Homework 9

C191A: Introduction to Quantum Computing, Fall 2025

Due: Monday, Nov. 10 2025, 10:00 pm

Instructions. Submit your homework to Gradescope (Entry Code: B3YK46) by 10:00 pm (Pacific time) on the due date listed above. No late submissions are accepted, since the solutions will be posted immediately after the deadline.

You are encouraged to collaborate with your peers on problem sets, as well as the usage of AI tools (e.g., ChatGPT, Bard, etc.) for learning purposes — such as asking for hints, clarifications, or alternative explanations — but not as a substitute for doing the problems yourself. If an AI system or a peer significantly helps you in your problem-solving process, you should acknowledge them in your submission (e.g., by listing their name or the tool you used on that problem).

However, one problem per homework (the first) is labeled "solve individually", so that you can honestly gauge your grasp of the material. Ultimately, you are responsible for engaging with the coursework in the way that helps you learn most effectively.

1 Stabilizer Formalism [Solve Individually]

1.1

Consider the code $C = \text{span}\left\{\frac{|000\rangle+|101\rangle}{\sqrt{2}}, \frac{|010\rangle+|111\rangle}{\sqrt{2}}\right\}$. Find two independent operators that stabilize this code.

1.2

What are the 4-qubit states stabilized by the operators $\{\mathbf{Z}_1\mathbf{X}_4, \mathbf{X}_2\mathbf{Z}_3\}$?

1.3

Would the code you found in **1.2** be able to detect the error $\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3\mathbf{Z}_4$? Would it be able to differentiate this error from a different error that acts on less qubits, like $\mathbf{X}_1\mathbf{X}_2$ and fix it accordingly? Why or why not?

2 Discretization of errors

Consider the phase flip code with logical qubit encoding

$$|\bar{0}\rangle = |+++\rangle$$
$$|\bar{1}\rangle = |---\rangle$$

A unitary error $\mathbf{E}$ is applied to the first qubit: $\mathbf{E}|0\rangle \to \frac{(1+i)}{\sqrt{2}}|0\rangle$ and $\mathbf{E}|1\rangle \to \frac{(1-i)}{\sqrt{2}}|1\rangle$.

2.1

Express this error $\mathbf{E}$ as a superposition of Pauli operators.

2.2

For an initial state $|\psi\rangle = |\bar{0}\rangle$, what is the state after the error?

2.3

What is the output distribution of the syndromes $\{\mathbf{X}_1\mathbf{X}_2, \mathbf{X}_2\mathbf{X}_3\}$? How would you correct the error for each possible scenario?

Now consider the Shor code encoded as

$$|\bar{0}\rangle = \left(\frac{|000\rangle + |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$
$$|\bar{1}\rangle = \left(\frac{|000\rangle - |111\rangle}{\sqrt{2}}\right)^{\otimes 3}$$

with stabilizer operators:

$$\{\mathbf{Z}_1\mathbf{Z}_2,\ \mathbf{Z}_2\mathbf{Z}_3,\ \mathbf{Z}_4\mathbf{Z}_5,\ \mathbf{Z}_5\mathbf{Z}_6,\ \mathbf{Z}_7\mathbf{Z}_8,\ \mathbf{Z}_8\mathbf{Z}_9,\ \mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6,\ \mathbf{X}_4\mathbf{X}_5\mathbf{X}_6\mathbf{X}_7\mathbf{X}_8\mathbf{X}_9\}$$

2.4

An error $\mathbf{E} = \frac{\mathbf{X}+\mathbf{Z}}{\sqrt{2}}$ happens on the first qubit. What are the possible results of the syndrome measurements? How do you correct the error in each outcome?

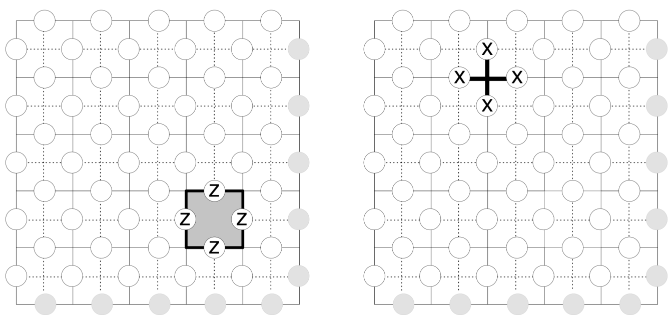
2.5

Now consider a general unitary single qubit error $\mathbf{E} = a_x\mathbf{X}_1 + a_y\mathbf{Y}_1 + a_z\mathbf{Z}_1$ on the first qubit of the Shor code, such that $a_i \in \mathbb{R}$, $\forall i \in \{x, y, z\}$ and $a_x^2 + a_y^2 + a_z^2 = 1$ (which is necessary and sufficient for $\mathbf{E}$ to be unitary).

What are the possible measurement results of the syndromes and how do you correct the error each case?

3 Toric code

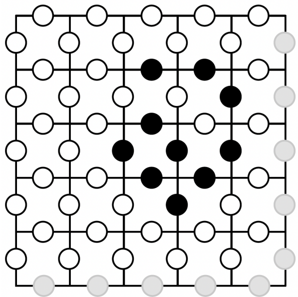
The toric code is a well-known family of two-dimensional stabilizer codes. It has two types of stabilizer generators: $\mathbf{Z}$ type stabilizers, which lie on "plaquettes" of the torus, and $\mathbf{X}$ type stabilizers, which lie on the "vertices". The figure below depicts a grid of qubits in a 5×5 toric code. Qubits are depicted by circles. Grey circles denote duplicated qubits due to the periodic (toric) boundary conditions (there are 50 physical qubits in total).



3.1

Consider the following toric code, where black circles denote the position of **Z** errors. Determine the location of the syndromes, i.e. the location of the vertex stabilizer generators which will return a -1 eigenvalue when measured.

Suggestion. Copy the figure into your solution.



3.2

Suppose we see the syndrome measurement results from 3.1 but don't know the location of the **Z** errors. The aim of a decoding algorithm for an error correcting code is to find a Pauli operator based on the syndrome which will return the corrupted state to the codespace without leading to a *logical error* on the encoded qubits.

Note: A logical error is a Pauli operator which commutes with all the stabilizers, but isn't generated by them. In the case of the toric code, it forms a closed loop around the torus.

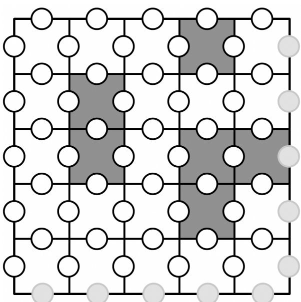
Determine the shortest string of Pauli **Z** operators which will return the lattice to the toric code space (the answer may not be unique). Will the application of this string of operators lead to a logical error? If so, identify the error.

Hint: Don't forget that periodic boundary conditions are in place.

3.3

In the following grid, shaded boxes indicate error syndromes – measured plaquette stabilizer generators which have returned a -1 eigenvalue. Determine the smallest set of **X** error operators, which could generate this error syndrome pattern.

Hint: there should be 4 **X** errors.



Acknowledgement

Problem 3 is adapted from Dan Browne's course https://sites.google.com/site/danbrowneucl/teaching/lectures-on-topological-codes-and-quantum-computation.