Financial Analytics — Imperial College Business School
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Assignment 4

Due: 11.59pm Sunday 30^{th} May 2021

Rules

1. This is a group assignment. (There are approximately 3 people per group and by now you should know your assigned group.)
2. You are free to use `R` or `Python` for the this assignment.
3. Within each group **I strongly encourage each person to attempt each question by his / herself first** before discussing it with other members of the group.
4. Students should **not** consult students in other groups when working on their assignments.
5. Late assignments will **not** be accepted and all assignments must be submitted through the Hub with one assignment submission per group. Your submission should include a PDF report with your answers to each question together with screenshots of any relevant code. Make sure your PDF clearly identifies each member of the group by CID and name.

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1. **Shrinkage Estimators of the Covariance Matrix (50 marks)**
   The purpose of this exercise[1] is to visualise how the covariance matrix gets distorted when it is estimated using a finite set of observations. The exercise also explores how a shrinkage technique of Ledoit and Wolf can mitigate this kind of distortion.

   (a) Assume n = 10 assets have returns that follow a multivariate normal distribution with expected returns equal to zero and *true* covariance matrix equal to the $n \times n$ diagonal matrix

$$\mathbf{V} = \begin{pmatrix} 0.8 & & & & \\ & 0.85 & & & \\ & & \ddots & & \\ & & & 1.2 & \\ & & & & 1.25 \end{pmatrix}.$$

   (The diagonal entries are equally spaced at 0.05 intervals.)
   Generate $T = 120$ samples $\mathbf{r}_t$, $t = 1, \ldots, T$, from this joint distribution. Each of these samples $\mathbf{r}_t \in \mathbb{R}^{10}$ is drawn from the ten-dimensional multivariate normal distribution $\mathrm{N}(\mathbf{0}, \mathbf{V})$. (You may find the *mvrnorm* function in `R` useful for doing this.)

[1] This exercise is taken from *Optimization Methods in Finance* (2^{nd} edition) by Cornuéjols, Peña and Tütüncü and published by Cambridge University Press.

(i) Use the T samples to estimate the sample covariance matrix $\widehat{\mathbf{V}}$ as follows. Let $\bar{\mathbf{r}} := (1/T)\sum_{t=1}^{T} \mathbf{r}_t$, $\mathbf{z}_t := \mathbf{r}_t - \bar{\mathbf{r}}$, $t = 1, \dots, T$, and

$$\widehat{\mathbf{V}} := \frac{1}{T} \sum_{t=1}^{T} \mathbf{z}_t \mathbf{z}_t^\top.$$

Plot the eigenvalues both of the true covariance matrix $\mathbf{V}$ and of the estimated covariance matrix $\widehat{\mathbf{V}}$ on the same plot. Do you observe anything peculiar? **(9 marks)**

(ii) Using the estimated covariance matrix $\widehat{\mathbf{V}}$, find the estimated minimum-risk fully invested portfolio $\widehat{\mathbf{x}}$. (Fully invested means that the portfolio weights should sum to 1.) Compute the *estimated* minimum variance $\widehat{\mathbf{x}}^\top \widehat{\mathbf{V}} \widehat{\mathbf{x}}$, the *actual* minimum variance $\widehat{\mathbf{x}}^\top \mathbf{V} \widehat{\mathbf{x}}$, and the *true* minimum variance $(\mathbf{x}^*)^\top \mathbf{V}(\mathbf{x}^*)$, where $\mathbf{x}^*$ is the true minimum-risk fully invested portfolio for $\mathbf{V}$. **(8 marks)**

(iii) Repeat parts (i) and (ii) several times (anywhere from a handful to a few thousand times). What do you observe? **(8 marks)**

(b) We will next apply the shrinkage technique of Ledoit and Wolf. To that end, let λ_i, $i = 1, \dots, n$, denote the eigenvalues of the covariance matrix $\widehat{\mathbf{V}}$ and $\bar{\lambda} := (1/n)\sum_{i=1}^{n} \lambda_i$. Define $\mathbf{C} := \bar{\lambda}\mathbf{I}_n$ where $\mathbf{I}_n$ is the $n \times n$ identity matrix, and

$$\alpha := \min\left(\frac{1}{T} . \frac{\frac{1}{T}\sum_{t=1}^{T} \operatorname{trace}((\mathbf{z}_t \mathbf{z}_t^\top - \widehat{\mathbf{V}})^2)}{\operatorname{trace}((\widehat{\mathbf{V}} - \mathbf{C})^2)}, 1 \right).$$

Finally, consider the shrunken matrix

$$\bar{\mathbf{V}} := (1 - \alpha)\widehat{\mathbf{V}} + \alpha \mathbf{C}.$$

(i) Plot the eigenvalues of the true covariance matrix $\mathbf{V}$, of the sample covariance $\widehat{\mathbf{V}}$, and of the shrunken covariance $\bar{\mathbf{V}}$ on the same plot. What do you observe now? **(9 marks)**

(ii) Using the shrunken covariance $\bar{\mathbf{V}}$ find the estimated minimum-risk fully invested portfolio $\bar{\mathbf{x}}$. Compute the *estimated* minimum variance $\bar{\mathbf{x}}^\top \bar{\mathbf{V}} \bar{\mathbf{x}}$, the *actual* minimum variance $\bar{\mathbf{x}}^\top \mathbf{V} \bar{\mathbf{x}}$, and the *true* minimum variance $(\mathbf{x}^*)^\top \mathbf{V}(\mathbf{x}^*)$. What do you observe? Are the results any different from part (a)(ii)? **(8 marks)**

(iii) Repeat parts (i) and (ii) several times (anywhere from a handful to a few thousand times). What do you observe? Are the results any different from part (a)(iii)? **(8 marks)**

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2. **A Leveraged Firm (20 marks)**
   A company[2] earns a rate of return of r_A and has beta β_A. A fraction w of the assets is owned by bondholders, and the remaining fraction $(1-w)$ is owned by equity holders. Every year the bondholders demand a riskless rate of return of r_B on their fraction of the assets, regardless of the actual rate of return r_A that was achieved that year. Beyond that, the equity holders take whatever is left after the bondholders have been paid.

   (a) What is the rate of return of the equity holders in terms of w, r_A and r_B? **(7 marks)**

   (b) What is the beta of the rate of return of the equity holders in terms of w and β_A? **(7 marks)**

   (c) Suppose β_A is positive and the expected rate of return on the market is greater than the risk-free rate. As w increases (that is, as the firm becomes more leveraged), what should happen to the expected rate of return on the equity of the firm? **(6 marks)**

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3. **Rough and Ready Calculations are Often Useful! (25 marks)**
   Returning to the Simplico gold mine example, we saw in the *Introduction to Real Options* lecture notes that the value of the lease (without the enhancement option) was \$24.1m. Without building a lattice, how could you quickly verify that this price was (approximately) correct? Or to put it another way, can you find a quick way to estimate the price of the lease without building a lattice and using backwards evaluation?

   Hints: Let S_t denote the price of gold at time t. How much is a security worth S_t at time t worth today at time 0? Recall also the annuity formula from Section 2.2 of the *Interest Rates and Deterministic Cash-Flows* notes for computing the value of a constant cash-flow over a fixed number of time periods.

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4. **Evaluating a More Complex Option on the Simplico Gold Mine (25 marks)**
   Do Exercise 1 in the *Introduction to Real Options* lecture notes. That is, compute the value of the enhancement option in the Simplice goldmine example when the enhancement costs \$5 million but raises the mine capability by 40% to $14,000$ ounces at an operating cost of \$240 per ounce. Moreover, due to technological considerations, you should assume that the enhancement (should it be required) will not be available until the beginning of the 5^{th} year.

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[2] This question is taken from Luenberger's *Investment Science*