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Practice Problems for *Linear Algebra Review*

Within Exam Scope

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1. **Subspaces and Spans**

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ are vectors in $\mathbb{R}^3$. Do you agree with the statement that these vectors span $\mathbb{R}^3$?

2. **Orthogonality**

   Which pairs are orthogonal among the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

3. **Linear Independence**

   If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are independent vectors, would the following vectors be independent?

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3.$$

4. **Range and Rank**

   For $\mathbf{A} \in \mathbb{R}^{n \times n}$, if $\text{rank}(\mathbf{A}) = m < n$, then what is the dimension of the null space?

5. **Matrix Inverse**

   We have matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Which of the following are true?

   (a) $\mathbf{A}$ is singular

   (b) $\mathbf{B}$ is invertible

   (c) $\mathbf{A} + \mathbf{B}$ is invertible.

6. **Subspaces**

   The smallest subspace of $\mathbb{R}^3$ containing the vectors $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ is the line whose equations are $x = a$ and $z = by$. What are the values of a and b?

7. **Matrix Inverse**

   For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, assume $\mathbf{A}$ and $\mathbf{B}$ are invertible:

   (a) (True/False) $\left(\mathbf{A}^{-1}\right)^\top = \left(\mathbf{A}^\top\right)^{-1}$,
   (b) (True/False) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$,
   (c) (True/False) If $\det(\mathbf{A}) = 2$, then $\det(\mathbf{A}^{-1}) = 2^{-1}$.
   (d) (True/False) If $\det(\mathbf{A}) = 2$ and $\alpha > 1$, then $\det(\alpha\mathbf{A}) = 2$.

8. **Linear Independence**

   Let $\mathbf{u} = \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix}$. What are possible values of λ that make $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly dependent?

9. **Matrix Product**

   For $\mathbf{A} = \begin{bmatrix} 1 & 1/3 \\ x & y \end{bmatrix}$, find the value of x and y such that $\mathbf{A}^2 = 0$.

10. **Inner Product**

    Consider the space of all matrices in $\mathbb{R}^{2\times 2}$. Define inner product as $\langle \mathbf{A}, \mathbf{B}\rangle = \mathrm{tr}(\mathbf{A}^\top\mathbf{B})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$. Let $\mathbf{U} = \begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} x^2 & x-1 \\ x+1 & -1 \end{bmatrix}$. Find all values of x such that $\mathbf{U} \perp \mathbf{V}$.

11. **Matrix Inverse**

    If a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^3 - 4\mathbf{A}^2 + 3\mathbf{A} - 2\mathbf{I}_n = \mathbf{0}$, what is $\mathbf{A}^{-1}$?

12. **Basis**

    Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix}$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that c is **not** equal to what value?

13. **Subspaces**

    Consider the set of points $(x, y, z) \in \mathbb{R}^3$. Which one of the following is a subspace of $\mathbb{R}^3$?

    (a) $x + 3y - 2z = 3$.
    (b) $x + y + z = 0$ and $x - y - z = 2$.
    (c) $\frac{x+1}{2} = \frac{y-2}{4} = \frac{z}{3}$.
    (d) $x^2 + y^2 = z$.
    (e) $x = -z$ and $x = z$.
    (f) $\frac{x}{3} = \frac{y+1}{2}$.

Beyond Exam Scope

1. **Solutions to Linear Equations**

   Consider the linear system of equations

   $$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{1}$$

   where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are known and $\mathbf{x} \in \mathbb{R}^n$ is unknown. Let $\mathbf{b} \in \mathcal{R}(\mathbf{A})$ so that (1) has at least one solution. Let $\mathbf{x}_1$ be one such solution. Show that *all* solutions to (1) can be written in the form $\mathbf{x}_1 + \mathbf{n}$ for some $\mathbf{n} \in \mathcal{N}(\mathbf{A})$ where $\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{A}\mathbf{x} = \mathbf{0}\}$ is the *nullspace* of $\mathbf{A}$. (You can check that $\mathcal{N}(\mathbf{A})$ is indeed a subspace of $\mathbb{R}^n$.)

2. **Adjacency Matrices I (Challenging - Not Examinable!)**

   If $\mathbf{A}$ is an adjacency matrix show that $\mathbf{A}^k_{ij} = \#$ paths from $i \to j$ in exactly k steps. *Hint:* Use induction. (If you're not familiar with induction that's ok and you can skip this question.)

3. **Adjacency Matrices II**

   Consider the adjacency matrix $\mathbf{A}$ below with rows and columns ordered according to the node labels {'A','B','C','D','E','F','G'}. Compute $\mathbf{A}^3$ and confirm that the value in $\mathbf{A}^3$ corresponding to paths from $E \to B$ is correct by explicitly writing out and counting all the paths from $E \to B$ that take exactly 3 steps.

   $$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. **Diagonalization**

   Suppose an $n \times n$ matrix $\mathbf{A}$ can be diagonalized and let $\mathbf{u}_k := \mathbf{A}^k\mathbf{u}_0$ for some initial vector $\mathbf{u}_0$. Show that for any $k \in \mathbb{N}$ we can write

   $$\mathbf{u}_k = c_1\lambda_1^k\mathbf{x}_1 + \cdots + c_n\lambda_n^k\mathbf{x}_n. \tag{2}$$

   (a) What are the c_i's, λ_i's and $\mathbf{x}_i$'s?

   (b) There are many problems of interest where $\mathbf{u}_k$ represents the state of some system (e.g. an economy, or web-browser in `Google's` PageRank model) at time k. Very often, we are interested in the behavior of this system in the *long-run*, i.e. when k gets very large. How might the representation in (2) be useful for determining this long-run behaviour?

5. **Cash-Flow Dynamics**

Multinational companies in the U.S., Japan and Europe have assets of \$4 trillion. At the start \$2 trillion are in the U.S.and \$2 trillion in Europe. Each year 1/2 the U.S. money stays home, and 1/4 goes to each of Europe and Japan. For Japan and Europe, 1/2 stays home and 1/2 is sent to the U.S.

(a) Find the 3×3 matrix $\mathbf{A}$ that that gives $\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k$ where $\mathbf{x}_k$ is the 3×1 vector containing the asset values in the US, Europe and Japan, respectively, at the end of year k.

(b) Find the eigenvalues and eigenvectors of $\mathbf{A}$.

(c) Find the cash-flow distribution at the end of year k.

(d) Find the limiting cash-flow distribution of the \$4 trillion as the world ends.

(This question is taken from Gilbert Strang's *Linear Algebra and Its Applications*.)