

MOCK Final Exam

Exam Duration: 2 Hours
Total Marks Available: 100

Instructions

1. This is a closed-book exam.
2. Keep your answers succinct and to the point. Long rambling answers with irrelevant details will work against you.
3. You may use a calculator or `Excel` for any calculations that you need to do.
4. Please read the entire exam carefully before starting on your answers!

Question 1. (25 marks)

(a) Let $C_0 = 3$ denote the time $t = 0$ price of a European call option on a non-dividend-paying stock with strike $K = 110$ and maturity $T = 1$ year. The current value of the stock price is $S_0 = 100$ and the discount factor is $d(0,T) = .99$. Can you compute the price P_0 of the European put option with the same strike and maturity? If so, what is the price? If not, why can't you compute it? (6 marks)

Solution: The price can be computed via put-call parity which yields

$$\begin{aligned} P_0 &= d(0,T)K + C_0 - S_0 \\ &= .99 \times 110 + 3 - 100 \\ &= 11.9. \end{aligned}$$

(b) Consider the pivot-table in Figure 1 at the end of this exam. It displays the P&L of an options and futures portfolio for various combinations of stresses to the underlying security and implied volatilities. You are told that the total dollar delta, i.e. ESP, and total dollar gamma for this portfolio are $-17,485$ and $-458,145$, respectively. What does a delta-gamma approximation suggest the P&L will be if the underlying increases by 10%? Is this consistent with the pivot-table? Explain your answer. (7 marks)

Solution: The delta-gamma approximation suggests

$$\begin{aligned} \text{P\&L} &\approx \text{ESP} \times \text{Return} + \$\ \text{Gamma} \times \text{Return}^2 \\ &= -17,485 \times (.1) + (-458,145) \times (.1)^2 \\ &= -1,749 - 4,581 \\ &= -6,330 \end{aligned}$$

The corresponding number in the pivot-table is $-6,128$ and so the numbers are quite close to and therefore consistent with one another.

(c) Suppose the CAPM holds and the expected return on a particular risky asset is significantly less than the risk-free rate of interest. True or False: such an asset should rarely be held in an agents optimal portfolio? Give a reason for your answer. (6 marks)

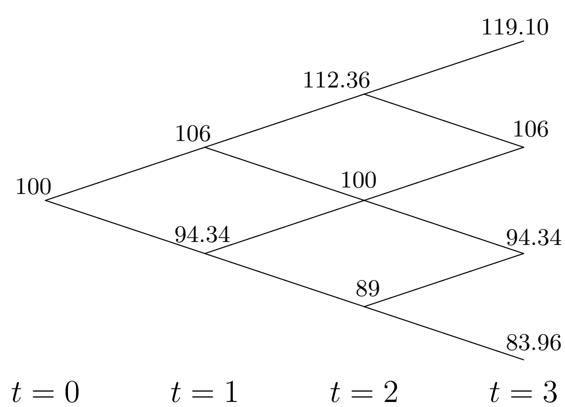
Solution: False. The asset is part of the market portfolio and everybody should therefore hold it in their optimal portfolio if the CAPM holds.

(d) You're a market-maker and have just sold a put option on some underlying stock. You decide to delta-hedge your risk and so you put on a short position in the stock immediately after selling the option. True or false: if the underlying stock rises then will need to buy some stock. Explain your answer. (6 marks)

Solution: This is true. The delta of the put option will increase as the stock price rises. (In absolute value it will decrease.) Therefore the size of the short position in the delta-hedge will also decrease. This is achieved by buying some of the stock to reduce your short position.

Question 2. (25 marks)

Consider the binomial lattice below with $S_0 = 100$, $u = 1.06$ and $d = 1/u$. It describes the evolution of a non-dividend paying stock in a 3-period world. You may also assume there is a cash account which pays a total return of R = 1.01 per period and that borrowing or lending at that rate is possible. In addition short-sales are allowed.



(a) Compute the price of an American put option on the stock with strike = 98 and expiration date $t = 3$. (10 marks)

<u>**Solution:**</u> The risk-neutral probability q is given by $q = (R - d)/(u - d) = (1.01 - .9434)/(1.06 - .9434) = 0.5712$ and the value of the American put option at $t = 0$ turns out to be 2.37.

(b) Is it ever optimal to early exercise the put option from part (a). If so, when and does this violate put-call parity? (5 marks)

Solution: It is optimal to early-exercise the option at $t = 2$ when the stock price is 89. This does not violate put-call parity since the latter only applies to European options.

(c) A **chooser option** gives the owner the right to choose at time $t = 1$ either a European call option or a European put option. The call and put options in question both have the same strike K and expiration at $T = 2$. Can you find an expression for the time $t = 0$ value of the chooser option for general values of K that can be written as the sum of the original call option and a put option with a different expiration and a different strike. (Put-call parity helps!) (10 marks)

Solution: Let V_t denote the value of the chooser option at time t. We then have

$$\begin{aligned} V_1 &= \max(C_1, P_1) \\ &= \max(C_1,\, C_1 + K/R - S_1) \quad (1) \\ &= C_1 + \max(0,\, K/R - S_1) \quad (2) \end{aligned}$$

where (1) follows from put-call-parity. We can now apply 1-period risk-neutral pricing to (2), i.e. divide across (2) by R and take risk-neutral expectations, to obtain

$$V_0 = C_0 + \mathrm{E}_0^Q \left[\frac{1}{R} \max(0, K/R - S_1) \right]. \quad (3)$$

It follows from (3) that the time $t = 0$ value of the chooser option is the value of the original call option and a put option that expires at $t = 1$ with strike $= K/R$.

Question 3. (10 marks)

A forward contract on a security is a contract agreed upon at date $t = 0$ to purchase or sell the security at date T for a price F that is specified at $t = 0$. The forward price F is set in such a way that the initial *value* f_0 of the forward contract satisfies $f_0 = 0$. At the maturity date T the value of the contract is then given by $f_T = \pm(S_T - F)$ where S_T is the time T value of the underlying security. (The time T payoff is $+(S_T - F)$ if you went long the forward at $t = 0$ and $-(S_T - F)$ if you went short the forward.)

Assuming an arbitrage-free binomial model for the dynamics of S_t, compute an expression for F. (You can assume that T corresponds to exactly n periods in the binomial model.)

Solution: We know how to price securities within the binomial model. Since the initial value of the contract is 0 and the final value is $f_T = \pm(S_T - F)$ we have

$$0 = \mathrm{E}_0^Q \left[\frac{1}{R^n} \pm (S_T - F) \right]$$

from which it follows (since R and F are constants) that

$$\begin{aligned}F &= \mathrm{E}_0^Q\left[S_T\right] \\ &= R^n\,\mathrm{E}_0^Q\left[\frac{1}{R^n}S_T\right] \\ &= R^n\,S_0.\end{aligned}$$

And so we see that the forward price F is equal to the futures price we obtained in the lecture notes / slides. (This is only true in the binomial model and is not true in general. Specifically it doesn't hold when the interest rate R is stochastic.)

Question 4. (20 marks)

In the Black-Scholes model the time T payoff of an Asian call option is given by

$$h(\mathbf{X}) := \max\left(0,\ \frac{\sum_{i=1}^m S_{iT/m}}{m} - K\right)$$

where $\mathbf{X} = (S_{T/m}, S_{2T/m}, \ldots, S_T)$ and where S_t denotes the time t price of the underlying security. In words, the Asian call option is a European call option on the *average* price of the stock with the average taken over the stock price at the m pre-specified times iT/m for $i = 1, \ldots, m$. The risk-neutral stock-price is assumed to satisfy $S_t \sim \mathrm{GBM}(r, \sigma)$ and the arbitrage-free value of the option is then given by $\theta := \mathrm{E}[e^{-rT}h(\mathbf{X})]$.

(a) Provide pseudo-code for a Monte-Carlo algorithm that estimates C_0 using n Monte-Carlo samples. Your pseudo-code should also calculate an approximate 95% confidence interval. (15 marks)

<u>**Solution:**</u> Since $S_t \sim \mathrm{GBM}(r, \sigma)$ it follows that

$$S_{i\Delta} = S_{(i-1)\Delta}\, e^{\left(r-\frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}\,Z_i} \tag{4}$$

for $i = 1, \ldots, m$ and where $\Delta = T/m$ and the Z_i's are IID $\mathrm{N}(0, 1)$. Based on (4) it should be clear that the following pseudo-code is correct:

Estimating C_0 Using n Samples

> **set** $\Delta = T/m$
> **for** $j = 1 : n$
>
> **for** $i = 1 : m$
>
> **generate** $Z_i \sim \mathrm{N}(0,1)$
> **set** $S_i = S_{i-1} \exp\left((r - \sigma^2/2)\,\Delta + \sigma\sqrt{\Delta}\,Z_i\right)$
>
> **set** $A = (\sum_{i=1}^m S_i)/m$
> **set** $C_j = e^{-rT} \max(A - K,\, 0)$
>
> **set** $\widehat{C}_0 = \frac{\sum_{j=1}^n C_j}{n}$
>
> **set** $\widehat{\sigma}^2 = \frac{\sum_{j=1}^n (C_j - \widehat{C}_0)^2}{n-1}$
>
> **set** Approx 95% CI $= \left[\widehat{C}_0 - 1.96\frac{\widehat{\sigma}}{\sqrt{n}},\quad \widehat{C}_0 + 1.96\frac{\widehat{\sigma}}{\sqrt{n}}\right]$

(b) What if anything can you say about the delta, gamma and vega of the Asian option? Will they be positive or negative? You don't have to prove anything here but you should give a 1 or 2 line answer to provide some intuition for your answers. (5 marks)

Solution: They will be positive and the reason is the same for regular European call options which are long delta, gamma and vega.

Question 5. (20 marks)

(a) A common compromise on long-only portfolios in the mean-variance setting are the so-called (100+L)/L portfolios. Such a portfolio allows the long positions to be worth up to (100+L)% of the portfolio value and the short positions to be worth at most L% of the portfolio. Note that the full investment constraint $\mathbf{w}^\top \mathbf{1} = 1$ is always imposed. Suppose then that we wish to allow such portfolios. This results in the "short" constraint

$$\sum_{j=1}^{n} \min(w_j, 0) \geq -L \;\Leftrightarrow\; \sum_{j=1}^{n} \max(-w_j, 0) \leq L. \tag{5}$$

Note that there is no need to include the "long" constraint since (5) together with the constraint $\mathbf{w}^\top \mathbf{1} = 1$ will ensure it is satisfied. The problem with (5) is that it is not linear and so if we add it to our mean-variance problem formulation we will no longer have a convex quadratic program.

Show how we can *linearize* (5), i.e. replace it with linear constraints and justify your answer. (10 marks)

Solution: Let $\mathbf{y} = [y_1 \cdots y_n]^\top$. The idea is that each y_j "represents" $\max(-w_j, 0)$. With this representation in mind we can replace (5) with

$$\begin{aligned} \mathbf{y} &\geq -\mathbf{w} && (6) \\ \mathbf{y} &\geq 0 && (7) \\ \mathbf{y}^\top \mathbf{1} &\leq L. && (8) \end{aligned}$$

In order to justify this new formulation, consider the mean-variance optimization problem with constraints (6) to (8) included. Let $\mathbf{w}^*$ and $\mathbf{y}^*$ be the optimal values. Suppose now that there is some index j for which $y_j^* \neq \max(-w_j^*, 0)$. Then it should be clear (after a little thought) that we can simply replace y_j^* with $\max(-w_j^*, 0)$ without altering the objective or the feasibility of any of the constraints. Hence $\mathbf{w}^*$ is the optimal solution to the mean-variance problem with the leverage constraint (5).

(b) The mean-variance optimization problem can be formulated as

$$\begin{aligned} \min_{w_1,\ldots,w_n} \quad & \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij} w_i w_j \\ \text{subject to} \quad & \sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} && (9) \\ & \sum_{i=1}^{n} w_i = 1 && (10) \end{aligned}$$

where $\sigma_{ij} = \mathrm{Cov}(r_i, r_j)$ and $\bar{r}$ is the target mean return of the portfolio. It can be shown that a solution to this problem is given by the solution $\{\mathbf{w} = (w_1, \ldots, w_n), \lambda, \mu\}$, to the n linear equations

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0, \quad \text{for } i = 1, \ldots, n \qquad (11)$$

together with (9) and (10). Suppose we have two known solutions $\{\mathbf{w}^1, \lambda^1, \mu^1\}$ and $\{\mathbf{w}^2, \lambda^2, \mu^2\}$ corresponding to different target returns $\bar{r}^1$ and $\bar{r}^2$, respectively. Is it possible to determine the solution for an arbitrary target return $\bar{r}^3$ using these solutions? Please justify your response and briefly discuss any possible implications for investing in the real world (assuming the mean-variance framework is true). (10 marks)

Solution: The answer is yes. To see this note that there exists some α for which

$$\bar{r}^3 = \alpha \bar{r}^1 + (1 - \alpha)\bar{r}^2.$$

(It's ok if you used $\bar{r}^3 = \alpha\bar{r}^1 + \beta\bar{r}^2$ and didn't recognize we could take $\beta = 1 - \alpha$.) You can now confirm that

$$\{\mathbf{w}^3, \lambda^3, \mu^3\} := \alpha\{\mathbf{w}^1, \lambda^1, \mu^1\} + (1 - \alpha)\{\mathbf{w}^2, \lambda^2, \mu^2\}$$

is a solution to (9) to (11) corresponding to the target return $\bar{r}^3$. This means that the portfolio $\alpha\mathbf{w}^1 + (1 - \alpha)\mathbf{w}^2$ is the optimal portfolio for the problem with target return $\bar{r}^3$. The implication for investing in the real world is that it suggests that everyone can achieve their optimal portfolio of risky assets by investing in just two risky funds. (Observations like this are credited with helping give rise to the mutual fund industry.)

Underlying	SX5E Index								
	Underlying and Volatility Stress Table								
Sum of PnL	Volatility Stress								
Underlying Stress	-10	-5	-2	-1	0	1	2	5	10
-20	7,234	(3,488)	(10,770)	(13,305)	(15,885)	(18,508)	(21,168)	(29,351)	(43,504)
-10	29,406	13,520	3,663	341	(2,995)	(6,345)	(9,706)	(19,845)	(36,862)
-5	35,537	17,725	6,928	3,319	(293)	(3,908)	(7,525)	(18,380)	(36,458)
-2	37,552	18,874	7,647	3,905	165	(3,574)	(7,310)	(18,505)	(37,095)
-1	37,948	19,032	7,684	3,905	129	(3,645)	(7,416)	(18,709)	(37,450)
0	38,207	19,079	7,623	3,810	0	(3,806)	(7,609)	(18,992)	(37,873)
1	38,330	19,017	7,463	3,619	(220)	(4,056)	(7,888)	(19,354)	(38,365)
2	38,318	18,846	7,207	3,335	(531)	(4,394)	(8,251)	(19,794)	(38,925)
5	37,495	17,698	5,868	1,934	(1,995)	(5,920)	(9,839)	(21,566)	(41,000)
10	33,611	13,757	1,824	(2,153)	(6,128)	(10,102)	(14,073)	(25,969)	(45,722)
20	17,546	(923)	(12,382)	(16,244)	(20,124)	(24,018)	(27,925)	(39,709)	(59,483)

Figure 1: P&L for an options portfolio on the Eurostoxx 50 index under stresses to the underlying and implied volatility