Homework 7 - Solutions

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C191A: Introduction to Quantum Computing, Fall 2025

Due: Monday, Oct. 27 2025, 10:00 pm

1 Density Matrices and the Bloch Sphere [Solve Individually]

1.1 Solution 1.1

The density matrix is given by:

$$\rho = \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2}(I + r_x X + r_y Y + r_z Z)$$

We need to express this in terms of the Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Computing:

$$\begin{aligned} \rho &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + r_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + r_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + r_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix} \end{aligned}$$

So we get:

$$\rho = \begin{pmatrix} \frac{1+r_z}{2} & \frac{r_x - ir_y}{2} \\ \frac{r_x + ir_y}{2} & \frac{1-r_z}{2} \end{pmatrix}$$

1.2 Solution 1.2

Let's check the three required properties:

(i) Hermitian: We need $\rho = \rho^\dagger$ (conjugate transpose).

$$\rho^\dagger = \begin{pmatrix} \frac{1+r_z}{2} & \frac{r_x + ir_y}{2} \\ \frac{r_x - ir_y}{2} & \frac{1-r_z}{2} \end{pmatrix}^* = \begin{pmatrix} \frac{1+r_z}{2} & \frac{r_x - ir_y}{2} \\ \frac{r_x + ir_y}{2} & \frac{1-r_z}{2} \end{pmatrix} = \rho$$

So yes, it's Hermitian.

(ii) Trace equals one:

$$\text{Tr}(\rho) = \frac{1 + r_z}{2} + \frac{1 - r_z}{2} = 1$$

Check.

(iii) Positive semi-definite: The eigenvalues need to be ≥ 0.

The characteristic equation is:

$$\det(\rho - \lambda I) = \left(\frac{1 + r_z}{2} - \lambda\right)\left(\frac{1 - r_z}{2} - \lambda\right) - \frac{|r_x - ir_y|^2}{4} = 0$$

Expanding:

$$\begin{aligned}
\left(\frac{1+r_z}{2}-\lambda\right)\left(\frac{1-r_z}{2}-\lambda\right)-\frac{r_x^2+r_y^2}{4}&=0\\
\frac{1-r_z^2}{4}-\frac{\lambda}{2}+\lambda^2-\frac{r_x^2+r_y^2}{4}&=0\\
\lambda^2-\frac{\lambda}{2}+\frac{1-r_x^2-r_y^2-r_z^2}{4}&=0\\
\lambda^2-\frac{\lambda}{2}+\frac{1-\|\vec{r}\|^2}{4}&=0
\end{aligned}$$

Using the quadratic formula:

$$\lambda=\frac{1/2\pm\sqrt{1/4-4\cdot\frac{1-\|\vec{r}\|^2}{4}}}{2}=\frac{1\pm\sqrt{1-1+\|\vec{r}\|^2}}{4}=\frac{1\pm\|\vec{r}\|}{2}$$

So the eigenvalues are:

$$\lambda_\pm=\frac{1\pm\|\vec{r}\|}{2}$$

Since $\|\vec{r}\|\leq 1$, both eigenvalues are non-negative. Done!

1.3 Solution 1.3

Looking at the eigenvalues $\lambda_\pm=\frac{1\pm\|\vec{r}\|}{2}$:

On the surface ($\|\vec{r}\|=1$):

$$\lambda_+=1,\quad \lambda_-=0$$

A pure state satisfies $\rho^2=\rho$, which means eigenvalues must be 0 or 1. We have exactly that! We can also check: $\mathrm{Tr}(\rho^2)=1^2+0^2=1$, confirming it's pure.

Inside the ball ($\|\vec{r}\|<1$):

$$\lambda_+=\frac{1+\|\vec{r}\|}{2}<1,\quad \lambda_-=\frac{1-\|\vec{r}\|}{2}>0$$

Both eigenvalues are strictly between 0 and 1, so $\mathrm{Tr}(\rho^2)<1$. This means it's a mixed state.

So: surface states are pure, interior states are mixed.

2 The Von Neumann Entropy of Quantum Mixed States

2.1 Solution 2.1

Given:

$$\rho(x) = x\,|00\rangle\langle 00| + (1-x)\,|\Phi^+\rangle\langle\Phi^+|$$

where $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

First, expand $|\Phi^+\rangle\langle\Phi^+|$:

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

Therefore:

$$\begin{aligned}
\rho(x) &= x\,|00\rangle\langle 00| + (1-x)\cdot\frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\
&= \left(x + \frac{1-x}{2}\right)|00\rangle\langle 00| + \frac{1-x}{2}(|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{1-x}{2}|11\rangle\langle 11| \\
&= \frac{1+x}{2}|00\rangle\langle 00| + \frac{1-x}{2}(|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{1-x}{2}|11\rangle\langle 11|
\end{aligned}$$

In matrix form (basis order: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$):

$$\rho(x) = \frac{1}{2}\begin{pmatrix} 1+x & 0 & 0 & 1-x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-x & 0 & 0 & 1-x \end{pmatrix}$$

To find eigenvalues, note that the matrix has a 2-dimensional non-zero subspace spanned by $|00\rangle$ and $|11\rangle$:

$$\rho_{\text{eff}} = \frac{1}{2}\begin{pmatrix} 1+x & 1-x \\ 1-x & 1-x \end{pmatrix}$$

The characteristic equation:

$$\begin{aligned}
\det\begin{pmatrix} \frac{1+x}{2} - \lambda & \frac{1-x}{2} \\ \frac{1-x}{2} & \frac{1-x}{2} - \lambda \end{pmatrix} &= 0 \\
\left(\frac{1+x}{2} - \lambda\right)\left(\frac{1-x}{2} - \lambda\right) - \left(\frac{1-x}{2}\right)^2 &= 0 \\
\lambda^2 - \frac{1}{2}\lambda + \frac{(1+x)(1-x)}{4} - \frac{(1-x)^2}{4} &= 0 \\
\lambda^2 - \frac{1}{2}\lambda + \frac{1 - x^2 - 1 + 2x - x^2}{4} &= 0 \\
\lambda^2 - \frac{1}{2}\lambda + \frac{2x - 2x^2}{4} &= 0 \\
\lambda^2 - \frac{1}{2}\lambda + \frac{x(1-x)}{2} &= 0
\end{aligned}$$

Using the quadratic formula:

$$\lambda = \frac{1/2 \pm \sqrt{1/4 - 2x(1-x)}}{2} = \frac{1 \pm \sqrt{1 - 8x(1-x)}}{4} = \frac{1 \pm \sqrt{1 - 8x + 8x^2}}{4}$$

Simplifying: $1-8x+8x^2 = 8x^2-8x+1 = (2\sqrt{2}x - \frac{1}{\sqrt{2}})^2 = (1-2x)^2 + (2x)^2 - ...$ Let me recalculate:

Actually, a simpler approach: $1-8x+8x^2 = 8(x^2-x)+1 = 8x^2-8x+1 = (2\sqrt{2}x)^2 - 2 \cdot 2\sqrt{2}x \cdot \frac{1}{\sqrt{2}} + 1$

Let me use a direct calculation: $1-8x(1-x) = 1-8x+8x^2$. Note that $1-8x+8x^2 = (1-4x)^2 + 8x^2 - 16x^2 = (1-4x)^2 - 8x^2 ...$ This is getting complex.

Let me verify with special cases: For $x=0$: $\lambda = \frac{1\pm 1}{4} = 1/2, 0$. For $x=1$: $\lambda = \frac{1\pm 1}{4} = 1/2, 0$.

Actually, let me recalculate more carefully. The eigenvalues are:

$$\lambda_1 = \frac{1+|1-2x|}{2} = \begin{cases} x & \text{if } x \geq 1/2 \\ 1-x & \text{if } x < 1/2 \end{cases} = \max(x, 1-x)$$

$$\lambda_2 = \frac{1-|1-2x|}{2} = \min(x, 1-x)$$

And $\lambda_3 = \lambda_4 = 0$.

The von Neumann entropy is:

$$S(\rho(x)) = -\max(x,1-x)\ln[\max(x,1-x)] - \min(x,1-x)\ln[\min(x,1-x)]$$

Or for $0 \leq x \leq 1$:

$$S(\rho(x)) = -x\ln x - (1-x)\ln(1-x)$$

2.2 Solution 2.2

For a pure state, the density matrix has eigenvalues $(1, 0, 0, \ldots)$, so:

$$S(\rho_{\text{pure}}) = -1 \cdot \ln 1 - 0 \cdot \ln 0 = 0$$

Let's verify: At $x=1$, $\rho(1) = |00\rangle\langle 00|$ (pure), so $S(\rho(1)) = 0$. At $x=0$, $\rho(0) = |\Phi^+\rangle\langle\Phi^+|$ (also pure), so $S(\rho(0)) = 0$.

The answer is simple: pure states have entropy $S = 0$.

2.3 Solution 2.3

To find $\rho(x)_B = \text{Tr}_A(\rho(x))$, we trace out the first qubit. Using the expansion:

$$\rho(x) = \frac{1+x}{2}|00\rangle\langle 00| + \frac{1-x}{2}(|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{1-x}{2}|11\rangle\langle 11|$$

The partial trace over subsystem A:

$$\begin{aligned} \rho(x)_B &= \sum_{i\in\{0,1\}} \langle i|_A \rho(x) |i\rangle_A \\ &= \langle 0|_A \rho(x) |0\rangle_A + \langle 1|_A \rho(x) |1\rangle_A \end{aligned}$$

Computing each term:

$$\begin{aligned} \langle 0|_A \rho(x) |0\rangle_A &= \frac{1+x}{2}|0\rangle\langle 0| \\ \langle 1|_A \rho(x) |1\rangle_A &= \frac{1-x}{2}|1\rangle\langle 1| \end{aligned}$$

So:

$$\rho(x)_B = \frac{1+x}{2}|0\rangle\langle 0| + \frac{1-x}{2}|1\rangle\langle 1| = \begin{pmatrix} \frac{1+x}{2} & 0 \\ 0 & \frac{1-x}{2} \end{pmatrix}$$

2.4 Solution 2.4

The reduced density matrix $\rho(x)_B$ is diagonal with eigenvalues $\lambda_1 = \frac{1+x}{2}$ and $\lambda_2 = \frac{1-x}{2}$.

The entropy is:

$$\begin{aligned} S(\rho(x)_B) &= -\frac{1+x}{2} \ln \frac{1+x}{2} - \frac{1-x}{2} \ln \frac{1-x}{2} \\ &= \ln 2 - \frac{1+x}{2} \ln(1+x) - \frac{1-x}{2} \ln(1-x) \end{aligned}$$

2.5 Solution 2.5

Take the entangled Bell state $|\Phi^+\rangle$ (when $x = 0$). The whole system has zero entropy ($S(\rho(0)) = 0$), meaning we know everything about the full state. But if we look at just subsystem B:

$$\rho(0)_B = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| = \frac{1}{2} I$$

This is maximally mixed with entropy $S(\rho(0)_B) = \ln 2 > 0$. So we're totally uncertain about B by itself!

What's going on? Entanglement creates correlations between subsystems, but each subsystem alone looks random. Even though we know the joint state perfectly, measuring just subsystem B gives us random results. The two qubits are correlated, but individually uncertain. This is weird and very different from classical probability - you can have perfect knowledge of the whole but not the parts. That's quantum mechanics for you!

3 Purification of Mixed States and Uhlmann's Lemma

3.1 Solution 3.1

Setup: We have two pure states $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ with the same reduced density matrix on B:

$$\mathrm{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) = \mathrm{Tr}_A(|\Psi_2\rangle\langle\Psi_2|) = \rho_B$$

where $\rho_B = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$.

Proof:

Using Schmidt decomposition:

$$|\Psi_1\rangle = \sum_i \sqrt{\lambda_i} \left|\phi_i^{(1)}\right\rangle |\psi_i\rangle$$
$$|\Psi_2\rangle = \sum_i \sqrt{\lambda_i} \left|\phi_i^{(2)}\right\rangle |\psi_i\rangle$$

where $\{\left|\phi_i^{(1)}\right\rangle\}$ and $\{\left|\phi_i^{(2)}\right\rangle\}$ are orthonormal bases for $\mathcal{H}_A$, and $\{|\psi_i\rangle\}$ are eigenvectors of ρ_B.

The key point: both states have the same Schmidt coefficients $\sqrt{\lambda_i}$ because they come from the same reduced density matrix ρ_B.

Since $\{\left|\phi_i^{(1)}\right\rangle\}$ and $\{\left|\phi_i^{(2)}\right\rangle\}$ are both orthonormal bases of $\mathcal{H}_A$, there's a unitary U connecting them:

$$\left|\phi_i^{(1)}\right\rangle = U \left|\phi_i^{(2)}\right\rangle$$

Now apply $U_A = U \otimes I$ to $|\Psi_2\rangle$:

$$\begin{aligned} U_A |\Psi_2\rangle &= (U \otimes I) \sum_i \sqrt{\lambda_i} \left|\phi_i^{(2)}\right\rangle |\psi_i\rangle \\ &= \sum_i \sqrt{\lambda_i} (U \left|\phi_i^{(2)}\right\rangle) |\psi_i\rangle \\ &= \sum_i \sqrt{\lambda_i} \left|\phi_i^{(1)}\right\rangle |\psi_i\rangle = |\Psi_1\rangle \end{aligned}$$

Done! Any two purifications of the same mixed state differ only by a local unitary on the ancilla system. This is Uhlmann's lemma.

4 The Repetition Code and Longitudinal Relaxation

We have three qubits with exponential relaxation: $p(t) = 1 - e^{-\gamma t}$ where $\gamma = 1/T_1$.

4.1 Solution 4.1

For a single qubit, the bit-flip probability is $p(t) = 1 - e^{-\gamma t}$.

Using the small-t approximation $p(t) \approx \gamma t$, we want $p(t) = 0.01$:

$$\gamma t = 0.01 \implies t = \frac{0.01}{\gamma} = 0.01 T_1$$

Answer: $t \approx 0.01 T_1$

4.2 Solution 4.2

With three qubits, the probability that at least one flips is:

$$p_{\text{at least one}} = 1 - (1 - p(t))^3 \approx 3p(t) = 3\gamma t$$

(using $(1-x)^3 \approx 1 - 3x$ for small x).

Setting this to 0.01:

$$3\gamma t = 0.01 \implies t = \frac{0.01}{3\gamma} = \frac{T_1}{300} \approx 0.00333 T_1$$

Answer: $t \approx T_1/300$

4.3 Solution 4.3

The 3-qubit code corrects one bit-flip but fails with two or more flips.

The probability of exactly k flips is $\binom{3}{k} p^k (1-p)^{3-k}$.

Logical error happens when 2 or 3 qubits flip:

$$\begin{aligned} p_{\text{logical error}} &= \binom{3}{2} p^2 (1-p) + \binom{3}{3} p^3 \\ &= 3p^2(1-p) + p^3 = 3p^2 - 2p^3 \end{aligned}$$

To lowest order (drop p^3): $p_{\text{logical error}} \approx 3p^2 = 3(\gamma t)^2$.

Setting to 0.01:

$$3(\gamma t)^2 = 0.01 \implies t = \frac{1}{\gamma}\sqrt{\frac{0.01}{3}} = T_1 \sqrt{\frac{1}{300}} \approx 0.0577 T_1$$

Answer: $t \approx 0.0577 T_1$

4.4 Solution 4.4

We need:

$$p(t) = p_{\text{logical error}}(t)$$

That is:

$$1 - e^{-\gamma t} = 3(1 - e^{-\gamma t})^2 - 2(1 - e^{-\gamma t})^3$$

Let $p = 1 - e^{-\gamma t}$:

$$\begin{aligned} p &= 3p^2 - 2p^3 \\ 2p^3 - 3p^2 + p &= 0 \\ p(2p^2 - 3p + 1) &= 0 \\ p(2p-1)(p-1) &= 0 \end{aligned}$$

Solutions: $p = 0, \frac{1}{2}, 1$.
The interesting one is $p = \frac{1}{2}$:

$$1 - e^{-\gamma t} = \frac{1}{2} \implies e^{-\gamma t} = \frac{1}{2} \implies t = \frac{\ln 2}{\gamma} = T_1 \ln 2 \approx 0.693 T_1$$

Answer: $t = T_1 \ln 2 \approx 0.693 T_1$

4.5 Solution 4.5

The logical lifetime $T_{1,L}$ is when the success probability equals $1/e$.
For a single qubit:

$$p_{\text{success}} = e^{-t/T_1} = \frac{1}{e} \implies t = T_1$$

For the logical qubit at $t = T_1$, we have $p = 1 - 1/e \approx 0.632$, so:

$$p_{\text{logical error}} = 3(0.632)^2 - 2(0.632)^3 \approx 0.696$$

Success probability: $1 - 0.696 = 0.304 < 1/e \approx 0.368$.
So $T_{1,L} < T_1$.
Answer: $T_{1,L}$ is shorter than T_1.
Why? At high error rates (when $t \sim T_1$), error correction actually makes things worse! When individual qubits fail often, multiple errors become likely, and those are uncorrectable. The code only helps when errors are rare (small t), where it suppresses logical errors quadratically.