Imperial College Business School

Candidate Name:	
CID Number:	

MSc Business Analytics Examinations 2020/2021

For internal Students of Imperial College of Science Technology and Medicine.
This paper also forms part of the examination for the Associateship.

OPTIMISATION AND DECISION MODELS

MOCK EXAM

CLOSED BOOK

Instructions

Answer all three questions.

Question 1: Linear Programming and Duality **(33 marks)**

Caribbean Sun produces canned fruits that are sold to supermarkets. To this end, the two factories A and B of *Caribbean Sun* can purchase pineapples from three suppliers:

Company	**Maximum Monthly Purchase Quantity**	**Price**
Sao Paolo Farms & Co	200t	£10/t
Costa Rica Pineapples Inc	300t	£11/t
Philippines Rural	400t	£9/t

Assume that 1 ton (t) of fresh pineapples results in 1t of canned fruit, and that the canned fruit is sold for £40 to supermarket chains worldwide. For each ton of canned fruit, the factories also encounter direct labour costs of £25 (factory A) and £20 (factory B). Assume that factory A has a production capacity of 500t per month, while factory B has a production capacity of 350t per month.

(a) Formulate a Linear Program that determines the profit-maximising production schedule. **(10 marks)**

Denote by SA, CA and PA the quantities purchased from *Sao Paolo Farms & Co*, *Costa Rica Pineapples Inc* and *Philippines Rural* for production in factory A, and by SB, CB and PB the respective quantities for production in factory B. The optimisation problem can then be formulated as follows.

$$
\begin{array}{lll}
\text{maximise} & 5\,SA + 4\,CA + 6\,PA + 10\,SB + 9\,CB + 11\,PB & \\
\text{subject to} & SA + SB \leq 200 & [S] \\
& CA + CB \leq 300 & [C] \\
& PA + PB \leq 400 & [P] \\
& SA + CA + PA \leq 500 & [A] \\
& SB + CB + PB \leq 350 & [B] \\
& SA, CA, PA, SB, CB, PB \geq 0 &
\end{array}
$$

Here, the “5 SA” result from 40 (sales price) – 25 (direct labour cost) – 10 (purchase price), for example. The first three constraints represent the maximum purchase amounts, and the two penultimate constraints reflect the maximum production quantities. (In brackets behind the constraints we list the dual variables for the next part.)

(b) Dualise the linear program from part (a), using either the direct or the indirect method. Interpret the dual problem. **(10 marks)**

The dual problem reads as follows:

$$
\begin{array}{lll}
\text{minimise} & 200\,S + 300\,C + 400\,P + 500\,A + 350\,B & \\
\text{subject to} & S + A \geq 5 & [SA] \\
& C + A \geq 4 & [CA] \\
& P + A \geq 6 & [PA] \\
& S + B \geq 10 & [SB]
\end{array}
$$

$C + B \geq 9$ [CB]
$P + B \geq 11$ [PB]
$S, C, P, A, B \geq 0$

The dual variables S, C, P can be interpreted as the profit contributions of one ton of pineapples coming from *Sao Paolo Farms & Co*, *Costa Rica Pineapples Inc* and *Philippines Rural*, respectively, and A and B can be interpreted as the values added by the operations in the factories A and B. The objective function aims to determine the smallest profits, while the constraints ensure that all "profit opportunities" (supplier-factory production pairs) are accounted for.

(c) Try to determine a good solution to the linear program from part (a) by inspection. Do the same for the dual problem from part (b). Looking at both solutions, what can you say about the profits of the optimal production plan for part (a)?
(You don't have to find optimal solutions, but you should justify why your solutions are "good".) **(13 marks)**

Looking at the primal problem from part (a), we can "guess" the following good solution based on the objective coefficients:

- first maximise PB: PB = 350
- then maximise SB, CB: SB = CB = 0
- then maximise PA: PA = 50
- then maximise SA: SA = 200
- then maximise CA: CA = 250

This solution has a primal objective value of

$$5 * 200 + 4 * 250 + 6 * 50 + 11 * 350 = £6,150.$$

Now let us look at the dual problem from part (b). There are many strategies that can be used to derive a "good" dual solution. One possible strategy is the following one:

- choose A large enough to satisfy all constraints involving it: A = 6
- choose B large enough to satisfy all constraints involving it: B = 11

The objective value of this solution is 500 * 6 + 350 * 11 = £6,850. Thus, the optimal production plan will result in profits between £6,150 and £6,850.

A better dual solution could be found as follows:

- choose A large enough to satisfy the second constraint involving it: A = 4
- choose B large enough to satisfy the second constraint involving it: B = 9
- choose the remaining variables accordingly: S = 1, P = 2

The objective value of this solution is 200 * 1 + 400 * 2 + 500 * 4 + 350 * 9 = £6,150. In this case, we see that the primal and dual solutions attain the same objective values, which imply (by the weak duality theorem) that both must be optimal. Thus, the optimal production plan will result in profits of *exactly* £6,150.

Question 2: Integer Programming **(35 marks)**

The marketing department of the *A. E. Ross Company (AER)* is considering several options for its next advertisement campaign. In particular, the group has identified the following options:

	Target audience reached	**Man-hours marketing required**	**Man-hours sales required**	**Cost (£)**
Specialist magazine	200,000	300	300	150,000
General magazine	150,000	200	300	100,000
Newspaper	300,000	400	350	300,000
Radio	450,000	600	450	400,000
Television	600,000	800	600	500,000
Promotion campaign	400,000	300	800	100,000

For simplicity, the marketing department assumes that the target audiences for the different campaign options do not overlap. The options can be combined freely as long as the following three conditions are met:

(i) The firm does not want to advertise in both specialist and general magazines.
(ii) If selected, the promotion campaign must be combined with a radio and/or TV advertisement in order to ensure its effectiveness.
(iii) The overall campaign must not consume more than 1,500 marketing man-hours, 1,200 sales man-hours and £1,500,000 cash.

(a) Formulate a binary optimisation problem that determines the campaign which reaches the widest audience (under the aforementioned constraints). Do *not* solve the problem! **(10 marks)**

Using the binary decision variables S, G, N, R, T and P for the specialist magazine, general magazine, newspaper, radio, TV and promotion campaigns, respectively, we obtain the following binary optimisation problem:

$$
\begin{array}{ll}
\text{maximise} & 200S + 150G + 300N + 450R + 600T + 400P \\
\text{subject to} & S + G \leq 1 \\
& P \leq R + T \\
& 300S + 200G + 400N + 600R + 800T + 300P \leq 1{,}500 \\
& 300S + 300G + 350N + 450R + 600T + 800P \leq 1{,}200 \\
& 150S + 100G + 300N + 400R + 500T + 100P \leq 1{,}500 \\
& S, G, N, R, T, P \in \{0, 1\}
\end{array}
$$

(b) The newspaper and television channel mentioned in the table above are both owned by UK Broadcasting (UKBC) Ltd. UKBC also offers the following combined advertisement package: AER can pay £250,000 for the newspaper plus £10,000/min for TV advertisements, as long as they order at least 20min of TV advertisements. AER estimates that they would reach a target audience of 300,000 from the newspaper advertisement, as well as 20,000 per min of TV advertisement. This package does not require any marketing or sales man-hours on AER's side. If AER chooses this package, they cannot select the newspaper or TV packages from the table. How would you include this option into the optimisation problem from part (a)? **(15 marks)**

We introduce an additional binary variable NT and an additional continuous variable M for the newspaper/TV package and the number of TV minutes, respectively, and adapt the problem as follows:

$$
\begin{array}{ll}
\text{maximise} & 200S + 150G + 300N + 450R + 600T + 400P + 300\,NT + 20\,M \\
\text{subject to} & S + G \le 1 \\
& P \le R + T \\
& 300S + 200G + 400N + 600R + 800T + 300P \le 1{,}500 \\
& 300S + 300G + 350N + 450R + 600T + 800P \le 1{,}200 \\
& 150S + 100G + 300N + 400R + 500T \\
& \qquad + 100P + 250\,NT + 10\,M \le 1{,}500 \\
& NT + N \le 1 \\
& NT + T \le 1 \\
& M \le L * NT \qquad \text{(where L is a large positive number)} \\
& M \ge 20 - L * (1 - NT) \qquad \text{(where L is a large positive number)} \\
& S, G, N, R, T, P, NT \in \{0, 1\},\ M \ge 0
\end{array}
$$

(c) A different radio channel (not listed in the table above) offers broadcasting packages of 20min, 50min and 100min air time for £4,000/min. At most one of these packages can be selected, but the package can be combined with the radio option from the table above. AER estimates that they would reach a target audience of 5,000 per min broadcasting time. Moreover, each min broadcasting time requires 5 marketing man-hours (but no sales man-hours). How would you include this option into the optimisation problem from part (a)? **(10 marks)**

We can employ the `*N* possible values' reformulation from the lecture and introduce auxiliary binary variables R_{20}, R_{50} and R_{100} for the 20min, 50min and 100min air time packages, respectively:

$$
\begin{array}{ll}
\text{maximise} & 200S + 150G + 300N + 450R + 600T + 400P \\
& \qquad + 5 * 20 * R_{20} + 5 * 50 * R_{50} + 5 * 100 * R_{100} \\
\text{subject to} & S + G \le 1 \\
& P \le R + T \\
& R_{20} + R_{50} + R_{100} \le 1 \\
& 300S + 200G + 400N + 600R + 800T + 300P \\
& \qquad + 5 * 20 * R_{20} + 5 * 50 * R_{50} + 5 * 100 * R_{100} \le 1{,}500 \\
& 300S + 300G + 350N + 450R + 600T + 800P \le 1{,}200 \\
& 150S + 100G + 300N + 400R + 500T + 100P \\
& \qquad + 4 * 20 * R_{20} + 4 * 50 * R_{50} + 4 * 100 * R_{100} \le 1{,}500 \\
& S, G, N, R, T, P, R_{20}, R_{50}, R_{100} \in \{0, 1\}
\end{array}
$$

Question 3: Nonlinear Programming **(32 marks)**

In class, we have discussed the Markowitz model, which is commonly used to decide how to optimally invest a limited budget into *n* assets.

(a) Formulate the Markowitz risk minimisation model, which minimises the portfolio risk subject to a lower bound on the acceptable portfolio return. Assume that short sales are not allowed. Explain all variables and constraints. Is the problem convex? Justify your answer! **(5 marks)**

Let *x* be the *n*-dimensional vector of asset weights, and assume that a normalised budget of 1 monetary unit should be allocated among the different assets. Assume that the mean value and the covariance matrix of the vector of uncertain asset return rates are given by μ and Σ, respectively. The Markowitz model can then be formulated as:

$$\begin{array}{ll} \text{minimise} & x^T \Sigma x \\ \text{subject to} & e^T x = 1 \\ & \mu^T x \geq \underline{\mu} \\ & x \geq 0 \end{array}$$

Here, $\underline{\mu}$ represents the minimum acceptable expected portfolio return. The objective function minimises the portfolio risk, as measured by the portfolio variance, and the constraints enforce the budget, the minimum acceptable portfolio return and the no-shortsales constraints (in turn).

The problem is convex since it minimises a convex function (Σ is positive semi-definite) over a convex feasible region (the intersection of a hyperplane with two halfspaces).

(b) Consider a variant of the model from part (a) where the selected portfolio should have a 2-norm distance of at most *D* from a given reference portfolio x^0, where the 2-norm between two vectors *x* and *y* is defined as:

$$\|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Formulate this problem variant as a *quadratically constrained quadratic program*, that is, an optimisation problem where the decision variables are continuous and the objective function and all constraints are either linear or quadratic in the decision variables!

Is the problem convex? Justify your answer! **(8 marks)**

We simply add the squared constraint $\|x - x^0\|_2 \leq D$, which amounts to:

$$(x_1 - x_1^0)^2 + \cdots + (x_n - x_n^0)^2 \leq D^2$$

Each individual term $(x_i - y_i)^2$ on the left-hand side of this constraint is convex as it is the composition of a convex function and an affine function. Since the sum of convex functions is convex, we conclude that the constraint left-hand side is convex. We thus have a less-than-or-equal-to constraint with a convex function on the left-hand side and a constant on the right-hand side, which is a convex constraint. Since the objective function and the other constraints are convex (see part (a)), we thus conclude that the problem remains convex.

(c) Consider a variant of the model from part (a) where the selected portfolio should have a 1-norm distance of at most *D* from a given reference portfolio x^0, where the 1-norm between two vectors *x* and *y* is defined as:

$$\|x - y\|_1 = |x_1 - y_1| + \cdots + |x_n - y_n|$$

Formulate this problem variant as a *quadratic program*, that is, an optimisation problem where the decision variables are continuous, the objective function is quadratic in the decision variables, and all constraints are linear in the decision variables!

Is the problem convex? Justify your answer! **(14 marks)**

The problem is equivalent to the problem from part (a) with the additional constraint

$$\left|x_1 - x_1^0\right| + \cdots + |x_n - x_n^0| \leq D.$$

This constraint is equivalent to the constraints

$$\theta_1 + \cdots + \theta_n \leq D$$
$$\theta_i = |x_i - x_i^0| \qquad \text{for all i = 1, …, n,}$$

where we have introduced additional auxiliary decision variables $\theta_1, \dots, \theta_n$. One readily verifies that this constraint set is satisfiable if and only if the constraint set

$$\theta_1 + \cdots + \theta_n \leq D$$
$$\theta_i \geq |x_i - x_i^0| \qquad \text{for all i = 1, …, n}$$

is satisfiable. The latter constraint set, finally, is equivalent to

$$\theta_i \geq \ x_i - x_i^0 \ \text{ and } \theta_i \geq \ x_i^0 - x_i \qquad \text{for all i = 1, …, n, as well as}$$
$$\theta_1 + \cdots + \theta_n \leq D,$$

which is a set of linear constraints. Together with the objective function and the remaining constraints from part (a), we thus obtain a quadratic program.

Since the objective function and the other constraints from part (a) are convex and we have only added linear constraints, we conclude that the problem remains convex.

(d) Take the model from part (a), and add the requirement that at most *K* assets can be invested in. Is the problem convex? Justify your answer! **(5 marks)**

We can model this requirement by adding *n* binary variables $y_i \in \{0,1\}$ and adding the constraints that $x_i \leq y_i$ for all *i* = 1, …, *n* and $\sum_{i=1}^{n} y_i \leq K$. One readily verifies that this requirement makes the problem non-convex (except for the trivial case where $K \geq n$): Take two portfolios that invest in *K* assets, not all the same, and combine half of each portfolio: the resulting portfolio is a convex combination that is not feasible as it invests in more than *K* assets.