Exponential smoothing (with no trend or seasonality)

$f_{n,h} = \hat{L}_n$ for any $h > 0$

1. Initialization: $\hat{L}_0 = \frac{1}{n} \sum_{i=1}^{n} X_i$, $\quad f_{0,1} = \hat{L}_0$,

2. Period 1: $E_1 = X_1 - \hat{L}_0$,

$$\hat{L}_1 = \alpha^* X_1 + (1 - \alpha^*) \hat{L}_0, \quad 0 < \alpha < 1,$$

$$f_{1,1} = \hat{L}_1$$

Period 2: $E_2 = X_2 - \hat{L}_1$,

$$\hat{L}_2 = \alpha^* X_2 + (1 - \alpha^*) \hat{L}_1$$

$\vdots$

Period n: $E_n = X_n - \hat{L}_{n-1}$

$$\hat{L}_n = \alpha X_n + (1 - \alpha) \hat{L}_{n-1}$$

$$\min_{\alpha} \frac{1}{n} \sum_{i=1}^{n} E_i^2 \quad \Rightarrow \quad \boxed{\alpha^*}$$

s.t. $0 < \alpha < 1$

$$\hat{L}_{t+1} = \alpha X_{t+1} + (1-\alpha) \cdot \hat{L}_t$$

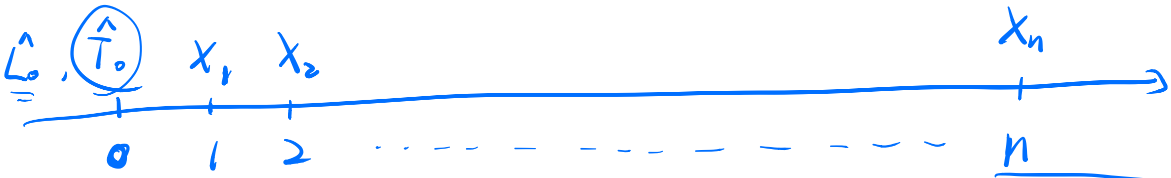
$$= \alpha X_{t+1} + (1-\alpha) \cdot [\alpha X_t + (1-\alpha) \hat{L}_{t-1}]$$

$$= \alpha X_{t+1} + (1-\alpha) \alpha X_t + (1-\alpha)^2 \cdot [\alpha X_{t-1} + (1-\alpha) \hat{L}_{t-2}]$$

$$= \alpha X_{t+1} + (1-\alpha)\alpha X_t + (1-\alpha)^2 \alpha X_{t-1} + (1-\alpha)^3 \hat{L}_{t-2}$$

$$(1-\alpha)^n \alpha$$

Holt's Model (with trend, but no seasonality).



1. Initialization: Regress X_t on t.

$$X_t = at + b + u_t$$

$$\hat{L}_0 = \hat{b}, \quad \hat{T}_0 = \hat{a}, \quad f_{0,1} = \hat{L}_0 + \hat{T}_0$$

$$\boxed{\begin{array}{l} \hat{L}_n, \hat{T}_n \\ f_{n,h} = \hat{L}_n + h \cdot \hat{T}_n \end{array}}$$

2. Period 1: $E_1 = X_1 - (\hat{L}_0 + \hat{T}_0)$

$$\begin{cases} \hat{L}_1 = \alpha X_1 + (1-\alpha)(\hat{L}_0 + \hat{T}_0), & 0 < \alpha < 1 \\ \hat{T}_1 = \beta(\hat{L}_1 - \hat{L}_0) + (1-\beta) \cdot \hat{T}_0, & 0 < \beta < 1 \end{cases}$$

$$f_{1,1} = \hat{L}_1 + \hat{T}_1$$

Period 2: $E_2 = X_2 - (\hat{L}_1 + \hat{T}_1)$

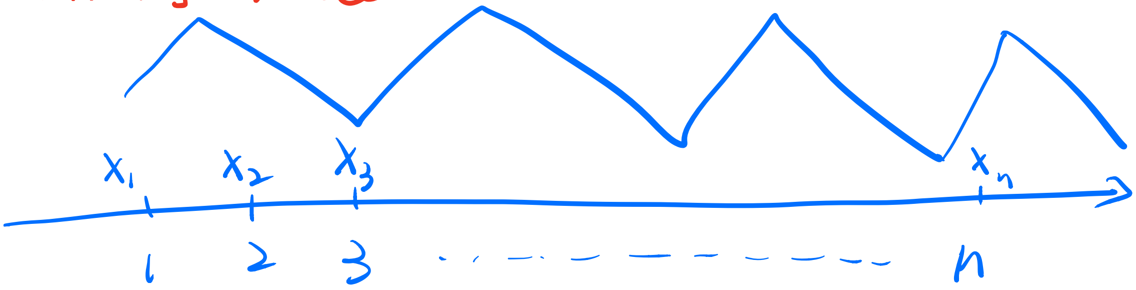
$\vdots$

$$\min_{\alpha, \beta} \frac{1}{n} \sum_{i=1}^{n} E_i^2 \quad \Rightarrow \quad \boxed{\alpha^*, \beta^*}$$

s.t. $0 < \alpha < 1$

$0 < \beta < 1$

Winter's model (with both trend and seasonality)



$\boxed{p=3}$

no trend

$x_1 \approx L_1 \cdot S_1$, $x_2 \approx L_2 S_2$, $x_3 \approx L_3 \cdot S_3$, ...

$\bar{S}_1$ $\quad S_1 = S_4 = S_7 = \cdots$, $\quad S_2 = S_5 = S_8 = \cdots$,

$S_3 = S_6 = S_9 = \cdots$,

Initialization:

$p=3$.

① deseasonalize time series.

$$\bar{X}_t = (X_{t-1} + X_t + X_{t+1}) / 3$$

$$\approx (L_{t-1} \cdot S_{t-1} + L_t \cdot S_t + L_{t+1} \cdot S_{t+1}) / 3$$

$$\approx L_t \cdot \left(\boxed{S_{t-1} + S_t + S_{t+1}}\right) / 3 = L_t \cdot \boxed{\bar{S}}$$

without loss of generality $\boxed{\bar{S} = 1}$.

$\tilde{L}_t = L_t \cdot \bar{S}$

② Regress $\bar{X}_t$ on t, obtain $\hat{L}_0$ (and $\hat{T}_0$)

③ $\bar{S}_t = X_t / \bar{X}_t \approx L_t \cdot S_t / L_t \approx S_t$

④ $\hat{S}_1$ is obtained by calculating the average of $(\bar{S}_1, \bar{S}_4, \bar{S}_7, \cdots)$

$\hat{S}_2$ .... $(\bar{S}_2, \bar{S}_5, \bar{S}_8, \cdots)$

$\hat{S}_3$ .... $(\bar{S}_3, \bar{S}_6, \bar{S}_9, \cdots)$

Assume we have $\hat{L}_0, \hat{T}_0, \hat{S}_1, \hat{S}_2, \hat{S}_3$ $\boxed{p=3}$

At period 0: $f_{0,1} = (\hat{L}_0 + \hat{T}_0) \cdot \hat{S}_1$.

At period 1: $E_1 = X_1 - f_{0,1}$.

$$\hat{L}_1 = \alpha \cdot X_1/\hat{S}_1 + (1-\alpha) \cdot (\hat{L}_0 + \hat{T}_0)$$

$$\hat{T}_1 = \beta (\hat{L}_1 - \hat{L}_0) + (1-\beta) \cdot \hat{T}_0$$

$\hat{S}_1 = \hat{S}_{1+p}$

$$\hat{S}_1 = \hat{S}_4 = \gamma \cdot X_1/\hat{L}_1 + (1-\gamma) \cdot \hat{S}_1$$

$$f_{1,1} = (\hat{L}_1 + \hat{T}_1) \cdot \hat{S}_2$$

At period 2: $\hat{E}_2 = X_2 - f_{1,1}$,

$\vdots$

$$\min_{\alpha, \beta, \gamma} \frac{1}{n} \sum_{i=1}^{n} E_i^2 \Rightarrow \alpha^*, \beta^*, \gamma^*$$

s.t. $0 < \alpha < 1$

$0 < \beta < 1$

$0 < \gamma < 1$.

at period n, $\hat{L}_n, \hat{T}_n, \hat{S}_n, \hat{S}_{n+1}, \hat{S}_{n+2}$

$$f_{n,h} = (\hat{L}_n + h \cdot \hat{T}_n) \cdot \hat{S}_{n+h}.$$