Assignment 1

Due: 11.59pm Friday 7^{th} May 2021

Rules

1. This is a group assignment. (There are approximately 3 people per group and by now you should know your assigned group.)
2. You are free to use `R`, `Python` or `Excel` for the this assignment. (In fact `Excel` is particularly suited to binomial lattices and presents an easy way to visualise price dynamics in these models.)
3. Within each group **I strongly encourage each person to attempt each question by his / herself first** before discussing it with other members of the group.
4. Students should **not** consult students in other groups when working on their assignments.
5. Late assignments will **not** be accepted and all assignments must be submitted through the Hub with one assignment submission per group. Your submission should include a PDF report with your answers to each question together with screenshots of any relevant code. Make sure your PDF clearly identifies each member of the group by CID and name.

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1. **Forward Rates (10 marks)**
   If the spot rates for 1 and 2 years are $s_1 = 6.3\%$ and $s_2 = 6.9\%$, what is the forward rate, $f_{1,2}$ assuming annual compounding? What do you get if you assume continuous compounding?

   Solution: With annual compounding we have

$$f_{1,2} = \frac{(1+s_2)^2}{(1+s_1)} - 1 = \frac{1.069^2}{1.063} - 1 \approx 7.503\%.$$

   With continuous compounding, we have

$$f_{1,2} = \log\left(\frac{e^{2s_2}}{e^{s_1}}\right) = \log\left(\frac{e^{2\times .069}}{e^{.063}}\right) = 7.5\%.$$

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2. **Pricing a Floating Rate Bond (15 marks)**
   A floating rate bond is the same as a regular bond except that the coupon payments at the end of a period are determined by the prevailing short rate at the beginning of the period. Specifically let the coupon payments be $c_1, \ldots, c_n$ occurring at times $t_1 < t_2 < \cdots < t_n$,

respectively. The face value F of the bond is also paid at time t_n with $c_i = r_{i-1}F$ where r_{i-1} is the short rate prevailing between times $i-1$ and i. While r_{i-1} is stochastic it is known at time $i-1$. Show that the arbitrage-free time $t=0$ value of the floating rate bond is F.

Hint: Consider working backwards in time. (The floating rate bond is that rare example of a stochastic cash-flow that we can price without needing a model!)

Solution: Consider the value of the bond at time t_{n-1}. At that time we know we will be receiving $r_{n-1}F + F = (1+r_{n-1})F$ at time t_n. At time t_{n-1} we know r_{n-1} and so this is a deterministic cash-flow whose value V_{n-1} at time t_{n-1} is given by

$$V_{n-1} = \frac{(1+r_{n-1})F}{1+r_{n-1}} = F.$$

We can now proceed to time t_{n-2} and use the same argument to see the bonds value at time must again be F. Continuing in this way we find that the initial value of the bond is $V_0 = F$.

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3. **Mortgage-Mathematics (30 marks)**
Consider the interest-only (IO) and principal-only (PO) securities in a deterministic world without prepayments and defaults. These securities have time k cash-flows of $P_k := B - cM_{k-1}$ and $I_k := cM_{k-1}$, respectively, for $k = 1, \ldots, n$ and where M_k (and all other notation) is defined in Section 4 of the *Interest Rates and Deterministic Cash-Flows* lecture notes.

   (a) Compute the present value V_0 of the PO security assuming a risk-free rate of r per period. (Note that r is the risk-free rate used to discount deterministic cash-flows so that $V_0 = \sum_{k=1}^{n} P_k/(1+r)^k$.) **(10 marks)**

   Solution: Substituting for M_{k-1} in $P_k := B - cM_{k-1}$ (using (2) from the lecture notes) we obtain

$$\begin{aligned} P_k &= B - c\left((1+c)^{k-1}M_0 - B\left[\frac{(1+c)^{k-1}-1}{c}\right]\right) \\ &= -c\,(1+c)^{k-1}M_0 + B\,(1+c)^{k-1} \\ &= (B - c\,M_0)\,(1+c)^{k-1}. \end{aligned} \tag{1}$$

   The present value, V_0, of the PO security is therefore given by

$$V_0 = \sum_{k=1}^{n} \frac{P_k}{(1+r)^k} = (B - cM_0)\,\frac{(1+r)^n - (1+c)^n}{(r-c)(1+r)^n} \tag{2}$$

   where r is the per-period risk-free interest rate. If we now substitute for B in (2) we obtain

$$V_0 = \frac{cM_0}{(1+c)^n - 1} \times \frac{(1+r)^n - (1+c)^n}{(r-c)(1+r)^n} \tag{3}$$

which in the case $r = c$ reduces to

$$V_0 \;=\; \frac{rn\, M_0}{(1+r)\,[(1+r)^n - 1]}. \tag{4}$$

(b) What happens to V_0 as $n \to \infty$? **(5 marks)**

<u>*Solution:*</u> It is easy to see from (3) that $V_0 \to 0$ as $n \to \infty$. This should be intuitively clear: if the number of payments is infinite, then this must imply that the borrower will never repay the principal.

(c) Compute the present value W_0 of the IO security. **(10 marks)**

<u>*Solution:*</u> The present value of the interest payment stream could be computed by evaluating

$$W_0 = \sum_{k=1}^{n} \frac{I_k}{(1+r)^k}$$

directly but it is much easier to recognize that the sum of the principal-only and interest-only streams must equal the total value of the mortgage F_0 as calculated in the lecture notes. Using the expression there for F_0 and (3) we therefore obtain

$$\begin{aligned} W_0 &= F_0 - V_0 \\ &= \frac{cM_0}{[(1+c)^n - 1](1+r)^n} \left[(1+c)^n \, \frac{(1+r)^n - 1}{r} - \frac{(1+r)^n - (1+c)^n}{r-c} \right]. \end{aligned} \tag{5}$$

Moreover when $r \to c$ it is easy to check that (5) reduces to

$$W_0 = M_0 - \frac{rn\, M_0}{(1+r)\,[(1+r)^n - 1]}$$

as expected from (4) and since $F_0 = M_0$ when $r = c$.

(d) The **duration** of a fixed-income security is a measure of how long the owner of the security must wait until the cash-flows associated with the security are received. More specifically, it is a *weighted average* of the cash-flow times with weights given by the cash-flow contributions to the overall value of the security. If we let D_P and D_I denote the durations of the PO and IO securities, respectively, then they are given by

$$\begin{aligned} D_P &= \frac{1}{12\, V_0} \sum_{k=1}^{n} \frac{k\, P_k}{(1+r)^k} \\ D_I &= \frac{1}{12 W_0} \sum_{k=1}^{n} \frac{k\, I_k}{(1+r)^k} \end{aligned}$$

where we divide by 12 to convert the duration into annual rather than monthly time units. Which of the two securities do you think has the longer duration? Justify your answer. (You don't need to do any calculations here!) **(5 marks)**

Solution: It should be clear that the PO security has a longer duration than the IO security. This follows because at the beginning of the mortgage interest must be paid on the total mortgage principal M_0 whereas at the end of the mortgage, the remaining principal is much smaller than M_0 and so the interest payments become negligible in comparison to the principal payments. Therefore, the weighted average of the payment times, i.e. the duration, is greater for the principal stream.

Remark: The concept of duration is a very important concept in the fixed income world. To get some sense of why this is the case, first note that we can extend the definition above to compute the duration of a portfolio. (After all a portfolio of fixed income securities can be viewed as a single fixed income security.) Now suppose, for example, that our fixed income portfolio has a duration of 4.5 years. This means (in the weighted average sense described above) that we need to wait on average 4.5 years to receive the cash-flows associated with the portfolio. Of course the portfolio will be exposed to changes in interest rates and will move up or down in value as interest rates fall or rise. Intuitively a portfolio with a larger (in absolute value) duration will be riskier than a portfolio with a smaller (in absolute value) duration.

Suppose, however, that we now add a security whose duration is -4.5 years to the portfolio. (This could be achieved, for example, by short-selling a bond whose duration was 4.5 years. In practice it could be achieved by trading an *interest-rate swap*.) Then the overall duration of the new portfolio would be 0 years and (in the duration sense) it would have no exposure to interest rate movements. As a result, we would be approximately *hedged* against changes in interest rates. All of this can be made precise mathematically. It's worth noting that as interest rates change the duration of a portfolio also changes so that the new portfolio would no longer have a duration of exactly zero after some initial interest rate movements.

In conclusion, the concept of duration is very important and is used every day in the fixed income markets to describe (one measure of) the risk of a fixed income portfolio.

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4. **Futures Hedging - from Luenberger's *Investment Science* (25 marks)**
Farmer D. Jones has a crop of grapefruit juice that will be ready for harvest and sale as $150,000$ pounds of grapefruit juice in 3 months. Jones is worried about possible price changes, so he is considering hedging. There is no futures contract for grapefruit juice, but there is a futures contract for orange juice. His son, Gavin, recently studied minimum-variance hedging and suggests it as a possible approach. Currently the spot prices are $1.20 per pound for orange juice and $1.50 per pound for grapefruit juice. The standard deviation of the prices of orange juice and grapefruit juice is about 20% per year, and the correlation coefficient between them is about .7. What is the minimum variance hedge for farmer Jones, and how effective is this

hedge as compared to no hedge?

Remark: When we say the standard deviation of the price of grapefruit juice is about 20% per year this means (in the language of the lecture notes) that $\sigma_{S_T} \approx 20\% \times S_0/\sqrt{4}$ where S_0 is the current price of grapefruit and we divide by $\sqrt{4}$ because the maturity is 3 months which is 1/4 of a year. Similarly you can assume $\sigma_{F_T} \approx (.2 \times 1.20)/\sqrt{4}$.

<u>Solution</u>: The minimum variance hedge is

$$\begin{aligned} h &= -\frac{\mathrm{Cov}(Z_T, F_T)}{\mathrm{Var}(F_T)} \\ &= -150{,}000 \, \frac{\mathrm{Cov}(S_T, F_T)}{\mathrm{Var}(F_T)} \\ &= -150{,}000 \, \rho \, \frac{\sigma_{S_T}\sigma_{F_T}}{\sigma_{F_T}^2} \end{aligned}$$

where $\rho = .7$, $\sigma_{S_T} = (.2 \times 1.50)/\sqrt{4}$ and $\sigma_{F_T} = (.2 \times 1.20)/\sqrt{4}$. This yields a value of $h = -131{,}250$ lbs of orange juice futures.

You can check that the new standard deviation stdev_{new} satisfies

$$\begin{aligned} \mathrm{stdev}_{new} &= \sqrt{1-\rho^2}\,\sigma_{old} \\ &= .714 \, \sigma_{old} \end{aligned}$$

where $\sigma_{old} = \sigma_{S_T}$ is the standard deviation of the un-hedged cash-flow

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5. **The Die Game (25 marks)**
   You are allowed to roll a fair 6-sided die a maximum of 3 times. After any throw you can elect to "stop". If you elect to stop after the i^{th} throw then you will receive $\$X_i$ where X_i is the result of the i^{th} throw, for $i = 1, 2$ or 3. For example, suppose you throw a 3 on your second throw and then elect to stop. You will then receive a payoff of \$3 and will not proceed with the 3^{rd} throw. You must stop after the 3^{rd} throw if you have not elected to stop after the earlier throws.

   (a) If you are risk-neutral what is the fair value of this game to you? *Hint:* Consider working backwards in time. e.g. If you have just one throw left what is the fair value of the game? **(10 marks)**

   <u>*Solution:*</u> Let V_t for $t = 1, 2, 3$ denote the value of the game at time t after just rolling the die at that time. Similarly, let X_t denote the value of the die roll at time t. We then have $V_3(X_3) = X_3$ since we must stop at time $t = 3$ if we have not already elected to stop after an earlier die roll. Then V_2 is a function of X_2 and satisfies (why?)

$$\begin{aligned} V_2(X_2) &= \max\{\underbrace{X_2}_{\text{stop}}, \underbrace{\mathrm{E}[V_3(X_3)]}_{\text{continue}}\} \\ &= \max\{X_2, \, 3.5\}. \end{aligned}$$

Note that the optimal strategy at time $t = 2$ is to stop if $X_2 \geq 4$ and continue otherwise. Now we can compute $V_1(X_1)$. It satisfies

$$\begin{aligned} V_1(X_1) &= \max\{\underbrace{X_1}_{\text{stop}}, \underbrace{\mathrm{E}[V_2(X_2)]}_{\text{continue}}\} \\ &= \max\{X_1,\, 4.25\}. \end{aligned}$$

Therefore the optimal strategy at time $t = 1$ is to stop if $X_1 \geq 5$ and otherwise continue. Finally we can compute the fair value of the game at time $t = 0$ as $V_0 = \mathrm{E}[V_1(X_1)] = 4\frac{2}{3}$.

(b) If you are risk-averse and have log utility what is the fair value of this game to you? **(15 marks)**

<u>*Solution:*</u> Using the same notation as in part (a), we have $V_3(X_3) = \log(X_3)$ and

$$\begin{aligned} V_2(X_2) &= \max\{\log(X_2),\, \mathrm{E}[\log(X_3)]\} \\ &= \max\{\log(X_2),\, 1.0965\}. \end{aligned}$$

Note that $\log(2) = 0.6931 < 1.0965 < \log(3) = 1.0986$ and so the optimal strategy at time $t = 2$ is to stop if $X_2 \geq 3$ and otherwise continue. Now we can compute $V_1(X_1)$. It satisfies

$$\begin{aligned} V_1(X_1) &= \max\{\log(X_1),\, \mathrm{E}[V_2(X_2)]\} \\ &= \max\{\log(X_1),\, 1.3465\}. \end{aligned}$$

We note that $\log(3) < 1.3465 < \log(4) = 1.3863$ and so the optimal strategy at time $t = 1$ is to stop if $X_1 \geq 4$ and continue otherwise. The expected utility at time $t = 0$ is given by

$$V_0 = \mathrm{E}[V_1(X_1)] = 1.4712.$$

So how much should you be prepared to play this game (assuming you have log utility). Well we must use the idea of certainty equivalents. In particular, the fair value is the value F satisfying $\log(F) = 1.4712$ and so we see $F = 4.3545$ which we note is less than the fair value you obtained in part (a). (The two answers are quite close because log utility models a very mild level of risk aversion.)

<u>*Remark:*</u> The technique used here to compute the fair values of the game is called dynamic programming. We will use this technique to price American options and some real options later in the course.

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6. **Brownian Motion and Geometric Brownian Motion (20 marks)**
Let B_t be a standard Brownian motion.

(a) What is $\mathrm{E}_0[B_{t+s}B_s]$? *Hint:* write $B_{t+s} = (B_{t+s} - B_s + B_s)$. **(10 marks)**

<u>*Solution:*</u> First note that

$$\begin{aligned} \mathrm{E}_0\left[B_{t+s}B_s\right] &= \mathrm{E}_0\left[\left(B_{t+s} - B_s + B_s\right) B_s\right] \\ &= \mathrm{E}_0\left[\left(B_{t+s} - B_s\right) B_s\right] \ + \ \mathrm{E}_0\left[B_s^2\right]. \qquad (6) \end{aligned}$$

We know $\mathrm{E}_0\left[B_s^2\right] = s$ since $B_s \sim \mathrm{N}(0, s)$. To calculate the first term on the r.h.s. of (6) we can argue

$$\begin{aligned} \mathrm{E}_0\left[\left(B_{t+s} - B_s\right) B_s\right] &= \mathrm{E}_0\left[\left(B_{t+s} - B_s\right)\right] \mathrm{E}_0\left[B_s\right] \qquad (7) \\ &= 0 \times 0 \qquad (8) \\ &= 0 \end{aligned}$$

where (7) follows from the independent increments property of Brownian motion and (8) follows because the increments of SBM have mean zero. We therefore obtain $\mathrm{E}_0[B_{t+s}B_s] = s$.

(b) The geometric Brownian motion (GBM) model for a security price assumes its time t price is given by

$$S_{t+s} = S_t\, e^{(\mu - \sigma^2/2)s \,+\, \sigma(B_{t+s} - B_t)}$$

where B_t is SBM. Compute $\mathrm{E}_t\left[S_{t+s}^2\right]$ where $\mathrm{E}_t[\cdot]$ denotes the expectation conditional on all time t information. *Hint:* You'll need to know the MGF of a normal random variable. **(10 marks)**

<u>*Solution:*</u> We have

$$\begin{aligned} \mathrm{E}_t\left[S_{t+s}^2\right] &= \mathrm{E}_t\left[S_t^2\, e^{2(\mu - \sigma^2/2)s \,+\, 2\sigma(B_{t+s} - B_t)}\right] \\ &= S_t^2\, e^{2(\mu - \sigma^2/2)s}\, \mathrm{E}_t\left[e^{2\sigma(B_{t+s} - B_t)}\right] \\ &= S_t^2\, e^{2(\mu - \sigma^2/2)s}\, \mathrm{E}\left[e^{2\sigma Z}\right] \qquad (9) \end{aligned}$$

where $Z \sim \mathrm{N}(0,\, s)$. But the moment-generating function (MGF) of a normal random variable implies $\mathrm{E}\left[e^{2\sigma Z}\right] = e^{2\sigma^2 s}$ and so we obtain from (9) that

$$\begin{aligned} \mathrm{E}_t\left[S_{t+s}^2\right] &= S_t^2\, e^{2(\mu - \sigma^2/2)s + 2\sigma^2 s} \\ &= S_t^2\, e^{(2\mu + \sigma^2)s}. \qquad (10) \end{aligned}$$

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7. **Tower Property of Conditional Expectations (25 marks)**
Demonstrate the tower property by showing that

$$\mathrm{E}_u\left[\mathrm{E}_t[S_v^2]\right] = \mathrm{E}_u[S_v^2] \qquad (11)$$

where $u \leq t$. (The result is clearly true (why?!) when $v \leq t$ so you only need show it for $v \geq t$. Your calculations in Question 6b should be useful!)

<u>*Solution:*</u> Consider (10) with $s = v - t$ and $t = u$. This yields

$$\mathrm{E}_u[S_v^2] = S_u^2\, e^{(2\mu+\sigma^2)(v-u)}. \tag{12}$$

Similarly, we can take (10) again with $s = v - t$ but this time with $t = t$ and obtain

$$\mathrm{E}_t[S_v^2] = S_t^2\, e^{(2\mu+\sigma^2)(v-t)}. \tag{13}$$

Let's now compute $\mathrm{E}_u[\mathrm{E}_t[S_v^2]]$. Using (13) we obtain

$$\begin{aligned}
\mathrm{E}_u\left[\mathrm{E}_t[S_v^2]\right] &= \mathrm{E}_u\left[S_t^2\, e^{(2\mu+\sigma^2)(v-t)}\right] \\
&= e^{(2\mu+\sigma^2)(v-t)}\, \mathrm{E}_u\left[S_u^2 e^{2(\mu-\sigma^2/2)(t-u)+2\sigma(B_t-B_u)}\right] \\
&= S_u^2\, e^{(2\mu+\sigma^2)(v-t)}\, e^{2(\mu-\sigma^2/2)(t-u)}\, \mathrm{E}_u\left[e^{2\sigma(B_t-B_u)}\right] \\
&= S_u^2\, e^{(2\mu+\sigma^2)(v-t)}\, e^{2(\mu-\sigma^2/2)(t-u)}\, e^{2\sigma^2(t-u)} \\
&= S_u^2\, e^{2(v-u)\mu+(v-u)\sigma^2}
\end{aligned}$$

which we recognize as $\mathrm{E}_u[S_v^2]$ from (12). And so we have shown (11).

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8. **Using Monte-Carlo Simulation to Price Asian Options (25 marks)**
Use Monte-Carlo simulation to estimate the price $\mathrm{E}_0^Q[e^{-rT}h(\mathbf{X})]$ of an Asian call option where the time T payoff $h(\mathbf{X})$ is given by

$$h(\mathbf{X}) := \max\left(0,\ \frac{\sum_{i=1}^m S_{iT/m}}{m} - K\right)$$

where $\mathbf{X} = (S_{T/m}, S_{2T/m}, \ldots, S_T)$. You may assume that under the probability distribution Q (the so-called *risk-neutral* probability distribution that we will encounter later in the course) that $S_t \sim \mathrm{GBM}(r, \sigma)$ where $r = .05$ and $\sigma = .25$. Other relevant parameters are $T = 1$ year, $S_0 = 100$ and $m = 11$. Simulate $10,000$ paths and estimate the option price (with approx 95% CIs) for $K = 90, 100, 110$ and 120.

<u>*Solution:*</u> The approximate 95% CI's are $[13.62, 13.67]$, $[7.33, 7.37]$, $[3.41, 3.44]$ and $[1.40, 1.42]$ for the options with strikes $K = 90, 100, 110$ and 120, respectively. See the `R Notebook` *Pricing_Asian_Options.Rmd* for the code that produces these results.
