the insurance company selects a driver at random. [From Jacod & Protter (2000): *Probability Essentials*]

   (a) What is the probability the selected driver makes a claim in two consecutive years?

   (b) What is the probability that a claimant is female?

   **Solution**: (a) $(\alpha^2 + \beta^2)/2$. (b) $\beta/(\alpha + \beta)$.

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3. **Mean and Variance**

   Show that

   $$\mathrm{Var}(X) := \mathrm{E}(X - \mathrm{E}(X))^2 = \mathrm{E}(X)^2 - \mathrm{E}(X^2).$$

   **Solution**:

   $$\begin{aligned} \mathrm{E}(X - \mathrm{E}(x))^2 &= \mathrm{E}(X^2 - 2X\mathrm{E}(X) + \mathrm{E}(X)^2) \\ &= \mathrm{E}(X^2) - 2\mathrm{E}(X\mathrm{E}(X)) + E(X)^2 \\ &= \mathrm{E}(X^2) - 2E(X)^2 + E(X)^2 \\ &= \mathrm{E}(X)^2 - \mathrm{E}(X^2). \end{aligned}$$

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4. **Mode of Binomial and Poisson Distributions**

For a discrete random variable, define mode as the value with greatest probability, i.e., $\text{argmax}_j\{\text{P}(X = j)\}$. Find the mode of Binomial distribution $B(n, p)$ and the mode of Poisson distribution Poisson($\lambda$).

**Solution**: By calculating $\frac{\text{P}(X=k)}{\text{P}(X=k-1)}$ over $k = 1, \ldots$, we can find the "turning point" which indicates the mode. The mode of $B(n, p)$ is $\lfloor (n+1)p \rfloor$ and the mode of Poisson($\lambda$) is $\lfloor \lambda \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of $x$, or floor of $x$.

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5. **Continuous Random Variable**

Let $X$ be a continuous random variable with density

$$f(x) = \begin{cases} 0 & x < -1 \\ x & -1 \leq x \leq 0 \\ ae^{-bx} & x \geq 0 \end{cases}$$

and expected value $\text{E}(X) = 1$.

(a) What are $a$ and $b$?

**Solution**: We know $\int_{-\infty}^{\infty} f(x)dx = 1$ and $\text{E}(X) = 1$, hence solving

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{0} xdx + a\int_{0}^{\infty} e^{-bx}dx = -\frac{1}{2} + \frac{a}{b},$$
$$1 = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{0} x^2dx + a\int_{0}^{\infty} xe^{-bx}dx = \frac{1}{3} + \frac{a}{b^2},$$

we get

$$a = \frac{27}{8}, \quad b = \frac{9}{4}.$$

(b) Find $\text{Var}(X)$.

**Solution**: Having known $\text{E}(X) = 1$, we just need $\text{E}(X^2)$.

$$\text{E}(X^2) = \int_{-1}^{0} x^3dx + \frac{27}{8}\int_{0}^{\infty} x^2e^{-\frac{9}{4}x}dx = -\frac{1}{4} + \frac{8}{27}.$$

Hence

$$\text{Var}(X) = \text{E}(X^2) - \text{E}(X)^2 = -\frac{1}{4} + \frac{8}{27} - 1 = -\frac{103}{108}.$$

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6. **Normal Distribution**

Answer the following questions:

(a) Suppose that the height ($X$) in inches, of 25-year-old men is a normal random variable with mean $\mu = 70$ inches. If $\mathrm{P}(X > 79) = 0.025$, what is the standard deviation of $X$?

(b) Suppose that the weight ($Y$) in pounds, of 40-year-old men is a normal random variable with standard deviation $\sigma = 20$ pounds. If 5% of this population weigh less than 160 pounds, what is the mean $\mu$ of $Y$?

(c) Find an interval that covers the midel 95% of $Z \sim \mathcal{N}(64, 8)$.

**Solution**:

(a) $1.96 = \frac{79-70}{\sigma} \Rightarrow \sigma = 4.59$.

(b) $-1.645 = \frac{160-\mu}{20} \Rightarrow \mu = 192.9$.

(c) We have 2.5% probability at each one of the two tails. Therefore

$$\pm 1.96 = \frac{x - 64}{8} \Rightarrow x = 64 \pm 1.96 \times 8 = 48.32 \text{ or } 79.68.$$

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7. **Exponential Distribution**

Let $X \sim \exp(\lambda)$ and $Y \sim \exp(\mu)$ be two independent exponential random variables.

(a) What is the distribution of $\min\{X, Y\}$?

**Solution**:

$$\begin{aligned} \mathrm{P}(\min\{X,Y\} > x) &= \mathrm{P}(X > x, Y > x) \\ \text{(By independence)} &= \mathrm{P}(X > x)\mathrm{P}(Y > x) \\ &= e^{-\lambda x} e^{-\mu x} \\ &= e^{-(\lambda+\mu)x}. \end{aligned}$$

Hence $\min\{X, Y\} \sim \exp(\lambda + \mu)$ is also an exponential random variable.

(b) What is the probability that $X$ is the smaller one between $X$ and $Y$? Under $\lambda = \mu$?

**Solution**:

$$\begin{aligned} \mathrm{P}(X = \min\{X,Y\}) = \mathrm{P}(X \le Y) &= \int_0^\infty \mathrm{P}(X \le Y \mid Y = y) f_Y(y) dy \\ &= \int_0^\infty (1 - e^{-\lambda y}) \mu e^{-\mu y} dy \\ &= \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

Clearly if $\lambda = \mu$, the probability is 1/2, which is intuitive.

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8. **Getting out of the Jungle**

   You are lost in a jungle and just arrive at a crossroads with 3 choices. One of them will bring you back to the same point after 1 hour of walk; the second will bring you back after 6 hours of walk; the last will lead you out of the jungle after 2 hours of walk. Facing no signs on the roads you have to choose one randomly each time you are there. What is the expected time until you get out of the jungle?

   **Solution**: Let $T$ be the time until you get out of the jungle and $R_i$ be the choice of road $i$ for $i = 1, 2, 3$. Using conditional expectation, we have

$$\begin{aligned} \mathrm{E}(T) &= \mathrm{E}(T \mid R_1)\mathrm{P}(R_1) + \mathrm{E}(T \mid R_2)\mathrm{P}(R_2) + \mathrm{E}(T \mid R_2)\mathrm{P}(R_2) \\ &= \frac{1}{3}[(1 + \mathrm{E}(T)) + (6 + \mathrm{E}(T)) + 2] \\ &= \frac{1}{3}[9 + 2\mathrm{E}(T))]. \end{aligned}$$

   Solving this we get $\mathrm{E}(T) = 9$.

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9. **Genetics**

   The geneticist Gregor Mendel analyzed the inheritance of traits from parents to their offspring. The basic principles are:

   - Each of the two parents has two genes on a given dimension (e.g. color), and each offspring gets one gene from each parent.
   - Each of a particular parent's two genes has an equal chance of being passed on to the offspring.
   - The genetic contributions from the two parents are independent.

   There are two different types of genes, *dominant* and *recessive*, associated with different traits. (For example, for the colours of peas, *yellow* is dominant and *green* is recessive.) If an offspring gets one dominant gene and one recessive one, then it will exhibit the dominant trait, as it will if it gets two dominant genes from its parents. Only if both genes are recessive will the recessive trait show up.

   Some questions:

   (i) Suppose that one parent pea plant has one yellow gene and one green one, i.e. is *hybrid*, and that the other has two green genes. If they "mate", what is the probability that their offspring (named Zelda) will be green? (Actually a pea plant is self-pollinating, and can mate with itself. But,as Mendel literally did, we will create male and female pea-plants for our discussion.)

   **Solution:** We are told the parents have genetic makeup $(Y, G)$ and $(G, G)$. Moreover the offspring will be green if and only if its genetic makeup is $(G, G)$. This will occur with probability $1 \times 1/2 = 1/2$ given the independence assumption.

(ii) Suppose that the overall population of pea plants is 25% all yellow, 50% hybrid yellow and green, and 25% all green in genetic makeup. If Zelda is green and picks a mate at random, what is the probability that they have a green offspring?

**Solution**: Note that Zelda must have genetic makeup $(G, G)$. We can calculate her probability of having a green offspring as

$$\begin{aligned}
\mathrm{P}(\text{green offspring}) &= \mathrm{P}(\text{green offspring} \mid \text{mate is } GG)\mathrm{P}(\text{mate is } GG) \\
&\quad + \mathrm{P}(\text{green offspring} \mid \text{mate is } YG)\mathrm{P}(\text{mate is } YG) \\
&\quad + \mathrm{P}(\text{green offspring} \mid \text{mate is } YY)\mathrm{P}(\text{mate is } YY) \\
&= (1 \times .25) + (.5 \times .5) + (0 \times .25) \\
&= 0.5.
\end{aligned}$$

(iii) Now suppose that Zelda is green and picks a mate at random from among those who are yellow (her favourite colour). What is the probability that their offspring will be green?

**Solution**: Zelda's mate must be either $(Y, Y)$ or $(Y, G)$ since they are the genetic make-ups that yield a yellow pea. We must first compute these probabilities conditional on being yellow. We have

$$\begin{aligned}
\mathrm{P}((Y,G) \mid \text{yellow}) &= \mathrm{P}((Y,G) \mid (Y,G) \text{ or } \ (Y,Y)) \\
&= \frac{\mathrm{P}((Y,G) \cap \{(Y,G) \text{ or } \ (Y,Y))\}}{\mathrm{P}((Y,G) \text{ or } \ (Y,Y))} \\
&= \frac{\mathrm{P}((Y,G)}{\mathrm{P}((Y,G) \text{ or } \ (Y,Y))} \\
&= \frac{.5}{.75} \\
&= \frac{2}{3}.
\end{aligned}$$

Therefore we also have $\mathrm{P}((Y,Y) \mid \text{yellow}) = 1/3$. We can now compute

$$\begin{aligned}
\mathrm{P}(\text{green offspring} \mid \text{yellow}) &= (0 \times 1/3) + (1/2 \times 2/3) \\
&= 1/3.
\end{aligned}$$

(iv) Assuming the 25/50/25 split among pea plants in a given generation, will green peas be less common in the next generation if mating is random. Are they on a path towards eventually dying out?

**Solution**: There are three routes to a green offspring:

- Both parents are $(G, G)$ and both parents donate a green gene to the offspring. This occurs with probability $(1/4) \times (1/4) \times 1 \times 1 = 1/16$.

- One parent is $(G, G)$ and the other parent is $(Y, G)$ and both parents donate a green gene to the offspring. This occurs with probability $2 \times (1/4) \times (1/2) \times 1 \times (1/2) = 2/16$. (The factor of 2 arises because we can have a $(Y, G)$ father and $(G, G)$ mother or we can have a $(Y, G)$ mother and $(G, G)$ father.)
- Both parents are $(Y, G)$ and both parents donate a green gene to the offspring. This occurs with probability $(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$.

Because these possibilities are mutually exclusive we have

$$\begin{aligned} \text{P(green offspring)} &= \frac{1}{16} + \frac{2}{16} + \frac{1}{16} \\ &= 1/4. \end{aligned}$$

So under random mating the next generation will also be 25% green and (with a little more work) we could conclude that there is no tendency for green pea plants to die out over time.

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10. **Mendel's Sure Thing**

Mendel has learned about a betting scheme for "double or nothing bets" that seems to ensure victory. On the first round, the player bets \$1. If he wins, he stops, having won \$2 and thus having achieved a net gain of 2-1=\$1. If he loses, he bets \$2 the next time. If that second bet is successful, he stops, having won \$4, and thus achieving a *net* gain of (4-2)-1 = \$1. If the second bet also fails, he bets \$8 on the third round; if he wins then, his net gain over the three rounds is (8-4)-(1+2) =\$1 and he stops. If he loses on the third round, he bets \$16 on the fourth round ...

In short, the strategy is: if the player gets to the $n^{th}$ bet (having lost the previous $n-1$ bets), he makes a bet of $2^{n-1}$ dollars and:

- If he wins, he collects $2^n$ dollars and stops.
- If he loses, he goes on to an $(n+1)^{st}$ bet and then bets $2^n$ dollars.

(i) If Mendel can play this game forever, is he assured of winning \$1?

**Solution**: If Mendel wins for the first time on the $n^{th}$ bet then his net winnings will be

$$\begin{aligned} \left(2^n - 2^{n-1}\right) - \left(1 + 2 + \cdots + 2^{n-2}\right) &= 2^n - \left(1 + 2 + \cdots + 2^{n-1}\right) \\ &= 2^n - (2^n - 1) \\ &= 1. \end{aligned}$$

This means that regardless of when Mendel first wins, his net winnings will be \$1. Will he eventually win at some point? Yes! To see this note that the only way that Mendel can fail to win at some point is that if he loses infinitely many independent bets each of which has probability 1/2. The probability of losing $n$ bets in a row is $(1/2)^n$ and so the probability of losing infinitely many of them is

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0.$$

(ii) Suppose that Mendel only has \$1,023, and thus cannot play forever. Under this restriction:

- What is the probability distribution for his net gain (or loss) until he stops playing?
- What is his expected gain (or loss) until he stops?

**Solution**: Note that $1{,}023 = 2^n - 1$ where $n = 10$. This means that Mendel can make at most 10 bets since if he loses all 10 of them (which will occur with probability $(1/2)^{10} = 1{,}024$) he will have lost

$$1 + 2 + \cdots + 2^9 = 2^{10} - 1 = 1{,}023$$

and therefore have no money left to gamble. If he wins on or before the $10^{th}$ bet then, as before, his net winnings will be \$1. Let $U$ then be his net winnings. It follows that the distribution of $U$ is

$$U = \begin{cases} -1023, & \text{w.p. } \frac{1}{1024} \\ 1, & \text{w.p. } \frac{1023}{1024}. \end{cases}$$

We can now also compute the expected gain / loss until Minerva stops. It satisfies

$$\begin{aligned} \mathrm{E}[U] &= -1023 \times \frac{1}{1024} + 1 \times \frac{1023}{1024} \\ &= 0 \end{aligned}$$

so that on average, Minerva wins nothing!

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11. **Traffic Congestion**

Mendel and Minerva go together to the Bureau of Motor Vehicles (BMV), where they must take care of matters concerning their vehicles and drivers' licenses. They know from experience that the time it will take to complete their business is quite unpredictable: in fact, the transaction time (entry to exit) is exponentially distributed with a mean of 20 minutes, i.e. $\lambda = 1/20$, for each of them. Because they are going on different lines and dealing with different people, their transaction times are independent. Some questions about their visit:

(i) What is the probability that neither of them will be finished within 20 minutes?

**Solution**: Let us use $X_1$ and $X_2$ to denote Mendel's and Minerva's service times, respectively. Because of independence we have

$$\begin{aligned} \mathrm{P}(\text{both times exceed } 20) &= \mathrm{P}(X_1 > 20) \times \mathrm{P}(X_2 > 20) \\ &= e^{-\lambda x} \times e^{-\lambda x} \\ &= e^{-2\lambda x}. \end{aligned}$$

where $x = 20$ and $\lambda = 1/20$. Simplifying we obtain $\mathrm{P}(\text{both times exceed } 20) = e^{-2} = 0.14$.

(ii) What is the probability that Minerva will be finished in 20 minutes but Mendel will not be done by then?

**Solution**: Again by independence we have (with $x = 20$ and $\lambda = 1/20$)

$$\begin{aligned} \mathrm{P}(\text{Minerva finished in 20 but Mendel not}) &= \mathrm{P}(X_2 \le 20) \times \mathrm{P}(X_1 > 20) \\ &= \left(1 - e^{-\lambda x}\right) \times e^{-\lambda x} \\ &= \left(1 - e^{-1}\right) \times e^{-1} \\ &= 0.23. \end{aligned}$$

(iii) What is the probability that Mendel's transaction time will be at least ten minutes longer than Minerva's?

**Solution**: In order for this to occur Minerva must finish first (which occurs with probability 1/2) and then Mendel must take at least 10 additional minutes. Let's define two events:

$A$: Minerva finishes first

$B$: Mendel takes at least 10 minutes longer than Minerva

We want $\mathrm{P}(B)$ and we can calculate it as follows.

$$\begin{aligned} \mathrm{P}(B) &= \mathrm{P}(A \cap B) \\ &= \mathrm{P}(B \mid A)\mathrm{P}(A) \\ &= e^{-\lambda \times 10} \times \frac{1}{2} \\ &= 0.31 \end{aligned}$$

where we have used the memoryless property of the exponential distribution is obtaining that $\mathrm{P}(B \mid A) = e^{-\lambda \times 10}$. (This is because at the moment Minerva finishes, Mendel's remaining time to completion has an $\mathrm{Exp}(\lambda)$ distribution.)

(iv) Zanzibar is driving over to pick Mendel and Minerva up after they are finished. Because no vehicle standing is allowed outside the BMV, he wants to arrive at time $T$ when the probability is 95% that both of them are finished (and thus are waiting outside). Assuming that they reach the BMV at 1pm, what time $T$ satisfies his requirement?

**Solution**: We want the number $\tau$ for which $\mathrm{P}(X_1,\, X_2 \le \tau) = 0.95$. By independence we have

$$\begin{aligned} \mathrm{P}(X_1,\, X_2 \le \tau) &= \mathrm{P}(X_1 \le \tau)\,\mathrm{P}(X_2 \le \tau) \\ &= \left(1 - e^{-\frac{\tau}{20}}\right)^2. \end{aligned} \tag{1}$$

We therefore set the r.h.s. of (1) equal to 0.95 and solve for $\tau$. We find $\tau = 73.8$ minutes.

(v) When Zanzibar arrives at that time $T$, what is the probability that Mendel and Minerva are waiting and both of them have waited more than 30 minutes?

**Solution**: This occurs with probability

$$\begin{aligned} \mathrm{P}(X_1,\, X_2 \le 43.8) &= \mathrm{P}(X_1 \le 43.8)\,\mathrm{P}(X_2 \le 43.8) \\ &= \left(1 - e^{-\frac{43.8}{20}}\right)^2 \\ &= 0.79. \end{aligned}$$

So there is a roughly 4 in 5 chance that *both* of them will wait at least half an hour!

(vi) Let random variable $Z$ be the amount of time until the first of them is finished. What is the probability distribution of $Z$?

**Solution**: We have

$$\begin{aligned} \mathrm{P}(Z \ge z) &= \mathrm{P}(X_1 \ge z, X_2 \ge z) \\ &= \mathrm{P}(X_1 \ge z)\mathrm{P}(X_2 \ge z) \\ &= e^{-z\lambda} \times e^{-z\lambda} \\ &= e^{-z(2\lambda)} \end{aligned}$$

and so

$$\mathrm{P}(Z \le z) = 1 - e^{-z(2\lambda)}$$

which we recognize as the CDF of an $\mathrm{Exp}(2\lambda)$ random variable. Therefore $Z \sim \mathrm{Exp}(2\lambda)$.

This is a special case of a very important and powerful result:

> If $X_1 \sim \mathrm{Exp}(\lambda_1)$, $X_2 \sim \mathrm{Exp}(\lambda_2)$ and $X_1$ and $X_2$ are independent, then $Z \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$ where $Z := \min\{X_1, X_2\}$.

(vii) How long on average will it be until both of them are finished?

**Solution**: Using the previous result we know the expected time until the first one to finish will be $\mathrm{E}[Z] = 1/(2\lambda) = 10$ minutes where $Z := \min\{X_1, X_2\}$. By the memoryless property, the second one to finish will then an additional $1/\lambda = 20$ minutes on average. So the expected completion time for both to finish is $10 + 20 = 30$ minutes.

12. **California Nightmare**

Some geological measurements suggest that maximum-strength earthquakes occur on the southern end of California's San Andreas fault every 160 years on average. But the individual intervals between "Big Ones" vary a bit around this mean. Suppose that an adequate model posits that, given that a huge earthquake has just occurred, the time $X$ until the next one follows a normal distribution with mean 160 and standard deviation 30 years, i.e. $X \sim \mathrm{N}(160, 30)$.

The last earthquake on the southern San Andreas Fault – which winds through populous Southern California – occurred in 1857. What is the probability that the next large quake will take place within the next decade? (It is mid-2014 as of this writing.)

**Solution**: We define two events:

$E$: $X > 157$, which corresponds to no earthquake from (mid) 1857 to mid-2014.

$F$: $157 < X < 167$, which entails a big earthquake between mid-2014 and mid-2024.

What we are seeking is $\mathrm{P}(F \mid E)$. This follows from Bayes' Theorem since

$$\mathrm{P}(F \mid E) = \frac{\mathrm{P}(E \mid F)\mathrm{P}(F)}{\mathrm{P}(E).}$$

We know $\mathrm{P}(E \mid F) = 1$ (why?) and we can use `R` to obtain that $\mathrm{P}(F) \approx .132$ and $\mathrm{P}(E) \approx .540$. We conclude that $\mathrm{P}(F \mid E) \approx 0.244$.