BUSI97277: Financial Analytics 2021 Imperial College Business School
Martin Haugh

Assignment 1

Rules

1. This is a group assignment. (There are approximately 3 people per group and by now you should know your assigned group.)
2. You are free to use `R`, `Python` or `Excel` for the this assignment. (In fact `Excel` is particularly suited to binomial lattices and presents an easy way to visualise price dynamics in these models.)
3. Within each group **I strongly encourage each person to attempt each question by his / herself first** before discussing it with other members of the group.
4. Students should **not** consult students in other groups when working on their assignments.
5. Late assignments will **not** be accepted and all assignments must be submitted through the Hub with one assignment submission per group. Your submission should include a PDF report with your answers to each question together with screenshots of any relevant code. Make sure your PDF clearly identifies each member of the group by CID and name.

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1. **Forward Rates (10 marks)**
   If the spot rates for 1 and 2 years are $s_1 = 6.3\%$ and $s_2 = 6.9\%$, what is the forward rate, $f_{1,2}$ assuming annual compounding? What do you get if you assume continuous compounding?

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2. **Pricing a Floating Rate Bond (15 marks)**
   A floating rate bond is the same as a regular bond except that the coupon payments at the end of a period are determined by the prevailing short rate at the beginning of the period. Specifically let the coupon payments be $c_1, \ldots, c_n$ occurring at times $t_1 < t_2 < \cdots < t_n$, respectively. The face value F of the bond is also paid at time t_n with $c_i = r_{i-1}F$ where r_{i-1} is the short rate prevailing between times $i-1$ and i. While r_{i-1} is stochastic it is known at time $i-1$. Show that the arbitrage-free time $t = 0$ value of the floating rate bond is F.

   Hint: Consider working backwards in time. (The floating rate bond is that rare example of a stochastic cash-flow that we can price without needing a model!)

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3. **Mortgage-Mathematics (30 marks)**
   Consider the interest-only (IO) and principal-only (PO) securities in a deterministic world without prepayments and defaults. These securities have time k cash-flows of $P_k := B - cM_{k-1}$ and $I_k := cM_{k-1}$, respectively, for $k = 1, \ldots, n$ and where M_k (and all other notation) is defined in Section 4 of the *Interest Rates and Deterministic Cash-Flows* lecture notes.

(a) Compute the present value V_0 of the PO security assuming a risk-free rate of r per period. (Note that r is the risk-free rate used to discount deterministic cash-flows so that $V_0 = \sum_{k=1}^{n} P_k/(1+r)^k$.) **(10 marks)**

(b) What happens to V_0 as $n \to \infty$? **(5 marks)**

(c) Compute the present value W_0 of the IO security. **(10 marks)**

(d) The **duration** of a fixed-income security is a measure of how long the owner of the security must wait until the cash-flows associated with the security are received. More specifically, it is a *weighted average* of the cash-flow times with weights given by the cash-flow contributions to the overall value of the security. If we let D_P and D_I denote the durations of the PO and IO securities, respectively, then they are given by

$$\begin{aligned} D_P &= \frac{1}{12\, V_0} \sum_{k=1}^{n} \frac{k\, P_k}{(1+r)^k} \\ D_I &= \frac{1}{12 W_0} \sum_{k=1}^{n} \frac{k\, I_k}{(1+r)^k} \end{aligned}$$

where we divide by 12 to convert the duration into annual rather than monthly time units. Which of the two securities do you think has the longer duration? Justify your answer. (You don't need to do any calculations here!) **(5 marks)**

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4. **Futures Hedging - from Luenberger's *Investment Science* (25 marks)**
Farmer D. Jones has a crop of grapefruit juice that will be ready for harvest and sale as $150,000$ pounds of grapefruit juice in 3 months. Jones is worried about possible price changes, so he is considering hedging. There is no futures contract for grapefruit juice, but there is a futures contract for orange juice. His son, Gavin, recently studied minimum-variance hedging and suggests it as a possible approach. Currently the spot prices are \$1.20 per pound for orange juice and \$1.50 per pound for grapefruit juice. The standard deviation of the prices of orange juice and grapefruit juice is about 20% per year, and the correlation coefficient between them is about .7. What is the minimum variance hedge for farmer Jones, and how effective is this hedge as compared to no hedge?

Remark: When we say the standard deviation of the price of grapefruit juice is about 20% per year this means (in the language of the lecture notes) that $\sigma_{S_T} \approx 20\% \times S_0/\sqrt{4}$ where S_0 is the current price of grapefruit and we divide by $\sqrt{4}$ because the maturity is 3 months which is 1/4 of a year. Similarly you can assume $\sigma_{F_T} \approx (.2 \times 1.20)/\sqrt{4}$.

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5. **The Die Game (25 marks)**
You are allowed to roll a fair 6-sided die a maximum of 3 times. After any throw you can elect to "stop". If you elect to stop after the i^{th} throw then you will receive $\$X_i$ where X_i is the result of the i^{th} throw, for $i = 1, 2$ or 3. For example, suppose you throw a 3 on your second throw and then elect to stop. You will then receive a payoff of \$3 and will not proceed with the 3^{rd} throw. You must stop after the 3^{rd} throw if you have not elected to stop after the earlier throws.

   (a) If you are risk-neutral what is the fair value of this game to you? *Hint:* Consider working backwards in time. e.g. If you have just one throw left what is the fair value of the game? **(10 marks)**

   (b) If you are risk-averse and have log utility what is the fair value of this game to you? **(15 marks)**

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6. **Brownian Motion and Geometric Brownian Motion (20 marks)**
Let B_t be a standard Brownian motion.

   (a) What is $\mathrm{E}_0[B_{t+s}B_s]$? *Hint:* write $B_{t+s} = (B_{t+s} - B_s + B_s)$. **(10 marks)**

   (b) The geometric Brownian motion (GBM) model for a security price assumes its time t price is given by
$$S_{t+s} = S_t\, e^{(\mu - \sigma^2/2)s + \sigma(B_{t+s} - B_t)}$$
where B_t is SBM. Compute $\mathrm{E}_t\left[S_{t+s}^2\right]$ where $\mathrm{E}_t[\cdot]$ denotes the expectation conditional on all time t information. *Hint:* You'll need to know the MGF of a normal random variable. **(10 marks)**

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7. **Tower Property of Conditional Expectations (25 marks)**
Demonstrate the tower property by showing that
$$\mathrm{E}_u\left[\mathrm{E}_t[S_v^2]\right] = \mathrm{E}_u[S_v^2] \tag{1}$$
where $u \leq t$. (The result is clearly true (why?!) when $v \leq t$ so you only need show it for $v \geq t$. Your calculations in Question 6b should be useful!)

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8. **Using Monte-Carlo Simulation to Price Asian Options (25 marks)**
Use Monte-Carlo simulation to estimate the price $\mathrm{E}_0^Q[e^{-rT}h(\mathbf{X})]$ of an Asian call option where the time T payoff $h(\mathbf{X})$ is given by
$$h(\mathbf{X}) := \max\left(0, \frac{\sum_{i=1}^m S_{iT/m}}{m} - K\right)$$
where $\mathbf{X} = (S_{T/m}, S_{2T/m}, \ldots, S_T)$. You may assume that under the probability distribution Q (the so-called *risk-neutral* probability distribution that we will encounter later in the course)

that $S_t \sim \text{GBM}(r, \sigma)$ where $r = .05$ and $\sigma = .25$. Other relevant parameters are $T = 1$ year, $S_0 = 100$ and $m = 11$. Simulate $10,000$ paths and estimate the option price (with approx 95% CIs) for $K = 90, 100, 110$ and 120.