STAT 443: Lab 10

Aronn Grant Laurel (21232475)

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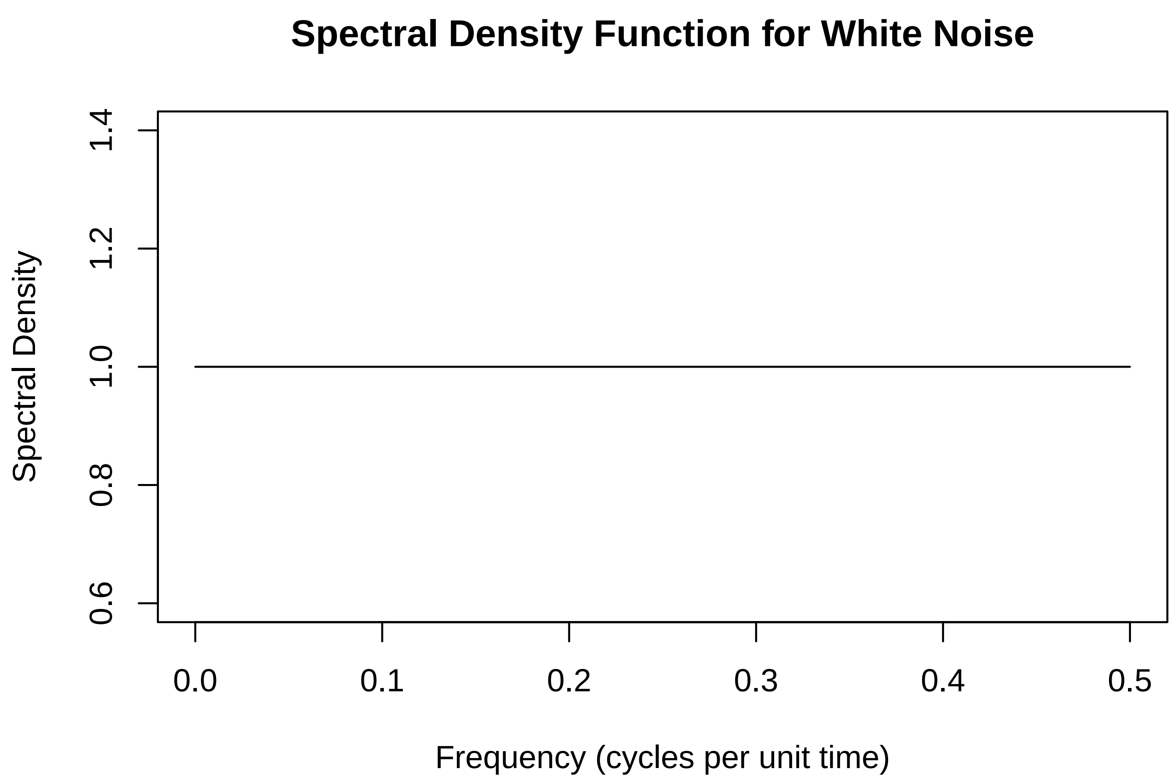
Question 1

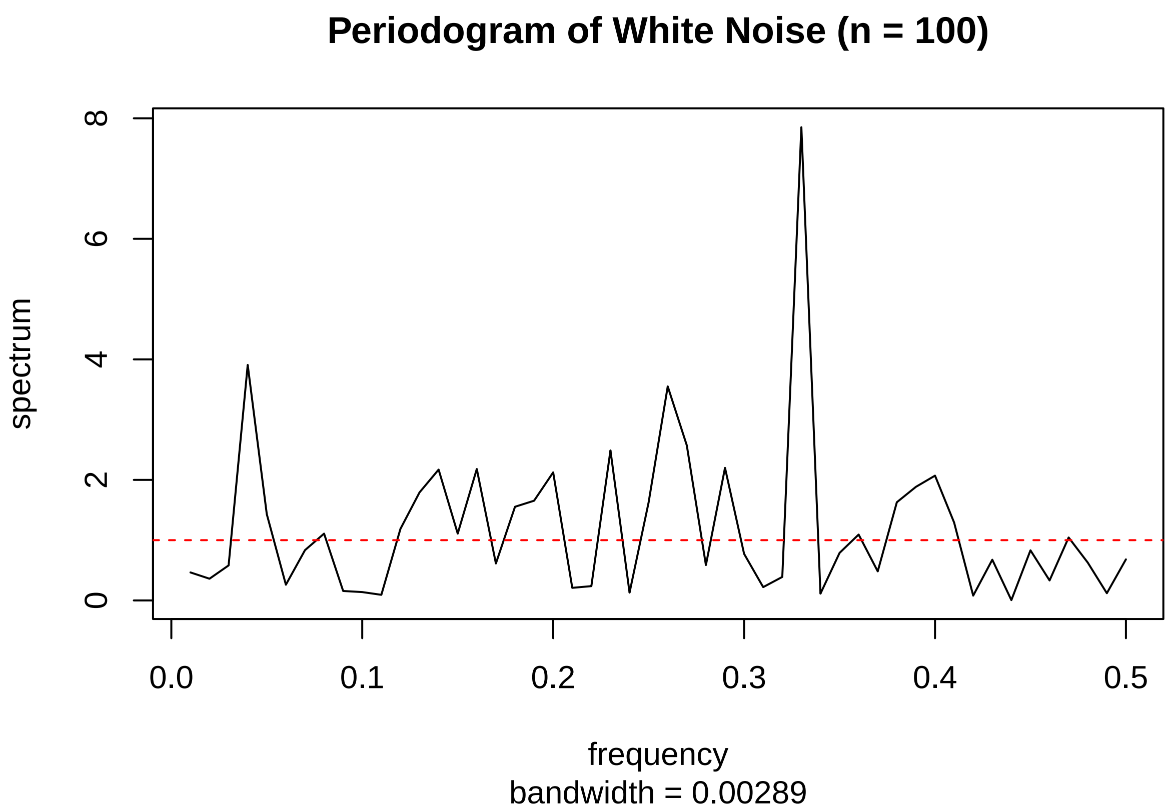
(a)

```r
# From Chapter 6 Lecture slides
freq <- seq(0, 0.5, length.out = 100)
spectral_density <- rep(1, length(freq))

plot(freq,
     spectral_density,
     xlab = "Frequency (cycles per unit time)",
     ylab = "Spectral Density",
     main = "Spectral Density Function for White Noise",
     type = "l")
```



(b)

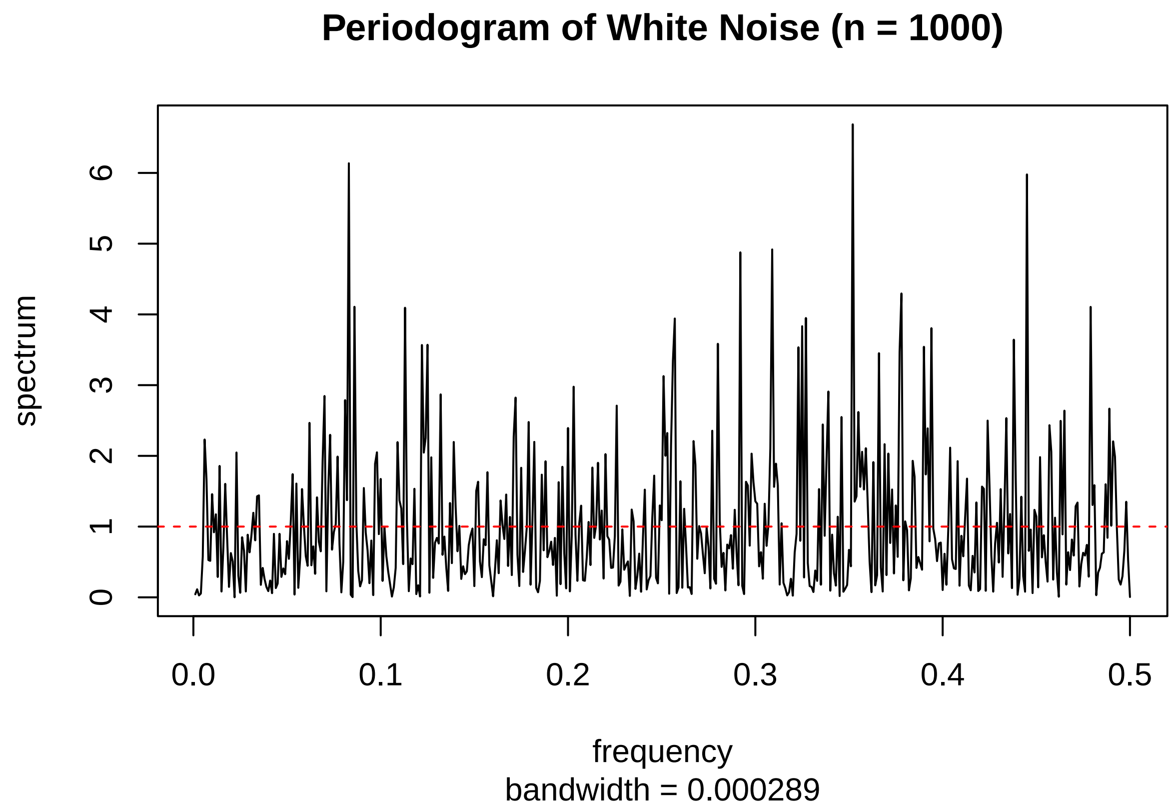
```r
q1b <- arima.sim(model = list(order = c(0, 0, 0)), n = 100, sd = 1)

spec.pgram(q1b,
           log = "no",
           main = "Periodogram of White Noise (n = 100)")

# True Value
abline(h =  1, col = "red", lty = 2)
```



We do not observe any pattern with the fluctuations and it having several frequencies that is at the true value = 1. While some frequency components show much higher power than the true value, others show lower power than the true value.

(c)

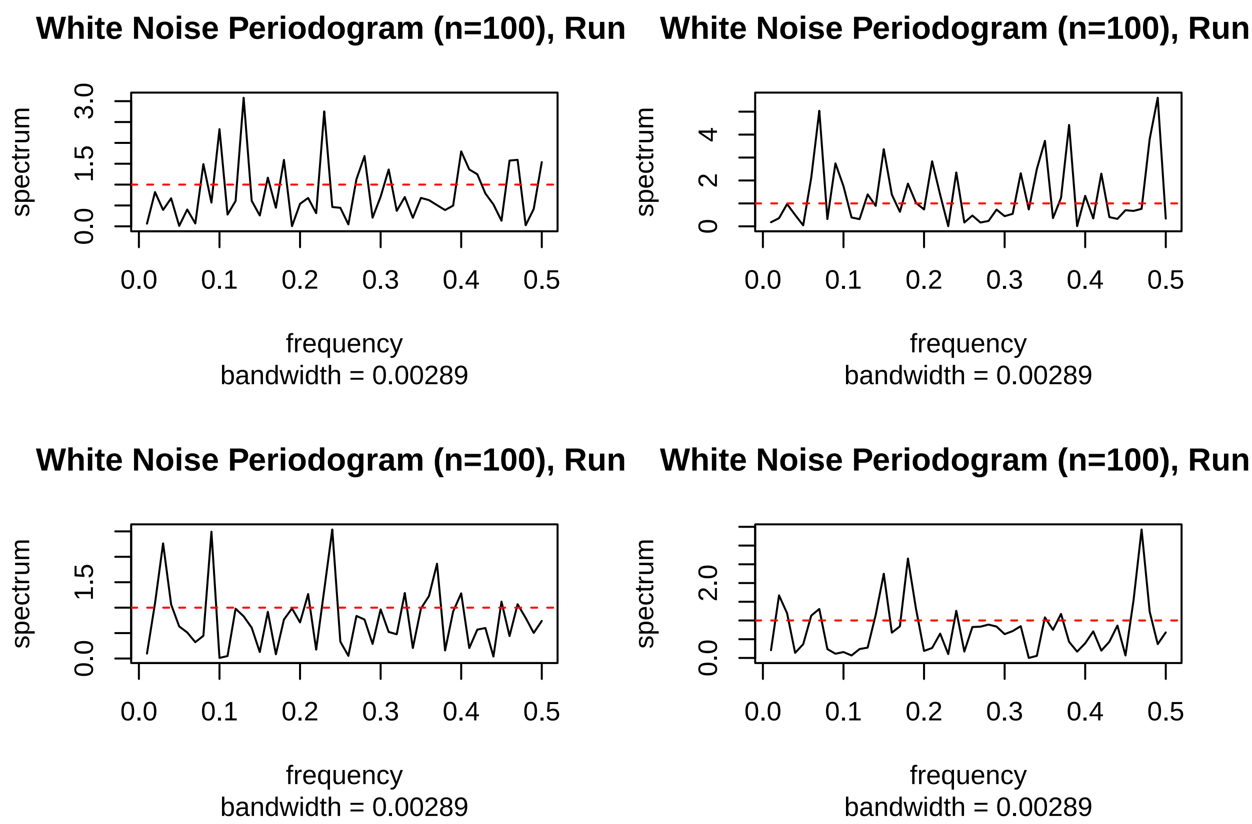
```r
q1b <- arima.sim(model = list(order = c(0, 0, 0)), n = 1000, sd = 1)

spec.pgram(q1b,
           log = "no",
           main = "Periodogram of White Noise (n = 1000)")
# True Value
abline(h =  1, col = "red", lty = 2)
```



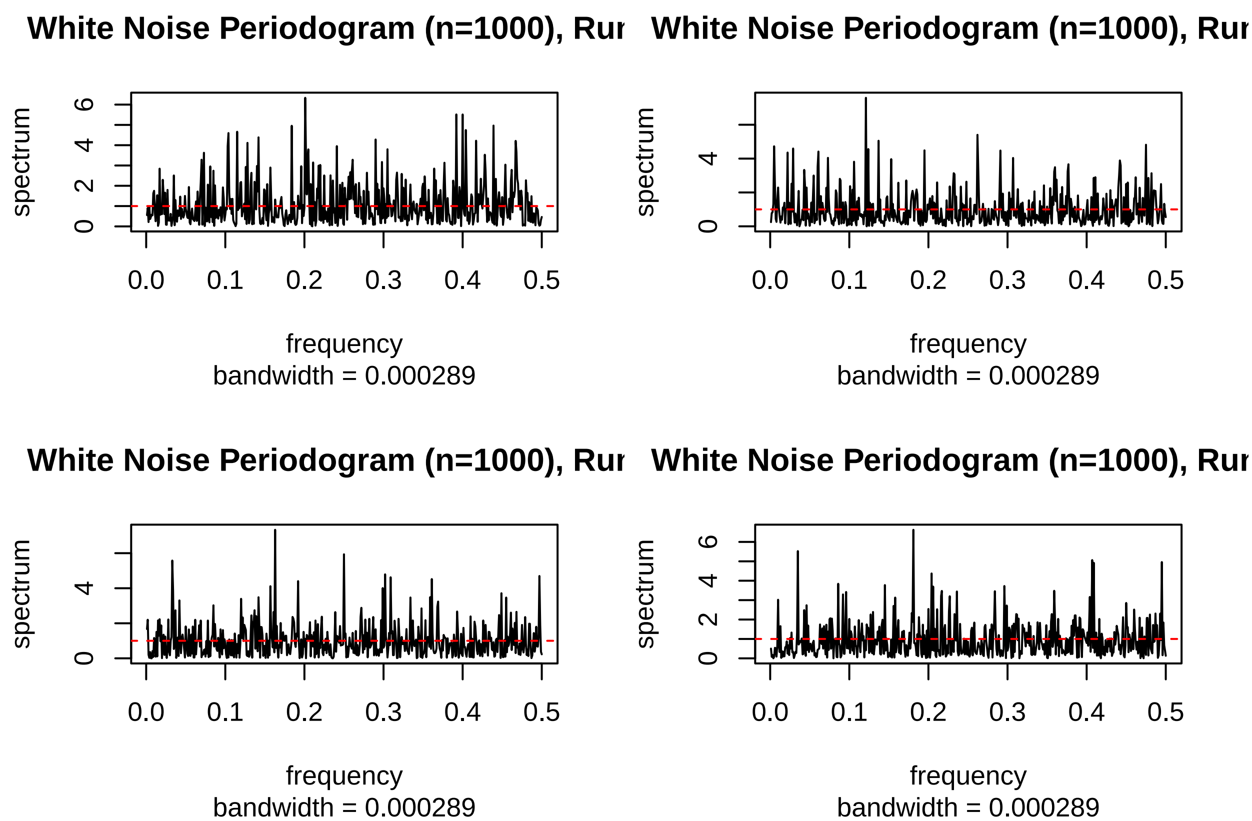
With an increase of sample size n = 1000, we observe more fluctuations spikes and more points around the true value.

(d)

```r
par(mfrow = c(2, 2))
for (i in 1:4) {

  set.seed(i)

  wn_series <- arima.sim(model = list(order = c(0,0,0)), n = 100)

  spec.pgram(wn_series,
             log = "no",
             main = paste("White Noise Periodogram (n=100), Run", i))

  abline(h = 1, col = "red", lty = 2)
}
```




```r
par(mfrow = c(2, 2))
for (i in 1:4) {
  set.seed(i)

  wn_series_long <- arima.sim(model = list(order = c(0,0,0)), n = 1000)

  spec.pgram(wn_series_long,
             log = "no",
             main = paste("White Noise Periodogram (n=1000), Run", i))

  abline(h = 1, col = "red", lty = 2)
}
```

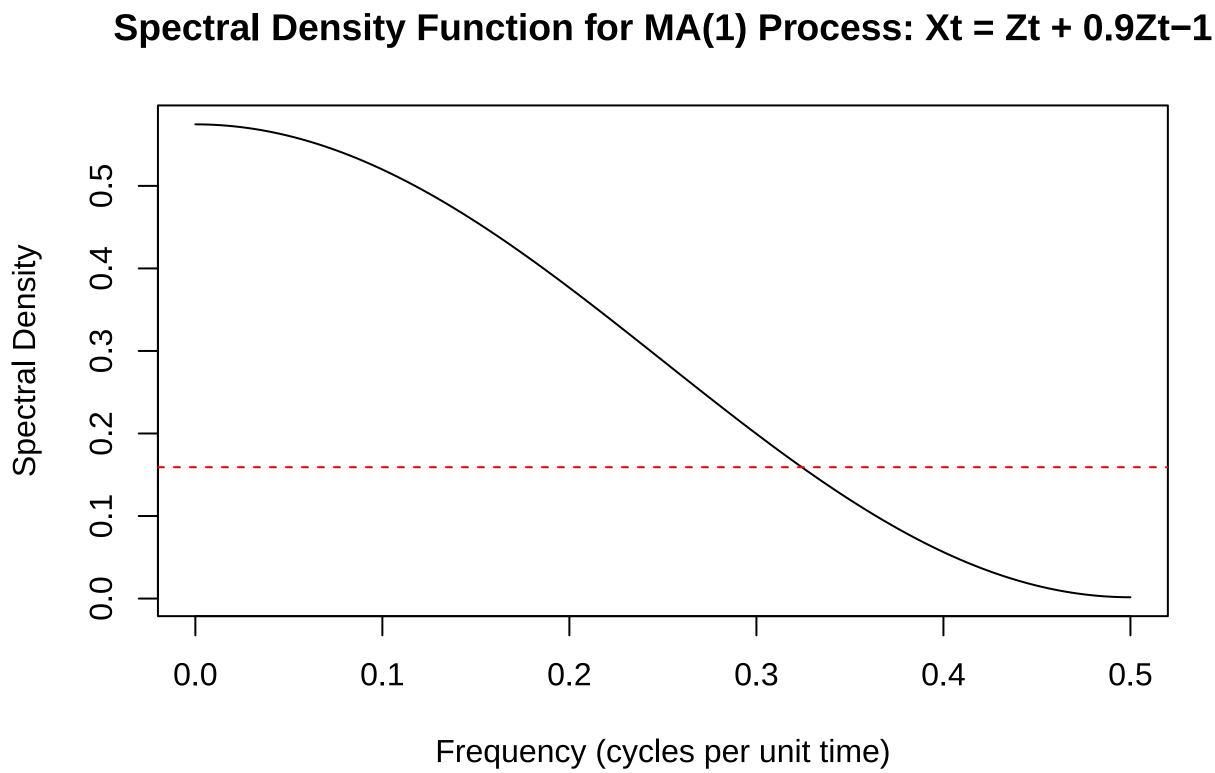


Similar to our findings in question b) and c), we can observe that higher sample size (n = 1000) has a relatively higher / more frequencies at which the spectral density is estimated.

Question 2

(a)

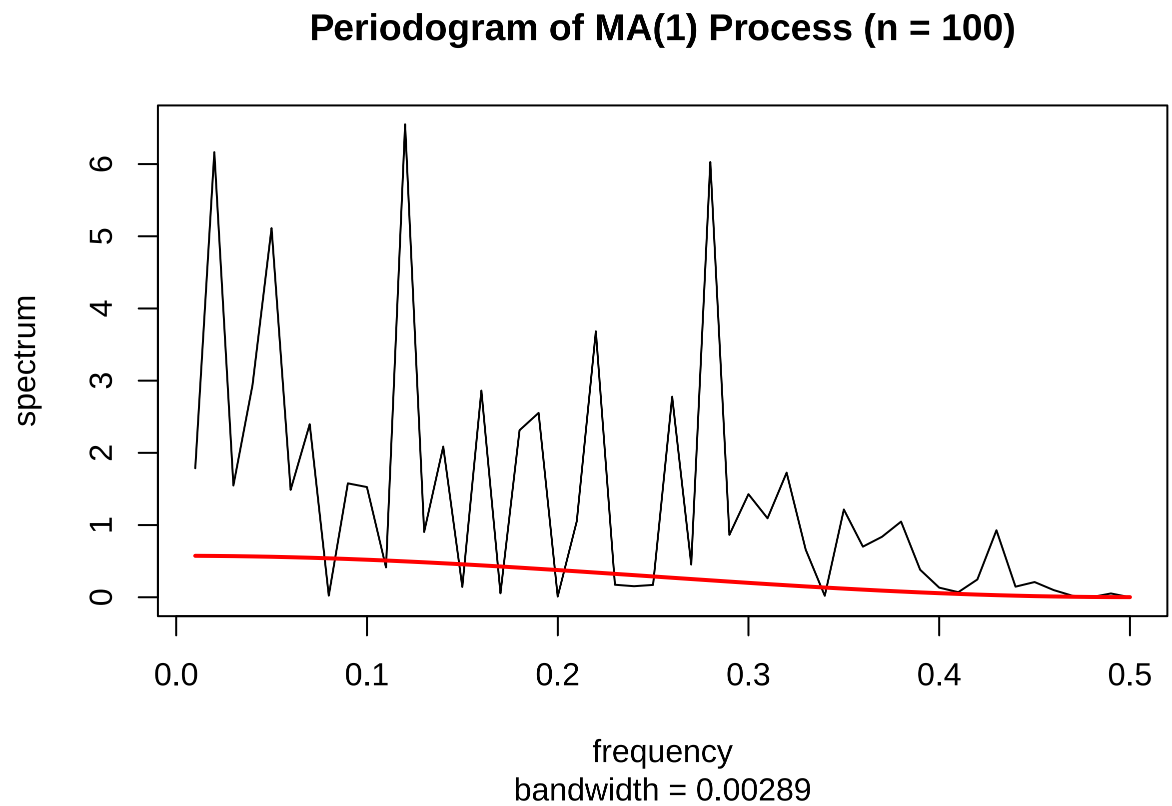
```r
# Omega
omega <- seq(0, pi, length.out = 1000)
freq <- omega / (2 * pi)

# Calculate spectral density [ MA(1) process ]
theta <- 0.9
spec_density <- (1 / (2 * pi)) * (1 + theta^2 + 2 * theta * cos(omega))

plot(freq, spec_density, type = "l",
     xlab = "Frequency (cycles per unit time)",
     ylab = "Spectral Density",
     main = "Spectral Density Function for MA(1) Process: Xt = Zt + 0.9Zt-1")

abline(h = 1/(2*pi), col = "red", lty = 2)
```



The spectral density for this MA(1) process is highest at lower frequencies and gradually decreases as the frequency increases.

(b)

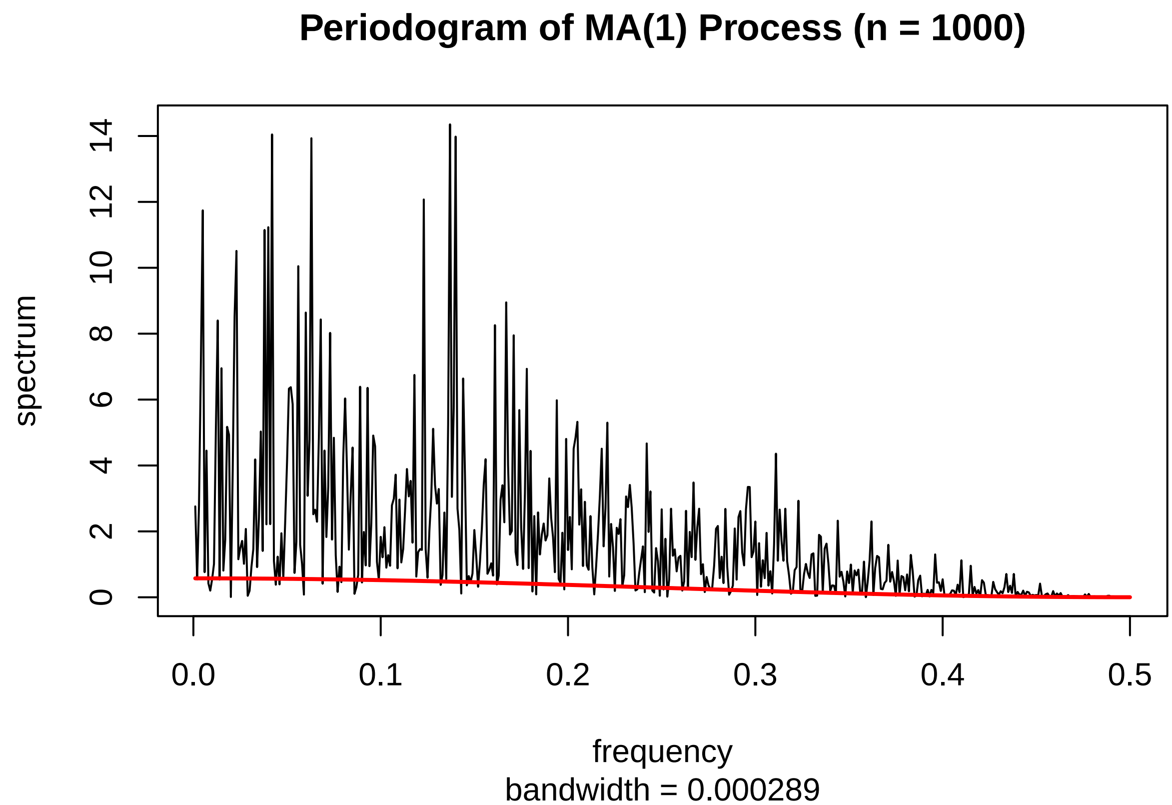
```
ma1_series_short <- arima.sim(model = list(order = c(0,0,1), ma = 0.9), n = 100)

periodogram_short <- spec.pgram(ma1_series_short,
                                log = "no",
                                main = "Periodogram of MA(1) Process (n = 100)")

omega <- 2 * pi * periodogram_short$freq
theta <- 0.9
true_spec <- (1 / (2 * pi)) * (1 + theta^2 + 2 * theta * cos(omega))

lines(periodogram_short$freq, true_spec, col = "red", lwd = 2)
```



From the periodogram at sample size = 100, we can see high fluctuations which decreases and approaches close to the true spectral density curve. The periodogram displays a high variance. Hence it is not reliable as an estimator of the spectral density.

(c)

```
ma1_series_long <- arima.sim(model = list(order = c(0,0,1), ma = 0.9), n = 1000)

periodogram_long <- spec.pgram(ma1_series_long,
                               log = "no",
                               main = "Periodogram of MA(1) Process (n = 1000)")

omega <- 2 * pi * periodogram_long$freq
theta <- 0.9
true_spec <- (1 / (2 * pi)) * (1 + theta^2 + 2*theta*cos(omega))

lines(periodogram_long$freq, true_spec, col = "red", lwd = 2)
```

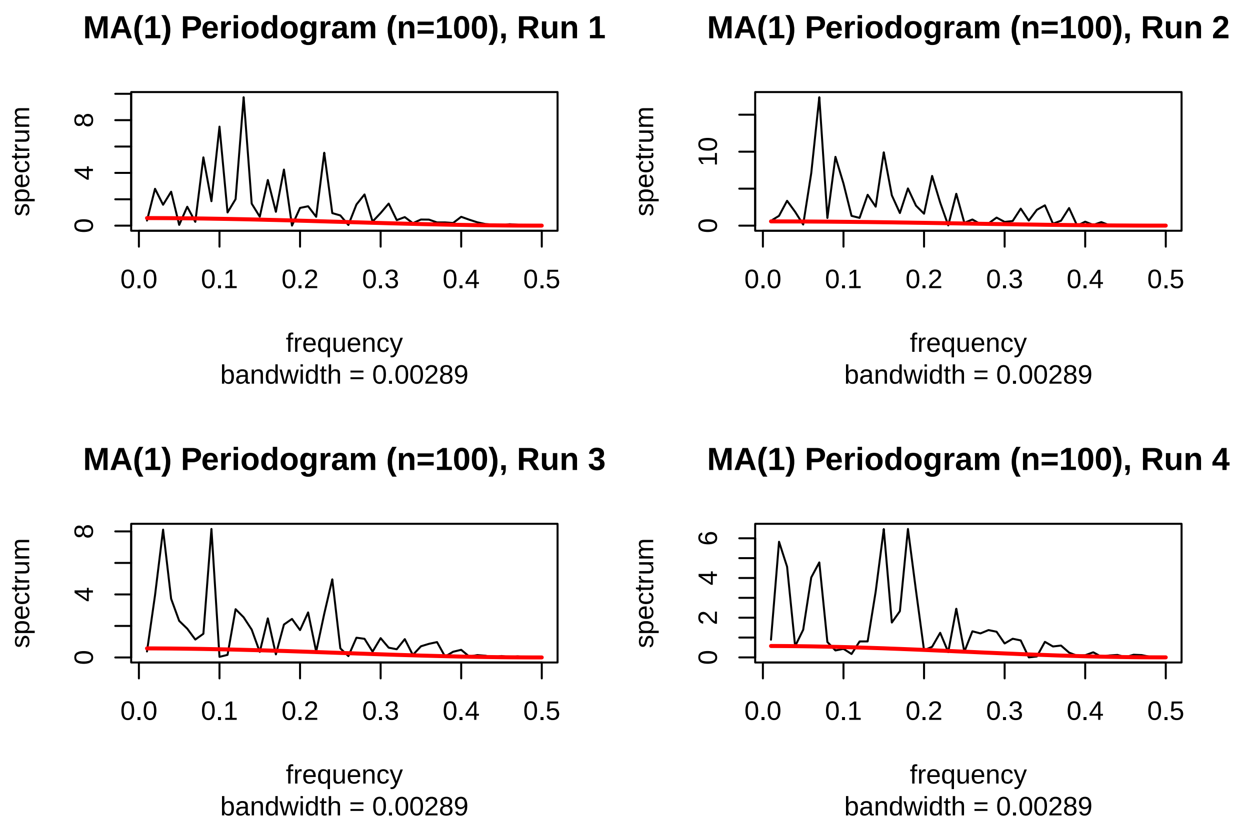


With sample size = 1000, we can also observe that our frequency approaches to the true spectral density curve as our frequency increases to 0.5. Increasing the sample size to 1000, it improved the estimate by gradually reducing the variance.

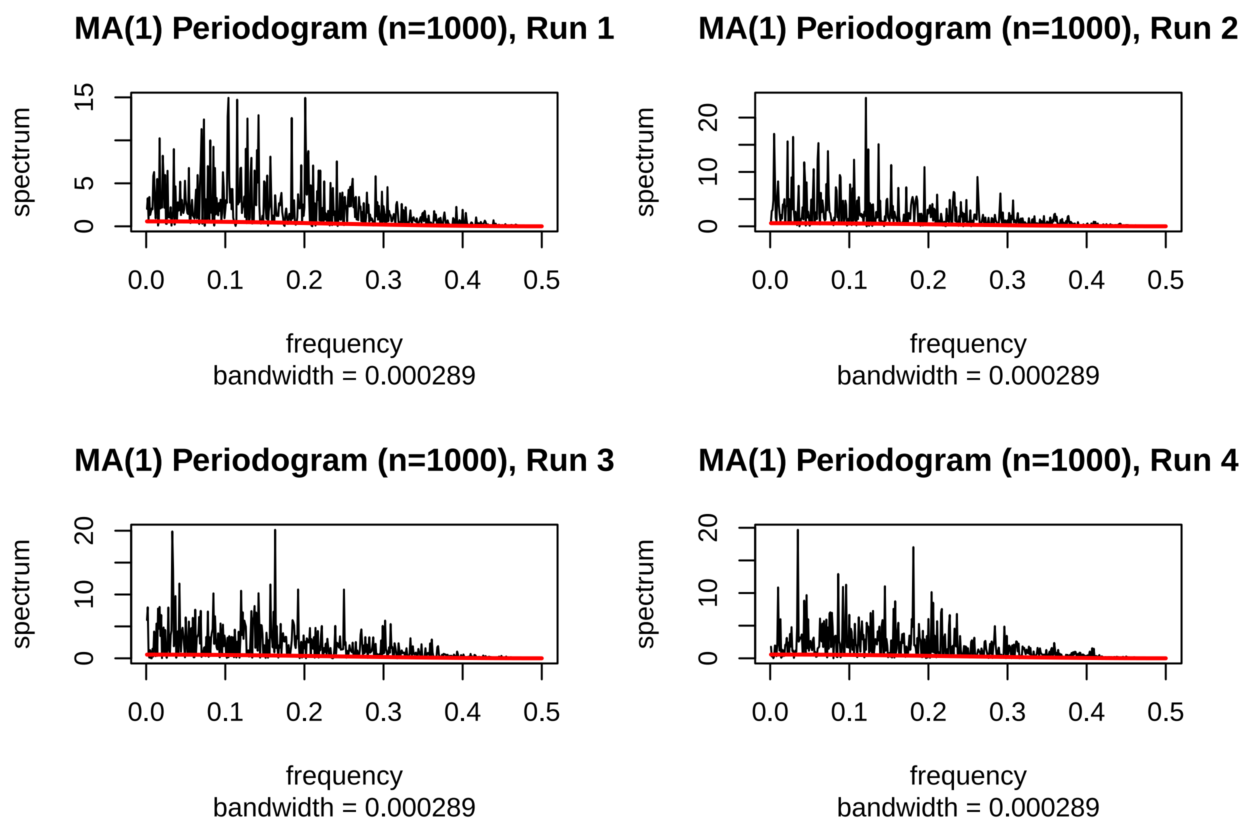
(d)

```
par(mfrow = c(2, 2))

get_true_spec <- function(freq) {
  omega <- 2 * pi * freq
  theta <- 0.9
  sigma_squared <- 1
  return((sigma_squared/(2*pi)) * (1 + theta^2 + 2*theta*cos(omega)))
}

for (i in 1:4) {
  set.seed(i)

  ma1_series <- arima.sim(model = list(order = c(0,0,1), ma = 0.9), n = 100)

  pgram <- spec.pgram(ma1_series, log = "no",
                      main = paste("MA(1) Periodogram (n=100), Run", i),
                      plot = TRUE)

  true_spec <- get_true_spec(pgram$freq)
```

```
  lines(pgram$freq, true_spec, col = "red", lwd = 2)
}
```




```
par(mfrow = c(2, 2))

for (i in 1:4) {
  set.seed(i)

  ma1_series_long <- arima.sim(model = list(order = c(0,0,1), ma = 0.9), n = 1000)

  pgram <- spec.pgram(ma1_series_long, log = "no",
                      main = paste("MA(1) Periodogram (n=1000), Run", i),
                      plot = TRUE)

  true_spec <- get_true_spec(pgram$freq)

  lines(pgram$freq, true_spec, col = "red", lwd = 2)
}
```

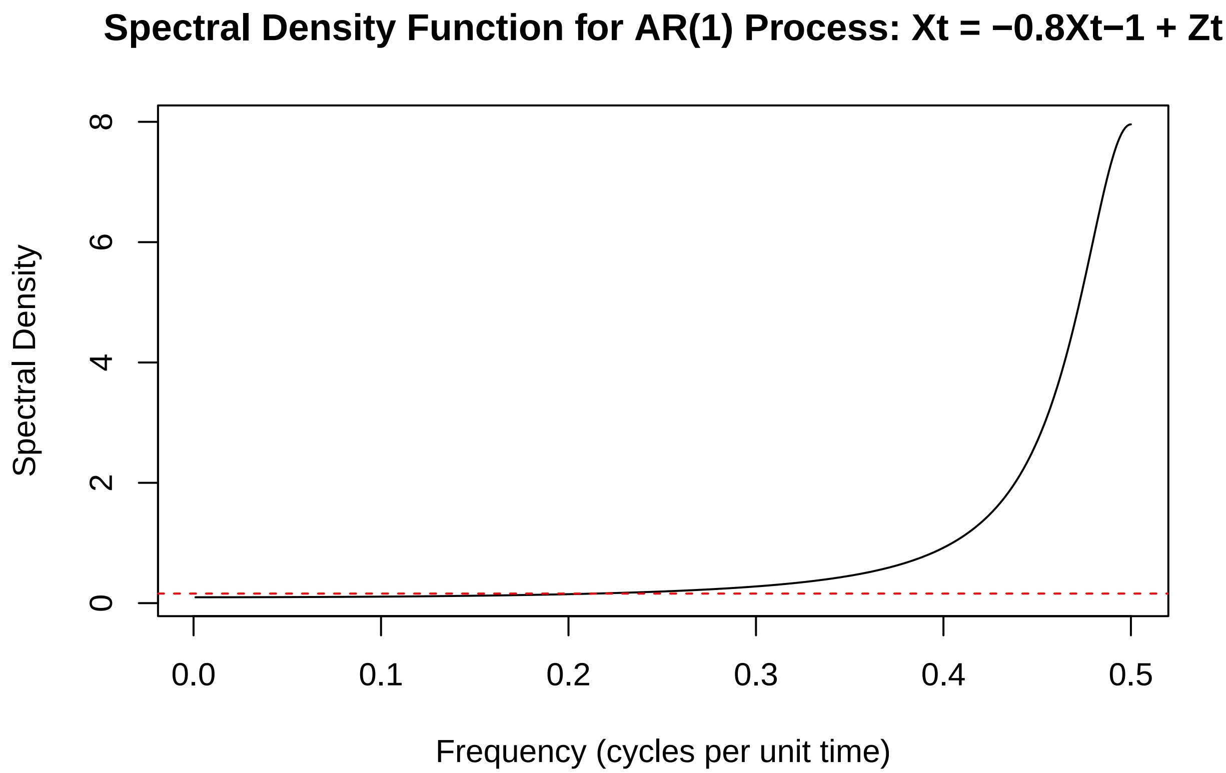


Across all simulations, we get similar findings to question 2 b) and c) where it is higher at low frequencies and gradually decreasing, but the 'peaks' at specific frequencies vary considerably between simulations. Overall, I believe it is not a consistent estimator, the periodograms remain quite noisy.

Question 3

(a)

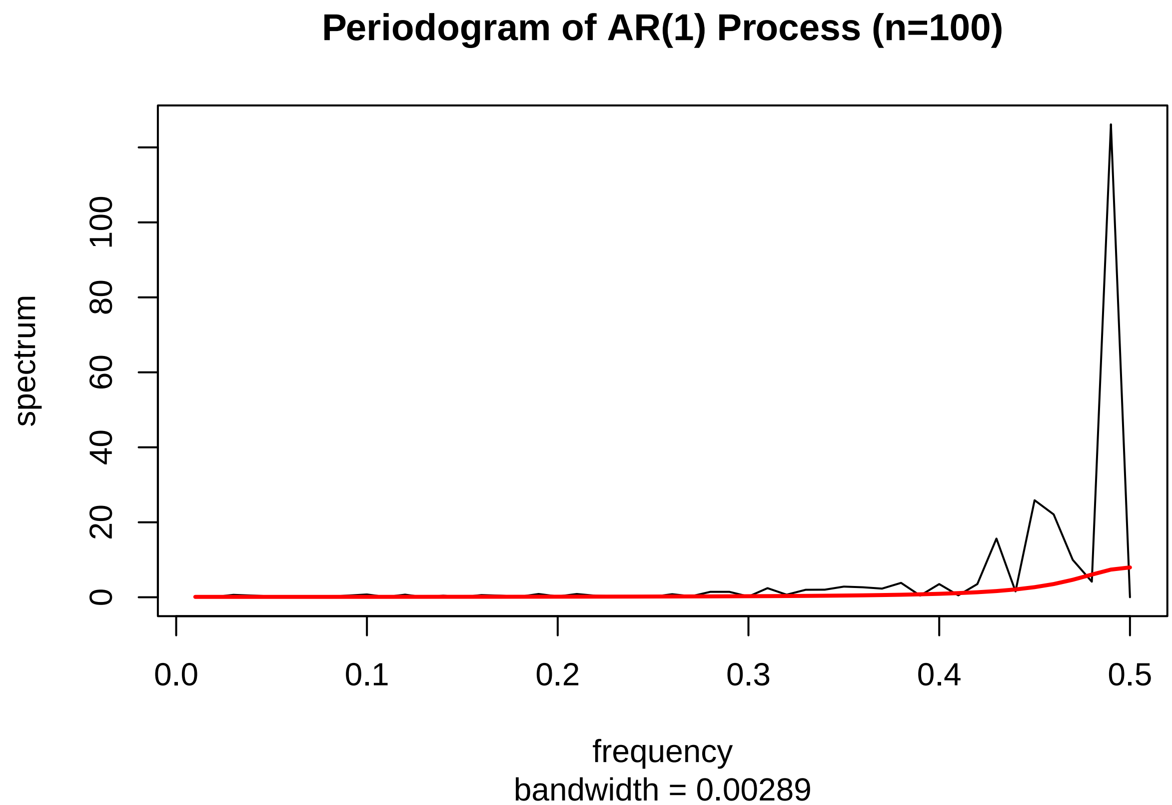
```r
freq <- omega/(2*pi)
spec_density <- 1/(pi * (1 + 1.6*cos(omega) + 0.8^2))

plot(freq,
     spec_density,
     type = "l",
     xlab = "Frequency (cycles per unit time)",
     ylab = "Spectral Density",
     main = "Spectral Density Function for AR(1) Process: Xt = -0.8Xt-1 + Zt")

abline(h = 1/(2*pi), col = "red", lty = 2)
```



Opposite to the MA(1) process we saw earlier, this AR(1) process ressembles an exponential graph with it's spectral density lowest at low frequencies and gradually increases at high frequencies.

(b)

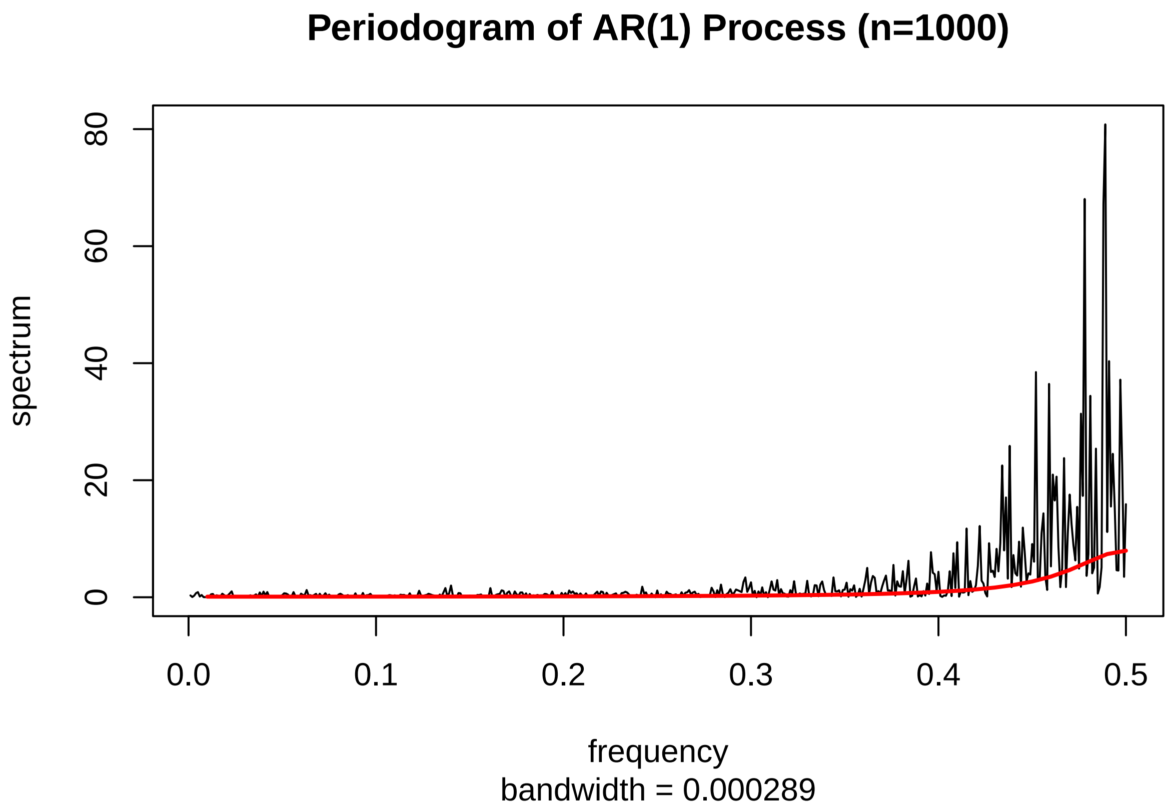
```
ar1_series <- arima.sim(model = list(order = c(1,0,0), ar = -0.8), n = 100)

ar1_periodogram <- spec.pgram(ar1_series,
                         log = "no",
                        main = "Periodogram of AR(1) Process (n=100)")

omega <- 2 * pi * ar1_periodogram$freq
true_spec <- 1/(pi * (1 + 1.6 * cos(omega) + 0.8^2))
lines(ar1_periodogram$freq, true_spec, col = "red", lwd = 2)
```



For sample size n = 100, the periodogram displays a high variance. Hence it is not reliable as an estimator of the spectral density

(c)

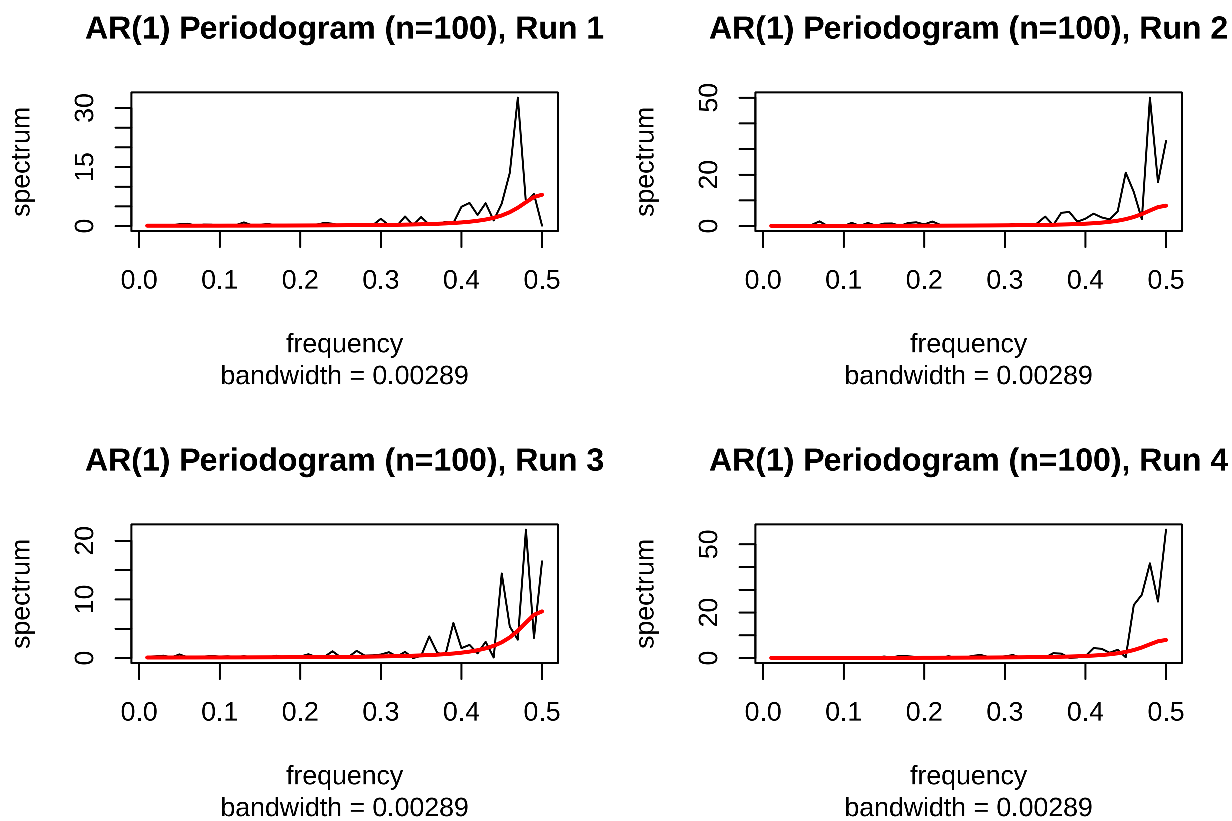
```
ar1_series_long <- arima.sim(model = list(order = c(1,0,0), ar = -0.8), n = 1000)

ar1_periodogram_long <- spec.pgram(ar1_series_long,
                                   log = "no",
                                   main = "Periodogram of AR(1) Process (n=1000)")

lines(ar1_periodogram$freq, true_spec, col = "red", lwd = 2)
```



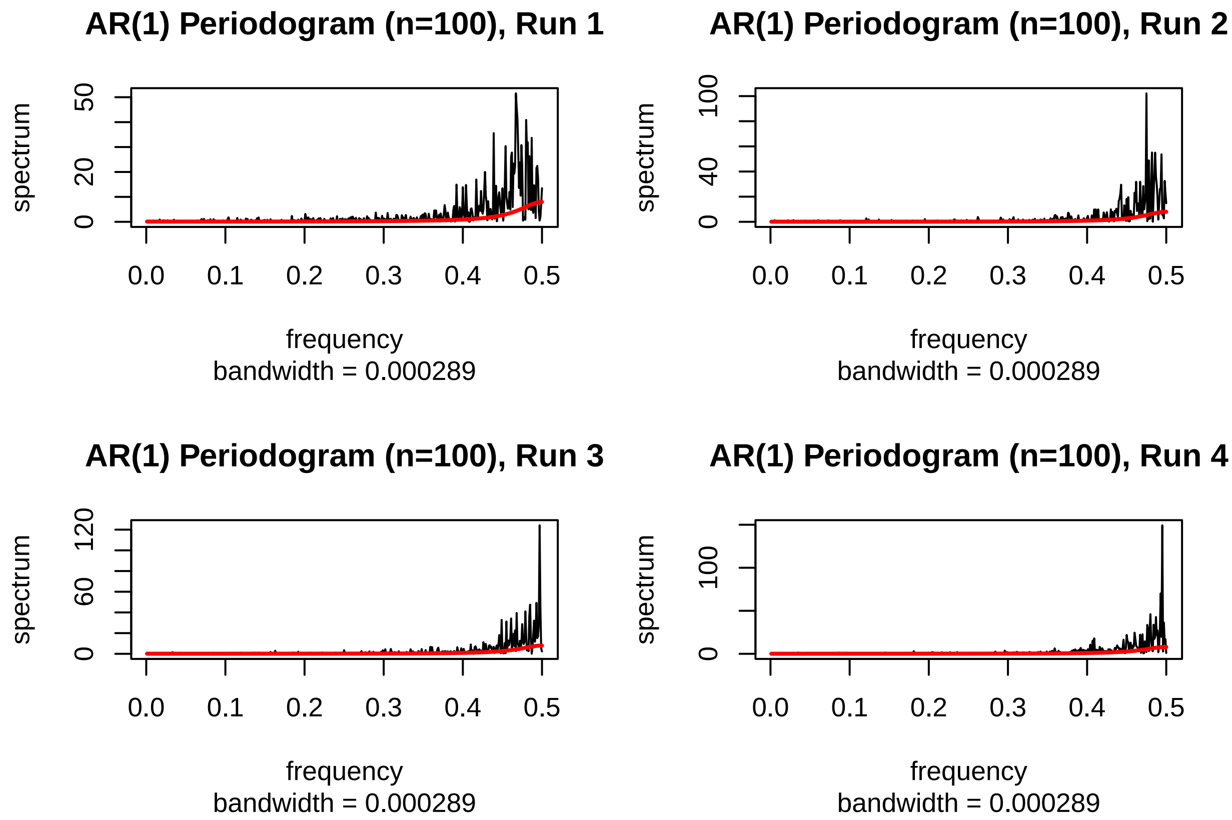
At increased sample size, we observe more fluctuates with similar variances. Hence it is not reliable as an estimator of the spectral density.

(d)

```
par(mfrow = c(2, 2))

get_true_spec <- function(freq) {
  omega <- 2 * pi * freq
  theta <- 0.9
  sigma_squared <- 1
  return ( 1 / (pi * (1 + 1.6 * cos(omega) + 0.8^2)) )
}

for (i in 1:4) {
  set.seed(i)

  ma1_series <- arima.sim(model = list(order = c(1,0,0), ar = -0.8), n = 100)

  pgram <- spec.pgram(ma1_series, log = "no",
                      main = paste("AR(1) Periodogram (n=100), Run", i),
                      plot = TRUE)

  true_spec <- get_true_spec(pgram$freq)

  lines(pgram$freq, true_spec, col = "red", lwd = 2)
}
```




```r
par(mfrow = c(2, 2))

for (i in 1:4) {
  set.seed(i)

  ma1_series <- arima.sim(model = list(order = c(1,0,0), ar = -0.8), n = 1000)

  pgram <- spec.pgram(ma1_series, log = "no",
                      main = paste("AR(1) Periodogram (n=100), Run", i),
                      plot = TRUE)

  true_spec <- get_true_spec(pgram$freq)

  lines(pgram$freq, true_spec, col = "red", lwd = 2)
}
```



Across all runs for both samples sizes, we can still observe noisy periodograms is not a consistent estimator of the spectral density because its variance does not decrease with more data.