Nonlinear Programming Solutions

Solution to (1) (a): The problem could look as follows:

$$\begin{array}{ll} \text{minimise} & \sum_{i=1}^{m} (y_i - b_0 - x_{i1}b_1 - \ldots - x_{in}b_n)^2 \\ \text{subject to} & b_0, b_1, \ldots b_n \in \mathbb{R} \end{array}$$

Solution to (1) (b): The problem is indeed convex for the following reasons:

- We minimise an objective function that is convex since (1) the mapping x —> x^2 is convex; (2) the mapping (b_0, …, b_n) —> y_i - b_0 - … is affine; (3) the composition of a convex function with an affine function is convex; and (4) the sum of convex functions is convex.

- The feasible region is convex since it's the entire space R^n.

Solution to (1) (c): We go through each requirement in turn:

- $b_i \geq -10$, $b_i \leq 10$ for all i = 1, …, n is a set of linear constraints; in particular, this implies that the feasible region remains convex.

- $b_1 \geq 2b_2$ is a linear constraint; in particular, this implies that the feasible region remains convex.

- $b_3 = b_4$ is a linear constraint; in particular, this implies that the feasible region remains convex.

- The resulting feasible region would no longer be convex! Indeed, we could take a feasible solution $(b_0', b_1', \ldots, b_n')$ with $b_5' = 1$ and another feasible solution $(b_0'', b_1'', \ldots, b_n'')$ with $b_5'' = 2$ but their midpoint $(b_0^\star, b_1^\star, \ldots, b_n^\star) = \frac{1}{2}(b_0', b_1', \ldots, b_n') + \frac{1}{2}(b_0'', b_1'', \ldots, b_n'')$ would satisfy $b_5^\star = 1.5$ and thus be infeasible. We could formulate this constraint by adding a binary variable y and requiring that b_5 <= 1 + My and b_5 >= 2 - M(1-y).

- We can formulate this by adding auxiliary variables y_1, …, y_{10} with $y_i \geq b_i$ and $y_i \geq -b_i$ as well as requiring that $y_1 + y_2 + \ldots + y_{10} \leq 10$. Since these are linear constraints, the feasible region remains convex.

- The resulting feasible region would no longer be convex! Indeed, we could take a feasible solution $(b_0', b_1', \ldots, b_n')$ that has the first 5 slopes nonzero and the rest zero as well as another feasible solution $(b_0'', b_1'', \ldots, b_n'')$ with the last 5 slopes nonzero and the rest zero but their midpoint $(b_0^\star, b_1^\star, \ldots, b_n^\star) = \frac{1}{2}(b_0', b_1', \ldots, b_n') + \frac{1}{2}(b_0'', b_1'', \ldots, b_n'')$ would have both the first 5 and the last 5 slopes nonzero and would thus be infeasible. We can formulate this constraint by a simple variant of the last homework assignment.

Solution to (2) (a):

$$\begin{array}{ll} \text{minimise} & 1/x_1 + 2/x_2 + | x_3 | \\ \text{subject to} & \max \{ x_1 + x_2, x_1 - x_3 \} >= 2 \\ & x_1, x_2 >= 0, x_3 \text{ unrestricted} \end{array}$$

The objective minimises a convex function: The terms $1/x_1$, $1/x_2$ and $| x_3 |$ are all convex functions in single variables (see Section 9.4). Likewise, the addition of convex functions, scaled with nonnegative weights (1, 2 and 1 in our case) is also convex by our convex calculus rules.

The feasible region is <u>not</u> convex: The maximum of affine functions is convex, but convex functions have to be on the left-hand side of <= constraints (or, equivalently, on the right-hand side of >= constraints), see Section 9.5.

Thus the problem is <u>not</u> convex.

Solution to (2) (b):

$$\begin{array}{ll} \text{maximise} & x_1 - x_2^2 \\ \text{subject to} & (2x_1 - x_2)^2 <= x_1 \\ & | 2 x_1 | <= 2 \end{array}$$

The objective function maximises a concave function: x_1 is linear and hence both convex and concave. x_2^2 is convex, and hence $-x_2^2$ is concave. The sum of two concave functions is concave, just as the sum of two convex functions is convex. (Reason: Take the negative of each function, which is convex. The sum of convex functions is convex. Then take the negative of the sum — it must be concave. However, taking the negative sum of the negative terms is equivalent to the original objective function!)

The feasible region is convex: The first constraint can be written as $(2x_1 - x_2)^2 - x_1 <= 0$. The first term is convex since it's a convex function pre-composed with an affine function of x_1 and x_2. The term $-x_1$ is linear and hence both convex and concave. The sum of convex functions is convex. The second constraint also has a convex left-hand side, by our elementary functions from Section 9.4. Note that here we again pre-compose the convex function $x \longrightarrow | x |$ with the affine function $x \longrightarrow 2 x$.

Thus the problem <u>is</u> convex.

Solution to (2) (c):

$$\begin{array}{ll} \text{maximise} & 3x_1 - 2x_2 + 5^2 \\ \text{subject to} & x_1 + x_2 <= 2 \\ & x_2 >= x_1 \\ & x_1, x_2 >= 0 \end{array}$$

The objective function maximises a concave function: The objective function is linear and hence both convex and concave. (Note that $5^2 = 25$ is just a constant.)

The feasible region is convex: It is actually the feasible region of a linear program!

Thus the problem <u>is</u> convex — it is actually a linear program!

Solution to (2) (d):

$$\begin{array}{ll} \text{minimise} & x_1 \\ \text{subject to} & x_1 x_2 >= 2 \\ & x_2 >= 4 \end{array}$$

The objective function minimises a convex function: It is actually a linear function.

The feasible region is convex: This should be obvious for the second constraint, which is actually linear. Surprisingly, the first constraint is also convex; here is the reason: $x_1 x_2 >= 2$ can be reformulated equivalently as $x_1 >= 2 / x_2$ since x_2 is guaranteed to be positive by the second constraint (otherwise, if we don't know the sign of x_2, we would not be allowed to multiply both sides by it!). Now the constraint is equivalent to $2 / x_2 - x_1 <= 0$, which is the weighted sum of the two convex $1 / x_2$ and $-x_1$ with weights 2 and 1, respectively.

Thus the problem <u>is</u> convex!