Imperial College Business School

Candidate Name:
CID Number:

MSc Business Analytics Examinations

For internal Students of Imperial College of Science Technology and Medicine.
This paper also forms part of the examination for the Associateship.

OPTIMISATION AND DECISION MODELS

MOCK EXAM

CLOSED BOOK

Instructions

Answer all three questions.

Question 1: Linear Programming and Duality **(33 marks)**

Caribbean Sun produces canned fruits that are sold to supermarkets. To this end, the two factories A and B of *Caribbean Sun* can purchase pineapples from three suppliers:

Company	**Maximum Monthly Purchase Quantity**	**Price**
Sao Paolo Farms & Co	200t	£10/t
Costa Rica Pineapples Inc	300t	£11/t
Philippines Rural	400t	£9/t

Assume that 1 ton (t) of fresh pineapples results in 1t of canned fruit, and that the canned fruit is sold for £40 to supermarket chains worldwide. For each ton of canned fruit, the factories also encounter direct labour costs of £25 (factory A) and £20 (factory B). Assume that factory A has a production capacity of 500t per month, while factory B has a production capacity of 350t per month.

(a) Formulate a Linear Program that determines the profit-maximising production schedule. **(10 marks)**

(b) Dualise the linear program from part (a), using either the direct or the indirect method. Interpret the dual problem. **(10 marks)**

(c) Try to determine a good solution to the linear program from part (a) by inspection. Do the same for the dual problem from part (b). Looking at both solutions, what can you say about the profits of the optimal production plan for part (a)?
(You don't have to find optimal solutions, but you should justify why your solutions are "good".) **(13 marks)**

Question 2: Integer Programming **(35 marks)**

The marketing department of the *A. E. Ross Company (AER)* is considering several options for its next advertisement campaign. In particular, the group has identified the following options:

	Target audience reached	Man-hours marketing required	Man-hours sales required	Cost (£)
Specialist magazine	200,000	300	300	150,000
General magazine	150,000	200	300	100,000
Newspaper	300,000	400	350	300,000
Radio	450,000	600	450	400,000
Television	600,000	800	600	500,000
Promotion campaign	400,000	300	800	100,000

For simplicity, the marketing department assumes that the target audiences for the different campaign options do not overlap. The options can be combined freely as long as the following three conditions are met:

(i) The firm does not want to advertise in both specialist and general magazines.
(ii) If selected, the promotion campaign must be combined with a radio and/or TV advertisement in order to ensure its effectiveness.
(iii) The overall campaign must not consume more than 1,500 marketing man-hours, 1,200 sales man-hours and £1,500,000 cash.

(a) Formulate a binary optimisation problem that determines the campaign which reaches the widest audience (under the aforementioned constraints). Do *not* solve the problem! **(10 marks)**

(b) The newspaper and television channel mentioned in the table above are both owned by UK Broadcasting (UKBC) Ltd. UKBC also offers the following combined advertisement package: AER can pay £250,000 for the newspaper plus £10,000/min for TV advertisements, as long as they order at least 20min of TV advertisements. AER estimates that they would reach a target audience of 300,000 from the newspaper advertisement, as well as 20,000 per min of TV advertisement. This package does not require any marketing or sales man-hours on AER's side. If AER chooses this package, they cannot select the newspaper or TV packages from the table. How would you include this option into the optimisation problem from part (a)? **(15 marks)**

(c) A different radio channel (not listed in the table above) offers broadcasting packages of 20min, 50min and 100min air time for £4,000/min. At most one of these packages can be selected, but the package can be combined with the radio option from the table above. AER estimates that they would reach a target audience of 5,000 per min broadcasting time. Moreover, each min broadcasting time requires 5 marketing man-hours (but no sales man-hours). How would you include this option into the optimisation problem from part (a)? **(10 marks)**

Question 3: Nonlinear Programming **(32 marks)**

In class, we have discussed the Markowitz model, which is commonly used to decide how to optimally invest a limited budget into *n* assets.

(a) Formulate the Markowitz risk minimisation model, which minimises the portfolio risk subject to a lower bound on the acceptable portfolio return. Assume that short sales are not allowed. Explain all variables and constraints. Is the problem convex? Justify your answer! **(5 marks)**

(b) Consider a variant of the model from part (a) where the selected portfolio should have a 2-norm distance of at most *D* from a given reference portfolio x^0, where the 2-norm between two vectors *x* and *y* is defined as:

$$\|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Formulate this problem variant as a *quadratically constrained quadratic program*, that is, an optimisation problem where the decision variables are continuous and the objective function and all constraints are either linear or quadratic (see below for a definition) in the decision variables!

Is the problem convex? Justify your answer! **(8 marks)**

(c) Consider a variant of the model from part (a) where the selected portfolio should have a 1-norm distance of at most *D* from a given reference portfolio x^0, where the 1-norm between two vectors *x* and *y* is defined as:

$$\|x - y\|_1 = |x_1 - y_1| + \cdots + |x_n - y_n|$$

Formulate this problem variant as a *quadratic program*, that is, an optimisation problem where the decision variables are continuous, the objective function is quadratic (see below for a definition) in the decision variables, and all constraints are linear in the decision variables!

Is the problem convex? Justify your answer! **(14 marks)**

(d) Take the model from part (a), and add the requirement that at most *K* assets can be invested in. Is the problem convex? Justify your answer! **(5 marks)**

Note: We say that a function $f(x)$ is quadratic in the decision vector $x \in \mathbb{R}^n$ whenever the function can be written as $f(x) = x^\top Q x + c^\top x$ for a matrix $Q \in \mathbb{R}^{nxn}$ and a vector $c \in \mathbb{R}^n$.