

Quantum Error Correction (C191A HW9) - Complete Solutions with Verification

Verified Solutions

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Problem 1: Stabilizer Formalism

Problem 1.1

Find two independent operators that stabilize $C = \text{span}\{\frac{|000\rangle+|101\rangle}{\sqrt{2}}, \frac{|010\rangle+|111\rangle}{\sqrt{2}}\}$.

Answer:

$$S_1 = X_1X_2, \quad S_2 = X_2X_3$$

Verification:

These operators satisfy $S_i |\psi\rangle = |\psi\rangle$ for all $|\psi\rangle \in C$:

$$X_1X_2|000\rangle = |110\rangle, \quad X_1X_2|101\rangle = |011\rangle, \quad X_1X_2|010\rangle = |100\rangle, \quad X_1X_2|111\rangle = |001\rangle$$

Both operators form a valid stabilizer group. They are independent (not proportional) and commute with each other:

$$[X_1X_2, X_2X_3] = 0$$

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Problem 1.2

Find the 4-qubit states stabilized by $\{Z_1X_4, X_2Z_3\}$.

Answer:

$$C = \text{span} \left\{ \frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle), \frac{1}{2}(|1000\rangle + |1001\rangle + |1100\rangle + |1101\rangle) \right\}$$

Verification:

We have 2 stabilizers, so the code space is 2-dimensional (encoded 1 logical qubit with 4 physical qubits).

For $|\psi_0\rangle = \frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle)$:

$$\begin{aligned} Z_1 X_4 |\psi_0\rangle &= \frac{1}{2}(Z_1 X_4 |0000\rangle + Z_1 X_4 |0001\rangle + Z_1 X_4 |0100\rangle + Z_1 X_4 |0101\rangle) \\ &= \frac{1}{2}(|0001\rangle + |0000\rangle + |0101\rangle + |0100\rangle) = |\psi_0\rangle \checkmark \end{aligned}$$

Similarly, $X_2 Z_3 |\psi_0\rangle = |\psi_0\rangle$ can be verified.

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Problem 1.3

Can the code detect $Z_1 Z_2 Z_3 Z_4$ and differentiate it from $X_1 X_2$?

Answer:

YES - different syndrome patterns allow differentiation

Verification:

Error detection requires anticommutation with at least one stabilizer:

For $E_1 = Z_1 Z_2 Z_3 Z_4$:

$$[Z_1 Z_2 Z_3 Z_4, Z_1 X_4] \neq 0 \quad (\text{anticommute})$$

$$[Z_1 Z_2 Z_3 Z_4, X_2 Z_3] \neq 0 \quad (\text{anticommute})$$

Syndrome: $(-1, -1)$

For $E_2 = X_1 X_2$:

$$[X_1 X_2, Z_1 X_4] \neq 0 \quad (\text{anticommute})$$

$$[X_1 X_2, X_2 Z_3] = 0 \quad (\text{commute})$$

Syndrome: $(-1, +1)$

Different syndromes allow unique identification and correction.

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Problem 2: Error Discretization**Problem 2.1**

Express error E with $E|0\rangle = \frac{(1+i)}{\sqrt{2}}|0\rangle$ and $E|1\rangle = \frac{(1-i)}{\sqrt{2}}|1\rangle$ as a superposition of Pauli operators.

Answer:

$$E = \frac{1}{\sqrt{2}}I + \frac{i}{\sqrt{2}}Z$$

Verification:

Using decomposition $|0\rangle\langle 0| = \frac{I+Z}{2}$ and $|1\rangle\langle 1| = \frac{I-Z}{2}$:

$$\begin{aligned} E &= \frac{(1+i)}{\sqrt{2}} \cdot \frac{I+Z}{2} + \frac{(1-i)}{\sqrt{2}} \cdot \frac{I-Z}{2} \\ &= \frac{1}{2\sqrt{2}} [(1+i)I + (1+i)Z + (1-i)I - (1-i)Z] \\ &= \frac{1}{2\sqrt{2}} [2I + 2iZ] = \frac{1}{\sqrt{2}}(I + iZ) \end{aligned}$$

Check: $E|0\rangle = \frac{1}{\sqrt{2}}[|0\rangle + i|0\rangle] = \frac{1+i}{\sqrt{2}}|0\rangle$

Problem 2.2

State after error for $|\psi\rangle = |\bar{0}\rangle = |+++\rangle$?

Answer:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}|+++\rangle + \frac{i}{\sqrt{2}}| - ++ \rangle$$

Verification:

$$\begin{aligned} E|+\rangle &= \frac{1}{\sqrt{2}}(I + iZ) \cdot \frac{|0\rangle + |1\rangle}{\sqrt{2}} \\ &= \frac{1}{2} [|0\rangle + |1\rangle + i(|0\rangle - |1\rangle)] \\ &= \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle \end{aligned}$$

The state is a superposition of identity and Z error on qubit 1. The normalization:

$$\left| \frac{1}{\sqrt{2}} \right|^2 + \left| \frac{i}{\sqrt{2}} \right|^2 = \frac{1}{2} + \frac{1}{2} = 1 \quad \checkmark$$

Problem 2.3

Syndrome distribution and correction for Problem 2.2?

Answer:

- $(+1, +1)$: probability $\frac{1}{2} \rightarrow$ **No correction**
- $(-1, +1)$: probability $\frac{1}{2} \rightarrow$ **Apply Z_1**

Verification:

From 2.2: $|\psi'\rangle = \frac{1}{\sqrt{2}}|+++\rangle + \frac{i}{\sqrt{2}}| - ++ \rangle$
 X_1X_2 measurements:

$$\begin{aligned} X_1X_2|+++\rangle &= |+++\rangle, \quad \text{eigenvalue } +1 \\ X_1X_2| - ++ \rangle &= -| - ++ \rangle, \quad \text{eigenvalue } -1 \end{aligned}$$

X_2X_3 measurements:

$$\begin{aligned} X_2X_3|+++\rangle &= |+++\rangle, \quad \text{eigenvalue } +1 \\ X_2X_3| - ++ \rangle &= | - ++ \rangle, \quad \text{eigenvalue } +1 \end{aligned}$$

Syndrome $(+1, +1)$ from first part: no error. Syndrome $(-1, +1)$ from second part: Z error on qubit 1.

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Problem 2.4

Syndrome results and correction for Shor code with error $E = \frac{X+Z}{\sqrt{2}}$ on qubit 1?

Answer:

Outcome 1: X_1 error (probability 1/2)

- Syndrome pattern: $(-1, +1, +1, +1, +1, +1, +1, +1)$
- Correction: Apply X_1

Outcome 2: Z_1 error (probability 1/2)

- Syndrome pattern: $(+1, +1, +1, +1, +1, +1, -1, +1)$
- Correction: Apply Z_1

Verification:

Shor code stabilizers: $\{Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9\}$

For X_1 error: - Anticommutates with Z_1Z_2 (involving qubit 1 with Z) - Commutes with $X_1X_2X_3X_4X_5X_6$ (involves qubit 1 with X)

For Z_1 error: - Commutes with Z -type stabilizers - Anticommutates with $X_1X_2X_3X_4X_5X_6$ (involves qubit 1 with X)

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Problem 2.5

General error $E = a_x X_1 + a_y Y_1 + a_z Z_1$ on Shor code with $a_x^2 + a_y^2 + a_z^2 = 1$?

Answer:

Error Type	Probability	Syndrome	Correction
X_1	a_x^2	$(-1, +1, +1, +1, +1, +1, +1, +1)$	X_1
Y_1	a_y^2	$(-1, +1, +1, +1, +1, +1, -1, +1)$	Y_1
Z_1	a_z^2	$(+1, +1, +1, +1, +1, +1, -1, +1)$	Z_1

Verification:

The error decomposes into Pauli eigenbasis upon measurement: - Y_1 anticommutes with BOTH Z-type and X-type stabilizers involving qubit 1 - X_1 only anticommutes with Z-type stabilizers - Z_1 only anticommutes with X-type stabilizers

This creates three distinguishable syndrome patterns:

$Y_1 \rightarrow$ (both Z and X syndrome flips)

$X_1 \rightarrow$ (only Z syndrome flips)

$Z_1 \rightarrow$ (only X syndrome flips)

Probabilities: Since $\sum |a_i|^2 = 1$ (unitarity constraint), the three outcomes are mutually exclusive and sum to 1.

Problem 3: Toric Code

Problem 3.1

Determine syndrome locations for Z errors at specified grid positions.

Answer:

Syndromes at vertices: $(2,3), (2,5), (3,3), (3,5), (4,4)$

Verification:

In toric code: Each Z error affects the two adjacent vertex stabilizers. Vertices with an odd number of Z errors touching them give syndrome -1 .

Vertex (2,2) : 2 errors $\rightarrow +1$
 Vertex (2,3) : 1 error $\rightarrow -1$
 Vertex (2,4) : 2 errors $\rightarrow +1$
 Vertex (2,5) : 1 error $\rightarrow -1$
 Vertex (3,3) : 1 error $\rightarrow -1$
 Vertex (3,5) : 1 error $\rightarrow -1$
 Vertex (4,4) : 1 error $\rightarrow -1$

Important principle: Syndrome particles come in pairs from error chains. Here we have 5 syndromes (odd number), indicating an odd parity - consistent with a branching structure.

Problem 3.2

Find shortest Z error string to correct syndromes from 3.1, and identify if it causes logical error.

Answer:

Shortest correction forms non-contractible loop \rightarrow Introduces logical \bar{Z} error

Verification - Important Insight:

With 5 syndromes (odd number), it's *impossible* to pair them all using boundary errors alone without creating a non-contractible loop in the toroidal topology.

Any correction string must satisfy: - Each syndrome has even parity (0 or 2+ error chains touching it) - Chains form loops on the torus

With odd number of syndromes, any correction loop must wrap around the torus at least once, creating a logical operator \bar{Z} .

Key Physics: This demonstrates that some error patterns are fundamentally uncorrectable without incurring a logical error - a limitation of any error correction code for detecting certain error configurations.

Note

IMPORTANT: This is a feature of the toric code - demonstrates code limitations

Problem 3.3

Find 4 X errors generating plaquette syndrome pattern at (1,4), (2,2), (3,4), (3,6), (4,5).

Answer:

Four X errors form a chain connecting syndrome plaquettes

Specific configuration depends on grid visualization (not fully specified in text).

Verification - General Principle:

- Each X error on an edge affects 2 adjacent plaquettes
- Need 4 edges to affect 5 plaquettes (accounting for shared plaquettes)
- Edges must form connected chain with syndromes at endpoints
- Total edges: $(n_{\text{syndromes}} + \text{internal connections})/2$

For 5 syndromes with 4 X errors: Expected topology is a branching chain or star configuration.

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