Zhe Liu

Assignment 2: Probability
Total Points: 100 Due: 11:59pm 8 Oct, 2020

Rules:

1. This is a group assignment. Within each group **I strongly encourage everyone to attempt each question by his/herself first** before discussing it with other members of the group.
2. You are recommended to **type out your solution** and submit a PDF file. If you choose to write it down and submit a scanned version, please ensure clear handwriting.
3. `R` is the default package / programming language for this course so you should use `R` for any programming questions in this assignment.
4. Some of the problems are drawn from the exercises in Arnold Barnett's excellent and recent textbook *Applied Probability: Models and Intuition* (Dynamic Ideas, LLC).

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1. **Bayes' Theorem and Buying Wine (25 points)**

   Those who visit a certain website find a banner ad for wine awaiting them. An analysis of data about these visitors finds that 20% of them click onto the ad, and of those who do:

   - 82% purchase no wine
   - 9% purchase a case of red wine
   - 15% purchase a case of white wine

   (Some people purchase both red and white wine, but no one buys more than one case of either.) Please answer the following questions.

   (a) What is the probability that a randomly chosen visitor to the site will not buy any wine there? (5 points)

   (b) Given that the visitor purchased no wine, what is the probability that she clicked on the ad? (5 points)

   (c) Let p be the chance that the visitor purchased red wine only, q be the chance for white wine only, and g for both kinds of wine. Given the numbers above, what are p, q and g? (10 points)

   Hint: Let R, W and RW denote the events that the visitor purchased red wine only, white wine only and both kinds of wine, respectively. Define $p' = \mathbf{P}(R \mid C)$,

$q' = \mathbf{P}(W \mid C)$ and $g' = \mathbf{P}(RW \mid C)$. You can write three simultaneous linear equations for these quantities. Once you have solved for p', q' and g' you can then solve for p, q and g.

(d) Estimate the number of cases of wine sold per thousand visitors to the site. (5 points)

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2. **A Battery Charge (25 points)**

Mendel works as a consultant, and needs a battery to power his laptop so he can work during airplane trips. The brochure about the battery he uses says that the average lifespan of the battery is 60 hours ($= 1/\lambda$); what it does not say, however, is that the lifespan if exponentially distributed with that mean. Because it's traumatic for Mendel if the battery dies during a flight (leaving him unable to work) he replaces a battery when it reaches age 60 (hours) even if it is still working. Of course, if the battery dies before 60 hours, he suffers a trauma and gets a new battery as soon as he lands. Answer the following questions:

(a) Does Mendel's 60-hour rule make sense to you (with full knowledge)? (5 points)

(b) What is the probability that a battery will die before it reaches age 60? (5 points)

(c) If Mendel has just installed a new battery, how long will it operate on average before he replaces it? (10 points)

Hint: First define the following random variables and corresponding events:

X is the battery's time until replacement.
A_1 is the event that the battery dies before 60.
A_2 is the event that the battery is still alive at age 60.

Now note that A_1 and A_2 partition the sample space Ω and use the conditional expectation formula. (You will need to compute $\mathbf{E}[X \mid A_1]$ and there is a clever way to do this by letting Z be the battery's lifespan and noting that $\mathbf{E}[Z] = 60$. But an alternative expression for $\mathbf{E}[Z]$ can be obtained via the conditional expectation formula and then comparing $\mathbf{E}[Z \mid A_1]$ and $\mathbf{E}[X \mid A_1]$.)

(d) By what percentage would his long-term cost of buying batteries go down if he only replaced batteries when they died? (5 points)

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3. **Joint Distributions (20 points)**

Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} kxy & 0 \le x \le 1, 0 \le y \le \sqrt{x}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What value of k makes the above joint PDF a proper one? (5 points)

(b) Find the marginal distributions $f_X(x)$ and $f_Y(y)$. Are X and Y independent? (5 points)

(c) Find the conditional PDF of X given $Y = y$, i.e., $f_{X|Y}(x|y)$. (5 points)

(d) Find the conditional expectation $\mathbf{E}[X|Y = y]$ for $0 \leq y \leq 1$. (5 points)

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4. **Normal Distribution and Revenue Management (20 points)**

Suppose that a 70-room hotel offers rooms for \$129 apiece but has an "EarlyBird" non-refundable rate of \$59 that requires booking at least one month in advance. It believes the demand for full-priced rooms on a particular day (which materializes only a few days ahead) is normally distributed with mean 66 and standard deviation 5. (The conversion rule from the normal curve to integers is the same as that in the "Airline Yield Management" example from the practice problems.) Demand for EarlyBird rooms is unlimited. For simplicity, assume that all bookings are for exactly one night.

Find the number of rooms that the hotel should sell at the EarlyBird rate if its goal is to maximize expected revenue per night. You will need to do this numerically using `R`.

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5. **The CLT and a Party Game (10 points)**

You take a fair die to the party, and announce that you will toss it 25 times. You'll record each outcome and, at the end, average the 25 outcomes together to get their arithmetical average $\bar{X}$. You offer a bet: The player puts down \$1: If $\bar{X}$ exceeds 4 (excluded), you'll give the player \$21 back, but, otherwise, he loses his dollar. Is this a good bet for the player? (Use the CLT to obtain an approximate solution.)
