Physics 151: Intro to QFT - Final Exam Answers

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Problem 1: Symmetries of the Free Theory

The action is:

$$S_0 = \int dt\, d^D x \left\{ i\phi^\dagger \partial_t \phi - \partial_i \phi^\dagger \partial_i \phi \right\} \tag{1}$$

(i) Conserved Current

For the phase symmetry $\phi \to e^{i\alpha}\phi$, we have $\delta\phi = i\alpha\phi$ and $\delta\phi^\dagger = -i\alpha\phi^\dagger$.

Using Noether's theorem with $\mathcal{L} = i\phi^\dagger \partial_t \phi - \partial_i \phi^\dagger \partial_i \phi$:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \frac{\delta\phi}{\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \frac{\delta\phi^\dagger}{\alpha} \tag{2}$$

For $\mu = 0$: $\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = i\phi^\dagger$, so

$$j^0 = (i\phi^\dagger)(i\phi) = -\phi^\dagger \phi \tag{3}$$

With the usual sign choice (so $Q > 0$ for particles):

$$j^0 = \phi^\dagger \phi \tag{4}$$

For $\mu = i$: $\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} = -\partial_i \phi^\dagger$, so

$$j^i = i(\phi^\dagger \partial_i \phi - \partial_i \phi^\dagger \cdot \phi) \tag{5}$$

Here j^0 is the particle number density, and j^i is the probability current.

(ii) Energy-Momentum Tensor

The energy-momentum tensor comes from spacetime translation symmetry $x^\mu \to x^\mu + \epsilon^\mu$:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \partial^\nu \phi^\dagger - \delta^{\mu\nu} \mathcal{L} \tag{6}$$

The components are:

Energy density ($\mu = \nu = 0$):

$$T^{00} = \partial_i \phi^\dagger \partial_i \phi \tag{7}$$

Momentum density ($\mu = i, \nu = 0$):

$$T^{i0} = i\phi^\dagger \partial_i \phi \tag{8}$$

Stress tensor ($\mu = i, \nu = j$):

$$T^{ij} = -\partial_i \phi^\dagger \partial_j \phi - \partial_i \phi \partial_j \phi^\dagger - \delta^{ij} \mathcal{L} \tag{9}$$

This tensor is conserved because of spacetime translation symmetry.

(iii) Canonical Quantization

The conjugate momentum is:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = i\phi^\dagger(x) \tag{10}$$

From $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^D(\vec{x}-\vec{y})$, we get:

$$[\phi(\vec{x}), \phi^\dagger(\vec{y})] = \delta^D(\vec{x}-\vec{y}) \tag{11}$$

In this non-relativistic theory, there are no antiparticles. The field expansion is:

$$\phi(t,\vec{x}) = \int \frac{d^D k}{(2\pi)^D}\, \hat{a}_{\vec{k}}\, e^{-i\omega_k t + i\vec{k}\cdot\vec{x}} \tag{12}$$

$$\phi^\dagger(t,\vec{x}) = \int \frac{d^D k}{(2\pi)^D}\, \hat{a}^\dagger_{\vec{k}}\, e^{i\omega_k t - i\vec{k}\cdot\vec{x}} \tag{13}$$

where $\omega_k = k^2$ and $[\hat{a}_{\vec{k}}, \hat{a}^\dagger_{\vec{k}'}] = (2\pi)^D \delta^D(\vec{k}-\vec{k}')$.

(iv) Conserved Charge

The conserved charge is:

$$Q = \int d^D x\, j^0 = \int d^D x\, \phi^\dagger \phi \tag{14}$$

Putting in the mode expansion and doing the space integral:

$$Q = \int \frac{d^D k}{(2\pi)^D}\, \hat{a}^\dagger_{\vec{k}} \hat{a}_{\vec{k}} = \hat{N} \tag{15}$$

This is the total particle number operator. In non-relativistic theory, particle number is conserved because there is no particle creation or annihilation at low energies.

Problem 2: Classical Scaling Dimensions

(i) Dynamical Critical Exponent z

The dispersion relation is $\omega = k^2$. Under scaling $x \to \lambda x$, we have $k \to \lambda^{-1} k$, so

$$\omega \sim k^2 \implies t \sim L^2 \tag{16}$$

Comparing with $t \sim L^z$:

$$z = 2 \tag{17}$$

(ii) Scaling Dimension of ϕ

We set $[\partial_t] = 1$ (energy units). From $\omega = k^2$:

$$[\partial_i] = \frac{1}{2}, \quad [d^D x] = -\frac{D}{2} \tag{18}$$

The action $\int dt\, d^D x\, i\phi^\dagger \partial_t \phi$ must be dimensionless:

$$[dt] + [d^D x] + 2[\phi] + [\partial_t] = 0 \tag{19}$$

$$-1 - \frac{D}{2} + 2[\phi] + 1 = 0 \implies [\phi] = \frac{D}{4} \tag{20}$$

Problem 3: Interactions and Renormalization

(i) Critical Dimension

From $[\phi] = D/4$, the interaction term has dimension:

$$[g^2(\phi^\dagger\phi)^2] = [g^2] + 4[\phi] = [g^2] + D \tag{21}$$

For the coupling to be dimensionless, we need $[g^2] = 0$, which requires:

$$D = 2 \tag{22}$$

At $D = 2$, higher terms like $(\phi^\dagger\phi)^3$ have positive dimension (irrelevant), so no new terms are generated. The theory is renormalizable at $D = 2$.

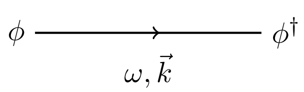
(ii) Feynman Rules

The Lagrangian is:

$$\mathcal{L} = i\phi^\dagger\partial_t\phi - \partial_i\phi^\dagger\partial_i\phi - m^2\phi^\dagger\phi - \frac{g^2}{2}(\phi^\dagger\phi)^2 \tag{23}$$

Propagator: The free part gives $(i\partial_t + \nabla^2 - m^2)\phi = 0$. In momentum space:

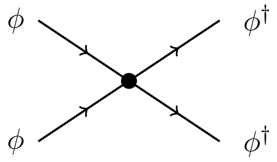
$$G(\omega, \vec{k}) = \frac{i}{\omega - k^2 - m^2 + i\epsilon} \tag{24}$$



$$= \frac{i}{\omega - k^2 - m^2 + i\epsilon}$$

Vertex: The interaction $-\frac{g^2}{2}(\phi^\dagger\phi)^2$ gives a 4-point vertex:

$$V = -ig^2 \tag{25}$$



$$= \quad -ig^2$$

(The factor 2 from pairing cancels the 1/2 in the Lagrangian.)

Feynman Rules:

1. Each propagator (line with arrow): $\dfrac{i}{\omega - k^2 - m^2 + i\epsilon}$
2. Each vertex (4 legs): $-ig^2$
3. Conserve energy-momentum at each vertex
4. Each loop: integrate $\displaystyle\int \frac{d\omega\, d^Dk}{(2\pi)^{D+1}}$
5. Divide by the symmetry factor

Problem 4: Spontaneous Symmetry Breaking

The potential is $V(\phi) = -\mu^2|\phi|^2 + \frac{g^2}{2}|\phi|^4$.

(i) Classical Ground State

Setting $\rho = |\phi|^2$:

$$\frac{dV}{d\rho} = -\mu^2 + g^2\rho = 0 \implies \rho_0 = \frac{\mu^2}{g^2} \tag{26}$$

The VEV is:

$$v = \sqrt{\rho_0} = \frac{\mu}{g} \tag{27}$$

We write the field as:

$$\phi(t, \vec{x}) = (v + \chi)e^{i\theta} \tag{28}$$

where χ is the amplitude fluctuation and θ is the phase.

(ii) Dispersion Relations

Expanding to quadratic order, the Lagrangian becomes:

$$\mathcal{L}^{(2)} = -2v\chi\dot{\theta} - (\nabla\chi)^2 - v^2(\nabla\theta)^2 - 2\mu^2\chi^2 \tag{29}$$

The equations of motion are:

$$v\dot{\theta} = 2\mu^2\chi - \nabla^2\chi \tag{30}$$

$$\dot{\chi} = v\nabla^2\theta \tag{31}$$

Combining these gives:

$$\ddot{\theta} = (2\mu^2 + k^2)k^2\theta \tag{32}$$

So the dispersion relation is:

$$\omega^2 = k^2(2\mu^2 + k^2) \tag{33}$$

$$\omega = k\sqrt{2\mu^2 + k^2} \tag{34}$$

For small k: $\omega \approx \sqrt{2}\mu \cdot k$ (linear, gapless mode).
For large k: $\omega \approx k^2$ (free particle).

By Goldstone's theorem, breaking the $U(1)$ symmetry gives one gapless mode. This mode has linear dispersion at low k, which is typical for non-relativistic systems.

Note: χ and θ are conjugate to each other, so there is only one real mode.

(iii) Large-N Expansion

For a theory with N field components and $SU(N)$ symmetry, we need to keep the combination

$$\lambda = g^2 N = \text{fixed} \tag{35}$$

as $N \to \infty$. This means $g^2 \sim 1/N$.

Problem 5: The Casimir Effect

(i) Fermionic Casimir Effect

For a boson with Dirichlet boundary conditions, the modes are $k_n = \pi n/d$. The zero-point energy is:

$$E_0^{\text{boson}} = \frac{1}{2}\sum_{n=1}^{\infty} \hbar\omega_n = \frac{\pi\hbar c}{2d}\sum_{n=1}^{\infty} n \tag{36}$$

Using zeta regularization ($\sum n = -1/12$):

$$E_0^{\text{boson}} = -\frac{\pi\hbar c}{24d} \tag{37}$$

The force is:

$$F_{\text{boson}} = -\frac{\partial E_0}{\partial d} = -\frac{\pi\hbar c}{24d^2} \quad \text{(attractive)} \tag{38}$$

For fermions, the zero-point energy has opposite sign due to anticommutation:

$$E_0^{\text{fermion}} = -\frac{1}{2}\sum_n \hbar\omega_n \tag{39}$$

With the same boundary conditions and including both components of the 1+1D Dirac fermion:

$$E_0^{\text{fermion}} = 2 \times \left(-\frac{1}{2}\right) \times \frac{\pi\hbar c}{2d} \times \left(-\frac{1}{12}\right) = +\frac{\pi\hbar c}{24d} \tag{40}$$

The force is:

$$F_{\text{fermion}} = +\frac{\pi\hbar c}{24d^2} \quad \text{(repulsive)} \tag{41}$$

The fermion gives a repulsive force with the same size as the boson's attractive force. This is because fermions have opposite-sign zero-point energy due to their statistics.