Solutions to Practice Problems for *Linear Algebra Review*

Within Exam Scope

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1. **Subspaces and Spans**

   Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ are vectors in $\mathbb{R}^3$. Do you agree with the statement that these vectors span $\mathbb{R}^3$?

   Solution: Not necessarily. They can be linearly dependent so that the dimension of their span is less than 3.

2. **Orthogonality**

   Which pairs are orthogonal among the vectors

   $$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

   Solution: $\mathbf{v}_1$ and $\mathbf{v}_3$; $\mathbf{v}_2$ and $\mathbf{v}_3$.

3. **Linear Independence**

   If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are independent vectors, would the following vectors be independent?

   $$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_3, \quad \mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{w}_3 = \mathbf{v}_2 + \mathbf{v}_3.$$

   Solution: Yes. Can check by definition. *Source: Strang, Gilbert (2006), Linear Algebra and Its Applications, CENGAGE.*

4. **Range and Rank**

   For $\mathbf{A} \in \mathbb{R}^{n \times n}$, if $\text{rank}(\mathbf{A}) = m < n$, then what is the dimension of the null space?

   Solution: $n - m$.

5. **Matrix Inverse**

   We have matrices $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Which of the following are true?

(a) $\mathbf{A}$ is singular

(b) $\mathbf{B}$ is invertible

(c) $\mathbf{A}+\mathbf{B}$ is invertible.

Solution: (1) and (3) true, (2) is false since $\mathbf{B}$ is singular.

6. **Subspaces**

The smallest subspace of $\mathbb{R}^3$ containing the vectors $\begin{pmatrix} 0 \\ -3 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ is the line whose equations are $x = a$ and $z = by$. What are the values of a and b?

Solution: $a = 0,\ b = -2$.

7. **Matrix Inverse**

For $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{n\times n}$ and $\alpha \in \mathbb{R}$, assume $\mathbf{A}$ and $\mathbf{B}$ are invertible:

(a) (True/False) $\left(\mathbf{A}^{-1}\right)^\top = \left(\mathbf{A}^\top\right)^{-1}$,

(b) (True/False) $(\mathbf{AB})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$,

(c) (True/False) If $\det(\mathbf{A}) = 2$, then $\det(\mathbf{A}^{-1}) = 2^{-1}$.

(d) (True/False) If $\det(\mathbf{A}) = 2$ and $\alpha > 1$, then $\det(\alpha\mathbf{A}) = 2$.

Solution: (a) True (b) False (c) True (d) False

8. **Linear Independence**

Let $\mathbf{u} = \begin{pmatrix} \lambda \\ 1 \\ 0 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 1 \\ \lambda \\ 1 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ \lambda \end{pmatrix}$. What are possible values of λ that make $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly dependent?

Solution: Solving $\det([\mathbf{u}, \mathbf{v}, \mathbf{w}]) = 0$ we get $\lambda = -\sqrt{2}, 0, \sqrt{2}$.

9. **Matrix Product**

For $\mathbf{A} = \begin{bmatrix} 1 & 1/3 \\ x & y \end{bmatrix}$, find the value of x and y such that $\mathbf{A}^2 = 0$.

Solution: $x = -3, y = -1$.

10. **Inner Product**

Consider the space of all matrices in $\mathbb{R}^{2\times 2}$. Define inner product as $\langle \mathbf{A}, \mathbf{B}\rangle = \operatorname{tr}(\mathbf{A}^\top \mathbf{B})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$. Let $\mathbf{U} = \begin{bmatrix} 1 & 4 \\ -3 & 5 \end{bmatrix}$ and $\mathbf{V} = \begin{bmatrix} x^2 & x-1 \\ x+1 & -1 \end{bmatrix}$. Find all values of x such that $\mathbf{U} \perp \mathbf{V}$.

Solution: $x = 3, -4$.

11. **Matrix Inverse**

If a non-singular matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^3 - 4\mathbf{A}^2 + 3\mathbf{A} - 2\mathbf{I}_n = \mathbf{0}$, what is $\mathbf{A}^{-1}$?

Solution: $\mathbf{A}^{-1} = \frac{1}{2}(\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I}_n)$.

12. **Basis**

Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ c \end{pmatrix}$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that c is **not** equal to what value?

Solution: $c \neq -3$.

13. **Subspaces**

Consider the set of points $(x, y, z) \in \mathbb{R}^3$. Which one of the following is a subspace of $\mathbb{R}^3$?

(a) $x + 3y - 2z = 3$.
(b) $x + y + z = 0$ and $x - y - z = 2$.
(c) $\frac{x+1}{2} = \frac{y-2}{4} = \frac{z}{3}$.
(d) $x^2 + y^2 = z$.
(e) $x = -z$ and $x = z$.
(f) $\frac{x}{3} = \frac{y+1}{2}$.

Solution: (e). Subspaces of $\mathbb{R}^3$ must contain the origin.