Financial Analytics

Some Topics in Credit Modeling

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Outline

Fixed Income Markets

- Fixed income markets are enormous and are in fact bigger than equity markets.
- According to *SIFMA*, by 2017 the total outstanding amount of US fixed income securities was $\$41$ trillion:

Government	\$14.5	35.2%
Municipal	\$3.9	9.5%
Mortgage-Backed Securities	\$9.3	22.6%
Corporate	\$9.0	21.9%
Agency	\$1.9	4.6%
Asset-Backed Securities	\$1.5	3.6%
Money Markets	\$1.0	2.4%
Total	\$41.1 tr	100%

  – in comparison, at end of 2017 size of US equity was approx $\$30$ trillion.

- Fixed income derivatives markets are also enormous
  - includes interest-rate and bond derivatives, credit derivatives etc.

Collateralized Debt Obligations (CDOs)

- Collateralized debt obligations (CDOs) are a class of asset-backed securities (ABS).
- Mortgage-backed securities (MBS) are also a type of ABS
  - but MBS market is so big we often used ABS to refer to securities backed by non-mortgage debt such as credit card debt, auto loans, student loans etc.
- A CDO is also a type of credit derivative.
- We'll briefly discuss a stylized CDO here in order to:
  - See one approach to securitization.
  - See how securitization can create securities with very different risk profiles.

Collateralized Debt Obligations (CDO's)

Will consider a simple 1-period CDO with the following characteristics:

- The maturity is 1 year.
- There are $N = 125$ bonds in the portfolio.
- Each bond pays a coupon of one unit after 1 year if it has not defaulted.
- The recovery rate on each defaulted bond is zero.
- There are 3 tranches of interest:
  1. The equity tranche with attachment points: 0-3 defaults
  2. The mezzanine tranche with attachment points: 4-6 defaults
  3. The senior tranche with attachment points: 7-125 defaults.

Assume probability q of defaulting within 1 year is identical across all bonds.

Collateralized Debt Obligations

- X_i is the normalized asset value of the i^{th} credit and we assume

$$X_i = \sqrt{\rho}M + \sqrt{1-\rho}\,Z_i \tag{1}$$

  where $M, Z_1, \ldots, Z_N$ are IID normal random variables

  - note correlation between each pair of asset values is identical.

- We assume the i^{th} credit defaults if $X_i \leq \bar{x}_i$.

Since probability q of default is identical across all bonds must therefore have

$$\bar{x}_1 = \cdots \bar{x}_N = \Phi^{-1}(q). \tag{2}$$

Then follows from (1) and (2) that

$$\begin{aligned} \mathsf{P}(i \text{ defaults} \mid M) &= \mathsf{P}(X_i \leq \bar{x}_i \mid M) \\ &= \mathsf{P}(\sqrt{\rho}M + \sqrt{1-\rho}\,Z_i \leq \Phi^{-1}(q) \mid M) \\ &= \mathsf{P}\left(Z_i \leq \frac{\Phi^{-1}(q) - \sqrt{\rho}M}{\sqrt{1-\rho}} \,\middle|\, M\right). \end{aligned}$$

Collateralized Debt Obligations

Therefore conditional on M, the total number of defaults is $\text{Bin}(N, q_M)$ where

$$q_M := \Phi\left(\frac{\Phi^{-1}(q) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right).$$

That is,

$$p(k \mid M) = \binom{N}{k} q_M^k (1 - q_M)^{N-k}.$$

Unconditional probabilities computed by numerically integrating the binomial probabilities with respect to M so that

$$\mathsf{P}(k \text{ defaults}) = \int_{-\infty}^{\infty} p(k \mid M)\phi(M)\, dM.$$

Can now compute expected (risk-neutral) loss on each of the three tranches:

Computing Expected Tranche Losses

$$
\begin{aligned}
\mathsf{E}\left[\text{Equity tranche loss}\right] &= 3 \times \mathsf{P}(3 \text{ or more defaults}) + \sum_{k=1}^{2} k\, \mathsf{P}(k \text{ defaults}) \\
\mathsf{E}\left[\text{Mezz tranche loss}\right] &= 3 \times \mathsf{P}(6 \text{ or more defaults}) + \sum_{k=1}^{2} k\, \mathsf{P}(k+3 \text{ defaults}) \\
\mathsf{E}\left[\text{Senior tranche loss}\right] &= \sum_{k=1}^{119} k\, \mathsf{P}(k+6 \text{ defaults}).
\end{aligned}
$$

Recalling Our Simple 1-Period CDO

- Maturity is 1 year and $N = 125$ bonds in the portfolio.
- Each bond pays a coupon of one unit after 1 year if it has not defaulted.
- Recovery rate is zero and there are 3 tranches of interest:
  1. The equity tranche with attachment points: 0-3 defaults
  2. The mezzanine tranche with attachment points: 4-6 defaults
  3. The senior tranche with attachment points: 7-125 defaults.
- Probability q of defaulting within 1 year is identical across all bonds.
- We showed

$$\mathrm{P}(k \text{ defaults}) = \int_{-\infty}^{\infty} p(k \mid M)\phi(M)\, dM.$$

where

$$\begin{aligned} p(k \mid M) &= \binom{N}{k} q_M^k (1 - q_M)^{N-k} \\ q_M &:= \Phi\left(\frac{\Phi^{-1}(q) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right). \end{aligned}$$

Total Portfolio Loss as a Function of ρ?

Question: How does the total expected loss in the portfolio vary with ρ?

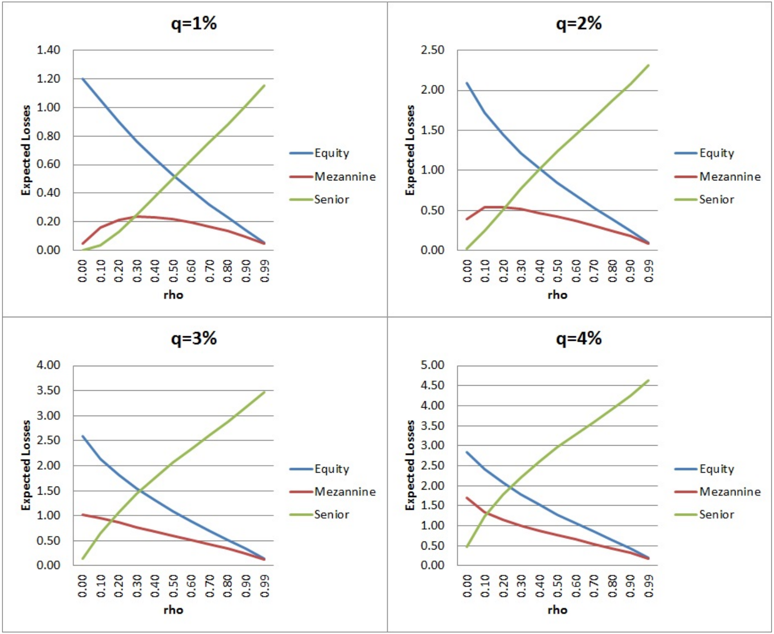
Tranche Losses as a Function of ρ

Regardless of the individual default probability q and correlation ρ we have:

$$\begin{aligned} \mathsf{E}\left[\%\ \text{Equity tranche loss}\right] &\geq \mathsf{E}\left[\%\ \text{Mezz tranche loss}\right] \\ &\geq \mathsf{E}\left[\%\ \text{Senior tranche loss}\right]. \end{aligned}$$

Some Facts

- Expected equity tranche loss always decreasing in ρ.
- Expected mezzanine tranche loss often relatively insensitive to ρ.
- Expected senior tranche loss (with upper attachment point of 100%) always increasing in ρ.



Expected Tranche Losses As a Function of ρ

The Gaussian-Copula Model

Recall the dependence structure we used to link the default events of the various bonds:

$$X_i = \sqrt{\rho} M + \sqrt{1-\rho}\, Z_i \tag{3}$$

where:

- X_i is the normalized asset value of the i^{th} credit
- $M, Z_1, \ldots, Z_N$ are IID normal random variables.

The model in (3) is the (in)famous Gaussian-copula model

- gained a lot of notoriety in the 2000's!

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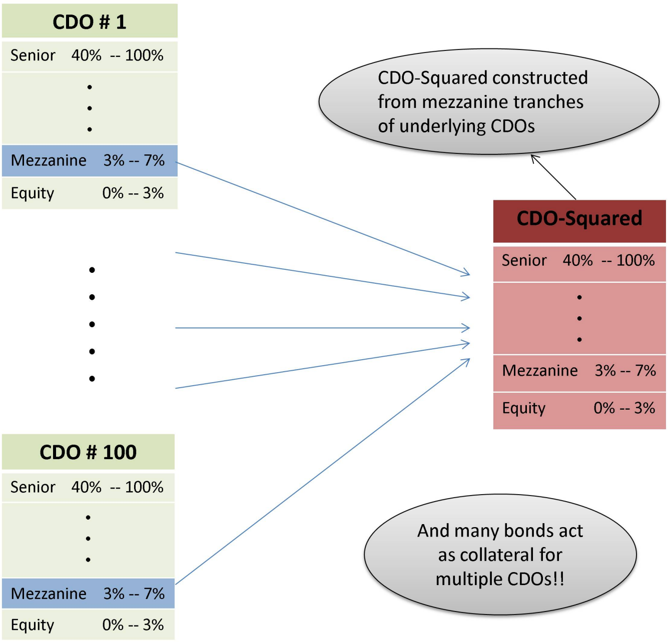
RECIPE FOR DISASTER: THE FORMULA THAT KILLED WALL STREET

In the mid-'80s, Wall Street turned to the quants—brainy financial engineers—to invent new ways to boost profits. Their methods for minting money worked brilliantly... until one of them devastated the global economy. *Photo: Jim Krantz/Gallery Stock*

An Extreme Example of Product Risk: CDO^2 and Beyond

- In practice CDO's are multi-period securities and can be cash or synthetic CDOs.
- Synthetic CDOs are constructed in a similar way to cash CDOs
  - but now the underlying bond portfolio is not a "real" physical portfolio.
- Now suppose we have 100 different (synthetic) CDO's.
- Can construct a new CDO using mezzanine tranches (for example) of these 100 CDOs
  - and obtain a **CDO-squared**!
- Note many of the same "bonds" act as collateral for many of the underlying CDOs.


CDO # 1
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
CDO-Squared constructed from mezzanine tranches of underlying CDOs
CDO-Squared
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
CDO # 100
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
And many bonds act as collateral for multiple CDOs!!

CDO-Squared's and Risk Management

Question: How would you price and risk-manage a CDO-squared?

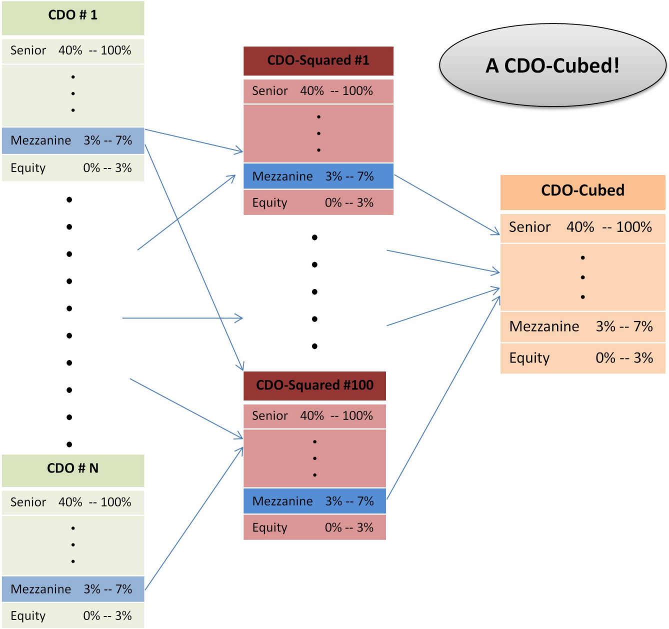
Some considerations:

- Legal contract governing each of the mezzanine tranches in the underlying portfolio of CDOs is ≈ 150 pages long
  - so only need to read $\approx 100 \times 150 = 15,000$ pages of legal documents
- Must also read the contract governing the CDO-squared, of course.
- How would you keep track of the performance of the CDO-squared?
  - many thousands of lines of computer code required!

More Specific Questions

1. How would we perform a scenario analysis?
2. Or estimate the VaR_α or CVaR_α of a CDO-Squared?

But why stop there!? There are also **CDO-cubed's!!**



CDO # 1
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
CDO # N
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
CDO-Squared #1
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
CDO-Squared #100
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%
A CDO-Cubed!
CDO-Cubed
Senior 40% -- 100%
Mezzanine 3% -- 7%
Equity 0% -- 3%

Securitization and CDOs
- Correlation Risk
- Synthetic CDOs and Beyond!

Structural Credit Models and a Little Corporate Finance
- The Merton Structural Model
- The Black-Cox Structural Model
- Some Incentive Problems in Corporate Finance

Structural Models

- Structural approach to credit modeling began with Merton (1974).
- It is based on the fundamental accounting equation:

$$\text{Assets} = \text{Debt} + \text{Equity} \tag{4}$$

  - states that asset-value of a firm equals value of firm's debt + equity
  - assuming no taxes.

(4) follows because all profits generated by a firm's assets will ultimately accrue to the debt- and equity-holders.

Structural Models

- The capital structure of a firm is such that debt-holders are senior to equity holders
  - implies that in event of bankruptcy debt-holders must be paid off in full before equity-holders can receive anything.

- This insight allowed Merton to write time T equity value E_T as a call option on firm value V_T with strike equal to face value of debt D_T.

Merton's model therefore implies

$$E_T = \max\left(0,\ V_T - D_T\right) \tag{5}$$

with default occurring if $V_T < D_T$.

Merton's Structural Model

- Note that (5) implicitly assumes the firm is wound up at time T and that default can only occur at that time
  - not very realistic assumptions
  - they have been relaxed in many directions since Merton's original work
  - nonetheless, can gain many insights from working with (5).

- Can take V_t to be value of a traded asset (why?) so that risk-neutral pricing applies.

- If firm does not pay dividends then could assume $V_t \sim \text{GBM}(r, \sigma)$
  - so $E_t =$ Black-Scholes price of a call option with maturity T, strike D_T and underlying security value V_t.

- Can compute other quantities such as the (risk-neutral) probability of default etc.

A Merton Lattice Model

e.g. Consider following example:

- $V_0 = 1,000$, $T = 7$ years, $\mu = 15\%$, $\sigma = 25\%$ and $r = 5\%$.
- $\#$ of time periods $= 7$.
- Face value of debt is 800 and coupon on the debt is zero.

First task is to construct lattice model for V_t. Can do this following by constructing a lattice:

- $\nu = (\mu - \sigma^2/2)$
- $\ln u = \sqrt{\sigma^2 \Delta t + (\nu \Delta t)^2}$
- $d = 1/u$
- Risk-neutral probability of an up-move is $q = (e^{r\Delta t} - d)/(u - d)$.

A Merton Lattice Model

Firm Price Lattice

							6940.6
						5262.6	3990.2
					3990.2	3025.5	2294.0
				3025.5	2294.0	1739.4	1318.9
			2294.0	1739.4	1318.9	1000.0	758.2*
		1739.4	1318.9	1000.0	758.2	574.9	435.9*
	1318.9	1000.0	758.2	574.9	435.9	330.5	250.6*
1000.0	758.2	574.9	435.9	330.5	250.6	190.0	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Note default only possible at time T.

Merton's Model

Now ready to price equity and debt, i.e. corporate bonds, of the firm.

- Price the equity first by simply viewing it as a regular call option on V_T with strike $K = 800$ and using risk-neutral backward evaluation.
- The bond or debt price can then be computed similarly or by simply observing that it must equal the difference between the firm-value and equity value at each time and state.

Equity Lattice

							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1127.8	687.1	358.4	128.3	0.0	0.0
	758.6	435.7	207.1	63.8	0.0	0.0	0.0
499.7	269.9	117.4	31.7	0.0	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Debt Lattice

							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		611.6	631.7	641.6	630.0	574.9	435.9*
	560.3	564.3	551.1	511.1	435.9	330.5	250.6*
500.3	488.3	457.5	404.2	330.5	250.6	190.0	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Merton's Model

- We see the initial values of the equity and debt are 499.7 and 500.3, respectively.
- The yield-to-maturity y of the bond satisfies $500.3 = e^{-yT} \times 800$ which implies $y = 6.7\%$.
- The credit spread is then given by $c = y - r = 1.7\%$ or 170 basis points.

Merton's Model

Can easily compute the true or risk-neutral probability of default by constructing an appropriate lattice.

Also easy to handle coupons:

- If debt pays a coupon of C per period, then have

$$E_T = \max(0,\ V_T - D_T - C).$$

- And in any earlier period we have

$$E_t = \max\left(0,\ \frac{1}{R}\left[qE^u + (1-q)E^d\right] - C\right)$$

where $R = e^{r\Delta t}$.

The Black-Cox Model

- Black-Cox model generalizes Merton model by allowing default to also occur before time T.

- In fact default occurs first time firm value falls below face value of debt.

- In that case we can compute equity value by placing 0 in those cells where default occurs

  - and updating other cells using usual backwards evaluation approach.

- Debt value at a given cell in the lattice given by difference between the firm and equity values in that cell.

- We obtain the following equity and debt lattices:

Equity Lattice

							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1115.8	660.7	300.1	0.0	0.0	0.0
	703.9	328.5	0.0	0.0	0.0	0.0	0.0
350.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Debt Lattice

							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		623.6	658.2	699.9	758.2*	574.9*	435.9*
	614.9	671.5	758.2*	574.9*	435.9*	330.5*	250.6*
650.0	758.2*	574.9*	435.9*	330.5*	250.6*	190.0*	144.1*
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

The Black-Cox Model

We see the debt-holders have benefitted from this new default regime: their value increased from 500.3 to 650.

Of course this increase has come at the expense of the equity holders whose value has fallen from 499.7 to 350.

In this case the credit spread on the bond is -200 basis points!

- unreasonable value of credit spread is evidence against the realism of default assumption made here.

Negative credit spreads generally not found in practice but have occurred here because the debt holders essentially own a down-and-in call option on the value of the firm with zero strike and barrier equal to the face value of the debt.

While it is true that a firm can default at any time, the barrier would generally be much lower than the face value of the long-term debt of 800.

Note that we could easily use a different and time-dependent default barrier to obtain a more realistic value of the credit spread.

Incentive Problems in Corporate Finance

Recall Merton's structural lattice model of firm default:

- $V_0 = 1,000$, $\mu = 15\%$ and $\sigma = 25\%$.
- $T = 7$ years $\#$ of time periods $= 7$.
- $r = 5\%$
- Face value of debt $= 800$ and coupon $= 0$.

```
Firm Price Lattice
                                                                6940.6
                                                        5262.6  3990.2
                                                3990.2  3025.5  2294.0
                                        3025.5  2294.0  1739.4  1318.9
                                2294.0  1739.4  1318.9  1000.0  758.2*
                        1739.4  1318.9  1000.0  758.2   574.9   435.9*
                1318.9  1000.0  758.2   574.9   435.9   330.5   250.6*
        1000.0  758.2   574.9   435.9   330.5   250.6   190.0   144.1*

        t = 0   t = 1   t = 2   t = 3   t = 4   t = 5   t = 6   t = 7
```

Equity and debt values are 499.7 and 500.3, respectively:

Equity Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
							6140.6
						4501.6	3190.2
					3266.4	2264.5	1494.0
				2336.9	1570.2	978.4	518.9
			1640.8	1054.7	603.6	258.0	0.0
		1127.8	687.1	358.4	128.3	0.0	0.0
	758.6	435.7	207.1	63.8	0.0	0.0	0.0
499.7	269.9	117.4	31.7	0.0	0.0	0.0	0.0

Debt Lattice

t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7
							800.0
						761.0	800.0
					723.9	761.0	800.0
				688.6	723.9	761.0	800.0
			653.2	684.7	715.3	742.0	758.2*
		611.6	631.7	641.6	630.0	574.9	435.9*
	560.3	564.3	551.1	511.1	435.9	330.5	250.6*
500.3	488.3	457.5	404.2	330.5	250.6	190.0	144.1*

Turning Down Good Investments

Suppose the firm is offered a great(!) investment opportunity:

Fair value of investment is $X = 100$ but cost to firm will only be 90.

But firm has no cash available and would therefore have to raise the 90 from current equity holders.

Question: Will the equity holders invest?

Turning Down Good Investments

Can model this situation by first adding X to the initial value of the firm and computing the resulting firm-value lattice:

```
Firm Value Lattice
                                                                7634.7
                                                        5788.8  4389.3
                                                4389.3  3328.1  2523.4
                                        3328.1  2523.4  1913.3  1450.7
                                2523.4  1913.3  1450.7  1100.0   834.1
                        1913.3  1450.7  1100.0   834.1   632.4   479.5*
                1450.7  1100.0  834.1   632.4    479.5   363.6   275.7*
        1100.0  834.1   632.4   479.5   363.6    275.7   209.0   158.5*

        t = 0   t = 1   t = 2   t = 3   t = 4   t = 5   t = 6   t = 7
```

Turning Down Good Investments

Can then compute the equity lattice in the usual manner:

Equity Lattice

							6834.7
						5027.8	3589.3
					3665.4	2567.1	1723.4
				2639.5	1799.6	1152.4	650.7
			1868.4	1224.8	726.9	339.0	34.1
		1296.5	809.4	441.4	176.2	16.9	0.0
	881.1	520.9	261.0	91.5	8.4	0.0	0.0
586.9	327.7	151.3	47.4	4.2	0.0	0.0	0.0
t = 0	t = 1	t = 2	t = 3	t = 4	t = 5	t = 6	t = 7

Note that equity value is now 586.9

- an increase of $586.9 - 499.7 = 87.2$ dollars
- but less than the 90 required to make the investment.

Question: What has happened?

Taking on Bad Investments

Other incentive problems can arise:

- Suppose fair value of investment is again 100 but now it costs 110.
- Clearly this is a bad investment and should not be made.
- But it may be rational for equity-holders to invest if the investment increases the volatility of the firm
  - recall equity-holders own a call option on the value of the firm
  - and the value of an option increases with volatility.

  So possible that increase in equity value due to the increase in volatility will exceed the decrease in equity value due to poor quality of investment
  - then it makes sense for equity holders (who control the firm) to invest.

Question: How might you model this situation using Merton's structural model?