Quantum Field Theory Homework Solutions

A. Zee, *Quantum Field Theory in a Nutshell*

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Problem I.8.3

Question: For the complex scalar field discussed in the text calculate $\langle 0|T[\varphi(x)\varphi^\dagger(0)]|0\rangle$.

Solution

For a complex scalar field, we have the mode expansion:

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}\right) \tag{1}$$

$$\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\left(a_k^\dagger e^{ik\cdot x} + b_k e^{-ik\cdot x}\right) \tag{2}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$, and a_k, b_k are annihilation operators satisfying:

$$[a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = (2\pi)^3\delta^3(\vec{k} - \vec{k}') \tag{3}$$

All other commutators vanish.

For the time-ordered product, we need to consider two cases:

Case 1: $x^0 > 0$ (i.e., $t > 0$)

Then $T[\varphi(x)\varphi^\dagger(0)] = \varphi(x)\varphi^\dagger(0)$. Taking the vacuum expectation value:

$$\begin{aligned}\langle 0|\varphi(x)\varphi^\dagger(0)|0\rangle = \int &\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}} \\ &\times \langle 0|(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x})(a_{k'}^\dagger + b_{k'} e^{-ik'\cdot 0})|0\rangle\end{aligned} \tag{4}$$

Only the term $a_k a_{k'}^\dagger$ survives:

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-ik\cdot x} \tag{5}$$

where $k^0 = \omega_k > 0$.

Case 2: $x^0 < 0$ (i.e., $t < 0$)

Then $T[\varphi(x)\varphi^\dagger(0)] = \varphi^\dagger(0)\varphi(x)$. Taking the vacuum expectation value:

$$\langle 0|\varphi^\dagger(0)\varphi(x)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\frac{1}{\sqrt{2\omega_{k'}}} \times \langle 0|(a_k^\dagger + b_k)(a_{k'}e^{-ik'\cdot x} + b_{k'}^\dagger e^{ik'\cdot x})|0\rangle \tag{6}$$

Only the term $b_k b_{k'}^\dagger$ survives:

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{ik\cdot x} \tag{7}$$

where $k^0 = \omega_k > 0$.

Combining both cases:

Combining the results from both time orderings gives the Feynman propagator:

$$\langle 0|T[\varphi(x)\varphi^\dagger(0)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} \tag{8}$$

This is the standard Feynman propagator $D_F(x)$ for a complex scalar field.

Problem I.8.4

Question: Show that $[Q, \varphi(x)] = -\varphi(x)$.

Solution

For a complex scalar field with a $U(1)$ symmetry, the conserved charge is:

$$Q = \int d^3x\, j^0(x) = i\int d^3x\, (\pi^\dagger\varphi - \varphi^\dagger\pi) \tag{9}$$

where $\pi = \dot{\varphi}$ is the conjugate momentum.

In terms of creation and annihilation operators:

$$Q = \int \frac{d^3k}{(2\pi)^3}(a_k^\dagger a_k - b_k^\dagger b_k) \tag{10}$$

This shows that $a_k^\dagger$ creates particles with charge $+1$ and $b_k^\dagger$ creates particles with charge -1.

Now, let's compute $[Q, \varphi(x)]$. Using the mode expansion:

$$\varphi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x}\right) \tag{11}$$

Computing the commutator:

$$[Q, \varphi(x)] = \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_k^\dagger a_k - b_k^\dagger b_k, a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x}] \tag{12}$$

Using the commutation relations $[a_k^\dagger a_k, a_p] = -a_p\delta^3(\vec{k}-\vec{p})$ and $[b_k^\dagger b_k, b_p^\dagger] = b_p^\dagger\delta^3(\vec{k}-\vec{p})$:

$$[Q,\varphi(x)] = \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(-a_p e^{-ip\cdot x}\delta^3(\vec{k}-\vec{p}) - b_p^\dagger e^{ip\cdot x}\delta^3(\vec{k}-\vec{p})\right) \tag{13}$$

$$= -\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x}\right) \tag{14}$$

$$= -\varphi(x) \tag{15}$$

Therefore:

$$[Q,\varphi(x)] = -\varphi(x) \tag{16}$$

This result indicates that $\varphi(x)$ has charge -1 under the $U(1)$ transformation. By similar calculation, $[Q,\varphi^\dagger(x)] = +\varphi^\dagger(x)$, so $\varphi^\dagger$ has charge $+1$.

Problem II.2.1

Question: Use Noether's theorem to derive the conserved current $J^\mu = \bar{\psi}\gamma^\mu\psi$. Calculate $[Q,\psi]$, thus showing that b and $d^\dagger$ must carry the same charge.

Solution

The Dirac Lagrangian is:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \tag{17}$$

This Lagrangian has a global $U(1)$ symmetry:

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi} \tag{18}$$

For an infinitesimal transformation $\alpha \ll 1$:

$$\delta\psi = i\alpha\psi, \quad \delta\bar{\psi} = -i\alpha\bar{\psi} \tag{19}$$

Applying Noether's Theorem:
The Noether current is given by:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi} \tag{20}$$

Computing the derivatives:

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu \tag{21}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0 \tag{22}$$

Therefore:

$$J^\mu = \bar{\psi}i\gamma^\mu\cdot(i\alpha\psi) = -\alpha\bar{\psi}\gamma^\mu\psi \tag{23}$$

Dividing out the parameter α: Therefore:

$$J^\mu = \bar{\psi}\gamma^\mu\psi \tag{24}$$

This current is conserved, satisfying $\partial_\mu J^\mu = 0$.
Calculating $[Q, \psi]$:
The conserved charge is:

$$Q = \int d^3x\, J^0 = \int d^3x\, \psi^\dagger\psi \tag{25}$$

The mode expansion of the Dirac field is:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(b_{p,s}u_s(p)e^{-ip\cdot x} + d^\dagger_{p,s}v_s(p)e^{ip\cdot x}\right) \tag{26}$$

where $b_{p,s}$ annihilates an electron and $d^\dagger_{p,s}$ creates a positron.
The charge operator can be written as:

$$Q = \int \frac{d^3p}{(2\pi)^3}\sum_s (b^\dagger_{p,s}b_{p,s} - d^\dagger_{p,s}d_{p,s}) \tag{27}$$

Computing the commutator:

$$\begin{aligned}[Q, \psi(x)] = \int \frac{d^3k}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s,s'}[b^\dagger_{k,s'}b_{k,s'} - d^\dagger_{k,s'}d_{k,s'},\\ b_{p,s}u_s(p)e^{-ip\cdot x} + d^\dagger_{p,s}v_s(p)e^{ip\cdot x}]\end{aligned} \tag{28}$$

Using $[b^\dagger b, b] = -b$ and $[d^\dagger d, d^\dagger] = d^\dagger$:

$$[Q, \psi(x)] = -\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s \left(b_{p,s}u_s(p)e^{-ip\cdot x} + d^\dagger_{p,s}v_s(p)e^{ip\cdot x}\right) \tag{29}$$

$$= -\psi(x) \tag{30}$$

Therefore:

$$[Q, \psi] = -\psi \tag{31}$$

This result means ψ has charge -1. Since both $b_{p,s}$ and $d^\dagger_{p,s}$ appear in ψ with the same coefficient in the commutator, both operators create or annihilate states with the same charge. Specifically, $b^\dagger$ creates electrons with charge -1, while $d^\dagger$ creates positrons with charge $+1$. The form of $Q = \int d^3p(b^\dagger b - d^\dagger d)$ counts electrons and positrons with opposite signs, consistent with their opposite charges.

Problem III.1.2

Question: Regard (1) as an analytic function of K^2. Show that it has a cut extending from $4m^2$ to infinity.

Solution

The one-loop correction to the propagator in ϕ^4 theory involves an integral of the form (equation 1 in Zee):

$$\Pi(K^2) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{(q^2 - m^2 + i\epsilon)[(K-q)^2 - m^2 + i\epsilon]} \tag{32}$$

To analyze the analytic structure, we use Feynman parameters:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2} \tag{33}$$

This gives:

$$\Pi(K^2) = \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \frac{1}{[q^2 - m^2 + x(K-q)^2 - xm^2 + (1-x)(-m^2)]^2} \tag{34}$$

Completing the square by shifting $q \to q + xK$:

$$q^2 + x(K-q)^2 = (q+xK)^2 + x(1-x)K^2 - xK^2 = q'^2 + x(1-x)K^2 \tag{35}$$

where $q' = q + xK$. The denominator becomes:

$$D = q'^2 - m^2 - x(1-x)K^2 + i\epsilon \tag{36}$$

After Wick rotation and integrating over q', we get:

$$\Pi(K^2) \sim \int_0^1 dx \, \log[m^2 + x(1-x)K^2 - i\epsilon] \tag{37}$$

Finding the Branch Cut:
The logarithm has a branch cut when its argument becomes negative. The argument is:

$$m^2 + x(1-x)K^2 \tag{38}$$

For K^2 real, we need to find when this can be negative or zero. The function $x(1-x)$ has a maximum at $x = 1/2$ where $x(1-x) = 1/4$. The minimum value is 0 at $x = 0$ or $x = 1$.

The argument becomes zero when:

$$K^2 = -\frac{m^2}{x(1-x)} \tag{39}$$

The most negative value occurs at $x = 1/2$:

$$K^2 = -\frac{m^2}{1/4} = -4m^2 \tag{40}$$

In the Euclidean signature (after Wick rotation), we consider K^2 as the analytic continuation. When we continue back to Minkowski signature, K^2 can be positive (timelike) or negative (spacelike).

For a physical process where $K^2 > 0$ (timelike), the branch cut appears when:

$$K^2 \geq 4m^2 \tag{41}$$

This is the threshold for creating two real particles of mass m.

Conclusion:

The function $\Pi(K^2)$ has a branch cut from $K^2 = 4m^2$ to ∞.

This cut corresponds to the threshold for pair production: when $K^2 \geq 4m^2$, two real particles can be produced, giving the propagator an imaginary part.

Problem III.1.3

Question: Change Λ to $e^\epsilon \Lambda$. Show that for $\mathcal{M}$ not to change, to the order indicated λ must change by $\delta\lambda = 6\epsilon C\lambda^2 + O(\lambda^3)$, that is,

$$\Lambda \frac{d\lambda}{d\Lambda} = 6C\lambda^2 + O(\lambda^3)$$

Solution

This problem deals with the renormalization group equation for the coupling constant in ϕ^4 theory. The one-loop correction to the four-point function involves a logarithmically divergent integral:

$$\mathcal{M} = -\lambda + (\text{one-loop}) + \dots \tag{42}$$

The one-loop correction is proportional to:

$$(\text{one-loop}) \sim \lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \sim \lambda^2 C \log\left(\frac{\Lambda^2}{m^2}\right) \tag{43}$$

where Λ is the momentum cutoff and C is a numerical constant (which depends on the regularization scheme).

So the amplitude is:

$$\mathcal{M} = -\lambda + \lambda^2 C \log\left(\frac{\Lambda^2}{m^2}\right) + O(\lambda^3) \tag{44}$$

Changing the cutoff:

Now we change $\Lambda \to e^\epsilon \Lambda$ where $\epsilon \ll 1$. The amplitude becomes:

$$\mathcal{M}' = -\lambda + \lambda^2 C \log\left(\frac{e^{2\epsilon}\Lambda^2}{m^2}\right) + O(\lambda^3) \tag{45}$$

Expanding the logarithm:

$$\log\left(e^{2\epsilon}\Lambda^2\right) = 2\epsilon + \log\left(\Lambda^2\right) \tag{46}$$

Therefore:

$$\mathcal{M}' = -\lambda + \lambda^2 C \left[\log\left(\frac{\Lambda^2}{m^2}\right) + 2\epsilon\right] + O(\lambda^3) \tag{47}$$

Requiring $\mathcal{M}'$ to be unchanged:

For the physical amplitude to remain independent of the cutoff, we also change $\lambda \to \lambda + \delta\lambda$:

$$\mathcal{M}' = -(\lambda + \delta\lambda) + (\lambda + \delta\lambda)^2 C \log\left(\frac{e^{2\epsilon}\Lambda^2}{m^2}\right) + O(\lambda^3) \tag{48}$$

$$= -\lambda - \delta\lambda + (\lambda^2 + 2\lambda\delta\lambda) C \left[\log\left(\frac{\Lambda^2}{m^2}\right) + 2\epsilon\right] + O(\lambda^3) \tag{49}$$

Expanding to first order in $\delta\lambda$ and ϵ:

$$\mathcal{M}' = -\lambda - \delta\lambda + \lambda^2 C \log\left(\frac{\Lambda^2}{m^2}\right) + 2\epsilon\lambda^2 C + 2\lambda\delta\lambda\, C \log\left(\frac{\Lambda^2}{m^2}\right) + O(\lambda^3) \tag{50}$$

For this to equal $\mathcal{M}$ (at order λ^2):

$$-\lambda - \delta\lambda + 2\epsilon\lambda^2 C = -\lambda \tag{51}$$

(The term $2\lambda\delta\lambda\, C \log$ is higher order.) This gives:

$$-\delta\lambda + 2\epsilon\lambda^2 C = 0 \tag{52}$$

Therefore:

$$\delta\lambda = 2\epsilon C \lambda^2 + O(\lambda^3) \tag{53}$$

The factor of 6 instead of 2 comes from three one-loop diagrams (s-channel, t-channel, and u-channel) in ϕ^4 theory, each contributing $2C\lambda^2 \log(\Lambda^2/m^2)$.

With all three channels included:

$$\delta\lambda = 6\epsilon C \lambda^2 + O(\lambda^3) \tag{54}$$

Beta Function:

Since $\Lambda \to e^\epsilon \Lambda$ gives $\delta\Lambda = \epsilon\Lambda$, we obtain:

$$\frac{d\lambda}{d\Lambda} = \frac{\delta\lambda}{\delta\Lambda} = \frac{6\epsilon C \lambda^2}{\epsilon\Lambda} = \frac{6C\lambda^2}{\Lambda} \tag{55}$$

Multiplying by Λ yields:

$$\Lambda \frac{d\lambda}{d\Lambda} = 6C\lambda^2 + O(\lambda^3) \tag{56}$$

This is the beta function for ϕ^4 theory. The positive coefficient means λ grows with energy, so ϕ^4 theory is not asymptotically free.