Practice Problems for Midterm 2

C191 Introduction to Quantum Computing, Fall 2025

1 Mixed States

1.1

Consider the quantum states $\rho = |+\rangle\langle+|$ and $\sigma = I/2$. Construct a (projective) measurement that produces different outcomes on ρ and σ.

1.2

Construct a purification for the mixed state $\rho = \frac{1}{14}|00\rangle\langle 00| + \frac{5}{14}|01\rangle\langle 01| + \frac{4}{7}|11\rangle\langle 11|$. That is, construct a pure state $|\psi\rangle_{AB}$ such that

$$\mathrm{Tr}_B\left[|\psi\rangle\langle\psi|\right] = \rho.$$

2 Decoherence and the Density Matrix.

Consider a qubit in the state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Suppose that some external physical process can apply a single Z gate at some time $t > 0$, but does so in a probabilistic manner. The probability distribution for having applied the Z gate by time t is given by $p(t) = \frac{1}{2} - \frac{1}{2}e^{-t/\tau}$.

2.1

Write down the density matrix $\rho(t)$ for the qubit at time $t \geq 0$, specifying all elements of the matrix.

2.2

Write down the expectation values $\langle X\rangle$ and $\langle Z\rangle$ for all times t.

3 The SWAP test.

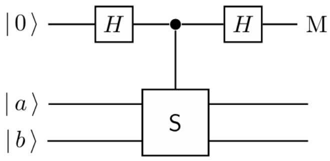
The swap gate S on two qubits is defined first on product vectors, $S : |a\rangle \otimes |b\rangle \mapsto |b\rangle \otimes |a\rangle$ and then extended to sums of product vectors by linearity.

3.1

Show that $P_{\pm} = \frac{1}{2}(\mathbb{1} \pm S)$ are two orthogonal projectors (i.e. $P_+^2 = P_+$, $P_-^2 = P_-$ and $P_+P_- = 0$.

Consider the following "swap-test" quantum circuit composed of two Hadamard gates, one controlled S operation and the measurement M in the computational basis,

The state vectors $|a\rangle$ and $|b\rangle$ of the target qubits are normalised but not orthogonal to each other.



3.2

Step through the execution of this circuit, writing down quantum states of the three qubits after each computational step. What are the probabilities of observing 0 or 1 when the measurement M is performed?

3.3

As the orthogonal projectors P_+ and P_- add to the identity, any vector decomposes into a sum of two vectors in the two subspaces V_+ and V_- onto which P_+ and P_- project. Verify that after the whole circuit above and measuring the first qubit as 0 or 1, the second and third register collapse to the normalized components of $|a\rangle \otimes |b\rangle$ in the two subspaces V_+ and V_-.

3.4

Does the measurement result $M = 0$ imply that $|a\rangle$ and $|b\rangle$ are identical? Does the measurement result $M = 1$ imply that $|a\rangle$ and $|b\rangle$ are not identical?

3.5

Suppose an efficient quantum algorithm encodes information about a complicated graph into a pure state of a qubit. Graphs which are isomorphic are mapped into the same state of the qubit. Given two complicated graphs your task is to check if they are isomorphic. You can run the algorithm as many times as you want and you can use the "swap-test" circuit. How would you accomplish this task?

3.6

Show that $\mathrm{Tr}\,(S(\rho_a \otimes \rho_b)) = \mathrm{Tr}(\rho_a\rho_b)$.

3.7

Instead of the state $|a\rangle \otimes |b\rangle$ the two target qubits are prepared in some mixed state $\rho_a \otimes \rho_b$. Show that the probability of getting the measurement outcome 0 is:

$$\frac{1}{2}\left(1 + \mathrm{Tr}\,\rho_a\rho_b\right)$$

3.8

Does the measurement result $M = 1$ imply that ρ_a and ρ_b are not identical?

4 Quantum error correction I

4.1

What is the maximum number of (mutually) independent stabilizers one can find on n qubits? Why?

4.2

Consider the stabilizers $\{Z_1 Z_2, Z_2 Z_3\}$ on 3 qubits. Construct a basis for the code subspace corresponding to these stabilizers.

4.3

Consider the stabilizer generators for a 7-qubit code

$$\begin{aligned} g_1 &= IIIXXXX \\ g_2 &= IXXIIXX \\ g_3 &= XIXIXIX \\ g_4 &= IIIZZZZ \\ g_5 &= IZZIIZZ \\ g_6 &= ZIZIZIZ \end{aligned}$$

What will the results of the syndrome measurements be if the error $Z_1 X_6$ happens?

4.4

Construct two undetectable Pauli errors for the above 7-qubit code that mutually anticommute (i.e., construct a logical X and logical Z for the code subspace). (Hint: Every stabilizer has an even number of Paulis and there is an odd number of qubits).

5 Quantum Error Correction II

5.1

Find a stabilizer for $|-\rangle$ and two independent stabilizers for $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

5.2

The 5-qubit code is defined via the following set of independent stabilizer generators:

$$g_1 = XZZXI, \quad g_2 = IXZZX, \quad g_3 = XIXZZ, \quad g_4 = ZXIXZ$$

Let $|\bar{\psi}\rangle$ be a state encoded in the 5-qubit code. What are the syndromes for the state $X_2 Z_3 |\bar{\psi}\rangle$?

5.3

Find a set of 2 independent stabilizer generators for the following stabilizer code: $\{\alpha |000\rangle + \beta |111\rangle : \alpha, \beta \in \mathbb{C}\}$.

5.4

Define a basis for the above code as $|\bar{0}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, $|\bar{1}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$. Find a logical Z gate for this code, i.e. a 3-qubit gate G such that $G(\alpha |\bar{0}\rangle + \beta |\bar{1}\rangle) = \alpha |\bar{0}\rangle - \beta |\bar{1}\rangle$. Also find a logical X gate for this code.

6 Quantum Error Correction III

Consider the four-qubit code (the [[4, 1, 2]] code) defined by the stabilizer generators

$$g_1 = XXXX, \quad g_2 = ZIZI, \quad g_3 = IZIZ.$$

6.1

Show that $|0_L\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ and $|1_L\rangle = \frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)$ are in the codespace.

The next parts of this question mainly focus on the analysis of Pauli X error, but similar arguments apply to other Pauli errors.

6.2

What are the resulting states starting from $|0_L\rangle$ and $|1_L\rangle$ if a Pauli X error occurred on the first qubit? What are the results of the syndrome measurements if this happened?

6.3

What are the resulting states starting from $|0_L\rangle$ and $|1_L\rangle$ if a Pauli X error occurred on the third qubit? What are the results of the syndrome measurements if this happened?

6.4

Write down a valid logical X (logical bitflip) operator for this code. Use this to show that the two pairs of states obtained from part (2) and (3) are related by a logical X error.

6.5

Argue why it is impossible for this four-qubit code to correct every possible single-qubit Pauli X error.

6.6

We have seen in class that at least five physical qubits are needed in order to have a quantum code that is able to correct every single-qubit Pauli error. Now let's prove this in a simpler situation, where the quantum code is non-degenerate, i.e. different noise operators in the set of correctable errors would bring the codewords to *orthogonal* error subspaces. In other words, $\langle i_L|E_l^\dagger E_k|i_L\rangle = 0$ for any different correctable error operators E_k and E_l, and any state $|i_L\rangle$ in the codespace. Now prove that in order for a quantum code encoding one logical qubit to be able to correct any single-qubit Pauli errors (X, Y, Z on any qubit), at least five physical qubits are needed.

(**Hint:** Use a dimension-counting argument to compare the dimensions available with n qubits and the dimensions needed to form orthogonal error subspaces.)

7 Quantum Fourier Transform

Recall that the Quantum Fourier Transform is an $N \times N$ matrix ($N = 2^n$ for n qubits) QFT_N where the entry at ith row and jth column ($i, j = 0, \ldots, N-1$) equals $\omega_N^{i \cdot j}/\sqrt{N}$ ($\omega_N = e^{2\pi i/N}$).

7.1

Show that $\mathrm{QFT}_2 = H$.

7.2

Does $\mathrm{QFT}_{2^n} |0^n\rangle = H^{\otimes n} |0^n\rangle$ when $n > 1$?

7.3

Does $\mathrm{QFT}_{2^n} = H^{\otimes n}$ when $n > 1$?

7.4

Show that QFT_N is a unitary.

7.5

Let $|\psi\rangle = \sum_{k=0}^{N-1} \alpha_k |k\rangle$ be a state. Suppose $\mathrm{QFT}_N |\psi\rangle = \sum_{k=0}^{N-1} \beta_k |k\rangle$. Show that the coefficients satisfy

$$\beta_k = \sum_{\ell} \alpha_\ell \frac{\omega_N^{k\ell}}{\sqrt{N}}.$$

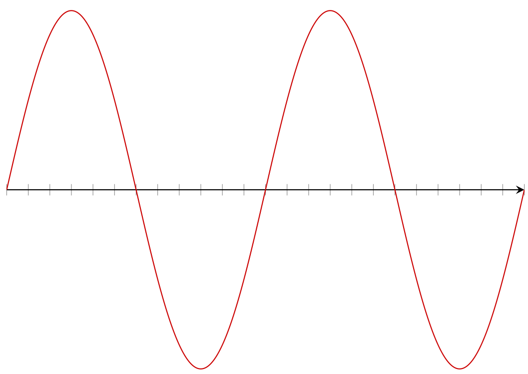
7.6

Let $|\psi + t\rangle = \sum_{k=0}^{N-1} \alpha_k |(k+t) \mod N\rangle$ be a shifted version of $|\psi\rangle$ ($t > 0$ is an integer). Show that

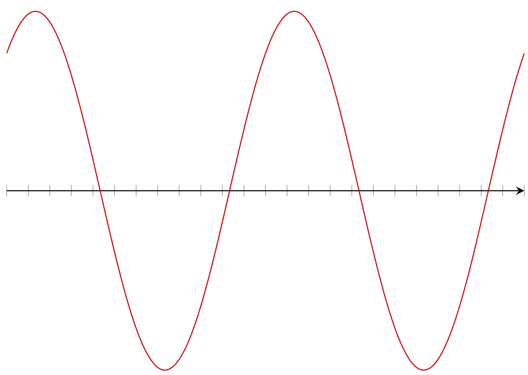
$$\mathrm{QFT}_N |\psi + t\rangle = \sum_{k=0}^{N-1} \omega_N^{kt} \beta_k |k\rangle .$$

Conclude that $\mathrm{QFT}_N |\psi\rangle$ and $\mathrm{QFT}_N |\psi + t\rangle$ generate the same measurement outcome distribution when measuring in the standard basis $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$.

The result above says that if the state $|\psi\rangle$ is like the following:



and the state $|\psi = t\rangle$ is a shifted version of $|\psi\rangle$:



then performing a Fourier transform on the two states and measuring in the computational basis will result in the same distribution of outcomes.

7.7

Show that if $|\psi\rangle$'s amplitudes are periodic with period t (i.e., $\alpha_k = \alpha_{(k+t) \mod N}$), then it is equal to the state $|\psi + t\rangle$:

$$|\psi + t\rangle = \sum_{k=0}^{N-1} \alpha_k \, |(k+t) \mod N\rangle \, .$$

7.8

Recall the N'th root of unity $\omega_N = e^{i\frac{2\pi}{N}}$. Show that $\omega_N^r = 1$ if and only if r is an integer multiple of N.

7.9

Recall (from part 7.6) that the quantum Fourier transforms of $|\psi\rangle$ and $|\psi + t\rangle$ are related in the following way. If

$$\mathrm{QFT}_N \, |\psi\rangle = \sum_{k=0}^{N-1} \beta_k \, |k\rangle \, ,$$

then

$$\mathrm{QFT}_N \, |\psi + t\rangle = \sum_{k=0}^{N-1} \omega_N^{kt} \beta_k \, |k\rangle \, .$$

Show that if $|\psi\rangle = |\psi + t\rangle$, then $\mathrm{QFT}_N \, |\psi\rangle$ has nonzero amplitudes only on integer multiples of N/t. That is, show that if $\beta_k \neq 0$, then $k = \ell(N/t)$ for some integer ℓ.

7.10

Argue how the previous parts imply the following fact about the quantum Fourier transform applied to quantum states with *periodic* amplitudes:

Suppose that you have a quantum state

$$|\psi\rangle = \sum_{k=0}^{N-1} \alpha_k \, |k\rangle$$

with periodic amplitudes ($\alpha_k = \alpha_{(k+t) \mod N}$ for some $1 \leq t \leq N-1$). Note that periodicity requires that t divides N.

Then

$$\mathrm{QFT}_N \, |\psi\rangle = \sum_{k=0}^{N-1} \beta_k \, |k\rangle$$

where $\beta_k \neq 0$ only if k is an integer multiple of N/t.