Homework 8

C191A: Introduction to Quantum Computing, Fall 2025

Due: Monday, Nov. 3 2025, 10:00 pm

Instructions. Submit your homework to Gradescope (Entry Code: B3YK46) by 10:00 pm (Pacific time) on the due date listed above. No late submissions are accepted, since the solutions will be posted immediately after the deadline.

You are encouraged to collaborate with your peers on problem sets, as well as the usage of AI tools (e.g., ChatGPT, Bard, etc.) for learning purposes — such as asking for hints, clarifications, or alternative explanations — but not as a substitute for doing the problems yourself. If an AI system or a peer significantly helps you in your problem-solving process, you should acknowledge them in your submission (e.g., by listing their name or the tool you used on that problem).

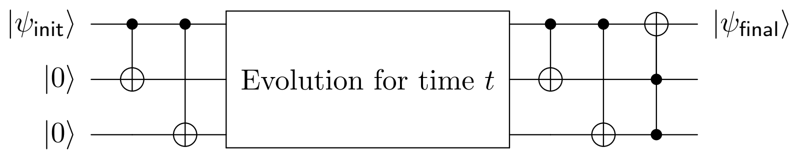
However, one problem per homework (the first) is labeled "solve individually", so that you can honestly gauge your grasp of the material. Ultimately, you are responsible for engaging with the coursework in the way that helps you learn most effectively.

1 The Repetition Code and Longitudinal Relaxation (Carried over from last week) (Solve Individually)

Suppose we have 3 identical physical qubits, that each experience exponential *longitudinal relaxation*, with lifetimes of $\mathsf{T}_1 = 1/\gamma$. What that means is that, as a function of time t, the probability of an $\mathbf{X}$ error (a bit-flip) being applied to a single qubit is $p(t) = 1 - e^{-\gamma t}$. For the sake of simplicity, we assume the relaxation process is symmetric w.r.t. $|0\rangle$ and $|1\rangle$.

Our goal is to protect a qubit $|\psi\rangle = |\psi_{\mathsf{init}}\rangle$ against this model of noise. For this purpose, we use the three-qubit bit-flip error correction code you learned about in lecture (see below). Subequently, we wait for time t, and then attempt to decode to obtain $|\psi_{\mathsf{final}}\rangle$.

Note: For the entire problem, you can assume perfect initial state preparation at the start of the circuit, and instantaneous and error free gates throughout the circuit.



(1)

Figure 1: A state $|\psi_{\mathsf{init}}\rangle$ is encoded into the 3-qubit repetition code. Then, a noise process is applied to each qubit independently. Finally, the code is decoded, resulting in the state $|\psi_{\mathsf{final}}\rangle$.

1.1

To lowest nonzero order, at approximately what wait time t would any single un-encoded physical qubit have a 1% probability of experiencing a bit-flip error?

Hint: "To lowest nonzero order", means, use the approximation $p(t) \approx \gamma t$.

1.2

To lowest nonzero order, at approximately what wait time t would the logical qubit (before decoding) have a 1% probability of experiencing a *physical* bit-flip error?

1.3

To lowest nonzero order, at approximately what wait time t will the corrected logical state (after decoding) $|\psi_{\mathsf{final}}\rangle$ have a 1% probability of having experienced a *logical* bit-flip error?

1.4

At what wait time t will the probability of a single physical qubit bit-flip error be equal to the probability of a bit-flip error on the decoded state $|\psi_{\mathsf{final}}\rangle$?

Note: since the probability is now large, the lowest non-zero approximation breaks down.

1.5

If we define the $\mathsf{T}_{1,\mathsf{L}}$ lifetime of the encoded *logical* qubit, in the same way it is defined for single qubits – meaning $\mathsf{T}_{1,\mathsf{L}}$ is the time at which the probability for $|\psi_{\mathsf{final}}\rangle$ to *not* have experienced a bit-flip is equal to $1/e$ – then is $\mathsf{T}_{1,\mathsf{L}}$ longer or shorter than the single qubit lifetime T_1?

2 Pauli Commutation Relations

This question will help you understand Pauli commutation relations, an important part of syndrome extraction analysis. First, we summarize a convenient and standard short-hand notation for n-qubit Pauli operators: subscript the non-trivial Pauli operators (i.e. X, Y, Z excluding I) with the qubit on which they act. For example on 5-qubits and indexing the first qubit at 1, $X_1 = X \otimes I \otimes I \otimes I \otimes I$, $X_1X_2 = X \otimes X \otimes I \otimes I \otimes I$, $X_2Y_4 = I \otimes X \otimes I \otimes Y \otimes I$, etc...

Recall that two operators, F_i and F_j, commute if $F_iF_j = F_jF_i$, and anti-commute if $F_iF_j = (-1)F_jF_i$. Consider a register of 8 qubits. Do the following pairs of operators commute or anti-commute? Show your work and/or explain your reasoning.

2.1

X_1, Y_1.

2.2

Z_1Z_2, Y_1Y_2.

2.3

$Z_1X_3Y_4$, $Y_1Z_2Y_4$.

2.4

$Z_1X_2Y_3Y_5Z_6X_7Y_8$, $X_1X_2Z_3Z_4Y_5Z_7X_8$.

3 9-qubit Shor code

This question will have you analyze properties of the 9-qubit Shor code. For convenience, its logical states and its stabilizers are provided below.

Logical states of the 9-qubit Shor code:

$$|0\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \quad (2)$$

$$|1\rangle_L = \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \quad (3)$$

Stabilizers of the 9-qubit Shor code:

$X_1X_2X_3X_4X_5X_6$	$X_4X_5X_6X_7X_8X_9$
Z_1Z_2	Z_2Z_3
Z_4Z_5	Z_5Z_6
Z_7Z_8	Z_8Z_9

Table 1: Stabilizers of the Shor code

3.1

Show that the 9-qubit Shor code is distance 3. That is, show that a three qubit error can take one codeword to the other codeword.

3.2

What are the syndromes of the following errors: Z_1, Z_2, Z_3?

3.3

What is the effect of the Z_1, Z_2, and Z_3 on the logical states of the Shor code? Provide a single correction operation that can correct each of the three individual error processes. This provides an example of why the Shor code is "degenerate;" some of the errors in the set of correctable errors map to the same syndrome string.

4 7-qubit Steane code

This question will have you investigate properties of the 7-qubit Steane code. For your convience, the Stabilizers are giving below in a useful geometric mnemonic in Figure 2. The stabilizers are denoted as $S_{x/z}^{(i)}$ for $i \in [1, 3]$ and there are 6 in total.

4.1

Tabulate the syndromes of all single qubit Z-type errors of the form Z_i for i in the range [1, 7]. Please use syndrome ordering so that the associated stabilizers appear in the order $S_x^{(1)}, S_x^{(2)}, S_x^{(3)}, S_z^{(1)}, S_z^{(2)}, S_z^{(2)}$

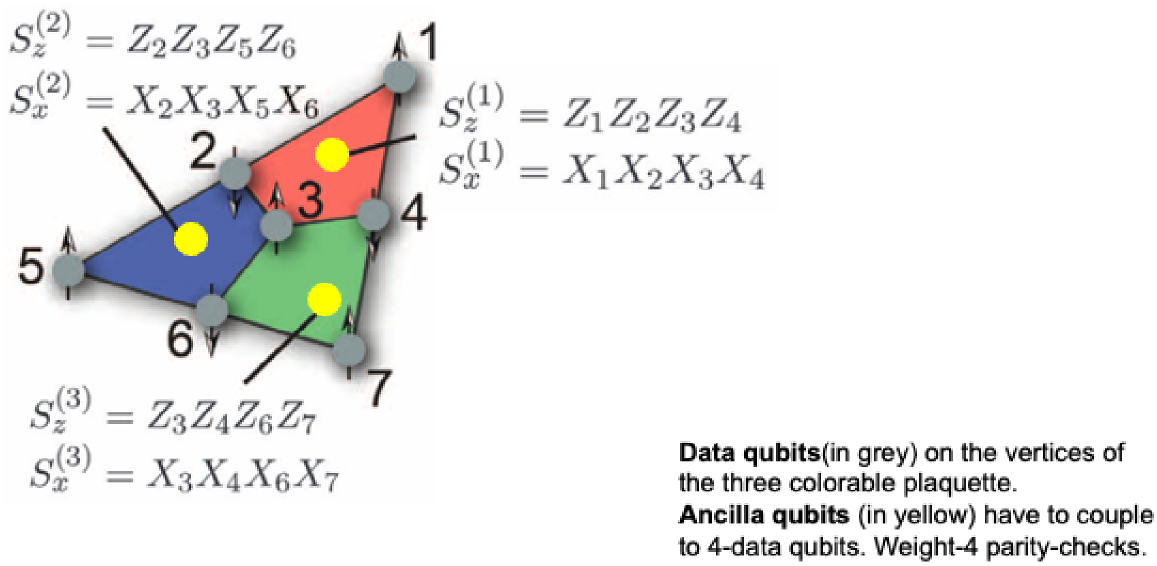


Figure 2: Geometric picture of the Steane code. This picture is particularly useful in extending the code to a large number of qubits via tessellation, which is studied in the theory of quantum color codes.

4.2

Tabulate the syndromes of all single qubit X-type errors of the form X_i for i in the range [1, 7]. Please use the same syndrome ordering as in part (a).

4.3

How many errors are there of the form

$$Z_i^a X_j^b$$

for $a \in \{0, 1\}, b \in \{0, 1\}, i \in [1, 7], j \in [1, 7]$?

4.4

How many unique syndromes are there in the Steane code?

4.5

Argue that the Steane code can correct all errors of the form

$$Z_i^a X_j^b$$

and no more (you may use the fact the Steane code is non-degenerate).