- 已知联合概率分布求边缘概率分布及条件概率分布.

已知：

P维（Data同上篇）

⇓

分两组：$x = \begin{pmatrix} x_a \\ x_b \end{pmatrix} \begin{matrix} \to m \\ \to n \end{matrix}$ $m+n=p$ $\mu = \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix}$ $\Sigma = \begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}$

求：$P(x_a)$，$P(x_b \mid x_a)$

$P(x_a)$，$P(x_a \mid x_b)$

方法：$\boxed{\text{配方}}$

引入定理：已知 $x \sim N(\mu, \Sigma)$

$y = Ax + B$

$\boxed{\text{结论：} y \sim N(A\mu + B, A\Sigma A^T)}$

$E(y) = E(Ax+B) = AE(x) + E(B)$

$= A\mu + B$

$Var(y) = Var(Ax+B) = Var(Ax) + \underbrace{Var(B)}_{0}$

$= A \cdot Var(x) \cdot A^T$

对应

eg：一维直观来看.

$x \sim N(\mu, \sigma^2)$

$y = ax + b$

$Var(y) = a^2 Var(x)$

配方：$x_a = (I_m \quad \underset{\text{零}}{O_n}) \begin{pmatrix} x_a \\ x_b \end{pmatrix}$

$\underbrace{(I_m \quad O_n)}_{A} \underbrace{\begin{pmatrix} x_a \\ x_b \end{pmatrix}}_{x} + 0$

把定理中的a对写下来：

- $E(x_a) = (I_m \quad 0) \begin{pmatrix} \mu_a \\ \mu_b \end{pmatrix} = \boxed{\mu_a}$

- $Var(x_a) = (I_m \quad 0)\begin{pmatrix}\Sigma_{aa} & \Sigma_{ab}\\ \Sigma_{ba} & \Sigma_{bb}\end{pmatrix}\begin{pmatrix} I_m \\ 0\end{pmatrix} = (\Sigma_{aa} \quad \Sigma_{ab})\begin{pmatrix}I_m\\ 0\end{pmatrix}$

$$= \boxed{\Sigma_{aa}}$$

$x_a \sim N(\mu_a, \Sigma_{aa})$

$x_b | x_a$: 令 $\begin{cases} x_{b\cdot a} = x_b - \Sigma_{ba}\Sigma_{aa}^{-1} x_a \\ \mu_{b\cdot a} = \mu_b - \Sigma_{ba}\Sigma_{aa}^{-1}\mu_a \\ \Sigma_{bb\cdot a} = \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab} \end{cases}$

配方: $x_{b\cdot a} = \underbrace{(-\Sigma_{ba}\Sigma_{aa}^{-1} \quad I_m)}_{A}\underbrace{\begin{pmatrix} x_a \\ x_b \end{pmatrix}}_{x} + 0$

- $E(x_{b\cdot a}) = (-\Sigma_{ba}\Sigma_{aa}^{-1} \quad I_m)\begin{pmatrix}\mu_a \\ \mu_b\end{pmatrix} = \mu_b - \Sigma_{ba}\Sigma_{aa}^{-1}\mu_a = \boxed{\mu_{b\cdot a}}$

- $Var(x_{b\cdot a}) = (-\Sigma_{ba}\Sigma_{aa}^{-1} \quad I_m)\begin{pmatrix}\Sigma_{aa} & \Sigma_{ab}\\ \Sigma_{ba} & \Sigma_{bb}\end{pmatrix}\begin{pmatrix}-\Sigma_{ba}\Sigma_{aa}^{-1} \\ I_m\end{pmatrix}$

$$= (0 \quad \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab})\begin{pmatrix}-\Sigma_{ba}\Sigma_{aa}^{-1} \\ I_m\end{pmatrix}$$

$$= \Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab}$$

$$\boxed{= \Sigma_{bb\cdot a}}$$

$x_{b\cdot a} \sim N(\mu_{b\cdot a}, \Sigma_{bb\cdot a})$

$x_b | x_a$

$x_b = x_{b\cdot a} + \Sigma_{ba}\Sigma_{aa}^{-1} x_a$

- $E(x_b | x_a) = \mu_{b\cdot a} + \Sigma_{ba}\Sigma_{aa}^{-1} x_a$

- $Var(x_b | x_a) = Var(x_{b \cdot a}) = \Sigma_{bb \cdot a}$

$$\boxed{x_b | x_a \sim N(\mu_{b \cdot a} + \Sigma_{ba}\Sigma_{aa}^{-1} x_a \,,\ \Sigma_{bb \cdot a})}$$