5.1

(30 分) (a) 振子能级 $\epsilon_n = (n + 1/2)h\nu$.
处于每一能级上的概率为 $P_n = \exp(-\beta\epsilon_n)$.
振子处于第一激发态与基态的概率之比:

$$\frac{P_1}{P_0} = \frac{\exp(-\beta\epsilon_1)}{\exp(-\beta\epsilon_0)} = \exp(-\beta(\epsilon_1 - \epsilon_0)) = \exp(-\beta h\nu).$$

(b) 平均能量

$$\bar{\epsilon} = \frac{\epsilon_0 P_0 + \epsilon_1 P_1}{P_0 + P_1}.$$

5.4

配分函数 $Z_N = (Z_1)^N$.

$$Z_1 = \frac{1}{h^3} \iint e^{-\frac{1}{2k_B T}(P_x^2 + P_y^2 + P_z^2)} dxdydzdP_x dP_y dP_z.$$

假设体积为 V,

$$Z_1 = \frac{V}{h^3} \int_0^\infty e^{-\frac{p^2}{2mk_B T}} 4\pi p^2 dp = \frac{V}{h^3}(2\pi m k_B T)^{3/2}.$$

$$Z_N = \left(\frac{V}{h^3}(2\pi m k_B T)^{3/2}\right)^N.$$

内能 $U = -\frac{\partial}{\partial\beta} \ln Z_N = -\frac{\partial}{\partial\beta} = N\frac{3}{2}k_B T$. 定容热容量 $C_V = \frac{\partial U}{\partial T}_V = \frac{3}{2}Nk_B$.

5.2

解: $Z = \frac{1}{h^3} \iint e^{-\frac{1}{2mk_B T}(\mathbf{p}^2) - \beta mgz} d^3p d^3r$. 假设 z 积分从 0 到 L:

$$Z_1 = \left(\frac{1}{h^3} \int e^{-\frac{\mathbf{p}^2}{2mk_B T}} d^3p\right) \left(\int_0^L e^{-\beta mgz} dz\right)$$

$$Z_1 = \frac{(2\pi m k_B T)^{3/2}}{h^3} \left[-\frac{1}{\beta mg} e^{-\beta mgz}\right]_0^L = \frac{(2\pi m k_B T)^{3/2}}{h^3} \frac{1}{\beta mg}(1 - e^{-\beta mgL}).$$

对于 N 个粒子, $Z = (Z_1)^N$. 内能 $U = -N\frac{\partial}{\partial\beta} \ln Z_1$.

$$\ln Z_1 = \frac{3}{2}\ln(2\pi m k_B) + \frac{3}{2}\ln T - 3\ln h + \ln V - \ln(\beta mg) + \ln\left(1 - e^{-\beta mgL}\right).$$

Using $\beta = 1/k_BT$,

$$\ln Z_1 = C + \frac{3}{2}\ln T + \ln T - \ln(mg/k_B) + \ln\bigl(1 - e^{-mgL/k_BT}\bigr).$$

$$U = Nk_BT^2\frac{\partial}{\partial T}\ln Z_1 = Nk_BT^2\left(\frac{3}{2T} + \frac{1}{T} + \frac{1}{1 - e^{-mgL/k_BT}}e^{-mgL/k_BT}\frac{mgL}{k_BT^2}\right).$$

$$U = N\left(\frac{5}{2}k_BT + \frac{mgLe^{-mgL/k_BT}}{1 - e^{-mgL/k_BT}}\right) = N\left(\frac{5}{2}k_BT + \frac{mgL}{e^{mgL/k_BT} - 1}\right).$$

图像中的公式为: $U = U_0 + NK_BT - \frac{NmgH}{e^{\beta mgH}-1}$. Assuming $H = L$ and U_0 includes $\frac{5}{2}NK_BT$, or there is a mistake in the image. I will transcribe the formula as it appears:

$$U = U_0 + NK_BT - \frac{NmgH}{e^{\beta mgH} - 1}.$$

气体热容量 $C_V = \frac{\partial U}{\partial T}_V$.

$$C_V = C_V^0 + NK_B - K_BT\frac{N(mgH)^2e^{\beta mgH}}{(e^{\beta mgH} - 1)^2}.$$

5.3

分子能量 $\epsilon = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$. 由能均分定理 $\bar{\epsilon} = \frac{3}{2}k_BT$. $\bar{\epsilon} = \overline{mgz}$.

$$\overline{mgz} = \frac{\int mgze^{-\beta\epsilon}d\tau}{\int e^{-\beta\epsilon}d\tau}.$$

积分代表全空间积分. Assuming integration over z from 0 to L.

$$\overline{mgz} = \frac{\int_0^L mgze^{-\beta mgz}dz}{\int_0^L e^{-\beta mgz}dz}.$$

图像中的结果是:

$$\overline{mgz} = k_BT\frac{mgL}{e^{\beta mgL} - 1}.$$

5.6

配分函数 $Z = \sum_{n=0}^{\infty} e^{-\beta\epsilon_n}$. $\epsilon_n = h\nu(n + 1/2)$.

$$Z = \sum_{n=0}^{\infty} e^{-\beta h\nu(n+1/2)} = e^{-\beta h\nu/2}\sum_{n=0}^{\infty}(e^{-\beta h\nu})^n = \frac{e^{-\beta h\nu/2}}{1 - e^{-\beta h\nu}}.$$

收至平均能量 $U = -\frac{\partial}{\partial\beta}\ln Z$.

$$\ln Z = -\frac{\beta h\nu}{2} - \ln\bigl(1 - e^{-\beta h\nu}\bigr).$$

$$U = \frac{h\nu}{2} + \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{2} + \frac{h\nu}{e^{\beta h\nu} - 1}.$$

$$U = h\nu\left(\frac{1}{2}\right) + \frac{h\nu}{e^{\beta h\nu} - 1}.$$

5.7

用: 哈密顿量 $H = \frac{1}{2I}P_\theta^2 + \frac{1}{2I\sin^2\theta}P_\phi^2$. 能谱面为: $P_\theta^2/(2IE) + P_\phi^2/(2IE\sin^2\theta) = 1$. 相体积为: $\Sigma(E) = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int\int_{H\le E} dP_\theta dP_\phi$.

$$\Sigma(E) = \int_0^\pi d\theta \int_0^{2\pi} d\phi(2\pi\sqrt{2IE}\sqrt{2IE\sin^2\theta}) = \int_0^\pi d\theta \int_0^{2\pi} d\phi(4\pi IE|\sin\theta|).$$

$$\Sigma(E) = (2\pi)(4\pi IE)\int_0^\pi \sin\theta d\theta = 8\pi^2 IE[-\cos\theta]_0^\pi = 8\pi^2 IE(1-(-1)) = 16\pi^2 IE.$$

Wait, the integral limit in the image is $H = E$, not $H \le E$. This suggests calculation of the area of the energy shell, not the volume of phase space up to E. Let's re-calculate based on the formula given in the image:

$$\Sigma(E) = \int_0^\pi d\theta \int_0^{2\pi} d\phi(2\pi IE|\sin\theta|).$$

$$\Sigma(E) = (2\pi)\int_0^\pi (2\pi IE\sin\theta)d\theta = 4\pi^2 IE\int_0^\pi \sin\theta d\theta = 4\pi^2 IE(2) = 8\pi^2 IE.$$

This matches the image. This $\Sigma(E)$ is related to the density of states. $\rho(E) = \frac{1}{h^2}\frac{d\Sigma}{dE} = \frac{8\pi^2 I}{h^2}$. 配分函数 $Z = \int_0^\infty e^{-\beta E}\rho(E)dE = \int_0^\infty e^{-\beta E}\frac{8\pi^2 I}{h^2}dE = \frac{8\pi^2 I}{h^2\beta}$. Alternatively, using the momentum integrals directly:

$$Z = \frac{1}{h^2}\int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{-\infty}^\infty dP_\theta \int_{-\infty}^\infty dP_\phi e^{-\beta(\frac{P_\theta^2}{2I} + \frac{P_\phi^2}{2I\sin^2\theta})}.$$

$$Z = \frac{1}{h^2}\int_0^\pi d\theta \int_0^{2\pi} d\phi\sqrt{\frac{2\pi I}{\beta}}\sqrt{\frac{2\pi I\sin^2\theta}{\beta}} = \frac{1}{h^2}\int_0^\pi d\theta \int_0^{2\pi} d\phi\frac{2\pi I|\sin\theta|}{\beta}.$$

$$Z = \frac{1}{h^2}(2\pi)\frac{2\pi I}{\beta}\int_0^\pi \sin\theta d\theta = \frac{4\pi^2 I}{h^2\beta}(2) = \frac{8\pi^2 I}{h^2\beta}.$$

This matches the image. 故 $U = NK_BT$. $C_V = NK_B$.

5.8

(a) $P \ \ = \frac{1}{Z}e^{\mu H/k_BT}$, $P \ \ = \frac{1}{Z}e^{-\mu H/k_BT}$. 配分函数 $Z = e^{\mu H/k_BT} + e^{-\mu H/k_BT}$. (b) 平均磁矩 $\bar{\mu} = P \ \ \cdot \mu - P \ \ \cdot \mu$.

$$\bar{\mu} = \frac{e^{\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}}\mu - \frac{e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}}\mu = \mu\frac{e^{\beta\mu H} - e^{-\beta\mu H}}{e^{\beta\mu H} + e^{-\beta\mu H}} = \mu\tanh\left(\frac{\mu H}{k_BT}\right).$$

(c) 磁化强度 $M = N_0\bar{\mu} = N_0\mu\tanh\left(\frac{\mu H}{k_BT}\right)$. 高温时, $\mu H \ll k_BT$. $\tanh(x) \approx x$. $M \approx N_0\mu\left(\frac{\mu H}{k_BT}\right) = N_0\frac{\mu^2 H}{k_BT}$. 图像中的结果是 $M \approx N_0\frac{2\mu H}{k_BT}$. I will transcribe the image result.

$$M \approx N_0\frac{2\mu H}{k_BT}.$$

低温时 $\tanh\left(\frac{\mu H}{k_BT}\right) \approx 1$. $M \approx N_0\mu$.