Financial Analytics

Introduction to Real Options

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Outline

Introduction to Real Options

Principal characteristics shared by real options problems:

1. They involve non-financial assets, e.g. factory capacity, oil leases, commodities, technology from R&D etc.

   Often the case, however, that financial uncertainty is also present.

2. Incomplete markets the natural setting since typically not possible to construct s.f. strategy that replicates the payoffs..

   e.g. Not possible to construct a s.f. strategy that replicates a payoff whose value depends on whether or not there is oil in a particular oilfield, or whether or not a particular manufacturing product will be popular with consumers.

   So use economic considerations to choose a good set of r.n. probabilities.

3. There are usually options available to the decision-maker.

   More generally, real options problems are usually control problems where the decision-maker can (partially) control some of the quantities under consideration.

A Real Options Example: The Simplico Gold Mine

Gold Price Lattice

$q = \frac{R-d}{u-d} = \frac{2}{3}$

$1-q = \frac{1}{3}$

									2063.9
								1719.9	1547.9
							1433.3	1289.9	1161.0
						1194.4	1075.0	967.5	870.7
					995.3	895.8	806.2	725.6	653.0
				829.4	746.5	671.8	604.7	544.2	489.8
			691.2	622.1	559.9	503.9	453.5	408.1	367.3
		576.0	518.4	466.6	419.9	377.9	340.1	306.1	275.5
	480.0	432.0	388.8	349.9	314.9	283.4	255.1	229.6	206.6
400.0	360.0	324.0	291.6	262.4	236.2	212.6	191.3	172.2	155.0
Date 0	1	2	3	4	5	6	7	8	9

- Current market price of gold is $\$400$ and it follows a binomial model:
  - it increases each year by a factor of 1.2 with probability $.75$ $= p$
  - or it decreases by a factor of $.9$ with probability $.25$.
- Interest rates are flat at $r = 10\%$ per year. $R = 1.1$

Luenberger's Simplico Gold Mine

- Gold can be extracted from the Simplico gold mine at a rate of up to $10,000$ ounces per year at a cost of $C = \$200$ per ounce.
- Want to compute price of a lease on the mine that expires after 10 years.
- Any gold that is extracted in a given year is sold at the end of the year at the price that prevailed at the beginning of the year.
- Gold is a traded commodity so we can obtain a unique risk-neutral price for any derivative security dependent upon its price process
  - risk-neutral probabilities are found to be $q = 2/3$ and $1 - q = 1/3$.
- Value of lease is then computed by working backwards in the lattice below
  - because the lease expires worthless the node values at $t = 10$ are all zero.

A Real Options Example: the Simplico Gold Mine

```
Lease Value (in millions)

                                                                  16.9
                                                           27.8   12.3
                                                    34.1   20.0    8.7
                                             37.1   24.3   14.1    6.1
                                      37.7   26.2   17.0    9.7    4.1
                               36.5   26.4   18.1   11.5    6.4    2.6
                        34.2   25.2   17.9   12.0    7.4    3.9    1.5
                 31.2   23.3   16.7   11.5    7.4    4.3    2.1    0.7
          27.8   20.7   15.0   10.4    6.7    4.0    2.0    0.7    0.1
   24.1   17.9   12.9    8.8    5.6    3.2    1.4    0.4    0.0    0.0

Date 0       1      2      3      4      5      6      7      8      9
```

e.g. Value of 16.9 on uppermost node at $t = 9$ is obtained by discounting the profits earned at $t = 10$ back to the beginning of the year:

$$16.94\text{m} = 10\text{k} \times (\underbrace{2,063.9 - 200}_{\text{net profit}})/1.1.$$

A Real Options Example: the Simplico Gold Mine

Lease Value (in millions)

										16.9
									27.8	12.3
								34.1	20.0	8.7
							37.1	24.3	14.1	6.1
						37.7	26.2	17.0	9.7	4.1
					36.5	26.4	18.1	11.5	6.4	2.6
				34.2	25.2	17.9	12.0	7.4	3.9	1.5
			31.2	23.3	16.7	11.5	7.4	4.3	2.1	0.7
		27.8	20.7	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.1	17.9	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Node value in any earlier year obtained by summing together discounted expected value of lease and profit (obtained at the end of year) back to beginning of year.

e.g. In year 6 central node has a value of 12.0 because:

$$12.0\text{m} = \frac{10\text{k} \times (503.9 - 200)}{1.1} + \frac{q \times 11.5\text{m} + (1-q) \times 7.4\text{m}}{1.1}.$$

discount back one year

Luenberger's Simplico Gold Mine

Lease Value (in millions)

$V_7(s)$

$V_0(S_0)$

										16.9
									27.8	12.3
								34.1	20.0	8.7
							37.1	24.3	14.1	6.1
						37.7	26.2	17.0	9.7	4.1
					36.5	26.4	18.1	11.5	6.4	2.6
				34.2	25.2	17.9	12.0	7.4	3.9	1.5
			31.2	23.3	16.7	11.5	7.4	4.3	2.1	0.7
		27.8	20.7	15.0	10.4	6.7	4.0	2.0	0.7	0.1
	24.1	17.9	12.9	8.8	5.6	3.2	1.4	0.4	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Of course never optimal to extract gold when price less than $\$200$.

Backwards evaluation therefore takes the form

$$V_t(s) = \frac{\underbrace{10\text{k} \times \max\{0, s - C\}}_{\text{the gold mined}} + \left(q V_{t+1}(us) + (1-q)\, V_{t+1}(ds)\right)}{1+r}$$

where $V_t(s) =$ time t value of lease when gold price is s and $C = 200$.

An Aside on Dynamic Programming (DP)

Recall how we priced an American put option in the binomial model:

- K is the strike.
- T is the maturity.

If $V_t(S) =$ option price at time t when underlying price is S then we saw

$$
\begin{aligned}
V_t(S) &= \max\left\{K - S,\ \frac{1}{R}\left[q \times V_{t+1}(uS) + (1-q) \times V_{t+1}(dS)\right]\right\} \\
&= \max\left\{\underbrace{K - S}_{\text{stop}},\ \underbrace{\frac{1}{R}\mathsf{E}_t^Q\left[V_{t+1}(S_{t+1})\right]}_{\text{continue}}\right\}. \qquad (1)
\end{aligned}
$$

- In fact (1) also holds in general and not just for the binomial model
  - it's an example of the Bellman equation from dynamic programming
- This particular problem is a so-called optimal stopping problem which is the simplest type of control problem.

An Aside on Dynamic Programming (DP)

Control problems occur throughout finance and examples include:

- Pricing American options.
- Dynamic portfolio optimization.
- Trading execution where the goal is to purchase or sell a block of shares in a fixed period of time.
- Pricing swing options which are common in the energy derivatives markets.

Control problem (often optimal stopping problems) often occur in real-options problems

- as we'll see again when we return to the Simplico goldmine and consider purchasing some new equipment.

The Bellman Equation for Swing Options

A swing option is similar to an American option but now the option can be exercised a total of n times out of $T > n$ periods in total.

If the swing option can be exercised at most once per period then the Bellman equation is

$V_T(S_T, m) = \begin{cases} h(S), & m > 0 \\ 0, & m = 0 \end{cases}$

payoff

discount value of one period ahead

$$V_t(S, m) = \max \left\{ \underbrace{h(S) + \frac{1}{R} \mathsf{E}_t^Q \left[V_{t+1}(S_{t+1}, m-1) \right]}_{\text{exercise}}, \underbrace{\frac{1}{R} \mathsf{E}_t^Q \left[V_{t+1}(S_{t+1}, m) \right]}_{\text{don't exercise}} \right\}$$

underlying price

exercise opportunity

one less exercising opportunity

where:

- $V_t(S, m) =$ time t value of the swing option when the underlying price is S and there are $m \leq n$ exercise opportunities remaining
- $h(S)$ is the exercise payoff, **e.g.** $h(S) = \max(0, S - K)$.

Control problems (often optimal stopping problems) often occur in real-options problems – as we'll see again when we return to the Simplico goldmine.

Should We Enhance the Simplico Goldmine?

- Suppose it's possible to enhance extraction rate to $12,500$ ounces per year by purchasing new equipment that costs $\$4$ million.
- Once new equipment in place then it remains in place for all future years.
- Moreover the extraction cost would also increase to $\$240$ per ounce with the enhancement in place.
- New equipment becomes property of original mine owner at end of lease.
- Lease owner therefore has an option to install the new equipment at any time
  - we want to determine value of this option!
- To do this, must first compute lease value assuming new equipment is in place at $t = 0$
  - done in exactly the same manner as before
  - values at each node and period are given in following lattice.

Should We Enhance the Simplico Goldmine?

Lease Value Assuming Enhancement in Place (in millions)

										20.7
									33.9	14.9
								41.4	24.1	10.5
							44.8	29.2	16.8	7.2
						45.2	31.2	20.0	11.3	4.7
					43.5	31.0	21.0	13.2	7.2	2.8
				40.4	29.3	20.4	13.4	8.0	4.1	1.4
			36.4	26.6	18.7	12.5	7.7	4.1	1.8	0.4
		31.8	23.3	16.3	10.8	6.5	3.4	1.3	0.2	0.0
	27.0	19.5	13.5	8.6	4.9	2.3	0.8	0.1	0.0	0.0
Date	0	1	2	3	4	5	6	7	8	9

Backwards evaluation therefore takes the form

$$V_t^{\mathsf{eq}}(s) = \frac{12.5\mathsf{k} \times \max\left\{0, s - C_{new}\right\} + \left(q V_{t+1}(us) + (1-q) V_{t+1}(ds)\right)}{1+r}$$

where $V_t^{\mathsf{eq}}(s)$ = time t value of lease when gold price = s and $C_{new} = 240$.

Should We Enhance the Simplico Goldmine?

- $V_t^{\text{eq}}(s) :=$ time t lease value when gold price $= s$ and new equip. in place
  - note the $\$4$ million cost of the new equipment has not been subtracted.
- We find $V_0^{\text{eq}}(400) = 27\text{m}$.
- Now let $U_t(s) :=$ time t price of lease when gold price $= s$ and with the option to enhance in place.
- Can then solve for $U_t(s)$ as follows:

$$U_t(s) = \max\left\{ V_t^{\text{eq}}(s) - 4\text{m}, \; \frac{10\text{k} \times \max\{0, s - \overset{=200}{C}\} + \left(qU_{t+1}(us) + (1-q)\,U_{t+1}(ds)\right)}{1+r} \right\}. \tag{2}$$

- We want $U_0(s)$ with $s = 400$ and we can compute this using (2) and working backwards from $U_9(s)$.

$U_{10}(s) = 0$

- We find $U_0(400) = \$24.6\text{m}$
  - slightly greater than lease value without the option.

Should We Enhance the Simplico Goldmine?

Lease Value with Option for Enhancement (in millions)

buy equipment

not possible, since purchased

										16.9
									29.9*	12.3
								37.4*	20.1*	8.7
							40.8*	25.2*	14.1	6.1
						41.2*	27.2*	17.0	9.7	4.1
					39.5*	27.0*	18.1	11.5	6.4	2.6
				36.4*	25.6	17.9	12.0	7.4	3.9	1.5
			32.6	23.5	16.7	11.5	7.4	4.3	2.1	0.7
		28.6	20.9	15.0	10.4	6.7	4.0	2.0	0.7	0.1
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Date	0	1	2	3	4	5	6	7	8	9

The equation used to compute the lease value:

$$U_t(s) = \max\left\{ V_t^{\text{eq}}(s) - 4\text{m},\ \frac{10\text{k} \times \max\{0, s - C\} + (qU_{t+1}(us) + (1-q)\,U_{t+1}(ds))}{1+r} \right\}$$

is a version of the Bellman equation for dynamic programming.

Zero-Level Pricing with Private Uncertainty

- How do we price real options when a unique set of risk-neutral (r.n.) probabilities not available?
- Zero-level pricing is one approach. The zero-level-price is the price that leaves decision-maker indifferent between purchasing / not purchasing an infinitesimal amount of the security.
- Say a source of uncertainty is private if it is independent of any uncertainty driving the financial markets
  - **e.g.** the success of an R&D project, the quantity of oil in an oilfield, the reliability of a vital piece of manufacturing equipment or the successful launch of a new product
  - could also include incidence of natural disasters etc. as sources of "private" uncertainty.

Economic considerations suggest that if we want to use zero-level pricing to compute real option prices when there is only private uncertainty involved, then we should use the true probabilities to do so and discount by the risk-free interest rate.

- If no arbitrage & market is complete (can repulicate everything)
then there exists a unique set of risk-neutral possibilities Q such that initial price C_0 of any (derivate) security can be calculated as

$$C_0 = E_0^Q\left[\sum_{i=1}^{n} \frac{F_{t_i}}{(1+r)^{t_i}}\right] \qquad (*)$$

where F_{t_i} = cash flow at time t_i that goes to owner of the securities

e.g. $\left.\begin{array}{l} F_T = \max(0, S_T - K) \\ F_{t_i} = 0,\ t_i < T \end{array}\right\}$ for a European call option

(1st fundamental theory of asset pricing.)

- If no arbitrage & market is complete ⇒ there exist many sets of r.n. possibility Q

And using $(*)$ ⇒ no arbitrage 2nd. F.T of a.p.

- If a cashflow can be replicated via a self-financing trading strategy then each possible Q will give the same pricing.
- If a cash flow $(F_{t_1}, F_{t_2} \ldots F_{t_n})$ can't be replicated then different Q's give different prices for that cash flow

Zero-Level Pricing with Private Uncertainty

Some intuition for this observation comes from the CAPM which states

$$\mathsf{E}^P[r_o] \;=\; r_f + \beta_o \left(\mathsf{E}^P[r_m] - r_f\right)$$

where:

- $\beta_o := \mathsf{Cov}(r_m, r_o)/\mathsf{Var}(r_m)$
- r_m is the return on the market portfolio.
- $\mathsf{E}^P[\cdot]$ denotes an expectation computed with the true probabilities.

If $r_o =$ return on an investment that is only exposed to private uncertainty then $\mathsf{Cov}(r_m, r_o) = 0$ and CAPM implies $\mathsf{E}^P[r_o] = r_f$. Value of investment / real option (in a CAPM world) then given by

$$P_0 \;=\; \frac{\mathsf{E}^{\mathbb{P}}[P_1]}{1 + r_f}$$

where P_1 is the terminal payoff of the investment.

This and other similar arguments are used to motivate practice of using:

- risk-neutral probabilities to price financial uncertainty of an investment and
- true probabilities to price the non-financial uncertainty.

Example: Valuing A Foreign Venture

- A particular investment gives you the rights to the monthly profits of a foreign venture for a fixed period of time.
- The first payment will be made one month from now and the final payment will be in 5 months time after which the investment will be worthless.
- The monthly payments are denominated in Euro, and are IID random variables with P-expectation μ.
- Payments also independent of returns in both domestic and foreign financial markets.
- Would like to determine the value of this investment!

Example: Valuing A Foreign Venture

- Let us first assume that the domestic, i.e. US, interest rate is 5% per annum, compounded monthly
  - implies a gross rate of 1.0042 per month.
- Annual interest rate in Euro zone, i.e. the foreign interest rate, is 10%, again compounded monthly
  - implies a per month gross interest rate of 1.0083.
- Can construct a binomial lattice for the \$/Euro exchange rate process if we view the foreign currency, i.e. the Euro, as an asset that pays "dividends", i.e. interest, in each period.
- Risk-neutral pricing in a binomial model for the exchange rate with up- and down-factors, u and d respectively, implies that

$$X_i = \mathsf{E}_i^Q \left[\frac{X_{i+1} + r_f X_i}{1 + r_d} \right] \tag{3}$$

where $X_i =$ \$/Euro exchange rate at time i.

Valuing A Foreign Venture

- Risk-neutral probability q of an up-move satisfies

$$q = \frac{1 + r_d - d - r_f}{u - d} \tag{4}$$

where r_d and r_f are domestic and foreign per-period interest rates, respectively.

- The binomial lattice is given below with $X_0 = 1.20$, $u = 1.05$ and $d = 1/u$.

Dollar/Euro Exchange Rate

					1.53
				1.46	1.39
			1.39	1.32	1.26
		1.32	1.26	1.20	1.14
	1.26	1.20	1.14	1.09	1.04
1.20	1.14	1.09	1.04	0.99	0.94
t=0	t=1	t=2	t=3	t=4	t=5

Valuing A Foreign Venture

Valuing the investment using zero-level pricing (and therefore using the the true probabilities for the non-financial uncertainty) is now straightforward:

- at each time-t node in the lattice we assume there is a cash-flow of μX_t.

These cash-flows are valued as usual by backwards evaluation using the risk-neutral probabilities computed in (4).

Question: Can you see an easy way (that does not require backwards evaluation) to value the cash-flows?