Data Structures and Algorithms

Live Class 4: Complexity

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Today

- **Search** algorithms
- How to analyse **algorithm complexity**
  - "How does my code slow as my data grows?"

Announcements

Tutors on the chat on the Hub:

- Weekdays 9-11am, 1-3pm, 5-7pm;
  weekends 9-11am, 1-3pm
- Help, advice, support, guidance, comfort
- Rooms for Session 3 and 4 questions

Guessing game

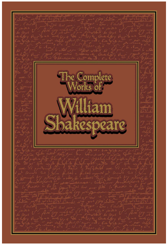
A friend is thinking of a number between 1-100 and you have to guess it.

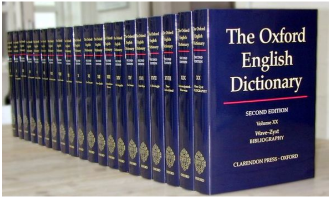
Whenever you make a guess, the friend tells you whether it is correct, too high, or too low.

How many guesses will you need?

Search algorithms

Search for the word “swagger”?





What is the worst we could do?

If the answers were “yes” and “no”?

- Linear search - worst case: go through everything

If the answers are “too low” and “too high”?

- Each time, discard half or remaining numbers
- Binary search - worst case?

Logarithms

Exponentials: $2^4 = 2 \times 2 \times 2 \times 2 = 16$

Logarithm flips the exponential:

- $\log_2 16 = 4$
- “How many 2s do we multiply to get 16?"
- “How many times do we divide 16 by 2 to get to 1?”

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- What about for numbers up to 1024? 2048? 1 000 000?

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Guess the number from 1 to n:

- **Linear search**: at most n guesses
- **Binary search**: at most $\log_2(n)$ guesses

Algorithm design: searching a list

Suppose we have a list `L`. We want to check whether it contains the number `13`.

What are computers good at?

1. Performing simple calculations
    - Arithmetic operations
    - Comparisons
    - Assignments
    - Accessing memory

2. Remembering the results

Goals in designing algorithms

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Efficiency:

- How much **time** will our computation take?
- How much **memory** will it need?

Example: linear search

Is x in list A?

```
def linear_search(A, x):
    for elem in A:
        if elem == x:
            return True
    return False
```

Efficiency:

- How much **time** will our computation take?
- How much **memory** will it need?

How much time will it take?

Simple: run and time it? But time depends on

1. Speed of computer
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- Step: **constant-time computer operation**
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For 3, **measure number of steps** **depending on the size of input**

Complexity and input

Searching for an item in a list?

```
def linear_search(A, x):
    # A is a list of length n
    for elem in A:
        if elem == x:
            return True
    return False
```

Complexity and input

Searching for an item in a list?

```python
def linear_search(A, x):
    # A is a list of length n
    for elem in A:
        if elem == x:
            return True
    return False
```

- x could be the first element of A
- x could not be in A
- **How to give a general complexity measure?**

Complexity cases

Cases for given input size (length of A):

- **Best case** — minimum time
- **Worst case** — maximum time
- **Average case** — average or expected time over all possible inputs

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Principle: focus on worst-case analysis

- Upper bound on running time
- Bonus: usually easier to analyze

Example

```
def sum_up_to(n):
    result = 0                  # 1 step
    while n > 0:                # 1 step, n times
        result = result + n     # 2 steps, n times
        n = n - 1               # 2 steps, n times
    return result               # 1 step
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Total: $5n + 2$ steps

- As n gets large, 2 is irrelevant
- Arguably, so is 5
    - It's the size of the problem that matters

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Principle: ignore constant factors and lower-order terms

- These depend on computer and program implementation
- They do not matter for large inputs
- Simplifies comparisons

Example

```python
def f(x):                       # Let's assume x is integer
    ans = 0                     # 1 step
    for i in range(100):
        ans += 1                # 200 steps
    for i in range(x):
        ans += 1                # 2*x
    for i in range(x):
        for j in range(x):
            ans -= 1            # 2*x^2
    return ans                  # 1 step for return
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Steps: $202 + 2x + 2x^2$

- x small $\rightarrow$ first loop dominates ($x = 3$)

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- Only need to consider last (nested) loop for large x
- Does the 2 in $2x^2$ matter? For large x, order of growth much more important

Asymptotic analysis

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   - Only large problems are interesting
   - What happens when size gets very large?

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Formal way to describe this approach:

- Big-O notation: upper bound on worst-case running time

Big-O: bound on runtime growth rate

Let $T(n)$ be the number of steps taken for input size n:

- Example: $T(n) = 202 + 2n + 2n^2$

Big-O: bound on runtime growth rate

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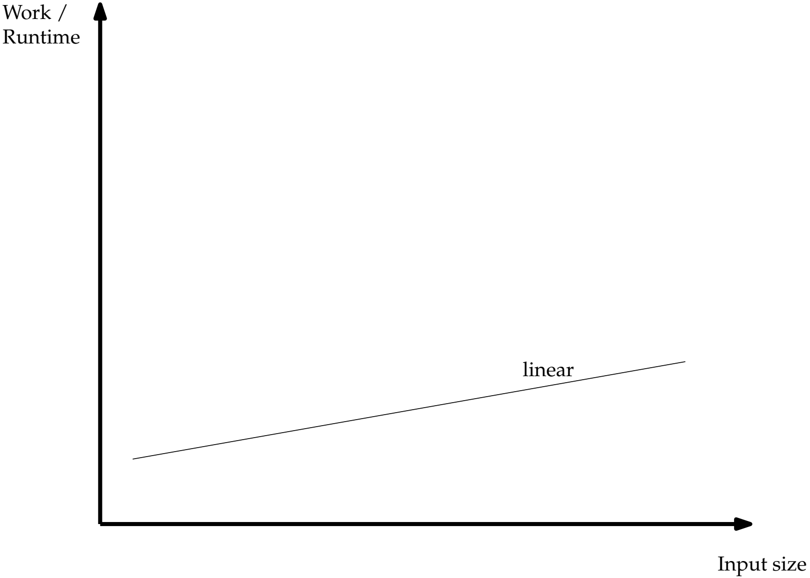
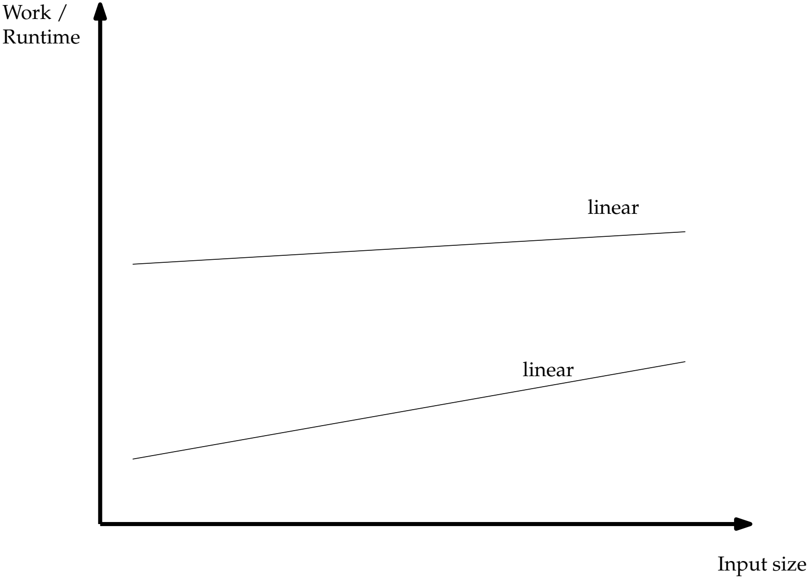
Example: $T(n)$ is $O(n^2)$ if for all large enough n, $T(n)$ is bounded above by a constant multiple of $f(n) = n^2$

- Gist: for high values of n, does $f(n)$ "grow at least as quickly"?
  - $cf(n) = cn^2$? (for any constant c)
  - $T(n) = 202 + 2n + 2n^2$?
- What if $f(n) = n$, that is $O(n)$?


Work /
Runtime
Input size


Work /
Runtime
linear
Input size


Work /
Runtime
linear
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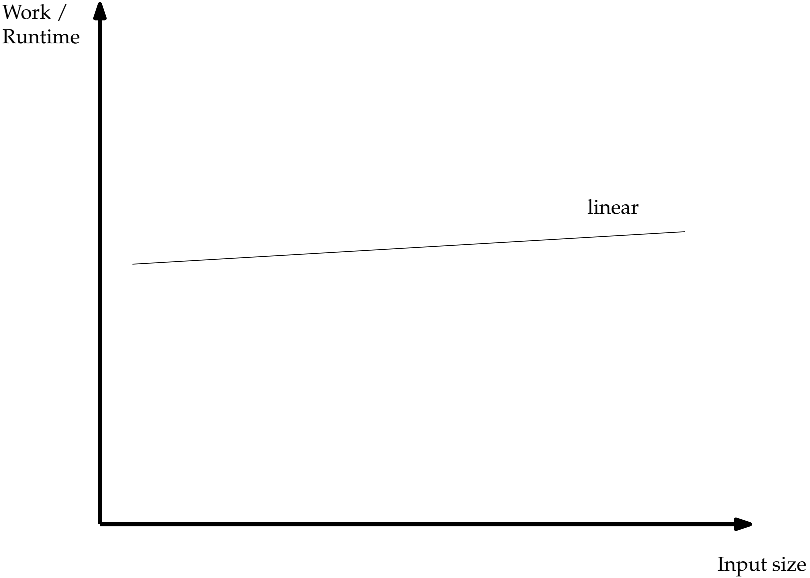
Input size


Work /
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linear
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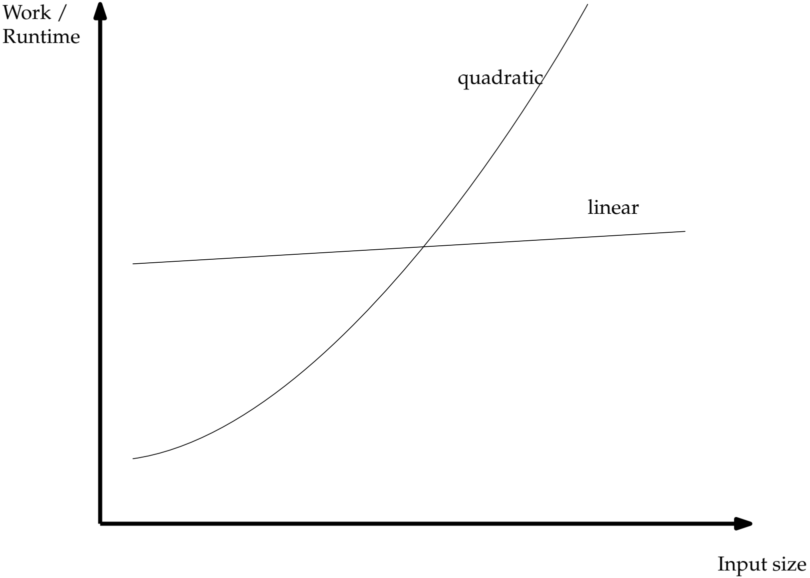
Work /
Runtime
quadratic
linear
Input size

Big O tells us how fast the algorithm is

Fast algorithm: worst-case running time grows slowly with input size

- $O(1)$: constant running time — primitive operations
- $O(\log n)$: logarithmic running time
- $O(n)$: linear running time — linear search
- $O(n \log n)$: log-linear time
- $O(n^c)$: polynomial running time
- $O(c^n)$: exponential running time
- $O(n!)$: factorial running time

Go to menti.com

This algorithm is...?

```
def fun_function(n):
    val = 0
    for i in range(n):
        val += 1
    for i in range(1000):
        val += 6
    return val
```

A. Constant time - $O(1)$

B. Linear time - $O(n)$

C. Quadratic time - $O(n^2)$

D. None of the above

E. I don't know

This algorithm is...?

```python
def fun_function(n):
    val = 0
    for i in range(n):
        val += 2
        for i in range(1000):
            val += 5
    return val
```

A. Constant time - $O(1)$

B. Linear time - $O(n)$

C. Quadratic time - $O(n^2)$

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This algorithm is...?

```python
def fun_function(n):
    val = 0
    for i in range(n):
        val += 2
        for i in range(n):
            val += 1
            for i in range(700):
                val += 5
    return val
```

A. Constant time - $O(1)$

B. Linear time - $O(n)$

C. Quadratic time - $O(n^2)$

D. None of the above

E. I don't know

This algorithm is...?

```python
def fun_function(n):
    val = 0
    for i in range(5, 30):
        val += 2
        for i in range(3):
            val += 1
            for i in range(700):
                val -= 2
    return val
```

A. Constant time - $O(1)$

B. Linear time - $O(n)$

C. Quadratic time - $O(n^2)$

D. None of the above

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Binary search on sorted list

Algorithm for finding x in sorted list L:

- Pick an index i roughly dividing L in half
- If $L[i] == x$, return True (if nothing left to search return False)
- If not:
  - If $L[i] > x$, repeat search on left half of L
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First iteration

9	24	32	**56**	57	59	61	99

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$L[i] = 56 > 24 \rightarrow$ discard right half and search left half

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Second iteration

9	**24**	32

56	57	59	61	99

$L[i] = 24 \rightarrow$ return True

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Complexity = **# of iterations** $\times$ **Constant time per iteration**

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- $\log_2(n)$ (but base of logarithm does not matter for big-O)
- Complexity $O(\log n)$!

Sorting algorithms

So if we have an unsorted list, should we sort it first?

- Suppose complexity $O(sort(n))$
- Is it less work to sort and do a binary search than do a linear search?
- In other words: is $sort(n) + \log(n) < n$?
- No...

In practice: what if we need to search repeatedly, say k times?

- Is $sort(n) + k\log(n) < kn$?
- Depends on k...

Complexity matters

You're planning a trip around the world visiting 10 cities.
What's the cheapest route?

Check all alternative routes?

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You're planning a trip around the world visiting 10 cities.
What's the cheapest route?

Check all alternative routes?

- There are $10 \times 9 \times 8 \times \cdots \times 2 \times 1 = 3628800$ possible routes
- Factorial complexity $O(n!)$
- Travelling salesperson problem

Review

Measuring algorithm time complexity:

- Number of basic steps taken
- Worst-case analysis
- Focus on large inputs

Searching and sorting (next session) are canonical algorithms problems

Review exercises:

- Big O practice
- Search algorithms