Homework 6
Fall 2025

Xiaoyang Zheng

October 13, 2025

Problem 1: Phase Estimation

1.1: What is $U^k|+\rangle$ written in the computational basis?

$$U^k |+\rangle = U^k \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \right) = \frac{1}{\sqrt{2}}(U^k |0\rangle + U^k |1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + e^{ik\theta} |1\rangle)$$

1.2: Identify two projective measurements $\langle m_c|$ and $\langle m_s|$

State after the operation: $|\psi_k\rangle = U^k |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{ik\theta} |1\rangle)$

For the first measurement,expect: $| \langle m_c|\psi_k\rangle |^2 = \frac{1+\cos(k\theta)}{2}$
Projecting onto $|+\rangle$ state:

$$\langle +|\psi_k\rangle = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle + e^{ik\theta} |1\rangle) = \frac{1}{2}(1 + e^{ik\theta})$$

The probability is:

$$| \langle +|\psi_k\rangle |^2 = \left| \frac{1}{2}(1 + \cos(k\theta) + i\sin(k\theta)) \right|^2 = \frac{1}{4}(2 + 2\cos(k\theta)) = \frac{1 + \cos(k\theta)}{2}$$

So: $\langle m_c| = \langle +| = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)$.

For the second measurement, expect: $| \langle m_s|\psi_k\rangle |^2 = \frac{1+\sin(k\theta)}{2}$
Projecting onto $|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i |1\rangle)$
The corresponding bra is $\langle +i| = \frac{1}{\sqrt{2}}(\langle 0| - i \langle 1|)$.

$$\langle +i|\psi_k\rangle = \frac{1}{\sqrt{2}}(\langle 0| - i \langle 1|) \cdot \frac{1}{\sqrt{2}}(|0\rangle + e^{ik\theta} |1\rangle) = \frac{1}{2}(1 - ie^{ik\theta})$$

The probability is the squared magnitude:

$$| \langle +i|\psi_k\rangle |^2 = \left| \frac{1}{2}(1 - i(\cos(k\theta) + i\sin(k\theta))) \right|^2 = \frac{1}{4}(1+2\sin(k\theta)+\sin^2(k\theta)+\cos^2(k\theta)) = \frac{1 + \sin(k\theta)}{2}$$

So, $\langle m_s| = \langle +i| = \frac{1}{\sqrt{2}}(\langle 0| - i \langle 1|)$.

$$\langle m_c| = \langle +| \quad \text{and} \quad \langle m_s| = \langle +i|$$

1.3: How can you estimate θ for $k = 1$? Why can we not just use a single measurement?

For $k = 1$,: Caculate probability $p_c = |\langle m_c|\psi_1\rangle|^2$ and $p_s = |\langle m_s|\psi_1\rangle|^2$

$$p_c = \frac{1+\cos(\theta)}{2} \implies \cos(\theta) = 2p_c - 1$$

$$p_s = \frac{1+\sin(\theta)}{2} \implies \sin(\theta) = 2p_s - 1$$

Estimated $\cos(\theta)$ and $\sin(\theta)$, the angle $\theta \in [0, 2\pi)$ is determined using : $\theta = \mathrm{atan2}(2p_s - 1, 2p_c - 1)$.

We cannot use a single measurement because:

- If we only measure $\cos(\theta)$, we cannot distinguish between θ and $2\pi - \theta$, as $\cos(\theta) = \cos(2\pi - \theta)$.
- If we only measure $\sin(\theta)$, we cannot distinguish between θ and $\pi - \theta$, as $\sin(\theta) = \sin(\pi - \theta)$.

1.4: What problems arise when trying to estimate θ for $k > 1$?

For $k > 1$, we measure $\cos(k\theta)$ and $\sin(k\theta)$, giving $k\theta \pmod{2\pi}$. Then:

$$k\theta = \phi + 2\pi m \implies \theta = \frac{\phi}{k} + \frac{2\pi m}{k}$$

There are k possible values ($m = 0, \ldots, k-1$) that produce unique $\theta \in [0, 2\pi)$. We cannot determine m from measurement alone.

1.5: How can you use information you gain from taking measurements at $k = 1$ to solve this problem at $k = 2$?

1. For $k = 1$: Get estimate $\hat{\theta}_1$ (unambiguous).
2. For $k = 2$: Get $\phi_2 = 2\theta \pmod{2\pi}$, giving two candidates: $\frac{\phi_2}{2}$ or $\frac{\phi_2}{2} + \pi$.
3. Choose the candidate closer to $\hat{\theta}_1$.

Problem 2: Grover's Algorithm on 2 qubits

2.1: Write out the 2-qubit Oracle unitary U_f as a 4×4 matrix.

Oracle: $U_f|x\rangle = (-1)^{f(x)}|x\rangle$, where $f(11) = 1$ and $f(x) = 0$ otherwise. Basis: $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

$$U_f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

This is a Controlled-Z gate.

2.2: Write $|u\rangle$ and how to generate it.

Equal superposition for 2 qubits ($N = 4$):

$$|u\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

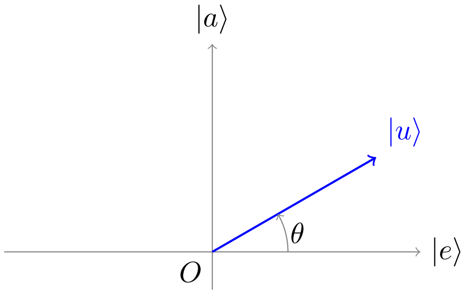
Generated by: $(H \otimes H)|00\rangle$

2.3: Draw the equal weight superposition $|u\rangle$ on a diagram.

State space basis: marked state $|a\rangle = |11\rangle$ and unmarked superposition $|e\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$.

$$|u\rangle = \frac{\sqrt{3}}{2}|e\rangle + \frac{1}{2}|a\rangle$$

Angle with $|e\rangle$ axis: $\theta = \arcsin(1/2) = \pi/6$.



2.4: What is the angle θ between the vectors $|e\rangle$ and $|u\rangle$?

$$\cos(\theta) = \langle u|e\rangle = \left(\frac{\sqrt{3}}{2}\langle e| + \frac{1}{2}\langle a|\right)|e\rangle = \frac{\sqrt{3}}{2}$$

$$\theta = \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

2.5: Action of the Oracle U_f on $|u\rangle$.

$$U_f|a\rangle = -|a\rangle, \quad U_f|e\rangle = |e\rangle$$

$$U_f|u\rangle = U_f\left(\frac{\sqrt{3}}{2}|e\rangle + \frac{1}{2}|a\rangle\right) = \frac{\sqrt{3}}{2}|e\rangle + \frac{1}{2}(-|a\rangle) = \frac{\sqrt{3}}{2}|e\rangle - \frac{1}{2}|a\rangle$$

Geometric: reflection about $|e\rangle$ axis.

2.6: Geometric interpretation of the action of U_u.

$U_u = I - 2|u\rangle\langle u|$ is a reflection operator.

$$U_u|u\rangle = |u\rangle - 2|u\rangle\langle u|u\rangle = -|u\rangle$$

$$U_u\left|u^\perp\right\rangle = \left|u^\perp\right\rangle - 2|u\rangle\left\langle u\middle|u^\perp\right\rangle = \left|u^\perp\right\rangle$$

Geometric: reflection about the hyperplane perpendicular to $|u\rangle$.

2.7: Apply one iteration of Grover's algorithm.

1. Start: $|u\rangle = \frac{\sqrt{3}}{2}|e\rangle + \frac{1}{2}|a\rangle$

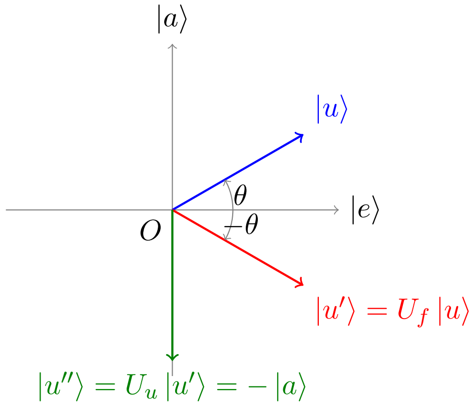
2. Apply U_f: $|u'\rangle = \frac{\sqrt{3}}{2}|e\rangle - \frac{1}{2}|a\rangle$

3. Apply U_u: $|u''\rangle = |u'\rangle - 2|u\rangle\langle u|u'\rangle$

$$\langle u|u'\rangle = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$|u''\rangle = |u'\rangle - |u\rangle = \left(\frac{\sqrt{3}}{2}|e\rangle - \frac{1}{2}|a\rangle\right) - \left(\frac{\sqrt{3}}{2}|e\rangle + \frac{1}{2}|a\rangle\right) = -|a\rangle$$

Final state: $-|a\rangle$. Measurement yields '11' with 100% probability.



Note: All TikZ diagrams in this document were generated by Gemini 2.5 Pro from hand-drawn sketches.