Minimizing the Expectations

The Balanced Loss Function

To compute the balanced policy, the following equation must be minimized.

$$g_s(q_s^B) = \max\{l_s^B(q_s), \pi_s^B(q_s)\}$$

where $l_s^B(q_s), \pi_s^B(q_s)$ are as they were defined previously. For general distributions, these equations can be difficult to solve. However, assuming the demand has a multivariate normal distribution, there is a closed form solution.

Holding Function $l_s^B(q_s)$

Unfortunately, $l_s^B(q_s)$ is the result of a $T-s$ dimensional integral, but it can be reduced a series of $T-s$ integrals of 1 dimension, where the later terms of the series are small. Letting Φ_{f_s} be the distribution of forecast f_s:

$$\begin{aligned}
l_s^B(q_s) &= \mathbb{E}[H_s^B(q_s) \mid f_s] \\
&= \mathbb{E}\left\{ \sum_{j=s}^{T} h_j \left[q_s - \left(\sum_{i=s}^{j} u_i - X_s \right)^+ \right]^+ \mid f_s \right\} \\
&= \int_{u_s}^{\infty} \int_{u_{(s+1)}}^{\infty} \cdots \int_{u_T}^{\infty} \sum_{j=s}^{T} h_j \left[q_s - \left(\sum_{i=s}^{j} u_i - X_s \right)^+ \right]^+ d\Phi(u_s, u_{s+1}, \dots u_T) \\
&= \sum_{j=s}^{T} \int_{u_s}^{\infty} \int_{u_{(s+1)}}^{\infty} \cdots \int_{u_j}^{\infty} h_j \left[q_s - \left(\sum_{i=s}^{j} u_i - X_s \right)^+ \right]^+ d\Phi(u_s, u_{s+1}, \dots u_T)
\end{aligned}$$

Given the assumption that the demands are multivariate normal, and letting $D_{[s,j]} = \sum_{i=s}^{j} D_s$, $D_{[s,j]} \sim N(\sum_{i=s}^{j} \mu_i, \sum_{i=s}^{j} \sigma_i^2)$, the expectation can be written as the sum of $(T-s)$ integrals.

$$l_s^B(q_s) = \sum_{j=s}^{T} \int_{u_j = X_s}^{X_s + q_s} h_j \left(q_s + X_s - u_j \right) d\Phi(u_j) \tag{1}$$

It's worth noting that for any individual sample path, there will be a period for which all periods beyond it have 0 holding cost. For simulated solutions, this can be immediately utilized as a stopping rule. For analytical solutions, this could be used for a further approximation (generalizing between the myopic and horizon forms).

This function has the following derivative.

$$\begin{aligned}
\frac{d}{dq_s} l_s^B(q_s) &= \sum_{j=s}^{T} \int_{u_j = X_s}^{X_s + q_s} h_j d\Phi(u_j) \\
&= \sum_{j=s}^{T} h_j (\Phi(X_s + q_s) - \Phi(X_s))
\end{aligned}$$

Penalty Function $\pi_s^B(q_s)$

$\pi_s^B(q_s)$ is a more manageable integral.

$$\begin{aligned}
\pi_s^B(q_s) &= \mathbb{E}[\Pi_t^B(q_s) \mid f_s] \\
&= \mathbb{E}\Big\{ p_s[D_s - (X_s^B + q_s)]^+ \mid f_s \Big\} \\
&= \int_{u_s=(X_s^B+q_s)}^{\infty} p_s(u_s - X_s^B - q_s) d\Phi_s(u_s) \qquad (2)
\end{aligned}$$

This function has the following derivative.

$$\begin{aligned}
\frac{d}{dq_s}\pi_s^B(q_s) &= -\int_{u_s=(X_s^B+q_s)}^{\infty} p_s d\Phi_s(u_s) \\
&= -p_s(1 - \Phi_s(X_s^B + q_s))
\end{aligned}$$

Loss Function

As an informal validation, the loss function was computed exactly, and using simulated results with 100 replicates. An arbitrary scenario was picked for this task. The results are plotted below: