# Financial Analytics

## An Introduction to Derivatives Pricing in the Binomial and Black-Scholes Models


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# Outline

# Introduction to Options

**Definition.** A European call option gives the right but not the obligation to purchase $1$ unit of the underlying security at a pre-specified price $K$ at a pre-specified time $T$.

**Definition.** An American call option gives the right but not the obligation to purchase $1$ unit of the underlying security at a pre-specified price $K$ at any time up to and including a pre-specified time $T$.

**Definition.** A European put option gives the right but not the obligation to sell $1$ unit of the underlying security at a pre-specified price $K$ at a specified time $T$.

**Definition.** An American put option gives the right but not the obligation to sell $1$ unit of the underlying security at a pre-specified price $K$ at any time up to and including a pre-specified time $T$.

$K$ and $T$ are called the strike and maturity or expiration, respectively.

# Payoff and Intrinsic Value of European Call and Puts

Payoff of a European call option at expiration $= \max\{S_T - K, 0\}$

Intrinsic value of a call option at time $t \leq T$ is $\max\{S_t - K, 0\}$

- In the money: $S_t > K$
- At the money: $S_t = K$
- Out of the money: $S_t < K$

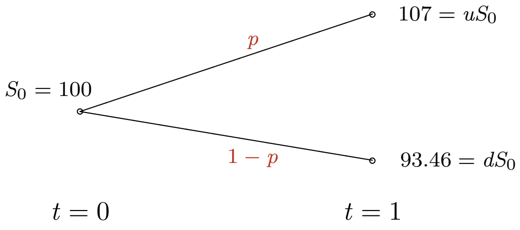
Payoff of a European put option at expiration $= \max\{K - S_T, 0\}$

Intrinsic value of a put option at time $t \leq T$ is $\max\{K - S_t, 0\}$

- In the money: $S_t < K$
- At the money: $S_t = K$
- Out of the money: $S_t > K$

Options have nonlinear payoffs so cannot price them without a model for the underlying security.

# The 1-Period Binomial Model




- Can borrow or lend at gross risk-free rate of $R$ per period
  - so \$1 in cash account at $t = 0$ is worth \$$R$ at $t = 1$
- Also assume that short-sales are allowed.

Some interesting questions now arise:

1. How much is a call option that pays $\max(S_1 - 107,\ 0)$ at $t = 1$ worth?
2. How much is a call option that pays $\max(S_1 - 92,\ 0)$ at $t = 1$ worth?

Pricing these options is easy but to price options in general we need concept of arbitrage.

# Type A Arbitrage

**Definition.** A type $A$ arbitrage is a security or portfolio that produces immediate positive reward at $t = 0$ and has non-negative value at $t = 1$.

**i.e.** a security with initial cost $V_0 < 0$ and time $t = 1$ value $V_1 \geq 0$.

**Example of type A arbitrage:**

- Somebody walks up to you on the street, gives you a positive amount of cash, and asks for nothing in return, either then or in the future.

# Type B Arbitrage

**Definition.** A type $B$ arbitrage is a security or portfolio that has a non-positive initial cost, has positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then.

**i.e.** a security with initial cost $V_0 \leq 0$, and $V_1 \geq 0$ but $V_1 \neq 0$.

**Examples of type B arbitrage:**

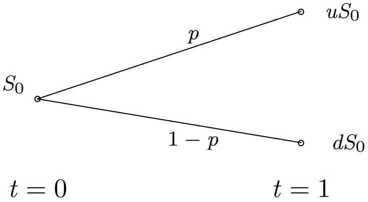
- A stock that costs nothing, but that will possibly generate dividend income in the future.
- A free lottery ticket.

# Arbitrage in the 1-Period Binomial Model

In finance we always assume that arbitrage opportunities do not exist since if they did, market forces would quickly act to dispel them.

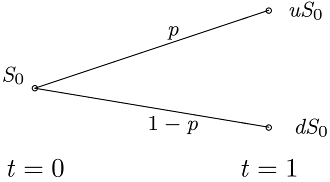
Absence of arbitrage is the key assumption that allows us to price derivative securities in many financial models.

**Theorem:** If there is no arbitrage in the 1-period binomial model then we must have $d < R < u$.




- Recall we can borrow or lend at gross risk-free rate $R$ per period.
- And short-sales are allowed.

# Arbitrage in the 1-Period Binomial Model



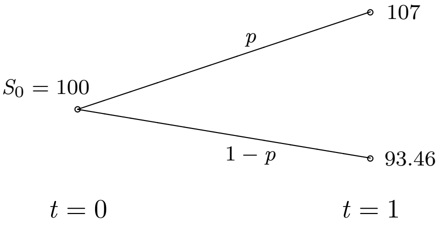

**Proof:** Suppose not the case that $d < R < u$. Then have two possibilities:

1. $R \leq d < u$: Then at $t = 0$ borrow $S_0$ and buy 1 share of the stock.

2. $d < u \leq R$: Then short-sell one share of stock at $t = 0$ and invest proceeds in cash-account.

In both cases we have a type B arbitrage and so the result follows.

# Option Pricing in the 1-Period Binomial Model




Assume now that $R = 1.01$.

1. How much is a call option that pays $\max(S_1 - 102,\ 0)$ at $t = 1$ worth?
2. How will the price vary as $p$ varies?

To answer these questions, we will construct a replicating portfolio.

# The Replicating Portfolio

- Consider buying $x$ shares and investing $\$y$ in cash at $t = 0$
- At $t = 1$ this portfolio is worth:

$$\begin{array}{rl} 107x + 1.01y & \text{when } S = 107 \\ 93.46x + 1.01y & \text{when } S = 93.46 \end{array}$$

- Can we choose $x$ and $y$ so that portfolio equals option payoff at $t = 1$?
- If so, then we must solve

$$\begin{aligned} 107x + 1.01y &= 5 \\ 93.46x + 1.01y &= 0 \end{aligned}$$

The solution is

$$\begin{aligned} x &= 0.3693 \\ y &= -34.1708 \end{aligned}$$

So yes, we can construct a replicating portfolio!

# The Replicating Portfolio

**Question:** What does a negative value of $y$ mean?

**Question:** What would a negative value of $x$ mean?

- The cost of this portfolio at $t = 0$ is

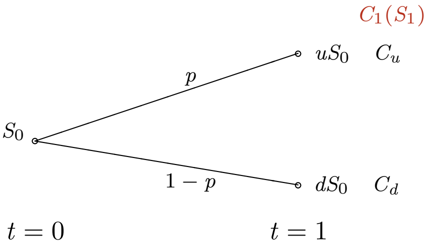
$$0.3693 \times 100 - 34.1708 \times 1 \approx 2.76$$

- So the fair value of the option is $2.76$
  - indeed $2.76$ is the arbitrage-free value of the option.

- Therefore option price does not directly depend on buyer's (or seller's) utility function.

- Nor does it depend on the true probabilities $p$ and $1 - p$ of up- and down-moves, respectively!

# Derivative Security Pricing




- Can use same replicating portfolio argument to find price $C_0$ of any derivative security with payoff function $C_1(S_1)$ at time $t=1$.
- Set up replicating portfolio as before:

$$\begin{aligned} uS_0x + Ry &= C_u \qquad (1) \\ dS_0x + Ry &= C_d \qquad (2) \end{aligned}$$

- Solve for $x$ and $y$ as before and then must have $C_0 = xS_0 + y$.

# Derivative Security Pricing

- After solving (1) and (2) can easily check(!)

$$\begin{aligned} C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\ &= \frac{1}{R}\left[q C_u + (1-q) C_d\right] \\ &= \frac{1}{R}\mathsf{E}_0^Q[C_1]. \end{aligned} \tag{3}$$

- Note that if $d < R < u$ then $q > 0$ and $1-q > 0$ and (3) implies (why?) there can be no-arbitrage.

# Derivative Security Pricing

Have therefore established converse to our earlier theorem:

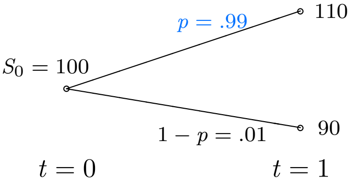
**Theorem.** If $d < R < u$ in the 1-period binomial model then there is no arbitrage.

We refer to (3) as risk-neutral pricing and $(q, 1-q)$ are the risk-neutral probabilities.

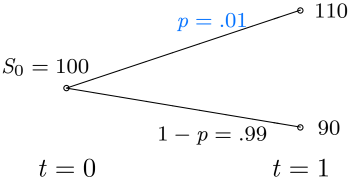
So we now know how to price any derivative security in this $1$-period model.

# Derivatives Prices Do Not Depend on $p$!

- Stock ABC




- Stock XYZ

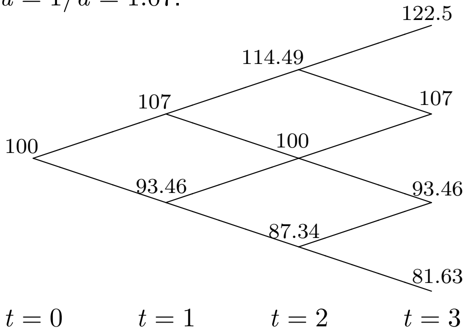


# Derivatives Prices Do Not Depend on $p$!

**Question:** How does the price of a call option on ABC with strike $K = \$100$ compare to the price of a call option on XYZ with strike $K = \$100$?

# A 3-period Binomial Model

Let $R = 1.01$ and $u = 1/d = 1.07$.

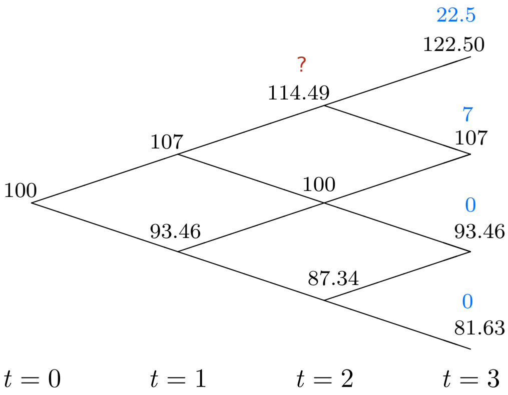



### Some Observations

- Multi-period binomial model is just a series of 1-period models spliced together!
- Just need to multiply 1-period probabilities along branches to get probabilities in multi-period model.
- Results from the 1-period model carry over to multi-period model.

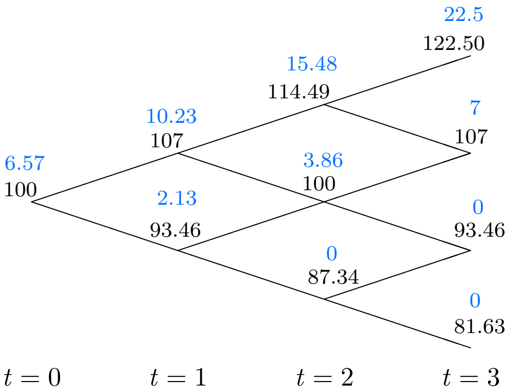
# Pricing a European Call Option

Assumptions: expiration at $t = 3$, strike $= \$100$ and $R = 1.01$.



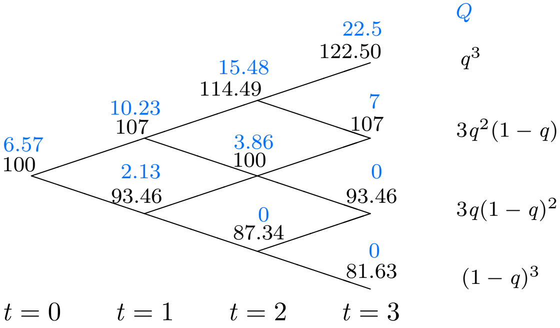

Can price option by working backwards in the lattice starting at $t = 3$ and using what we know about 1-period binomial models to obtain price at each prior node.

# Pricing a European Call Option




Do this repeatedly until we obtain arbitrage-free price at $t = 0$. The price of the option at each node is displayed above the underlying stock price in the binomial model to the right. Note that we repeatedly used our single-period option pricing formula to obtain these prices.

# Pricing a European Call Option




But can also calculate the price directly as

$$C_0 = \frac{1}{R^3} \mathsf{E}_0^{Q} \left[\max(S_T - 100,\ 0)\right] \tag{4}$$

- This is risk-neutral pricing in the binomial model
- Avoids having to calculate the price at every node.

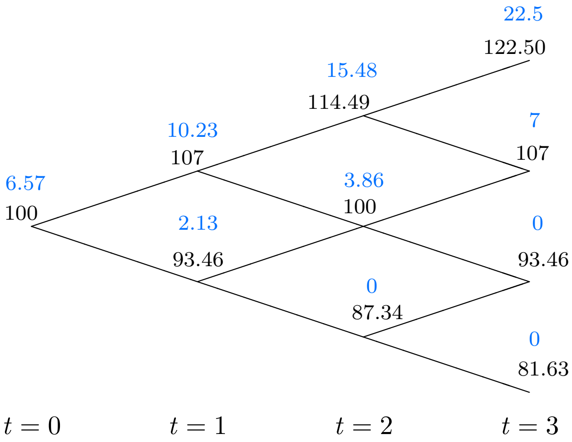
# Replicating Strategies

**Definition.** A self-financing (s.f.) trading strategy over the time interval $[0, T]$ is one where any gains or losses in the value of the portfolio are due entirely to gains or losses due to trading. In particular, no new cash is added to or withdrawn from the portfolio at any time.

**Definition.** A replicating strategy for a particular security with payoff $C_T$ at time $T$ is a s.f. strategy whose value at time $T$ is equal to $C_T$, i.e. the strategy replicates $C_T$.

**Question:** How would you find a replicating strategy for the option on the previous slide?

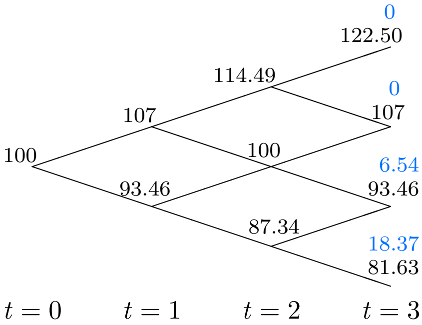
# Pricing a European Call Option



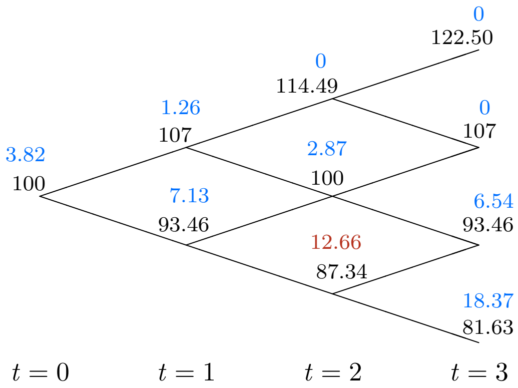
# Pricing American Options

- Can also price American options in same way as European options
  - but must also check if it's optimal to early exercise at each node.

**Example:** Price American put: expiration at $t = 3$, $K = \$100$ and $R = 1.01$.



# Pricing American Options




Price option by working backwards in the binomial lattice.

**e.g.** $$12.66 = \max\left\{12.66,\ \frac{1}{R}\left(q \times 6.54 + (1-q) \times 18.37\right)\right\}$$

where $12.66 = 100 - 87.34$ is the intrinsic value of the option at that node.

# Pricing American Options

More generally, the value, $V_t(S)$, of the American put option at any time $t$ node when the underlying price is $S$ can be computed according to

$$\begin{aligned} V_t(S) &= \max\left\{K - S,\ \frac{1}{R}\left[q \times V_{t+1}(uS) + (1-q) \times V_{t+1}(dS)\right]\right\} \\ &= \max\left\{K - S,\ \frac{1}{R}\mathsf{E}_t^Q\left[V_{t+1}(S_{t+1})\right]\right\}. \end{aligned}$$

The problem of pricing an American option is an optimal stopping problem which itself is a specific type dynamic programming (DP) problem.

# Complete versus Incomplete Markets

Assume there are no arbitrage opportunities.

**Definition.** If every security $C_T$ can be replicated (via a s.f. strategy), then we say that we have a complete market. Otherwise we have an incomplete market.

Complete markets are very convenient because every derivative security in a complete market can be priced

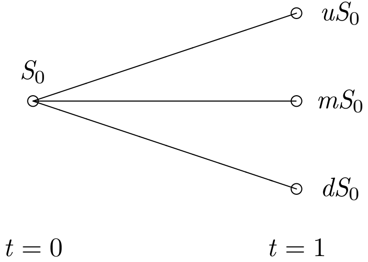
- by simply computing initial value of the s.f. strategy that replicates it.

**e.g.** The binomial model is a complete model as indeed is the famous Black-Scholes model.

In the real world, however, markets are incomplete but derivatives prices are often found by assuming they are complete.

# An Incomplete Market - the Trinomial Model

The 1-period trinomial model is identical to the 1-period binomial model except that there are now three possibilities for the time $t = 1$ security price $S_1$. As before, no-arbitrage implies $d < R < u$.




Option prices cannot in general be computed because three linear equations in two unknowns do not (in general) have a unique solution

- so cannot replicate time $t = 1$ payoff of an option in the trinomial model.

In arbitrage-free incomplete markets there are many sets of r.n. probabilities

- pricing of derivatives often done by choosing one particular set by calibrating model prices to market prices.

# Put-Call Parity

- $p_E(t; K, T)$ and $c_E(t; K, T)$ denote prices of European put and call.
- $p_A(t; K, T)$ and $c_A(t; K, T)$ denote prices of American put and call.
- $d(t, T)$ is the discount factor for discounting cash-flows from $T$ back to $t$.

**Theorem:** European put-call parity at time $t$ for non-dividend paying stock:

$$p_E(t; K, T) + S_t = c_E(t; K, T) + Kd(t, T)$$

**Proof:** Consider following trading strategy:

- At time $t$ buy European call with strike $K$ and expiration $T$
- At time $t$ sell European put with strike $K$ and expiration $T$
- At time $t$ (short) sell $1$ unit of underlying and buy at time $T$
- At time $t$ lend $d(t, T)K$ dollars until time $T$

Cash flow at time $T$: $\max\{S_T - K, 0\} - \max\{K - S_T, 0\} - S_T + K = 0$.

So value at time $t$ must also (why?) equal 0:

$$-c_E(t; K, T) + p_E(t; K, T) + S_t - Kd(t, T) = 0.$$

# Bounds on European Option Prices

- Suppose underlying security does not pay dividends.
- Suppose also the events $\{S_T > K\}$ and $\{S_T < K\}$ have strictly positive probability – a very reasonable assumption which implies

$$\begin{aligned} c_E(t; K, T) &> 0 && (5) \\ p_E(t; K, T) &> 0 && (6) \end{aligned}$$

Can then use put-call parity to obtain

$$\begin{aligned} c_E(t; K, T) &= S_t + p_E(t; K, T) - Kd(t, T) \\ &> S_t - Kd(t, T) \qquad \text{by (6)} \\ &\geq S_t - K \quad \text{since } R \geq 0. && (7) \end{aligned}$$

Consider now corresponding American call option:

$$\begin{aligned} c_A(t, K, T) &\geq c_E(t; K, T) \\ &> \max\left\{S_t - K, 0\right\} \quad \text{by } (5) \text{ and } (7). \end{aligned}$$

# Bounds on European Option Prices

So price of American call option on non-dividend paying stock strictly greater than intrinsic value when events $\{S_T > K\}$ and $\{S_T < K\}$ have strictly positive probability.

We have shown ...

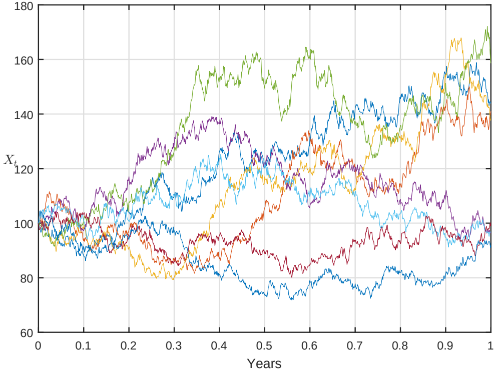
**Theorem.** It's never optimal to early-exercise an American call on a non-dividend paying stock and so $c_A(t; K, T) = c_E(t, K, T)$.

No such result holds for American put options.

# The Black-Scholes Model

The famous Black-Scholes model for option pricing assumes:

- The stock price $S_t$ satisfies $S_t \sim \text{GBM}(\mu, \sigma)$
- The stock continuously pays a dividend of $cS_t\,dt$ at each time $t$
- Continuous and frictionless trading
- There is a cash account available so that borrowing / lending at the (continuously-compounded) risk-free rate $r$ is possible.



# The Black-Scholes Option Price

Black and Scholes (1973) then showed that the arbitrage-free price of a European call option with strike $K$ and maturity $T$ is given by

$$C(S,t) \;=\; e^{-c(T-t)} S \Phi(d_1) \;-\; e^{-r(T-t)} K \Phi(d_2) \tag{8}$$

$$\text{where} \quad d_1 \;=\; \frac{\log\left(\frac{S}{K}\right) + (r - c + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 \;=\; d_1 - \sigma\sqrt{T-t}.$$

and where $\Phi(\cdot)$ is the standard normal CDF.

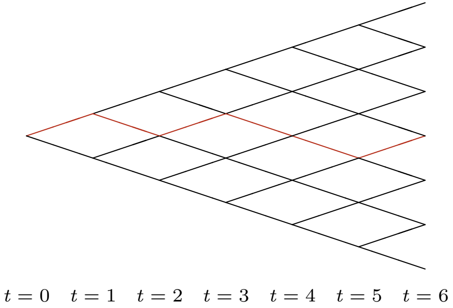
Price of European put-option easily computed from put-call parity and (8).

# The Black-Scholes Option Price

- Most interesting feature of the Black-Scholes formula is that $\mu$ does not appear anywhere!

- This is completely analogous to binomial option-price not depending on probability of an up-move $p$.

- In fact can obtain the Black-Scholes formula as the limit of (suitably calibrated) binomial option-price when $\Delta t \to 0$.



# Calibrating the Binomial Model to GBM

- Often wish to calibrate binomial model so that its dynamics match GBM.
- One common way of doing this is to set

$$
\begin{aligned}
p &= \frac{e^{\mu \Delta t} - d}{u - d} \\
u &= \exp(\sigma \sqrt{\Delta t}) \\
d &= 1/u \; = \; \exp(-\sigma \sqrt{\Delta t})
\end{aligned}
\tag{9}
$$

  where $\Delta t$ is the length of a period.
- Note then that

$$
\begin{aligned}
\mathsf{E}[S_{i+1} \mid S_i] &= puS_i + (1-p)\, dS_i \\
&= S_i \exp(\mu \Delta t)
\end{aligned}
$$

  as desired. Can also check that $\mathsf{E}[S_{i+1}^2 \mid S_i]$ also matches GBM.
- Gross risk-free rate per period $R$ chosen according to

$$
R = e^{r \Delta t}.
$$

# Calibrating the Binomial Model to GBM

- Recall from the binomial model that the true probability of an up-move $p$ has no bearing upon the risk-neutral probability $q$ and therefore it does not directly affect how securities are priced.
- From our calibration of the binomial model, we see that $\mu$, which enters the calibration only through $p$, does not impact security prices.
- On the other hand, $u$ and $d$ depend on $\sigma$ which therefore does impact security prices.

More typically the case that we wish to calibrate a binomial model to the risk-neutral dynamics of a stock following a GBM model.

If the stock has a continuous dividend yield of $c$ so that a dividend of size $cS_t\,dt$ is paid at time instant $t$ then risk-neutral dynamics of the stock satisfy

$$S_{t+s} = S_t\, e^{(r-c-\sigma^2/2)s+\sigma(B_{t+s}-B_t)}. \tag{10}$$

Corresponding $q$ for the binomial model can be obtained from (9) with $\mu$ replaced by $r-c$ and with $u$ and $d$ unchanged.

# The Volatility Surface

The Black-Scholes model is elegant but it does not perform well in practice:

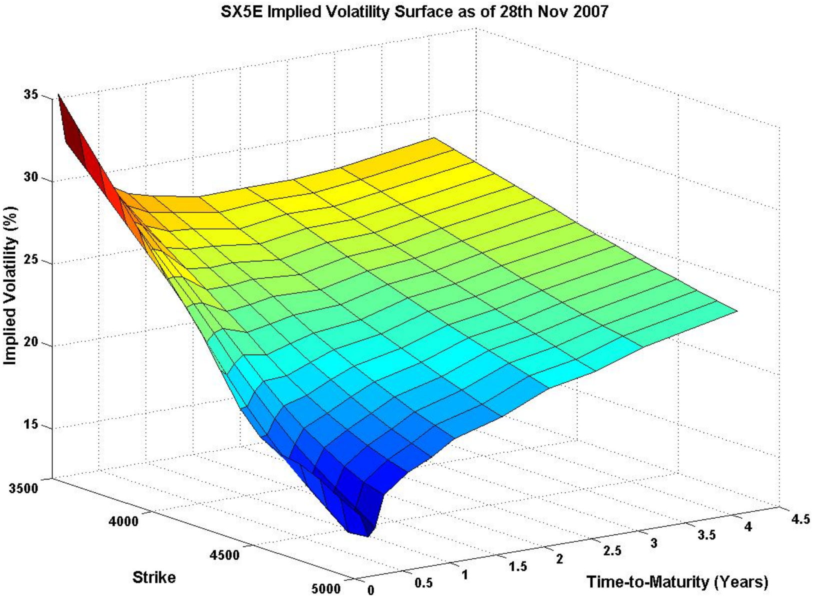
- Well known that stock prices can jump and do not always move in continuous manner predicted by GBM
- Stock prices also tend to have fatter tails than predicted by GBM.
- In practice volatility $\sigma$ is stochastic.

# The Volatility Surface

- If B-S model correct then should have a flat implied volatility surface.
- The implied volatility $\sigma(K,T)$ is defined implicitly by

$$C(S,K,T) = \mathsf{BS}\left(S,T,r,c,K,\sigma(K,T)\right) \tag{11}$$

  where $C(S,K,T) =$ market price of call option and $\mathsf{BS}(\cdot) =$ B-S price.
- There will always (why?) be a unique solution $\sigma(K,T)$ to (11).
- A plot of $\sigma(K,T)$ against $K$ and $T$ is called the implied volatility surface.
- If B-S model correct then volatility surface would be flat with $\sigma(K,T) = \sigma$.
- In practice, however, volatility surface is not flat and moves randomly
  - So the B-S model is very wrong!
  - But the language of B-S is pervasive in derivatives industry.


SX5E Implied Volatility Surface as of 28th Nov 2007
Implied Volatility (%)
35
30
25
20
15
Strike
3500
4000
4500
5000
Time-to-Maturity (Years)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5

# Why is there a Skew?

For stocks and stock indices there is generally a skew so that for any fixed maturity $T$ the implied volatility decreases with the strike $K$.

Most pronounced at shorter expirations for two reasons:

1. Risk aversion – can appear in many guises:
   (i) Stocks do not follow GBM but instead often jump. Jumps to downside tend to be larger and more frequent than jumps to upside.
   (ii) As markets go down, fear sets in and volatility goes up.
   (iii) Supply and demand: investors like to protect their portfolio by purchasing OTM puts so there is more demand for options with lower strikes.

2. The leverage effect. Based on fact that total value of company assets is a more natural candidate to follow GBM.

   In this case equity volatility should increase as equity value decreases.

# The Leverage Effect

- Let $V$, $E$ and $D$ denote value of firm, firm's equity and firm's debt.
- Then fundamental accounting equation states $V = D + E$.
- Let $\Delta V$, $\Delta E$ and $\Delta D$ be change in values of $V$, $E$ and $D$.

Then $V + \Delta V = (E + \Delta E) + (D + \Delta D)$ so that

$$\begin{aligned} \frac{V+\Delta V}{V} &= \frac{E+\Delta E}{V} + \frac{D+\Delta D}{V} \\ &= \frac{E}{V}\left(\frac{E+\Delta E}{E}\right) + \frac{D}{V}\left(\frac{D+\Delta D}{D}\right). \end{aligned} \tag{12}$$

# The Leverage Effect

If equity component is substantial so that debt is not too risky, then (12) implies

$$\sigma_V \approx \frac{E}{V}\,\sigma_E$$

where $\sigma_V$ and $\sigma_E$ are the firm value and equity volatilities.

Therefore have

$$\sigma_E \approx \frac{V}{E}\,\sigma_V. \tag{13}$$

# The Leverage Effect

**Example:** Suppose $V = 1$, $E = .5$ and $\sigma_V = 20\%$

- Then (13) implies $\sigma_E \approx 40\%$.

- Suppose $\sigma_V$ remains unchanged but that firm loses $20\%$ of its value over time.

- Almost all of this loss is borne by equity so (13) now implies $\sigma_E \approx 53\%$.

- $\sigma_E$ has therefore increased despite the fact that $\sigma_V$ has remained constant!

**Remark:** There was little or no skew in market before Wall Street crash of $1987$

- but then the market woke up to the flaws of GBM!

# Recalling Put-Call Parity

- The "Greeks" measure the sensitivity of the option price to changes in various parameters.

- They're usually computed using the B-S formula
  - despite fact that B-S model known to be a poor approximation to reality.

  Before discussing the Greeks we first recall put-call parity:

$$e^{-r(T-t)} K + \text{ Call Price}_t \; = \; e^{-c(T-t)} S_t + \text{ Put Price}_t. \qquad (14)$$

  where call and put have same strike $K$, maturity $T$, and $c =$ dividend yield.

- Put-call parity very useful for:
  1. Calculating Greeks. e.g. it implies that Vega(Call) = Vega(Put)
  2. For calibrating dividends or borrow rate
  3. Constructing the volatility surface.

# The Greeks: Delta

**Definition:** The delta of an option is the sensitivity of the option price to a change in the price of the underlying security.

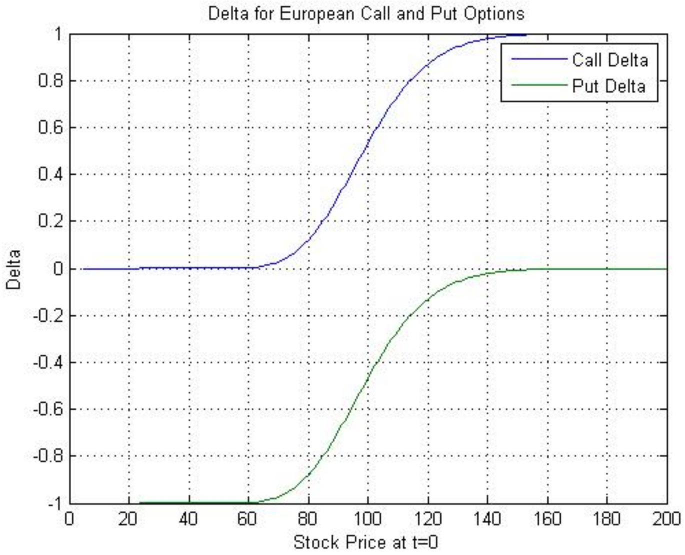
Delta of a European call option (in B-S model) is

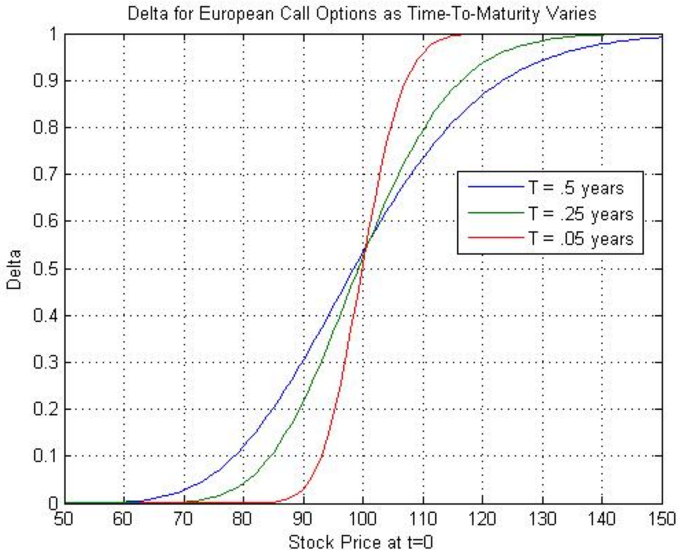
$$\text{delta} = \frac{\partial C}{\partial S} = e^{-c(T-t)}\Phi(d_1).$$

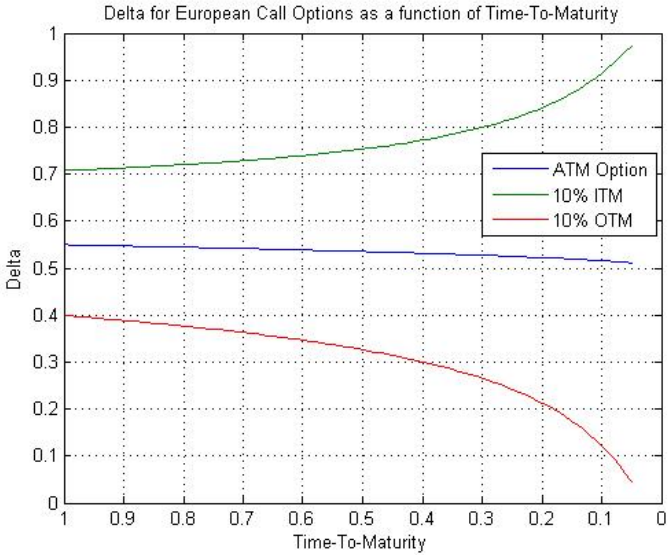
- By put-call parity, have

$$\text{delta}_{\text{put}} = \text{delta}_{\text{call}} - e^{-c(T-t)}.$$

- In following figures we assumed $r = c = 0$ and $K = 100$.
- Note that delta becomes steeper around $K$ when time-to-maturity decreases.


Delta for European Call and Put Options
Call Delta
Put Delta
Delta
Stock Price at t=0
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
20
40
60
80
100
120
140
160
180
200


Delta for European Call Options as Time-To-Maturity Varies
Delta
Stock Price at t=0
T = .5 years
T = .25 years
T = .05 years
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
50
60
70
80
90
100
110
120
130
140
150


Delta for European Call Options as a function of Time-To-Maturity
Delta
Time-To-Maturity
ATM Option
10% ITM
10% OTM
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

# The Greeks: Gamma

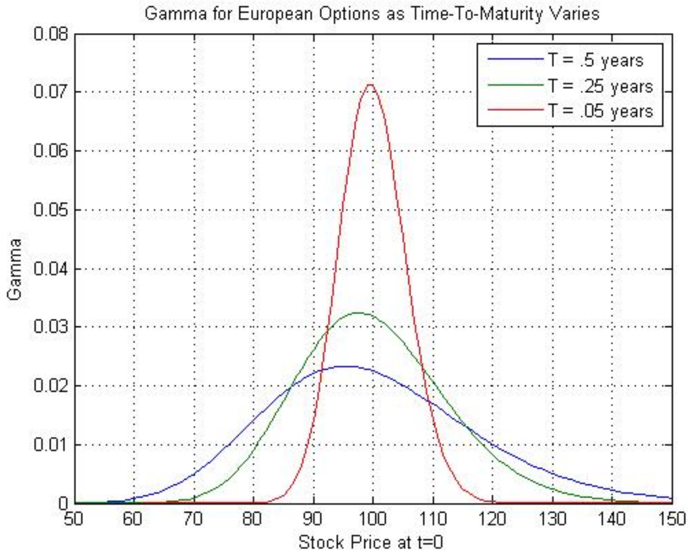
**Definition:** The gamma of an option is the sensitivity of the option's delta to a change in the price of the underlying security.

- The gamma of a European call option satisfies

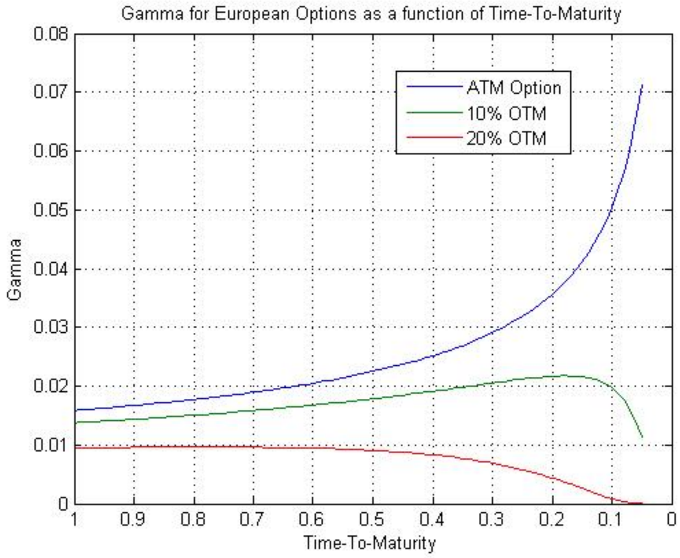
$$\text{gamma} \;=\; \frac{\partial^2 C}{\partial S^2} \;=\; e^{-c(T-t)} \frac{\phi(d_1)}{\sigma S \sqrt{T-t}}$$

  where $\phi(\cdot)$ is the standard normal PDF.

- By put-call parity, gamma of European call $=$ gamma of European put with same strike and maturity.



Gamma for European Options as Time-To-Maturity Varies
T = .5 years
T = .25 years
T = .05 years
Gamma
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
50
60
70
80
90
100
110
120
130
140
150
Stock Price at t=0


Gamma for European Options as a function of Time-To-Maturity
ATM Option
10% OTM
20% OTM
Gamma
Time-To-Maturity
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

# The Greeks: Vega

**Definition:** The vega of an option is the sensitivity of the option price to a change in volatility.

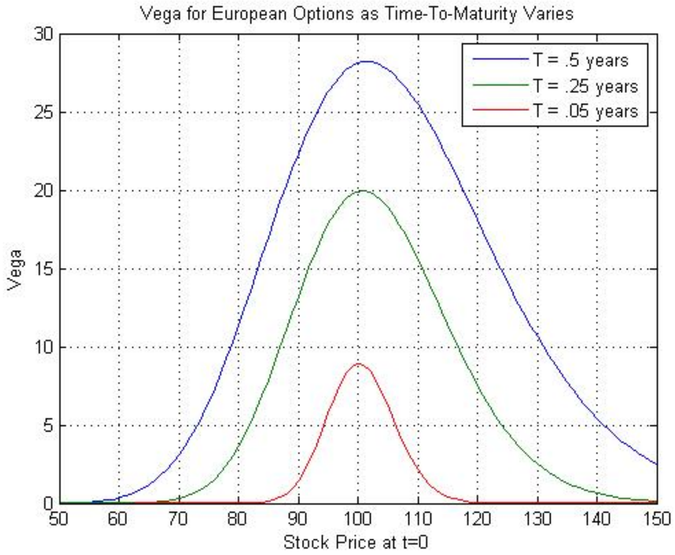
The vega of a European call option satisfies

$$\text{vega} \;=\; \frac{\partial C}{\partial \sigma} \;=\; e^{-c(T-t)} S \sqrt{T-t}\, \phi(d_1).$$

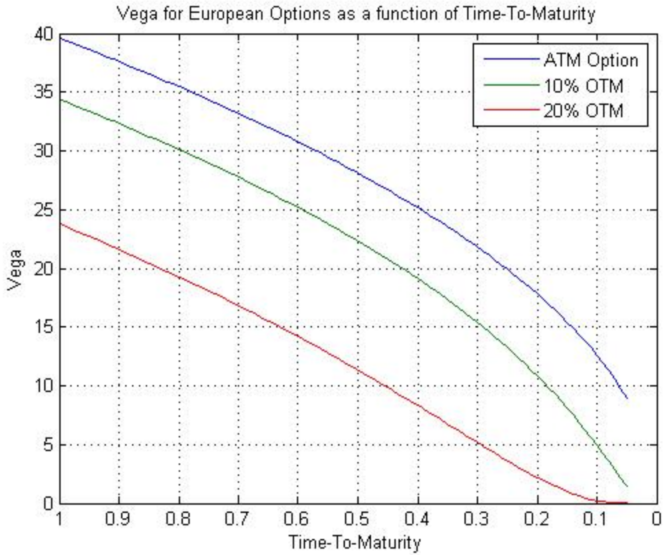
- Put-call parity implies vega of European call $=$ vega of European put with same strike and maturity.
- In following figures we assumed $K = 100$ and that $r = c = 0$.

**Question:** Why does vega increase with time-to-maturity?

**Question:** For a given time-to-maturity, why is vega peaked near the strike?



Vega for European Options as Time-To-Maturity Varies
T = .5 years
T = .25 years
T = .05 years
Vega
Stock Price at t=0
30
25
20
15
10
5
0
50
60
70
80
90
100
110
120
130
140
150


Vega for European Options as a function of Time-To-Maturity
ATM Option
10% OTM
20% OTM
Vega
Time-To-Maturity
40
35
30
25
20
15
10
5
0
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

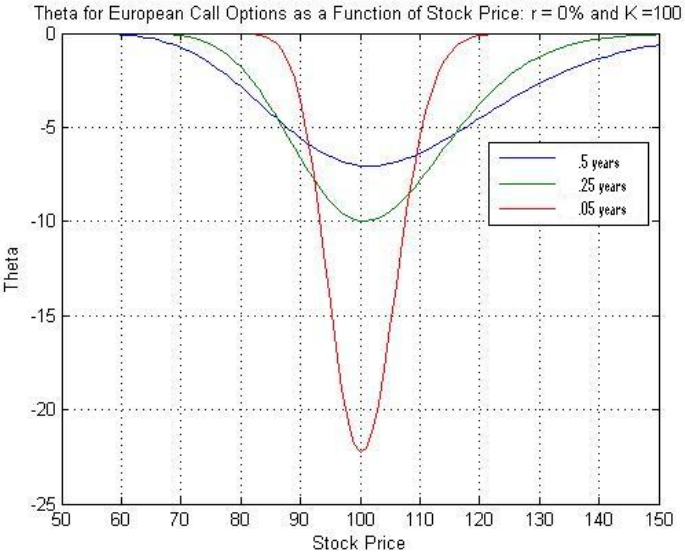
# The Greeks: Theta

**Definition:** The theta of an option is the sensitivity of the option price to a negative change in time-to-maturity.

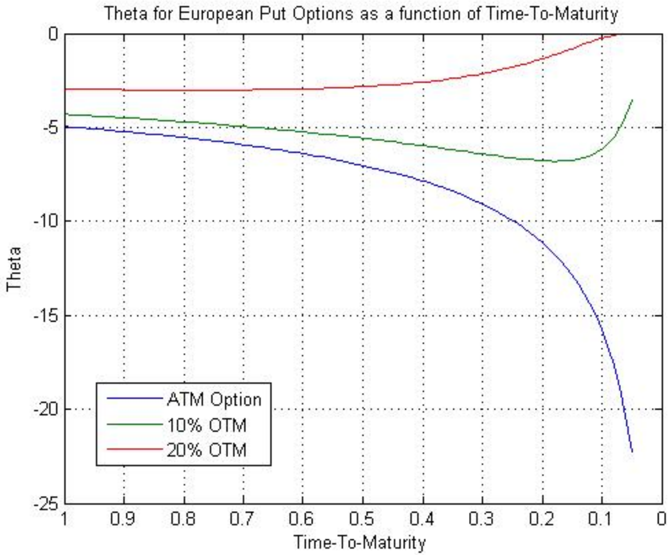
The theta of a European call option satisfies

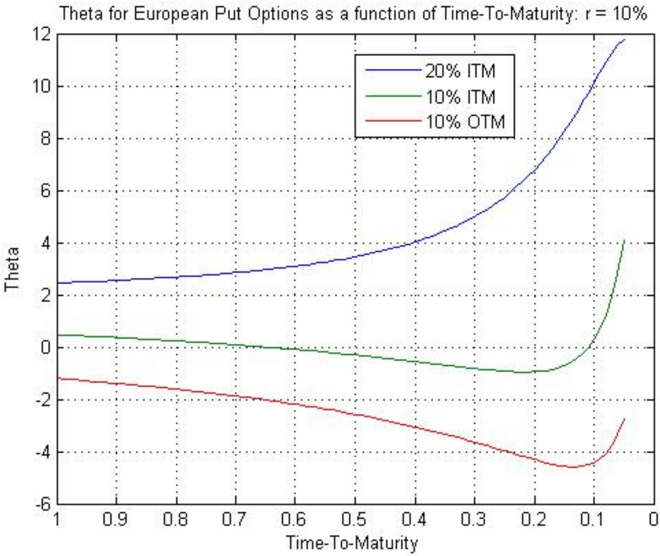
$$\begin{aligned}\text{theta} &= -\frac{\partial C}{\partial T} \\ &= -e^{-c(T-t)} S\phi(d_1)\frac{\sigma}{2\sqrt{T-t}} + ce^{-c(T-t)} S\Phi(d_1) - rKe^{-r(T-t)}\Phi(d_2).\end{aligned}$$

- The theta of a European put option can be obtained from put-call parity.
- In following figures have assumed $r = c = 0\%$ and $K = 100$.



Theta for European Call Options as a Function of Stock Price: r = 0% and K =100
Theta
Stock Price
0
-5
-10
-15
-20
-25
50
60
70
80
90
100
110
120
130
140
150
.5 years
.25 years
.05 years


Theta for European Put Options as a function of Time-To-Maturity
0
-5
-10
-15
-20
-25
Theta
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Time-To-Maturity
ATM Option
10% OTM
20% OTM




Still have $c = 0$ but now $r = 10\%$. Note theta positive for ITM put. Why?

Can also obtain positive theta for call options when $c$ is large.

# Delta-Hedging

- Delta-hedging is the act of continuously re-balancing a portfolio so that it is always delta-neutral, i.e. it has a total delta of zero.

- Not practical of course to hedge continuously
  - so instead we hedge periodically
  - this results in some replication error.

**e.g.** Suppose we have sold a call option with strike $K$ and maturity $T$ at time $t = 0$ and that we wish to delta-hedge this negative position.

- Let $C_0$ = initial value of the option
  - we receive this at $t = 0$. Why?
  - will use this to fund our s.f. trading / hedging strategy.

- Let $P_t$ denote time $t$ value of this strategy.

# Delta-Hedging

Delta-hedging strategy then given by

$$
\begin{aligned}
P_0 &:= C_0 && (15) \\
P_{t_{i+1}} &= P_{t_i} + (P_{t_i} - \delta_{t_i} S_{t_i})\, r\Delta t \;+\; \delta_{t_i}\left(S_{t_{i+1}} + cS_{t_i}\Delta t - S_{t_i}\right) && (16)
\end{aligned}
$$

- $\Delta t := t_{i+1} - t_i$ is the length of time between re-balancing
- $r =$ annual risk-free interest rate (assuming per-period compounding)
- $\delta_{t_i}$ is the B-S delta at time $t_i$
- $c$ is the dividend yield so that $cS_{t_i}\Delta t$ is the dividend paid in interval $(t_i, t_{i+1}]$

- Note that (15) and (16) respect the self-financing condition.
- Immediately after trading at time $t$ we have $(P_{t_i} - \delta_{t_i} S_{t_i})$ invested in the cash-account and $\delta_{t_i}$ units of the stock
  - so delta of portfolio offsets short delta position from call option.
- As $\Delta t \to 0$, the final value $P_T \to C_T$, the terminal value of the option
  - so the delta-hedging strategy becomes the replicating strategy.

# Delta-Hedging

- Recall $\delta_{t_i}$ satisfies

$$\delta_{t_i} = \frac{\partial C}{\partial S} \;=\; e^{-c(T-t)}\;\Phi(d_1)$$

where

$$d_1 := \frac{\log\left(\frac{S_{t_i}}{K}\right) + (r + \sigma^2_{\text{imp}}/2)(T - t_i)}{\sigma_{\text{imp}}\sqrt{T - t_i}}$$

- Stock prices simulated assuming $S_t \sim \text{GBM}(\mu, \sigma)$ so that

$$S_{t+\Delta t} = S_t \, e^{(\mu - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\, Z}$$

where $Z \sim \text{N}(0, 1)$.

- At maturity our replication error / trading P&L given by

$$\text{P\&L} \;:=\; P_T - (S_T - K)^+.$$

- In real world we don't know $\sigma$ and so it will not be $\sigma_{\text{imp}}$
  - in fact even if we assume B-S model, true dynamics of $S_t$ will not be GBM
  - has interesting implications for the distribution of P&L!

# Delta-Hedging

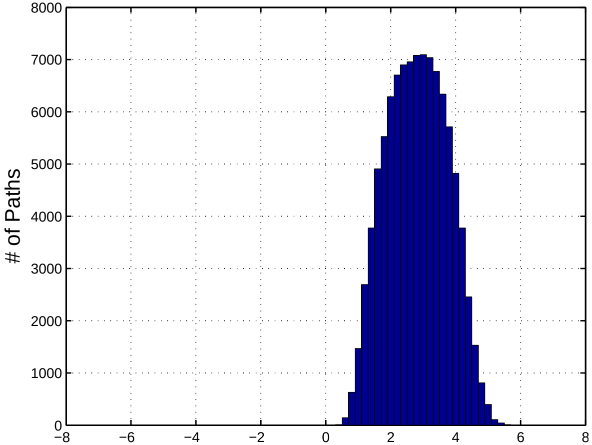
Many interesting questions now arise even if we assume $S_t \sim \text{GBM}(\mu, \sigma)$:

**Question:** What typically happens the total P&L if $\sigma < \sigma_{\text{imp}}$?

**Question:** What typically happens the total P&L if $\sigma > \sigma_{\text{imp}}$?

**Question:** If $\sigma = \sigma_{\text{imp}}$ what typically happens the total P&L as the number of re-balances increases?

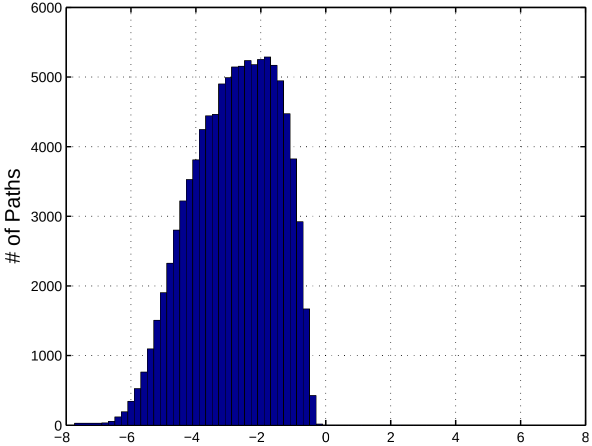
# Parameter Uncertainty and Hedging in Black-Scholes




Histogram of delta-hedging P&L with 100,000 sample paths, $S_0 = K = \$100$, $\sigma = 30\%$ and $\sigma_{\text{imp}} = 40\%$.

**Question:** Option hedger makes substantial gains. Why?

# Parameter Uncertainty and Hedging in Black-Scholes




Histogram of delta-hedging P&L with 100,000 sample paths, $S_0 = K = \$100$, $\sigma = 30\%$ and $\sigma_{\text{imp}} = 20\%$.

**Question:** Option hedger makes substantial loses. Why?

# Delta-Hedging

- Recall that fair price of an option increases as volatility increases.

- Therefore if $\sigma > \sigma_{\text{imp}}$ we expect to lose money on average when we delta-hedge an option that we sold.

- Similarly, expect to make money when we delta-hedge if $\sigma < \sigma_{\text{imp}}$.

- In general the payoff from delta-hedging an option is path-dependent.

# Parameter Uncertainty and Hedging in Black-Scholes

- Returning to s.f. trading strategy of (15) and (16), note that we can choose any model we like for the true security price dynamics
  - **e.g.** other diffusion or jump-diffusion models.

- Interesting to simulate these alternative models and observe what happens to the replication error / P&L from (15) and (16)
  - P&L $\rightarrow 0$ as $\Delta t \rightarrow 0$ only if price dynamics are GBM$(\mu, \sigma_{\text{imp}})$

- When using a model to price and hedge a derivative security (or portfolio) it's common to simulate alternative models to understand replication error when true model is not the assumed model.

  Goal then is to understand how robust the hedging strategy (based on the assumed model) is to alternative price dynamics that might prevail in practice.

- Given the appropriate data, one can also back-test the performance of a model on realized historical price data to assess its hedging performance.

# Scenario Analysis for Derivatives Portfolios

The scenario approach to risk management defines a number of scenarios where in each scenario various risk factors, e.g. the price of the underlying security and implied volatility, are assumed to have moved by some fixed amounts.

**e.g.** A scenario might assume all stock prices have fallen by $10\%$ and all implied volatilities have increased by $5$ percentage points.

Risk of portfolio could then be determined by computing P&L in each scenario and then considering the results, taking action (to reduce risk) where appropriate.

- Figure on next slide shows P&L under various scenarios of an options portfolio with the S&P 500 as the underlying security.
- Vertical axis represents percentage shifts in the price of the underlying security, i.e. the S&P 500.
- Horizontal axis represents absolute changes in implied volatility of each option in the portfolio.

# Scenario Analysis for Options Portfolios

| Underlying | SPX Index |
|---|---|

**Underlying and Volatility Stress Table**

| Sum of PnL | Vol Stress | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Underlying Stress | -10 | -5 | -2 | -1 | 0 | 1 | 2 | 5 | 10 |
| -20 | 13,938 | 11,774 | 10,631 | 10,277 | 9,936 | 9,608 | 9,293 | 8,419 | 7,183 |
| -10 | 6,109 | 4,946 | 4,436 | 4,291 | 4,158 | 4,035 | 3,922 | 3,634 | 3,296 |
| -5 | 1,831 | 1,652 | 1,637 | 1,643 | 1,654 | 1,670 | 1,689 | 1,766 | 1,946 |
| -2 | (314) | 89 | 356 | 447 | 539 | 631 | 723 | 1,001 | 1,461 |
| -1 | (920) | (338) | 15 | 132 | 248 | 363 | 478 | 816 | 1,361 |
| 0 | (1,463) | (714) | (280) | (139) | 0 | 137 | 273 | 668 | 1,293 |
| 1 | (1,939) | (1,035) | (527) | (363) | (203) | (45) | 110 | 559 | 1,259 |
| 2 | (2,346) | (1,300) | (723) | (539) | (359) | (182) | (9) | 489 | 1,258 |
| 5 | (3,125) | (1,744) | (1,003) | (769) | (541) | (318) | (102) | 518 | 1,460 |
| 10 | (2,921) | (1,297) | (423) | (146) | 123 | 385 | 641 | 1,372 | 2,483 |
| 20 | 2,344 | 3,559 | 4,272 | 4,506 | 4,738 | 4,967 | 5,194 | 5,860 | 6,919 |

Figure : P&L for an options portfolio on the S&P 500 under stresses to underlying and implied volatility

# Scenario Analysis for Options Portfolios

**e.g.** If the S&P 500 falls by $20\%$ and implied volatilities all rise by $5$ percentage points, then P&L would be $\$8.419$m.

- When constructing scenario tables can use (18) (on slide 75) to check for internal consistency and help identify possible software bugs.
- While scenario tables are a valuable source of information there are many potential pit-falls associated with using them. These include:
  1. Identifying the relevant risk factors
  2. Identifying "reasonable" shifts for these risk factors
- A key role of any risk manager then is to understand what scenarios are plausible and what scenarios are not.

  **e.g.** In a crisis would expect any drop in underlying security price to be accompanied by a rise in implied volatilities. Would therefore pay considerably more attention to the corresponding numbers in scenario table.

# Delta-Gamma-Vega Approximations to Option Prices

A simple application of Taylor's Theorem yields

$$
\begin{aligned}
C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\
&= C(S,\sigma) + \Delta S \times \delta + \frac{1}{2}(\Delta S)^2 \times \Gamma + \Delta\sigma \times \text{vega}
\end{aligned}
$$

where $C(S,\sigma) =$ price of a derivative security as a function of $S$ and $\sigma$.

Therefore obtain

$$
\begin{aligned}
\text{P\&L} &= C(S+\Delta S, \sigma+\Delta\sigma) - C(S,\sigma) \\
&\approx \delta\Delta S + \frac{\Gamma}{2}(\Delta S)^2 + \text{vega } \Delta\sigma \qquad (17) \\
&= \text{delta P\&L} + \text{gamma P\&L} + \text{vega P\&L}
\end{aligned}
$$

# Delta-Gamma-Vega Approximations to Option Prices

When $\Delta\sigma = 0$, obtain the well-known delta-gamma approximation

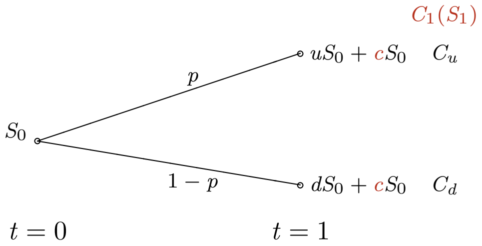
- often used in historical Value-at-Risk (VaR) calculations for portfolios that include options.

Can also use (17) to write

$$\begin{aligned} \text{P\&L} &= \delta S \left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\ &= \text{ESP} \times \text{Return} + \text{\$Gamma} \times \text{Return}^2 + \text{vega } \times \Delta\sigma \end{aligned} \qquad (18)$$

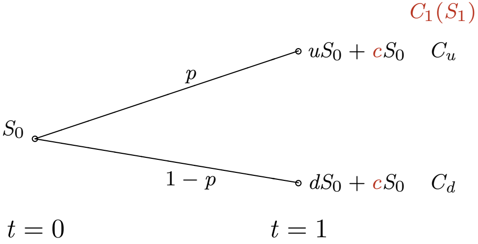
where ESP denotes the equivalent stock position or “dollar” delta.

# Appendix: Including Dividends




- Consider again 1-period model and assume stock pays a proportional dividend of $cS_0$ at $t = 1$.
- No-arbitrage conditions are now $d + c < R < u + c$.

# Appendix: Including Dividends




- Can use same replicating portfolio argument to find price, $C_0$, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$.
- Set up replicating portfolio as before:

$$uS_0x + cS_0x + Ry = C_u \quad (19)$$
$$dS_0x + cS_0x + Ry = C_d \quad (20)$$

- Solve for $x$ and $y$ as before and then must have $C_0 = xS_0 + y$.

# Appendix: Derivative Security Pricing with Dividends

- Solving (19) and (20) we obtain:

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d-c}{u-d}C_u + \frac{u+c-R}{u-d}C_d\right] \qquad (21) \\
&= \frac{1}{R}\left[q\,C_u + (1-q)\,C_d\right] \\
&= \frac{1}{R}\mathsf{E}_0^Q[C_1].
\end{aligned}
$$

  where now $q = (R-d-c)/(u-d)$.

- So now can price any derivative security in this $1$-period model with dividends!

- In a multi-period binomial model, would use these new r.n. probabilities for branches leading into nodes where a dividend is paid and the original r.n. probabilities from slide 14 otherwise.