Assignment 2

Due: 11.59pm Monday 17^{th} May 2021

Rules

1. This is a group assignment. (There are approximately 3 people per group and by now you should know your assigned group.)
2. You are free to use `R`, `Python` or `Excel` for the this assignment. (In fact `Excel` is particularly suited to binomial lattices and presents an easy way to visualise price dynamics in these models.)
3. Within each group **I strongly encourage each person to attempt each question by his / herself first** before discussing it with other members of the group.
4. Students should **not** consult students in other groups when working on their assignments.
5. Late assignments will **not** be accepted and all assignments must be submitted through the Hub with one assignment submission per group. Your submission should include a PDF report with your answers to each question together with screenshots of any relevant code. Make sure your PDF clearly identifies each member of the group by CID and name.

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1. **Option Pricing in the Binomial Model (25 marks)**
   Build a 15-period binomial model with the following parameters: $u = 1.0395$, $d = 1/u$, $S_0 = 100$ and the gross risk-free rate per period is $R = 1.000333$. (This risk-free rate per period corresponds to an annualized continuously-compounded risk-free rate of 2% and a period length of .01667 years. The total maturity is then $15 \times .01667 = 0.25$ corresponding to an option maturity of 3 months.)

   Now answer the following questions:

   (a) Compute the price of an American call option with strike $K = 110$ and maturity $T = .25$ years. **(5 marks)**

   Solution: 2.67

   (b) Compute the price of an American put option with strike $K = 110$ and maturity $T = .25$ years. **(5 marks)**

   Solution: 12:25

   (c) Is it ever optimal to early exercise the put option of part (b)? **(5 marks)**

   Solution: Yes

(d) If your answer to part (c) is "Yes", when is the earliest period at which it might be optimal to early exercise? **(5 marks)**

Solution: When $t = 4$

(e) Do the call and put option prices of parts (a) and (b) satisfy put-call parity? Why or why not? **(5 marks)**

Solution: No. American options do not satisfy put-call parity.

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2. **Pricing Futures Contracts (25 marks)**

Consider the binomial model where S_k denotes the time k price of a non-dividend-paying security. Let F_k denote the time k price of a futures contract written on the underlying security and assume that the contract expires after n periods. Then we know that $F_n = S_n$, i.e., at expiration the futures price and the security price must coincide. Show that the fair price of the futures contract is $F_0 = \mathrm{E}_0^Q[S_n] = R^n S_0$. *Hint: Show that* $F_k = \mathrm{E}_k^Q[F_{k+1}]$ *for any* $0 \leq k < n$.

Solution: We can compute the futures price at $t = n - 1$ by recalling that anytime we enter a futures contract, the initial value of the contract is 0. Risk-neutral pricing then implies

$$0 = \frac{1}{R}\mathrm{E}_{n-1}^Q[F_n - F_{n-1}]$$

which implies $0 = \mathrm{E}_{n-1}^Q[S_n - F_{n-1}]$. But F_{n-1} is known at time $n-1$ and so we can take F_{n-1} outside the expectation to obtain

$$F_{n-1} = \mathrm{E}_{n-1}^Q[F_n].$$

By the same argument, we also have more generally that $F_k = \mathrm{E}_k^Q[F_{k+1}]$ for $0 \leq k < n$. We therefore have

$$\begin{aligned}
F_0 &= \mathrm{E}_0^Q[F_1] \\
&= \mathrm{E}_0^Q\left[\mathrm{E}_1^Q[F_2]\right] \\
&= \mathrm{E}_0^Q\left[\mathrm{E}_1^Q\left[\mathrm{E}_2^Q[F_3]\right]\right] \\
&\vdots \qquad \vdots \\
&= \mathrm{E}_0^Q\left[\mathrm{E}_1^Q\left[\mathrm{E}_2^Q\left[\cdots \mathrm{E}_{n-1}^Q[F_n]\right]\right]\right] \\
&= \mathrm{E}_0^Q[F_n] \qquad (1)
\end{aligned}$$

where (1) follows from the tower property of conditional expectations. Since $F_n = S_n$ we also obtain

$$\begin{aligned}
F_0 &= \mathrm{E}_0^Q[S_n] \qquad (2) \\
&= R^n \mathrm{E}_0^Q\left[\frac{S_n}{R^n}\right] \\
&= R^n S_0 \qquad (3)
\end{aligned}$$

where (3) follows applying risk-neutral pricing to the underlying security itself. More generally it follows that $F_t = R^{n-t} S_t$ in the binomial model.

Remark: The expression in (2) is a very general expression that holds in essentially all arbitrage-free models and not just the binomial model. That said this expression needs to be adjusted for securities that pay dividends or require costly storage, e.g. oil!

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3. **Convergence of Binomial Model Option Prices to Black-Scholes Prices (30 marks)**
   (a) Write a function to compute the prices of European call options in the Black-Scholes framework. Your code should take as inputs all of the parameters that are required by the Black-Scholes formula. Run your code to compute the price of a call option with the following parameters: $S_0 = 100$, $\sigma = 30\%$, $r = 2\%$, dividend-yield $c = 0$, maturity $T = 1$ year and strike $K = 100$. **(10 marks)**

   Solution: The Black-Scholes call option price is 12.82.

   (b) Write a similar function to price the same option but now in the binomial model framework with $n = 10$ time periods. You should use the calibration outlined in Section 3.1 of the *An Introduction to Derivatives Pricing* lecture notes. What option price do you get? **(10 marks)**

   Solution: The binomial option price is ≈ 12.53.

   (c) What happens to the option price in part (b) as you let n get very large? In particular, does it converge to the Black-Scholes price you reported in part (a)? Produce a graph displaying the binomial option price as a function of n for $n = 10, 25, 50, 100, 500$, and $1,000$. On your graph you should also display (as a horizontal line) the Black-Scholes option price. **(10 marks)**

   Solution: The binomial option price should converge to the Black-Scholes option price!

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4. **Dynamic Hedging in the Black-Scholes Model (25 marks)**
   Write a Monte-Carlo simulation code to replicate Figures 8(a) and 8(b) of the *An Introduction to Derivatives Pricing* lecture notes. Be sure you understand why the histograms look as they do! What sort of histogram do you get if the true volatility is equal to the implied volatility?

   Remark: You can adapt your function from Exercise 3 so that it returns both the Black-Scholes option price and delta. You should also *vectorize* this function so that a vector of underlying prices can be input and vectors of option prices and deltas are returned. Doing this will enable you to vectorize your Monte-Carlo, i.e. simulate many paths at once rather than using a for-loop (which is generally much slower) to simulate one path at a time. This is generally much more efficient and means you would only need a single for-loop to iterate through time.

<u>*Solution:*</u> See the `R Notebook` *Delta-Hedging-Black-Scholes.Rmd*. If you take the true volatility equal to the implied volatility then your histogram should be centered around zero.
