

热学: 第 1 次作业

Due on 2024.3.4

周欣 *Section A*

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Problem 1

道尔顿提出一种温标：规定理想气体体积的相对增量正比于温度的增量，在标准大气压下，规定水的冰点温度为零度，沸水温度为 100 度。试用摄氏度 t 来表示道尔顿温标的温度 τ

Solution

设大气压强为 P_{atm} , $T_0 = 273.15K$ 为摄氏 0 度, $T_{100} = 373.15K$ 为摄氏 100 度, t 为摄氏度, τ 为道尔顿温标的温度。 V_0 为理想气体在摄氏 0 度下的体积。

由题意可得：

$$\begin{aligned} P_{atm}V_0 &= \nu RT_0 \\ P_{atm}V_{100} &= \nu RT_{100} \end{aligned}$$

由定义：

$$\tau = \frac{V - V_0}{V_{100} - V_0} \times 100$$

并带入气体体积和摄氏度的关系：

$$V = \frac{\nu R(t + 273.15)}{P_{atm}}$$

得到道尔顿温度与摄氏温度的转化关系：

$$\begin{aligned} \tau &= \frac{\frac{\nu R(t+273.15)}{P_{atm}} - V_0}{V_{100} - V_0} \times 100 \\ &= \frac{\frac{\nu R(t+273.15)}{P_{atm}} - \nu RT_0}{\nu RT_{100} - \nu RT_0} \times 100 \\ &= \frac{t}{100} \times 100 = t \end{aligned}$$

Problem 2

国际实用温标 (1990 年) 规定: 用于 13.803 (平衡氢三相点) 到 961.78°C (银在 0.101MPa 下的凝固点) 的标准测量仪器是铂电阻温度计。设铂电阻在 0°C 及 °C 时电阻的值分别为 R_0 及 $R(t)$, 定义 $W(t) = R(t)/R_0$, 且在不同测温区内 $W(t)$ 对 t 的函数关系是不同的, 在上述测温范围内大致有 $W(t) = 1 + At + Bt^2$ 若在 0.101MPa 下, 对应于冰的熔点、水的沸点、硫的沸点 (温度为 444.67°C) 电阻的阻值分别为 11.000Ω、15.247Ω、28.887Ω, 试确定上式中的常数 A 和 B。(正确标注常数 A 和 B 的单位)

Solution

由题意可得:

$$W(0) = 1$$

$$W(100) = 1 + 100^\circ\text{C} \cdot A + 10000^\circ\text{C}^2 \cdot B$$

$$W(444.67) = 1 + 444.67^\circ\text{C} \cdot A + (444.67^\circ\text{C})^2 \cdot B$$

同时代入电阻的阻值:

$$11/11 = R_0/R_0 = 1$$

$$15.247/11 = R_{100}/R_0 = 1 + 100^\circ\text{C} \cdot A + 10000^\circ\text{C}^2 \cdot B$$

$$28.887/11 = R_{444.67}/R_0 = 1 + 444.67^\circ\text{C} \cdot A + (444.67^\circ\text{C})^2 \cdot B$$

得到 A、B、C 的解:

$$\begin{cases} A = 3.9201^\circ\text{C}^{-1} \\ B = -5.9205 \times 10^{-7}^\circ\text{C}^{-2} \end{cases}$$

Problem 3

Write part of **Quick-Sort**($list, start, end$)

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1: function QUICK-SORT( $list, start, end$ )
2:   if  $start \geq end$  then
3:     return
4:   end if
5:    $mid \leftarrow \text{PARTITION}(list, start, end)$ 
6:   QUICK-SORT( $list, start, mid - 1$ )
7:   QUICK-SORT( $list, mid + 1, end$ )
8: end function
```

Algorithm 1: Start of QuickSort

Problem 4

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \dots, n$, $E[e_i] = 0$, and $\text{Var}[e_i] = \sigma_e^2$ and $\text{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares estimator for $\hat{\beta}_1$ for the slope β_1 .

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^n (Y_i - \hat{Y}_i)^2 \\ &= \sum_{i=1}^n (Y_i - \hat{\beta}_1 x_i)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^n x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^n x_i Y_i - \sum_{i=1}^n \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^n x_i Y_i - \hat{\beta}_1 \sum_{i=1}^n x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1 :

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

Solution

For the bias, we need to calculate the expected value $E[\hat{\beta}_1]$:

$$\begin{aligned} E[\hat{\beta}_1] &= E\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i E[Y_i]}{\sum x_i^2} \\ &= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\ &= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\ &= \beta_1 \end{aligned}$$

Thus since our estimator's expected value is β_1 , we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned} \text{Var}[\hat{\beta}_1] &= \text{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \text{Var}[Y_i] \\ &= \frac{1}{\sum x_i^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x_i^2} \end{aligned}$$

Problem 5

Prove a polynomial of degree k , $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \dots a_0$ are nonnegative constants.

证明. To prove that $a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0, c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, c_1 and a_k .

Taking the second inequality, we prove it in the following way. By summation, $\sum_{i=0}^k a_i$ will give us a new constant, A . By taking this value of A , we can then do the following:

$$\begin{aligned} a_k n^k + a_{k-1} n^{k-1} + \dots + a_1 n^1 + a_0 n^0 &= \\ &\leq (a_k + a_{k-1} \dots a_1 + a_0) \cdot n^k \\ &= A \cdot n^k \\ &\leq c_2 \cdot n^k \end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. □

Problem 18

Evaluate $\sum_{k=1}^5 k^2$ and $\sum_{k=1}^5 (k-1)^2$.

Problem 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem 6

Evaluate the integrals $\int_0^1 (1-x^2)dx$ and $\int_1^\infty \frac{1}{x^2}dx$.