Zee QFT Chapter Problems for HW 10

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Problem 1: Parity Violation (II.1.7)

Show explicitly that equation (25) violates parity:

$$\mathcal{L} = G\bar{\psi}_{1L}\gamma^\mu\psi_{2L}\bar{\psi}_{3L}\gamma_\mu\psi_{4L} \quad (25)$$

Transformation of Left-Handed Fields under Parity

The left-handed projection is defined as:

$$\psi_L = P_L\psi = \frac{1-\gamma^5}{2}\psi$$

Under a parity transformation ($\vec{x} \to -\vec{x}$), a Dirac spinor transforms as:

$$\psi \xrightarrow{P} \gamma^0\psi$$

Using the anticommutation relation $\{\gamma^0, \gamma^5\} = 0$, we compute the transformation of ψ_L:

$$\psi_L \xrightarrow{P} \frac{1-\gamma^5}{2}(\gamma^0\psi) = \gamma^0\frac{1+\gamma^5}{2}\psi = \gamma^0\psi_R$$

This shows that parity transforms a left-handed field into a right-handed field.

Parity Transformation of the Lagrangian

Applying the parity transformation to the Lagrangian in (25), which consists entirely of left-handed fields ($LLLL$ interaction):

$$\mathcal{L}(\vec{x}) \sim (\ldots\psi_L\ldots\psi_L) \xrightarrow{P} \mathcal{L}'(-\vec{x}) \sim (\ldots\psi_R\ldots\psi_R)$$

Since the transformed Lagrangian $\mathcal{L}'$ describes interactions between right-handed particles and contains no left-handed terms to preserve the form of the original $\mathcal{L}$, we have:

$$\mathcal{L}(\vec{x}) \neq \mathcal{L}(-\vec{x})$$

Therefore, the theory explicitly violates parity.

Problem 2: The Rarita-Schwinger Field (II.3.4)

Show that a spin $\frac{3}{2}$ particle can be described by a vector-spinor $\Psi_{\alpha\mu}$. Find the corresponding equations of motion.

Representation Analysis

A vector-spinor $\Psi_{\alpha\mu}$ carries both a spinor index α (4 components) and a Lorentz vector index μ (4 components), totaling 16 components.

In group theory terms, the tensor product of spin 1 and spin 1/2 decomposes as:

$$1 \otimes \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2}$$

A massive spin-$\frac{3}{2}$ particle requires $2s + 1 = 4$ degrees of freedom. To isolate the spin-$\frac{3}{2}$ part and remove the spin-$\frac{1}{2}$ component and unphysical degrees of freedom, we impose constraints.

The Rarita-Schwinger Equation

The equation of motion combines the Dirac equation with projection constraints:

$$\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu\partial_\rho\Psi_\sigma - m\Psi^\mu = 0$$

This equation implies the following constraints that reduce the degrees of freedom to 4:

1. $(i\partial\!\!\!/ - m)\Psi_\mu = 0$ (Dirac equation for each vector component)
2. $\gamma^\mu\Psi_\mu = 0$ (Vanishing gamma trace to remove spin-1/2 sector)
3. $\partial^\mu\Psi_\mu = 0$ (Vanishing divergence)

Problem 3: Feynman Amplitude for Scalar Theory (II.5.1)

Write down the Feynman amplitude for the diagram in Figure II.5.1 for the scalar theory.

Setup

The diagram represents the self-energy correction to a fermion via a scalar loop. The interaction Lagrangian is:

$$\mathcal{L}_{int} = f\varphi\bar{\psi}\psi$$

This yields a vertex factor of if.

Feynman Rules

Using the standard Feynman rules:

- **Vertices:** Two vertices, each contributing if.
- **Scalar Propagator:** Momentum k, mass μ: $D(k) = \frac{i}{k^2-\mu^2}$.
- **Fermion Propagator:** Momentum $p+k$, mass m: $S_F(p+k) = \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}$.
- **Loop Integral:** $\int \frac{d^4k}{(2\pi)^4}$.

Calculation of the Amplitude

The amplitude $-i\Sigma(p)$ is:

$$-i\Sigma(p) = \int \frac{d^4k}{(2\pi)^4}(if)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(if)\frac{i}{k^2-\mu^2}$$

Simplifying the constants ($i^3 = -i$):

$$\Sigma(p) = -if^2 \int \frac{d^4k}{(2\pi)^4}\frac{\not{p}+\not{k}+m}{((p+k)^2-m^2)(k^2-\mu^2)}$$

Problem 4: Feynman Amplitude for Vector Theory (II.5.2)

Show that the amplitude for the diagram in Figure II.5.3 (Vector theory) matches equation (26).

Setup

We consider the self-energy of a fermion interacting with a massive vector boson (Proca field).

The relevant Feynman rules are:

- **Vertex:** $ie\gamma^\mu$.
- **Vector Propagator:** $D_{\mu\nu}(k) = \frac{-i(g_{\mu\nu}-k_\mu k_\nu/\mu^2)}{k^2-\mu^2}$.

Construction of the Amplitude

Tracing backwards along the fermion line, the amplitude is:

$$\mathcal{M} = \int \frac{d^4k}{(2\pi)^4}\left[\bar{u}(p)(ie\gamma^\nu)\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(ie\gamma^\mu)u(p)\right]\left[\frac{-i(g_{\mu\nu}-\frac{k_\mu k_\nu}{\mu^2})}{k^2-\mu^2}\right]$$

Simplification to Match Eq. (26)

To match the form of Eq (26), we manipulate the vector propagator term. Absorbing the minus sign from the numerator into the tensor term:

$$-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{\mu^2}\right) = i\left(\frac{k_\mu k_\nu}{\mu^2} - g_{\mu\nu}\right)$$

Collecting the constants:

$$(ie) \times (ie) \times i \text{ (fermion)} \times i \text{ (vector)} = (ie)^2 i^2$$

Substituting back yields the target expression:

$$\mathcal{M} = (ie)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - \mu^2} \left(\frac{k_\mu k_\nu}{\mu^2} - g_{\mu\nu}\right) \bar{u}(p) \gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \gamma^\mu u(p)$$

This matches the form given in equation (26).