Homework 8 Solutions (Concise)

C191A: Introduction to Quantum Computing, Fall 2025

Xiaoyang Zheng

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1 The Repetition Code and Longitudinal Relaxation

1.1

We know that $p(t) = 1 - e^{-\gamma t} \approx \gamma t$ when t is small. If we want $p(t) = 0.01$, then we get $\gamma t = 0.01$, so $t = 0.01/\gamma$, which means $t = 0.01 T_1$.

1.2

The error probability is $P_{\text{err}} = P(\text{at least 1 flip}) = 1 - P(\text{no flips}) = 1 - (1 - p(t))^3$. Because $p(t) \approx \gamma t$, we can write $P_{\text{err}} \approx 1 - (1 - \gamma t)^3 \approx 1 - (1 - 3\gamma t) = 3\gamma t$. Setting $P_{\text{err}} = 0.01$ gives us $3\gamma t = 0.01$, so $t = \frac{0.01}{3\gamma} = \frac{0.01}{3} T_1$.

1.3

The logical error probability is $P_L = P(\text{2 flips}) + P(\text{3 flips}) = \binom{3}{2} p^2(1-p) + p^3$, where we let $p = p(t)$. Using the approximation $p \approx \gamma t$, we get $P_L \approx 3(\gamma t)^2(1 - \gamma t) + (\gamma t)^3 \approx 3(\gamma t)^2$. If we set $P_L = 0.01$, then $3(\gamma t)^2 = 0.01$, which gives $\gamma t = \sqrt{0.01/3} = 0.1/\sqrt{3}$. Therefore, $t = \frac{0.1}{\sqrt{3}\gamma} \approx 0.0577 T_1$.

1.4

We need to set $p = P_L$, which means $p = 3p^2 - 2p^3$. If $p \neq 0$, we can divide both sides by p to get $1 = 3p - 2p^2$, or $2p^2 - 3p + 1 = 0$. This factors as $(2p-1)(p-1) = 0$, so $p = 1$ (trivial solution) or $p = 1/2$. Using $p(t) = 1/2$, we have $1 - e^{-\gamma t} = 1/2$, which gives $e^{-\gamma t} = 1/2$. Taking the natural logarithm: $-\gamma t = -\ln(2)$, so $t = \frac{\ln(2)}{\gamma} = \ln(2) T_1$.

1.5

For the physical qubit, the no-flip probability is $P_S(t) = e^{-\gamma t}$. By definition, $P_S(T_1) = 1/e$. For the logical qubit, the no-flip probability is $P_L(t) = 1 - (3p^2 - 2p^3)$ where $p = 1 - e^{-\gamma t}$. When t is small: $P_S(t) \approx 1 - \gamma t$ (decays proportional to t). $P_L(t) \approx 1 - 3(\gamma t)^2$ (decays proportional to t^2). Since $3(\gamma t)^2 \ll \gamma t$ for small t, we can see that $P_L(t)$ decays much more slowly than $P_S(t)$. This means that $T_{1,L}$ (the time when $P_L(t)$ reaches $1/e$) is **longer** than T_1.

2 Pauli Commutation Relations

We use this rule: If there's an even number of anti-commuting (AC) pairs, then the operators commute. If there's an odd number of AC pairs, they anti-commute.

2.1 X_1, Y_1

On qubit 1, we have AC. Total: 1 (Odd), so they **Anti-commute**.

2.2 Z_1Z_2, Y_1Y_2

On Q1 we have AC, on Q2 we have AC. Total: 2 (Even), so they **Commute**.

2.3 $Z_1X_3Y_4, Y_1Z_2Y_4$

Q1 (AC), Q2 (C), Q3 (C), Q4 (C). Total: 1 (Odd), so they **Anti-commute**.

2.4 $Z_1 \dots Y_8, X_1 \dots X_8$

Q1(AC), Q2(C), Q3(AC), Q4(C), Q5(C), Q6(C), Q7(AC), Q8(AC). Total: 4 (Even), so they **Commute**.

3 9-qubit Shor code

3.1

The distance is $d = \min(\text{weight}(L))$ for any logical operator L. Let's consider $Z_L = Z_1Z_4Z_7$. We need to check if $[Z_L, S_i] = 0$ for all stabilizers S_i:

- $[Z_L, Z\text{-stabs}] = 0$ (all Z operators commute).
- $[Z_L, S_7 = X_1 \dots X_6] = 0$ (AC on X_1, X_4 means 2 ACs, so they commute).
- $[Z_L, S_8 = X_4 \dots X_9] = 0$ (AC on X_4, X_7 means 2 ACs, so they commute).

Yes, so Z_L is a logical operator. Since $\text{weight}(Z_L) = 3$, we have $d \leq 3$. The code can correct $t = 1$ error, which means $d \geq 2t + 1 = 3$. Therefore, $d = 3$. Also, $Z_1Z_4Z_7|0\rangle_L \to C \cdot (|0\rangle_a - |1\rangle_a)(|0\rangle_b - |1\rangle_b)(|0\rangle_c - |1\rangle_c) = |1\rangle_L$.

3.2

The syndrome has $s_i = 1$ if the error anti-commutes with S_i. The order is $(S_1 \dots S_6, S_7, S_8)$. The errors Z_1, Z_2, Z_3 all commute with $S_1 \dots S_6$ (because they're all Z-type).

- Z_1: anti-commutes with S_7 (on X_1), but commutes with S_8.
- Z_2: anti-commutes with S_7 (on X_2), but commutes with S_8.

- Z_3: anti-commutes with S_7 (on X_3), but commutes with S_8.

So all three errors Z_1, Z_2, Z_3 give the same syndrome: **00000010**.

3.3

Effect: The errors Z_1, Z_2, Z_3 all act on block 1, which is $(|000\rangle + |111\rangle)$. Each acts as Z, mapping it to $(|000\rangle - |111\rangle)$. So they have the same effect on $|\psi_L\rangle$. **Correction:** The syndrome '00000010' tells us to apply correction $C = Z_1$.

- If the error is $E = Z_1$, then $CE = Z_1 Z_1 = I$. (This corrects the error)
- If the error is $E = Z_2$, then $CE = Z_1 Z_2 = S_1$. (This is correct because S_1 is a stabilizer)
- If the error is $E = Z_3$, then $CE = Z_1 Z_3 = (Z_1 Z_2)(Z_2 Z_3) = S_1 S_2$. (This is correct because $S_1 S_2$ is also a stabilizer)

This phenomenon is called "degeneracy": multiple different errors can map to the same syndrome.

4 7-qubit Steane code

The syndrome order is: $(S_x^{(1)}, S_x^{(2)}, S_x^{(3)}, S_z^{(1)}, S_z^{(2)}, S_z^{(3)})$.

4.1 Syndromes for Z_i

For Z_i errors, we have $s_z^{(k)} = 0$. We need to check if they anti-commute with $S_x^{(j)}$.

Error	Syndrome
Z_1	**100000**
Z_2	**110000**
Z_3	**111000**
Z_4	**101000**
Z_5	**010000**
Z_6	**011000**
Z_7	**001000**

4.2 Syndromes for X_i

For X_i errors, we have $s_x^{(j)} = 0$. We need to check if they anti-commute with $S_z^{(k)}$.

Error	Syndrome
X_1	**000100**
X_2	**000110**
X_3	**000111**
X_4	**000101**
X_5	**000010**
X_6	**000011**
X_7	**000001**

4.3

Let's define $S = \{Z_i^a X_j^b \mid a, b \in \{0,1\}, i, j \in [1,7]\}$.

- When $a = 0, b = 0$, we get I, which is 1 operator.
- When $a = 1, b = 0$, we get Z_i, which gives us 7 operators.
- When $a = 0, b = 1$, we get X_j, which gives us 7 operators.
- When $a = 1, b = 1$, we get $Z_i X_j$, which gives us $7 \times 7 = 49$ operators.

The total number is $1 + 7 + 7 + 49 = \mathbf{64}$.

4.4

We have $n = 7$ qubits and $k = 1$ logical qubit, which means $n - k = 6$ stabilizers. The number of syndromes is $2^{n-k} = 2^6 = \mathbf{64}$.

4.5

I think the question has a typo. The set $Z_i^a X_j^b$ has 64 operators, which is not the same as the correctable set (must be ≤ 64). Let me assume the question meant single-qubit errors: $E_{1Q} = \{Z_i^a X_i^b \mid a, b \in \{0,1\}, i \in [1,7]\}$. This set is just $\{I, X_i, Z_i, Y_i\}$ and has $1 + 7 + 7 + 7 = 22$ errors. The Steane code is $[[7,1,3]]$, which means $d = 3$ and $t = 1$. So it can correct all single-qubit errors.

- $S(I) =$ '000000'
- $S(X_i)$ gives 7 unique non-zero syndromes (from part 4.2)
- $S(Z_i)$ gives 7 unique non-zero syndromes (from part 4.1)
- $S(Y_i) = S(X_i) + S(Z_i)$ gives 7 unique non-zero syndromes (for example, $S(Y_1) =$ '100100')

All 22 errors in E_{1Q} map to 22 unique syndromes. Because the code is non-degenerate, this one-to-one mapping means all of them can be corrected. "No more" means we cannot correct all weight-2 errors. For example, $S(X_1X_5) = S(X_1) + S(X_5) =$ '000100' + '000010' = '000110'. But this is the same as $S(X_2)$. The correction algorithm will see syndrome '000110' and apply X_2. If the actual error was X_1X_5, then applying X_2 gives $X_2X_1X_5$, which is a logical error. So the correction fails.

Note: This document was laid out by Gemini 2.5 based on hand-written homework.