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**Solutions to Practice Problems for *Linear Algebra Review***

**Within Exam Scope**

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1. **Subspaces and Spans**

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ are vectors in $\mathbb{R}^3$. Do you agree with the statement that these vectors span $\mathbb{R}^3$?

   **Solution:** Not necessarily. They can be linearly dependent so that the dimension of their span is less than 3.

2. **Orthogonality**

   Which pairs are orthogonal among the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

   **Solution:** $\mathbf{v}_1$ and $\mathbf{v}_3$; $\mathbf{v}_2$ and $\mathbf{v}_3$.

3. **Linear Independence**

   If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are independent vectors, would the following vectors be independent?

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3.$$

   **Solution:** Yes. Can check by definition. *Source: Strang, Gilbert (2006), Linear Algebra and Its Applications, CENGAGE.*

4. **Range and Rank**

   For $\mathbf{A} \in \mathbb{R}^{n \times n}$, if $\text{rank}(\mathbf{A}) = m < n$, then what is the dimension of the null space?

   **Solution:** $n - m$.

5. **Matrix Inverse**

   We have matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Which of the following are true?

(a) $\mathbf{A}$ is singular

(b) $\mathbf{B}$ is invertible

(c) $\mathbf{A}+\mathbf{B}$ is invertible.

**Solution:** (1) and (3) true, (2) is false since $\mathbf{B}$ is singular.

6. **Subspaces**

The smallest subspace of $\mathbb{R}^3$ containing the vectors $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ is the line whose equations are $x = a$ and $z = by$. What are the values of $a$ and $b$?

**Solution:** $a = 0,\ b = -2$.

7. **Matrix Inverse**

For $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, assume $\mathbf{A}$ and $\mathbf{B}$ are invertible:

(a) (True/False) $\left(\mathbf{A}^{-1}\right)^\top = \left(\mathbf{A}^\top\right)^{-1}$,

(b) (True/False) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$,

(c) (True/False) If $\det(\mathbf{A}) = 2$, then $\det(\mathbf{A}^{-1}) = 2^{-1}$.

(d) (True/False) If $\det(\mathbf{A}) = 2$ and $\alpha > 1$, then $\det(\alpha\mathbf{A}) = 2$.

**Solution:** (a) True (b) False (c) True (d) False

8. **Linear Independence**

Let $\mathbf{u} = \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix}$. What are possible values of $\lambda$ that make $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly dependent?

**Solution:** Solving $\det([\mathbf{u}, \mathbf{v}, \mathbf{w}]) = 0$ we get $\lambda = -\sqrt{2}, 0, \sqrt{2}$.

9. **Matrix Product**

For $\mathbf{A} = \begin{bmatrix} 1 & 1/3 \\ x & y \end{bmatrix}$, find the value of $x$ and $y$ such that $\mathbf{A}^2 = 0$.

**Solution:** $x = -3, y = -1$.

10. **Inner Product**

Consider the space of all matrices in $\mathbb{R}^{2\times 2}$. Define inner product as $\langle \mathbf{A}, \mathbf{B}\rangle = \mathrm{tr}(\mathbf{A}^\top \mathbf{B})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$. Let $\mathbf{U} = \begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} x^2 & x-1 \\ x+1 & -1 \end{bmatrix}$. Find all values of $x$ such that $\mathbf{U} \perp \mathbf{V}$.

**Solution:** $x = 3, -4$.

11. **Matrix Inverse**

If a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^3 - 4\mathbf{A}^2 + 3\mathbf{A} - 2\mathbf{I}_n = \mathbf{0}$, what is $\mathbf{A}^{-1}$?

**Solution:** $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I}_n)$.

12. **Basis**

Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix}$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that $c$ is **not** equal to what value?

**Solution:** $c \neq -3$.

13. **Subspaces**

Consider the set of points $(x, y, z) \in \mathbb{R}^3$. Which one of the following is a subspace of $\mathbb{R}^3$?

(a) $x + 3y - 2z = 3$.
(b) $x + y + z = 0$ and $x - y - z = 2$.
(c) $\frac{x+1}{2} = \frac{y-2}{4} = \frac{z}{3}$.
(d) $x^2 + y^2 = z$.
(e) $x = -z$ and $x = z$.
(f) $\frac{x}{3} = \frac{y+1}{2}$.

**Solution:** (e). Subspaces of $\mathbb{R}^3$ must contain the origin.

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1. **Solutions to Linear Equations**

   Consider the linear system of equations

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

   where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are known and $\mathbf{x} \in \mathbb{R}^n$ is unknown. Let $\mathbf{b} \in \mathcal{R}(\mathbf{A})$ so that (1) has at least one solution. Let $\mathbf{x}_1$ be one such solution. Show that *all* solutions to (1) can be written in the form $\mathbf{x}_1 + \mathbf{n}$ for some $\mathbf{n} \in \mathcal{N}(\mathbf{A})$ where $\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \, : \, \mathbf{Ax} = \mathbf{0}\}$ is the *nullspace* of $\mathbf{A}$. (You can check that $\mathcal{N}(\mathbf{A})$ is indeed a subspace of $\mathbb{R}^n$.)

   **Solution:** First note that

$$\mathbf{A}(\mathbf{x}_1 + \mathbf{n}) = \mathbf{Ax}_1 + \mathbf{An} = \mathbf{b} + \mathbf{0} = \mathbf{b}$$

   so $\mathbf{x}_1 + \mathbf{n}$ is indeed a solution to (1) for any $\mathbf{n} \in \mathcal{N}(\mathbf{A})$.

   Now let's check that *all* solutions to (1) are of this form. Specifically, let $\mathbf{y}$ also solve (1). Then $\mathbf{A}(\mathbf{y} - \mathbf{x}_1) = \mathbf{b} - \mathbf{b} = \mathbf{0}$ so $\mathbf{y} - \mathbf{x}_1 \in \mathcal{N}(\mathbf{A})$. Therefore $\mathbf{y} - \mathbf{x}_1 = \mathbf{n}$ for some $\mathbf{n} \in \mathcal{N}(\mathbf{A})$ and so $\mathbf{y} = \mathbf{x}_1 + \mathbf{n}$ as desired.

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2. **Adjacency Matrices I (Challenging - Not Examinable!)**

   If $\mathbf{A}$ is an adjacency matrix show that $\mathbf{A}_{ij}^k = \#$ paths from $i \rightarrow j$ in exactly $k$ steps. *Hint:* Use induction. (If you're not familiar with induction that's ok and you can skip this question.)

   **Solution:** We can prove this by induction. It is clearly true when $k = 1$. Now assume it is true for all values $k \leq n - 1$. We must now show that it is true for $n$. We have

$$\begin{aligned} \mathbf{A}_{ij}^n &= (\mathbf{A}^{n-1}\,\mathbf{A})_{ij} \\ &= \sum_{k=1}^{n} \mathbf{A}_{ik}^{n-1}\mathbf{A}_{kj} \end{aligned} \tag{2}$$

   But:

   - By assumption $\mathbf{A}_{ik}^{n-1} = \#$ paths from $i \rightarrow k$ in exactly $n - 1$ steps.
   - $\mathbf{A}_{kj} = \#$ paths from $k \rightarrow j$ in exactly 1 step.

   So $\mathbf{A}_{ik}^{n-1}\mathbf{A}_{kj} = \#$ paths from $i \rightarrow j$ in exactly $n$ steps that pass through node $k$ on step $n-1$. So (2) equals $\#$ paths from $i \rightarrow j$ in exactly $n$ steps.

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3. **Adjacency Matrices II**

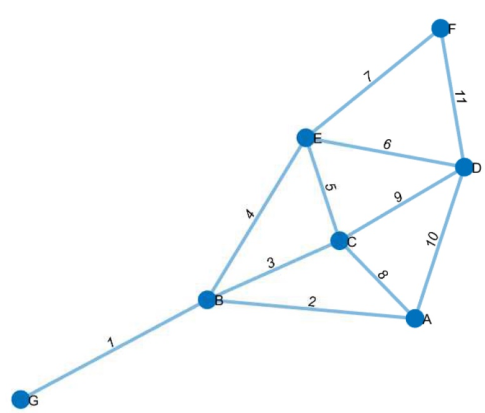
Consider the adjacency matrix $\mathbf{A}$ below with rows and columns ordered according to the node labels {'A','B','C','D','E','F','G'}. Compute $\mathbf{A}^3$ and confirm that the value in $\mathbf{A}^3$ corresponding to paths from $E \to B$ is correct by explicitly writing out and counting all the paths from $E \to B$ that take exactly 3 steps.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

**Solution:**You can check using `R` (no need to do it yourself!) that

$$\mathbf{A}^3 = \begin{pmatrix} 4 & 9 & 8 & 9 & 5 & 4 & 1 \\ 9 & 4 & 9 & 5 & 10 & 4 & 4 \\ 8 & 9 & 8 & 10 & 10 & 4 & 2 \\ 9 & 5 & 10 & 6 & 10 & 6 & 3 \\ 5 & \mathbf{10} & 10 & 10 & 6 & 6 & 1 \\ 4 & 4 & 4 & 6 & 6 & 2 & 1 \\ 1 & 4 & 2 & 3 & 1 & 1 & 0 \end{pmatrix}$$

and the value (highlighted in bold font) corresponding to paths from $E \to B$ is 10. We can confirm that this is indeed correct by calculating all the paths that go from $E \to B$ in exactly 3 steps. To help with this we include the graph of the network below.




We have the following 3-step paths from $E \to B$:

(a) $E \to B \to E \to B$
(b) $E \to F \to E \to B$
(c) $E \to D \to E \to B$
(d) $E \to C \to E \to B$
(e) $E \to B \to G \to B$

(f) $E \to B \to C \to B$

(g) $E \to B \to A \to B$

(h) $E \to D \to C \to B$

(i) $E \to D \to A \to B$

(j) $E \to C \to A \to B$

and we see that there indeed 10 such paths.

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4. **Diagonalization**

Suppose an $n \times n$ matrix $\mathbf{A}$ can be diagonalized and let $\mathbf{u}_k := \mathbf{A}^k \mathbf{u}_0$ for some initial vector $\mathbf{u}_0$. Show that for any $k \in \mathbb{N}$ we can write

$$\mathbf{u}_k = c_1 \lambda_1^k \mathbf{x}_1 + \cdots + c_n \lambda_n^k \mathbf{x}_n. \tag{3}$$

(a) What are the $c_i$'s, $\lambda_i$'s and $\mathbf{x}_i$'s?

**Solution:** We are told that $\mathbf{A}$ can be diagonalized so let $\mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$ be the diagonalization where $\mathbf{D}$ is a diagonal $n \times n$ matrix with the eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly repeated) of $\mathbf{A}$ along its diagonal and let $\mathbf{S}$ be the $n \times n$ matrix whose $i^{th}$ column $\mathbf{x}_i$ is the eigenvector of $\mathbf{A}$ corresponding to $\lambda_i$. Then

$$\begin{aligned}
\mathbf{u}_k &= \mathbf{A}^k \mathbf{u}_0 \\
&= \left(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}\right)^k \mathbf{u}_0 \\
&= \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}\mathbf{u}_0 \\
&= \mathbf{S}\mathbf{D}^k\mathbf{c} \\
&= \begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} \lambda_1^k & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \\
&= c_1 \lambda_1^k \mathbf{x}_1 + \cdots + c_n \lambda_n^k \mathbf{x}_n.
\end{aligned}$$

where $\mathbf{c} := \mathbf{S}^{-1}\mathbf{u}_0$.

(b) There are many problems of interest where $\mathbf{u}_k$ represents the state of some system (e.g. an economy, or web-browser in `Google's` PageRank model) at time $k$. Very often, we are interested in the behavior of this system in the *long-run*, i.e. when $k$ gets very large. How might the representation in (3) be useful for determining this long-run behaviour?

**Solution:** We see that the only terms in (3) that depend on $k$ are $\lambda_i^k$ for $i = 1, \ldots, n$. Clearly then if $|\lambda_i| > 1$ for any $i$ the $\mathbf{u}_k$'s will diverge. Similarly if $|\lambda_i| < 1$ for all $i$ then $\mathbf{u}_k \to 0$ as $k \to \infty$. A particularly interesting case is where $|\lambda_i| \leq 1$ for all $i$ but with one eigen-value (say $\lambda_j$) equal to 1. In that case $\mathbf{u}_k \to c_j \mathbf{x}_j$ as $k \to \infty$. It's also of interest when $|\lambda_i| \leq 1$ for all $i$ and one or more of the eigenvalues have absolute value equal to 1. For example suppose $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_i < 1$ for $i = 3, \ldots, n$. In that case we have $\mathbf{u}_k = c_1\mathbf{x}_1 + c_2(-1)^k\mathbf{x}_2$ for large $k$ and so $\mathbf{u}_k$ will oscillate between $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ when $k$ is large and even, and $c_1\mathbf{x}_1 - c_2\mathbf{x}_2$ when $k$ is large and odd.

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5. **Cash-Flow Dynamics**

   Multinational companies in the U.S., Japan and Europe have assets of \$4 trillion. At the start \$2 trillion are in the U.S.and \$2 trillion in Europe. Each year 1/2 the U.S. money stays home, and 1/4 goes to each of Europe and Japan. For Japan and Europe, 1/2 stays home and 1/2 is sent to the U.S.

   (a) Find the $3 \times 3$ matrix $\mathbf{A}$ that that gives $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$ where $\mathbf{x}_k$ is the $3 \times 1$ vector containing the asset values in the US, Europe and Japan, respectively, at the end of year $k$.

   (b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

   (c) Find the cash-flow distribution at the end of year $k$.

   (d) Find the limiting cash-flow distribution of the \$4 trillion as the world ends.

   (This question is taken from Gilbert Strang's *Linear Algebra and Its Applications*.)

   **Solution:** See the `R Notebook` *Multinational_Cashflows.Rmd*. You might also note how your results are consistent with the results of the previous question.

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