Maths and Statistics Foundations
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Solutions to Practice Problems for *Statistics Review*

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1. **Maximum Likelihood Estimation**

   Suppose that $X_1, \ldots, X_n$ form a random sample from a uniform distribution on the interval $(0, \theta)$, where $\theta > 0$ but is unknown. Find the MLE of θ.

   Solution: The pdf of each observation has the following form

$$f(x \mid \theta) = \begin{cases} 1/\theta, & \text{for } 0 \leq x \leq \theta \\ 0, & \text{otherwise} \end{cases}$$

   Therefore the likelihood function has the form

$$L(\theta) = \begin{cases} 1/\theta^n, & \text{for } 0 \leq x_i \leq \theta \ (i = 1, \ldots, n) \\ 0, & \text{otherwise} \end{cases}$$

   It can be seen that the MLE of θ must be a value of θ for which $\theta \geq x_i$ for $i = 1, \ldots, n$ and which maximizes $1/\theta^n$ among all such values. Since $1/\theta^n$ is a decreasing function of θ, the estimate will be the smallest possible value of θ such that $\theta \geq x_i$ for $\imath = 1, \ldots, n$. This value is $\theta = \max(x_1, \ldots, x_n)$, it follows that the MLE of θ is $\hat{\theta} = \max(X_1, \ldots, X_n)$.

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2. **Confidence Intervals**

   You want to rent an unfurnished one-bedroom apartment in London next year. The mean monthly rent for a random sample of 60 apartments advertised on RightMove (a website that lists apartments and houses for buy or rent) is £1300. Assume a normal population with standard deviation of £200.

   (a) Construct a 95% confidence interval of the average monthly rent.

   Solution:

$$\bar{x} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 1300 \pm 1.96 \frac{200}{\sqrt{60}} = (1249.39, 1350.61).$$

   (b) How large a sample of one-bedroom apartments above would be needed to estimate the population mean within a margin of error of £50 with 90% confidence?

   Solution:

$$50 = z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = 1.645 \frac{200}{\sqrt{n}} \quad \Rightarrow \quad n = 43.3 \approx 44.$$

(c) True or false: With all else constant, an increase in population standard deviation will shorten the length of a confidence interval.

Solution: False. It will lengthen the CI.

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3. **Hypothesis Testing**

(a) (**Two-tailed test**) An inventor has developed a new, energy-efficient lawn mower engine. He claims that the engine will run continuously for 5 hours (300 minutes) on a single gallon of regular gasoline. From his stock of 2000 engines, the inventor selects a simple random sample of 50 engines for testing. The engines run for an average of 295 minutes, with a standard deviation of 20 minutes. Test the null hypothesis that the mean run time is 300 minutes against the alternative hypothesis that the mean run time is not 300 minutes. Use a 0.05 level of significance. (Assume that run times for the population of engines are normally distributed.)

Solution: The t-statistic is

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{295 - 300}{20/\sqrt{50}} = -1.77$$

Since

$$|t| < t_{n-1}^{\alpha} = t_{49}^{0.025} = 2.01,$$

we cannot reject the null hypothesis.

(b) (**One-tailed test**) Bon Air Elementary School has 1000 students. The principal of the school thinks that the average IQ of students at Bon Air is at least 110. To prove her point, she administers an IQ test to 20 randomly selected students. Among the sampled students, the average IQ is 108 with a standard deviation of 10. Based on these results, should the principal accept or reject her original hypothesis? Assume a significance level of 0.01. (Assume that test scores in the population of engines are normally distributed.)

Solution: The t-statistic is

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{108 - 110}{10/\sqrt{20}} = -0.894$$

Hence the p-value for this one-tailed test is

$$\mathrm{P}(t_{n-1} < t) = \mathrm{P}(t_{19} < -0.894) = 0.19.$$

Since $p > 0.01$, we cannot reject the null hypothesis.