Indirect Method:

Maximisation method

In the form of

$$\underset{x}{\text{maximise}}\ c^T x$$

$$\text{subject to } Ax \leq b$$

$$x \in \mathbb{R}^n_+$$

- replacing $=$'s with $\leq / \geq$'s
- replacing $\geq$'s with $\leq$'s
- replacing unrestricted variables with restricted ones

Minimisation method

In the form of

$$\underset{y}{\text{minimise}}\ b^T y$$

$$\text{subject to } A^T y \geq c$$

$$y \in \mathbb{R}^m_+$$

- replacing $=$'s with $\leq / \geq$'s
- replacing $\geq$'s with $\leq$'s
- replacing unrestricted variables with restricted ones

Example:

$$\text{minimise } 3x - y$$

$$\text{subject to } 2x + y = 5$$

$$x \geq 0,\ y \in \mathbb{R}$$

①. Replace "=" constraints

subject to $2x + y \leq 5$

$2x + y \geq 5 \rightarrow -2x - y \leq -5$

②. Replace $y \in \mathbb{R}$

$$\left.\begin{aligned} &\text{minimise } 3x - y^+ + y^- \\ &\text{subject to } 2x + y^+ - y^- \geq 5 \\ &\qquad -2x - y^+ + y^- \geq -5 \\ &\qquad x, y^+, y^- \geq 0 \end{aligned}\right\} \text{standard form}$$

$$b = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} 5 \\ -5 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$$

③. Dual form:

$$\text{maximise } 5x_1 - 5x_2$$

$$\begin{aligned} \text{subject to } & 2x_1 - 2x_2 \leq 3 \\ & x_1 - x_2 \leq -1 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example:

minimise $3x_1 + 5x_2 - x_3$

subject to $x_1 + x_3 = 4$

$x_2 - 2x_3 \leq 2$

$x_1, x_2 \geq 0$, x_3 unrestricted

$$\Downarrow$$

minimise $3x_1 + 5x_2 - x_3^+ + x_3^-$

$x_1 + (x_3^+ - x_3^-) \geq 4$

$-x_1 - (x_3^+ - x_3^-) \geq -4$

$-x_2 + 2x_3 \geq -2 \longrightarrow -x_2 + 2(x_3^+ - x_3^-) \geq -2$

$x_1, x_2, x_3^+, x_3^- \geq 0$

$$\Downarrow$$

$$\left.\begin{aligned} \text{minimise} \quad & 3x_1 + 5x_2 - x_3^+ + x_3^- \\ & x_1 + x_3^+ - x_3^- \geq 4 \\ & -x_1 - x_3^+ + x_3^- \geq -4 \\ & -x_2 + 2x_3^+ - 2x_3^- \geq -2 \end{aligned}\right\} \text{standard form}$$

$x_1, x_2, x_3^+, x_3^- \geq 0$

$$b = \begin{pmatrix} 3 \\ 5 \\ -1 \\ 1 \end{pmatrix} \qquad c = \begin{pmatrix} 4 \\ -4 \\ -2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 0 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -2 \end{pmatrix}$$

$$\Downarrow$$

Dual form:

maximise $4y_1 - 4y_2 - 2y_3$

subject to $y_1 - y_2 \leq 3$

$-y_3 \leq 5$

$y_1 - y_2 + 2y_3 \leq -1$

$-y_1 + y_2 - 2y_3 \leq 1$

$y_1, y_2, y_3 \geq 0$

Direct Method:

Example:

minimise $3x - y$

subject to $2x + y = 5$

$x \geq 0,\ y \in \mathbb{R}$

one constraint, two decision variables for primal

$\Downarrow$

one decision variable, two constraints for dual

maximise $5z$

subject to $2z \leq 3$

$1z \geq -1$

$z \in \mathbb{R}$

1 Correspondence variables $\Longleftrightarrow$ constraints:

- For every *primal constraint*, create a *dual variable*.
- For every *primal variable*, create a *dual constraint*.

2 Constraint matrix, objective function and right-hand sides:

- Transpose the *primal constraint matrix* $\boldsymbol{A}$ to get the *dual constraint matrix* $\boldsymbol{A}^T$.
- The *primal objective coefficients* $\boldsymbol{c}$ become the *dual right-hand side coefficients*.
- The *primal right-hand side coefficients* $\boldsymbol{b}$ become the *dual objective coefficients*.

3 Objective & constraint directions, variable ranges:

If the primal problem is a *maximisation problem*:

- The dual problem is a *minimisation problem*.

i-th primal *constraint*	i-th dual *variable*	
$a_i^T\boldsymbol{x} \leq b_i$	$y_i \geq 0$	("sensible")
$a_i^T\boldsymbol{x} = b_i$	y_i unrestricted	("odd")
$a_i^T\boldsymbol{x} \geq b_i$	$y_i \leq 0$	("bizarre")

i-th primal *variable*	i-th dual *constraint*	
$x_i \geq 0$	$a_i^T\boldsymbol{y} \geq c_i$	("sensible")
x_i unrestricted	$a_i^T\boldsymbol{y} = c_i$	("odd")
$x_i \leq 0$	$a_i^T\boldsymbol{y} \leq c_i$	("bizarre")

3 Objective & constraint directions, variable ranges:

If the primal problem is a *minimisation problem*:

- The dual problem is a *maximisation problem*.

i-th primal *constraint*	i-th dual *variable*	
$a_i^T\boldsymbol{x} \geq b_i$	$y_i \geq 0$	("sensible")
$a_i^T\boldsymbol{x} = b_i$	y_i unrestricted	("odd")
$a_i^T\boldsymbol{x} \leq b_i$	$y_i \leq 0$	("bizarre")

i-th primal *variable*	i-th dual *constraint*	
$x_i \geq 0$	$a_i^T\boldsymbol{y} \leq c_i$	("sensible")
x_i unrestricted	$a_i^T\boldsymbol{y} = c_i$	("odd")
$x_i \leq 0$	$a_i^T\boldsymbol{y} \geq c_i$	("bizarre")

Example:

maximise $x_1 - x_3$ — three decision variables (primal)

s.t. $x_1 + x_2 = 4$ — two constraints

$x_3 \leq 2$ — $\downarrow$

$x_1, x_2 \leq 0$, x_3 unrestricted — two decision variables (dual)

three constraints

$$A^T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

minimise $\underline{4}\, y_1 + \underline{2}\, y_2$

s.t.
$$\underline{y_1 \quad \leq} \quad 1$$
$$\underline{y_1 \quad \leq} \quad 0$$
$$\underline{y_2 \quad =} \quad -1$$

y_1 unrestricted. $y_2 \geq 0$

minimise $4y_1 + 2y_2$

subject to $y_1 \leq 1$

$y_1 \leq 0$

$y_2 = -1$

y_1 unrestricted. $y_2 \geq 0$

Duality Theorem

①. Both problems are feasible and bounded

②. One problem is feasible + unbounded, the other infeasible

③. Both problems are infeasible

④. A duel price is reported for each constraint.

⑤ The dual price is only positive when a constraint is binding.

⑤∴ The dual price gives the improvement in the objective function is relaxed by one unit.