STAT 443: Homework 4

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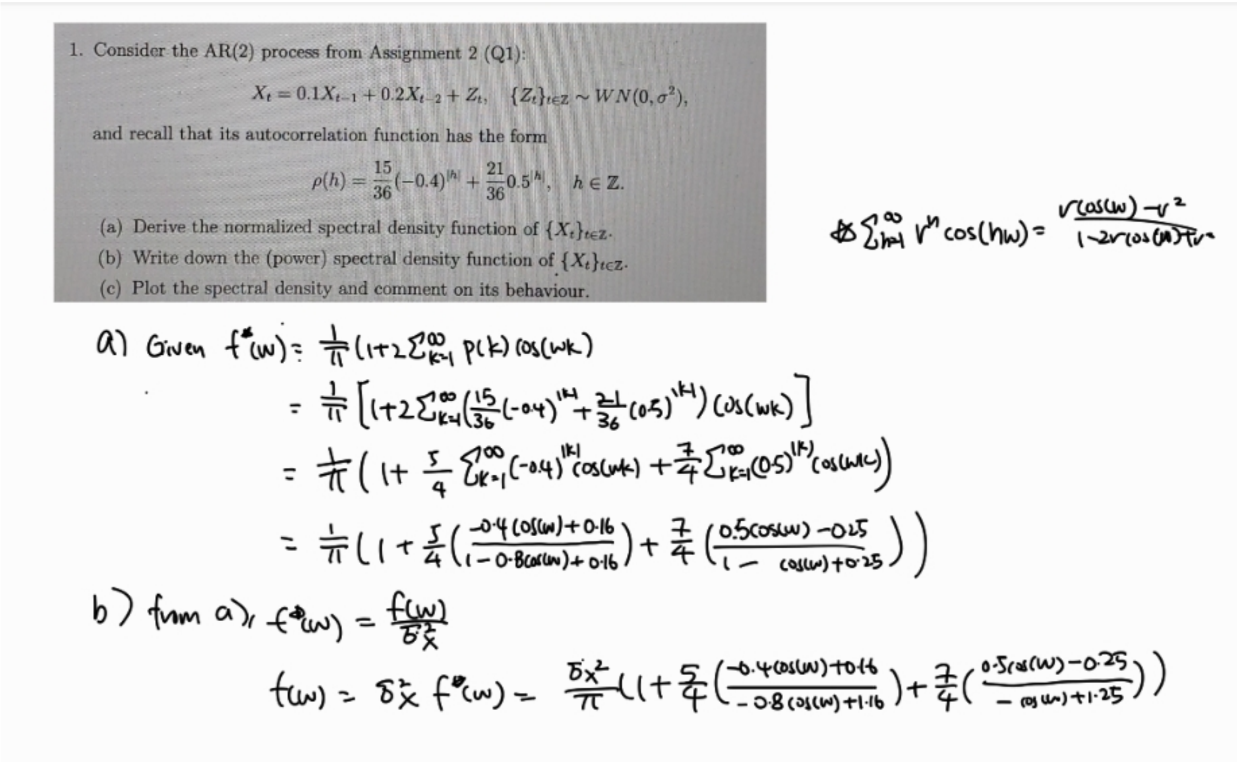
Question 1

(a & b)

```r
knitr::include_graphics("q1a.jpg")
```



1. Consider the AR(2) process from Assignment 2 (Q1):

$$X_t = 0.1X_{t-1} + 0.2X_{t-2} + Z_t, \quad \{Z_t\}_{t\in\mathbb{Z}} \sim WN(0, \sigma^2),$$

and recall that its autocorrelation function has the form

$$\rho(h) = \frac{15}{36}(-0.4)^{|h|} + \frac{21}{36}0.5^{|h|}, \quad h \in \mathbb{Z}.$$

(a) Derive the normalized spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$.

(b) Write down the (power) spectral density function of $\{X_t\}_{t\in\mathbb{Z}}$.

(c) Plot the spectral density and comment on its behaviour.

$$\sum_{h=1}^{\infty} r^h \cos(hw) = \frac{r\cos(w) - r^2}{1 - 2r\cos(w) + r^2}$$

a) Given $f^*(w) = \frac{1}{\pi}(1 + 2\sum_{k=1}^{\infty} \rho(k)\cos(wk)$

$$= \frac{1}{\pi}\left[(1 + 2\sum_{k=1}^{\infty}\left(\frac{15}{36}(-0.4)^{|k|} + \frac{21}{36}(0.5)^{|k|}\right)\cos(wk)\right]$$

$$= \frac{1}{\pi}\left(1 + \frac{5}{4}\sum_{k=1}^{\infty}(-0.4)^{|k|}\cos(wk) + \frac{7}{4}\sum_{k=1}^{\infty}(0.5)^{|k|}\cos(wk)\right)$$

$$= \frac{1}{\pi}\left(1 + \frac{5}{4}\left(\frac{-0.4\cos(w) + 0.16}{1 - 0.8\cos(w) + 0.16}\right) + \frac{7}{4}\left(\frac{0.5\cos(w) - 0.25}{1 - \cos(w) + 0.25}\right)\right)$$

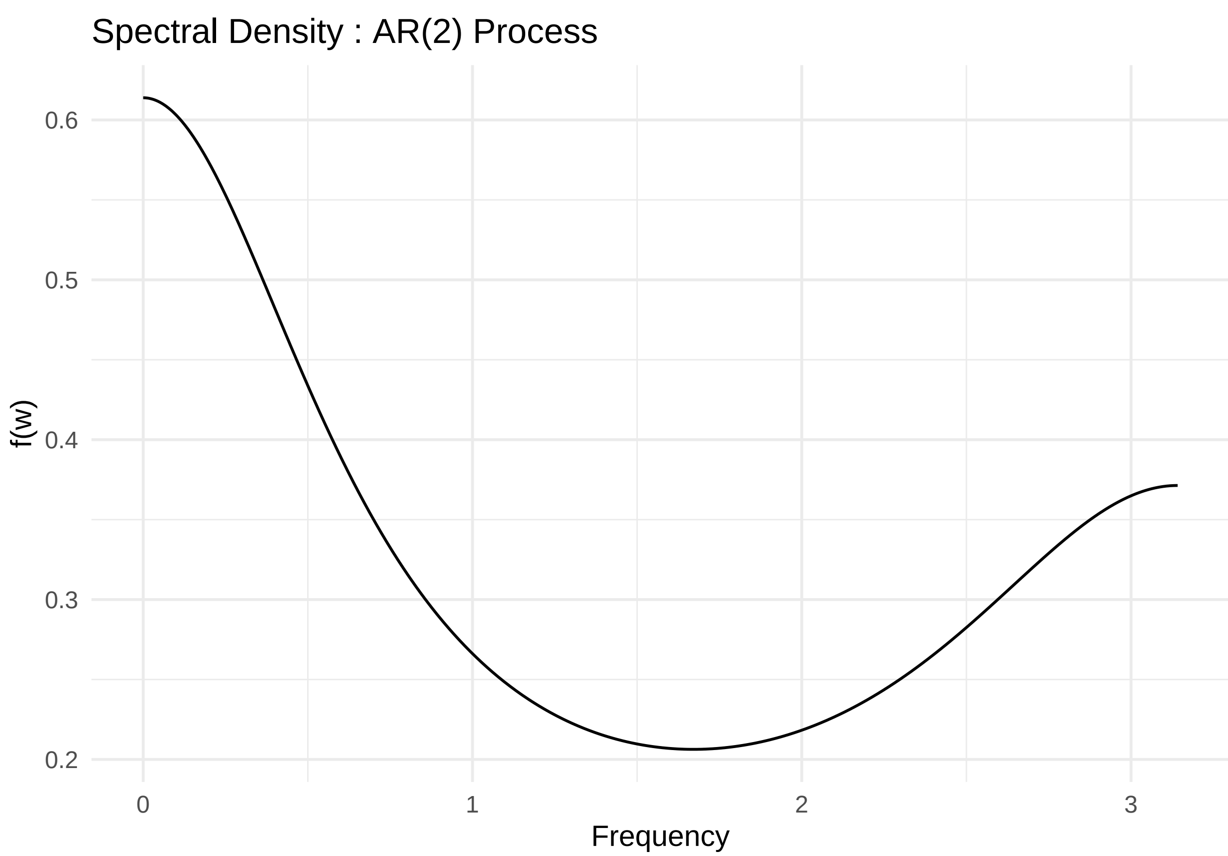
b) from a), $f^*(w) = \frac{f(w)}{\sigma_X^2}$

$$f(w) = \sigma_X^2 f^*(w) = \frac{\sigma_X^2}{\pi}\left(1 + \frac{5}{4}\left(\frac{-0.4\cos(w) + 0.16}{-0.8\cos(w) + 1.16}\right) + \frac{7}{4}\left(\frac{0.5\cos(w) - 0.25}{-\cos(w) + 1.25}\right)\right)$$

(c)

```r
# Calculate the spectral density
spectral_density <- function(omega) {
  term2 <- (15/18) * ((-0.4*cos(omega) - 0.16) / (1.16 + 0.8 * cos(omega)))
  term3 <- (21/18) * ((0.5*cos(omega) - 0.25) / (1.25 - cos(omega)))

  return((1/pi) * (1 + term2 + term3))
}
```

```r
# Create a sequence of frequencies from 0 to pi
omega <- seq(0, pi, length.out = 1000)

# Calculate the spectral density for each frequency
spec_values <- sapply(omega, spectral_density)

# Create a data frame for plotting
spec_df <- data.frame(
  omega = omega,
  frequency = omega/(2*pi),
  spectral_density = spec_values
)


# Plot using ggplot2
ggplot(spec_df, aes(x = omega, y = spec_values)) +
  geom_line() +
  labs(title = "Spectral Density : AR(2) Process",
       x = "Frequency",
       y = "f(w)") +
  theme_minimal()
```

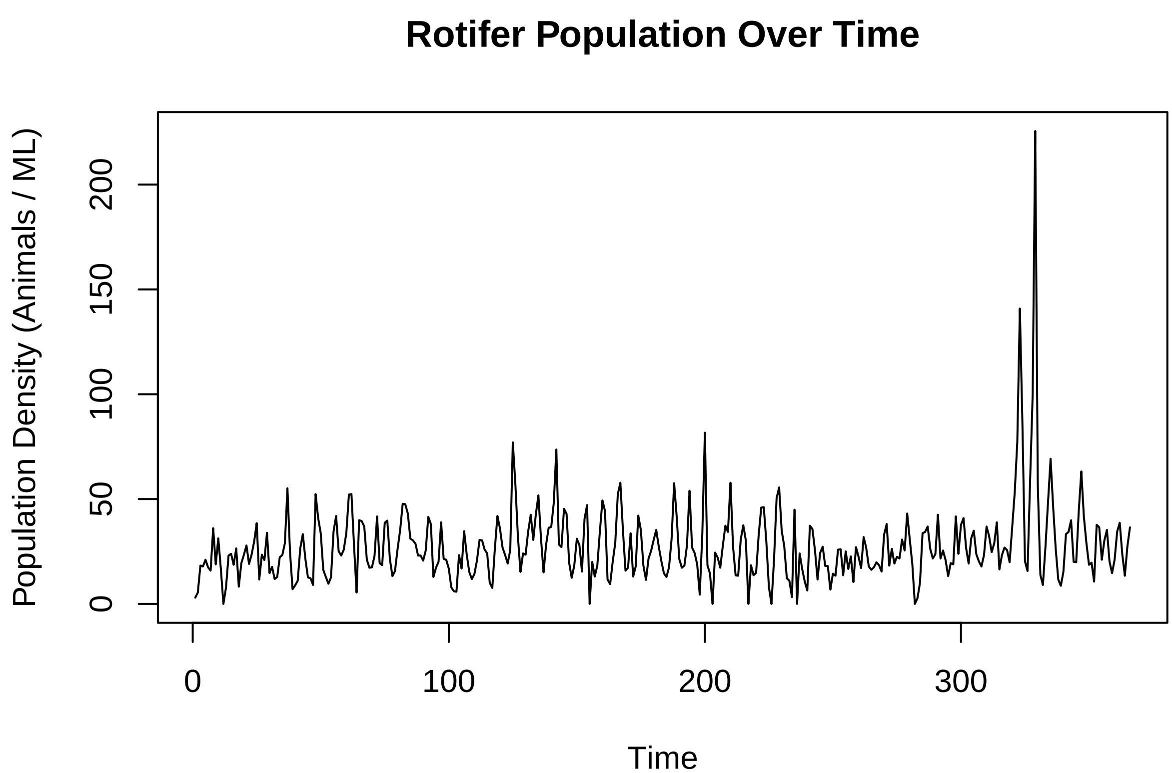


We can the spectral density being dominated by low frequencies, with the spectral density is at highest at frequency = 0 as it's peak. As we move further away from frequency 0, we observe a dip / decreasing spectral density and another smaller peaks at approximately frequency 3. Overall, this suggests that our AR(2) process has long term dependencies.

Question 3

(a)

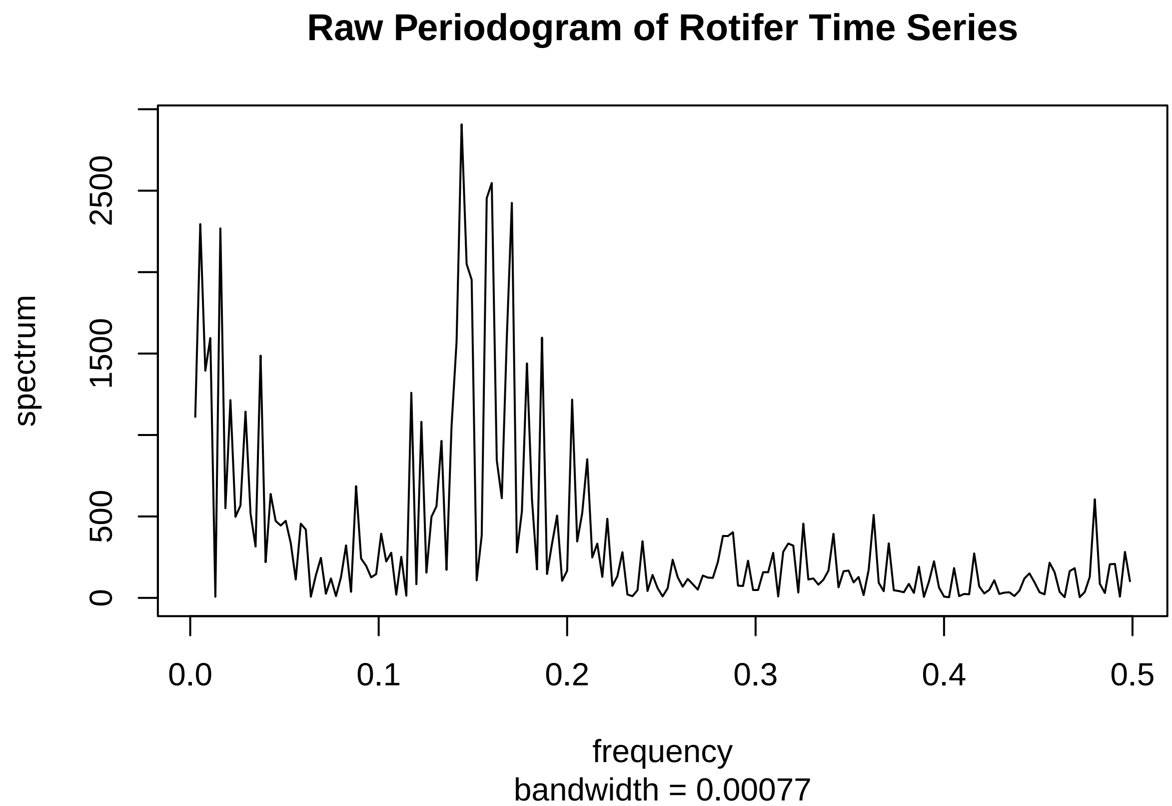
```
predator<- read.csv("predator_prey.csv")

rot_ts <- ts(predator$rotifers..animals.ml.)

plot(rot_ts,
     main = "Rotifer Population Over Time",
     ylab = "Population Density (Animals / ML)"
     )
```



We observe a consistent fluctuation until at time ~350 where we observe a large spike / peak before dropping back to the consistent fluctuation.

(b)

```
raw_spec <- spectrum(rot_ts,
                     log = "no",
                     main = "Raw Periodogram of Rotifer Time Series")
```




```r
frequencies <- raw_spec$freq
spectrum_values <- raw_spec$spec

# Get the max
max_index <- which.max(spectrum_values)

dominant_freq <- frequencies[max_index]
dominant_freq
```

```
## [1] 0.144
```

```r
angular_freq <- 2 * pi * dominant_freq
angular_freq
```

```
## [1] 0.9047787
```

```r
wavelength_days <- 1 / dominant_freq
wavelength_days
```

```
## [1] 6.944444
```

For our dominating frequency 0.144, we get wavelength (days) ~ 6.944 and angular frequency ~ 0.905

(c)

```r
N <- length(rot_ts)

num_cycles <- N / wavelength_days
num_cycles
```

```
## [1] 52.704
```

(d)

```r
# this is where your R code goes
```

(e)

```r
# this is where your R code goes
```

(f)

```r
# this is where your R code goes
```

(g)

```r
# this is where your R code goes
```

(h)

```r
# this is where your R code goes
```