Zee QFT Chapter II.1 Problems

Xiaoyang Zheng

November 9, 2025

Problem II.1.1

Problem: Show that the bilinears $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\sigma^{\mu\nu}\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$, and $\bar{\psi}\gamma^5\psi$ transform as scalar, vector, tensor, axial vector, and pseudoscalar respectively.

Solution:
Under Lorentz: $\psi \to (1 - \frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma})\psi$, $\bar{\psi} \to \bar{\psi}(1 + \frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma})$ where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.
Under parity: $\psi \to \gamma^0\psi$, $\bar{\psi} \to \bar{\psi}\gamma^0$.

1. Scalar: $\bar{\psi}\psi$

Lorentz invariant to first order. Under parity: $\bar{\psi}\psi \to \bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi$.

2. Vector: $\bar{\psi}\gamma^\mu\psi$

Using $[\sigma^{\rho\sigma}, \gamma^\mu] = 2i(g^{\rho\mu}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho)$, transforms as $\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi$ under Lorentz.

Under parity: $\gamma^0\gamma^i\gamma^0 = -\gamma^i$ gives $\bar{\psi}\gamma^0\psi \to +\bar{\psi}\gamma^0\psi$ and $\bar{\psi}\gamma^i\psi \to -\bar{\psi}\gamma^i\psi$.

3. Tensor: $\bar{\psi}\sigma^{\mu\nu}\psi$

Since $\sigma^{\mu\nu}$ generates Lorentz transformations:

$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\bar{\psi}\sigma^{\rho\sigma}\psi \tag{1}$$

Transforms as antisymmetric rank-2 tensor.

4. Axial Vector: $\bar{\psi}\gamma^\mu\gamma^5\psi$

Since $[\sigma^{\rho\sigma}, \gamma^5] = 0$: $\bar{\psi}\gamma^\mu\gamma^5\psi \to \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\gamma^5\psi$ under Lorentz.

Using $\gamma^0\gamma^5\gamma^0 = -\gamma^5$: under parity $\bar{\psi}\gamma^0\gamma^5\psi \to -\bar{\psi}\gamma^0\gamma^5\psi$ and $\bar{\psi}\gamma^i\gamma^5\psi \to +\bar{\psi}\gamma^i\gamma^5\psi$. This is a pseudovector.

5. Pseudoscalar: $\bar{\psi}\gamma^5\psi$

Lorentz invariant. Under parity: $\bar{\psi}\gamma^5\psi \to \bar{\psi}\gamma^0\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^5\psi$.

Problem II.1.6

Problem: Solve the massless Dirac equation.

Solution:
The massless Dirac equation $i\not{\partial}\psi = 0$ becomes for plane waves:

$$\not{p}\, u(p) = 0 \tag{2}$$

In the Weyl basis with $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ and $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, the equation decouples:

$$\bar{\sigma}^\mu \partial_\mu \psi_R = 0 \tag{3}$$
$$\sigma^\mu \partial_\mu \psi_L = 0 \tag{4}$$

Left and right-handed Weyl fermions are independent when $m = 0$. For momentum along z-axis with $E = |\vec{p}|$, the helicity eigenstates are:

$$u_R = \sqrt{E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ (right-handed)}, \quad u_L = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{ (left-handed)} \tag{5}$$

For massless fermions, helicity equals chirality and is conserved.

Problem II.1.11

Problem: Work out the Dirac equation in (1+1)-dimensional spacetime.

Solution:

Need two γ matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g^{\mu\nu} = \mathrm{diag}(1, -1)$. Use Pauli matrices:

$$\gamma^0 = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6}$$

The Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$ gives for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$:

$$i\partial_t \psi_2 + i\partial_x \psi_2 = m\psi_1 \tag{7}$$

$$i\partial_t \psi_1 - i\partial_x \psi_1 = m\psi_2 \tag{8}$$

In light-cone coordinates $x^\pm = t \pm x$ with $\partial_\pm = \partial_t \pm \partial_x$:

$$i\partial_+ \psi_2 = \frac{m}{2}\psi_1, \quad i\partial_- \psi_1 = \frac{m}{2}\psi_2 \tag{9}$$

For $m = 0$, the components decouple: $\psi_1 = \psi_1(x^+)$ (right-mover), $\psi_2 = \psi_2(x^-)$ (left-mover). In (1+1)D, the two components represent chirality, not spin.

Problem II.1.12

Problem: Work out the Dirac equation in (2+1)-dimensional spacetime. Show that the mass term violates parity and time reversal.

Solution:
Use three Pauli matrices for γ matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g^{\mu\nu} = \mathrm{diag}(1, -1, -1)$:

$$\gamma^0 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^2 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10}$$

The Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$ gives for $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$:

$$i\partial_t \psi_1 - \partial_x \psi_2 + i\partial_y \psi_2 = m\psi_1 \tag{11}$$

$$-i\partial_t \psi_2 - \partial_x \psi_1 - i\partial_y \psi_1 = m\psi_2 \tag{12}$$

Parity and Time Reversal Violation

The mass term is $m\bar{\psi}\psi = m\psi^\dagger \gamma^0 \psi$. With $\gamma^0 = \sigma^3$:

$$m\bar{\psi}\psi = m(|\psi_1|^2 - |\psi_2|^2) \tag{13}$$

This is a Chern-Simons-type topological mass. Under parity $(t, x, y) \to (t, -x, y)$ with $\psi \to P\psi$ where $P = \gamma^0$, we verify $P\gamma^1 P^{-1} = -\gamma^1$ but the two spinor components transform oppositely. The mass term changes sign:

$$m\bar{\psi}\psi \to -m\bar{\psi}\psi \tag{14}$$

Similarly under time reversal. Thus the mass term violates both $\mathcal{P}$ and $\mathcal{T}$ symmetry. This is the parity anomaly in (2+1)D - the mass acts like a magnetic flux through the plane.