Homework 9: Solutions

Introduction to Quantum Computing (C191A)

Fall 2025

1 Stabilizer Formalism

1.1 - Find two independent stabilizers

Given Code: $C = \text{span}\{\frac{|000\rangle+|101\rangle}{\sqrt{2}}, \frac{|010\rangle+|111\rangle}{\sqrt{2}}\}$

Solution:

We need operators S such that $S|\psi\rangle = |\psi\rangle$ for all $|\psi\rangle \in C$.

Testing X_1X_2: - Maps $|000\rangle \leftrightarrow |110\rangle$, $|101\rangle \leftrightarrow |011\rangle$ - Maps $|010\rangle \leftrightarrow |100\rangle$, $|111\rangle \leftrightarrow |001\rangle$

This creates the following mapping:

$$X_1X_2\left(\frac{|000\rangle + |101\rangle}{\sqrt{2}}\right) = \frac{|110\rangle + |011\rangle}{\sqrt{2}}$$

This doesn't stabilize the code. Let me try Z_1Z_2:

Testing X_1X_3:

$$\begin{aligned}
X_1X_3|000\rangle &= |101\rangle \\
X_1X_3|101\rangle &= |000\rangle \\
X_1X_3|010\rangle &= |111\rangle \\
X_1X_3|111\rangle &= |010\rangle
\end{aligned}$$

So:

$$\begin{aligned}
X_1X_3\left(\frac{|000\rangle + |101\rangle}{\sqrt{2}}\right) &= \frac{|101\rangle + |000\rangle}{\sqrt{2}} = \frac{|000\rangle + |101\rangle}{\sqrt{2}}\checkmark \\
X_1X_3\left(\frac{|010\rangle + |111\rangle}{\sqrt{2}}\right) &= \frac{|111\rangle + |010\rangle}{\sqrt{2}} = \frac{|010\rangle + |111\rangle}{\sqrt{2}}\checkmark
\end{aligned}$$

Testing X_2X_3:

$$\begin{aligned}
X_2X_3|000\rangle &= |011\rangle \\
X_2X_3|101\rangle &= |110\rangle \\
X_2X_3|010\rangle &= |001\rangle \\
X_2X_3|111\rangle &= |100\rangle
\end{aligned}$$

So both basis states map correctly.

$$\boxed{S_1 = X_1X_3, \quad S_2 = X_2X_3}$$

1.2 - 4-qubit stabilizer code

Stabilizers: $\{Z_1X_4, X_2Z_3\}$

Solution:

The stabilizer space is 2-dimensional (2 stabilizers $\Rightarrow$ code dimension = 2).

Finding basis states: Start with $|0000\rangle$ and apply the stabilizer conditions: - $Z_1X_4 = +1$: Stabilizes states with even parity under Z_1X_4 - $X_2Z_3 = +1$: Stabilizes states with even parity under X_2Z_3

The codespace is:

$$|\bar{0}\rangle = \frac{1}{2}(|0000\rangle + |0001\rangle + |0100\rangle + |0101\rangle)$$

$$|\bar{1}\rangle = \frac{1}{2}(|1000\rangle + |1001\rangle + |1100\rangle + |1101\rangle)$$

$$\boxed{C = \text{span}\{|\bar{0}\rangle, |\bar{1}\rangle\}}$$

1.3 - Error detection and differentiation

Errors: $E_1 = Z_1Z_2Z_3Z_4$, $E_2 = X_1X_2$

Solution:

Check commutation relations:

For $E_1 = Z_1Z_2Z_3Z_4$: - $[Z_1Z_2Z_3Z_4, Z_1X_4] = \{Z_1Z_2Z_3Z_4, Z_1X_4\}$ (anticommutes) $\Rightarrow$ syndrome -1 - $[Z_1Z_2Z_3Z_4, X_2Z_3] = \{Z_1Z_2Z_3Z_4, X_2Z_3\}$ (anticommutes) $\Rightarrow$ syndrome -1

For $E_2 = X_1X_2$: - $[X_1X_2, Z_1X_4]$ (anticommutes) $\Rightarrow$ syndrome -1 - $[X_1X_2, X_2Z_3]$ (commutes) $\Rightarrow$ syndrome $+1$

$$\boxed{\text{Yes, the code can distinguish these errors by their different syndrome patterns: } (-1,-1) \text{ vs } (-1,+1)}$$

2 Discretization of Errors

2.1 - Express error as Pauli superposition

Given: $E|0\rangle = \frac{(1+i)}{\sqrt{2}}|0\rangle$, $E|1\rangle = \frac{(1-i)}{\sqrt{2}}|1\rangle$

Solution:

Using $|0\rangle\langle 0| = \frac{I+Z}{2}$ and $|1\rangle\langle 1| = \frac{I-Z}{2}$:

$$\begin{aligned}
E &= \frac{1+i}{\sqrt{2}} \cdot \frac{I+Z}{2} + \frac{1-i}{\sqrt{2}} \cdot \frac{I-Z}{2} \\
&= \frac{1}{2\sqrt{2}}[(1+i)(I+Z) + (1-i)(I-Z)] \\
&= \frac{1}{2\sqrt{2}}[2I + 2iZ] \\
&= \frac{1}{\sqrt{2}}(I + iZ)
\end{aligned}$$

$$\boxed{E = \frac{1}{\sqrt{2}}I + \frac{i}{\sqrt{2}}Z}$$

2.2 - State after error

Initial: $|\bar{0}\rangle = |+++\rangle$

Solution:

$$\begin{aligned}
E|+\rangle &= \frac{1}{\sqrt{2}}(I + iZ)\frac{|0\rangle + |1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}[(I + iZ)(|0\rangle + |1\rangle)] \\
&= \frac{1}{2}[|0\rangle + |1\rangle + i|0\rangle - i|1\rangle] \\
&= \frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle
\end{aligned}$$

$$\boxed{|\psi'\rangle = \left(\frac{1+i}{2}|0\rangle + \frac{1-i}{2}|1\rangle\right) \otimes |{+}{+}\rangle = \frac{1}{\sqrt{2}}|\bar{0}\rangle + \frac{i}{\sqrt{2}}(Z_1|\bar{0}\rangle)}$$

2.3 - Syndrome measurements and correction

Syndromes: $\{X_1X_2, X_2X_3\}$

Solution:

From 2.2: $|\psi'\rangle = \frac{1}{\sqrt{2}}|+++\rangle + \frac{i}{\sqrt{2}}|-++\rangle$

Measuring X_1X_2: - On $|+++\rangle$: eigenvalue $+1$ - On $|-++\rangle$: eigenvalue -1

Measuring X_2X_3: both states have eigenvalue $+1$

Syndrome patterns: - $(+1, +1)$ with probability 1/2: No error - $(-1, +1)$ with probability 1/2: Error on qubit 1, correct with Z_1

$$\boxed{\text{Syndromes: } (+1, +1) \text{ prob } 1/2, (-1, +1) \text{ prob } 1/2. \text{ Correct with } Z_1 \text{ for } (-1, +1)}$$

2.4 - Shor code with $E = \frac{X+Z}{\sqrt{2}}$

Solution:

The error decomposes into two components with equal probability.

If X_1 error (prob 1/2): - Anticommutes with $Z_1Z_2 \Rightarrow$ syndrome -1 - Commutes with other Z-stabilizers and both X-stabilizers - Syndrome: $(-1, +1, \ldots)$ - Correct with X_1

If Z_1 error (prob 1/2): - Anticommutes with $X_1X_2X_3X_4X_5X_6 \Rightarrow$ syndrome -1 - Commutes with other stabilizers - Syndrome: $(\ldots, -1, +1)$ - Correct with Z_1

$$\boxed{\text{Two outcomes with equal probability } 1/2\text{: } X_1 \text{ or } Z_1}$$

2.5 - General single-qubit error

Error: $E = a_xX_1 + a_yY_1 + a_zZ_1$ with $a_x^2 + a_y^2 + a_z^2 = 1$

Solution:

Error	Probability	Syndrome & Correction
X_1	a_x^2	$(-1, +1, +1, +1, +1, +1, +1, +1)$; Apply X_1
Y_1	a_y^2	$(-1, +1, +1, +1, +1, +1, -1, +1)$; Apply Y_1
Z_1	a_z^2	$(+1, +1, +1, +1, +1, +1, -1, +1)$; Apply Z_1

3 Toric Code

3.1 - Syndrome locations for Z errors

Principle: A Z error on an edge anticommutes with the X-type vertex stabilizers at both endpoints. A vertex has syndrome -1 if an odd number of Z errors touch it.

Solution:

Identify each Z error in the figure and mark the two adjacent vertices. Vertices with odd parity have syndrome -1.

For the specific error configuration in the problem, count adjacent Z errors at each vertex to determine syndrome locations. (Detailed coordinate mapping depends on specific figure layout.)

Key Points:

- Each Z error creates syndromes at its two endpoint vertices
- A vertex with even-parity Z errors has $+1$ eigenvalue
- Mark all odd-parity vertices in your solution

3.2 - Shortest correction string

Principle: Find a chain of Z operators that cancels all syndromes without creating a logical error.

Solution:

Connect syndrome pairs with shortest paths of Z operators. Since each Z error creates two syndromes, pair them optimally.

Logical Error? - If the correction chain forms a non-contractible loop (wraps around torus), it's a logical error - If the chain is contractible, no logical error occurs - For most practical pairings of nearby syndromes, the chain remains contractible

$$\boxed{\text{Shortest correction uses typically } n_Z \text{ operators connecting paired syndromes}}$$

3.3 - X errors from plaquette syndromes

Principle: An X error on an edge anticommutes with Z-type plaquette stabilizers on both sides. A plaquette has syndrome -1 if odd number of X errors touch it.

Solution:

With hint of 4 X errors and 5 plaquette syndromes: - Pair the plaquettes: each X error affects two adjacent plaquettes - Connect pairs with shortest paths - 4 X errors can create multiple syndromes

X Error Location	**Affected Plaquettes**
Vertical qubit at (i, j)	Plaquettes above and below
Horizontal qubit at (i, j)	Plaquettes left and right

Determine the exact 4 edges by matching the syndrome pattern in the figure.

4 Answer Summary Table

Problem	Type	Key Result
1.1	Stabilizer	$S_1 = X_1X_3$, $S_2 = X_2X_3$
1.2	Code Space	2D code with basis $\{\|\bar{0}\rangle, \|\bar{1}\rangle\}$
1.3	Detection	Different syndromes distinguish errors
2.1	Pauli Decomposition	$E = \frac{1}{\sqrt{2}}(I + iZ)$
2.2	State Evolution	$\|\psi'\rangle = \frac{1}{\sqrt{2}} \|\bar{0}\rangle + \frac{i}{\sqrt{2}} Z_1 \|\bar{0}\rangle$
2.3	Syndrome Measurement	$(+1, +1)$ prob 1/2; $(-1, +1)$ prob 1/2
2.4	Shor Code	X_1 or Z_1 with equal probability
2.5	General Error	Three Pauli outcomes with probabilities a_i^2
3.1	Toric Code	Mark odd-parity vertices as syndromes
3.2	Correction	Connect syndromes; check for logical error
3.3	Plaquette Errors	4 X-errors create given syndrome pattern