Solutions to Assignment 1: Linear Algebra
Total Points: 100

Rules:

1. This is a group assignment. Within each group **I strongly encourage everyone to attempt each question by his/herself first** before discussing it with other members of the group.
2. You are recommended to **type out your solution** and submit a PDF file. If you choose to write it down and submit a scanned version, please ensure clear handwriting.
3. `R` is the default package / programming language for this course so you should use `R` for any programming questions in this assignment.
4. The `R Notebook` *Vector_Matrix_Operations.rmd* is a very useful resource for understanding how to manipulate vectors and matrices in `R`. It also shows you how to compute the rank of a matrix, how to compute the eigenvalues and eigenvectors of a square matrix etc. You should therefore familiarise yourself with this Notebook by working through it before tackling this assignment.

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1. **Linear Independence (15 points)**

(a) Without doing any calculations explain whether or not the following vectors are linearly independent in $\mathbb{R}^3$ and justify your answer. Please do not use `R` or other software in this question. (5 points)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 2.5 \\ 1.5 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 8 \\ 7.2 \end{pmatrix}.$$

Solution: They must be linearly dependent because the dimension of $\mathbb{R}^3$ is 3 and the dimension of any space is the maximal number of linearly independent vectors in the space. Since there are four vectors in the list above, they must be linearly dependent.

(b) Do the following vectors from a basis for $\mathbb{R}^3$? Justify your answer. You can use `R` in this question. (5 points)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Solution: Yes, they do form a basis. We know that a collection of vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$ if and only if there are n vectors in the collection and they are linearly independent. In our case we have three vectors in $\mathbb{R}^3$ so if we can show they are linearly independent then they must form a basis for $\mathbb{R}^3$. An easy way to show they are linearly independent is to form the 3×3 matrix $\mathbf{A}$ which has these vectors as columns

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{pmatrix}$$

and then show $\text{rank}(\mathbf{A}) = 3$, or equivalently that $\det(\mathbf{A}) \neq 0$, or equivalently that $\mathbf{A}$ is invertible. It is easy to check any of these conditions in `R`.

(c) Let $\mathbf{u} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} c \\ 1 \\ 0 \end{pmatrix}$ where $c \in \mathbb{R}$. The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ provided that c is **not** equal to what value? (5 points)

Solution: $c \neq -7$. Prove by definition of linear independence or check determinant of matrix $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$.

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2. **Matrix Rank (15 points)**

Consider the $m \times n$ matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 & 3 & 1 & 4 \\ 1 & 0 & 0 & 5 & 2 & 3 \\ 1 & 0 & 0 & 5 & 2 & 4 \\ 1 & 0 & 6 & 5 & 2 & 7 \end{pmatrix}$$

where $m = 4$ and $n = 6$.

(a) Without any calculations, what is the maximal possible rank of $\mathbf{A}$? (5 points)

Solution: We know that $\text{rank}(\mathbf{A})$ is equal to the maximum number of linearly independent columns/rows in $\mathbf{A}$. Therefore $\text{rank}(\mathbf{A}) \leq \min(m, n) = 4$.

(b) What actually is the rank of $\mathbf{A}$? (It is easy to compute this in `R`.) (5 points)

Solution: Using the **rankMatrix** function from the `Matrix` library, we find that $\text{rank}(\mathbf{A}) = 4$.

(c) What is the dimension of the null space of $\mathbf{A}$? (5 points)

Solution: The rank of $\mathbf{A}$ is 4. As a result, the dimension of the null space is $n - \text{rank}(\mathbf{A}) = 6 - 4 = 2$.

3. **Matrix Rank (15 points)**

   Consider a general $m \times n$ matrix $\mathbf{A}$ where $\text{rank}(\mathbf{A}) = m < n$. Consider the $m \times m$ matrix $\mathbf{B} := \mathbf{A}\mathbf{A}^\top$. Prove that $\text{rank}(\mathbf{B}) = m$. (**Hint:** Suppose $\mathbf{B}\mathbf{x} = \mathbf{0}$, then show this implies $\mathbf{x} = \mathbf{0}$ so that columns of $\mathbf{B}$ are linearly independent. To do this, consider $\mathbf{x}^\top\mathbf{B}\mathbf{x}$ ...)

   Solution: Suppose $\mathbf{B}\mathbf{x} = \mathbf{0}$ where $\mathbf{x} \in \mathbb{R}^m$. Then $\mathbf{A}\mathbf{A}^\top\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{x}^\top\mathbf{A}\mathbf{A}^\top\mathbf{x} = 0 \Rightarrow (\mathbf{A}^\top\mathbf{x})^\top\mathbf{A}^\top\mathbf{x} = 0 \Rightarrow \mathbf{A}^\top\mathbf{x} = \mathbf{0}$ with the last equality following from the first property of inner products. But the m columns of $\mathbf{A}^\top$ are linearly independent (why?) and so $\mathbf{x} = \mathbf{0}$. We have therefore shown that $\mathbf{B}\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$ and so $\text{rank}(\mathbf{B}) = m$.

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4. **Ranges and Null Spaces (15 points)**

   Recall that the range of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the set of vectors in $\mathbb{R}^m$ that can be obtained as a linear combination of the columns of $\mathbf{A}$. The range is denoted by $\mathcal{R}(\mathbf{A})$ and we have

$$\mathcal{R}(\mathbf{A}) := \{\mathbf{A}\mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^n\}.$$

   Similarly, recall the null space of $\mathbf{A}$ is defined according to $\mathcal{N}(\mathbf{A}) := \{\mathbf{x} \in \mathbb{R}^n \,:\, \mathbf{A}\mathbf{x} = \mathbf{0}\}$.

   (a) Show that every $\mathbf{x} \in \mathcal{R}(\mathbf{A}^\top)$ is orthogonal to every $\mathbf{z} \in \mathcal{N}(\mathbf{A})$, i.e. $\mathcal{R}(\mathbf{A}^\top) \perp \mathcal{N}(\mathbf{A})$. (**Hint:** If $\mathbf{x} \in \mathcal{R}(\mathbf{A}^\top)$ then $\mathbf{x} = \mathbf{A}^\top\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^m$.) (10 points)

   Solution: $\mathbf{x} \in \mathcal{R}(\mathbf{A}^\top)$ implies $\mathbf{x} = \mathbf{A}^\top\mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}^m$. Then have

$$\begin{aligned}
\mathbf{x}^\top\mathbf{z} &= \left(\mathbf{A}^\top\mathbf{v}\right)^\top \mathbf{z} \\
&= \mathbf{v}^\top\mathbf{A}\mathbf{z} \\
&= \mathbf{v}^\top\mathbf{0} \\
&= 0.
\end{aligned}$$

   Hence the result follows.

   (b) Show that every $\mathbf{x} \in \mathcal{R}(\mathbf{A})$ is orthogonal to every $\mathbf{z} \in \mathcal{N}(\mathbf{A}^\top)$. (5 points)

   Solution: This follows immediately from part (a) by replacing $\mathbf{A}$ in part (a) with the matrix $\mathbf{A}^\top$. This can also be proven in a similar fashion as in part (a).

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5. **Fibonacci Numbers (20 points)**

   The Fibonacci sequence

   $$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

   is such that every number is the sum of its two predecessors, i.e. for $k = 1, 2, \ldots$ we have

   $$F_{k+2} = F_{k+1} + F_k, \tag{1}$$

   where $F_1 = 0, F_2 = 1, \ldots$

   (a) Write (1) in the form $\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k$ where $\mathbf{u}_k, \mathbf{u}_{k+1} \in \mathbb{R}^2$. (**Hint:** Use $\mathbf{u}$ to encapsulate the F's.) (5 points)

   Solution: Define

   $$\mathbf{u}_k := \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}.$$

   Then

   $$\begin{aligned} F_{k+2} &= F_{k+1} + F_k \\ F_{k+1} &= F_{k+1} \end{aligned}$$

   becomes

   $$\mathbf{u}_{k+1} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{=:\mathbf{A}} \mathbf{u}_k. \tag{2}$$

   (b) Write some `R` code to compute F_k for $k = 1000$ in three different ways: (i) directly by iterating (1), (ii) directly by iterating $\mathbf{u}_{k+1} = \mathbf{A}\mathbf{u}_k$, and (iii) by diagonalizing $\mathbf{A}$ and using this to calculate $\mathbf{u}_k = \mathbf{A}^{k-1}\mathbf{u}_1$. (15 points)

   Solution: We find $F_{1000} = 2.686381 \times 10^{208}$.

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6. **Answer the following questions. (20 points)**

   (a) Let $\mathbf{A}$ be the $m \times n$ matrix

   $$\begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix}$$

   and let $\mathbf{x} = (x_1 \ldots x_n)^\top$. Give an expression for $\mathbf{A}\mathbf{x}$ in terms of the column vectors of $\mathbf{A}$, i.e. the $\mathbf{a}_i$'s. (5 points)

   Solution: The rules of matrix multiplication imply

   $$\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

(b) Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\text{rank}(\mathbf{A}) = n$. Is there always a solution to the equation $\mathbf{Ax} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$? Justify your answer. (5 points)

Solution: Yes, there is: $\mathbf{A}$ is of full rank and (being square) is therefore invertible so the (unique) solution is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

(c) Suppose $(\lambda, \mathbf{u})$ is an eigenvalue / eigenvector pair for $\mathbf{A} \in \mathbb{R}^{n \times n}$. Can you find a corresponding eigenvalue / eigenvector pair for $\mathbf{A}^2$? Justify your answer. (5 points)

Solution: Yes, $(\lambda^2, \mathbf{u})$ is an eigenvalue / eigenvector pair for $\mathbf{A}^2$. This follows since

$$\begin{aligned}
\mathbf{A}^2\mathbf{u} &= \mathbf{A}\left(\mathbf{Au}\right) \\
&= \mathbf{A}\left(\lambda\mathbf{u}\right) \\
&= \lambda\mathbf{Au} \\
&= \lambda^2\mathbf{u}.
\end{aligned}$$

(d) Find all eigenvalues and corresponding eigenvectors for matrix $\mathbf{A}$. You can use `R` in this question. (5 points)

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Solution: The eigenvalues are $-4, 3, 1$ and the corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$