Zee QFT Chapter VI.8 Problems 1 and 2

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November 3, 2025

Problem VI.8.1

Problem: Show that the solution of $dg/dt = -bg^3 + \cdots$ is given by

$$\frac{1}{\alpha(t)} = \frac{1}{\alpha(0)} + 8\pi bt + \cdots$$

where we defined $\alpha(t) = g(t)^2/4\pi$.

Solution:
Given $\alpha(t) = g^2/4\pi$, differentiate with respect to t:

$$\frac{d\alpha}{dt} = \frac{g}{2\pi}\frac{dg}{dt} = \frac{g}{2\pi}(-bg^3) = -\frac{bg^4}{2\pi} \tag{1}$$

Since $g^4 = (4\pi\alpha)^2 = 16\pi^2\alpha^2$:

$$\frac{d\alpha}{dt} = -8\pi b\alpha^2 \tag{2}$$

Separating variables and integrating:

$$\int_{\alpha(0)}^{\alpha(t)} \frac{d\alpha'}{\alpha'^2} = -8\pi b\int_0^t dt' \quad \Rightarrow \quad -\frac{1}{\alpha(t)} + \frac{1}{\alpha(0)} = -8\pi bt \tag{3}$$

Therefore:

$$\boxed{\frac{1}{\alpha(t)} = \frac{1}{\alpha(0)} + 8\pi bt + \cdots} \tag{4}$$

Interpretation: For $b > 0$, α decreases with t (**asymptotic freedom**); for $b < 0$, α grows with t.

Problem VI.8.2

Problem: Study the threshold effect when the renormalization scale μ crosses a particle mass m. In the crude approximation, particles with $m < \mu$ contribute to RG flow (set $m = 0$) while those with $m > \mu$ decouple (set $m = \infty$). Investigate how particles actually decouple smoothly as μ decreases below m.

Solution (QED):

Beta Function with Threshold Effects

For QED with N_f fermions of mass m_i, the beta function is:

$$\beta(e) = \frac{e^3}{12\pi^2} \sum_{i=1}^{N_f} f(m_i/\mu) \tag{5}$$

The threshold function $f(r)$ with $r = m/\mu$ satisfies:

$$f(r) \to \begin{cases} 1 & r \to 0 \quad (\mu \gg m) \\ 0 & r \to \infty \quad (\mu \ll m) \end{cases} \tag{6}$$

Explicit Form

From vacuum polarization calculations:

$$f(r) \approx \begin{cases} 1 - \frac{10}{3}r^2 + \mathcal{O}(r^4) & r \ll 1 \\ \text{smooth drop to 0} & r \sim \frac{1}{2} \\ 0 & r > \frac{1}{2} \end{cases} \tag{7}$$

Physical Regimes

- **High energy** ($\mu \gg m$): Fermion effectively massless, full contribution to vacuum polarization, $f \approx 1$.
- **Threshold** ($\mu \sim 2m$): Pair production becomes kinematically suppressed, smooth transition.
- **Low energy** ($\mu \ll m$): Fermion decouples, virtual pairs cannot be produced, $f \approx 0$ (**decoupling theorem**).

Crude vs. Smooth Approximation

Crude: $f(r) = \Theta(1/r - 1)$ (step function at $\mu = m$)

Reality: Smooth interpolation with characteristic width $\Delta\mu \sim m$ around threshold.

Running Coupling

For QED with leptons (m_e, m_μ, m_τ):

$$\frac{d\alpha}{d\ln\mu} = \frac{\alpha^2}{3\pi}\sum_{\ell=e,\mu,\tau} f(m_\ell/\mu) \tag{8}$$

In crude approximation between thresholds:

$$\mu < m_e : \quad \beta = 0 \tag{9}$$

$$m_e < \mu < m_\mu : \quad \beta = \frac{\alpha^2}{3\pi} \tag{10}$$

$$m_\mu < \mu < m_\tau : \quad \beta = \frac{2\alpha^2}{3\pi} \tag{11}$$

$$\mu > m_\tau : \quad \beta = \frac{\alpha^2}{\pi} \tag{12}$$

QCD Application

Critical for strong coupling $\alpha_s(\mu)$. The number of active quark flavors changes at mass thresholds:

$$\beta_{\text{QCD}} = -\frac{\alpha_s^2}{2\pi}\left(\frac{33 - 2N_f}{12}\right) \tag{13}$$

N_f changes: 6 flavors at $\mu > m_t$, down to 3 flavors (u, d, s) at $\mu \sim 1$ GeV. Each threshold modifies the running.

Matching Condition

At threshold $\mu = m_i$, continuity requires:

$$\alpha^{(N_f)}(m_i^-) = \alpha^{(N_f-1)}(m_i^+) + \mathcal{O}(\alpha^2) \tag{14}$$

Higher-order corrections come from threshold loop diagrams.