C191A

Fall 2025

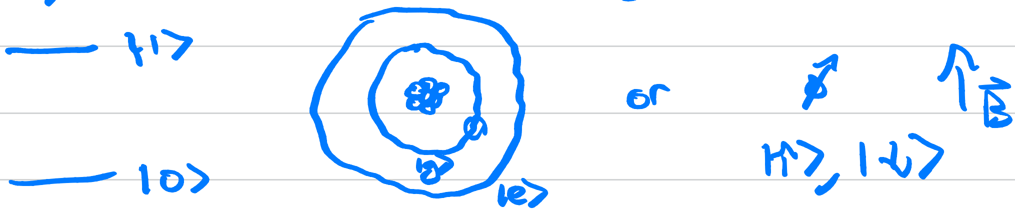
Lecture 14

Subjects for today:

- Bloch sphere
- Rotations
- Mixed states (time permitting)

The Bloch Sphere:

Let's consider the possible states of a qubit, or two level system:



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Only requirement is that $\alpha^2 + \beta^2 = 1$.

If α and β were only real numbers, then this equation would describe a circle. But α and β can be complex, so instead, this equation describes the surface of a sphere.

In particular, we can rewrite $|\psi\rangle$ as

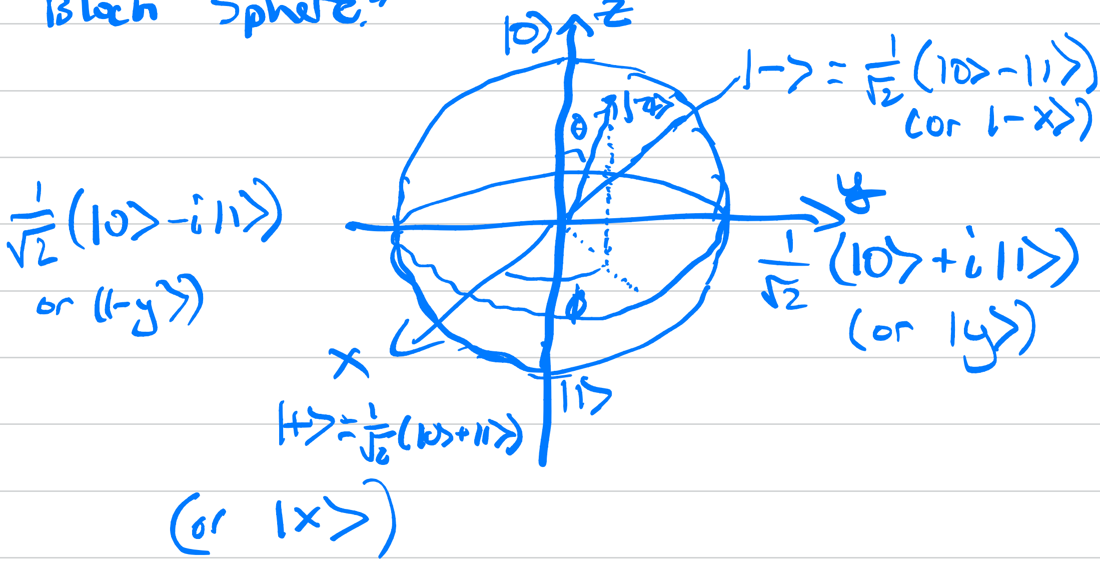
$$|\psi\rangle = e^{i\gamma}\left(\cos\left(\tfrac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\tfrac{\theta}{2}\right)|1\rangle\right)$$

Where θ, ϕ, and γ are real numbers, and
$0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \gamma \leq 2\pi$.

γ is a global phase, so it has no observable effect and we can ignore it.

$$\Rightarrow \quad |\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

θ and ϕ are polar angles for a state living somewhere on the surface of the unit sphere, which is called the "Bloch Sphere."



- Note that states along orthogonal axes are not orthogonal:

$$\langle x | y \rangle \neq 0, \text{ but } \langle x | -x \rangle = 0$$

- States on opposites sides of the Bloch sphere are orthogonal to each other:

$$\langle -z | z \rangle = 0$$

Question for the class:
- What about the states of two qubits?

Now 4 basis states instead of 2:

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle, \text{ or } \underbrace{|\Psi^+\rangle, |\Psi^-\rangle, |\Phi^+\rangle, |\Phi^-\rangle}_{\text{Bell states.}}$$

Need four numbers, $\alpha, \beta, \gamma, \delta$ to describe arbitrary state, with $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.
These are states on a 4D hypersphere.

Rotations:

- Now that we can visualize/describe an arbitrary single qubit state, let's move on to how to understand the operations we can perform on it.

- Any state on the Bloch sphere can be transformed into any other state through the appropriate single qubit rotation.

- An arbitrary rotation by angle θ around vector $\hat{n}$ is given by:

$$R_{\hat{n}}(\theta) = e^{-i\theta \hat{n} \cdot \frac{1}{2} \vec{\sigma}}$$

where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- However, it's not very convenient or practical to rotate around arbitrary axes, so instead we can start by considering rotations around $\hat{x}$, $\hat{y}$, and $\hat{z}$.

$$R_x(\theta) = e^{-i\theta\sigma_x/2}$$

If $\vec{A}\cdot\vec{A} = \vec{I}$ (which is true for the Pauli matrices $\vec{\sigma}$, then

$$e^{i\theta A} = \cos(\theta) I + i\sin(\theta) A$$

$$\Rightarrow R_x(\theta) = e^{-i\theta\sigma_x/2} = \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2}\sigma_x$$

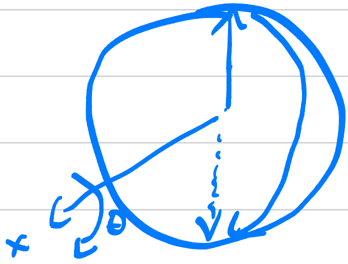
$$\Rightarrow R_x(\theta) = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

Similarly, $R_y(\theta) = \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$= e^{-i\theta/2} \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

- Let's consider rotating around $\hat{x}$ by $\theta = \pi$:



$$R_x(\theta) = \begin{bmatrix} \cos \pi/2 & -i \sin \frac{\pi}{2} \\ -i \sin \pi/2 & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -iX$$

where $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. so to within a global phase, $R_x(\pi) = X$.

- This is a useful way to get some intuition about what a gate does to a given state.

$X|0\rangle \to |1\rangle \qquad X|1\rangle \to |0\rangle$

what about $X|+\rangle$?

$$X\left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) = \frac{1}{\sqrt{2}}(X|0\rangle + X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) = |+\rangle$$

- But we could have guessed this already from the fact that rotating a vector around itself doesn't change it, and $|+\rangle$ points along $\hat{x}$.

- Question for class:

  What does X do to the state $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$?

A: $|y\rangle \to |-y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

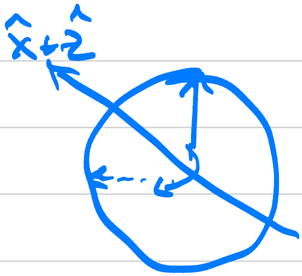
Similarly, the gates Y and Z are equivalent to rotations by angle $\theta = \pi$ around $\hat{y}$ and $\hat{z}$ respectively, to within a global phase.

- Question for the class:

  What about H?
  (Hadamard gate)

A: $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

This corresponds to a rotation by $\theta = \pi$ around the vector $\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$, or a rotation by $\theta = \pi/2$ around $\hat{y}$, followed by a rotation about $\hat{x}$ of $\theta = \pi$.



- To get from any point on the Bloch sphere to any other point on the Bloch sphere, we could either do arbitrary rotations around any two orthogonal axes, or it is sufficient to have discrete sets of gates, such as:

Note: $T = \sqrt{S}$
$S = \sqrt{Z}$

- $R_x(\pi/2)$, $R_z(\pi/2)$, $R_z(\pi/4)$ or
- H, S, and T, where $S = R_z(\pi/2)$, $T = R_z(\pi/4)$

Solovay-Kitaev theorem:

Any single qubit gate can be approximated using a finite sequence of universal gates with at most error ε with length $O(\log^c(1/\varepsilon))$, where c is a positive constant. (And this sequence is easy to find.)

Question for the class:
Why not just use arbitrary rotations?

A: Can't program an arbitrary set of gates into the computer, particularly in a fault tolerant/error-corrected manner. Much better to have a finite set of gates you can calibrate and make sure you can implement without errors spreading.

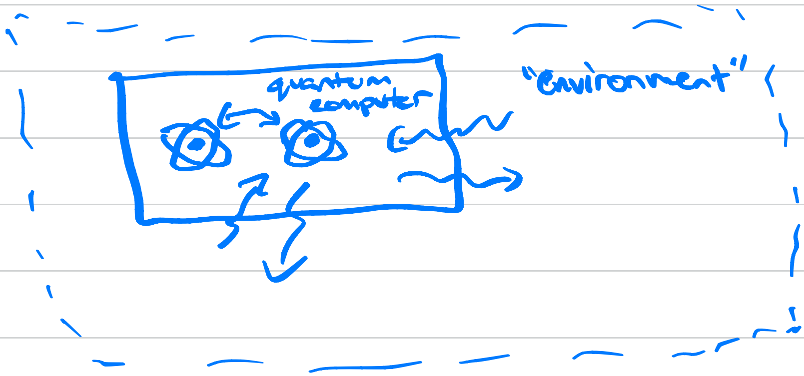
Mixed States:

- So far, we have only considered unitary, reversible dynamics of a perfectly isolated quantum system.

This cannot capture a number of important processes that occur in a real quantum computer, such as:

- Spontaneous emission
- noise from the environment
- errors in gates, state preperation, and measurement.
- interactions with other uncontrolled/macroscopic systems.

These processes are all examples of "open quantum systems," or in other words dynamics that arise from interactions between the quantum computer and the rest of the universe.



- To handle this, you either need to expand your system to include the environment and all its degrees of freedom, which is usually impossible, or you have to introduce the formalism of open quantum systems and live with non-unitary dynamics.

- To some extent, error correction is all about recovering reversible, unitary dynamics in the presence of interactions w/ the outside world, and also imperfect gates.

- In the next lecture, we will introduce the concept of the density matrix, which treats this formally.

- For now, it's helpful to just consider an example:

Suppose you prepare a qubit the state

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, and then

you let your friend borrow the qubit for a while. While they have it, your friend measures the qubit in the $|0\rangle, |1\rangle$ basis, but doesn't record the outcome of the measurement. They then give you back the qubit. What state is it in?

We know with probability 50% the qubit wound up in $|0\rangle$, and w/ 50% the qubit wound up in $|1\rangle$, but we don't know which.

- This is called a "mixed state," which reflects our imperfect knowledge of the state, and is a statistical mixture of quantum states rather than a coherent superposition.

- There is **NO** useful or meaningful unitary operation you could do w/ this qubit, because this particular qubit is in a maximally mixed state, with **no** coherence left. Any single qubit gate will still leave you in a 50/50 mixture of $|0\rangle$ and $|1\rangle$.

- In contrast, $|0\rangle$, $|1\rangle$, $|+\rangle$, etc are all examples of "pure states," with no statistical mixture.

- In any real quantum system, you will always be somewhere in between pure and maximally mixed, the goal is usually to stay as close to pure as possible. In the next lecture, we will quantify all this.