热学: 第 1 次作业

Due on 2024.3.4

周欣 *Section A*

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Problem 1

道尔顿提出一种温标：规定理想气体体积的相对增量正比于温度的增量，在标准大气压下，规定水的冰点温度为零度，沸水温度为 100 度。试用摄氏度 t 来表示道尔顿温标的温度 τ

Solution

设大气压强为 P_{atm},$T_0 = 273.15K$ 为摄氏 0 度，$T_{100} = 373.15K$ 为摄氏 100 度，t 为摄氏度，τ 为道尔顿温标的温度。V_0 为理想气体在摄氏 0 度下的体积。

由题意可得：

$$P_{atm}V_0 = \nu RT_0$$
$$P_{atm}V_{100} = \nu RT_{100}$$

由定义：

$$\tau = \frac{V - V_0}{V_{100} - V_0} \times 100$$

并带入气体体积和摄氏度的关系：

$$V = \frac{\nu R(t + 273.15)}{P_{atm}}$$

得到道尔顿温度与摄氏温度的转化关系：

$$\begin{aligned}
\tau &= \frac{\frac{\nu R(t+273.15)}{P_{atm}} - V_0}{V_{100} - V_0} \times 100 \\
&= \frac{\frac{\nu R(t+273.15)}{P_{atm}} - \nu RT_0}{\nu RT_{100} - \nu RT_0} \times 100 \\
&= \frac{t}{100} \times 100 = t
\end{aligned}$$

Problem 2

国际实用温标（1990 年）规定：用于 13.803 (平衡氢三相点) 到 961.78°C（银在 0.101MPa 下的凝固点）的标准测量仪器是铂电阻温度计。设铂电阻在 0°C 及 °C 时电阻的值分别为 R_0 及 $R(t)$，定义 $W(t) = R(t)/R_0$，且在不同测温区内 $W(t)$ 对 t 的函数关系是不同的，在上述测温范围内大致有 $W(t) = 1 + At + Bt^2$ 若在 0.101MPa 下，对应于冰的熔点、水的沸点、硫的沸点（温度为 444.67°C）电阻的阻值分别为 11.000Ω、15.247Ω、28.887Ω，试确定上式中的常数 A 和 B。（正确标注常数 A 和 B 的单位）

Solution

由题意可得：

$$\begin{aligned} W(0) &= 1 \\ W(100) &= 1 + 100^\circ C \cdot A + 10000^\circ C^2 \cdot B \\ W(444.67) &= 1 + 444.67^\circ C \cdot A + (444.67^\circ C)^2 \cdot B \end{aligned}$$

同时代入电阻的阻值：

$$\begin{aligned} 11/11 &= R_0/R_0 = 1 \\ 15.247/11 &= R_{100}/R_0 = 1 + 100^\circ C \cdot A + 10000^\circ C^2 \cdot B \\ 28.887/11 &= R_0 = 1 + 444.67^\circ C \cdot A + (444.67^\circ C)^2 \cdot B \end{aligned}$$

得到 A、B、C 的解：

$$\begin{cases} A = 3.9201^\circ C^{-1} \\ B = -5.9205 \times 10^{-7^\circ} C^{-2} \end{cases}$$

Problem 3

Write part of **Quick-Sort(**$list, start, end$**)**

function QUICK-SORT($list, start, end$)
    if $start \geq end$ **then**
        return
    end if
    $mid \leftarrow$ PARTITION($list, start, end$)
    QUICK-SORT($list, start, mid - 1$)
    QUICK-SORT($list, mid + 1, end$)
end function

Algorithm 1: Start of QuickSort

Problem 4

Suppose we would like to fit a straight line through the origin, i.e., $Y_i = \beta_1 x_i + e_i$ with $i = 1, \ldots, n$, $\mathrm{E}[e_i] = 0$, and $\mathrm{Var}[e_i] = \sigma_e^2$ and $\mathrm{Cov}[e_i, e_j] = 0, \forall i \neq j$.

Part A

Find the least squares esimator for $\hat{\beta}_1$ for the slope β_1.

Solution

To find the least squares estimator, we should minimize our Residual Sum of Squares, RSS:

$$\begin{aligned} RSS &= \sum_{i=1}^{n} \left(Y_i - \hat{Y}_i\right)^2 \\ &= \sum_{i=1}^{n} \left(Y_i - \hat{\beta}_1 x_i\right)^2 \end{aligned}$$

By taking the partial derivative in respect to $\hat{\beta}_1$, we get:

$$\frac{\partial}{\partial \hat{\beta}_1}(RSS) = -2 \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) = 0$$

This gives us:

$$\begin{aligned} \sum_{i=1}^{n} x_i (Y_i - \hat{\beta}_1 x_i) &= \sum_{i=1}^{n} x_i Y_i - \sum_{i=1}^{n} \hat{\beta}_1 x_i^2 \\ &= \sum_{i=1}^{n} x_i Y_i - \hat{\beta}_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Solving for $\hat{\beta}_1$ gives the final estimator for β_1:

$$\hat{\beta}_1 = \frac{\sum x_i Y_i}{\sum x_i^2}$$

Problem 4 continued on next page...

Part B

Calculate the bias and the variance for the estimated slope $\hat{\beta}_1$.

Solution

For the bias, we need to calculate the expected value $\mathrm{E}[\hat{\beta}_1]$:

$$\begin{aligned}
\mathrm{E}[\hat{\beta}_1] &= \mathrm{E}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i \mathrm{E}[Y_i]}{\sum x_i^2} \\
&= \frac{\sum x_i (\beta_1 x_i)}{\sum x_i^2} \\
&= \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1 \frac{\sum x_i^2 \beta_1}{\sum x_i^2} \\
&= \beta_1
\end{aligned}$$

Thus since our estimator's expected value is β_1, we can conclude that the bias of our estimator is 0.

For the variance:

$$\begin{aligned}
\mathrm{Var}[\hat{\beta}_1] &= \mathrm{Var}\left[\frac{\sum x_i Y_i}{\sum x_i^2}\right] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{\sum x_i^2}{\sum x_i^2 \sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \mathrm{Var}[Y_i] \\
&= \frac{1}{\sum x_i^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum x_i^2}
\end{aligned}$$

Problem 5

Prove a polynomial of degree k, $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ is a member of $\Theta(n^k)$ where $a_k \ldots a_0$ are nonnegative constants.

证明. To prove that $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$, we must show the following:

$$\exists c_1 \exists c_2 \forall n \geq n_0,\ c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

For the first inequality, it is easy to see that it holds because no matter what the constants are, $n^k \leq a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0$ even if $c_1 = 1$ and $n_0 = 1$. This is because $n^k \leq c_1 \cdot a_k n^k$ for any nonnegative constant, c_1 and a_k.

Taking the second inequality, we prove it in the following way. By summation, $\sum\limits_{i=0}^{k} a_i$ will give us a new constant, A. By taking this value of A, we can then do the following:

$$\begin{aligned}
a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n^1 + a_0 n^0 = \\
&\leq (a_k + a_{k-1} \ldots a_1 + a_0) \cdot n^k \\
&= A \cdot n^k \\
&\leq c_2 \cdot n^k
\end{aligned}$$

where $n_0 = 1$ and $c_2 = A$. c_2 is just a constant. Thus the proof is complete. □

Problem 18

Evaluate $\sum_{k=1}^{5} k^2$ and $\sum_{k=1}^{5}(k-1)^2$.

Problem 19

Find the derivative of $f(x) = x^4 + 3x^2 - 2$

Problem 6

Evaluate the integrals $\int_0^1 (1 - x^2)\mathrm{d}x$ and $\int_1^\infty \frac{1}{x^2}\mathrm{d}x$.