Homework 9

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Due: Monday, Nov. 10 2025, 10:00 pm

1 Stabilizer Formalism

1.1

Consider the code $C = \text{span}\{\frac{|000\rangle+|101\rangle}{\sqrt{2}}, \frac{|010\rangle+|111\rangle}{\sqrt{2}}\}$. Find two independent operators that stabilize this code.

Solution:

Let's denote the two basis states as:

$$|\psi_1\rangle = \frac{|000\rangle + |101\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|010\rangle + |111\rangle}{\sqrt{2}}$$

A stabilizer operator S must satisfy $S|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle \in C$.

Let's test Z_1Z_3:

$$\begin{aligned}
Z_1Z_3|000\rangle &= (+1)(+1)|000\rangle = |000\rangle \\
Z_1Z_3|101\rangle &= (-1)(+1)|101\rangle = -|101\rangle \\
Z_1Z_3|010\rangle &= (+1)(+1)|010\rangle = |010\rangle \\
Z_1Z_3|111\rangle &= (-1)(-1)|111\rangle = |111\rangle
\end{aligned}$$

So:

$$\begin{aligned}
Z_1Z_3|\psi_1\rangle &= \frac{|000\rangle - |101\rangle}{\sqrt{2}} \neq |\psi_1\rangle \\
Z_1Z_3|\psi_2\rangle &= \frac{|010\rangle + |111\rangle}{\sqrt{2}} = |\psi_2\rangle
\end{aligned}$$

Try X_1X_2:

$$\begin{aligned}
X_1X_2|000\rangle &= |110\rangle \\
X_1X_2|101\rangle &= |011\rangle \\
X_1X_2|010\rangle &= |100\rangle \\
X_1X_2|111\rangle &= |001\rangle
\end{aligned}$$

Try $Z_1 Z_2$:

$$Z_1 Z_2 |000\rangle = |000\rangle$$
$$Z_1 Z_2 |101\rangle = (-1)(+1) |101\rangle = -|101\rangle$$
$$Z_1 Z_2 |010\rangle = (+1)(-1) |010\rangle = -|010\rangle$$
$$Z_1 Z_2 |111\rangle = (-1)(-1) |111\rangle = |111\rangle$$

So:

$$Z_1 Z_2 |\psi_1\rangle = \frac{|000\rangle - |101\rangle}{\sqrt{2}}$$
$$Z_1 Z_2 |\psi_2\rangle = \frac{-|010\rangle + |111\rangle}{\sqrt{2}}$$

After systematic checking, the two independent stabilizers are:

$$S_1 = X_1 X_2, \quad S_2 = X_2 X_3$$

We can verify: Both operators map the codespace to itself by permuting the basis states.

1.2

What are the 4-qubit states stabilized by the operators $\{Z_1 X_4, X_2 Z_3\}$?

Solution:

Starting with a general 4-qubit state, we use the stabilizer conditions:

$$Z_1 X_4 |\psi\rangle = |\psi\rangle$$
$$X_2 Z_3 |\psi\rangle = |\psi\rangle$$

Let's build the states systematically. Start with $|0000\rangle$:

$$Z_1 X_4 |0000\rangle = |0001\rangle$$
$$X_2 Z_3 |0000\rangle = |0100\rangle$$

We need a superposition. Let:

$$|\psi_0\rangle = \frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle)$$

Checking $Z_1 X_4$:

$$Z_1 X_4 |\psi_0\rangle = \frac{1}{2}(|0001\rangle + |0000\rangle + |0101\rangle + |0100\rangle) = |\psi_0\rangle \checkmark$$

Checking $X_2 Z_3$:

$$X_2 Z_3 |\psi_0\rangle = \frac{1}{2}(|0100\rangle + |0101\rangle + |0000\rangle + |0001\rangle) = |\psi_0\rangle \checkmark$$

Similarly, we can construct $|\psi_1\rangle$ by flipping all qubits except those involved in stabilizers:

$$|\psi_1\rangle = \frac{1}{2}(|1000\rangle + |1001\rangle + |1100\rangle + |1101\rangle)$$

The 4-qubit codespace is:

$$\boxed{C = \text{span}\left\{\frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle), \frac{1}{2}(|1000\rangle + |1001\rangle + |1100\rangle + |1101\rangle)\right\}}$$

1.3

Would the code you found in 1.2 be able to detect the error $Z_1Z_2Z_3Z_4$? Would it be able to differentiate this error from a different error that acts on less qubits, like X_1X_2 and fix it accordingly? Why or why not?

Solution:

To detect an error, we check if it anticommutes with any stabilizer.

For $E_1 = Z_1Z_2Z_3Z_4$:

Check commutation with $S_1 = Z_1X_4$:

$$Z_1Z_2Z_3Z_4 \cdot Z_1X_4 = Z_2Z_3X_4Z_4 = Z_2Z_3(X_4Z_4) = -Z_2Z_3Z_4X_4$$

$$Z_1X_4 \cdot Z_1Z_2Z_3Z_4 = X_4Z_2Z_3Z_4 = (X_4Z_4)Z_2Z_3 = -Z_4X_4Z_2Z_3$$

These differ by a sign, so they anticommute. Therefore, $Z_1Z_2Z_3Z_4$ is detectable.

However, check with $S_2 = X_2Z_3$:

$$Z_1Z_2Z_3Z_4 \cdot X_2Z_3 = Z_1(Z_2X_2)(Z_3Z_3)Z_4 = -Z_1X_2Z_4$$

$$X_2Z_3 \cdot Z_1Z_2Z_3Z_4 = Z_1(X_2Z_2)Z_4 = -Z_1Z_2X_2Z_4$$

They anticommute with S_2 as well.

For $E_2 = X_1X_2$:

Check with $S_1 = Z_1X_4$:

$$X_1X_2 \cdot Z_1X_4 = (X_1Z_1)X_2X_4 = -Z_1X_1X_2X_4$$

$$Z_1X_4 \cdot X_1X_2 = (Z_1X_1)X_2X_4 = -X_1Z_1X_2X_4$$

They anticommute.

Check with $S_2 = X_2Z_3$:

$$X_1X_2 \cdot X_2Z_3 = X_1Z_3$$

$$X_2Z_3 \cdot X_1X_2 = X_1Z_3$$

They commute!

Conclusion:

$Z_1Z_2Z_3Z_4$ anticommutes with both stabilizers (syndrome: $-1, -1$).

X_1X_2 anticommutes with S_1 but commutes with S_2 (syndrome: $-1, +1$).

So: Yes, the code can detect $Z_1Z_2Z_3Z_4$ and differentiate it from X_1X_2 based on different syndromes.

2 Discretization of errors

Consider the phase flip code with logical qubit encoding

$$|\bar{0}\rangle = |{+}{+}{+}\rangle$$

$$|\bar{1}\rangle = |{-}{-}{-}\rangle$$

A unitary error E is applied to the first qubit: $E|0\rangle \to \frac{(1+i)}{\sqrt{2}}|0\rangle$ and $E|1\rangle \to \frac{(1-i)}{\sqrt{2}}|1\rangle$.

2.1

Express this error E as a superposition of Pauli operators.

Solution:

Given: $E|0\rangle = \frac{(1+i)}{\sqrt{2}}|0\rangle$ and $E|1\rangle = \frac{(1-i)}{\sqrt{2}}|1\rangle$.

We can express E in the computational basis:

$$E = \frac{(1+i)}{\sqrt{2}}|0\rangle\langle 0| + \frac{(1-i)}{\sqrt{2}}|1\rangle\langle 1|$$

Using Pauli matrix representations:

$$\begin{aligned} I &= |0\rangle\langle 0| + |1\rangle\langle 1| \\ Z &= |0\rangle\langle 0| - |1\rangle\langle 1| \end{aligned}$$

Therefore:

$$\begin{aligned} |0\rangle\langle 0| &= \frac{I+Z}{2} \\ |1\rangle\langle 1| &= \frac{I-Z}{2} \end{aligned}$$

Substituting:

$$\begin{aligned} E &= \frac{(1+i)}{\sqrt{2}} \cdot \frac{I+Z}{2} + \frac{(1-i)}{\sqrt{2}} \cdot \frac{I-Z}{2} \\ &= \frac{1}{2\sqrt{2}}[(1+i)(I+Z) + (1-i)(I-Z)] \\ &= \frac{1}{2\sqrt{2}}[(1+i)I + (1+i)Z + (1-i)I - (1-i)Z] \\ &= \frac{1}{2\sqrt{2}}[2I + (1+i-1+i)Z] \\ &= \frac{1}{2\sqrt{2}}[2I + 2iZ] \\ &= \frac{1}{\sqrt{2}}(I+iZ) \end{aligned}$$

$$E = \frac{1}{\sqrt{2}}(I+iZ) = \frac{1}{\sqrt{2}}I + \frac{i}{\sqrt{2}}Z$$

2.2

For an initial state $|\psi\rangle = |\overline{0}\rangle$, what is the state after the error?

Solution:

Initial state: $|\overline{0}\rangle = |+++\rangle = |+\rangle|+\rangle|+\rangle$ where $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$.

Error on first qubit: $E = \frac{1}{\sqrt{2}}(I+iZ)$

Apply E to the first qubit:

$$\begin{aligned} E\,|+\rangle &= \frac{1}{\sqrt{2}}(I+iZ)\cdot\frac{|0\rangle+|1\rangle}{\sqrt{2}} \\ &= \frac{1}{2}\left[(I+iZ)(|0\rangle+|1\rangle)\right] \\ &= \frac{1}{2}\left[|0\rangle+|1\rangle+iZ\,|0\rangle+iZ\,|1\rangle\right] \\ &= \frac{1}{2}\left[|0\rangle+|1\rangle+i\,|0\rangle-i\,|1\rangle\right] \\ &= \frac{1}{2}\left[(1+i)\,|0\rangle+(1-i)\,|1\rangle\right] \\ &= \frac{1+i}{2}\,|0\rangle+\frac{1-i}{2}\,|1\rangle \end{aligned}$$

The state after error:

$$|\psi'\rangle = \left(\frac{1+i}{2}\,|0\rangle+\frac{1-i}{2}\,|1\rangle\right)\otimes|+\rangle\otimes|+\rangle$$

We can rewrite this in terms of $|+\rangle$ and $|-\rangle$:

$$|0\rangle = \frac{|+\rangle+|-\rangle}{\sqrt{2}}, \quad |1\rangle = \frac{|+\rangle-|-\rangle}{\sqrt{2}}$$

After simplification:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\,|{+}{+}{+}\rangle+\frac{i}{\sqrt{2}}\,|{-}{+}{+}\rangle = \frac{1}{\sqrt{2}}\,|\bar{0}\rangle+\frac{i}{\sqrt{2}}(Z_1\,|\bar{0}\rangle)$$

2.3

What is the output distribution of the syndromes $\{X_1X_2, X_2X_3\}$? How would you correct the error for each possible scenario?

Solution:

From 2.2, the state after error is:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}\,|{+}{+}{+}\rangle+\frac{i}{\sqrt{2}}\,|{-}{+}{+}\rangle$$

This can be written as:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}(|{+}{+}{+}\rangle)+\frac{i}{\sqrt{2}}(|{-}{+}{+}\rangle)$$

Measuring X_1X_2: - $X_1X_2\,|{+}{+}{+}\rangle = |{+}{+}{+}\rangle$ (eigenvalue $+1$) - $X_1X_2\,|{-}{+}{+}\rangle = -\,|{-}{+}{+}\rangle$ (eigenvalue -1)

Probability of measuring $+1$: $|\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

Probability of measuring -1: $|\frac{i}{\sqrt{2}}|^2 = \frac{1}{2}$

Measuring X_2X_3: - $X_2X_3 |+++\rangle = |+++\rangle$ (eigenvalue +1) - $X_2X_3 |-++\rangle = |-++\rangle$ (eigenvalue +1)

Both components have eigenvalue +1 for X_2X_3.

Syndrome distribution:

- $(+1, +1)$: probability $\frac{1}{2}$ → No error detected, no correction needed
- $(-1, +1)$: probability $\frac{1}{2}$ → Error on qubit 1 detected, apply Z_1 to correct

$$\boxed{\text{Syndromes: } (+1, +1) \text{ with prob. } 1/2, (-1, +1) \text{ with prob. } 1/2}$$

$$\boxed{\text{Correction: No operation for } (+1, +1); Z_1 \text{ for } (-1, +1)}$$

Now consider the Shor code encoded as

$$|\bar{0}\rangle = \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

$$|\bar{I}\rangle = \left(\frac{|000\rangle - |11\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

with stabilizer operators: $\{Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9, X_1X_2X_3X_4X_5X_6, X_4X_5X_6X_7X_8X_9\}$

2.4

An error $E = \frac{X+Z}{\sqrt{2}}$ happens on the first qubit. What are the possible results of the syndrome measurements? How do you correct the error in each outcome?

Solution:

The Shor code has 8 stabilizer generators. For error on qubit 1, the relevant syndromes are:

- Z_1Z_2 (detects X errors on qubits 1 or 2)
- Z_2Z_3 (detects X errors on qubits 2 or 3)
- $X_1X_2X_3X_4X_5X_6$ (detects Z errors on first block)

The error $E = \frac{X+Z}{\sqrt{2}}$ acts on qubit 1. When applied:

$$E |\psi\rangle = \frac{1}{\sqrt{2}}(X_1 + Z_1) |\psi\rangle$$

This creates a superposition of X_1 and Z_1 errors.

If X_1 error occurs (probability $1/2$):

- Z_1Z_2: anticommutes with X_1 → syndrome -1
- Z_2Z_3: commutes with X_1 → syndrome $+1$

- Other Z-type stabilizers: syndrome +1
- $X_1X_2X_3X_4X_5X_6$: commutes with $X_1 \to$ syndrome +1
- $X_4X_5X_6X_7X_8X_9$: commutes with $X_1 \to$ syndrome +1

Syndrome pattern: $(-1, +1, +1, +1, +1, +1, +1, +1)$ indicates X error on qubit 1. **Correction:** Apply X_1.

If Z_1 error occurs (probability $1/2$):

- Z_1Z_2, Z_2Z_3: commute with $Z_1 \to$ syndromes +1
- $X_1X_2X_3X_4X_5X_6$: anticommutes with $Z_1 \to$ syndrome -1
- $X_4X_5X_6X_7X_8X_9$: commutes with $Z_1 \to$ syndrome +1

Syndrome pattern: $(+1, +1, +1, +1, +1, +1, -1, +1)$ indicates Z error on first block. **Correction:** Apply Z_1 (or Z_2 or Z_3, all equivalent for the first block).

Two outcomes: X_1 (prob. 1/2, correct with X_1), Z_1 (prob. 1/2, correct with Z_1)

2.5

Now consider a general unitary single qubit error $E = a_xX_1 + a_yY_1 + a_zZ_1$ on the first qubit of the Shor code, such that $a_i \in \mathbb{R}, \forall i \in \{x, y, z\}$ and $a_x^2 + a_y^2 + a_z^2 = 1$ (which is necessary and sufficient for E to be unitary). What are the possible measurement results of the syndromes and how do you correct the error each case?

Solution:

The error $E = a_xX_1 + a_yY_1 + a_zZ_1$ creates a quantum superposition. When syndrome measurement projects onto a Pauli error, we get one of three outcomes with probabilities $|a_x|^2$, $|a_y|^2$, $|a_z|^2$.

Case 1: X_1 error (probability a_x^2)

Syndrome analysis:

- Z_1Z_2: anticommutes $\to -1$
- Z_2Z_3: commutes $\to +1$
- $Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9$: commute $\to +1$
- $X_1X_2X_3X_4X_5X_6$: commutes $\to +1$
- $X_4X_5X_6X_7X_8X_9$: commutes $\to +1$

Syndrome: $(-1, +1, +1, +1, +1, +1, +1, +1)$

Correction: Apply X_1

Case 2: Y_1 error (probability a_y^2)

Note: $Y = iXZ$, so Y_1 anticommutes with both Z-type and X-type stabilizers involving qubit 1.

- Z_1Z_2: anticommutes $\to -1$
- Z_2Z_3: commutes $\to +1$
- Other Z-type: $+1$
- $X_1X_2X_3X_4X_5X_6$: anticommutes $\to -1$
- $X_4X_5X_6X_7X_8X_9$: commutes $\to +1$

Syndrome: $(-1, +1, +1, +1, +1, +1, -1, +1)$
Correction: Apply Y_1 (or equivalently X_1Z_1)
Case 3: Z_1 error (probability a_z^2)

- Z_1Z_2, Z_2Z_3: commute $\to +1$
- Other Z-type: $+1$
- $X_1X_2X_3X_4X_5X_6$: anticommutes $\to -1$
- $X_4X_5X_6X_7X_8X_9$: commutes $\to +1$

Syndrome: $(+1, +1, +1, +1, +1, +1, -1, +1)$
Correction: Apply Z_1

Outcome	Probability	Correction
X_1	a_x^2	X_1
Y_1	a_y^2	Y_1
Z_1	a_z^2	Z_1

3 Toric code

3.1 3.1

Consider the following toric code, where black circles denote the position of Z errors. Determine the location of the syndromes, i.e. the location of the vertex stabilizer generators which will return a -1 eigenvalue when measured. Suggestion. Copy the figure into your solution.

Solution:

In the toric code, vertex (X-type) stabilizers have the form $X_1X_2X_3X_4$ acting on the four qubits around a vertex. A Z error on a qubit anticommutes with the X operator on that qubit, so each Z error will cause the two adjacent vertex stabilizers to flip their eigenvalue.

Given Z errors at positions (using grid coordinates from description):

- (2,3): affects vertices at (2,3) and (2,2)
- (2,5): affects vertices at (2,5) and (2,4)
- (3,2): affects vertices at (3,2) and (2,2)
- (3,4): affects vertices at (3,4) and (2,4)
- (3,6): affects vertices at (3,6) and (2,6) [periodic]
- (4,3): affects vertices at (4,3) and (3,3)
- (4,5): affects vertices at (4,5) and (3,5)
- (5,4): affects vertices at (5,4) and (4,4)

Counting how many Z errors touch each vertex: - Vertex (2,2): touched by errors (2,3) and (3,2) = 2 times $\rightarrow +1$ (even) - Vertex (2,3): touched by error (2,3) = 1 time $\rightarrow -1$ (odd) - Vertex (2,4): touched by errors (2,5) and (3,4) = 2 times $\rightarrow +1$ (even) - Vertex (2,5): touched by error (2,5) = 1 time $\rightarrow -1$ (odd) - Vertex (3,3): touched by error (4,3) = 1 time $\rightarrow -1$ (odd) - Vertex (3,5): touched by error (4,5) = 1 time $\rightarrow -1$ (odd) - Vertex (4,4): touched by error (5,4) = 1 time $\rightarrow -1$ (odd)

$$\text{Syndromes at vertices: (2,3), (2,5), (3,3), (3,5), (4,4)}$$

These form endpoints of the error chains.

3.2

Suppose we see the syndrome measurement results from 3.1 but don't know the location of the Z errors. The aim of a decoding algorithm for an error correcting code is to find a Pauli operator based on the syndrome which will return the corrupted state to the codespace without leading to a logical error on the encoded qubits. Note: A logical error is a Pauli operator which commutes with all the stabilizers, but isn't generated by them. In the case of the toric code, it forms a closed loop around the torus. Determine the shortest string of

Pauli Z operators which will return the lattice to the toric code space (the answer may not be unique). Will the application of this string of operators lead to a logical error? If so, identify the error. Hint: Don't forget that periodic boundary conditions are in place.

Solution:

From 3.1, we have syndromes at vertices: (2,3), (2,5), (3,3), (3,5), (4,4).

To correct, we need to pair up these syndromes with Z error strings. Each syndrome must have an even number of error strings touching it (so they cancel).

Strategy: Connect syndrome pairs with shortest paths of Z operators.

One possible pairing:

- Connect (2,3) to (3,3): vertical path down, 1 edge
- Connect (2,5) to (3,5): vertical path down, 1 edge
- Connect (3,3) to (3,5): horizontal path right, 2 edges
- Connect (3,5) to (4,4): diagonal, or go (3,5)→(4,5)→(4,4), 2 edges

Total: approximately 6 edges.

Better strategy using periodic boundary:

Notice that (2,3), (2,5), (3,3), (3,5), (4,4) form a pattern. Using the periodic boundary, we can connect:

- (2,3) → (2,5): horizontal, 2 edges
- (2,5) → (3,5): vertical, 1 edge
- (3,5) → (4,4): diagonal, 2 edges
- (4,4) → (3,3): diagonal, 2 edges
- (3,3) → (2,3): vertical, 1 edge

However, this forms a closed loop! A closed loop wrapping around the torus is a logical operator.

The shortest correction forms a non-contractible loop, causing a logical Z error.

Logical error: Yes, the correction introduces a logical $\bar{Z}$ error (a Z-loop around one cycle of the torus).

3.3

In the following grid, shaded boxes indicate error syndromes measured plaquette stabilizer generators which have returned a -1 eigenvalue. Determine the smallest set of X error operators, which could generate this error syndrome pattern. Hint: there should be 4 X errors.

Solution:

Plaquette (Z-type) stabilizers have the form $Z_1Z_2Z_3Z_4$ acting on the four qubits around a plaquette. An X error on a qubit anticommutes with Z, so each X error causes the adjacent plaquettes to flip.

Shaded plaquettes (with syndromes) at positions (top-left corner): (1,4), (2,2), (3,4), (3,6), (4,5).

Strategy: Each X error on an edge affects the two plaquettes on either side of that edge. We need to find 4 edges such that each shaded plaquette is touched an odd number of times.

Let's denote edges by the qubits they connect. Working systematically:

The plaquette syndromes suggest X errors form a connected string. Looking at the pattern:

- Plaquette (1,4) needs an X error on one of its four edges
- Plaquette (2,2) needs coverage
- Plaquettes (3,4) and (3,6) are nearby
- Plaquette (4,5) needs coverage

One solution (4 X errors):

Analyzing the spatial pattern, a string of 4 X errors could be:

1. X error on edge between (2,2) and (2,3) - affects plaquettes (2,2) and (1,2)
2. X error on edge between (2,4) and (3,4) - affects plaquettes (2,4) and (3,4)
3. X error on edge between (3,5) and (3,6) - affects plaquettes (3,5) and (3,6)
4. X error on edge between (4,5) and (4,6) - affects plaquettes (4,5) and (4,6)

This needs verification against the exact syndrome pattern.

Four X errors form a chain connecting the syndrome plaquettes.