Solutions to Physics 151 Mid-Term Examination
Introduction to Quantum Field Theory

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1 Problem 1: Vacuum Energy in ϕ^4 Theory

We consider the action for a real scalar field $\phi(x^\mu)$ in four spacetime dimensions:

$$S = \int d^4x \left\{ \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \right\}. \tag{1}$$

1.1 Part (i) & (ii): Connected Vacuum Diagrams and Symmetry Factors

Vacuum diagrams are Feynman diagrams with no external legs. We construct these by contracting all fields at interaction vertices. The symmetry factor S of a diagram is defined as the number of ways the diagram remains invariant under permutations of its elements.

Order λ^1: Figure-Eight Diagram

Topology: At first order in λ, there will be a single 4-point vertex from the ϕ^4 interaction. All four fields must be contracted in pair to form two loops sharing a common vertex, creating a "8" structure.

Diagram: Two propagator loops attached to the same vertex.

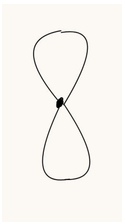

Figure 1: Vacuum diagram (order λ^1).

Symmetry Factor: We count all permutations leaving the diagram invariant:

- Interchange the two loops: factor of 2
- Within each loop, the two propagators can be swapped: factor of $2 \times 2 = 4$

Total symmetry factor: $S = 2 \times 2 \times 2 = 8$.

Order λ^2: Two Distinct Topologies

At second order in λ, we have two vertices with eight half-lines total. All must be contracted to form vacuum diagrams.

Diagram A:

Topology: Two vertices connected by two propagators, with each vertex having an additional self-loop.

Diagram:

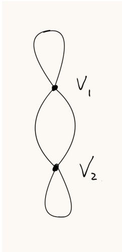

Figure 2: Vacuum diagram A (order λ^2).

Symmetry Factor:

- Interchange the two vertices: factor of 2
- Swap the two propagators connecting the vertices: factor of 2
- Swap the two propagators in the loop at vertex 1: factor of 2
- Swap the two propagators in the loop at vertex 2: factor of 2

Total symmetry factor: $S = 2^4 = 16$.

(b) Diagram B:

Topology: Two vertices connected by four parallel propagators (no self-loops).

Diagram:

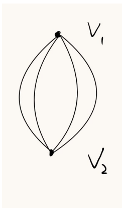

Figure 3: Vacuum diagram B (order λ^2).

Symmetry Factor:

- Interchange the two vertices: factor of 2
- Permute all four propagators: factor of $4! = 24$

Total symmetry factor: $S = 2 \times 24 = 48$.

1.2 Part (iii): Mass Dimension of λ in d Dimensions

In natural units ($\hbar = c = 1$), the action S must be dimensionless. We work in $d = D + 1$ spacetime dimensions.

- The integration measure $d^d x$ has dimension $[d^d x] = -d$ (since $[x] = -1$ in energy units).
- The Lagrangian density must satisfy $[\mathcal{L}] = d$ to make $S = \int d^d x \, \mathcal{L}$ dimensionless.

- From the kinetic term $(\partial_\mu\phi)^2$: with $[\partial_\mu] = 1$, we have

$$[(\partial\phi)^2] = 2([\partial] + [\phi]) = 2(1 + [\phi]) = d. \tag{2}$$

  This gives $[\phi] = \frac{d-2}{2}$.

- From the interaction term $\lambda\phi^4$:

$$[\lambda] + 4[\phi] = d \implies [\lambda] = d - 4[\phi] = d - 4 \cdot \frac{d-2}{2} = d - 2(d-2) = 4 - d. \tag{3}$$

Summarize: In d spacetime dimensions, $[\lambda] = 4 - d$. Specifically:

- In $d = 4$: $[\lambda] = 0$ (dimensionless, marginal coupling)
- In $d < 4$: $[\lambda] > 0$ (relevant coupling)
- In $d > 4$: $[\lambda] < 0$ (irrelevant coupling)

1.3 Part (iv): Mass Dimension of g for ϕ^3 Interaction

If we add a cubic interaction $g\phi^3$ to the Lagrangian, we apply the same dimensional analysis:

In $d = 4$ dimensions:

$$[g\phi^3] = [g] + 3[\phi] = 4 \implies [g] = 4 - 3 \times 1 = 1. \tag{4}$$

General formula in d dimensions:

$$[g] = d - 3[\phi] = d - 3 \cdot \frac{d-2}{2} = d - \frac{3(d-2)}{2} = \frac{2d - 3d + 6}{2} = \frac{6-d}{2}. \tag{5}$$

In $d = 4$ dimensions, $[g] = 1$ (dimension of mass). In general d dimensions, $[g] = \frac{6-d}{2}$.

2 Problem 2: Scaling Dimensions of Free Fields

We compare the scaling properties of relativistic and non-relativistic free scalar field theories in $d = D + 1$ spacetime dimensions.

2.1 Part (i): Scaling Dimension of Relativistic Field ϕ

The relativistic free field has action:

$$S_{\text{rel}} = \frac{1}{2} \int d^d x \, (\partial_\mu\phi)(\partial^\mu\phi). \tag{6}$$

Dimensional Analysis:

- The measure $d^d x$ has dimension $[d^d x] = -d$.
- For dimensionless action, we require $[\mathcal{L}_{\text{rel}}] = d$.
- The derivative has $[\partial_\mu] = 1$ (energy units).
- From $[(\partial\phi)^2] = d$, we obtain:

$$2([\partial] + [\phi]) = d \implies 2(1 + [\phi]) = d \implies [\phi] = \frac{d-2}{2}. \tag{7}$$

In d spacetime dimensions, the relativistic scalar field has scaling dimension

$$[\phi] = \frac{d-2}{2}$$

For example, in $d = 4$: $[\phi] = 1$ (dimension of mass).

2.2 Part (ii): Scaling Dimension of Non-Relativistic Field Φ

The non-relativistic (Lifshitz) scalar has action:

$$S_{\text{nonrel}} = \frac{1}{2}\int dt\, d^D x \left\{ (\dot{\Phi})^2 - (\partial_i\partial_i\Phi)(\partial_j\partial_j\Phi) \right\}. \tag{8}$$

The action S_{nonrel} exhibits anisotropic scaling where time and space scale differently.

The two kinetic terms must have the same dimension. Setting $[\partial_t] = 1$ and $[\partial_i] = z$, because $E = -i\partial_t$:

$$[(\dot{\Phi})^2] = [(\nabla^2\Phi)^2] \implies 2(1 + [\Phi]) = 2(2z + [\Phi]) \implies z = 1/2. \tag{9}$$

This implies $[t] = -1$ and $[x^i] = -1/2$.

Field Dimension: The action must be dimensionless. The integration measure gives $[dt\, d^D x] = -1 - D/2$, so we need $[\mathcal{L}_{\text{nonrel}}] = 1 + D/2$. Equating this to the dimension of the time-derivative term:

$$2(1 + [\Phi]) = 1 + D/2 \implies [\Phi] = \frac{D-2}{4}. \tag{10}$$

he non-relativistic Lifshitz scalar has scaling dimension $[\Phi] = \frac{D-2}{4}$ in $d = D + 1$ dimensions. For example, in $d = 4$ (i.e., $D = 3$): $[\Phi] = 1/4$.

2.3 Part (iii): Feynman Propagator for the Lifshitz Scalar

Derivation from the Equation of Motion:

The Euler-Lagrange equation from S_{nonrel} gives:

$$\frac{\delta S}{\delta \Phi} = 0 \implies -\partial_t^2\Phi + \nabla^4\Phi = 0. \tag{11}$$

In momentum space with Fourier transform $\Phi(t, \vec{x}) = \int \frac{d\omega\, d^D k}{(2\pi)^{D+1}} e^{-i\omega t + i\vec{k}\cdot\vec{x}} \tilde{\Phi}(\omega, \vec{k})$:

- $\partial_t \to -i\omega$
- $\nabla^2 \to -k^2$ where $k = |\vec{k}|$

The equation becomes:

$$\omega^2 - k^4 = 0. \tag{12}$$

Propagator with $i\epsilon$ Prescription:

The Feynman propagator is obtained by inverting the kinetic operator with the standard $i\epsilon$ prescription for time-ordering:

$$D_F(\omega, \vec{k}) = \frac{i}{\omega^2 - k^4 + i\epsilon}. \tag{13}$$

Normalization: The numerator is normalized to i (not $-i$) for consistency with the canonical commutation relations. The scalar field Φ has a single degree of freedom (one independent polarization), justifying the simple numerator.

Position Space Form:

$$D_F(t - t', \vec{x} - \vec{x}') = \int \frac{d\omega\, d^D k}{(2\pi)^{D+1}} \frac{i\, e^{-i\omega(t-t') + i\vec{k}\cdot(\vec{x}-\vec{x}')}}{\omega^2 - k^4 + i\epsilon}. \tag{14}$$

The Feynman propagator is $D_F(\omega, \vec{k}) = \frac{i}{\omega^2 - k^4 + i\epsilon}$, with numerator normalized to unity (times i) reflecting the single polarization of the scalar field.

3 Problem 3: Renormalization Group Flow in Lifshitz Theory

We extend the non-relativistic Lifshitz scalar theory by adding two Gaussian (quadratic) terms:

$$\hat{S}_{\text{nonrel}} = \frac{1}{2}\int dt\, d^3x \left\{ (\dot{\Phi})^2 - (\partial_i\partial_i\Phi)(\partial_j\partial_j\Phi) - c^2(\partial_i\Phi)(\partial_i\Phi) - m^2\Phi^2 \right\}. \tag{15}$$

We now analyze this theory in $d = 4$ spacetime dimensions (i.e., $D = 3$ spatial dimensions) from the perspective of renormalization group (RG) flow.

3.1 Part (i): Relevance of the Couplings c^2 and m^2

From Problem 2(ii), we established that for the Lifshitz scalar in D spatial dimensions:

- Dynamical exponent: $z = 2$
- Field dimension: $[\Phi] = \frac{D-2}{4}$
- Lagrangian density dimension: $[\mathcal{L}] = 2 + D$

For $D = 3$, we have:

- $[\Phi] = \frac{3-2}{4} = \frac{1}{4}$
- $[\mathcal{L}] = 2 + 3 = 5$
- $[\partial_i] = 1$ (spatial derivative)

Dimension of m^2:
From the mass term $m^2\Phi^2$, requiring $[m^2\Phi^2] = 5$:

$$[m^2] + 2[\Phi] = 5 \implies [m^2] + 2 \times \frac{1}{4} = 5 \implies [m^2] = 5 - \frac{1}{2} = \frac{9}{2}. \tag{16}$$

Since $[m^2] = \frac{9}{2} > 0$, the coupling m^2 is **relevant**.
Dimension of c^2:
From the gradient term $c^2(\partial_i\Phi)^2$, requiring $[c^2(\partial_i\Phi)^2] = 5$:

$$[c^2] + 2([\partial_i] + [\Phi]) = 5 \implies [c^2] + 2\left(1 + \frac{1}{4}\right) = 5 \implies [c^2] + \frac{5}{2} = 5. \tag{17}$$

Therefore:

$$[c^2] = 5 - \frac{5}{2} = \frac{5}{2}. \tag{18}$$

Since $[c^2] = \frac{5}{2} > 0$, the coupling c^2 is also **relevant**.
Both couplings are **relevant**, meaning they grow under RG flow toward the infrared (low energies).

3.2 Part (ii): Qualitative RG Flow Patterns

Fixed Point Structure:
The point $(c^2, m^2) = (0, 0)$ corresponds to the pure Lifshitz theory without the additional perturbations. This is a critical point of the RG flow.
Flow Behavior:

- Since both c^2 and m^2 are relevant couplings with positive mass dimensions, they grow as we flow toward the infrared (decreasing energy scale).

- The origin $(0, 0)$ is an **ultraviolet (UV) unstable fixed point**—the Lifshitz fixed point is attractive in the UV direction but repulsive in the IR.
- Starting from any point $(c^2, m^2) \neq (0, 0)$ in the parameter space, RG flow moves away from the origin as energy decreases.

Flow Pattern Description:

- Near the origin, both couplings increase under RG flow toward lower energies.
- The flow is radially outward from the Lifshitz point $(0, 0)$.
- The eigendirections of the flow are determined by the RG eigenvalues $\lambda_{c^2} = [c^2] = 5/2$ and $\lambda_{m^2} = [m^2] = 9/2$.
- Since $\lambda_{m^2} > \lambda_{c^2}$, the m^2 direction is "more relevant" and grows faster.

The RG flow exhibits a UV-stable (IR-unstable) fixed point at the origin. All flow lines move radially outward from $(0, 0)$ toward the infrared, with m^2 growing faster than c^2.

3.3 Part (iii): Emergent Relativistic Lorentz Symmetry

Question: Is there an RG fixed point exhibiting emergent relativistic symmetry?

Analysis:

For relativistic symmetry to emerge, the dispersion relation must become $\omega^2 \sim k^2$ (light-cone structure) rather than $\omega^2 \sim k^4$ (Lifshitz scaling).

In the full action $\hat{S}_{\text{nonrel}}$, the equation of motion in momentum space is:

$$\omega^2 - k^4 - c^2 k^2 - m^2 = 0. \tag{19}$$

At high momenta (UV limit, $k \to \infty$):

- The k^4 term dominates: $\omega^2 \approx k^4$, giving Lifshitz scaling.
- This corresponds to the Lifshitz fixed point $(c^2, m^2) = (0, 0)$.

At low momenta (IR limit, $k \to 0$):

- If $c^2 > 0$ and $c^2 \gg m^2/k^2$, the $c^2 k^2$ term dominates over both k^4 and m^2.
- The dispersion becomes $\omega^2 \approx c^2 k^2$, which is relativistic with effective "speed of light" c.
- The k^4 term becomes negligible in comparison (higher-derivative/irrelevant operator).

Fixed Point Location:

The emergent relativistic symmetry does not appear at a traditional fixed point but rather in the **infrared regime** where c^2 and m^2 have grown large.

More precisely, we can identify an effective IR theory:

- In the regime where $c^2 k^2 \gg k^4$ and $m^2 \ll c^2 k^2$, the theory reduces to:

$$S_{\text{eff}} \approx \frac{1}{2} \int dt\, d^3x \left\{ (\dot{\Phi})^2 - c^2 (\partial_i \Phi)^2 \right\}, \tag{20}$$

  which is manifestly relativistic with Lorentz symmetry and speed c.
- This corresponds to a "Lorentz-invariant Gaussian fixed point" in the extended theory.

So yes, emergent relativistic Lorentz symmetry appears in the **infrared (low-energy) limit**. As c^2 grows relevant under RG flow, the theory flows toward an effective relativistic theory where the dispersion relation becomes $\omega^2 = c^2 k^2 + m^2$. The Lifshitz k^4 term becomes an irrelevant higher-derivative correction in this regime.

This is an example of **emergent Lorentz symmetry**, where a fundamentally non-relativistic UV theory flows to a relativistic IR fixed point. The phenomenon occurs not at the UV Lifshitz point $(0, 0)$, but in the IR regime where $c^2 \to \infty$ (in dimensionless RG flow).

4 Problem 4: Propagators and Green's Functions

We return to the relativistic ϕ^4 theory from Problem 1, focusing on tree-level propagators.

4.1 Part (i): Advanced and Feynman Propagators in Momentum Space

The free Klein-Gordon equation is:

$$(\Box + m^2)\phi(x) = 0 \quad \text{where} \quad \Box = \partial_\mu \partial^\mu. \tag{21}$$

The propagators are Green's functions satisfying:

$$(\Box_x + m^2)D(x-y) = -i\delta^{(4)}(x-y). \tag{22}$$

In momentum space, with Fourier convention:

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \tilde{D}(k), \tag{23}$$

we have $\Box \to -k^2 = -k^\mu k_\mu = -(k_0^2 - \vec{k}^2)$.

The equation becomes:

$$(-k^2 + m^2)\tilde{D}(k) = -i \implies (k^2 - m^2)\tilde{D}(k) = i. \tag{24}$$

(a) Feynman Propagator $D_F(x-y)$:

The Feynman propagator time-orders the fields and uses the $+i\epsilon$ prescription in the denominator to specify contour integration around the poles:

$$\boxed{D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}} \tag{25}$$

Alternatively, in terms of energy-momentum:

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik^0(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2 - \vec{k}^2 - m^2 + i\epsilon}. \tag{26}$$

The $i\epsilon$ prescription places both poles slightly below the real axis in the complex k^0 plane, ensuring causality and correct time-ordering.

(b) Advanced Propagator $D_A(x-y)$:

The advanced propagator vanishes for $x^0 < y^0$ and propagates disturbances backward in time. The $i\epsilon$ prescription is modified:

$$\boxed{D_A(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2 - m^2 - i\epsilon}} \tag{27}$$

Equivalently:

$$D_A(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik^0(x^0-y^0)+i\vec{k}\cdot(\vec{x}-\vec{y})}}{k_0^2 - \vec{k}^2 - m^2 - i\epsilon}. \tag{28}$$

The sign flip in $i\epsilon$ places both poles slightly above the real k^0 axis, giving the advanced boundary condition.

4.2 Part (ii): Differential Equation Satisfied by the Propagators

Question: Do $D_F(x-y)$ and $D_A(x-y)$ satisfy the same differential equation?

Yes, both propagators satisfy the same inhomogeneous Klein-Gordon equation:

$$\boxed{(\Box_x + m^2)D(x-y) = -i\delta^{(4)}(x-y)} \tag{29}$$

Proof:

We apply the Klein-Gordon operator to either propagator in momentum space:

$$(\Box_x + m^2)D(x-y) = \int \frac{d^4k}{(2\pi)^4}(-k^2+m^2)\tilde{D}(k)e^{-ik\cdot(x-y)}. \tag{30}$$

For both the Feynman and advanced propagators:

$$\tilde{D}(k) = \frac{i}{k^2 - m^2 \pm i\epsilon}, \tag{31}$$

we compute:

$$(-k^2+m^2)\tilde{D}(k) = (-(k^2-m^2)) \cdot \frac{i}{k^2-m^2 \pm i\epsilon}. \tag{32}$$

In the limit $\epsilon \to 0^+$ (after integration), this becomes:

$$= \frac{-(k^2-m^2)\cdot i}{k^2-m^2 \pm i\epsilon} \to -i, \tag{33}$$

independent of the sign of $i\epsilon$. The $\pm i\epsilon$ prescription affects *how* we integrate (contour choice), not the formal algebraic result.

Therefore:

$$(\Box_x + m^2)D(x-y) = \int \frac{d^4k}{(2\pi)^4}(-i)e^{-ik\cdot(x-y)} = -i\delta^{(4)}(x-y). \tag{34}$$

So both D_F and D_A are Green's functions for the same differential operator $(\Box + m^2)$, satisfying identical equations. They differ only in their boundary conditions (time-ordering vs. advanced causality), which is encoded in the $i\epsilon$ prescription and manifests in different analytic properties, but not in the formal differential equation itself.