Zee QFT Chapter Problems for HW 11

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Problem 1: Large N Expansion for Real Symmetric Matrices

Consider a random matrix ϕ that is real symmetric rather than Hermitian. Show that the Feynman rules become more complicated but the density of eigenvalues remains the Wigner Semicircle Law in the large N limit.

Modified Feynman Rules

For a Hermitian matrix, the propagator is $\langle \phi_{ij}\phi_{kl} \rangle \propto \delta_{il}\delta_{jk}$. However, for a real symmetric matrix ($\phi_{ij} = \phi_{ji}$ with $\phi_{ij} \in \mathbb{R}$), the propagator admits two distinct contractions:

$$\langle \phi_{ij}\phi_{kl} \rangle = \frac{1}{Nm^2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})$$

In the double-line formalism:

- The first term ($\delta_{ij}\delta_{kl}$) represents the standard planar propagator (untwisted ribbon).
- The second term ($\delta_{il}\delta_{kj}$) connects indices with a "twist" (representing non-orientable topology, like a Möbius strip).

This structure makes the diagrammatic expansion more complicated, as it now includes contributions from non-orientable surfaces.

The Large N Limit

We evaluate the self-energy $\Sigma(z)$ appearing in the Dyson equation $G(z) = [z - \Sigma(z)]^{-1}$.

- **Planar Diagrams (Untwisted):** The loop summation over the internal index contributes a factor of N, canceling the $1/N$ suppression from the propagator. This contribution is of order $O(1)$.
- **Twisted Diagrams:** The delta functions in the twisted term force index crossings that eliminate free index loops. Consequently, there is no factor of N to compensate for the propagator's $1/N$. These diagrams are of order $O(1/N)$.

Thus, in the limit $N \to \infty$, twisted diagrams are suppressed.

Density of Eigenvalues

The Dyson equation reduces to the same quadratic form as in the Hermitian case:

$$G(z) = \frac{1}{z - \frac{1}{m^2}G(z)} \implies m^2 z G(z) - G(z)^2 - m^2 = 0$$

Solving for $G(z)$ and extracting the imaginary part discontinuity across the real axis yields the density of states $\rho(E)$:

$$\rho(E) = -\frac{1}{\pi}\mathrm{Im}\, G(E + i\epsilon) = \frac{m^2}{2\pi}\sqrt{\frac{4}{m^2} - E^2}$$

Setting $a = 2/m$, we recover the **Wigner Semicircle Law**:

$$\rho(E) = \frac{2}{\pi a^2}\sqrt{a^2 - E^2}$$

Problem 2: Axial Gauge Fixing

Show that for any given gauge potential $A_\mu(x)$ and a fixed 4-vector n^μ, there exists a gauge transformation $U(x)$ such that the transformed field A'_μ satisfies the axial gauge condition $n \cdot A'(x) = 0$.

Gauge Transformation Law

Under a local gauge transformation $U(x)$, the non-Abelian gauge field transforms as:

$$A'_\mu = U A_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$$

We seek to impose the condition $n^\mu A'_\mu = 0$.

Deriving the Differential Equation

Contracting the transformation law with n^μ:

$$0 = n^\mu A'_\mu = U(n \cdot A)U^{-1} + \frac{i}{g}(n \cdot \partial U)U^{-1}$$

Multiplying from the right by U, we obtain a first-order partial differential equation for $U(x)$:

$$n^\mu \partial_\mu U(x) = ig\, U(x)\, (n^\mu A_\mu(x))$$

Solution via Wilson Line

We parameterize spacetime along lines parallel to the vector n. Let $x = x_0 + \sigma n$, where x_0 is a point on a boundary surface perpendicular to n. The operator $n \cdot \partial$ becomes the derivative with respect to the parameter σ:

$$\frac{d}{d\sigma} U(\sigma) = ig\, U(\sigma)\, [n \cdot A(x_0 + \sigma n)]$$

This is a linear ODE. Its solution is the path-ordered exponential (Wilson line):

$$U(x) = \mathcal{P} \exp\left(ig \int_0^{\sigma} ds\, n \cdot A(x_0 + sn) \right)$$

Since this integral can always be constructed (provided A is regular), a gauge transformation $U(x)$ satisfying the axial gauge condition always exists.