Q1

Question 1

(a).

(i).

```
library(Matrix)
V=Diagonal(x = seq(0.8,1.25,0.05))
r=rep(0, 10)
```

```
set.seed(12345)
X0 <- MASS::mvrnorm(n=120, mu = r, Sigma = V)
```

```
colMeans(X0)
```

```
##  [1]  0.06049362 -0.11921312 -0.22629882  0.16626835  0.04674217  0.01787684
##  [7]  0.06585541  0.07328721  0.03908269  0.23791583
```

```
r_mean=colMeans(X0)
z=X0-r_mean
```

```
res=0
for (r in 1:nrow(z))
  res=res+z[r,]%*% t(z[r,])
  res=res/nrow(z)
V2=res
```

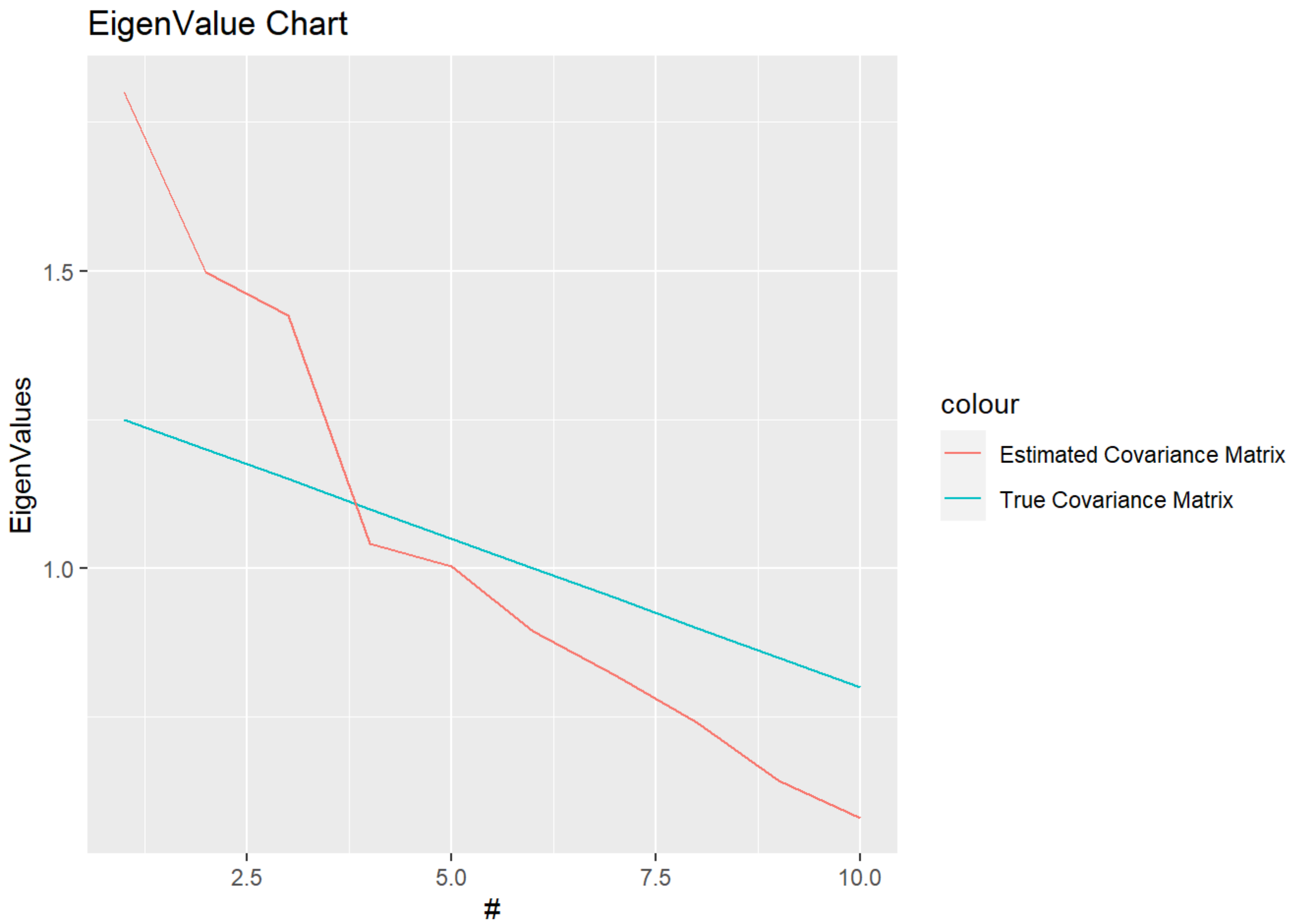
```
e <- eigen(V)
e$values
```

```
##  [1] 1.25 1.20 1.15 1.10 1.05 1.00 0.95 0.90 0.85 0.80
```

```
e2 <- eigen(V2)
e2$values
```

```
##  [1] 1.8008256 1.4972401 1.4256470 1.0413507 1.0038687 0.8939928 0.8202445
##  [8] 0.7409669 0.6428633 0.5812950
```

```
library(ggplot2)
library(RColorBrewer)
cbPalette <- brewer.pal(10, name = "Paired")
ggplot() +
    geom_line(mapping = aes(x = seq(1,10,1), y = eigen(V)$values, color = "True Covariance Matrix")) +
    geom_line(mapping = aes(x = seq(1,10,1), y = eigen(V2)$values, color = "Estimated Covariance Matrix")) +
    labs(x = "#", y = "EigenValues", title = "EigenValue Chart")
```



The eigenvalue is distorted when using only a finite set of observations

(ii).

```
library(CVXR)
```

```
##
## Attaching package: 'CVXR'
```

```
## The following object is masked from 'package:stats':
##
##     power
```

```
NumAssets=10
w <- Variable(NumAssets)   # decision variables
risk <- quad_form(w, V2)  # This is w' Sample_Cov w
constraints <- list(w >= 0, sum(w) == 1)
prob <- Problem(Minimize(risk), constraints)
result <- solve(prob)
MinVar <- result$getValue(risk)
w_MinVar <- result$getValue(w)
return_Min <- t(r_mean) %*% w_MinVar
```

```
expected=t(w_MinVar) %*%V2 %*%w_MinVar
actual=t(w_MinVar) %*%V %*%w_MinVar
```

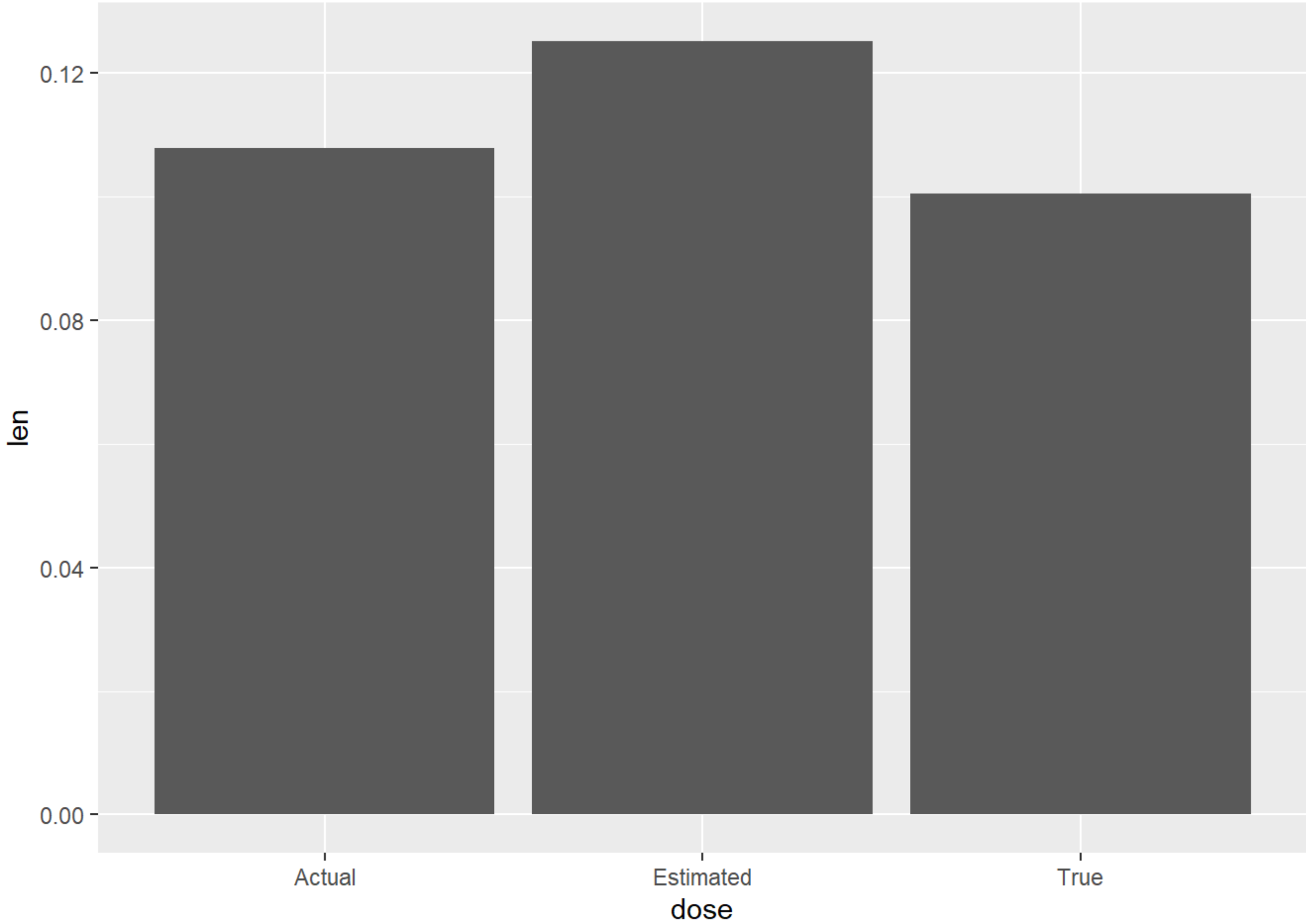
```
library(CVXR)

NumAssets=10
w <- Variable(NumAssets)   # decision variables
risk <- quad_form(w, V)  # This is w' Sample_Cov w
constraints <- list(w >= 0, sum(w) == 1)
prob <- Problem(Minimize(risk), constraints)
result <- solve(prob)
MinVar <- result$getValue(risk)
w_MinVar2 <- result$getValue(w)
return_Min <- t(r_mean) %*% w_MinVar
```

```
true=t(w_MinVar2) %*%V %*%w_MinVar2
expected
```

```
##           [,1]
## [1,] 0.1251664
```

```
df <- data.frame(dose=c("Estimated", "Actual", "True"),
                len=c(expected[1], actual[1], true[1]))
p<-ggplot(data=df, aes(x=dose, y=len)) +
  geom_bar(stat="identity")
p
```



(iii).

The actual variance is always larger than the true variance For the estimated variance, it is really unstable and there is not an obvious trend.

b.

(i).

```
lamda=eigen(V2)$values
lamda
```

```
##  [1] 1.8008256 1.4972401 1.4256470 1.0413507 1.0038687 0.8939928 0.8202445
##  [8] 0.7409669 0.6428633 0.5812950
```

```
lamda_mean=mean(lamda)
```

```
C=lamda_mean*diag(10)
```

```
library(psych)
```

```
##
## Attaching package: 'psych'
```

```
## The following object is masked from 'package:CVXR':
##
##     logistic
```

```
## The following objects are masked from 'package:ggplot2':
##
##     %+%, alpha
```

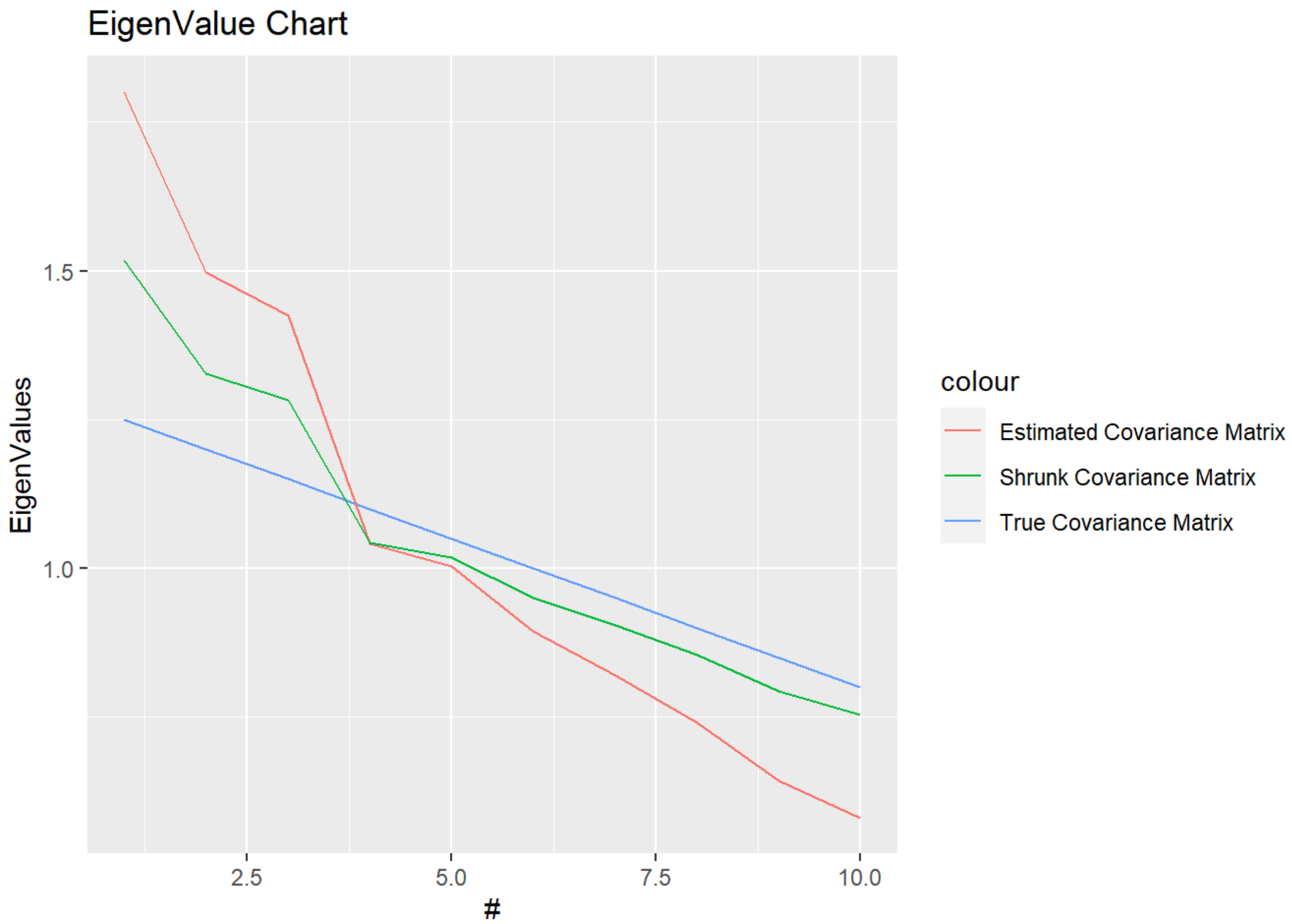
```
res=0
for (r in 1:nrow(z))
  res=res+tr((z[r,]%*% t(z[r,])-V2)^2)
  res=res/nrow(z)
alpha=min((res/tr((V2-C)^2))/nrow(z),1)
```

```
V3=(1-alpha)*V2+alpha*C
```

```
eigen(V2)$values
```

```
##  [1] 1.8008256 1.4972401 1.4256470 1.0413507 1.0038687 0.8939928 0.8202445
##  [8] 0.7409669 0.6428633 0.5812950
```

```
library(RColorBrewer)
cbPalette <- brewer.pal(10, name = "Paired")
ggplot() +
    geom_line(mapping = aes(x = seq(1,10,1), y = eigen(V)$values, colour = "True Covariance Matrix")) +
    geom_line(mapping = aes(x = seq(1,10,1), y = eigen(V2)$values, colour = "Estimated Covariance Matrix")) +
    geom_line(mapping = aes(x = seq(1,10,1), y = eigen(V3)$values, colour = "Shrunk Covariance Matrix")) +
    labs(x = "#", y = "EigenValues", title = "EigenValue Chart")
```



The eigenvalue with shrinkage method, compared with the one only using sample data, is much closer to the actual one

(ii).

```
library(CVXR)

NumAssets=10
w <- Variable(NumAssets)   # decision variables
risk <- quad_form(w, V3)  # This is w' Sample_Cov w
constraints <- list(w >= 0, sum(w) == 1)
prob <- Problem(Minimize(risk), constraints)
result <- solve(prob)
MinVar <- result$getValue(risk)
w_MinVar3 <- result$getValue(w)
return_Min <- t(r_mean) %*% w_MinVar
```

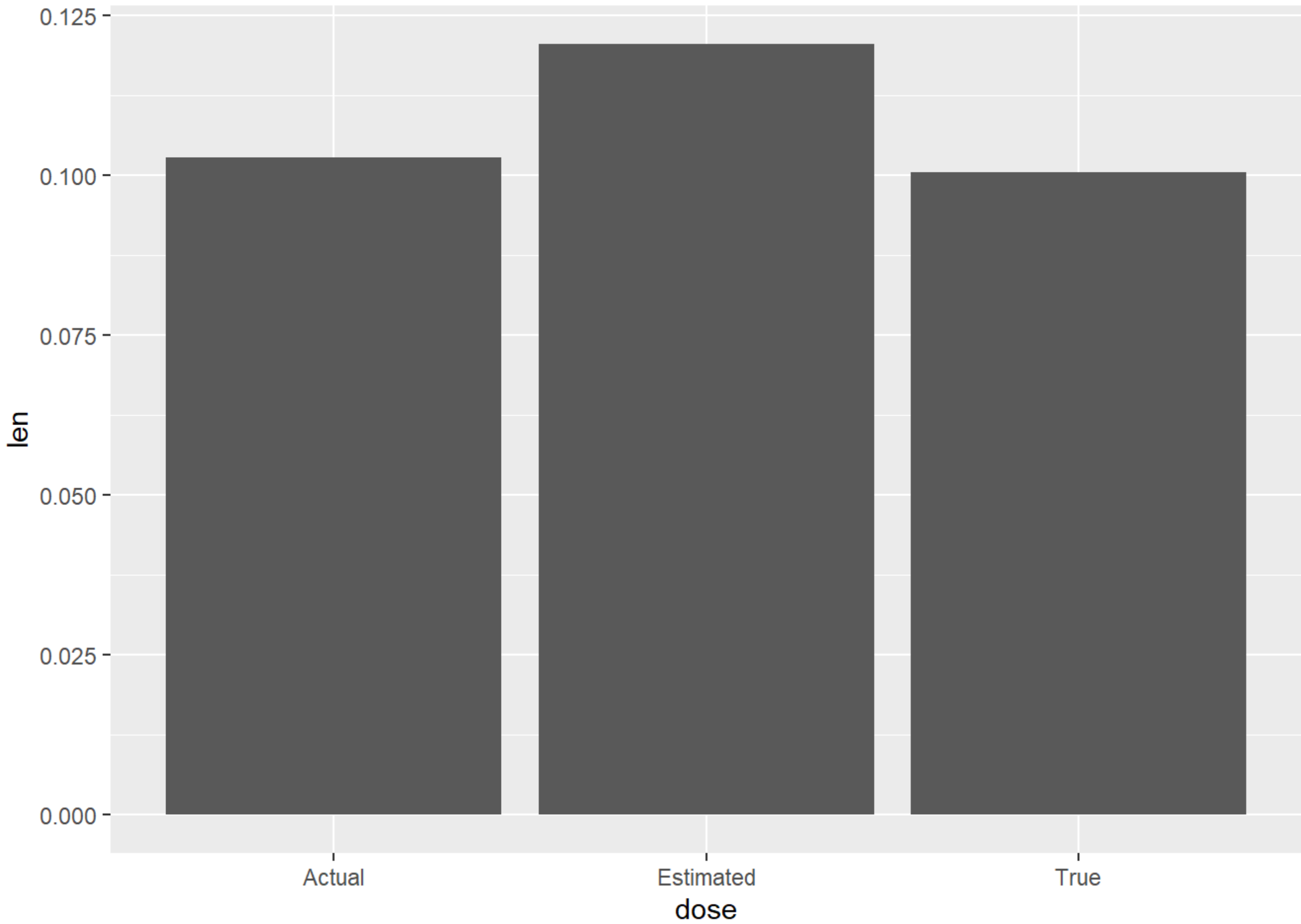
```
expected=t(w_MinVar3) %*%V3 %*%w_MinVar3
actual=t(w_MinVar3) %*%V %*%w_MinVar3
```

```
library(CVXR)

NumAssets=10
w <- Variable(NumAssets)   # decision variables
risk <- quad_form(w, V)  # This is w' Sample_Cov w
constraints <- list(w >= 0, sum(w) == 1)
prob <- Problem(Minimize(risk), constraints)
result <- solve(prob)
MinVar <- result$getValue(risk)
w_MinVar2 <- result$getValue(w)
return_Min <- t(r_mean) %*% w_MinVar
```

```
true=t(w_MinVar2) %*%V %*%w_MinVar2
```

```
df <- data.frame(dose=c("Estimated", "Actual", "True"),
                len=c(expected[1], actual[1], true[1]))
p<-ggplot(data=df, aes(x=dose, y=len)) +
  geom_bar(stat="identity")
p
```



(iii).

The actual variance is much closer to the true variance. Also, comparing to part b, now the estimated variance is less unstable.

Assignment 4

Group 3

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2. A Leveraged Firm

(a) Let r_E represents the rate of return of the equity holders.

Then, we have

$$r_A = wr_B + (1-w)r_E$$

Therefore,

$$r_E = \frac{r_A - wr_B}{1-w}$$

(b) Let β_E represents the beta of the return of the equity holders.

Then, we have

$$\beta_E = \frac{Cov(r_E, r_M)}{\sigma_M^2}$$

Because

$$r_E = \frac{r_A - wr_B}{1-w}$$

Therefore, we have

$$\beta_E = \frac{Cov(\frac{r_A - wr_B}{1-w}, r_M)}{\sigma_M^2} = \frac{\frac{1}{1-w}[Cov(r_A, r_M) - Cov(wr_B, r_M)]}{\sigma_M^2}$$

Since $Cov(wr_B, r_M) = 0$, we have

$$\beta_E = \frac{\frac{1}{1-w}[Cov(r_A, r_M)]}{\sigma_M^2} = \frac{\beta_A}{1-w}$$

(c) Based on CAPM, we have

$$E[r_E] = r_B + \beta_E(E[r_M] - r_B)$$

Since we know

$$\beta_E = \frac{\beta_A}{1-w}$$

Therefore, we have

$$E[r_E] = r_B + \frac{\beta_A}{1-w}(E[r_M] - r_B)$$

Because β_A is positive and $E[r_M] - r_B > 0$, we know that:

As w increases, $E[r_E]$ increases.
If w is close to 1, then $E[r_E]$ will be close to ∞.

3. Rough and Ready Calculations

To answer this question, we use the idea of the annuity formula to discount all the future cashflow and calculate the PV of all future cashflow.
The table below is the binomial model of gold:

The Gold Price										
										2476.7
									2063.9	1857.5
								1719.9	1547.9	1393.1
							1433.3	1289.9	1161.0	1044.9
						1194.4	1075.0	967.5	870.7	783.6
					995.3	895.8	806.2	725.6	653.0	587.7
				829.4	746.5	671.8	604.7	544.2	489.8	440.8
			691.2	622.1	559.9	503.9	453.5	408.1	367.3	330.6
		576.0	518.4	466.6	419.9	377.9	340.1	306.1	275.5	247.9
	480.0	432.0	388.8	349.9	314.9	283.4	255.1	229.6	206.6	186.0
400.0	360.0	324.0	291.6	262.4	236.2	212.6	191.3	172.2	155.0	139.5

Given a gold price, the cashflow will be price – extraction cost. When the cashflow is negative, the gold will not be extracted from the Simplico gold mine and the cashflow is 0. Therefore, the binomial model of cashflow is:

The CashFlow										
										2276.69457
									1863.91214	1657.52093
								1519.92678	1347.93411	1193.1407
							1233.27232	1089.94509	960.950579	844.855521
						994.3936	874.95424	767.458816	670.712934	583.641641
					795.328	695.7952	606.21568	525.594112	453.034701	387.731231
				629.44	546.496	471.8464	404.66176	344.195584	289.776026	240.798423
			491.2	422.08	359.872	303.8848	253.49632	208.146688	167.332019	130.598817
		376	318.4	266.56	219.904	177.9136	140.12224	106.110016	75.4990144	47.949113
	280	232	188.8	149.92	114.928	83.4352	55.09168	29.582512	6.6242608	0
200	160	124	91.6	62.44	36.196	12.5764	0	0	0	0

In each time period but the t_0, there is more than one possible gold price. Therefore, we need to calculate the expected cash flow of each time period. The binomial model of the possibility of each gold price is:

Binomial Model										
										0.05631
									0.07508	0.18771
								0.10011	0.22525	0.28157
							0.13348	0.26697	0.30034	0.25028
						0.17798	0.31146	0.31146	0.2336	0.146
					0.2373	0.35596	0.31146	0.20764	0.1168	0.0584
				0.31641	0.39551	0.29663	0.17303	0.08652	0.03893	0.01622
			0.42188	0.42188	0.26367	0.13184	0.05768	0.02307	0.00865	0.00309
		0.5625	0.42188	0.21094	0.08789	0.03296	0.01154	0.00385	0.00124	0.00039
	0.75	0.375	0.14063	0.04688	0.01465	0.00439	0.00128	0.00037	0.0001	2.9E-05
1	0.25	0.0625	0.01563	0.00391	0.00098	0.00024	6.1E-05	1.5E-05	3.8E-06	9.5E-07

With the table above, we could then calculate the expected value of cashflow at each time period:

Expeced value of Cashflow										
200	250	306.25	369.53125	440.72266	520.81299	610.914612	712.279468	826.31423	954.603203	1098.92887

And we could discount the expected value of cashflow by the risk-free rate R

Discounted Average CashFlow										
200	227.27273	253.09917	277.6343	301.0195	323.38389	344.845372	365.511992	385.481687	404.844945	423.684651

Summing than together, the PV of all future cashflow is 3506.77. Multiplying the PV by 10K, the max extraction rate of each time period, the price of lease is 35067782.39 or 35.1M, which is larger than 24.1M. However, it can still prove that 24.1M is a reasonable value of the lease.

4. Evaluating a More Complex Option on the Simplico Gold Mine

Based on known parameters, we can calculate the risk-neutral probability:

$$q = \frac{R-d}{u-d} = \frac{1.1-0.9}{1.2-0.9} = \frac{2}{3}$$

$$1-q = \frac{1}{3}$$

Therefore, we can obtain the gold price according to binomial model:

Gold Price

	0	1	2	3	4	5	6	7	8	9	10
10											2476.69
9										2063.91	1857.52
8									1719.93	1547.93	1393.14
7								1433.27	1289.95	1160.95	1044.86
6							1194.39	1074.95	967.46	870.71	783.64
5						995.33	895.80	806.22	725.59	653.03	587.73
4					829.44	746.50	671.85	604.66	544.20	489.78	440.80
3				691.20	622.08	559.87	503.88	453.50	408.15	367.33	330.60
2			576.00	518.40	466.56	419.90	377.91	340.12	306.11	275.50	247.95
1		480.00	432.00	388.80	349.92	314.93	283.44	255.09	229.58	206.62	185.96
0	400.00	**360.00**	324.00	291.60	262.44	236.20	212.58	191.32	172.19	154.97	139.47

Then, we can calculate the lease price before introducing the new equipment:

$$V_t(s) = \frac{10k * \max\{0, s-200\} + [\frac{2}{3}V_{t+1}(us) + \frac{1}{3}V_{t+1}(ds)]}{1.1}$$

And we get the binomial tree of lease value with $V_{10} = 0$:

Lease Value (in millions)

10											0.0
9										16,944,655.8	0.0
8									27,800,321.7	12,253,946.4	0.0
7								34,115,541.1	19,982,472.7	8,735,914.4	0.0
6							37,092,763.7	24,343,229.8	14,119,085.9	6,097,390.3	0.0
5						37,660,608.3	26,234,640.0	17,013,996.4	9,721,545.8	4,118,497.3	0.0
4					36,531,660.4	26,350,062.8	18,091,047.3	11,517,071.3	6,423,390.8	2,634,327.5	0.0
3				34,248,616.9	25,221,115.0	17,867,153.7	11,983,352.7	7,394,377.5	3,949,774.5	1,521,200.2	0.0
2			31,221,056.7	23,252,253.3	16,738,205.9	11,504,971.9	7,402,581.8	4,302,357.1	2,094,562.3	686,354.7	0.0
1		27,754,679.6	20,748,329.4	15,004,980.5	10,376,024.1	6,733,335.6	3,967,003.7	1,983,341.8	703,153.1	60,220.6	0.0
0	24,074,547.5	17,936,647.4	12,894,277.5	8,821,154.5	5,609,761.9	3,172,343.2	1,448,845.1	437,213.2	36,497.3	**0.0**	0.0

Because we can only add new equipment from the beginning at year 5, with $V_{10} = 0$, we have:

From year 5 to year 9:

$$U_t(s) = \max\left\{\frac{14k * \max\{0, s-240\} + \left[\frac{2}{3}V_{t+1}(us) + \frac{1}{3}V_{t+1}(ds)\right]}{1.1} - 5million, \frac{10k * \max\{0, s-200\} + \left[\frac{2}{3}U_{t+1}(us) + \frac{1}{3}U_{t+1}(ds)\right]}{1.1}\right\}$$

$$(t = 5, 6, \dots, 9)$$

From year 0 to year 4:

$$V_t(s) = \frac{10k * \max\{0, s - 200\} + \left[\frac{2}{3}V_{t+1}(us) + \frac{1}{3}V_{t+1}(ds)\right]}{1.1}, \quad (t = 0, 1, \dots, 4)$$

And we get the binomial tree of lease value if we can add new equipment:

Uts											
10											0.0
9										**18213427.25**	0.0
8									**32948549.54**	**12253946.41**	0.0
7								**41369120.44**	**22003560.92**	**8735914.36**	0.0
6							**45154744.46**	**27687884.66**	**14119085.91**	**6097390.31**	0.0
5						**45602011.00**	**29953371.37**	**17426957.82**	**9721545.84**	**4118497.28**	0.0
4					42380142.23	**29767247.36**	**18552341.55**	**11517071.29**	**6423390.78**	**2634327.51**	0.0
3				38446405.83	27376854.77	**18146726.02**	**11983352.74**	**7394377.47**	**3949774.49**	**1521200.17**	0.0
2			34176642.02	24610107.02	16907643.62	**11504971.92**	**7402581.83**	**4302357.11**	**2094562.27**	**686354.68**	0.0
1		29804749.45	21602389.13	15107670.09	10376024.06	**6733335.55**	**3967003.65**	**1983341.84**	**703153.11**	**60220.55**	0.0
0	25479581.10	18473118.73	12956513.55	8821154.55	5609761.90	**3172343.16**	**1448845.11**	**437213.19**	**36497.30**	**0.00**	0.0

We obtain the lease value of $25479581.1 at t=0, with the option for adding the new equipment, which is larger than the lease value without the option.