Financial Analytics

Introduction and Mathematical Preliminaries

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Outline

The St. Petersburg Paradox

- Suppose a fair coin is tossed repeatedly until the first head appears.
- If the first head appears on the n^{th} toss, then you will receive $\$2^n$.
- Let X be the amount you are willing to pay in order to play the game.

Question: What is X?

Note the expected payoff satisfies

$$\begin{aligned} \mathrm{E}[\text{Payoff}] &= \sum_{n=1}^{\infty} 2^n \mathrm{P}(1^{st} \text{ head on } n^{th} \text{ toss}) \\ &= \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} \\ &= \infty. \end{aligned}$$

Question: So is $X = \infty$?

This is the St. Petersburg paradox - the mistaken belief that the fair value of a game is its expected value.

Utility Functions

The 17^{th} century Swiss mathematician Daniel Bernoulli resolved this paradox by introducing the idea of a utility function $u(\cdot)$.

$u(x)$ measures how much utility someone obtains from having x units of wealth.

A utility function should have the following properties:

1. It should be increasing in x, i.e. $u'(x) > 0$ where $u'(x)$ is the derivative of u w.r.t. x.
   - reflecting that people prefer more money to less money.
2. It should be concave, i.e. $u''(x) < 0$ where $u''(x)$ is the second derivative of u w.r.t. x.
   - reflecting the fact that marginal benefit of an additional dollar decreases in the wealth level x.

A risk averse individual / investor will have an increasing concave utility function.

Resolving the St. Petersburg Paradox

Common examples of such utility functions include:

- Log utility with $u(x) = \log(x)$
- Power utility with $u(x) = x^{1-\gamma}/(1-\gamma)$ for $\gamma > 0$ and $\gamma \neq 1$
- Exponential utility with $u(x) = -e^{-\alpha x}/\alpha$ for $\alpha \geq 0$.

Bernoulli recognized that an individual with $\log$ utility has expected utility

$$
\begin{aligned}
\mathrm{E}[u(\text{Payoff})] &= \sum_{n=1}^{\infty} \log(2^n)\mathrm{P}(1^{st} \text{ head on } n^{th} \text{ toss}) \\
&= \log(2) \sum_{n=1}^{\infty} \frac{n}{2^n} \\
&< \infty.
\end{aligned}
$$

Certainty Equivalents

Question: What's the fair value of the game to you if you have log utility?

Definition. The certainty equivalent is the fixed amount of money x_{ce} that makes you indifferent between playing the game and receiving x_{ce} for certain. That is

$$u(x_{\text{ce}}) = \mathsf{E}[u(\text{Payoff})].$$

In the St. Petersburg game this implies ...

$$x_{\text{ce}} = e^{\log(2) \sum_{n=1}^{\infty} n/2^n}.$$

Certainty Equivalents

- Many problems in finance require us to model the decision-maker's preferences.
- Often assume she is risk-averse and endow her with an appropriate utility function.
- But different individuals will have different utility functions reflecting different levels of risk aversion
  - so they will have different certainty equivalents for random cash-flows.

Question: So how then should random cash-flows be valued?

This is a central problem in financial economics and in general equilibrium models are required to do this.

One exception (where equilibrium models are not required) is the case of derivative securities.

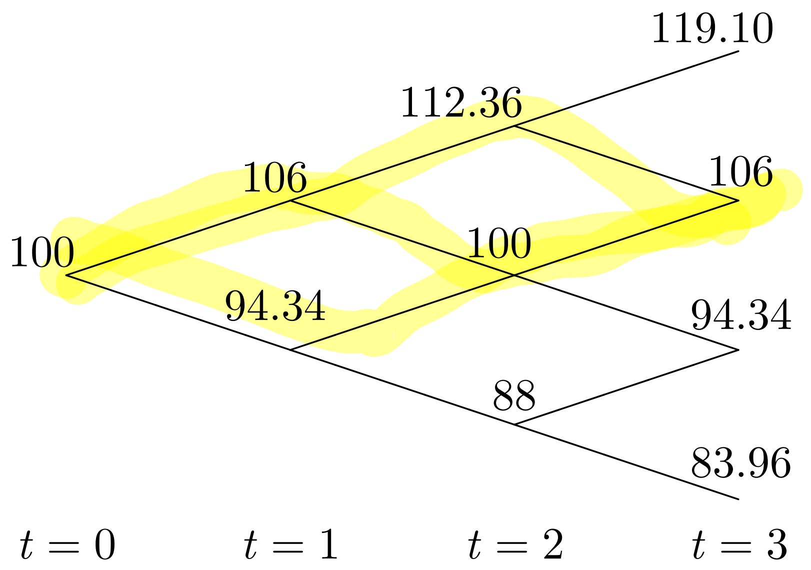
The Binomial Model

The binomial model is a classic workhorse model in finance:

- There are n time periods.
- There is one risky asset.
- At any time t the value S_t of the risky asset over the next period will either:
  - Increase by a factor u with probability p
  - Or decrease by a factor d with probability $1-p$.
- There is a risk-free asset called the cash account:
  - $\$1$ invested in the cash account at $t=0$ will be worth R^t dollars at time t.

A 3-Period Binomial Model

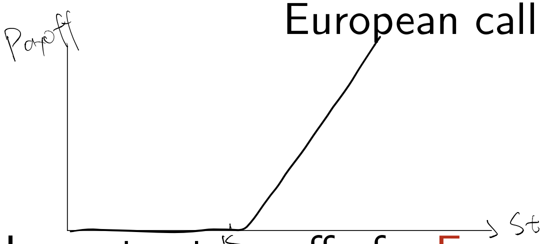
- A 3-period binomial model with $u = 1.06$, $d = 1/u$ and $S_0 = 100$.



The Binomial Model

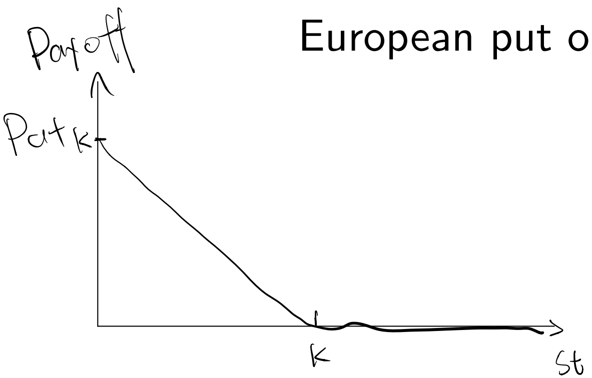
Would like to compute price of a European call option in this market.

The option expires at $t = 3$ and has a strike of 95 so it has a time $t = 3$ payoff of



$$\begin{aligned}\text{European call option payoff} \quad &:= \quad (S_3 - 95)^+ \\ &:= \quad \max(0, S_3 - 95).\end{aligned}$$

In contrast payoff of a European put option with same strike and maturity is



$$\begin{aligned}\text{European put option payoff} \quad &:= \quad (95 - S_3)^+ \\ &:= \quad \max(0, 95 - S_3).\end{aligned}$$

Some Interesting Questions

Some interesting questions / observations arise:

1. Do we have enough information to compute the option prices?
2. As with the earlier coin-tossing game, shouldn't the option prices somehow depend on the utility functions of the buyer and seller?
3. Will the price depend on the probability of an up-move in each period?

not depend on the probability

In fact, will see later that we can indeed compute unique arbitrage-free prices for all securities in the binomial model

- including in particular European call and put option prices.

These prices will be computed via the concept of a self-financing (s.f.) trading strategy that replicates the payoff of the security we wish to price.

Trading Strategies

A trading strategy is simply a rule telling us exactly what positions to hold in the stock and cash account at each time t and at each node

- A positive position in the stock means we hold the stock
- A negative position in the stock means we have short-sold the stock
- A positive position in the cash-account means we have lent to the "bank"
- A negative position in the cash-account means we have borrowed from the "bank".

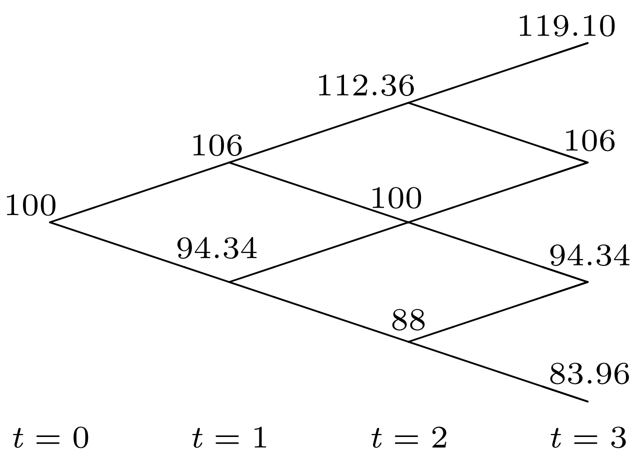
Trading Strategies

The s.f. property implies value of the strategy immediately after trading at any node is identical to its value immediately before trading at the node

- so no cash added to or withdrawn from the strategy / portfolio at any node.

Finally the rule telling us what to hold at each node can only depend on information that is available to us at that node.

e.g. Recall our 3-period binomial model:



The Mechanics of Short-Selling a Stock

Mechanics of short-selling are:

1. Find a broker who will lend the stock to you
   - broker demands a (usually) small fee for this.
2. Sell the stock in the market.
3. At some point buy the stock back in the market and return it to the broker.

   (Occasionally the broker may demand the stock be returned. In this case you will have to buy the stock at that point or find an alternative broker who will lend the stock to you.)

Short-selling allows you to gain a negative exposure to the stock

- since a stock's price is unbounded, your potential losses from a short-sale are also unbounded
- makes short-selling a very risky activity!

Short-Selling

Short-selling plays an extremely important in financial markets

- for "price discovery"
- for the pricing and hedging of derivatives securities.

  - Fair / Fundamental value of the stock is \$50
  - Current price is \$100
  - Short selling → drive the price down

- Fraud Discovery

A Futures Markets on a Cricket Match

Best way to understand mechanics of a futures market is by example and so we'll use an imaginary futures market based on the outcome of a cricket test match.

- Consider a futures contracts written on the total number of runs that are scored by the two teams in a the match.
- The market opens before the cricket match takes place and expires at the conclusion of the match.
- Table on following slide presents one possible evolution of the market between June 3 and June 19.
- Initial position is long 100 contracts and it's assumed this position is held until test match ends on June 19.
- Initial balance of $\$10,000$ is assumed and this balance earns interest at a rate of $.005\%$ per day.
- Also important to note that when the futures position is initially adopted the cost is zero, i.e. initially there is no exchange of cash.

CRICKET FUTURES CONTRACTS

Date	F_t Price	Position	Profit	Interest	Balance Margin Accounts	
June 3	$F_1 =$ 720.00	100	0	0	10,000	
June 4	$F_2 =$ 721.84	100	184	1	10,184	
June 5	$F_3 =$ 721.52	100	-31	1	10,153	
June 6	711.88	100	-964	1	9,190	
June 7	716.67	100	479	0	9,669	
June 8	720.04	100	337	0	10,006	
June 9	672.45	100	-4,759	1	5,248	Explanation?
June 10	673.25	100	80	0	5,328	
June 11	687.04	100	1,379	0	6,708	
June 12	670.56	100	-1,648	0	5,060	
June 13	656.25	100	-1,431	0	3,630	
June 14	647.14	100	-912	0	2,718	
June 15	665.57	100	1,843	0	4,561	Match Begins
June 16	673.48	100	791	0	5,353	
June 17	672.88	100	-60	0	5,293	
June 18	646.63	100	-2,625	0	2,669	
June 19	$F_T \mid F_N =$ 659.00	100	1,294	0	3,963	Match Ends
Total			-6,042		3,963	

A Futures Markets on a Cricket Match

- Futures market opens on June 3rd and test match begins on June 15th.
- Market closes when match is completed on June 19th.
- Final price of futures contract is by definition set equal to total number of runs scored in the test match.
- The closing "price" on the first day of the market was 720
  - Can be interpreted as market forecast for total number of runs that will be scored by both teams in the test match
  - This forecast varies through time as new information becomes available

    e.g. information regarding player selection and fitness, current form of players, weather forecast updates, umpire selection, condition of field etc.

1. Eliminate "double coincidence of wants" problem

2. Reduce counterparty risk

Mechanics of Futures Markets

- The contract size is $\$1$ so:

  e.g. If you go long one contract and price increases by two then your balance increases by $2 \times 1 \times \$1 = \$2$.
  e.g. But if you go short 5 contracts and price decreases by 8 then your balance increases by $(-8) \times (-5) \times \$1 = \$40$.

- This process of marking-to-market is usually done on a daily basis.
- So value of futures position immediately after marking-to-market is always zero.

Convenient to think about a futures contract with price process F_t as follows:

- Futures contract is a security that is always worth zero but it pays a daily "dividend". (could be positive or negative)
- The dividend per contract at the end of day t is $\pm(F_t - F_{t-1})$.
- These dividends are random and can be positive or negative.

Margin requirements

Margin requirements intended to protect futures exchange against default risk.

- A typical margin requirement would be that the futures trader maintain a minimum balance in her trading account.
- This minimum balance will often be a function of the contract value (perhaps 5% to 10%) multiplied by the position size.
- If the balance drops below the minimum level a margin call is made and the trader must deposit enough funds so as to meet the balance requirement.
- Failure to satisfy margin call will result in futures position being closed out.

Strengths and Weaknesses of Futures Markets

Futures markets have a number of strengths:

- Easy to take a position using futures markets without having to purchase the underlying asset. Indeed, often impossible to buy the underlying asset.

  e.g. equity indices, interest rates, cricket matches, presidential elections, ...
- Futures markets allow you to leverage your position.
- They are designed to eliminate counter-party risk and the double-coincidence-of-wants problem.
- Mechanics of futures markets are generally independent of the underlying 'security' so they are easy to operate and easily understood by investors.

Futures markets also have some weaknesses:

- The fact that they are so useful for leveraging a position also makes them dangerous for unsophisticated and / or rogue investors.
- Futures prices are (more or less) linear in price of underlying security. This limits the types of risks that can be perfectly hedged using futures markets.

A Perfect Hedge

Consider the following example:

- A wheat producer knows he will have $100,000$ bushels of wheat available to sell in three months time.
- He is concerned the spot price of wheat will move against him, i.e. fall, in the meantime.
- So he decides to lock in the sale price now by hedging in the futures markets.
- Each wheat futures contract is for $5,000$ bushels, so he decides to sell 20 three-month futures contracts.
- Note that the wheat producer now has a perfectly hedged position.

F_T: contract price

S_t: spot price

- At $t=0$, short 20 contracts : cash flow $= 0$

  - Future position at maturity $T =$ 3 months

$$\underbrace{(F_0 - F_1)}_{\text{Day 1}} + \underbrace{(F_1 - F_2)}_{\text{Day 2}} + \cdots \underbrace{(F_{T-1} - F_T)}_{\text{day } T} = F_0 - F_T$$

Overall Cashflow at time T: $100k \times S_T \times 20[F_0 - F_T] = 20F_0$

$$F_T = 5k \cdot S_T \qquad \approx 100k S_0$$

$100k\, S_T$: uncertain cashflow if no hedge

$100k\, S_0$: rough cashflow if hedge

Perfect Hedges

In general, perfect hedges are not available for a number of reasons:

1. None of the expiration dates of available futures contracts may exactly match the expiration date of the payoff Z_T that we want to hedge.
2. Z_T may not correspond exactly to an integer number of futures contracts.
3. The security underlying the futures contract may be different to the security underlying Z_T. e.g.: airline hedge jet fuel prices
4. Z_T may be a non-linear function of the security price underlying the futures contract.
5. Combinations of all the above are also possible.

When perfect hedges are not available can use the minimum-variance hedge.

Constructing Minimum-Variance Hedges

Let Z_T = date T cash flow that we wish to hedge and F_t = time t price of futures contract.

At date $t = 0$ we adopt a position of h in the futures contract and hold this position until time T.

$\text{Var}(aX+bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\,\text{Cov}(X,Y)$

Since initial cost of a futures position is zero, can write terminal cash-flow as

$$Y_T = Z_T + h(F_T - F_0).$$

Our objective then is to minimize

$$\text{Var}(Y_T) = \text{Var}(Z_T) + h^2\text{Var}(F_T) + 2h\text{Cov}(Z_T,\ F_T)$$

$$\frac{d\,\text{Var}(Y_T)}{dh} = 2h\text{Var}(Z_T) + 2\text{Cov}(Z_T, F_T) = 0 \quad \text{at } h^*$$

Constructing Minimum-Variance Hedges

Find that minimizing h and minimum variance are given by

$$h^* = -\frac{\mathrm{Cov}(Z_T, F_T)}{\mathrm{Var}(F_T)}$$

The larger $\mathrm{Cov}(Z_T, F_T)$, the better hedge

$$\mathrm{Var}(Y_T^*) = \mathrm{Var}(Z_T) - \frac{\mathrm{Cov}(Z_T,\ F_T)^2}{\mathrm{Var}(F_T)}.$$

Such static hedging strategies are often used in practice

- but dynamic hedging strategies are capable of achieving a smaller variance.

The Monte-Carlo Framework

Suppose we want to estimate $\theta := \mathbb{E}[h(\mathbf{X})]$ for $\mathbf{X} \in \mathbb{R}^m$ via Monte-Carlo. We proceed as follows:

1. Generate $\mathbf{X}_1, \ldots, \mathbf{X}_n$ IID from the distribution of $\mathbf{X}$.
2. Let $Y_i := h(\mathbf{X}_i)$ and set

$$\widehat{\theta}_n = \frac{Y_1 + \ldots + Y_n}{n}.$$

Strong Law of Large Numbers (SLLN) implies

$$\widehat{\theta}_n \to \theta \quad \text{as } n \to \infty \text{ with probability } 1.$$

Assumption: Already know how to simulate IID r.vars from a given distribution.

Recall The Central Limit Theorem

Question: How large n should be so that we can have confidence in $\hat{\theta}_n$?

Answer: Can figure this out through the use of confidence intervals

- but first need the the Central Limit Theorem.

Theorem. (Central Limit Theorem)

- Suppose $Y_1, \ldots, Y_n$ are IID and $\mathbb{E}[Y_i^2] < \infty$.
- Let $\theta := \mathbb{E}[Y_i]$ and $\sigma^2 :=$ Var(Y_i). $\mathrm{Var}(\hat{\theta}_n) = \frac{1}{n^2}\mathrm{Var}(\sum Y_i) = \frac{1}{n^2} \times n\sigma^2 = \frac{\sigma^2}{n}$
- Define $\widehat{\theta}_n = \frac{\sum_{i=1}^n Y_i}{n}$.

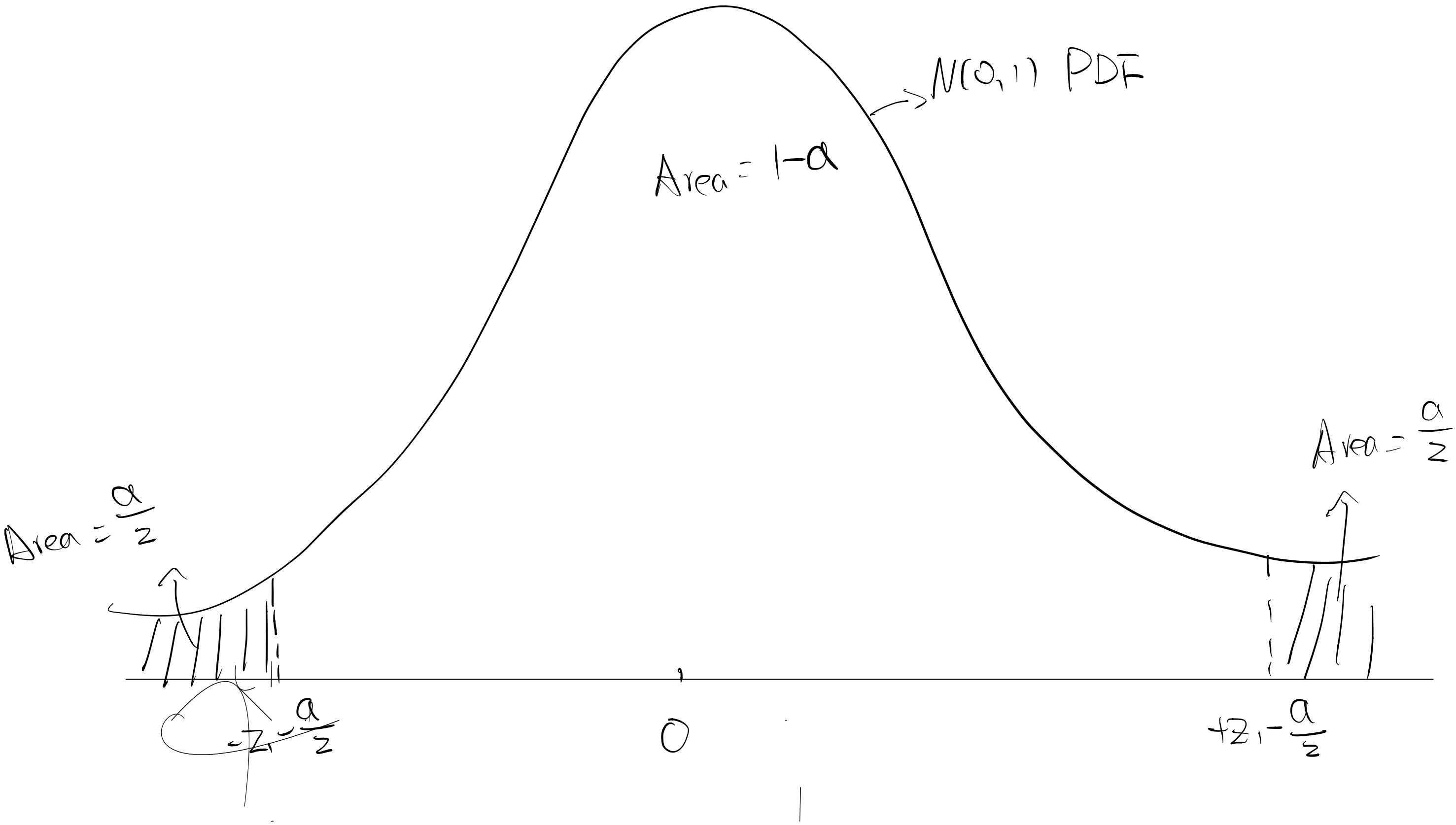
Then

$$\frac{\widehat{\theta}_n - \theta}{\sigma/\sqrt{n}} \Rightarrow \mathrm{N}(0,1) \quad \text{as} \quad n \to \infty.$$

Using the CLT to Construct Confidence Intervals

Let $z_{1-\alpha/2}$ be the $(1-\alpha/2)$ percentile point of the $\mathsf{N}(0,1)$ distribution so that

$$P(-z_{1-\alpha/2} \leq \mathsf{N}(0,1) \leq z_{1-\alpha/2}) = 1-\alpha.$$



Using the CLT to Construct Confidence Intervals

Recall the CLT implies $\sqrt{n}\left(\widehat{\theta}_n - \theta\right)/\sigma \sim \mathsf{N}(0,1)$ for large n.

Therefore

$$P\left(-z_{1-\alpha/2} \;\leq\; \frac{\sqrt{n}(\widehat{\theta}_n - \theta)}{\sigma} \;\leq\; z_{1-\alpha/2}\right) \quad\approx\quad 1-\alpha$$

$$\Rightarrow\; P\left(\widehat{\theta}_n - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \;\leq\; \theta \;\leq\; \widehat{\theta}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right) \quad\approx\quad 1-\alpha.$$

Using the CLT to Construct Confidence Intervals

Approx. $100(1-\alpha)\%$ CI for θ therefore given by

$$[L(\mathbf{Y}),\ U(\mathbf{Y})] \ = \ \left[\widehat{\theta}_n - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}},\ \ \widehat{\theta}_n + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right]. \tag{1}$$

Problem: We don't usually know σ^2.

Solution: Estimate σ^2 with

$$\hat{\sigma}_n^2 = \frac{\sum_{i=1}^n (Y_i - \hat{\theta}_n)^2}{n-1}.$$

Can show $\hat{\sigma}_n^2$ an unbiased estimator of σ^2 and that $\hat{\sigma}_n^2 \rightarrow \sigma^2$ w.p. 1 as $n \rightarrow \infty$.

So now replace σ with $\hat{\sigma}_n$ in (1) to obtain

$$[L(\mathbf{Y}),\ U(\mathbf{Y})] = \left[\widehat{\theta}_n - z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}},\ \ \widehat{\theta}_n + z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}\right] \tag{2}$$

as our approximate $100(1-\alpha)\%$ CI for θ when n is "large".

Brownian Motion

Definition. A random process X_t is a Brownian motion with parameters (μ, σ) if:

1. For $0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$

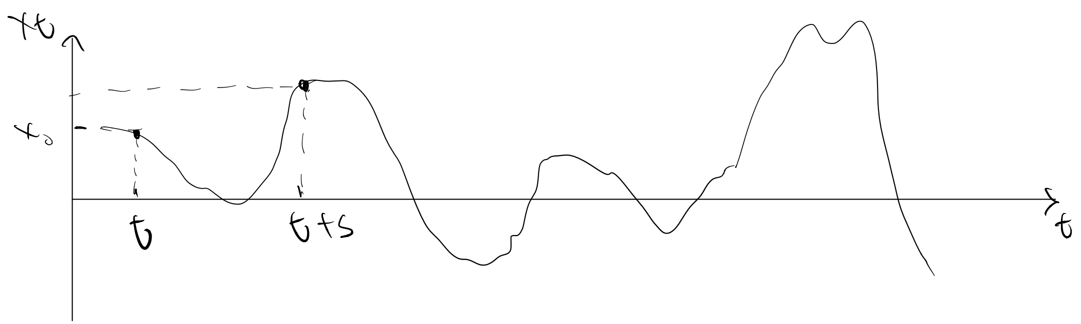
$$(X_{t_1} - X_0),\ (X_{t_2} - X_{t_1}),\ (X_{t_3} - X_{t_2}),\ \ldots,\ (X_{t_n} - X_{t_{n-1}})$$

   are mutually independent.

2. For $s > 0$, $X_{t+s} - X_t \sim \mathsf{N}(\mu s,\ \sigma^2 s)$.

3. X_t is a continuous function of t with probability 1.

Say that X_t is a $B(\mu, \sigma)$ Brownian motion with drift μ and volatility σ.



Brownian Motion

When $X_0 = 0$, $\mu = 0$ and $\sigma = 1$ we have a standard Brownian motion (SBM)

- will use B_t to denote an SBM.

- From part 1 of the definition it follows that for an SBM

$$B_{t_1},\ (B_{t_2} - B_{t_1}),\ (B_{t_3} - B_{t_2}),\ \ldots,\ (B_{t_n} - B_{t_{n-1}})$$

  are mutually independent for any $0 < t_1 < t_2 < \ldots < t_{n-1} < t_n$.

- From part 2 of the definition it follows that for an SBM

$$B_{t+s} - B_t \sim \mathsf{N}(0,\ s)$$

① . $E_0[B_s^2] = Var(B_s) + E_0[B_s]^2$

$E_0[B_s^2] = Var(B_s) + E_0[B_s]^2$

$= s + 0^2$

$= s + 0^2$

$= s$

$\left[\begin{array}{l} Var(x) = E[x^2] - E[x]^2 \\ \Rightarrow E[x^2] = Var[x] + E[x^2] \end{array} \right]$

② . $E_t[B_s^2]$ $\quad t < s$ $\quad B_s - B_t \sim N(0, s-t)$

$E_t[B_s^2] = E_t[(B_s - B_t + B_t)^2]$

$= E_t[(B_s - B_t)^2 + B_t^2 + 2B_t(B_s - B_t)]$

$= \underbrace{E_t[(B_s - B_t)^2]}_{\text{Term 1}} + \underbrace{E_t[B_t^2]}_{\text{Term 2}} + \underbrace{E_t[2B_t(B_s - B_t)]}_{\text{Term 3}}$

Term 2 $= B_t^2$

Term 3 $= 2B_t \, E_t[\underbrace{(B_s - B_t)}] = 2B_t \times 0 = 0$

$\hookrightarrow N(0, s-t)$ conditional on time t

$B_s - B_t \,|\, t \sim N(0, s-t)$

Term 1 $= s - t$

$\Rightarrow E_t[B_s^2] = B_t^2 + s - t$

Simulating a Brownian Motion

$0 \quad t_1 \quad t_2$

Simulating a Standard Brownian Motion at Times $t_1 < t_2 < \ldots < t_n$

> **set** $t_0 = 0$, $B_{t_0} = 0$
> **for** $i = 1$ to n
>
> Variance
>
> **generate** $Z \sim \mathrm{N}(0, t_i - t_{i-1})$
> **set** $B_{t_i} = B_{t_{i-1}} + Z$

help to simulate any Brownian Motion

$B_{t_i} = B_{t_{i-1}} + \underbrace{(B_{t_i} - B_{t_{i-1}})}_{\to Z}$

If $X \sim B(\mu, \sigma)$ and $X_0 = x$ then can write

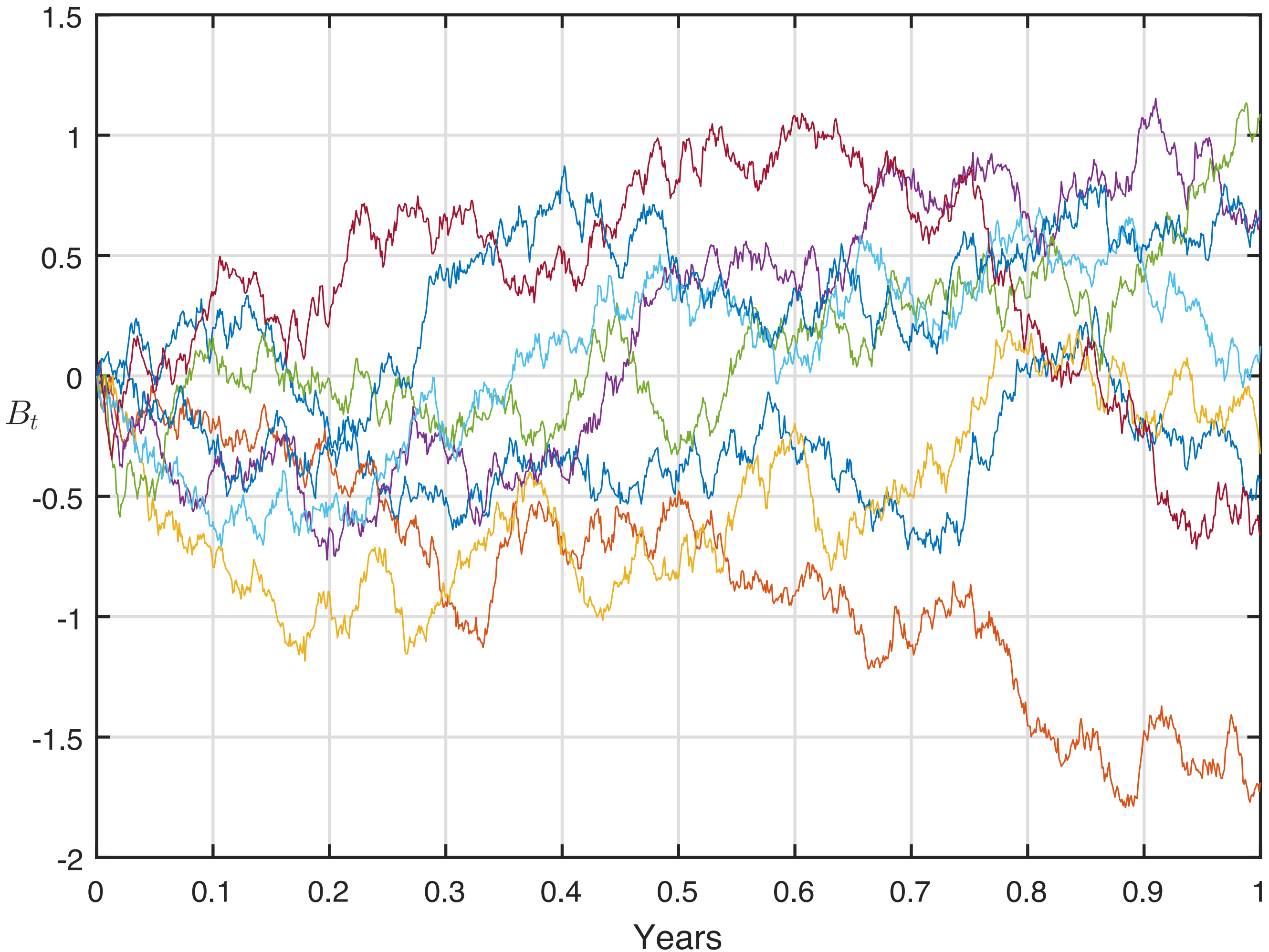
$$X_t = x + \mu t + \sigma B_t$$

where B_t is an SBM.

$X_{t+s} - X_t = (x + \mu(t+s) + \sigma B_{t+s}) - (x + \mu t + \sigma B_t)$

$= \mu s + \sigma(B_{t+s} - B_t) \sim N(\mu s, \sigma^2 s)$

Question: How might you simulate a $B(\mu, \sigma)$ process?



- Some simulated paths of Brownian motion.

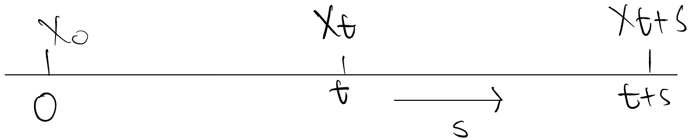
Geometric Brownian Motion

Definition. We say that a stochastic process X_t is a geometric Brownian motion (GBM) if for all $t \geq 0$

$$X_t = X_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t}$$

where B_t is a standard Brownian motion.

We call μ the drift, σ the volatility and write $X_t \sim \text{GBM}(\mu, \sigma)$.

Note that 

$\sigma B_{t+s} = \sigma B_t + \sigma (B_{t+s} - B_t)$

$$\begin{aligned} X_{t+s} &= X_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right)(t+s) + \sigma B_{t+s}} \\ &= X_0 \, e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) s + \sigma (B_{t+s} - B_t)} \\ &= X_t \, e^{\left(\mu - \frac{\sigma^2}{2}\right) s + \sigma (B_{t+s} - B_t)} \end{aligned} \tag{3}$$

$\longrightarrow N(0, s)$

– a representation that is very useful for simulating GBM.

Geometric Brownian Motion

Question: Suppose $X_t \sim \mathrm{GBM}(\mu, \sigma)$. What is $\mathsf{E}_t[X_{t+s}]$?

Answer: From (3) we have

$$
\begin{aligned}
\mathsf{E}_t[X_{t+s}] &= \mathsf{E}_t\left[X_t\, e^{\left(\mu-\frac{\sigma^2}{2}\right)s+\sigma(B_{t+s}-B_t)}\right] \\
&= X_t\, e^{\left(\mu-\frac{\sigma^2}{2}\right)s}\, \mathsf{E}_t\left[e^{\sigma(B_{t+s}-B_t)}\right] \\
&= X_t\, e^{\left(\mu-\frac{\sigma^2}{2}\right)s}\, e^{\frac{\sigma^2}{2}s} \\
&= e^{\mu s} X_t
\end{aligned}
$$

Fact (MGF of a normal r.v)

Suppose $Z \sim N(\mu, \sigma^2)$

$\Longrightarrow E[e^{sZ}] = e^{\mu s + \frac{1}{2}\sigma^2 s^2}$

$= E[e^{Z}] \quad Z \sim N(0, \sigma^2 s)$

– so the expected growth rate of X_t is μ.

Geometric Brownian Motion

The following properties of GBM follow immediately from the definition of BM:

1. Fix $t_1, t_2, \ldots, t_n$. Then $\frac{X_{t_2}}{X_{t_1}}, \ \frac{X_{t_3}}{X_{t_2}}, \ \ldots, \ \frac{X_{t_n}}{X_{t_{n-1}}}$ are mutually independent.

2. Paths of X_t are continuous as a function of t, i.e., they do not jump.

3. For $s > 0$, $\log\left(\frac{X_{t+s}}{X_t}\right) \ \sim \ \mathrm{N}\left((\mu - \frac{\sigma^2}{2})s, \ \sigma^2 s\right)$.

Question: How would you simulate X_{t_i} for $t_1 < t_2 < \ldots < t_n$?

$$\frac{X_{t+s}}{X_t} = e^{(\mu - \frac{\sigma^2}{2})s + \sigma(B_{t+s} - B_t)}$$

$$\Rightarrow \log\left(\frac{X_{t+s}}{X_t}\right) = (\mu - \frac{\sigma^2}{2})s + \sigma(B_{t+s} - B_t)$$

$$\sim N\left((\mu - \frac{\sigma^2}{2})s, \sigma^2 s\right)$$

log return

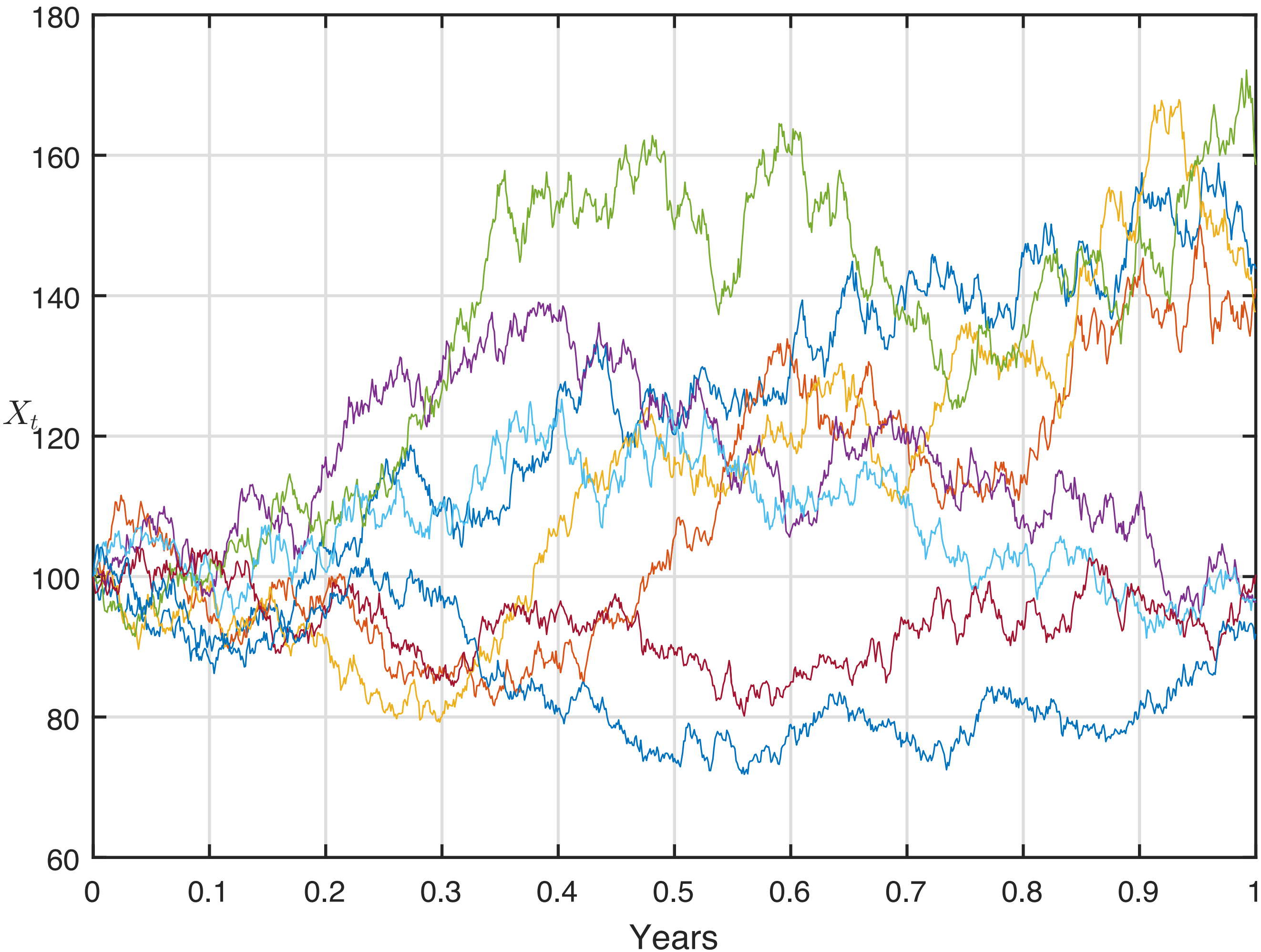
Modeling Stock Prices as GBM

Suppose $X_t \sim \text{GBM}(\mu, \sigma)$. Then clear that:

1. If $X_t > 0$, then X_{t+s} is always positive for any $s > 0$.
   - so limited liability of stock price is not violated.
2. The distribution of X_{t+s}/X_t only depends on s and not on X_t

   $\log\left(\frac{X_{t+s}}{X_t}\right)$: log return

- These properties suggest GBM might be a reasonable model for stock prices.
- Indeed GBM is the underlying model for the famous Black-Scholes option pricing formula.



- Some simulated paths of GBM with $X_t = X_0 \, e^{(\mu - \sigma^2/2)t + \sigma B_t}$ where $X_0 = 100$, $\mu = 10\%$ and $\sigma = 30\%$.

Multivariate GBM

e.g. Let $\mathbf{S}_t := (S_{1,t}, \ldots, S_{m,t})$ be a vector of m stock prices at time t.

The multivariate GBM model assumes

$$S_{i,t} = S_{i,0}\, e^{\left(r - \frac{\sigma_i^2}{2}\right)t + \sigma_i \sqrt{t}\, X_i}$$

where:

- r is the continuously compounded risk-free interest rate
- $\mathbf{X} \sim \mathrm{MN}_m(\mathbf{0}, \mathbf{\Sigma})$ where $\mathbf{X} := (X_1, \ldots, X_m)$
- $\mathbf{\Sigma}$ is a correlation matrix so that each X_i is standard normal.

$$\Sigma = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \\ & & & 1 \end{bmatrix}$$

Pricing Exotic Options via Monte-Carlo

Goal: Estimate price C_0 of the option which expires at time $T = 1$ with payoff

$$(\max \mathbf{S}_T - K)^+ \quad = \max(0, \max_i S_{i,T} - k)$$

where $\max \mathbf{S}_T := \max_i S_{i,T}$.

m=3 (MSFT. Amazon. Tesla)

This is a call-on-the-max option and C_0 satisfies

$$C_0 = \mathsf{E}\left[e^{-rT} (\max \mathbf{S}_T - K)^+\right]$$

– have to estimate via Monte-Carlo.

Estimating C_0 Using n Monte-Carlo Samples

for $j = 1 : n$ n= number of path

  generate $\mathbf{X} \sim \mathsf{MN}_m(\mathbf{0}, \mathbf{\Sigma})$

  for $i = 1 : m$ m= number of periods

    set $S_{i,T} = S_{i,0} \exp\left(\left(r - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T}\, X_i\right)$

  set $V_j = \max\{S_{1,T}, \ldots, S_{m,T}\}$

  set $h_j = e^{-rT}\,(V_j - K)^+$

set $\widehat{C}_0 = \frac{\sum_{j=1}^n h_j}{n}$

set $\hat{\sigma}_n^2 = \frac{\sum_{j=1}^n (h_j - \widehat{C}_0)^2}{n-1}$

set Approx. 95% CI for C_0 to be

$$[L, U] = \left[\widehat{C}_0 - z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}},\quad \widehat{C}_0 + z_{1-\alpha/2}\frac{\hat{\sigma}_n}{\sqrt{n}}\right]$$

The Tower Property of Conditional Expectations

Let I_t denote all available information up to and including time t.

Then $\mathsf{E}_t[X] := \mathsf{E}[X \mid I_t]$ denotes the expected value of X conditional on I_t.

$\mathsf{E}_t[X]$ also a random variable when viewed from any time $u \leq t$.

e.g. $X = S_v$ where $S \sim GBM(\mu, \sigma)$ and $v \geq t$. Then

$$\begin{aligned}
\mathsf{E}_t[S_v] &= \mathsf{E}_t\left[S_t e^{(\mu-\sigma^2/2)(v-t)+\sigma(B_v-B_t)}\right] \qquad \text{by (3) with } s = v - t \\
&= S_t e^{(\mu-\sigma^2/2)(v-t)}\, \mathsf{E}_t\left[e^{\sigma(B_v-B_t)}\right] = e^{\frac{\sigma^2}{2}(v-t)} \quad \text{(MGF result)} \\
&= S_t e^{\mu(v-t)} \qquad (4)
\end{aligned}$$

Now $\mathsf{E}_t[S_v] = S_t e^{\mu(v-t)}$ is itself a random variable when viewed from time $u \leq t$.

So can take its expectation conditional on I_u to obtain ...

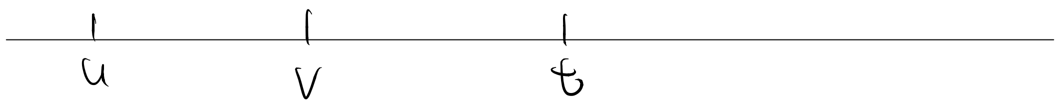
The Tower Property of Conditional Expectations

t: future time

u: earlier time

$$
\begin{aligned}
\mathsf{E}_u[\mathsf{E}_t[S_v]] &= \mathsf{E}_u[S_t e^{\mu(v-t)}] && \text{by (4)} \\
&= e^{\mu(v-t)}\, \mathsf{E}_u[S_u e^{(\mu-\sigma^2/2)(t-u)+\sigma(B_t-B_u)}] && \text{by (3)} \\
&= S_u\, e^{\mu(v-u)}\, e^{-\sigma^2(t-u)/2}\, \underbrace{\mathsf{E}_u[e^{\sigma(B_t-B_u)}]}_{e^{\frac{\sigma^2}{2}(t-u)}} \\
&= S_u\, e^{\mu(v-u)} && \text{by MGF of a normal r.var} \\
&= \mathsf{E}_u[S_v] && \text{by (4).}
\end{aligned}
$$

(but now taking = u)



The Tower Property of Conditional Expectations

Have therefore shown $\mathsf{E}_u[\mathsf{E}_t[S_v]] = \mathsf{E}_u[S_v]$ for any $u \leq t$.

This is not an accident and indeed for any random variable X and $u \leq t$

$$\mathsf{E}_u[\mathsf{E}_t[X]] = \mathsf{E}_u[X].$$

This is an example of the tower property of conditional expectations

- useful for pricing options and futures later in course.