Zhe Liu

**Assignment 1: Linear Algebra**
**Total Points: 100 Due: 11:59pm 29 Sept, 2020**

**Rules:**

1. This is a group assignment. Within each group **I strongly encourage everyone to attempt each question by his/herself first** before discussing it with other members of the group.
2. You are recommended to **type out your solution** and submit a PDF file. If you choose to write it down and submit a scanned version, please ensure clear handwriting.
3. `R` is the default package / programming language for this course so you should use `R` for any programming questions in this assignment.
4. The `R Notebook` *Vector_Matrix_Operations.rmd* is a very useful resource for understanding how to manipulate vectors and matrices in `R`. It also shows you how to compute the rank of a matrix, how to compute the eigenvalues and eigenvectors of a square matrix etc. You should therefore familiarise yourself with this Notebook by working through it before tackling this assignment.

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1. **Linear Independence (15 points)**

   (a) Without doing any calculations explain whether or not the following vectors are linearly independent in $\mathbb{R}^3$ and justify your answer. Please do not use `R` or other software in this question. (5 points)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2.5 \\ 1.5 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 8 \\ 7.2 \end{pmatrix}.$$

   (b) Do the following vectors from a basis for $\mathbb{R}^3$? Justify your answer. You can use `R` in this question. (5 points)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

   (c) Let $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that $c$ is **not** equal to what value? (5 points)

2. **Matrix Rank (15 points)**

   Consider the $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 4 \\ 1 & 0 & 0 & 5 & 2 & 3 \\ 1 & 0 & 0 & 5 & 2 & 4 \\ 1 & 0 & 6 & 5 & 2 & 7 \end{pmatrix}$$

   where $m = 4$ and $n = 6$.

   (a) Without any calculations, what is the maximal possible rank of $\mathbf{A}$? (5 points)

   (b) What actually is the rank of $\mathbf{A}$? (It is easy to compute this in `R`.) (5 points)

   (c) What is the dimension of the null space of $\mathbf{A}$? (5 points)

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3. **Matrix Rank (15 points)**

   Consider a general $m \times n$ matrix $\mathbf{A}$ where $\text{rank}(\mathbf{A}) = m < n$. Consider the $m \times m$ matrix $\mathbf{B} := \mathbf{A}\mathbf{A}^\top$. Prove that $\text{rank}(\mathbf{B}) = m$. (**Hint:** Suppose $\mathbf{B}\mathbf{x} = \mathbf{0}$, then show this implies $\mathbf{x} = \mathbf{0}$ so that columns of $\mathbf{B}$ are linearly independent. To do this, consider $\mathbf{x}^\top\mathbf{B}\mathbf{x}$ ...)

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4. **Ranges and Null Spaces (15 points)**

   Recall that the range of $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the set of vectors in $\mathbb{R}^m$ that can be obtained as a linear combination of the columns of $\mathbf{A}$. The range is denoted by $\mathcal{R}(\mathbf{A})$ and we have

$$\mathcal{R}(\mathbf{A}) := \{\mathbf{A}\mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^n\}.$$

   Similarly, recall the null space of $\mathbf{A}$ is defined according to $\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{A}\mathbf{x} = \mathbf{0}\}$.

   (a) Show that every $\mathbf{x} \in \mathcal{R}(\mathbf{A}^\top)$ is orthogonal to every $\mathbf{z} \in \mathcal{N}(\mathbf{A})$, i.e. $\mathcal{R}(\mathbf{A}^\top) \perp \mathcal{N}(\mathbf{A})$. (**Hint:** If $\mathbf{x} \in \mathcal{R}(\mathbf{A}^\top)$ then $\mathbf{x} = \mathbf{A}^\top\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^m$.) (10 points)

   (b) Show that every $\mathbf{x} \in \mathcal{R}(\mathbf{A})$ is orthogonal to every $\mathbf{z} \in \mathcal{N}(\mathbf{A}^\top)$. (5 points)

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5. **Fibonacci Numbers (20 points)**

The Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

is such that every number is the sum of its two predecessors, i.e. for $k = 1, 2, \ldots$ we have

$$F_{k+2} = F_{k+1} + F_k, \tag{1}$$

where $F_1 = 0, F_2 = 1, \ldots$

(a) Write (1) in the form $\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k$ where $\mathbf{u}_k, \mathbf{u}_{k+1} \in \mathbb{R}^2$. (**Hint:** Use $\mathbf{u}$ to encapsulate the $F$'s.) (5 points)

(b) Write some `R` code to compute $F_k$ for $k = 1000$ in three different ways: (i) directly by iterating (1), (ii) directly by iterating $\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k$, and (iii) by diagonalizing $\mathbf{A}$ and using this to calculate $\mathbf{u}_k = \mathbf{A}^{k-1}\mathbf{u}_1$. (15 points)

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6. **Answer the following questions. (20 points)**

(a) Let $\mathbf{A}$ be the $m \times n$ matrix

$$\begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

and let $\mathbf{x} = (x_1 \ldots x_n)^\top$. Give an expression for $\mathbf{Ax}$ in terms of the column vectors of $\mathbf{A}$, i.e. the $\mathbf{a}_i$'s. (5 points)

(b) Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\text{rank}(\mathbf{A}) = n$. Is there always a solution to the equation $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$? Justify your answer. (5 points)

(c) Suppose $(\lambda, \mathbf{u})$ is an eigenvalue / eigenvector pair for $\mathbf{A} \in \mathbb{R}^{n \times n}$. Can you find a corresponding eigenvalue / eigenvector pair for $\mathbf{A}^2$ ? Justify your answer. (5 points)

(d) Find all eigenvalues and corresponding eigenvectors for matrix $\mathbf{A}$. You can use `R` in this question. (5 points)

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$