

STAT 443: Time Series and Forecasting

Lab 10: Spectral density estimation

- The lab must be completed in R Markdown. Display all the R code used to perform your analysis.
- Create a pdf file and use it as your lab submission.
- Please ensure that the file you submit is in good order (e.g., not corrupted and contains the work you intend to submit). No late (re-)submissions will be accepted.

We have met the idea of the *spectral density function*, $f(\omega)$, of a stochastic process, this being the Fourier transform of the autocovariance function for the process. As we will see, there is a tool known as the *periodogram* which can estimate the spectral density function given a sample from the stochastic process. In effect, the periodogram is a histogram, and can be created in R using the command `spec.pgram()`. By default, this command plots the periodogram on the log scale, so in the following use the option `log="no"` for better comparisons. R uses $\omega/2\pi$ as frequency (per unit time), so the horizontal axis in the plot of the periodogram ranges from 0 to 0.5 rather than 0 to π . Moreover, as an estimate of the spectral density, the periodogram from R should be divided by π to be consistent with definitions provided in class. In what follows though it is the shape of the periodogram that is of most interest.

In the following, suppose that $\{Z_t\}_{t \in \mathbb{Z}}$ is white noise with mean 0 and variance 1.

1. Here we will compare the spectral density of white noise with the periodograms obtained from simulated white noise samples.
 - (a) Plot the spectral density function for $\{Z_t\}_{t \in \mathbb{Z}}$.
 - (b) Use the `arma.sim()` command, or otherwise, to simulate a series of length 100 from $\{Z_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
 - (c) Now simulate a series of length 1000 from $\{Z_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
 - (d) Repeat parts (b) and (c) several times. Comment on how the periodogram behaves as an estimator of the spectral density function based on what you have observed.

2. Let $\{X_t\}_{t \in \mathbb{Z}}$ be defined by

$$X_t = Z_t + 0.9Z_{t-1}.$$

- (a) Plot the spectral density function for $\{X_t\}_{t \in \mathbb{Z}}$; see in-class activity “Examples of Spectral Densities” (Question 1).
- (b) Use the `arma.sim()` command to simulate a series of length 100 from $\{X_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
- (c) Use the `arma.sim` command to simulate a series of length 1000 from $\{X_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
- (d) Repeat parts (b) and (c) several times. Comment on how the periodogram from R behaves as an estimator of the spectral density function based on what you have observed.

3. The process

$$X_t = -0.8X_{t-1} + Z_t$$

has spectral density function

$$f(\omega) = \frac{1}{\pi(1 + 1.6 \cos(\omega) + 0.8^2)}, \quad \omega \in (0, \pi).$$

- (a) Plot the spectral density function for $\{X_t\}_{t \in \mathbb{Z}}$.
- (b) Use the `arma.sim()` command to simulate a series of length 100 from $\{X_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
- (c) Use the `arma.sim()` command to simulate a series of length 1000 from $\{X_t\}_{t \in \mathbb{Z}}$. Use `spec.pgram()` to create and plot the periodogram for your sample. Comment on what you observe in regard to the true spectrum and its estimate here.
- (d) Repeat parts (b) and (c) several times. Comment on how the periodogram from R behaves as an estimator of the spectral density function, based on what you have observed.