Homework 5

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Problem I.7.4

For both internal particles in the loop to be real (on-shell), we need $p^2 = m^2$ for each.

Let the two particles have 4-momenta k and $k_1 + k_2 - k$. Then:

$$k^2 = m^2$$
$$(k_1 + k_2 - k)^2 = m^2$$

In the CM frame, $P = k_1 + k_2 = (E, \vec{0})$. Let $k = (k^0, \vec{k})$, then $k' = (E - k^0, -\vec{k})$.

From the on-shell conditions:

$$(k^0)^2 - |\vec{k}|^2 = m^2 \quad \Rightarrow \quad k^0 = \sqrt{|\vec{k}|^2 + m^2}$$
$$(E - k^0)^2 - |\vec{k}|^2 = m^2 \quad \Rightarrow \quad E - k^0 = \sqrt{|\vec{k}|^2 + m^2}$$

Therefore $E = 2k^0 = 2\sqrt{|\vec{k}|^2 + m^2}$.

Since $|\vec{k}|^2 \geq 0$, the minimum energy is at $|\vec{k}|^2 = 0$:

$$E_{\text{min}} = 2m$$

Thus $\boxed{E \geq 2m}$.

Problem I.8.1

(a) Starting from the momentum-space propagator:

$$D_F(x) = i \int \frac{d^{D+1}k}{(2\pi)^{D+1}} \frac{e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon}$$

Do the k^0 integral first:

$$I_0 = \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{-ik^0 t}}{(k^0)^2 - \omega_k^2 + i\epsilon}$$

where $\omega_k = \sqrt{|\vec{k}|^2 + m^2}$.

The poles are at $k^0 \approx \pm\omega_k$ with the $i\epsilon$ prescription shifting them slightly off the real axis.

For $t > 0$: close contour in lower half-plane, pick up pole at $k^0 = \omega_k$:

$$I_0 = -i\frac{e^{-i\omega_k t}}{2\omega_k}$$

For $t < 0$: close contour in upper half-plane, pick up pole at $k^0 = -\omega_k$:

$$I_0 = i\frac{e^{i\omega_k t}}{-2\omega_k}$$

Combining with step functions:

$$\boxed{D_F(x) = \int \frac{d^D k}{(2\pi)^D 2\omega_k}\left[\theta(t)e^{-ik\cdot x} + \theta(-t)e^{ik\cdot x}\right]}$$

(b) Start with the manifestly Lorentz invariant integral:

$$I = \int d^{D+1}k\, \delta(k^2 - m^2)\theta(k^0)$$

Using $\delta((k^0)^2 - \omega_k^2) = \frac{1}{2\omega_k}[\delta(k^0 - \omega_k) + \delta(k^0 + \omega_k)]$ and $\theta(k^0)$ kills the negative pole:

$$I = \int d^D k \int dk^0\, \frac{1}{2\omega_k}\delta(k^0 - \omega_k) = \int \frac{d^D k}{2\omega_k}$$

Since we started with a Lorentz invariant quantity, the measure $d^D k/(2\omega_k)$ is Lorentz invariant.

(c) Define $a_O(\vec{k}) = a_Z(\vec{k})/\sqrt{2\omega_k}$ where a_Z are Zee's operators.
Lorentz transformation: We know $U(\Lambda)a_Z^\dagger(\vec{k})U(\Lambda)^\dagger = a_Z^\dagger(\Lambda\vec{k})$. Then:

$$\begin{aligned}
U(\Lambda)a_O^\dagger(\vec{k})U(\Lambda)^\dagger &= \frac{1}{\sqrt{2\omega_k}}U(\Lambda)a_Z^\dagger(\vec{k})U(\Lambda)^\dagger \\
&= \frac{1}{\sqrt{2\omega_k}}a_Z^\dagger(\Lambda\vec{k}) = \sqrt{\frac{\omega_{\Lambda k}}{\omega_k}}a_O^\dagger(\Lambda\vec{k})
\end{aligned}$$

Commutation relation: Using $[a_Z(\vec{k}), a_Z^\dagger(\vec{p})] = (2\pi)^D 2\omega_k \delta^D(\vec{k} - \vec{p})$:

$$\begin{aligned}
[a_O(\vec{k}), a_O^\dagger(\vec{p})] &= \frac{1}{\sqrt{4\omega_k\omega_p}}[a_Z(\vec{k}), a_Z^\dagger(\vec{p})] \\
&= \frac{(2\pi)^D 2\omega_k \delta^D(\vec{k} - \vec{p})}{2\sqrt{\omega_k\omega_p}} \\
&= (2\pi)^D \delta^D(\vec{k} - \vec{p})
\end{aligned}$$

where in the last step the delta function sets $\omega_p = \omega_k$.

Problem I.8.2

Calculate $\langle k'|H|k\rangle$ where $|k\rangle = a^\dagger(\vec{k})|0\rangle$.
The Hamiltonian is:

$$H = \int \frac{d^D p}{(2\pi)^D 2\omega_p}\omega_p a^\dagger(\vec{p})a(\vec{p})$$

Act H on $|k\rangle$:

$$H|k\rangle = \int \frac{d^D p}{(2\pi)^D 2\omega_p}\omega_p a^\dagger(\vec{p})a(\vec{p})a^\dagger(\vec{k})|0\rangle$$

Using $a(\vec{p})a^\dagger(\vec{k}) = a^\dagger(\vec{k})a(\vec{p}) + [a(\vec{p}), a^\dagger(\vec{k})]$:

$$\begin{aligned} H|k\rangle &= \int \frac{d^D p}{(2\pi)^D 2\omega_p} \omega_p a^\dagger(\vec{p})(2\pi)^D 2\omega_k \delta^D(\vec{p} - \vec{k})|0\rangle \\ &= \omega_k a^\dagger(\vec{k})|0\rangle = \omega_k |k\rangle \end{aligned}$$

So $|k\rangle$ is an eigenstate with eigenvalue ω_k.
Now:

$$\langle k'|H|k\rangle = \omega_k \langle k'|k\rangle$$

Calculate the inner product:

$$\begin{aligned} \langle k'|k\rangle &= \langle 0|a(\vec{k}')a^\dagger(\vec{k})|0\rangle \\ &= \langle 0|[a(\vec{k}'), a^\dagger(\vec{k})]|0\rangle \\ &= (2\pi)^D 2\omega_k \delta^D(\vec{k}' - \vec{k}) \end{aligned}$$

Therefore:

$$\boxed{\langle k'|H|k\rangle = \omega_k (2\pi)^D 2\omega_k \delta^D(\vec{k}' - \vec{k})}$$