Problem Set 5: Gate Errors

5.1: Bounding the Error of an Imperfect Rotation

Proof. We are asked to find an upper bound for the inner product $\langle\psi|U_\theta^\dagger U_{\theta+\delta}|\psi\rangle$. This quantity measures the fidelity between the state we wanted to produce, $U_\theta|\psi\rangle$, and the state we actually produced, $U_{\theta+\delta}|\psi\rangle$. A more informative measure of error is how much this inner product deviates from 1 (the ideal case with no error, $\delta = 0$). We will therefore find an upper bound for $|1 - \langle\psi|U_\theta^\dagger U_{\theta+\delta}|\psi\rangle|$.

First, let's simplify the operator product. The unitary operator is $U_\alpha = e^{i\alpha Z}$. Its adjoint is $U_\alpha^\dagger = (e^{i\alpha Z})^\dagger = e^{-i\alpha Z}$.

$$U_\theta^\dagger U_{\theta+\delta} = e^{-i\theta Z} e^{i(\theta+\delta)Z}$$

Since the operators are functions of the same matrix Z, they commute, and we can add the exponents:

$$U_\theta^\dagger U_{\theta+\delta} = e^{(-i\theta+i\theta+i\delta)Z} = e^{i\delta Z} = U_\delta$$

So the expression we need to analyze is $|1 - \langle\psi|U_\delta|\psi\rangle|$.

Let an arbitrary single-qubit state be $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|\alpha|^2 + |\beta|^2 = 1$. The matrix for U_δ is:

$$U_\delta = \begin{bmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{bmatrix}$$

Applying this to $|\psi\rangle$:

$$U_\delta|\psi\rangle = \alpha e^{i\delta}|0\rangle + \beta e^{-i\delta}|1\rangle$$

Now, we can compute the inner product $\langle\psi|U_\delta|\psi\rangle$:

$$\langle\psi|U_\delta|\psi\rangle = (\alpha^*\langle 0| + \beta^*\langle 1|)(\alpha e^{i\delta}|0\rangle + \beta e^{-i\delta}|1\rangle) = |\alpha|^2 e^{i\delta} + |\beta|^2 e^{-i\delta}$$

Now we analyze the error term. Using the identity $1 = |\alpha|^2 + |\beta|^2$:

$$\begin{aligned} 1 - \langle\psi|U_\delta|\psi\rangle &= (|\alpha|^2 + |\beta|^2) - (|\alpha|^2 e^{i\delta} + |\beta|^2 e^{-i\delta}) \\ &= |\alpha|^2(1 - e^{i\delta}) + |\beta|^2(1 - e^{-i\delta}) \end{aligned}$$

To find the bound, we take the magnitude and apply the triangle inequality:

$$\begin{aligned} |1 - \langle\psi|U_\delta|\psi\rangle| &= \big||\alpha|^2(1 - e^{i\delta}) + |\beta|^2(1 - e^{-i\delta})\big| \\ &\leq |\alpha|^2|1 - e^{i\delta}| + |\beta|^2|1 - e^{-i\delta}| \end{aligned}$$

We note that $|1 - e^{-i\delta}| = |(e^{i\delta} - 1)(-e^{-i\delta})| = |e^{i\delta} - 1| \cdot |-e^{-i\delta}| = |e^{i\delta} - 1|$. The two magnitude terms are equal.

$$\begin{aligned} |1 - \langle\psi|U_\delta|\psi\rangle| &\leq |\alpha|^2|e^{i\delta} - 1| + |\beta|^2|e^{i\delta} - 1| \\ &= (|\alpha|^2 + |\beta|^2)|e^{i\delta} - 1| \\ &= |e^{i\delta} - 1| \end{aligned}$$

Using the hint provided, $|e^{i\delta} - 1| \leq |\delta|$, we arrive at the final bound for the deviation from unity:

$$|1 - \langle\psi|U_\theta^\dagger U_{\theta+\delta}|\psi\rangle| \leq |\delta|$$

This shows that the error in the final state scales linearly with the error in the rotation angle. □

5.2: Compounded Errors and the Nature of Quantum Computation

Required Precision for Sequential Rotations When we compose a sequence of s imperfect gates, each with a small error δ, these errors accumulate. While the total error is a complex sum, a reasonable first-order approximation suggests that the errors add up. The result from 5.1 shows that the error introduced by one gate is bounded by $|\delta|$. For a sequence of s such gates, the total accumulated error, E_{total}, will be approximately bounded by the sum of individual errors:

$$E_{total} \lesssim s|\delta|$$

For the total error to be "not significant," it must be less than some small constant, say $\epsilon \ll 1$.

$$s|\delta| < \epsilon \implies |\delta| < \frac{\epsilon}{s}$$

This result is highly significant: it means that for an algorithm's length (or depth) s to double, the required precision of each individual gate must also double (i.e., the error magnitude δ must be halved). The required hardware precision scales inversely with the length of the desired computation.

Implications for the Nature of Quantum Computing This finding directly addresses whether quantum computing is fundamentally analog or digital.

- **Analog Computation:** An analog computer encodes information in continuous physical quantities (e.g., voltage, rotation angle). A defining characteristic of analog systems is that small, continuous errors from physical components accumulate over time and are not corrected. The linear accumulation of error ($E \propto s\delta$) is the hallmark of an analog computational model. The "weak noise regime" described for near-term processors, where δ must shrink as s grows, indicates that these devices are operating as **analog computers**.
- **Digital Computation:** A digital computer encodes information in discrete states (e.g., 0 and 1). Gates are designed to be restorative, meaning they map a range of imperfect inputs (e.g., a voltage close to 0V) to a discrete, ideal output (exactly 0V), thus preventing the accumulation of small errors. This fault tolerance is what allows for extremely deep and complex computations.

Conclusion: The discussion suggests that quantum computing is fundamentally **analog at the physical level**. Qubits are represented by continuous physical states, and gates are continuous physical processes, both of which are susceptible to the linear accumulation of errors.

The eventual goal of creating "error-corrected, fault-tolerant quantum computers" that can tolerate a constant error δ represents a paradigm shift from analog to digital. Quantum Error Correction (QEC) is the process that "digitizes" the computation. By encoding a single logical qubit into many physical qubits and constantly checking for and correcting errors, QEC creates a restorative digital abstraction over the noisy analog hardware. This transition is essential for building scalable quantum computers capable of solving large-scale problems.