Financial Analytics | Imperial College Business School
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Assignment 4

Due: 11.59pm Sunday 30^{th} May 2021

Rules

1. This is a group assignment. (There are approximately 3 people per group and by now you should know your assigned group.)
2. You are free to use `R` or `Python` for the this assignment.
3. Within each group **I strongly encourage each person to attempt each question by his / herself first** before discussing it with other members of the group.
4. Students should **not** consult students in other groups when working on their assignments.
5. Late assignments will **not** be accepted and all assignments must be submitted through the Hub with one assignment submission per group. Your submission should include a PDF report with your answers to each question together with screenshots of any relevant code. Make sure your PDF clearly identifies each member of the group by CID and name.

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1. **Shrinkage Estimators of the Covariance Matrix (50 marks)**
   The purpose of this exercise[1] is to visualise how the covariance matrix gets distorted when it is estimated using a finite set of observations. The exercise also explores how a shrinkage technique of Ledoit and Wolf can mitigate this kind of distortion.

   (a) Assume n = 10 assets have returns that follow a multivariate normal distribution with expected returns equal to zero and *true* covariance matrix equal to the $n \times n$ diagonal matrix

$$\mathbf{V} = \begin{pmatrix} 0.8 & & & & \\ & 0.85 & & & \\ & & \ddots & & \\ & & & 1.2 & \\ & & & & 1.25 \end{pmatrix}.$$

   (The diagonal entries are equally spaced at 0.05 intervals.)
   Generate $T = 120$ samples $\mathbf{r}_t$, $t = 1, \ldots, T$, from this joint distribution. Each of these samples $\mathbf{r}_t \in \mathbb{R}^{10}$ is drawn from the ten-dimensional multivariate normal distribution $\mathrm{N}(\mathbf{0}, \mathbf{V})$. (You may find the *mvrnorm* function in `R` useful for doing this.)

[1] This exercise is taken from *Optimization Methods in Finance* (2^{nd} edition) by Cornuéjols, Peña and Tütüncü and published by Cambridge University Press.

(i) Use the T samples to estimate the sample covariance matrix $\widehat{\mathbf{V}}$ as follows. Let $\bar{\mathbf{r}} := (1/T)\sum_{t=1}^{T}\mathbf{r}_t$, $\mathbf{z}_t := \mathbf{r}_t - \bar{\mathbf{r}}$, $t = 1, \ldots, T$, and

$$\widehat{\mathbf{V}} := \frac{1}{T}\sum_{t=1}^{T}\mathbf{z}_t\mathbf{z}_t^\top.$$

Plot the eigenvalues both of the true covariance matrix $\mathbf{V}$ and of the estimated covariance matrix $\widehat{\mathbf{V}}$ on the same plot. Do you observe anything peculiar? **(9 marks)**

(ii) Using the estimated covariance matrix $\widehat{\mathbf{V}}$, find the estimated minimum-risk fully invested portfolio $\widehat{\mathbf{x}}$. (Fully invested means that the portfolio weights should sum to 1.) Compute the *estimated* minimum variance $\widehat{\mathbf{x}}^\top\widehat{\mathbf{V}}\widehat{\mathbf{x}}$, the *actual* minimum variance $\widehat{\mathbf{x}}^\top\mathbf{V}\widehat{\mathbf{x}}$, and the *true* minimum variance $(\mathbf{x}^*)^\top\mathbf{V}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the true minimum-risk fully invested portfolio for $\mathbf{V}$. **(8 marks)**

(iii) Repeat parts (i) and (ii) several times (anywhere from a handful to a few thousand times). What do you observe? **(8 marks)**

(b) We will next apply the shrinkage technique of Ledoit and Wolf. To that end, let λ_i, $i = 1, \ldots, n$, denote the eigenvalues of the covariance matrix $\widehat{\mathbf{V}}$ and $\bar{\lambda} := (1/n)\sum_{i=1}^{n}\lambda_i$. Define $\mathbf{C} := \bar{\lambda}\mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and

$$\alpha := \min\left(\frac{1}{T}.\frac{\frac{1}{T}\sum_{t=1}^{T}\operatorname{trace}((\mathbf{z}_t\mathbf{z}_t^\top - \widehat{\mathbf{V}})^2)}{\operatorname{trace}((\widehat{\mathbf{V}} - \mathbf{C})^2)}, 1\right).$$

Finally, consider the shrunken matrix

$$\bar{\mathbf{V}} := (1-\alpha)\widehat{\mathbf{V}} + \alpha\mathbf{C}.$$

(i) Plot the eigenvalues of the true covariance matrix $\mathbf{V}$, of the sample covariance $\widehat{\mathbf{V}}$, and of the shrunken covariance $\bar{\mathbf{V}}$ on the same plot. What do you observe now? **(9 marks)**

(ii) Using the shrunken covariance $\bar{\mathbf{V}}$ find the estimated minimum-risk fully invested portfolio $\bar{\mathbf{x}}$. Compute the *estimated* minimum variance $\bar{\mathbf{x}}^\top\bar{\mathbf{V}}\bar{\mathbf{x}}$, the *actual* minimum variance $\bar{\mathbf{x}}^\top\mathbf{V}\bar{\mathbf{x}}$, and the *true* minimum variance $(\mathbf{x}^*)^\top\mathbf{V}(\mathbf{x}^*)$. What do you observe? Are the results any different from part (a)(ii)? **(8 marks)**

(iii) Repeat parts (i) and (ii) several times (anywhere from a handful to a few thousand times). What do you observe? Are the results any different from part (a)(iii)? **(8 marks)**

<u>*Solution:*</u> See the `R Notebook` *Solution_Shrinkage_CovarianceMatrix.Rmd.*

Note that in this question we are shrinking the sample covariance matrix towards the matrix $\mathbf{C}$ which is the identity matrix *scaled* by $\bar{\lambda}$. The sum of the variances in $\mathbf{C}$ equals $n\bar{\lambda} =$

$\sum_{i=1}^{n} \lambda_i = \text{trace}(\widehat{\mathbf{V}})$ which equals the sum of the estimated variances in $\widehat{\mathbf{V}}$. And so we are shrinking $\widehat{\mathbf{V}}$ towards a variance-covariance matrix $\mathbf{C}$ whose sum-of-variances equals the sum-of-variances in $\widehat{\mathbf{V}}$. The shrinkage therefore leaves the total "variation" unchanged but it shrinks the covariances, and in particular it shrinks the eigen-values of $\widehat{\mathbf{V}}$ towards $\bar{\lambda}$.

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2. **A Leveraged Firm (20 marks)**
A company[2] earns a rate of return of r_A and has beta β_A. A fraction w of the assets is owned by bondholders, and the remaining fraction $(1-w)$ is owned by equity holders. Every year the bondholders demand a riskless rate of return of r_B on their fraction of the assets, regardless of the actual rate of return r_A that was achieved that year. Beyond that, the equity holders take whatever is left after the bondholders have been paid.

   (a) What is the rate of return of the equity holders in terms of w, r_A and r_B? **(7 marks)**

   Solution: It must be the case that $r_A = wr_B + (1-w)r_E$ where r_E is the rate of return earned by the equity-holders. We therefore have $r_E = (r_A - wr_B)/(1-w)$.

   (b) What is the beta of the rate of return of the equity holders in terms of w and β_A? **(7 marks)**

   Solution: Let β_E denote the beta for the equity-holders. Then

$$\begin{aligned}
\beta_E &= \frac{\text{Cov}(r_E, r_M)}{\sigma_M^2} \\
&= \frac{\text{Cov}\left(\frac{r_A - wr_B}{1-w}, r_M\right)}{\sigma_M^2} && \text{by part } (a) \\
&= \frac{1}{1-w} \frac{\text{Cov}(r_A, r_M)}{\sigma_M^2} && \text{since } r_B \text{ is deterministic} \\
&= \frac{\beta_A}{1-w}. && (1)
\end{aligned}$$

   (c) Suppose β_A is positive and the expected rate of return on the market is greater than the risk-free rate. As w increases (that is, as the firm becomes more leveraged), what should happen to the expected rate of return on the equity of the firm? **(6 marks)**

   Solution: The CAPM and (1) implies that $\bar{r}_A \to \infty$ as $w \to 1$. (Note that this should make it clear why it makes no sense for companies (and banks in particular) to boast about their high return on equity (and therefore justify their large bonuses!) without first accounting for leverage!

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[2]This question is taken from Luenberger's *Investment Science*

3. **Rough and Ready Calculations are Often Useful! (25 marks)**
Returning to the Simplico gold mine example, we saw in the *Introduction to Real Options* lecture notes that the value of the lease (without the enhancement option) was \$24.1m. Without building a lattice, how could you quickly verify that this price was (approximately) correct? Or to put it another way, can you find a quick way to estimate the price of the lease without building a lattice and using backwards evaluation?

   Hints: Let S_t denote the price of gold at time t. How much is a security worth S_t at time t worth today at time 0? Recall also the annuity formula from Section 2.2 of the *Interest Rates and Deterministic Cash-Flows* notes for computing the value of a constant cash-flow over a fixed number of time periods.

   <u>*Solution*</u>: This is actually covered in the (very) short Section 14.10 of Luenberger (2nd edition). The argument proceeds as follows:

   (a) Each year there is an option to mine or not but given the cost of extraction is only \$200, the option will be exercised at all times and nodes except for a few in the bottom right corner of the gold price lattice. Because those nodes will occur with very small (risk-neutral) probability we won't lose much at all if we simply assume that the option to mine is exercised at **all** times and nodes.

   (b) We therefore see that the income obtained from the mine in each period, t, is $10,000 \times S_t(1+r)$ where we divide by $1+r = 1.1$ to note that the income is received at the end of period t rather than the beginning of the period.

   (c) The fair value *today* of that income at period t is (why?) $10,000 \times S_0/(1+r) = 10,000 \times 400/1.1 = \$3,636,363$. There are 10 such periods so the fair value today of the entire income stream is \$36.636m.

   (d) The costs of operating the mine are $10,000 \times 200 = \$2$m per annum. Using the standard annuity formula we know the value today of that cash-flow is \$12.289m.

   (e) Therefore we estimate the value of the mine to be \$36.636 - \$12.289 = \$24.074m. which is very close to the true price.

   It is a very good idea in general to perform these kinds of quick and dirty approximations as they provide a sanity check on the more complex math and code you used to construct the more accurate price. Experienced managers / investors will often think this way! But note that with the enhancement options etc. in place these kinds of calculations can become much harder.

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4. **Evaluating a More Complex Option on the Simplico Gold Mine (25 marks)**
Do Exercise 1 in the *Introduction to Real Options* lecture notes. That is, compute the value of the enhancement option in the Simplice goldmine example when the enhancement costs \$5 million but raises the mine capability by 40% to 14,000 ounces at an operating cost of

\$240 per ounce. Moreover, due to technological considerations, you should assume that the enhancement (should it be required) will not be available until the beginning of the 5^{th} year.

Solution: See the `Excel` spreadsheet *SimplicoNewEnhancementSolution.xlsx*. We see that the value of the lease with the enhancement option is \$25.98m so that the value of the option is 25.48 - 24.1 = \$1.38m.
