

An Introduction to Derivatives Pricing

In these lecture notes we discuss the pricing of European and American options via the concept of no-arbitrage in the binomial model. We also derive some model-independent bounds on options prices and introduce other important concepts including self-financing strategies, replicating strategies and complete and incomplete markets. We also introduce the famous Black-Scholes model which assumes stock price have geometric Brownian motion (GBM) dynamics and we note how GBM can be obtained as the limiting process of the binomial model when the period length goes to zero. In this limit, option prices in the binomial model converge to option price in the Black-Scholes model. We also introduce the concepts of implied volatilities and the Greeks, and we conclude with a brief discussion of derivatives pricing more generally.

1 Introduction to Options and the Binomial Model

We first define the main types of options, namely European and American call and put options.

Definition 1 *A European call (put) option gives the right, but not the obligation, to buy (sell)* 1 *unit of the underlying security at a pre-specified price,* K*, at a pre-specified time,* T*.*

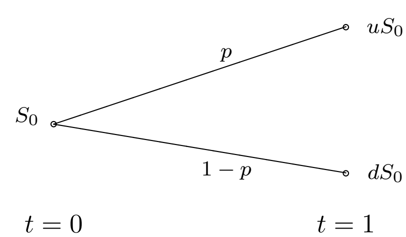
Definition 2 *An American call (put) option gives the right, but not the obligation, to buy (sell)* 1 *unit of the underlying security at a pre-specified price,* K*, at any time up to and including a pre-specified time,* T*.*

K and T are called the strike and maturity / expiration of the option, respectively. Let S_t denote the price of the underlying security at time t. Then, for example, if $S_T < K$ a European call option will expire worthless and the option will not be *exercised*. A European put option, however, would be exercised and the payoff would be $K - S_T$. More generally, the payoff at maturity of a European call option is $\max\{S_T - K, 0\}$ and its **intrinsic value** at any time $t < T$ is given by $\max\{S_t - K, 0\}$. The payoff of a European put option at maturity is $\max\{K - S_T, 0\}$ and its **intrinsic value** at any time $t < T$ is given by $\max\{K - S_t, 0\}$.

The binomial model is a classic model in finance that was originally introduced for the pricing of options on stocks but it has since found many applications throughout finance. We will begin with the 1-period binomial model.

1.1 The 1-Period Binomial Model

Consider the 1-period binomial model where the underlying security has a value of $S_0 = 100$ at $t = 0$ and increases by a factor of u or decreases by a factor of d in the following period. We also assume that borrowing or lending at a gross risk-free rate of R is possible. This means that \$1 in the cash account at $t = 0$ will be worth R at $t = 1$. We also assume that **short-sales** are allowed.



Suppose now that $S_0 = 100$, $R = 1.01$, $u = 1.07$ and $d = 1/u = .9346$. Some interesting questions now arise:

1. How much is a call option that pays $\max(S_1 - 107,\ 0)$ at $t = 1$ worth?

2. How much is a call option that pays $\max(S_1 - 92,\ 0)$ at $t = 1$ worth?

Pricing these options is easy but to price options in general we need more general definitions of arbitrage.

Definition 3 *A type A arbitrage is a security or portfolio that produces immediate positive reward at $t = 0$ and has non-negative value at $t = 1$.* **i.e.** *a security with initial cost, $V_0 < 0$, and time $t = 1$ value $V_1 \geq 0$.*

An example of a type A arbitrage would be somebody walking up to you on the street, giving you a positive amount of cash, and asking for nothing in return, either then or in the future.

Definition 4 *A type B arbitrage is a security or portfolio that has a non-positive initial cost, has positive probability of yielding a positive payoff at $t = 1$ and zero probability of producing a negative payoff then.* **i.e.** *a security with initial cost, $V_0 \leq 0$, and $V_1 \geq 0$ but $V_1 \neq 0$.*

An example of a type B arbitrage would be a stock that costs nothing, but that will possibly generate dividend income in the future. Another example would be a free lottery ticket. In finance we **always** assume that arbitrage opportunities do not exist since if they did, market forces would quickly act to dispel them. The absence of arbitrage is the key assumption that allows us to price derivative securities in many financial models. We now have the following result.

Theorem 1 *If there is no arbitrage in the 1-period binomial model then we must have $d < R < u$.*

Proof: Suppose it is not the case that $d < R < u$. Then (since $d < u$) we must have either (i) $R < d < u$ or (ii) $d < u < R$. If case (i) holds then at $t = 0$ we should borrow S_0 and purchase one unit of the stock. If case (ii) holds then we should short-sell one unit of the stock at $t = 0$ and invest the proceeds in cash-account. In both cases we have (why?) a type B arbitrage and so the result follows. ■

We will soon see the other direction, i.e. if $d < R < u$, then there can be no-arbitrage. Let's return to our earlier numerical example and consider the following questions:

1. How much is a call option that pays $\max(S_1 - 102,\ 0)$ at $t = 1$ worth?

2. How will the price vary as p varies?

To answer these questions, we will construct a *replicating* portfolio. Let us buy x shares and invest y in the cash account at $t = 0$. At $t = 1$ this portfolio is worth:

$$\begin{array}{rl} 107x + 1.01y & \quad \text{when } S = 107 \\ 93.46x + 1.01y & \quad \text{when } S = 93.46 \end{array}$$

Can we choose x and y so that portfolio equals the option payoff at $t = 1$? We can indeed by solving

$$\begin{aligned} 107x + 1.01y &= 5 \\ 93.46x + 1.01y &= 0 \end{aligned}$$

and the solution is $x = 0.3693$ and $y = -34.1708$. Note that the cost of this portfolio at $t = 0$ is

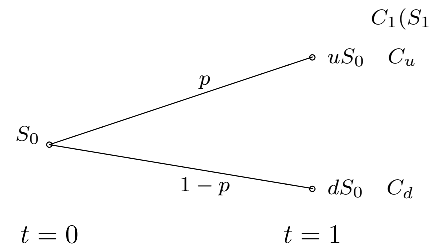
$$0.3693 \times 100 - 34.1708 \times 1 \approx 2.76.$$

This implies the fair or arbitrage-free value of the option is 2.76.

Exercise 1 *Convince yourself that if the time $t = 0$ price of the option was not 2.76 then there would be an arbitrage.* Hint: *Let C_0 be the time $t = 0$ price of the option and consider two cases: (i) $C_0 > 2.76$ and (ii) $C_0 < 2.76$.*

Derivative Security Pricing in the 1-Period Binomial Model

Can we use the same replicating portfolio argument to find the price, C_0, of any derivative security with payoff function, $C_1(S_1)$, at time $t = 1$? Yes we can by setting up replicating portfolio as before and solving the following two linear equations for x and y



$$
\begin{aligned}
uS_0x + Ry &= C_u \qquad (1)\\
dS_0x + Ry &= C_d \qquad (2)
\end{aligned}
$$

The arbitrage-free time $t = 0$ price of the derivative must (Why?) then be $C_0 := xS_0 + y$. Solving (1) and (2) then yields

$$
\begin{aligned}
C_0 &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right]\\
&= \frac{1}{R}\left[qC_u + (1-q)C_d\right]\\
&= \frac{1}{R}\mathrm{E}_0^Q[C_1] \qquad (3)
\end{aligned}
$$

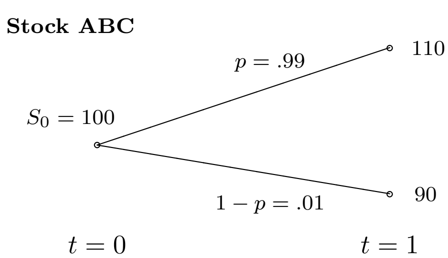
where $q := (R-d)/(u-d)$ so that $1 - q = (u-R)/(u-d)$. Note that if $d < R < u$ then $q > 0$ and $1 - q > 0$ and so by (3) there can be (why?) no-arbitrage. We state this as the converse to Theorem 1.

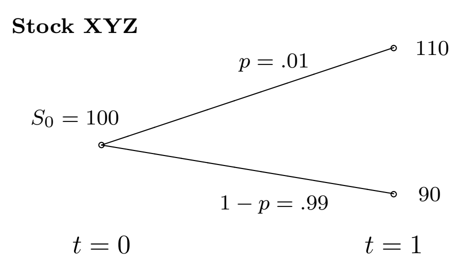
Theorem 2 *If $d < R < u$ in the 1-period binomial model then there is no arbitrage.*

We refer to (3) as **risk-neutral pricing** and $(q, 1-q)$ are the **risk-neutral (r.n.) probabilities**. So we now know how to price any derivative security in this 1-period binomial model via a replication argument. Moreover this replication argument is equivalent to pricing using risk-neutral probabilities.

Derivatives Prices Do Not Depend on p!

From (3) we also note that the price of the derivative does not depend on p! This at first appears very surprising. To understand this result further consider the following two stocks, ABC and XYZ:





Note that the probability of an up-move for ABC is $p = .99$ whereas the probability of an up-move for XYZ is $p = .01$. Consider now the following two questions:

Question: What is the price of a call option on ABC with strike $K = \$100$?

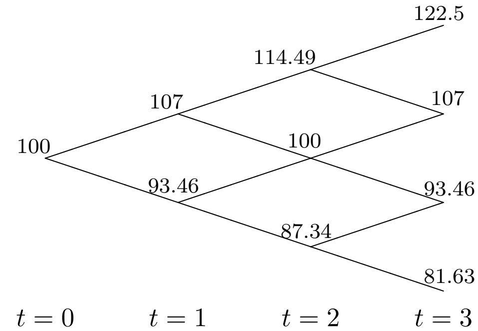
Question: What is the price of a call option on XYZ with strike $K = \$100$?

You should be surprised by your answers. But then if you think a little more carefully you'll realize that the answers are actually not surprising *given* the premise that two stocks like ABC and XYZ actually exist side-by-side in the market.

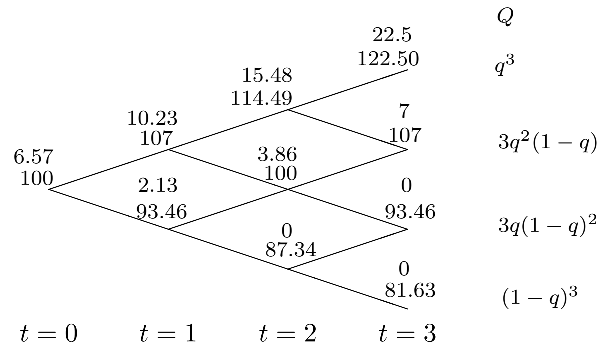
Exercise 2 *Be sure you understand why the call options on ABC and XYZ have the same price.*

1.2 The Multi-Period Binomial Model

Consider the multi-period binomial model displayed to the right where as before we have assumed $u = 1/d = 1.07$. The important thing to notice is that the multi-period model is just a series of 1-period models spliced together! This implies all the results from the 1-period model apply and that we just need to multiply 1-period probabilities along branches to get probabilities in the multi-period model.



Example 1 (Pricing a European Call Option)
Suppose now that we wish to price a European call option with expiration at $t = 3$ and strike = \$100. As before we assume a gross risk-free rate of $R = 1.01$ per period. We can do this by working backwards in the lattice starting at time $t = 3$ and using what we know about 1-period binomial models to obtain the price at each prior node. We do this repeatedly until we obtain the arbitrage-free price at $t = 0$. The price of the option at each node is displayed above the underlying stock price in the binomial model to the right. Note that we repeatedly used (3) to obtain these prices.



For example, the upper node at $t = 1$ has a value of 10.23. This is the value of the derivative security that pays either 15.48 (after an up-move) or 3.88 (after a down-move) 1 period later. It is not hard to see (using the tower property of conditional expectations) that the process of backwards evaluation that we just described is equivalent to pricing the option as

$$C_0 = \frac{1}{R^3} \mathrm{E}_0^Q \left[\max(S_T - 100,\ 0) \right] \tag{4}$$

and we note the risk-neutral probabilities for S_T are displayed at the far right in the binomial lattice above. Risk-neutral pricing pricing via (4) has the advantage of not needing to calculate the option price at every intermediate node. ∎

1.2.1 Self-Financing and Replicating Trading Strategies

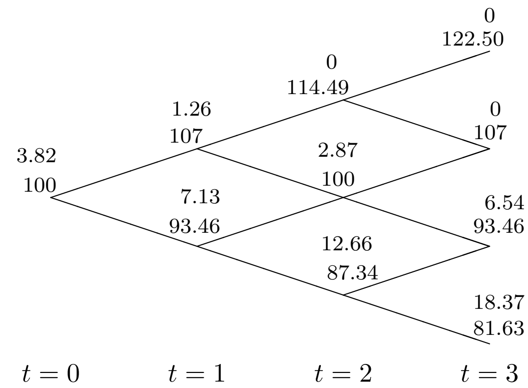
Definition 5 *A* **self-financing** *(s.f.) trading strategy over the time interval $[0, T]$ is one where any gains or losses in the value of the portfolio are due entirely to gains or losses due to trading. In particular, no new cash is added to or withdrawn from the portfolio at any time.*

Definition 6 *A* **replicating strategy** *for a particular security with payoff C_T at time T is a s.f. strategy whose value at time T is equal to C_T, i.e. the strategy replicates C_T.*

It's important to note that our approach to pricing options (and all derivative securities) in the binomial model is via the imposition of no-arbitrage and the construction of a replicating strategy for the option in question. Specifically, if we know the initial value of the s.f. strategy that replicates the payoff of the option then this value must be the value of the option. Otherwise there would be an arbitrage!

Exercise 3 *How would you find a* **replicating strategy** *for the call option of Example 1?*

Example 2 (Pricing an American Put Option)
We can price American options in the same way as European options only now at each node we must also check to see if it's optimal to early exercise there. We will see in Section 2, however, that it's never optimal to early exercise an American call option on a non-dividend paying stock. So instead we will price an American put option with expiration at $t = 3$ and strike $K = \$100$. Once again we assume $R = 1.01$. The American option price at each node is displayed in the lattice to the left. As before we start at expiration $t = 3$ where we know the value of the option. We then work backwards in the lattice and at each node we set the value equal to the maximum of the intrinsic value and the (risk-neutral) expected discounted value one period ahead.



For example, the value of the option at the lower node at time $t = 2$ is given by

$$12.66 = \max\left[12.66,\ \frac{1}{R}\left(q \times 6.54 + (1-q) \times 18.37\right)\right]$$

where $12.66 = 100 - 87.34$ is the intrinsic value of the option at that node. More generally, the value, $V_t(S)$, of the American put option at any time t node when the underlying price is S can be computed according to

$$\begin{aligned} V_t(S) &= \max\left[K - S,\ \frac{1}{R}\left[q \times V_{t+1}(uS) + (1-q) \times V_{t+1}(dS)\right]\right] \\ &= \max\left[K - S,\ \frac{1}{R}\mathrm{E}_t^Q\left[V_{t+1}(S_{t+1})\right]\right]. \end{aligned}$$

■

Remark 1 *Pricing an American option is an example of a* **dynamic programming** *problem. These problems are typically solved by working backwards in time as we have done here.*

1.3 Complete versus Incomplete Markets

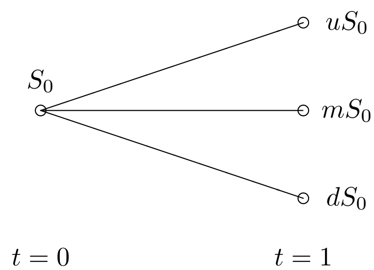
Assume there are no arbitrage opportunities. We have the following definition.

Definition 7 *If every security C_T can be replicated (via a s.f. strategy), then we say that we have a* **complete market***. Otherwise we have an* **incomplete market***.*

Complete markets are very convenient because every derivative security in a complete market can be priced (using the unique set of risk neutral probabilities). This price can be found by simply computing the initial value of the s.f. strategy that replicates it. The binomial model is a complete model (why?) as indeed is the famous **Black-Scholes** model of Section 3. In the real world, however, markets are incomplete but derivatives prices are often found by *assuming* they are complete.

Remark 2 *It's worth emphasizing that in a complete market,* **every security** *(and not just options) can be priced using the (unique) risk-neutral probabilities. This is just another way of saying that every security can be replicated via a s.f. trading strategy. (This all follows from our analysis of the 1-period binomial model in Section 1.1 where the replicating portfolio led to the arbitrage-free price which we then interpreted as pricing via risk-neutral probabilities.)*

Example 3 (An Incomplete Market - the Trinomial Model)
Consider the 1-period trinomial model in the figure below. It is identical to the 1-period binomial model except that there are now three possibilities for the time $t = 1$ security price S_1. In particular, it can take on the values dS_0, mS_0 and uS_0 with $d < m < u$ and where S_0 is the initial, i.e. $t = 0$, price of the security. As before, we insist that $d < R < u$.



Option prices cannot in general be computed in this model for the simple reason that three linear equations in two unknowns do not (in general) have a unique solution. This means we cannot (in general) replicate the time $t = 1$ payoff of an option in the trinomial model. ∎

Remark 3 *In arbitrage-free incomplete markets there are many sets of r.n. probabilities. Pricing of derivative securities is often done by choosing one particular set with the choice being guided by economic considerations or (more commonly) by calibrating model prices to observable market prices.*

---

2 Model-Free Bounds for Option Prices

Because the underlying security price process S_t is stochastic and the option payoffs are non-linear functions of the underlying security price, we cannot price options without a model. We can, however, obtain some **model-free** bounds for options prices. We let $c_E(t; K, T)$ and $p_E(t; K, T)$ denote the time t prices of a European call and put, respectively, with strike K and expiration T. Similarly, we let $c_A(t; K, T)$ and $p_A(t; K, T)$ denote the time t prices of an American call and put, respectively, with strike K and expiration T. It should be clear that the price of an American option is greater than or equal to the price of the corresponding European option.

2.1 Put-Call Parity

The most important model-free result on option pricing is put-call parity for European options. This result holds irrespective of the underlying model and only relies on the absence of arbitrage.

Theorem 3 (Put-Call Parity)
Suppose the underlying security does not pay dividends. We then have

$$p_E(t; K, T) + S_t = c_E(t; K, T) + Kd(t, T) \tag{5}$$

where $d(t, T)$ is the discount factor used to discount cash-flows from time T back to time t.

Proof: Consider the following trading strategy:

- At time t buy one European call with strike K and expiration T
- At time t sell one European put with strike K and expiration T
- At time t sell short 1 unit of the underlying security and buy it back at time T

- At time t lend $K\,d(t,T)$ dollars up to time T

Regardless of the underlying security price, it is easy to see that the cash-flow at time T corresponding to this strategy will be zero. No-arbitrage then implies that the value of this strategy at time t must therefore also be zero. We therefore obtain $-c_E(t;K,T)+p_E(t;K,T)+S_t-Kd(t,T)=0$ which is (5). ■

When the underlying security does pay dividends then a similar argument can be used to obtain

$$p_E(t;K,T)+S_t-D=c_E(t;K,T)+Kd(t,T) \tag{6}$$

where D is the present value of all dividends until maturity.

2.2 Early Exercise of American Call Options?

Suppose now the underlying security does not pay dividends and that the events $\{S_T>K\}$ and $\{S_T<K\}$ have strictly positive probability so that (why?) $c_E(t;K,T)>0$ and $p_E(t;K,T)>0$. We can then use put-call parity to obtain

$$c_E(t;K,T)=S_t+p_E(t;K,T)-Kd(t,T)>S_t-Kd(t,T). \tag{7}$$

Consider now the corresponding American call option. We obtain

$$c_A(t,K,T)\geq c_E(t;K,T)>\max\Big\{S_t-K,\,0\Big\}.$$

Therefore the price of an American call on a non-dividend-paying stock is always strictly greater than the intrinsic value of the call option when the events $\{S_T>K\}$ and $\{S_T<K\}$ have strictly positive probability. We have thus shown the following result.

Theorem 4 *It is never optimal to early-exercise an American call on a non-dividend paying stock. Therefore* $c_A(t;K,T)=c_E(t,K,T)$.

Unfortunately there is no such result relating American put options to European put options. Indeed it is sometimes optimal to early exercise an American put option even when the underlying security does not pay a dividend.

3 The Black-Scholes Model

The Black-Scholes model assume the stock price S_t follows a GBM so that $S_t\sim \text{GBM}(\mu,\sigma)$ and that the continuously compounded risk-free interest rate r is constant so that 1 unit of currency invested in the cash account at time 0 will be worth $\exp(rt)$ at time t. We will denote by $C(S,t)$ the value of a European call option at time t with maturity T and strike K. The model assumes that trading in the stock and cash account can take place continuously and is frictionless, i.e. there are no transactions costs. In their seminal 1973 paper Black and Scholes showed via PDE methods that the arbitrage-free price of the option is given by

$$C(S,t) \;=\; e^{-c(T-t)}S_t\Phi(d_1)\;-\;e^{-r(T-t)}K\Phi(d_2) \tag{8}$$

$$\begin{aligned}\text{where}\;\; d_1 &= \frac{\log\left(\frac{S_t}{K}\right)+(r-c+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\\ d_2 &= d_1-\sigma\sqrt{T-t}.\end{aligned}$$

and where $\Phi(\cdot)$ is the CDF of the standard normal distribution and c is the dividend yield as described above. The price of a European put-option can also now be easily computed from put-call parity (see (12) below) and (8).

The most interesting feature of the Black-Scholes formula is that μ does not appear in it. This is completely analogous to the probability of an up-move p in the binomial model having no bearing on the option-price in that model. In fact, one can also obtain the Black-Scholes formula as the limit when $\Delta t\to 0$ of the option price in the binomial model.

3.1 Calibrating the Binomial Model to GBM

We often wish to **calibrate** the binomial model so that its dynamics match that of GBM. To do this end we need to choose u, d and p, the real-world probability of an up-move, appropriately. There are many possible ways of doing this but one of the more common[1] choices is to set

$$\begin{aligned} p &= \frac{e^{\mu\Delta t} - d}{u - d} \\ u &= \exp(\sigma\sqrt{\Delta t}) \\ d &= 1/u \;=\; \exp(-\sigma\sqrt{\Delta t}) \end{aligned} \tag{9}$$

where Δt is the length of a period. Note then, for example, that

$$\mathrm{E}[S_{i+1} \mid S_i] = puS_i + (1-p)dS_i = S_i \exp(\mu\Delta t)$$

as desired. We will choose the gross risk-free rate per period R so that it corresponds to a continuously-compounded rate r in continuous time. We therefore have

$$R \;=\; e^{r\Delta t}.$$

Remark 4 *Recall from our study of option-pricing in the binomial model that the true probability of an up-move, p, has no bearing upon the risk-neutral probability q and therefore it does not* directly *affect how securities are priced. From our calibration of the binomial model, we therefore see that μ, which enters the calibration only through p, does not impact security prices. On the other hand, u and d depend on σ which therefore does impact security prices. This is a recurring theme in derivatives pricing.*

It is more typically the case, however, that we wish to calibrate a binomial model to the **risk-neutral dynamics**[2] of a stock following a GBM model. In that case, if the stock has a continuous dividend yield of c so that a dividend of size $cS_t\,dt$ is paid at time t then the risk-neutral dynamics of the stock can be shown to satisfy

$$S_{t+s} \;=\; S_t\; e^{(r-c-\sigma^2/2)s \;+\; \sigma(B_{t+s}-B_t)} \tag{10}$$

where B_t is now a standard Brownian motion under the risk-neutral distribution. The corresponding q for the binomial model can be obtained from (9) with μ replaced by $r - c$ and with u and d unchanged.

3.2 Black-Scholes in Practice and Implied Volatilities

The Black-Scholes model is an elegant model but it does not perform very well in practice. For example, it is well known that stock prices jump on occasions and do not always move in the continuous manner predicted by the GBM motion model. Stock prices also tend to have fatter tails than those predicted by GBM. Finally, if the Black-Scholes model were correct then we should have a flat *implied volatility surface*. The volatility surface is a function of strike, K, and time-to-maturity, T, and is defined implicitly

$$C(S,K,T) := \mathsf{BS}\left(S,T,r,c,K,\sigma(K,T)\right) \tag{11}$$

where $C(S,K,T)$ denotes the current **market price** of a call option with time-to-maturity T and strike K, and $\mathsf{BS}(\cdot)$ is the Black-Scholes formula for pricing a call option. In other words, $\sigma(K,T)$ is the volatility that, when substituted into the Black-Scholes formula, gives the market price, $C(S,K,T)$. Because the Black-Scholes formula is continuous and increasing in σ, there will always be a unique solution $\sigma(K,T)$ to (11). If the Black-Scholes model were correct then the volatility surface would be flat with $\sigma(K,T) = \sigma$ for all K and T. In practice, however, not only is the volatility surface not flat but it actually moves randomly, and often significantly, with time. The Black-Scholes model is therefore not just wrong (like all models!) but seriously wrong. Indeed it has been said that the implied volatility is the wrong number going into the wrong model to create the right price!

[1] This calibration becomes more accurate as $\Delta t \to 0$.

[2] The term "risk-neutral dynamics" refers to the behavior of the process when the risk-neutral probabilities are used.

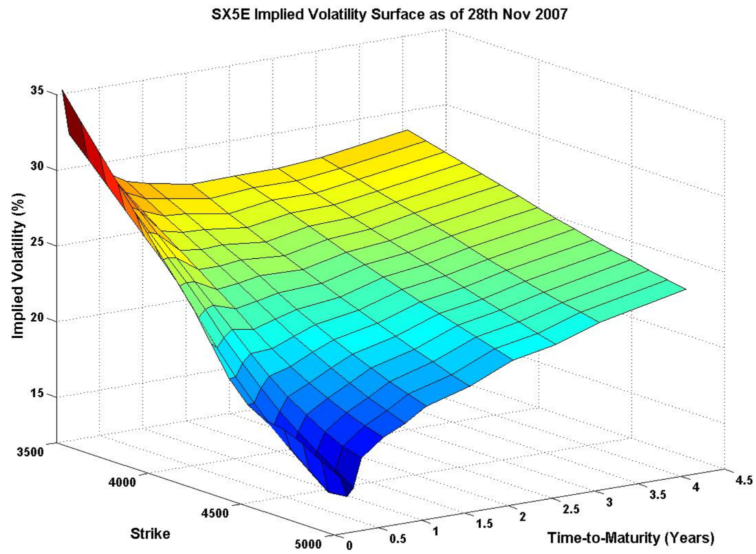


Figure 1: The Volatility Surface

In Figure 1 above we see a snapshot of the[3] volatility surface for the Eurostoxx 50 index on November 28^{th}, 2007. The principal features of the volatility surface is that options with lower strikes tend to have higher implied volatilities. For a given maturity, T, this feature is typically referred to as the volatility **skew** or **smile**. For a given strike, K, the implied volatility can be either increasing or decreasing with time-to-maturity. In general, however, $\sigma(K,T)$ tends to converge to a constant as $T \to \infty$. For T small, however, we often observe an inverted volatility surface with short-term options having much higher volatilities than longer-term options. This is particularly true in times of market stress.

Clearly then the Black-Scholes model is far from accurate and market participants are well aware of this. However, the language of Black-Scholes is pervasive. Every trading desk computes the Black-Scholes implied volatility surface and the Greeks they compute and use are Black-Scholes Greeks.

3.3 The Greeks

We now turn to the sensitivities of the option prices to the various model parameters. These sensitivities, known as the **Greeks**, are usually computed using the Black-Scholes formula and are used throughout the finance industry for hedging options and derivatives portfolios. (This is despite the fact that the Black-Scholes model is known to be a poor approximation to reality!)

Consider a European call and put options, each with the same strike K and maturity T. Assuming a continuous dividend yield, c, then put-call parity[4] at time t states

$$e^{-r(T-t)}\,K + \text{ Call-Price}_t = e^{-c(T-t)}\,S_t + \text{ Put-Price}_t. \tag{12}$$

Put-call parity is useful for calculating Greeks. For example[5], it implies that Vega(Call) = Vega(Put) and that Gamma(Call) = Gamma(Put). It is also extremely useful for **calibrating dividends** and constructing the so-called volatility surface in practice. The principal Greeks for European call options are described below. The Greeks for put options can be calculated in the same manner or via put-call parity.

[3]Note that by put-call parity the implied volatility $\sigma(K,T)$ for a given European call option will be also be the implied volatility for a European put option of the same strike and maturity. Hence we can talk about "the" implied volatility surface.

[4]This is a version of (6) but in continuous time. In this case the dividend term D in (6) is $\left(1 - e^{-c(T-t)}\right) S_t$.

[5]See below for definitions of vega and gamma.

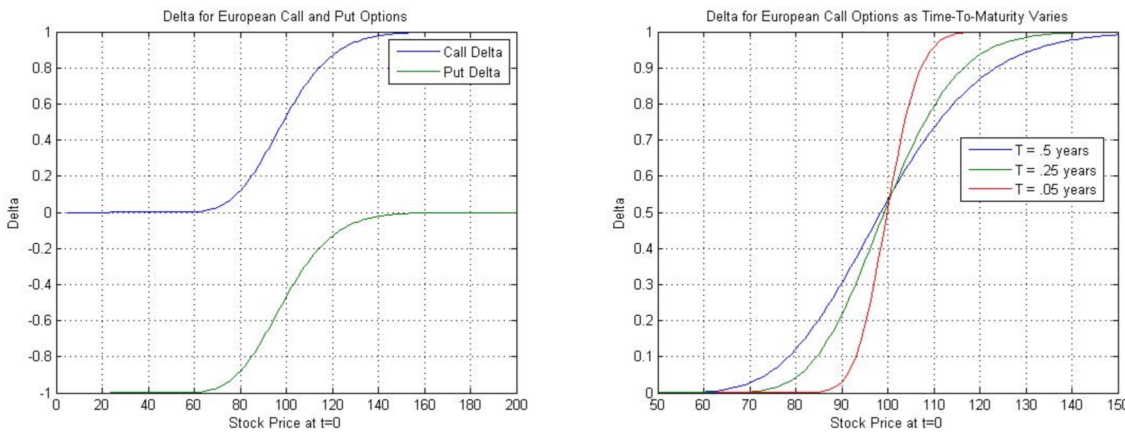
3.3.1 Delta

Definition: The **delta** of an option is the sensitivity of the option price to a change in the price of the underlying security.

The delta of a European call option satisfies

$$\text{delta}_{\text{call}} = \frac{\partial C}{\partial S} = e^{-c(T-t)}\ \Phi(d_1).$$

By put-call parity, we have $\text{delta}_{\text{put}} = \text{delta}_{\text{call}} - e^{-c(T-t)}$. Figure 2(a) shows the delta for a call and put



(a) Delta for European Call and Put Options (b) Delta for Call Options as Time-To-Maturity Varies

Figure 2: Delta for European Options

option, respectively, as a function of the underlying stock price. In Figure 2(b) we show the delta for a call option as a function of the underlying stock price for three different times-to-maturity. It was assumed $r = c = 0$. What is the strike K? Note that the delta becomes steeper around K when time-to-maturity decreases. Note also that $\text{delta}_{\text{call}} = \Phi(d_1)$ is often loosely interpreted as the (risk-neutral) probability that the option will expire in the money.

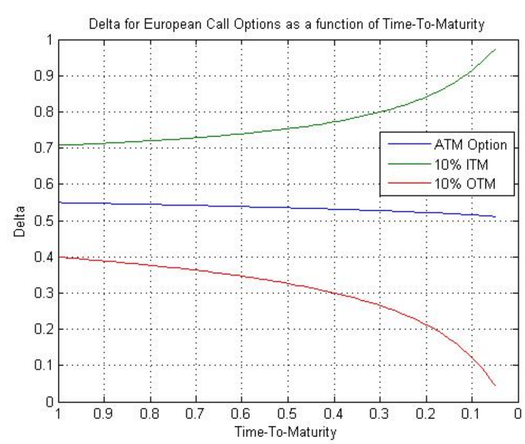


Figure 3: Delta for European Call Options as a Function of Time-To-Maturity

In Figure 3 we show the delta of a call option as a function of time-to-maturity for three options of different **moneyness**. Are there any surprises here? What would the corresponding plot for put options look like?

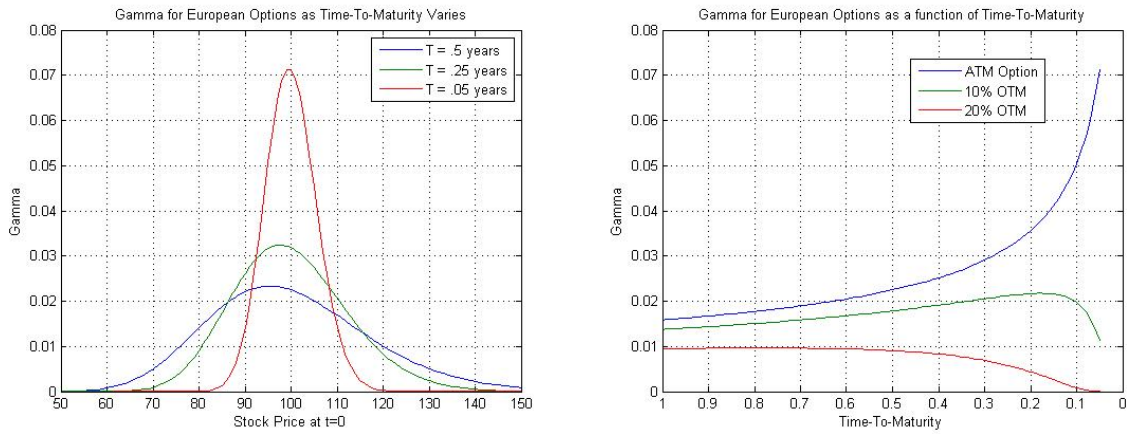
3.3.2 Gamma

Definition: The **gamma** of an option is the sensitivity of the option's delta to a change in the price of the underlying security.

The gamma of a call option satisfies

$$\text{gamma} \;=\; \frac{\partial^2 C}{\partial S^2} \;=\; e^{-c(T-t)}\frac{\phi(d_1)}{\sigma S\sqrt{T-t}}$$

where $\phi(\cdot)$ is the standard normal PDF.



(a) Gamma as a Function of Stock Price (b) Gamma as a Function of Time-to-Maturity

Figure 4: Gamma for European Options

In Figure 4(a) we show the gamma of a European option as a function of stock price for three different time-to-maturities. Note that by put-call parity, the gamma for European call and put options with the same strike are equal. Gamma is always positive due to option **convexity**. Traders who are long gamma can make money by gamma *scalping*. Gamma scalping is the process of regularly re-balancing your options portfolio to be delta-neutral. However, you must pay for this long gamma position up front with the option premium. In Figure 4(b), we display gamma as a function of time-to-maturity. Can you explain the behavior of the three curves in Figure 4(b)?

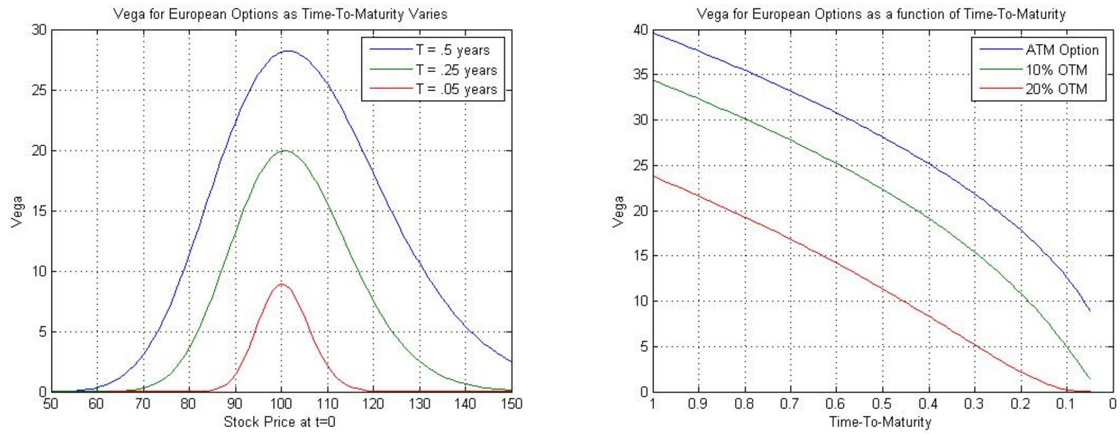
3.3.3 Vega

Definition: The **vega** of an option is the sensitivity of the option price to a change in volatility.

The vega of a call option satisfies

$$\text{vega} \;=\; \frac{\partial C}{\partial \sigma} \;=\; e^{-c(T-t)}S\sqrt{T-t}\;\phi(d_1).$$

In Figure 5(b) we plot vega as a function of the underlying stock price. We assumed $K = 100$ and that $r = c = 0$. Note again that by put-call parity, the vega of a call option equals the vega of a put option with the same strike. Why does vega increase with time-to-maturity? For a given time-to-maturity, why is vega peaked



(a) Vega as a Function of Stock Price

(b) Vega as a Function of Time-to-Maturity

Figure 5: Vega for European Options

near the strike? Turning to Figure 5(b), note that the vega decreases to 0 as time-to-maturity goes to 0. This is consistent with Figure 5(a). It is also clear from the expression for vega.

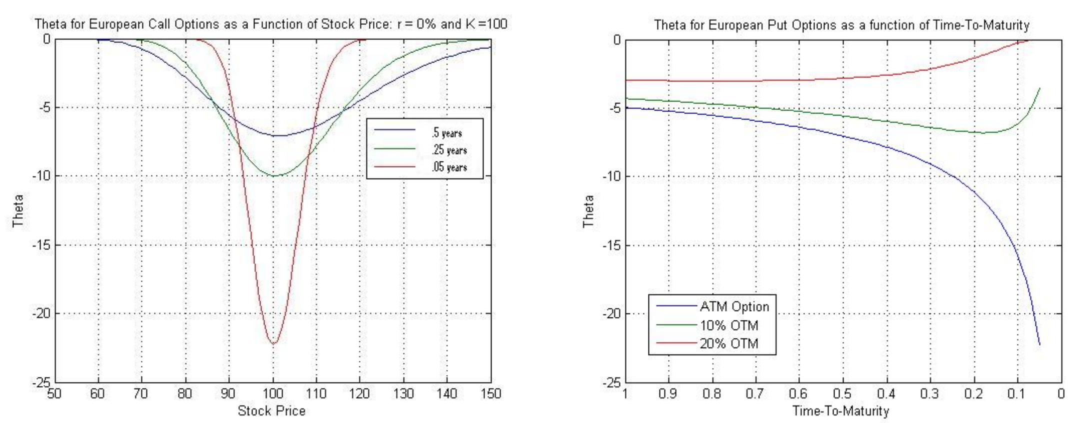
Question: Is there any "inconsistency" to talk about vega when we use the Black-Scholes model?

3.3.4 Theta

Definition: The **theta** of an option is the sensitivity of the option price to a *negative* change in time-to-maturity.

The theta of a call option satisfies

$$\text{theta} \;=\; -\frac{\partial C}{\partial T} \;\;=\;\; -e^{-c(T-t)} S\phi(d_1)\frac{\sigma}{2\sqrt{T-t}} \;+\; ce^{-c(T-t)} S\mathsf{N}(d_1) \;-\; rKe^{-r(T-t)}\mathsf{N}(d_2).$$



(a) Theta as a Function of Stock Price

(b) Theta as a Function of Time-to-Maturity

Figure 6: Theta for European Options

In Figure 6(a) we plot theta for three call options of different times-to-maturity as a function of the underlying stock price. We have assumed that $r = c = 0$. Note that the call option's theta is always negative. Can you explain why this is the case? Why does theta become more negatively peaked as time-to-maturity decreases to 0?

In Figure 6(b) we again plot theta for three call options of different moneyness, but this time as a function of time-to-maturity. Note that the ATM option has the most negative theta and this gets more negative as time-to-maturity goes to 0. Can you explain why?

Options Can Have Positive Theta: In Figure 7 we plot theta for three put options of different moneyness as a function of time-to-maturity. We assume here that $c = 0$ and $r = 10\%$. Note that theta can be positive for in-the-money put options. Why? We can also obtain positive theta for call options when c is large. In typical scenarios, however, theta for both call and put options will be negative.

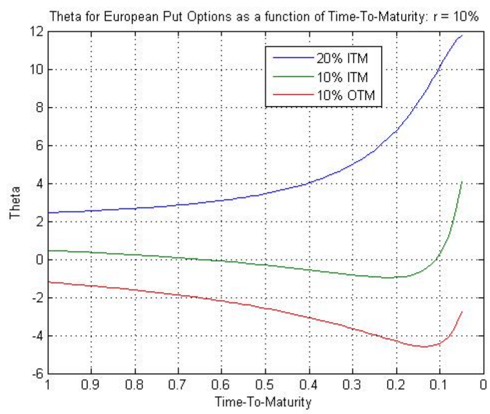


Figure 7: Positive Theta is Possible

4 Risk-Management of Derivatives Portfolios

Risk management is a key problem in finance and so here we briefly discuss some approaches tailored to the risk management of derivatives portfolios. By way of example, we will only consider portfolios of equity options and futures but the methods we discuss apply to derivatives portfolios in other asset classes, e.g. fixed income, currencies, credit and commodities, as well. Moreover, the scenario analysis approach that we outline in Section 4.2 can easily be applied to equity and fixed income portfolios via the use of *factor models* for those asset classes.

4.1 Delta Hedging in the Black-Scholes Model

Consider, for example, the use of the Black-Scholes model to delta-hedge a vanilla European call option. Moreover, we will assume that the assumptions of Black-Scholes are correct so that the security price has GBM dynamics, it is possible to trade continuously at no cost and borrowing and lending at the risk-free rate are also possible. It is then possible to dynamically *replicate* the payoff of the call option using a self-financing (s.f.) trading strategy. The initial value of this s.f. strategy is the Black-Scholes price of the option. The s.f. replication strategy requires the continuous *delta-hedging* of the option but of course it is not practical to do this and so instead we hedge periodically. (Periodic or discrete hedging then results in some *replication error* but this error goes to 0 as the time interval between re-balancing goes to 0 *if* the true stock price dynamics are GBM. Of course in practice true stock price dynamics are not GBM dynamics.)

Towards this end, we assume we have sold the option at time $t = 0$ and we then use the resulting payment of

C_0 to fund the s.f. strategy. Its time t value P_t therefore satisfies

$$P_0 := C_0 \tag{13}$$

$$P_{t_{i+1}} = P_{t_i} + (P_{t_i} - \delta_{t_i} S_{t_i}) r\Delta t + \delta_{t_i}\left(S_{t_{i+1}} - S_{t_i} + cS_{t_i}\Delta t\right) \tag{14}$$

where

- $\Delta t := t_{i+1} - t_i$ is the length of time between re-balancing (assumed constant for all i)
- r is the annual risk-free interest rate (assuming per-period compounding)
- c is the annual dividend yield so that holding the stock between times t_i and t_{i+1} yields a dividend payment of $cS_{t_i}\Delta t$ in that time interval
- δ_{t_i} is the Black-Scholes delta at time t_i. This delta is a function of S_{t_i} and some assumed implied volatility, σ_{imp} say.

Note that (13) and (14) respect the self-financing condition. Stock prices are simulated assuming $S_t \sim \mathsf{GBM}(\mu, \sigma)$ so that

$$S_{t+\Delta t} = S_t e^{(\mu - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} Z}$$

where $Z \sim \mathsf{N}(0,1)$. In the case of our short position in a call option with strike K and maturity T, the final trading P&L is then given by

$$\text{P\&L} := P_T - (S_T - K)^+ \tag{15}$$

where P_T is the terminal value of the replicating strategy in (14). In the Black-Scholes world we have $\sigma = \sigma_{\mathsf{imp}}$ and the P&L will be 0 along every price path in the limit as $\Delta t \to 0$.

In practice, however, we do not know σ and so the market (and hence the option hedger) has no way to ensure a value of σ_{imp} such that $\sigma = \sigma_{imp}$. This has interesting implications for the trading P&L and it means in particular that we cannot exactly replicate the option even if all of the assumptions of Black-Scholes are correct. In Figure 8 we display histograms of the P&L in (15) that results from simulating 100,000 sample paths of the underlying price process with $S_0 = K = \$100$. (Other parameters and details are given below the figure.) In the case of the first histogram the true volatility was $\sigma = 30\%$ with $\sigma_{\mathsf{imp}} = 20\%$ and the option hedger makes (why?) substantial losses. In the case of the second histogram the true volatility was $\sigma = 30\%$ with $\sigma_{\mathsf{imp}} = 40\%$ and the option hedger makes (why?) substantial gains.

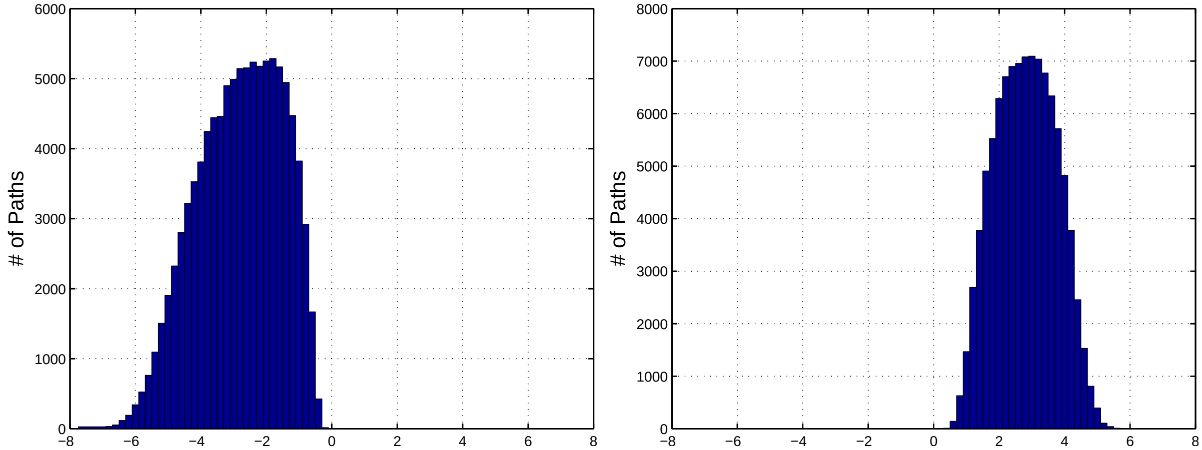
Clearly then this is a situation where substantial errors in the form of non-zero hedging P&L's are made and this can only be due to the use of incorrect model parameters. This example is intended to highlight the importance of not just having a good model but also having the correct model parameters.

Exercise 4 *Make sure you understand why losses are made in Figure 8(a) while gains are made in Figure 8(b).*

Returning to the self-financing trading strategy of (13) and (14), note that we can choose any model we like for the security price dynamics. In particular, we are not restricted to choosing GBM and other models could be used instead. It is interesting to simulate these alternative models and to then observe what happens to the replication error when the δ_{t_i}'s are computed assuming (incorrectly) GBM price dynamics. Note that it is common to perform simulation experiments like this when using a model to price and hedge a particular security. The goal then is to understand how robust the hedging strategy (based on the given model) is to alternative price dynamics that might prevail in practice. Given the appropriate data, one can also **backtest** the performance of a model on realized historical price data to assess its hedging performance. This back-testing is sometimes called a **historical simulation**.

4.2 Scenario Analysis

The scenario approach to risk management defines a number of scenarios where in each scenario various risk factors, e.g. the price of the underlying security and implied volatility, are assumed to have moved by some fixed amounts. For example, a scenario might assume that all stock prices have fallen by 10% and all implied



(a) Delta-hedging P&L: true vol. = 30%, imp. vol. = 20%. (b) Delta-hedging P&L: true vol. = 30%, imp. vol. = 40%.

Figure 8: Histogram of P&L from simulating 100,000 paths where we hedge a short call position with $S_0 = K = \$100$, $T = 6$ months, true volatility $\sigma = 30\%$, and $r = c = 1\%$. A time step of $dt = 1/2,000$ was used so hedging P&L due to discretization error is negligible. The hedge ratio, i.e. delta, was calculated using the implied volatility that was used to calculate the initial option price.

volatilities have increased by 5 percentage points. Another scenario might assume the same movements but with an additional steepening of the volatility surface. The risk of a portfolio could then be determined by computing the P&L in each scenario and then considering the results, taking action (to reduce risk) where appropriate.

Underlying	SPX Index								
	Underlying and Volatility Stress Table								
Sum of PnL	Vol Stress								
Underlying Stress	-10	-5	-2	-1	0	1	2	5	10
-20	13,938	11,774	10,631	10,277	9,936	9,608	9,293	8,419	7,183
-10	6,109	4,946	4,436	4,291	4,158	4,035	3,922	3,634	3,296
-5	1,831	1,652	1,637	1,643	1,654	1,670	1,689	1,766	1,946
-2	(314)	89	356	447	539	631	723	1,001	1,461
-1	(920)	(338)	15	132	248	363	478	816	1,361
0	(1,463)	(714)	(280)	(139)	0	137	273	668	1,293
1	(1,939)	(1,035)	(527)	(363)	(203)	(45)	110	559	1,259
2	(2,346)	(1,300)	(723)	(539)	(359)	(182)	(9)	489	1,258
5	(3,125)	(1,744)	(1,003)	(769)	(541)	(318)	(102)	518	1,460
10	(2,921)	(1,297)	(423)	(146)	123	385	641	1,372	2,483
20	2,344	3,559	4,272	4,506	4,738	4,967	5,194	5,860	6,919

Figure 9: P&L for an Options Portfolio on the S&P 500 under Stresses to Underlying and Implied Volatility

Figure 9 shows the P&L under various scenarios of an options portfolio with the S&P 500 as the underlying security. The vertical axis represents *percentage* shifts in the price of the underlying security, i.e. the S&P 500, whereas the horizontal axis represents *absolute* changes in the implied volatility of each option in the portfolio. For example, we see that if the S&P 500 were to fall by 20% and implied volatilities were to all rise by 5 percentage points, then the portfolio would gain 8.419 million dollars (assuming that the numbers in Figure 9 are expressed in units of $1,000$ dollars).

While scenario tables are a valuable source of information there are many potential pit-falls associated with using them. These include:

1. *Identifying the relevant risk factors*

   While it is usually pretty clear what the main risk factors for a particular asset class are, it is quite possible that a portfolio has been constructed so that it is approximately neutral to changes in those risk factors. Such a portfolio might then only have (possibly very large) exposures to secondary risk factors. It is important then to include shifts in these secondary factors in any scenario analysis. The upshot of this observation is that the relevant risk factors *depend* on the specific portfolio under consideration rather than just the asset class of the portfolio.

2. *Identifying "reasonable" shifts for these risk factors*

   For example, we may feel that a shift of -10% is plausible for the S&P 500 because we know from experience that such a move, while extreme, is indeed possible in a very volatile market. But how do we determine plausible shifts for less transparent risk factors? The answer typically lies in the use of statistical techniques such as PCA, extreme-value theory, time series methods, common sense(!) etc.

A key role of any risk manager then is to understand what scenarios are plausible and what scenarios are not. For example, in a crisis we would expect any drop in the price of the underlying security to be accompanied by a rise in implied volatilities. We would therefore pay considerably less attention to the numbers in the upper left quadrant of Figure 9.

4.3 Delta-Gamma-Vega Approximations to Derivatives Prices

A simple application of Taylor's Theorem says

$$
\begin{aligned}
C(S+\Delta S, \sigma+\Delta\sigma) &\approx C(S,\sigma) + \Delta S \frac{\partial C}{\partial S} + \frac{1}{2}(\Delta S)^2 \frac{\partial^2 C}{\partial S^2} + \Delta\sigma \frac{\partial C}{\partial \sigma} \\
= C(S,\sigma) &+ \Delta S \times \delta + \frac{1}{2}(\Delta S)^2 \times \Gamma + \Delta\sigma \times \text{vega}
\end{aligned}
$$

where $C(S,\sigma)$ is the price of a derivative security as a function of the current stock price S and the volatility parameter σ. We therefore obtain[6]

$$
\begin{aligned}
\text{P\&L} &= C(S+\Delta S, \sigma+\Delta\sigma) - C(S,\sigma) \\
&\approx \delta \Delta S + \frac{\Gamma}{2}(\Delta S)^2 + \text{vega } \Delta\sigma \\
&= \text{delta P\&L} + \text{gamma P\&L} + \text{vega P\&L} .
\end{aligned} \tag{16}
$$

Using (16) we can also write

$$
\begin{aligned}
\text{P\&L} &= \delta S \left(\frac{\Delta S}{S}\right) + \frac{\Gamma S^2}{2}\left(\frac{\Delta S}{S}\right)^2 + \text{vega } \Delta\sigma \\
&= \text{ESP} \times \text{Return} + \text{\$Gamma} \times \text{Return}^2 + \text{vega } \times \Delta\sigma
\end{aligned} \tag{17}
$$

where ESP denotes the **equivalent stock position** or **"dollar" delta**.

When constructing scenario tables as in Figure 9 we can use approximations like (17) to check for internal consistency and to help identify possible bugs in the software. For example, suppose the S&P falls by 10% and all implied volatilities increase by 5 volatility points. Then according to Figure 9, the portfolio in question should gain \$3.634m. The delta-gamma-vega approximation of (17) suggests that the P&L should be

$$
\text{P\&L} \approx \text{ESP} \times (-10\%) + \text{\$Gamma} \times (-10\%)^2 + \text{vega } \times 5
$$

[6]When $\Delta\sigma = 0$, we obtain the well-known *delta-gamma* approximation. This approximation is often used, for example, in historical **Value-at-Risk** (VaR) calculations for portfolios that include options. Unfortunately we don't have time to discuss VaR in this course but VaR (and the related CVaR) are important measures of risk that are used throughout the financial services.

which should be close to the true number \$3.634m. Knowing the dollar delta, \$Gamma and vega of the portfolio is therefore enough to estimate any entry in the scenario table. As a result, (and depending on the particular asset class) investors / traders / risk managers should always know their exposure to the Greeks, i.e. dollar delta, dollar gamma and vega etc.

It is very important to note, however, that approximations such as (17) are **local** approximations as they are based (via Taylor's Theorem) on relatively "small" moves in the risk factors. These approximations can and indeed do break down in violent markets where changes in the risk factors can be very large.