Imperial College London
BUSINESS SCHOOL

MSc Business Analytics 2020/21
Optimisation and Decision Models
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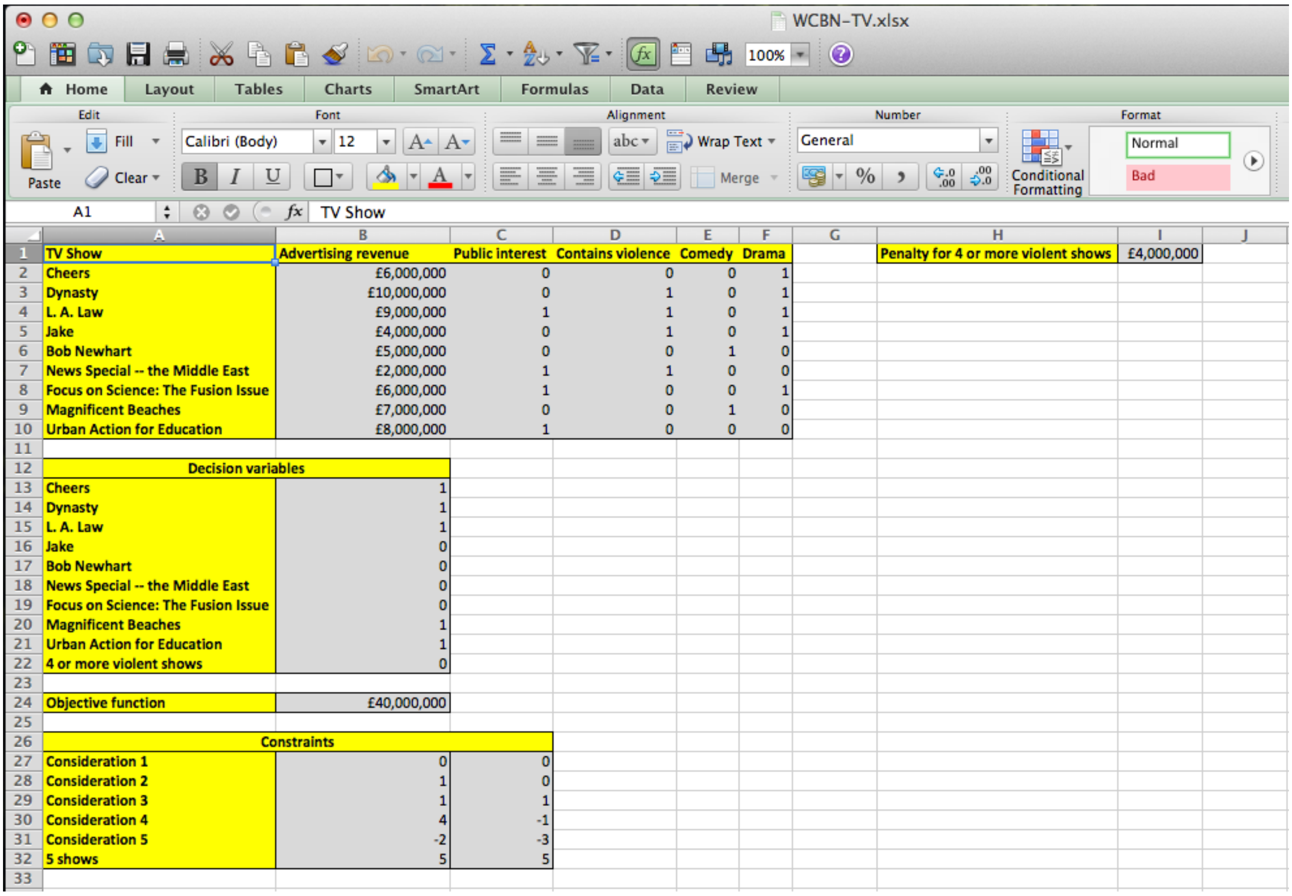
Solutions 3

Solution to (1) (a): We introduce the binary decision variables C, D, L, J, B, N, M and U for the different TV shows, with the interpretation that $C = 1$ if "Cheers" is scheduled and $C = 0$ otherwise (analogously for the other variables). We also introduce a variable V that attains the value 1 if 4 or more shows in the "contains violence" category are scheduled. WCBN-TV's scheduling problem can then be formulated as the following binary optimisation problem:

$$
\begin{array}{ll}
\text{maximise} & 6C + 10D + 9L + 4J + 5B + 2N + 6F + 7M + 8U - 4V \\
\text{subject to} & L + N + F + U \geq D + L + J + N \\
& F \leq J + L \\
& F + U \leq 1 \\
& C + B + M - 1 \leq 2 * (C + D + L + J + F) \\
& D + L + J + N - 3 \leq V \\
& C + D + L + J + B + N + F + M + U = 5 \\
& C,\ D,\ L,\ J,\ B,\ N,\ F,\ M,\ U,\ V \in \{0, 1\}
\end{array}
$$

Here, the first five constraints correspond to the five scheduling considerations from the question. In view of the fourth constraint, note that the left-hand side is positive if and only if at least two of the three variables C, B and M attain the value 1. In that case, the constraint requires at least one of the variables C, D, L, J or F on the right-hand side to attain the value 1.

Solution to (1) (b): The Excel model could look as follows (also provided separately):



	A	B	C	D	E	F	G	H	I
1	TV Show	Advertising revenue	Public interest	Contains violence	Comedy	Drama		Penalty for 4 or more violent shows	£4,000,000
2	Cheers	£6,000,000	0	0	0	1			
3	Dynasty	£10,000,000	0	1	0	1			
4	L. A. Law	£9,000,000	1	1	0	1			
5	Jake	£4,000,000	0	1	0	1			
6	Bob Newhart	£5,000,000	0	0	1	0			
7	News Special -- the Middle East	£2,000,000	1	1	0	0			
8	Focus on Science: The Fusion Issue	£6,000,000	1	0	0	1			
9	Magnificent Beaches	£7,000,000	0	0	1	0			
10	Urban Action for Education	£8,000,000	1	0	0	0			
11									
12	Decision variables								
13	Cheers	1							
14	Dynasty	1							
15	L. A. Law	1							
16	Jake	0							
17	Bob Newhart	0							
18	News Special -- the Middle East	0							
19	Focus on Science: The Fusion Issue	0							
20	Magnificent Beaches	1							
21	Urban Action for Education	1							
22	4 or more violent shows	0							
23									
24	Objective function	£40,000,000							
25									
26	Constraints								
27	Consideration 1	0	0						
28	Consideration 2	1	0						
29	Consideration 3	1	1						
30	Consideration 4	4	-1						
31	Consideration 5	-2	-3						
32	5 shows	5	5						
33									

Thus, the optimal solution is to schedule Cheers, Dynasty, L. A. Law, Magnificent Beaches and Urban Action for Education, leading to revenues of £40m.

Solution to (2) (a): We introduce the binary variables A, B, C, L, S and W to indicate whether (value 1) or not (value 0) we place an ATM machine in Arlington, Belmont, Cambridge, Lexington, Somerville and Winchester, respectively. The optimisation problem can then be cast as follows:

$$
\begin{array}{ll}
\text{minimise} & A + B + C + L + S + W \\
\text{subject to} & A + B + C \geq 1 \\
& A + B + C + L \geq 1 \\
& A + B + C + W \geq 1 \\
& B + L + S \geq 1 \\
& L + S \geq 1 \\
& C + W \geq 1 \\
& A, B, C, L, S, W \in \{0, 1\}
\end{array}
$$

Solution to (2) (b): The AMPL model could look like this:

```
var A binary;
var B binary;
var C binary;
var L binary;
var S binary;
var W binary;

minimize objective: A + B + C + L + S + W;

subject to Arlington: A + B + C >= 1;
subject to Belmont: A + B + C + L >= 1;
subject to Cambridge: A + B + C + W >= 1;
subject to Lexington: B + L + S >= 1;
subject to Somerville: L + S >= 1;
subject to Winchester: C + W >= 1;
```

The optimal solution installs ATMs in Cambridge and Somerville. Another optimal solution would be to install ATMs in Cambridge and Lexington, resulting in the same optimal value.

Solution to (3) (a): We introduce the binary variables $x_1, \ldots, x_4$ with the interpretation that $x_i = 1$ if product line 1 is produced and $x_i = 0$ otherwise. We also introduce the continuous variables $y_1, \ldots, y_4$ that represent the production quantities for the four product lines. Finally, we introduce a binary variable z for the third consideration in the question. The problem can then be formulated as the following mixed-integer optimisation problem:

$$
\begin{array}{ll}
\text{maximise} & 70y_1 + 60y_2 + 90y_3 + 80y_4 \\
& - 50{,}000x_1 - 40{,}000x_2 - 70{,}000x_3 - 60{,}000x_4 \\
\text{subject to} & y_1 \leq \mathrm{M}x_1, \quad y_2 \leq \mathrm{M}x_2, \quad y_3 \leq \mathrm{M}x_3, \quad y_4 \leq \mathrm{M}x_4, \\
& y_1 \leq 10{,}000, \;\; y_2 \leq 15{,}000, \; y_3 \leq 12{,}500, \; y_4 \leq 9{,}000 \\
& x_1 + x_2 + x_3 + x_4 \leq 2 \\
& x_1 \leq x_2 + x_3 \\
& y_1 + y_3 \leq 20{,}000 + \mathrm{M}z \\
& y_2 + y_4 \leq 20{,}000 + \mathrm{M}(1-z) \\
& y_1, y_2, y_3, y_4 \geq 0
\end{array}
$$

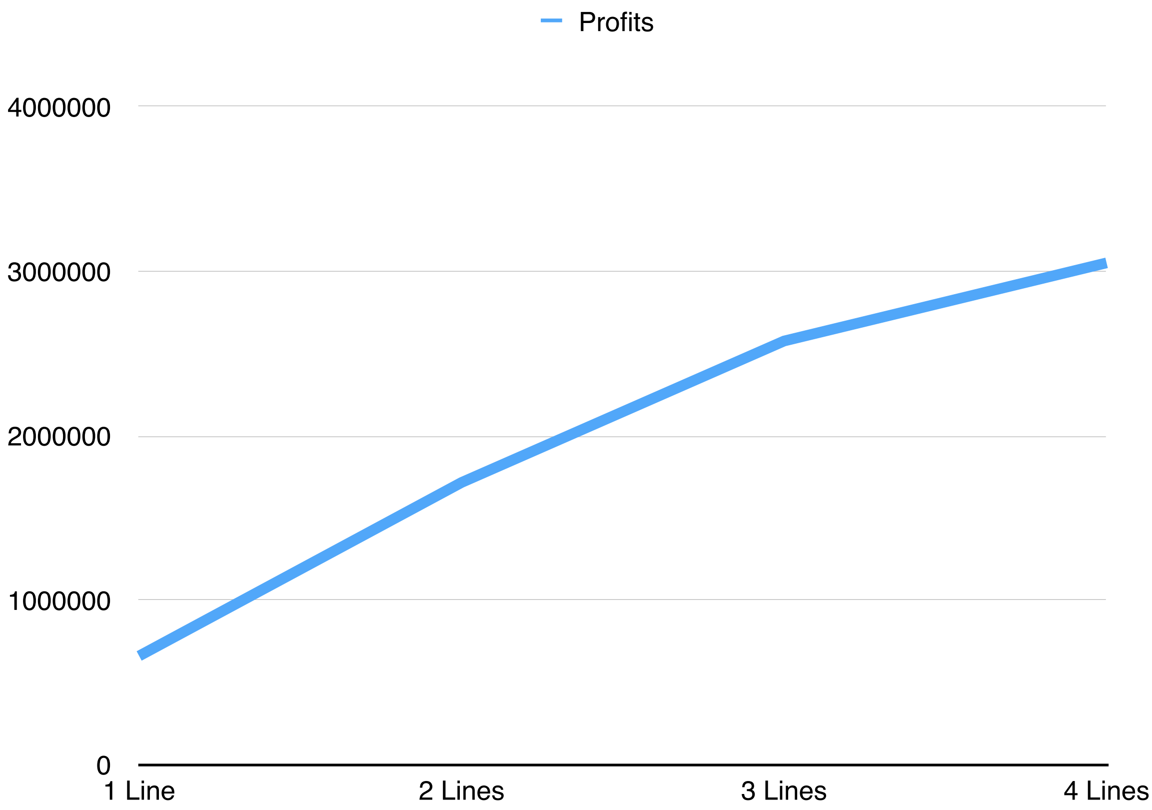
Here, the first constraint ensures that we only produce positive amounts ($y_i > 0$) if the corresponding variable x_i attains the value 1. Due to this constraint, the objective function evaluates the total profit, that is, it correctly accounts for the start-up costs of the various product lines. The second constraint ensures that we do not produce more than the demand for the products. The third constraint ensures that at most two product lines are being produced. The third constraint implies that x_1 can only attain the value 1 if either $x_2 = 1$ or $x_3 = 1$ (or both), that is, it represents the second consideration from the question. The last two constraints, finally, represents an either/or reformulation for the third consideration that is similar to the one from the lecture slides. Here, M denotes a sufficiently large positive number.

Solution to (3) (b): The Excel model can be found in a separate file. The optimal production plan is to produce 15,000 units of product 2 and 12,500 units of product 3, leading to overall profits of £1.915m.

Solution to (3) (c): In the simplest case, we can check how the optimal value and the optimal solution change if we remove each consideration individually:

- Removing consideration 1 results in a substantially larger objective value of £3,050m with all four product lines being produced.
- Removing consideration 2 does not affect the optimal objective value or solution.
- Removing consideration 3 does not affect the optimal objective value or solution either.

In conclusion, only the first consideration really affects the objective value. We can go one step further and calculate the optimal objective values for one and three product lines being produced. These objective values turn out to be £660k and £2,575m. One can then produce a graph with the number of product lines on the x-axis and the profits on the y-axis:



The graph has the expected “decreasing benefits of additional product lines” shape. We could add to this graph which product lines are selected in each setting. This would be a good decision tool for the board of the Progressive Company.

Solution to (4) (a): Remember that the linear program for the 1-norm regression is

$$\begin{array}{ll} \text{minimise} & \text{theta}_1 + \ldots + \text{theta}_{100} \\ \text{subject to} & \text{theta}_i \geq \text{Chance}_i - \text{GRE}_i * \text{beta\_GRE} - \ldots - \text{Res}_i * \text{beta\_Res} \\ & \qquad - \text{intercept} \qquad i = 1,\ldots,100 \\ & \text{theta}_i \geq \text{GRE}_i * \text{beta\_GRE} + \ldots + \text{Res}_i * \text{beta\_Res} + \text{intercept} \\ & \qquad - \text{Chance}_i \qquad i = 1,\ldots,100 \end{array}$$

We now add binary decision variables x_GRE, …, x_Res with the interpretation that x_GRE = 1 if the slope beta_GRE is allowed to be nonzero; x_GRE = 0 otherwise, and similar for the other independent variables. We then obtain the following additional constraints:

-M * x_GRE <= beta_GRE <= M * x_GRE
(…)
-M * x_Res <= beta_Res <= M * x_Res
x_GRE + … + x_Res <= K
x_GRE, …, x_Res in {0, 1}

Here, M is a large number.

Solution to (4) (b): The AMPL file could look like this:

```
set NUM ordered;                   # candidate number

param GRE {NUM};                   # GRE Score
param TOEFL {NUM};                      # TOEFL Score
param Univ {NUM};                       # University Rating
param SOP {NUM};                   # Statement of Purpose Strength
param LOR {NUM};                   # Letter of Recommend. Strength
param CGPA {NUM};                       # Undergraduate GPA
param Res {NUM};                   # Research Experience
param Chance {NUM};                     # Chance of Admission

data Graduate_Admissions.dat;

var theta {NUM};
var beta_GRE;
var beta_TOEFL;
var beta_Univ;
var beta_SOP;
var beta_LOR;
var beta_CGPA;
var beta_Res;
var intercept;

var x_GRE binary;
var x_TOEFL binary;
var x_Univ binary;
var x_SOP binary;
var x_LOR binary;
```

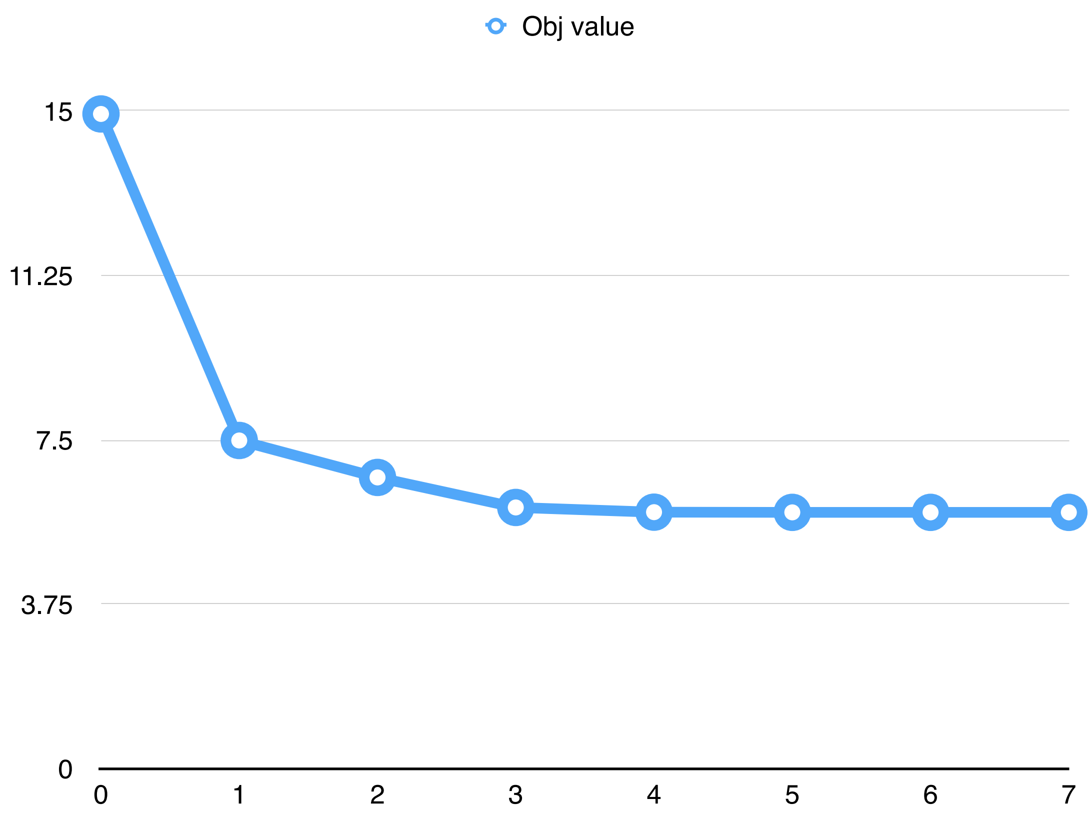
```
var x_CGPA binary;
var x_Res binary;

minimize objective: sum {i in NUM} theta[i];

subject to abs1 {i in NUM}: theta[i] >= Chance[i] - GRE[i] * beta_GRE
                                        - TOEFL[i] * beta_TOEFL
                                        - Univ[i] * beta_Univ
                                        - SOP[i] * beta_SOP
                                        - LOR[i] * beta_LOR
                                        - CGPA[i] * beta_CGPA
                                        - Res[i] * beta_Res
                                        - intercept;


subject to abs2 {i in NUM}: theta[i] >= -Chance[i] + GRE[i] * beta_GRE
                                         + TOEFL[i] * beta_TOEFL
                                         + Univ[i] * beta_Univ
                                         + SOP[i] * beta_SOP
                                         + LOR[i] * beta_LOR
                                         + CGPA[i] * beta_CGPA
                                         + Res[i] * beta_Res
                                         + intercept;

subject to bin_GRE1: beta_GRE >= -1000 * x_GRE;
subject to bin_GRE2: beta_GRE <= 1000 * x_GRE;

subject to bin_TOEFL1: beta_TOEFL >= -1000 * x_TOEFL;
subject to bin_TOEFL2: beta_TOEFL <= 1000 * x_TOEFL;

subject to bin_Univ1: beta_Univ >= -1000 * x_Univ;
subject to bin_Univ2: beta_Univ <= 1000 * x_Univ;

subject to bin_SOP1: beta_SOP >= -1000 * x_SOP;
subject to bin_SOP2: beta_SOP <= 1000 * x_SOP;

subject to bin_LOR1: beta_LOR >= -1000 * x_LOR;
subject to bin_LOR2: beta_LOR <= 1000 * x_LOR;

subject to bin_CGPA1: beta_CGPA >= -1000 * x_CGPA;
subject to bin_CGPA2: beta_CGPA <= 1000 * x_CGPA;

subject to bin_Res1: beta_Res >= -1000 * x_Res;
subject to bin_Res2: beta_Res <= 1000 * x_Res;

subject to sparsity: x_GRE + x_TOEFL + x_Univ + x_SOP + x_LOR + x_CGPA +
x_Res <= 1;
```

(Here, 1 needs to be replaced with 0, 1, 2, …, 7.)

I obtained the following objective values:

```
K = 0    14.93
K = 1    7.480512821
     - the model chooses CGPA
K = 2    6.637182847
     - the model chooses CGPA + GRE
K = 3    6.134
     - the model chooses TOEFL, Univ and LOR
K = 4    5.96751606
     - the model chooses GRE, TOEFL, Univ and LOR
K = 5    5.867923725
K = 6    5.844671944
K = 7    5.840174093
```

as well as the following graph:



A couple of comments are in place:

- Similar to the problem of the Progressive Company, we observe a “decreasing effect of additional flexibility” effect: The more features we include, the smaller the decrease in the prediction error. This is expected.
- The features selected do not normally form a set-inclusive hierarchy. GRE, for example, is selected for K = 2 but not for K = 3.
- It is important to look at AMPL’s output. For my big-M constant, AMPL warned me that there were some round-off errors, which required me to use the comment

```
option cplex_options 'integrality = 1e-06';
```

This command was not discussed in class, but it can be readily found when searching for AMPL's warning message online.