Dummit and Foote chap 4 solution

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4.1 Group action and permutation presentation

Missing exercise number : 5

Problem 1. Let G act on the set A. Prove that if $a, b \in A$ and for some $b = g \cdot a$ for some $g \in G$, then $G_b = gG_ag^{-1}$. Deduce that if G acts transitively on A then the kernel of the action is $\cap_{g\in G} gG_ag^{-1}$

Proof.

1. Since $G_b = \{x \in G \mid x \cdot b = b\} = \{x \in G \mid x \cdot (g \cdot a) = g \cdot a\} = \{x \in G \mid (g^{-1}xg) \cdot a = a\}$, we see that $g^{-1}G_bg = G_a$, hence $G_b = gG_ag^{-1}$.

2. If G acts transitively on A, then for every $b \in A$, there exists $g \in G$ such that $b = g \cdot a$. Therefore, $\ker(\phi) = \cap_{b\in A} G_b = \cap_{g\in G} gG_ag^{-1}$, where ϕ denotes the permuation presentation afforded by this action.

□

Problem 2. Let G be a permutation group on the set A(i.e. $G \leq S_A$), let $\sigma \in G$, and let $a \in A$. Prove that $\sigma G_a \sigma^{-1} = G_{\sigma(a)}$. Deduce that if G acts transitive on A, then

$$\bigcap_{\sigma\in G} \sigma G_a \sigma^{-1} = 1$$

Proof.

1. By the previous problem, let $b = \sigma \cdot a = \sigma(a)$, we see that $G_{\sigma(a)} = \sigma G_a \sigma^{-1}$.

2. It remains to show that $\ker(\phi) = 1$, which is trivial given that any non-zero permutation will permute elements in A.

□

Problem 3. Assume that G is an abelian, transitive subgroup of S_A. Show that $\sigma(a) \neq a$ for all $\sigma \in G - \{1\}$. Deduce that $|G| = |A|$.

Proof. 1. Suppose G is an abelian group, then $G_b = \sigma G_a \sigma^{-1} = G_a$ for all $a, b \in A$. By way of contradiction, suppose $\sigma(a) = a$ for some $\sigma \in G - \{1\}$, then $\sigma \in G_a$, and hence $\sigma \in G_a = \cap_{\sigma\in G} \sigma G_a \sigma^{-1} = \ker(\phi) = 1$, contradict problem 2 corollary. Therefore, there is some element in A not fixed by G.

2. Hence $G_a = \{1\}$ for all $a \in A$, and by obrit-stabilizer theorem, since $|A| = |G : G_a|$, then $|A| = |G|$.

□

Problem 4. Let S_3 act on the set Ω of ordered pairs: $\{(i,j) \mid 1 \leq i,j \leq 3\}$ by $\sigma((i,j)) = (\sigma(i), \sigma(j))$. Find orbits of S_3 on Ω. For each $\sigma \in S_3$ find cycle decomposition of σ under this action

Proof. 1. The orbits of S_3 on Ω by direct calculation is

$$\{(1,1),(2,2),(3,3)\}, \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$$

2. Fix labeling by letting

$$1 = (1,1), 2 = (1,2), 3 = (1,3), 4 = (2,1), 5 = (2,2), 6 = (2,3), 7 = (3,1), 8 = (3,2), 9 = (3,3)$$

Then the cycle decomposition is

$$(1\,2) = (1\,5)(2\,4)(3\,6)(7\,8)$$

□

Problem 6. Let R be the set of all polynomials with integer coefficients in the independent variables x_1, x_2, x_3, x_4 and let S_4 act on R by permitting the indices of the four variables:

$$\sigma \cdot p(x_1, x_2, x_3, x_4) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

for all $\sigma \in S_4$.

(a) Find the polynomials in the orbit of S_4 on R containing $x_1 + x_2$.

(b) Find the polynomial in the orbit of S_4 on R containing $x_1x_2 + x_3x_4$.

(c) Find the polynomial in the orbit of S_4 on R containing $(x_1 + x_2)(x_3 + x_4)$

Proof. (a) Let $q(x_1, x_2, x_3, x_4) = x_1 + x_2$, then $G_q = \langle (1,2), (3,4) \rangle$. So the orbit containing q is

$$\{x_1 + x_2, x_1 + x_3, x_1 + x_4, x_2 + x_3, x_2 + x_4, x_3 + x_4\}$$

(b) Let $r(x_1, x_2, x_3, x_4) = x_1x_2 + x_3x_4$, then $G_r = \langle (1,2), (3,4), (1,3)(2,4) \rangle$. So the orbit containing r is

$$\{x_1x_2 + x_3x_4, x_2x_3 + x_1x_4, x_1x_3 + x_2x_4\}$$

(c) Having the exact structure of (b), we immediately see that

$$\{(x_1 + x_2)(x_3 + x_4), (x_1 + x_3)(x_2 + x_4), (x_1 + x_4)(x_2 + x_3)\}$$

□

Problem 7. Let G be a transitive permutation group on a finite set A. A block is a nonempty subset B of A such that for all $\sigma \in G$ either $\sigma(B) = B$ or $\sigma(B) \cap B = \emptyset$.

(a) Prove that if B is a block containing the element a of A, then the set G_B define by $G_B = \{\sigma \in G \mid \sigma(B) = B\}$ is a subgroup containing G_a.

(b) Show that if B is a block and $\sigma_1(B), ..., \sigma_n(B)$ is distinct images of B under elements in G, then these forms a partition of A.

(c) A (transitive) group B on a set A is said to be primitive if the only blocks in A are the trivial ones: the set of size 1 and itself. Show that S_4 is primitive on $A = \{1, 2, 3, 4\}$. Show that D_8 is not primitive as a permutation group on the four vertices of a square.

(d) Prove that transitive group G is primitive on A if and only if for each $a \in A$, the only subgroups of G containing G_a are G_a and G (i.e. G_a is the maximal subgroup in G).

Proof. (a) Let $\sigma_1, \sigma_2 \in G_B$, since $1 \in G_B$, then $G_B \neq \emptyset$ and if $\sigma_2(B) = B$, then $B = \sigma_2^{-1}(B)$, so that $\sigma_1\sigma_2^{-1}(B) = \sigma_1(B) = B$, hence $\sigma_1\sigma_2^{-1} \in G_B$. Therefore, G_B is a subgroup, and suppose $a \in B$, then if $\sigma \in G_a$, then $\sigma(a) = a \in B$, since B is a block, then $a \in \sigma(B) \cap B$, thus $\sigma(B) \cap B \neq \emptyset$, so it forces $\sigma(B) = B$, then $\sigma \in G_B$.

(b) Suppose $\sigma_i(B) \cap \sigma_j(B) \neq \emptyset$. Let $a \in \sigma_i(B) \cap \sigma_j(B)$, then $a = \sigma_i(b_i) = \sigma_j(b_j)$, then $b_i = \sigma_i^{-1}\sigma_j(b_j)$, hence $\sigma_i^{-1}\sigma_j(B) = B$ so that $\sigma_i(B) = \sigma_j(B)$ contradicts that they are distinct.

(c) Let B be a subset of A such that $1 < |B| < |A|$, then choose $x \in B$ and $y \in B^c$. Let $\sigma = (x\,y) \in S_4$, then $y \in \sigma(B)$. Let $z \in B \setminus \{x\}$, then $z \in \sigma(B)$ hence $\sigma(B) \cap B \neq \emptyset$ and $\sigma(B) \cap B^c \neq \emptyset$. This forces B not to be a block, and thus S_4 is primitive.
Consider $B = \{1, 3\}$, then $r(B) = \{2, 4\} \cap \{1, 3\} =$ and $s(B) = \{1, 3\} = B$. Therefore, B is non-primitive.

(d) Suppose that G is primitive, and by way of contradiction suppose $G_a < H < G$ for some subgroup H, construct a set $B = \{h \cdot a \mid h \in H\}$, suppose $\tau \in G_B$, then

$$\tau(h_1(a)) = h_2(a) \implies h_2^{-1}\tau h_1(a) = a \implies h_2^{-1}\tau h_1 \in G_a \subseteq H$$

Let $h_3 = h_2^{-1}\tau h_1 \in H$, then $\tau = h_2 h_3 h_1^{-1} \in H$, therefore $G_B \subseteq H$. By construction, it is trivial that $H \subseteq G_B$. Then $H = G_B$. Let $g \in G \setminus H$, then $g(b_1) \neq b_2$ for all $b_1, b_2 \in B$ by above argument, therefore, $g(B) \cap B = \emptyset$ and hence B is a block and $|B| < |A|$ and $|B| > 1$ for $H \neq G_a$, and this contradict with the fact that G is primitive.
Suppose that G_a is maximal for all $a \in A$, by way of contradiction suppose that G is not primitive, let B be a non-trivial block in A, then $G_a \leq G_B \leq G$. Suppose $G_B = G$, then $\sigma(B) = B$ for all $\sigma \in G$, and since G acts transitively on A, then $\sigma(B) = A = B$. Otherwise when $G_B = G_a$, for all $a \in B$, since G is transitive, there exists $g' \in G$ such that $g'(a) = b$ for all $a, b \in B$. Since

$$b = g'(a) \in g'(B) \cap B$$

, then $g'(B) = B$ and thus $g' \in G_B$. But since $G_a = G = G_b$, then $b = g' \cdot a = a$. Then $B = \{a\}$.

□

Problem 8. A transitive permutation group G on a set A is called doubly transitive if for any $a \in A$ the subgroup G_a is transitive on the set $A - \{a\}$.

(a) Prove that S_n is doubly transitive on $\{1, 2, ..., n\}$ for all $n \geq 2$.

(b) Prove that a doubly transitive group is primitive. Deduce that D_8 is not doubly transitive in its action on the 4 vertices of a square.

Proof. 1. Let $i \in A$, then $G_i \simeq S_{n-1}$ which is transitive on $A - \{i\}$.

2. Suppose B be a non-trivial block in A, let $x, z \in B$ and $y \in B^c$, $x \neq z$, let $\sigma \in G_x$, then $\sigma(x) = x \in B$, then $\sigma(B) \subseteq B$, since $|\sigma(B)| = |B|$, then $\sigma(B) = B$. But since G_x is transitive on $A - \{x\}$, $\sigma(z) = y$ for some $\sigma \in G_x$, and this is a contradiction, given that $y \notin B = \sigma(B)$ and $\sigma(z) \in \sigma(B)$. It follows that G is primitive.
   Since D_8 is not primitive, then it is not doubly transitive.

□

Problem 9. Assumes G acts transitively on the finite set A and let H be a normal subgroup of G. Let $\mathcal{O}_1, ..., \mathcal{O}_r$ be distinct orbits of H on A.

(a) Prove that G permutes the sets $\mathcal{O}_1, ..., \mathcal{O}_r$ in the sense that for each $g \in G$ and each $i \in \{1, ..., r\}$ there is a j such that $g\mathcal{O}_i = \mathcal{O}_j$, where $g\mathcal{O} = \{g \cdot a \mid a \in \mathcal{O}\}$ (i.e. the sets $\mathcal{O}_1, ..., \mathcal{O}_r$ are blocks). Prove that G is transitive on $\{\mathcal{O}_1, ..., \mathcal{O}_r\}$. Deduce that all orbits have the same cardinality.

(b) Prove that if $a \in \mathcal{O}_1$, then $|\mathcal{O}_1| = |H : H \cap G_a|$ and prove that $r = |G : HG_a|$.

Proof. (a) Let x and y be representative of $\mathcal{O}_i$ and $\mathcal{O}_j$, then since G is transitive, there exists $g \in G$ such that $g \cdot x = y$, let $z \in \mathcal{O}_i$, then there exists $h \in H$ such that $h \cdot x = z$, then and $g \cdot z = g \cdot (h \cdot x) = h'g \cdot x = h' \cdot y \in \mathcal{O}_j$. Note also that $h \cdot y = hg \cdot x = gh' \cdot x = g\mathcal{O}_i$ for all $h \in H$ .Therefore, $\mathcal{O}_j = g\mathcal{O}_i$ and this proves that G is transitive on $\{\mathcal{O}_1, ..., \mathcal{O}_r\}$. Let $\phi_g : A \to A$ defined by $\phi_g(x) = g \cdot x$

(b) By orbit stabilizer theorem,

$$|\mathcal{O}_1| = |H : H_a| = |H : G_a \cap H|$$

and that

$$r = \frac{|A|}{|H : G_a \cap H|} = \frac{|G : G_a|}{|H : G_a \cap H|} = \frac{|G : G_a|}{|HG_a : G_a|} = |G : HG_a|$$

□

Problem 10. Let H and K be subgroups of the group G. For each $x \in G$ define the HK double coset of x in G to be the set

$$HxK = \{hxk \mid h \in H, k \in K\}$$

(a) Prove that HxK is the union of the left cosets $x_1K, ..., x_nK$ where $\{x_1K, ..., x_nK\}$ is the orbit containing xK of H acting by left multiplication on the set of left cosets of K.

(b) Prove that HxK is the union of the right cosets of H

(c) Show that HxK and HyK are either the same set or are disjoint for all $x, y \in G$. Show that the set of HK double coset partitions G.

(d) Prove that $|HxK| = |K| \cdot |H : H \cap xKx^{-1}|$.

(e) Prove that $|HxK| = |H| \cdot |K : K \cap x^{-1}Hx|$

Proof. (a) Suppose H acts on the set of left cosets of K by left multiplication, then

$$HxK = \cup_{h\in H} hxK = \cup_{h\in H} h \cdot xK = \cup_{i=1}^{n} x_i K$$

where $\{x_1K, ..., x_nK\}$ is the orbit containing xK.

(b) Similar to (a), $HxK = \cup_{k\in K} Hxk$.

(c) Suppose $HxK \cap HyK \neq \emptyset$, take $g \in HxK \cap HyK$, then $g = h_1xk_1 = h_2yk_2$, then $(h_2^{-1}h_1)xk_2k_2^{-1} = y$ and thus

$$hyk = (hh_2^{-1}h_1)x(k_2k_2^{-1}k) \in HxK$$

, then $HyK \subseteq HxK$, similarly, $HxK \subseteq HyK$, then $HxK = HyK$.

(d) Since $|HxK| = |HxKx^{-1}|$, then by second isomorphism theorem we have that

$$|HxK| = \frac{|H||xKx^{-1}|}{|H \cap xKx^{-1}|} = |K||H : H \cap xKx^{-1}|$$

(e) Similar to (d) we have that

$$|HxK| = |x^{-1}HxK| = \frac{|H||K|}{|x^{-1}Hx \cap K|} = |H||K : x^{-1}Hx \cap K|$$

□

4.2 Group acting on themself by left multiplication - Cayley's theorem

Missing exercise number : 2, 3, 4, 5, 6, 7

Problem 1. Let $G = \{1, a, b, c\}$ be the Klein 4-group whose group table is written out in Section 2.5.

(a) Label $1, a, b, c$ with integer $1, 2, 4, 3$, respectively, and prove under left regular representation of G into S_4 the non-identity elements is mapped as follows:

$$a \mapsto (1\,2)(3\,4) \qquad b \mapsto (1\,4)(2\,3) \qquad c \mapsto (1\,3)(2\,4)$$

(b) Relabel $1, a, b, c$ as $1, 4, 2, 3$ respectively, and compute the image of each element if G under the left regular expression of G into S_4. Show that the image of each element of G in S_4 under this labeling is the same subgroup as the image of G in part (a).

Proof. (a) Since

$$\begin{array}{lll}
a \cdot 1 = a, \quad \sigma_a(1) = 2 & b \cdot 1 = b, \quad \sigma_b(1) = 4 & c \cdot 1 = c, \quad \sigma_c(1) = 3 \\
a \cdot a = 1, \quad \sigma_a(2) = 1 & b \cdot a = c, \quad \sigma_b(2) = 3 & c \cdot a = b, \quad \sigma_c(2) = 4 \\
a \cdot b = c, \quad \sigma_a(4) = 3 & b \cdot b = 1, \quad \sigma_b(4) = 1 & c \cdot b = a, \quad \sigma_c(4) = 2 \\
a \cdot c = b, \quad \sigma_a(3) = 4 & b \cdot c = a, \quad \sigma_b(3) = 2 & c \cdot c = 1, \quad \sigma_c(3) = 1
\end{array}$$

It follows that a, b, c maps as above.

(b) Since

$$\begin{array}{llllll}
a\cdot 1 = a, & \sigma_a(1)=4 & b\cdot 1 = b, & \sigma_b(1)=2 & c\cdot 1 = c, & \sigma_c(1)=3\\
a\cdot a = 1, & \sigma_a(4)=1 & b\cdot a = c, & \sigma_b(4)=3 & c\cdot a = b, & \sigma_c(4)=2\\
a\cdot b = c, & \sigma_a(2)=3 & b\cdot b = 1, & \sigma_b(2)=1 & c\cdot b = a, & \sigma_c(2)=4\\
a\cdot c = b, & \sigma_a(3)=2 & b\cdot c = a, & \sigma_b(3)=4 & c\cdot c = 1, & \sigma_c(3)=1
\end{array}$$

So that left regular representation of G into S_4 the non-identity elements are mapped as follows:

$$a \mapsto (1\,4)(2\,3) \qquad b \mapsto (1\,2)(3\,4) \qquad c \mapsto (1\,3)(2\,4)$$

Then they generate the same subgroup.

□

Problem 8. Prove that if H has finite index n then there is a normal subgroup K of G with $K \leq H$ and $|G : K| \leq n!$

Proof. Let G acts on the set of left cosets of H, then $\ker\varphi = K \leq H$, $K \trianglelefteq G$ and let $\varphi : G \to S_n$ be permuation presentation afford by this action, then

$$G/K \hookrightarrow S_n \Rightarrow |G : K| \leq n!$$

□

Problem 9. Prove that if p is a prime and G is a group of order p^α for some $\alpha \in \mathbb{Z}^+$, then every subgroup of index p is normal in G. Deduce that every group of order p^2 has a normal subgroup of order p.

Proof. The statement follows directly from Corollary 5 and the existence of the order p subgroup follows from Cauchy's theorem. □

Problem 10. Prove that every non-abelian group of order 6 has a nonnormal subgroup of order 2.

Proof. By way of contradiction, suppose that there exists some non-abelian group G of order 6 and that every order 2 subgroup is normal. Let N be some subgroup of order 2, the existence of this group is guaranteed by Cauchy theorem, let G act on the set of left coset of N by left multiplication, then

$$G/N \overset{\varphi}{\hookrightarrow} S_3$$

Let $N = \{1, a\}$, since N is normal subgroup of G,

$$gHg^{-1} = \{1, gag^{-1}\} = \{1, a\} \quad \text{, for all } g \in G$$

Then $gag^{-1} = a$, for all $g \in G$, then $a \in Z(G)$. Since $G/N = \langle xN \rangle$ where $x^3 \in N$, then $g = x^m a$ for all $g \in G.$, and thus G is an abelian group. Since G has no normal group of order 2, then $\ker\phi = 1$ and hence

$$G \overset{\phi}{\hookrightarrow} S_3$$

Since $|G| = |S_3| = 6$, it follows that $G \simeq S_3$. For abelian group G, $G \simeq Z_6$ and $G \simeq Z_2 \times Z_3$. □

Problem 11. Let G be a finite group and let $\pi : G \to S_G$ be the left regular representation. Prove that if x is an element of G of order n and $|G| = nm$, then $\pi(x)$ is a product of m n-cycles. Deduce that $\pi(x)$ is an odd permutation if and only if $|x|$ is even and $\frac{|G|}{|x|}$ is odd.

Proof. 1. Since π is injective function, we have that $|\pi(x)| = |x| = n$. Let $H = \langle \pi(x) \rangle \leq S_G$, let H acts on G by function evaluation, since

$$\pi(x)^i(a) = a \Rightarrow x^i a = a \Rightarrow x^i = 1 \Rightarrow i \equiv 0 \pmod n$$

then $H_a = 1$ for all $a \in G$. So by orbit-stabilizer theorem, $|\mathcal{O}| = |H : H_a| = |H| = n$ for all orbit $\mathcal{O}$ and $a \in \mathcal{O}$. Let $\mathcal{O}_1, ..., \mathcal{O}_r$, since $|\mathcal{O}_i| = n$ and $G = \sqcup_{i=1}^r \mathcal{O}_i$, then

$$r = \frac{|G|}{|\mathcal{O}_1|} = \frac{mn}{n} = m$$

Since each orbit of H corresponds to a cycle of $\pi(x)$, then $\pi(x)$ is product of m n-cycles.

2. Suppose $\pi(x)$ is an odd permutation, let $\pi(x) = \sigma_1...\sigma_m$ where σ_i is a n-cycle, then

$$\epsilon(\pi(x)) = \epsilon(\sigma_1...\sigma_m) = \epsilon(\sigma_1)...\epsilon(\sigma_r) = \epsilon(\sigma_1)^m = -1$$

Then m is odd and $\epsilon(\sigma_1) = -1$ and thus n is even. Suppose n is even and m is odd, then we immediately see that $\pi(x)$ is an odd permutation.

□

Problem 12. Let G and π be as in the preceding exercise. Prove that if $\pi(G)$ contains an odd permutation, then G has a subgroup of index 2.

Proof. Suppose $\pi(G)$ contains an odd permutation, then

$$\pi(G)/(A_G \cap \pi(G)) \simeq A_G\pi(G)/A_G$$

and since $A_G\pi(G) > A_G$, then $A_G\pi(G) = S_G$. Therefore,

$$G/(\pi^{-1}(A_G \cap \pi(G))) \simeq \pi(G)/(A_G \cap \pi(G)) \simeq S_G/A_G$$

and $\pi^{-1}(A_G \cap \pi(G))$ is a subgroup of index 2. □

Problem 13. Prove that if $|G| = 2k$ where k is odd then G has an index 2 subgroup.

Proof. By Cauchy theorem, since there exists an order 2 element and thus G has an index 2 subgroup by the previous problem. □

Problem 14. Let G be a finite group of composite order n with the property that G has a subgroup of order k for each positive integer k dividing n. Prove that G is not simple.

4.3 Group action on themselves by conjugation

Problem 1. Suppose G has a left action on A, denotes $g \cdot a$ for all $g \in G$ and $a \in A$. Denotes the corresponding right action on A by $a \cdot g$. Prove that the (corresponding) relation $\sim$ and $\sim'$ defined by

$$a \sim b \quad \text{if and only if} \quad a = g \cdot b \quad \text{for some } g \in G$$

and

$$a \sim' b \quad \text{if and only if} \quad a = b \cdot g \quad \text{for some } g \in G$$

are the same relation.

Proof. Since $a \sim b$ if and only if $a = g \cdot b$ if and only if $a = b \cdot g^{-1}$ if and only if $a \sim' b$. □

Problem 2. Find all conjugacy class and their sizes in the following groups:

(a) D_8

(b) Q_8

(c) A_5

Proof. (a) Since D_8 has 3 abelian index 2 subgroup then the conjugacy class are

$$\{1\} \qquad \{r, r^3\} \qquad \{r^2\} \qquad \{sr, sr^3\} \qquad \{s, sr^2\}$$

The corresponding size is 1, 2, 1, 2, 2

(b) Since $|Q_8 : \langle i \rangle| = 2$, $\langle i \rangle \leq C_{Q_8}(i)$ and $i \notin Z(Q_8)$, then

$$\{1\} \qquad \{-1\} \qquad \{\pm i\} \qquad \{\pm j\} \qquad \{\pm k\}$$

the corresponding size is 1, 1, 2, 2, 2

(c) All 3-cycles are conjugates in A_5 and so does $(2, 2)$-cycles, and 5-cycles is either conjugates to $(1\,2\,3\,4\,5)$ or $(1\,3\,2\,4\,5)$.

□

Problem 4. Prove that if $S \subseteq G$ and $g \in G$ then $gN_G(S)g^{-1} = N_G(gSg^{-1})$ and $gC_G(S)g^{-1} = C_G(gSg^{-1})$.

Proof. Let $h \in N_G(S)$, then $hSh^{-1} = S$,

$$(ghg^{-1})(gSg^{-1})(ghg^{-1})^{-1} = ghSh^{-1}g^{-1} = gSg^{-1}$$

therefore, $gN_G(S)g^{-1} \subseteq N_G(gSg^{-1})$, then we have that $g^{-1}N_G(gSg^{-1})g \subseteq N_G(S)$ thus $N_G(gSg^{-1}) \subseteq gN_G(S)g^{-1}$, hence $gN_G(S)g^{-1} = N_G(gSg^{-1})$.
Let $h \in C_G(S)$, then $hsh^{-1} = s$ for all $s \in S$,

$$(ghg^{-1})(gsg^{-1})(ghg^{-1})^{-1} = ghsh^{-1}g^{-1} = gsg^{-1}$$

then $gC_G(S)g^{-1} \subseteq C_G(gSg^{-1})$, hence $gC_G(S)g^{-1} = C_G(gSg^{-1})$ □

Problem 5. If the center of G is of index n, prove that every conjugacy class has at most n element.

Proof. Since $Z(G) \leq C_G(g)$ for all $g \in G$, then

$$n = |G : Z(G)| \geq |G : C_G(g)|$$

therefore, conjugacy class has at most n elements. $\square$

Problem 6. Assume G is a non-abelian group of order 15. Prove that $Z(G) = 1$. Use the fact that $\langle g \rangle \leq C_G(g)$ to show that there is at most one possible class equation for G.

Proof. Since G is non-abelian, $Z(G) \neq G$, then the only possible order of $Z(G)$ are $\{1, 3, 5\}$. Since $|G/Z(G)| = 3, 5$ if $|Z(G)| = 3, 5$, then $G/Z(G)$ is cyclic, thus G is abelian, which is a contradiction. Hence $|Z(G)| = 1$. Let a and b be the number of conjugacy class in G with order 3 and 5, then

$$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(g_i)| = 1 + 3a + 5b$$

where $g_1, ..., g_r$ are representatives of distinct conjugacy classes of G not contained in the $Z(G)$. Since the only integral solution is $(a, b) = (3, 1)$, then the last statement is established. $\square$

Problem 7. For $n = 2, 4, 6$ and 7 make lists of the partitions of n and give representatives for the corresponding conjugacy classes of S_n.

Proof. For $n = 2$, the only possible partition are $(1, 1)$ and 2, then

Partition of 2	**Representative of Conjugacy Class**
1, 1	1
2	(1 2)

For $n = 4$, the possible partition are $(1, 1, 1, 1)$, $(1, 1, 2)$, $(1, 3)$, 4 $(2, 2)$,then

Partition of 4	**Representative of Conjugacy Class**
1, 1, 1, 1	1
1, 1, 2	(1 2)
1, 3	(1 2 3)
4	(1 2 3 4)
2, 2	(1 2)(3 4)

For $n = 6$, the possible partition are $(1, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 4)$, $(1, 5)$, 6, $(1, 1, 2, 2)$, $(1, 2, 3)$, $(2, 2, 2)$, $(2, 4)$, $(3, 3)$ then
For $n = 7$, the possible partition are $(1, 1, 1, 1, 1, 1, 1)$, $(1, 1, 1, 1, 1, 2)$, $(1, 1, 1, 1, 3)$, $(1, 1, 1, 4)$, $(1, 1, 5)$, $(1, 6)$, 7, $(1, 1, 1, 2, 2)$, $(1, 1, 2, 3)$, $(1, 2, 2, 2)$, $(1, 2, 4)$, $(1, 3, 3)$, $(2, 2, 3)$, $(2, 5)$, $(3, 4)$ then

Partition of 6	Representative of Conjugacy Class
1, 1, 1, 1, 1, 1	1
1, 1, 1, 1, 2	(1 2)
1, 1, 1, 3	(1 2 3)
1, 1, 4	(1 2 3 4)
1, 5	(1 2 3 4 5)
6	(1 2 3 4 5 6)
1, 1, 2, 2	(1 2)(3 4)
1, 2, 3	(1 2)(3 4 5)
2, 2, 2	(1 2)(3 4)(5 6)
2, 4	(1 2)(3 4 5 6)
3, 3	(1 2 3)(4 5 6)

Partition of 7	Representative of Conjugacy Class
1, 1, 1, 1, 1, 1, 1	1
1, 1, 1, 1, 1, 2	(1 2)
1, 1, 1, 1, 3	(1 2 3)
1, 1, 1, 4	(1 2 3 4)
1, 1, 5	(1 2 3 4 5)
1, 6	(1 2 3 4 5 6)
7	(1 2 3 4 5 6 7)
1, 1, 1, 2, 2	(1 2)(3 4)
1, 1, 2, 3	(1 2)(3 4 5)
1, 2, 2, 2	(1 2)(3 4)(5 6)
1, 2, 4	(1 2)(3 4 5 6)
1, 3, 3	(1 2)(3 4)(5 6)
2, 2, 3	(1 2)(3 4)(5 6 7)
2, 5	(1 2)(3 4 5 6 7)
3, 4	(1 2 3)(4 5 6 7)

□

Problem 8. Prove that $Z(S_n) = 1$ for $n \geq 3$.

Proof. Let $\tau \in S_n - \{1\}$, suppose $\sigma_1 ... \sigma_t$ be cycle decomposition of τ with order $2 \leq |\sigma_1| \leq |\sigma_2| ... \leq |\sigma_t|$, suppose $\sum_{i=1}^t |\sigma_i| < n$, then there exists some $a \in \{1, ..., n\}$ such that $a \notin \mathcal{O}$ where $\mathcal{O}$ is an orbit corresponds to σ_1. Let $\gamma = (a\, b)$ where $b \in \mathcal{O}$, then $\gamma \tau \gamma^{-1} \neq \tau$. Otherwise, choose $a \in \mathcal{O}_1$ and $b \in \mathcal{O}_2$ where $\mathcal{O}_i$ correspond to σ_i, then let $\gamma = (a\, b)$, then $\gamma \tau \gamma^{-1} \neq \tau$. □

Problem 9. Show that $|C_{S_n}((1\,2)(3\,4))| = 8 \cdot (n-4)!$ for all $n \geq 4$. Determine the elements in this centralizer explicitly.

Proof. Let A be the set of elements in S_n with cycle type $(2, 2)$, then

$$|A| = \frac{n(n-1)(n-2)(n-3)}{8}$$

therefore,

$$|C_{S_n}((1\,2)(3\,4))| = \frac{|G|}{|A|} = \frac{n!}{\frac{n(n-1)(n-2)(n-3)}{8}} = 8 \cdot (n-4)!$$

since $C_{S_4}((1\,2)(3\,4)) = \{1, (1\,2), (3\,4), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,3\,2\,4), (1\,4\,2\,3)\}$ which has order 8 and every element in S_{n-4} commutes with $(1\,2)(3\,4)$, it follows that

$$C_{S_n}((1\,2)(3\,4)) = \{\tau\sigma \mid \tau \in C_{S_4}((1\,2)(3\,4)), \sigma \in S_{n-4}\}$$

. □

Problem 10. Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5)$ in S_5. In each of (a) to (c), find an explicit element $\tau \in S_5$ which accomplishes the specified conjugation:

(a) $\tau\sigma\tau^{-1} = \sigma^2$

(b) $\tau\sigma\tau^{-1} = \sigma^{-1}$

(c) $\tau\sigma\tau^{-1} = \sigma^{-2}$

Proof. (a) $\sigma^2 = (1\,3\,5\,2\,4)$, so choose $\tau = (2\,3\,5\,4)$.

(b) $\sigma^{-1} = (1\,5\,4\,3\,2)$, so choose $\tau = (2\,5)(3\,4)$.

(c) $\sigma^{-2} = (1\,4\,2\,5\,3)$, so choose $\tau = (2\,4\,5\,3)$.

□

Problem 11. In each of (a)–(d), determine whether σ_1 and σ_2 are conjugate. If they are, give an explicit permutation τ such that $\tau\sigma_1\tau^{-1} = \sigma_2$:

(a) $\sigma_1 = (1\ 2)(3\ 4\ 5)$ and $\sigma_2 = (1\ 2\ 3)(4\ 5)$

(b) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = (3\ 7\ 5\ 10)(4\ 9)(13\ 11\ 2)$

(c) $\sigma_1 = (1\ 5)(3\ 7\ 2)(10\ 6\ 8\ 11)$ and $\sigma_2 = \sigma_1^3$

(d) $\sigma_1 = (1\ 3)(2\ 4\ 6)$ and $\sigma_2 = (3\ 5)(2\ 4)(5\ 6)$

Proof.

(a) $\tau = (1\,4\,2\,5\,3)$

(b) $\tau = (1\,4)(3\,13\,12\,8\,5\,9\,6\,7\,11\,10)$

(c) $\sigma_1^3 = (1\,2)$ so that σ_1 and σ_2 have different cycle types and hence do not conjugate.

(d) Since σ_1 and σ_2 have different cycle types and hence not conjugate with each other.

□

Problem 12. Find a representative for each conjugacy class of elements of order 4 in S_8 and S_{12}.

Proof. Since the conjugacy class is determined by the cycle type, and since all order 4 elements must be product of at least one 4-cycle and possibly 2-cycle, the representative conjugacy classes in S_8 are

$$(1\,2\,3\,4), \quad (1\,2\,3\,4)(5\,6), \quad (1\,2\,3\,4)(5\,6)(7\,8), \quad (1\,2\,3\,4)(5\,6\,7\,8)$$

and the representative conjugacy classes in S_{12} are

$$(1\,2\,3\,4), \quad (1\,2\,3\,4)(5\,6), \quad (1\,2\,3\,4)(5\,6)(7\,8), \quad (1\,2\,3\,4)(5\,6\,7\,8), \quad (1\,2\,3\,4)(5\,6\,7\,8)(9\,10),$$

$$(1\,2\,3\,4)(5\,6\,7\,8)(9\,10)(11\,12), \quad (1\,2\,3\,4)(5\,6\,7\,8)(9\,10\,11\,12)$$

□

Problem 13. Find all finite group which have exactly 2 conjugacy class.

Proof. Let G be a group of order n, suppose since $\{1\}$ is a conjugacy class then the other conjugacy class has order $n-1$, then let g be some representative of this conjugacy class, then

$$|G : C_G(g)| = n - 1$$

hence $n - 1 \mid n$, thus $n = 2$. It follows directly that $H \simeq Z_2$. □

Problem 14. In Exercise 1 of Section 2 two labellings of the elements $\{1, a, b, c\}$ of the Klein 4-group V were chosen to give two versions of the left regular representation of V into S_4. Let π_1 be the version of regular representation obtained in part (a) of that exercise and let π_2 be the version obtained via the labelling in part (b). Let $\tau = (2\ 4)$. Show that $\tau \circ \pi_1(g) \circ \tau^{-1} = \pi_2(g)$ for each $g \in V$ (i.e., conjugation by τ sends the image of π_1 to the image of π_2 elementwise).

Proof. This can be checked directly by brute force calculation, or since $\tau = (2\,4)$ is the label transformation of the elements, then it serves as a change of basis function, from 1 to 2 and hence

$$\tau \circ \pi_1(g) \circ \tau^{-1} = \pi_2(g) \qquad \text{for all } g \in V$$

□

Problem 17. Let A be a nonempty set and let X be any subset of S_A. Let

$$F(X) = \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in X\} \quad \text{— the fixed set of } X.$$

Let $M(X) = A - F(X)$ be the elements which are moved by some element of X. Let

$$D = \{\sigma \in S_A \mid |M(\sigma)| < \infty\}.$$

Prove that D is a normal subgroup of S_A.

Proof. Let $\sigma \in D$ and $\tau \in S_A$. Given $a \in A$, there exists $b \in A$ such that $a = \tau(b)$. Suppose $b \in M(\sigma)$, let $c = \sigma(b) \neq b$, then

$$\tau \circ \sigma \circ \tau^{-1}(a) = \tau \circ \sigma(b) = \tau(c) \neq \tau(b) = a$$

hence $a \in M(\tau\sigma\tau^{-1})$. Suppose $b \in F(\sigma)$, then $\sigma(b) = b$, then

$$\tau \circ \sigma \circ \tau^{-1}(a) = \tau \circ \sigma(b) = \tau(b) = a$$

hence $a \in F(\tau\sigma\tau^{-1})$. Consequently,

$$\tau(M(\sigma)) = M(\tau\sigma\tau^{-1})$$

and then

$$|M(\sigma)| = |M(\tau\sigma\tau^{-1})| < \infty$$

, thus $\tau\sigma\tau^{-1} \in D$. Therefore, $\tau D\tau^{-1} \subseteq D$, thus D is a normal subgroup of S_A. □

Problem 18. Let A be a set, let H be a subgroup of S_A and let $F(H)$ be the fixed points of H on A as defined in the preceding exercise. Prove that if $\tau \in N_{S_A}(H)$ then τ stabilizes the set $F(H)$ and its complement $A - F(H)$.

Proof. Let $\tau \in N_{S_A}(H)$, then

$$\tau H\tau^{-1} = H$$

let $a \in F(H)$, then $\sigma(a) = a$ for all $\sigma \in H$, then

$$\sigma(\tau(a)) = \tau(\tau^{-1}\sigma(\tau(a))) = \tau(a)$$

hence $\tau(a) \in F(H)$ and since τ is bijective we have that

$$\tau(F(X)) = F(X)$$

Let $a \in A - F(X)$,

$$\sigma(\tau(a)) = \tau(\tau^{-1}\sigma(\tau(a))) = \tau(b)$$

where $b \neq a$, then $\tau(a) \in A - F(X)$. Hence $\tau(A - F(X)) = A - F(X)$. □

Problem 19. Assume H is a normal subgroup of G, $\mathcal{K}$ is a conjugacy class of G contained in H, and $x \in \mathcal{K}$. Prove that $\mathcal{K}$ is a union of k conjugacy classes of equal size in H, where $k = |G : HC_G(x)|$. Deduce that a conjugacy class in S_n, which consists of even permutations, is either a single conjugacy class under the action of A_n, or is a union of two classes of the same size in A_n. [Let $A = C_G(x)$ and $B = H$, so $A \cap B = C_H(x)$. Draw the lattice diagram associated to the Second Isomorphism Theorem and interpret the appropriate indices. See also Exercise 9, Section 1.]

Proof. Let H acts on $\mathcal{K}$ by conjugation, then $\mathcal{K} = \cup_{i=1}^{k}\mathcal{K}_i$ where $\mathcal{K}_i$ are orbits H acting on $\mathcal{K}$, let a be representative of $\mathcal{K}_i$, since $a \in \mathcal{K}$, then $a = gxg^{-1}$ for some $g \in G$. Since

$$\begin{aligned} C_H(gxg^{-1}) &= \{h \in H \mid hgxg^{-1}h^{-1} = gxg^{-1}\} \\ &= \{h \in H \mid (g^{-1}hg)x(g^{-1}hg)^{-1} = x\} \\ &= \{h \in H \mid (g^{-1}hg) \in C_G(x)\} \\ &= \{h \in H \mid h \in gC_G(x)g^{-1}\} = gC_G(x)g^{-1} \cap H \end{aligned}$$

then

$$\begin{aligned} |\mathcal{K}_i| &= |H : C_H(gxg^{-1})| = |H : gC_G(x)g^{-1} \cap H| \\ &= |HgC_G(x)g^{-1} : gC_G(x)g^{-1}| = |gHC_G(x)g^{-1} : gC_G(x)g^{-1}| \\ &= |HC_G(x) : C_G(x)| \end{aligned}$$

so that

$$k = \frac{|G : C_G(x)|}{|HC_G(x) : C_G(x)|} = |G : HC_G(x)|$$

Let $G = S_n$ and $H = A_n$ and $x \in A_n$, since $S_n \geq A_n C_G(x) \geq A_n$

$$k = |S_n : A_n C_G(x)| = 1, 2$$

then $k = 1$ if and only if $C_G(x)$ contains at least one odd permuation, and otherwise $k = 2$ if and only if $C_G(x) \leq A_n$. $\square$

Problem 20. Let $\sigma \in A_n$. Show that all elements in the conjugacy class of σ in S_n (i.e., all elements of the same cycle type as σ) are conjugate in A_n if and only if σ commutes with an odd permutation. [Use the preceding exercise.]

Proof. By the previous theorem, all elements in the conjugacy class of σ in S_n are conjugate in A_n if and only if $A_n C_G(x) = S_n$, then $C_G(x)$ contains odd permutation, that is σ commutes with an odd permutation. $\square$

Problem 21. Let $\mathcal{K}$ be a conjugacy class in S_n, and assume that $\mathcal{K} \subseteq A_n$. Show $\sigma \in S_n$ does not commute with any odd permutation if and only if the cycle type of σ consists of distinct odd integers. Deduce that $\mathcal{K}$ consists of two conjugacy classes in A_n if and only if the cycle type of an element of $\mathcal{K}$ consists of distinct odd integers. [Assume first that $\sigma \in \mathcal{K}$ does not commute with any odd permutation. Observe that σ commutes with each individual cycle in its cycle decomposition—use this to show that all its cycles must be of odd length. If two cycles have the same odd length, k, find a product of k transpositions which interchanges them and commutes with σ. Conversely, if the cycle type of σ consists of distinct integers, prove that σ commutes *only* with the group generated by the cycles in its cycle decomposition.]

Proof. Assumes that σ does not commutes with any odd permutation. Let $\sigma_1 \dots \sigma_t$ be cycle decompostion of σ, then

$$\sigma_i \sigma \sigma_i^{-1} = \sigma$$

so that σ_i can have only odd length. Suppose that two cycle have the same odd length k, let $\sigma_i = (a_1 \, \dots \, a_k)$ and $\sigma_j = (b_1 \, \dots \, b_k)$ be the corresponding cycle, then define $\tau = (a_1 \, b_1) \dots (a_k \, b_k)$. Observe that

$$\tau \sigma \tau^{-1} = \sigma$$

then since k is odd, it follows that τ is an odd permutation, which is a contradiction.
Conversely, suppose the cycle type of σ is $n_1 < n_2 < \cdots < n_k$, let $\sigma_1 \dots \sigma_k$ be cycle decomposition of σ, and let $\tau \in C_{S_n}(\sigma)$, then

$$\tau \sigma \tau^{-1} = (\tau \sigma_1 \tau^{-1}) \dots (\tau \sigma_k \tau^{-1}) = \sigma_1 \dots \sigma_k$$

so that $\sigma_i = \tau \sigma_i \tau^{-1}$ by comparing the cycle type. If follows that $\tau \in C_{S_n}(\sigma_i) = \langle \sigma_i \rangle S_{n-n_i}$, and

$$\tau \in \langle \sigma_1, \dots \sigma_k \rangle \leq A_n$$

given that σ_i are all even permutations. $\square$

Problem 22. Show that if n is odd, then the set of all n-cycles consists of two conjugacy classes of equal size in A_n.

Proof. Since the cycle type of a n-cycle in S_n is $(1, n)$ and n is odd, then it does not commutes with odd permuations, and by problem 19, we have that all n-cycles consists of two conjugacy classes of equal size in A_n $\square$

Problem 23. Recall (cf. Exercise 16, Section 2.4) that a proper subgroup M of G is called *maximal* if whenever $M \leq H \leq G$, either $H = M$ or $H = G$. Prove that if M is a maximal subgroup of G, then either $N_G(M) = M$ or $N_G(M) = G$. Deduce that if M is a maximal subgroup of G that is not normal in G, then the number of nonidentity elements of G that are contained in conjugates of M is at most $(|M| - 1)|G : M|$.

Proof. By definition, since $M \leq N_G(M) \leq G$, and M is maximal, either $N_G(M) = M$ or $N_G(M) = G$. Suppose further than M is nonnormal, then $N_G(M) = M$. Since

$$|G : N_G(M)| = |G : M|$$

then there are $|G : M|$ conjugates and since each conjugates shares a 1, so that

$$\left| \bigcup_{g \in G} gHg^{-1} - \{1\} \right| \leq (|M| - 1)|G : M|$$

$\square$

Problem 24. Assume H is a proper subgroup of the finite group G. Prove $G \neq \bigcup_{g \in G} gHg^{-1}$, i.e., G is not the union of the conjugates of any proper subgroup. [Put H in some maximal subgroup and use the preceding exercise.]

Proof. Suppose that $G = \cup_{g \in G} gHg^{-1}$, let M be some maximal subgroup containing H, if M is nonnormal, then $G = \cup_{g \in G} gMg^{-1}$, hence

$$|G| \leq (|M| - 1)|G : M| < |G|$$

which is a contradiction. Otherwise, if M is normal, then

$$G = \cup_{g \in G} gMg^{-1} = M$$

contradicts M maximal. $\square$

Problem 25. Let $G = GL_2(\mathbb{C})$ and let

$$H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a, b, c \in \mathbb{C},\ ac \neq 0 \right\}.$$

Prove that every element of G is conjugate to some element of the subgroup H and deduce that G is the union of conjugates of H. [Show that every element of $GL_2(\mathbb{C})$ has an eigenvector.]

Proof. Let $A \in GL_2(\mathbb{C})$, let $p_A(x)$ be characteristic polynomial of A, then there exists at least 1 complex root of $p_A(x)$, and thus this root λ is an eigenvalue, let b_1 some eigenvector corresponds to this eigenvalue, then by basis extension theorem, we can extend to a basis $\{b_1, b_2\}$ of $M_2(\mathbb{C})$, then

$$A = \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} b_1 & b_2 \end{pmatrix}^{-1}$$

$\square$

Therefore G is union of conjugates of H.

Problem 26. Let G be a transitive permutation group on the finite set A with $|A| > 1$. Show that there is some $\sigma \in G$ such that $\sigma(a) \neq a$ for all $a \in A$ (such an element σ is called *fixed point free*).

Proof. Since $|G/G_a| = |A|$ for all $a \in A$, then

$$|G| = |A||G_a| = \sum_{a \in A} |G_a| \geq (|A| - 1) + |G| > |G|$$

which is a contradiction. □

Problem 27. Let $g_1, g_2, \ldots, g_r$ be representatives of the conjugacy classes of the finite group G and assume these elements pairwise commute. Prove that G is abelian.

Proof. Since g_i commutes with g_j for $i, j \in \{1, \ldots r\}$, so that

$$|C_G(g_i)| \geq r$$

then by orbit-stablizer theorem,

$$|G| = \sum_{i=1,\, g_i \neq 1}^{r} |G : C_G(g_i)| + 1 \leq \sum_{i=1,\, g_i \neq 1}^{r} \frac{|G|}{r} + 1 = \frac{r-1}{r}|G| + 1 \implies \frac{1}{r}|G| \leq 1 \implies |G| \leq r$$

Since representatives of G are all distinct, then $r \leq |G|$, hence $|G| = r$, then all conjugacy class has only 1 element, and thus G is abelian. □

Problem 29. Let p be a prime and let G be a group of order p^α. Prove that G has a subgroup of order p^β, for every β with $0 \leq \beta \leq \alpha$.

Proof. We show the statement by induction on α. When $|G| = p$, then subgroup 1 and G satisfy the statement. Suppose that the statement is true for $\alpha < k$. When $\alpha = k$, since $|G|$ is a prime power, $Z(G)$ is nontrivial, and thus $|Z(G)| = p^\epsilon$ where $1 \leq \epsilon \leq \alpha$, by Cauchy's theorem, let $H \leq Z(G)$ be such that $|H| = p$, then $H \trianglelefteq G$ and

$$|G/H| = p^{\alpha - 1}$$

so by the induction hypothesis, there exists a subgroup $\overline{K} \leq G/H$ with $|\overline{K}| = p^k$ for all $0 \leq k \leq \alpha - 1$, then by the fourth isomorphism theorem,

$$|K| = |\overline{K}||H| = p^{k+1}$$

for all $0 \leq k \leq \alpha - 1$. Then since $\{1\}$ is a subgroup of G, the statement is established. □

Problem 30. If G is a group of odd order, prove for any nonidentity element $x \in G$ that x and x^{-1} are not conjugate in G.

Proof. By way of contradiction, suppose that x and x^{-1} are conjugates, let $\mathcal{C}$ be conjugacy class containing x. Since $|G|$ is odd, and thus $|G : G_x| = |\mathcal{C}| = 2m + 1 < \infty$ is odd. For any element $y \in \mathcal{C}$, since $y = gxg^{-1}$ for some $g \in G$, then

$$y^{-1} = gx^{-1}g^{-1} \in \mathcal{C}$$

note that since no $y \in \mathcal{C}$ would have $y = y^{-1}$, otherwise $y^2 = 1$ and $|G|$ is even. Since every element in $\mathcal{C}$ maps to its inverse in $\mathcal{C}$, then $|\mathcal{C}|$ is even which is a contradiction. □

Problem 31. Using the usual generators and relations for the dihedral group D_{2n} (cf. Section 1.2) show that for $n = 2k$ an even integer, the conjugacy classes in D_{2n} are the following:

$$\{1\}, \{r^{\pm 1}\}, \{r^{\pm 2}\}, \dots, \{r^{\pm(k-1)}\}, \{sr^{2b} \mid b = 1, \dots, k\}, \text{ and } \{sr^{2b-1} \mid b = 1, \dots, k\}.$$

Give the class equation for D_{2n}.

Proof. Let $\mathcal{C}(x)$ be the conjugacy class contained in D_{2n} and let $x \in \mathcal{C}(x)$. Suppose $x = r^i$ for $0 \leq i < n$, r^j commutes with x obviously and since

$$sr^j r^i (sr^j)^{-1} = sr^j r^i r^{-j} s = sr^i s = r^{-i} = r^{n-i}$$

therefore $\mathcal{C}(x) = \{r^{\pm i}\}$. Suppose $x = sr^i$ for $0 \leq i < n$, then

$$r^j sr^i r^{-j} = sr^{i-2j}$$

and

$$sr^j sr^i (sr^j)^{-1} = r^{i-j} r^{-j} s = sr^{2j-i}$$

for $0 \leq j < n$. Therefore,

$$\mathcal{C}(x) = \begin{cases} \{sr^{2b} \mid b = 1, ..., k\} \\ \{sr^{2b-1} \mid b = 1, ..., k\} \end{cases}$$

□

Problem 32. For $n = 2k + 1$, an odd integer, show that the conjugacy classes in D_{2n} are

$$\{1\}, \{r^{\pm 1}\}, \dots, \{r^{\pm k}\}, \{sr^b \mid b = 1, \dots, n\}.$$

Give the class equation for D_{2n}

Proof. Slightly modify the proof of problem 31, let $\mathcal{C}(x)$ be a conjugacy class contained in D_{2n}, and let $x \in \mathcal{C}(x)$. Suppose $x = r^i$ for $0 \leq i < n$, then

$$\mathcal{C}(x) = \{r^{\pm i}\}$$

Suppose $x = sr^i$, since

$$sr^m = \begin{cases} sr^{i+2(\frac{m-i}{2})} & \text{if } m - i \equiv 0 \pmod 2 \\ sr^{(-i)+2(\frac{m+i-2k-1}{2})} & \text{if } m - i \equiv 1 \pmod 2 \end{cases}$$

then $\mathcal{C}(x) = \{sr^b \mid b = 1, ..., n\}$ □

Problem 33. This exercise gives a formula for the size of each conjugacy class in S_n. Let σ be a permutation in S_n, and let $m_1, m_2, \dots, m_s$ be the distinct integers which appear in the cycle type of σ (including 1-cycles). For each $i \in \{1, 2, \dots, s\}$, assume σ has k_i cycles of length m_i (so that $\sum_{i=1}^{s} k_i m_i = n$). Prove that the number of conjugates of σ is

$$\frac{n!}{(k_1! m_1^{k_1})(k_2! m_2^{k_2}) \cdots (k_s! m_s^{k_s})}.$$

Proof. Since there are k_i m_i-cycles to permute and m_i-cycle themself can be permuted cyclic, then the number of conjugates of σ is

$$\frac{n!}{(k_1!m_1^{k_1})\dots(k_s!m_s^{k_s})}$$

□

Problem 34. Prove that if p is a prime and P is a subgroup of S_p of order p, then $|N_{S_p}(P)| = p(p-1)$.

Proof. Since all p-cycle are conjugates with each other, then if σ is a p-cycle, then $\sigma \in gPg^{-1}$ for some $g \in S_p$ and since gPg^{-1} is a subgroup of order p, and hence intersection of two conjugates can only be trivial group given that p is a prime. Since there are exactly $p!/p = (p-1)!$ p-cycle in S_p and $p-1$ p-cycles for each conjugates of P, then

$$(p-1)n_p = (p-1)! \implies n_p = (p-2)!$$

where n_p denotes the number of conjugates of P. Therefore, $|N_G(P)| = |G|/n_p = p!/(p-2)! = p(p-1)$ □

Problem 35. Let p be a prime. Find a formula for the number of conjugacy classes of elements of order p in S_n (using the greatest integer function).

Proof. Since the only possible order p element is the product of p-cycle, and each cycle type corresponds to a conjugacy class, then the number of conjugacy classes is $\lfloor \frac{n}{p} \rfloor$. □

Problem 36.

Let $\pi : G \to S_G$ be the left regular representation afforded by the action of G on itself by left multiplication. For each $g \in G$, denote the permutation $\pi(g)$ by σ_g, so that $\sigma_g(x) = gx$ for all $x \in G$. Let $\lambda : G \to S_G$ be the permutation representation afforded by the corresponding right action of G on itself, and for each $h \in G$, denote the permutation $\lambda(h)$ by τ_h. Thus $\tau_h(x) = xh^{-1}$ for all $x \in G$ (λ is called the *right regular representation* of G).

(a) Prove that σ_g and τ_h commute for all $g, h \in G$. (Thus the centralizer in S_G of $\pi(G)$ contains the subgroup $\lambda(G)$, which is isomorphic to G.)

(b) Prove that $\sigma_g = \tau_g$ if and only if g is an element of order 1 or 2 in the center of G.

(c) Prove that $\sigma_g = \tau_h$ if and only if g and h lie in the center of G. Deduce that $\pi(G) \cap \lambda(G) = \pi(Z(G)) = \lambda(Z(G))$.

Proof. (a) Since

$$\sigma_g \circ \tau_h(a) = gah^{-1} = \tau_h \circ \sigma_g(a) \qquad \text{for all } a \in G$$

then σ_g and τ_h commute for all $g, h \in G$

(b) Suppose $\sigma_g = \tau_g$, then

$$\sigma_g(a) = \tau_g(a) \implies ga = ag^{-1} \qquad \text{for all } a \in G$$

so in particular, when $a = 1$, $g = g^{-1}$, then $|g| = 2$. Since

$$ga = ag^{-1} = ag \qquad \text{for all } a \in G$$

then $g \in Z(G)$. Conversely, let $g^2 = 1$ and $g \in Z(G)$, then

$$\sigma_g(a) = ga = ag = ag^{-1} = \tau_g(a) \qquad \text{for all } a \in G$$

hence $\sigma_g = \tau_g$

(c) Suppose $\sigma_g = \tau_h$, then

$$\sigma_g(a) = \tau_h(a) \implies ga = ah^{-1} \qquad \text{for all } a \in G$$

so in particular, when $a = 1$, $g = h^{-1}$. Since

$$ga = ah^{-1} = ag \qquad \text{for all } a \in G$$

then $g \in Z(G)$. Conversely, let $g = h^{-1}$ and $g \in Z(G)$, then

$$\sigma_g(a) = ga = ag = ah^{-1} = \tau_h(a) \qquad \text{for all } a \in G$$

hence $\sigma_g = \tau_h$. Consequently, if $u \in \pi(G) \cap \lambda(G)$, then $u = \sigma_g = \tau_h$ for some $g, h \in G$, then $g \in Z(G)$, therefore, $u = \sigma_g \in \pi(Z(G))$ and $u = \tau_g \in \lambda(Z(G))$. For $u \in \pi(Z(G))$, then $u = \sigma_g = \tau_{g^{-1}}$, hence $u \in \pi(G) \cap \lambda(G)$. Similar case follows for $u \in \lambda(G)$. Therefore, $\pi(G) \cap \lambda(G) = \pi(Z(G)) = \lambda(Z(G))$

□

4.4 Automorphism

Problem 1. If $\sigma \in \operatorname{Aut}(G)$ and φ_g is conjugation by g prove $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)$.

Proof. Since

$$(\sigma\varphi_g\sigma^{-1})(x) = \sigma\varphi_g(\sigma^{-1}(x)) = \sigma(g\sigma^{-1}(x)g^{-1}) = \sigma(g)x\sigma(g)^{-1} = \phi_{\sigma(g)}(x)$$

for all $x \in G$. Then $\sigma\varphi_g\sigma^{-1} = \varphi_{\sigma(g)} \in \operatorname{Inn}(G)$ for all $\varphi_g \in \operatorname{Inn}(G)$ and $\sigma \in \operatorname{Aut}(G)$. Hence $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)$. □

Problem 2. Prove that if G is an abelian group of order pq, where p, q are distinct primes, then G is cyclic.

Proof. According to Cauchy's Theorem, there exist elements x, and y in G have orders p and q. By Lagrange's Theorem, $G = \langle x, y\rangle$. Define $\phi : \langle x, y\rangle \to \langle x\rangle \times \langle y\rangle$ by $\phi(x^n y^m) = (x^n, y^m)$. Surjectivity of ϕ is trivial by construction. Since both domain and codomain are finite, the map ϕ is bijective. Since

$$\phi(x^n y^m x^k y^l) = \phi(x^n x^l y^m y^l) = (x^n x^k, y^m y^l) = (x^n, y^m)(x^k, y^l) = \phi(x^n y^m)\phi(x^k y^l)$$

, ϕ is a group homomorphism. Then $G = \langle x, y\rangle \simeq \langle x\rangle \times \langle y\rangle \simeq Z_p \times Z_q$. Since (x, y) has order pq, then $G \simeq Z_p \times Z_q \simeq Z_{pq}$. □

Problem 3. Prove that under automorphism r has at most 2 possible images and s has at most 4 possible images. Deduce that $|\operatorname{Aut}(D_8)| \leq 8$

Proof. Let $\sigma \in \operatorname{Aut}(D_8)$, then $|r| = |\sigma(r)| = 4$, and thus $\sigma(r) = r, r^3$. Similarly, since $|s| = |\sigma(s)| = 2$, we have $\sigma(s) = s, sr, sr^2, sr^3$. Since every element is uniquely defined by the generators r and s, basic combinatorics deduces that $|\operatorname{Aut}(D_8)| \leq 2 \cdot 4 = 8$. □

Problem 4. Use arguments similar to those in the preceding exercise to show $|\operatorname{Aut}(Q_8)| \leq 24$

Proof. Let $\sigma \in \operatorname{Aut}(Q_8)$. Since $Q_8 = \langle i, j\rangle$, we only have to check the possible image of i and j. Since $|i| = |\sigma(i)| = 4$, then $\sigma(i) = \pm i, \pm j, \pm k$. Since $j \neq i$ and $|j| = |\sigma(j)| = 4$, then $\sigma(j) = \pm i, \pm j, \pm k$ and $\sigma(j) \neq \pm\sigma(i)$. Basic combinatorics deduces that $|\operatorname{Aut}(Q_8)| \leq 6 \cdot 4 = 24$. □

Problem 5. Use the fact that $D_8 \trianglelefteq D_{16}$ to prove that $\operatorname{Aut}(D_8) \simeq D_8$.

Proof. Since $D_8 \trianglelefteq D_{16}$, the permutation presentation φ conjugation action on elements in D_8 is a group homomorphism from D_{16} to $\operatorname{Aut}(D_8)$, then

$$D_{16}/C_{D_{16}}(D_8) \overset{\varphi}{\hookrightarrow} \operatorname{Aut}(D_8)$$

Since $Z(D_{16}) = \langle r^4 \rangle = C_{D_{16}}(D_8)$, then

$$|D_{16}/C_{D_{16}}(D_8)| = |D_{16}/\langle r^4 \rangle| = 16/2 = 8 \leq |\operatorname{Aut}(D_8)| \leq 8$$

, hence $D_{16}/C_{D_{16}}(D_8) \simeq \operatorname{Aut}(D_8)$. Since $D_{16}/\langle r^4 \rangle \simeq D_8$, then $D_8 \simeq \operatorname{Aut}(D_8)$. $\square$

Problem 6. Prove that characteristic subgroups are normal. Give an example of a normal subgroup that is not characteristic.

Proof. Let $H \operatorname{char} G$. Since the conjugation map $\varphi_g : G \to G$ is an automorphism in G, it follows that

$$gHg^{-1} = \varphi_g(H) = H$$

then $H \trianglelefteq G$.
Suppose $\phi : \mathbb{Q} \to \mathbb{Q}$ by $\phi(x) = \frac{x}{2}$ is an automorphism, then $1 \in \mathbb{Z}$ but $\phi(1) = \frac{1}{2} \notin \mathbb{Z}$, so that $\mathbb{Z}$ is not a characteristic of $\mathbb{Q}$. Since $\mathbb{Q}$ is abelian, every subgroup is normal, so in particular, $\mathbb{Z} \trianglelefteq \mathbb{Q}$.
Suppose $\phi : V_4 \to V_4$ by $\phi(a) = b$ and $\phi(b) = a$ is a group homomorphism. Since $\phi(c) = \phi(ab) = \phi(a)\phi(b) = ba = c$ and $\phi(1) = 1$, then $\phi \in \operatorname{Aut}(V_4)$ but since $\phi(\langle a \rangle) = \langle b \rangle$, then $\langle a \rangle$ is not a characteristic subgroup of V_4, but since V_4 is abelian, $\langle a \rangle \trianglelefteq V_4$. $\square$

Problem 7. If H is the unique subgroup of a given order in a group G, prove H is characteristic in G.

Proof. Let $\sigma \in \operatorname{Aut}(G)$. We first prove that $\sigma(H)$ is a subgroup by checking subgroup criterion. Since $1 = \sigma(1) \in \sigma(H)$, then $\sigma(H) \neq \emptyset$. Since

$$\sigma(x)^{-1}\sigma(y) = \sigma(x^{-1})\sigma(y) = \sigma(x^{-1}y) \in \sigma(H)$$

then $\sigma(H) \leq G$. Since $|\sigma(H)| = |H|$ given that σ is bijective and H is the only normal subgroup in G with this order, it follows that $\sigma(H) = H$. Therefore, $H \operatorname{char} G$. $\square$

Problem 8. Let G be a group with subgroups H and K with $H \leq K$.

(a) Prove that if H is characteristic in K and K is normal in G, then H is normal in G.

(b) Prove that if H is characteristic in K and K is characteristic in G, then H is characteristic in G. Use this to prove that the Klein 4-group V_4 is characteristic in S_4.

(c) Give an example to show that if H is normal in K and K is characteristic in G, then H need not be normal in G.

Proof. (a) Let $H \operatorname{char} K \trianglelefteq G$, then $\varphi_g \in \operatorname{Aut}(K)$ where φ_g is conjugation by g, and thus

$$gHg^{-1} = \varphi_g(H) = H$$

hence $H \trianglelefteq G$.

(b) Suppose $H \operatorname{char} K \operatorname{char} G$. For all $\sigma \in \operatorname{Aut}(G)$, $\sigma(K) = K$ implies that $\sigma' \in \operatorname{Aut}(K)$ where σ' is σ restricted domain to K. Since $H \operatorname{char} K$, then $\sigma(H) = \sigma'(H) = H$. Then $H \operatorname{char} G$.
Since $A_4 \operatorname{char} S_4$ and $V_4 \operatorname{char} A_4$ given that they are the only subgroup of that order in their supergroup, then $V_4 \operatorname{char} S_4$.

(c) Since V_4 is abelian, then $\langle(1\,2)(3\,4)\rangle \trianglelefteq V_4$ and $V_4 \operatorname{char} A_4$ by previous problem, however

$$(1\,2\,3)(1\,2)(3\,4)(1\,3\,2) = (2\,3)(1\,4) \notin \langle(1\,2)(3\,4)\rangle$$

then $\langle(1\,2)(3\,4)\rangle$ is not a characteristic subgroup of A_4.

$\square$

Problem 9. If r, s are the usual generators for the dihedral group D_{2n}, use the preceding two exercises to deduce that every subgroup of $\langle r\rangle$ is normal in D_{2n}.

Proof. Let $H \leq \langle r\rangle$. Since $\langle r\rangle$ is cyclic, then H is the unique subgroup of this order and thus $H \operatorname{char}\langle r\rangle$. Since $|D_{2n} : \langle r\rangle| = 2$, then $\langle r\rangle \trianglelefteq D_{2n}$. $\square$

Problem 10. Let G be a group, let A be an abelian normal subgroup of G, and write $\overline{G} = G/A$. Show that $\overline{G}$ acts (on the left) by conjugation on A by $\overline{g} \cdot a = gag^{-1}$, where g is any representative of the coset $\overline{g}$ (in particular, show that this action is well defined). Give an explicit example to show that this action is not well defined if A is non-abelian.

Proof. Suppose A is abelian group. Let $\overline{G}$ acts on A by $\overline{g} \cdot a = gag^{-1}$. Let $g_1, g_2 \in \overline{g}$, then $g_1^{-1}g_2 \in A$, so that

$$g_1^{-1}g_2ag_2^{-1}g_1 = a \qquad \text{for all } a \in A$$

then $g_2ag_2^{-1} = g_1ag_1^{-1}$ for all $a \in A$, therefore, the action is well defined.
Let $\overline{g_1}, \overline{g_2} \in \overline{G}$, then

$$\overline{g_1g_2} \cdot a = g_1g_2ag_2^{-1}g_1^{-1} = g_1\overline{g_2} \cdot ag_1^{-1} = \overline{g_1} \cdot (\overline{g_2} \cdot a)$$

and that

$$\overline{1} \cdot a = 1a1^{-1} = a \qquad \text{for all } a \in A$$

Let $G = D_8$ and $A = \langle r\rangle$, $r, r^2 \in \overline{r}$, but $rsr^{-1} = sr^2$ and $r^2sr^{-2} = s$, then the action is not well defined. $\square$

Problem 11. If p is a prime and P is a subgroup of S_p of order p, prove

$$N_{S_p}(P)/C_{S_p}(P) \cong \operatorname{Aut}(P).$$

[Use Exercise 34, Section 3.]

Proof. Let $P = \langle y\rangle$, for some $|y| = p$, then $\operatorname{Aut}(P) \simeq \operatorname{Aut}(Z_p) \simeq (\mathbb{Z}/p\mathbb{Z})^\times$. Since $|N_{S_p}(P)| = p(p-1)$ and $C_{S_p}(P) = p!/(p!/p) = p$, then $|N_{S_p}(P)/C_{S_p}(P)| = (p-1) = |\operatorname{Aut}(P)|$. Since $N_{S_p}(P)/C_{S_p}(P) \hookrightarrow \operatorname{Aut}(P)$, then $N_{S_p}(P)/C_{S_p}(P) \simeq \operatorname{Aut}(P)$. $\square$

Problem 12. Let G be a group of order 3825. Prove that if H is a normal subgroup of order 17 in G, then $H \leq Z(G)$.

Proof. Let $H \trianglelefteq G$ be a subgroup of order 17. Since $3825 = 3^2 \cdot 5^2 \cdot 17$ and

$$G/C_G(H) \hookrightarrow \mathrm{Aut}(H) \cong (\mathbb{Z}/17\mathbb{Z})^\times$$

then

$$|G/C_G(H)| \mid |(\mathbb{Z}/17\mathbb{Z})^\times| = 16 = 2^4$$

and thus $|G/C_G(H)| = 1$ since every prime factor is coprime with 2. Therefore, $G = C_G(H)$, and hence $H \leq Z(G)$. ☐

Problem 13. Let G be a group of order 203. Prove that if H is a normal subgroup of order 7 in G, then $H \leq Z(G)$. Deduce that G is abelian in this case.

Proof. Let $H \trianglelefteq G$ be a subgroup of order 7. Since $203 = 7 \cdot 29$, and

$$G/C_G(H) \hookrightarrow \mathrm{Aut}(H) \simeq (\mathbb{Z}/7\mathbb{Z})^\times$$

then

$$|G/C_G(H)| \mid |(\mathbb{Z}/7\mathbb{Z})^\times| = 6 = 2 \cdot 3$$

and thus $|G/C_G(H)| = 1$ since every prime factor is coprime with 2 and 3. Therefore, $C_G(H) = G$, and hence $H \leq Z(G)$. It follows that $Z(G)$ is non-trivial, suppose otherwise that G is non-abelian, then $|Z(G)| = 17$, thus

$$G/Z(G) \simeq Z_{29}$$

is cyclic, and thus G is abelian group, which is a contradiction. Therefore, G is abelian. ☐

Problem 14. Let G be a group of order 1575. Prove that if H is a normal subgroup of order 9 in G, then $H \leq Z(G)$.

Proof. Let $H \trianglelefteq G$ be a subgroup of order 9. Since $1575 = 3^2 \cdot 5^2 \cdot 7$, and

$$G/C_G(H) \hookrightarrow \mathrm{Aut}(H) \simeq (\mathbb{Z}/9\mathbb{Z})^\times$$

and then

$$|G/C_G(H)| \mid |(\mathbb{Z}/9\mathbb{Z})^\times| = 6 = 2 \cdot 3$$

and since $|H| = 3^2$, H is abelian and hence $H \leq C_G(H)$ and then $|G/C_G(H)||5^2 \cdot 7$, since both 5 and 7 are comprime with 2 and 3, then $|G/C_G(H)| = 1$. Therefore, $G = C_G(H)$, and hence $H \leq Z(G)$. ☐

Problem 15. Prove that each of the following (multiplicative) groups is cyclic:

$$(\mathbb{Z}/5\mathbb{Z})^\times,\ (\mathbb{Z}/9\mathbb{Z})^\times, \text{ and } (\mathbb{Z}/18\mathbb{Z})^\times.$$

Proof.

(a) Since $|(\mathbb{Z}/5\mathbb{Z})^\times| = 4$, and

$$2^1 = 2, \qquad 2^2 = 4, \qquad 2^3 = 3$$

then $(\mathbb{Z}/5\mathbb{Z})^\times = \langle 2 \rangle$.

(b) Since $|(\mathbb{Z}/9\mathbb{Z})^\times| = 6$, and

$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 7$$

then $(\mathbb{Z}/9\mathbb{Z})^\times = \langle 2 \rangle$.

(c) Since $|(\mathbb{Z}/18\mathbb{Z})^\times| = 6$ and

$$5^1 = 2, \quad 5^2 = 7, \quad 5^3 = 17, \quad 5^4 = 13$$

then $(\mathbb{Z}/18\mathbb{Z})^\times = \langle 5 \rangle$.

□

Problem 16. Prove that $(\mathbb{Z}/24\mathbb{Z})^\times$ is an elementary abelian group of order 8. (We shall see later that $(\mathbb{Z}/n\mathbb{Z})^\times$ is an elementary abelian group if and only if $n \mid 24$.)

Proof. Since $|(\mathbb{Z}/24\mathbb{Z})^\times| = 8$, and since

$$1^2 = 1, \quad 5^2 = 1, \quad 7^2 = 1, \quad 11^2 = 1, \quad 13^2 = 1, \quad 17^2 = 1, \quad 19^2 = 1, \quad 23^2 = 1$$

then every non identity element is of order 2, and hence

$$(\mathbb{Z}/24\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$$

□

Problem 17. Let $G = \langle x \rangle$ be a cyclic group of order n. For $n = 2, 3, 4, 5, 6$, write out the elements of $\mathrm{Aut}(G)$ explicitly (by Proposition 16 above, we know $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, so for each element $a \in (\mathbb{Z}/n\mathbb{Z})^\times$, write out explicitly what the automorphism ψ_a does to the elements $\{1, x, x^2, \ldots, x^{n-1}\}$ of G).

Proof. Let $G = \langle x \rangle$ be a cyclic group of order n. The elements of G are $\{1, x, x^2, \ldots, x^{n-1}\}$, and G satisfies $x^n = 1$. An automorphism of G is a homomorphism $\psi : G \to G$ that is bijective. Since G is cyclic, every automorphism ψ is determined by the image of the generator x, which must also be a generator of G. Hence, $\psi(x) = x^a$, where $a \in \mathbb{Z}$ satisfies $\gcd(a, n) = 1$, ensuring that x^a is a generator of G. The set of all such integers a modulo n forms the group $(\mathbb{Z}/n\mathbb{Z})^\times$ under multiplication. Thus, the automorphism group of G, denoted $\mathrm{Aut}(G)$, is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.

For $n = 2, 3, 4, 5, 6$, we compute $(\mathbb{Z}/n\mathbb{Z})^\times$ and explicitly describe the automorphisms:

- For $n = 2$, $(\mathbb{Z}/2\mathbb{Z})^\times = \{1\}$. The only automorphism is the identity map: $\psi(x) = x$.
- For $n = 3$, $(\mathbb{Z}/3\mathbb{Z})^\times = \{1, 2\}$. The automorphisms are $\psi_1(x) = x$ and $\psi_2(x) = x^2$.
- For $n = 4$, $(\mathbb{Z}/4\mathbb{Z})^\times = \{1, 3\}$. The automorphisms are $\psi_1(x) = x$ and $\psi_3(x) = x^3$.
- For $n = 5$, $(\mathbb{Z}/5\mathbb{Z})^\times = \{1, 2, 3, 4\}$. The automorphisms are $\psi_1(x) = x$, $\psi_2(x) = x^2$, $\psi_3(x) = x^3$, and $\psi_4(x) = x^4$.
- For $n = 6$, $(\mathbb{Z}/6\mathbb{Z})^\times = \{1, 5\}$. The automorphisms are $\psi_1(x) = x$ and $\psi_5(x) = x^5$.

Each automorphism ψ_a maps $x^k \mapsto (x^a)^k = x^{ak}$ for all $k \in \mathbb{Z}$. Therefore, $\mathrm{Aut}(G) \cong (\mathbb{Z}/n\mathbb{Z})^\times$ is explicitly constructed for the given values of n. □

Problem 18. This exercise shows that for $n \neq 6$, every automorphism of S_n is inner. Fix an integer $n \geq 2$ with $n \neq 6$.

(a) Prove that the automorphism group of a group G permutes the conjugacy classes of G, i.e., for each $\sigma \in \mathrm{Aut}(G)$ and each conjugacy class $\mathcal{K}$ of G, the set $\sigma(\mathcal{K})$ is also a conjugacy class of G.

(b) Let $\mathcal{K}$ be the conjugacy class of transpositions in S_n, and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_n that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$. Deduce that any automorphism of S_n sends transpositions to transpositions. [See Exercise 33 in Section 3.]

(c) Prove that for each $\sigma \in \mathrm{Aut}(S_n)$,

$$\sigma : (1\,2) \mapsto (a\,b_2), \quad \sigma : (1\,3) \mapsto (a\,b_3), \quad \ldots, \quad \sigma : (1\,n) \mapsto (a\,b_n),$$

for some distinct integers $a, b_2, b_3, \ldots, b_n \in \{1, 2, \ldots, n\}$.

(d) Show that $(1\,2), (1\,3), \ldots, (1\,n)$ generate S_n, and deduce that any automorphism of S_n is uniquely determined by its action on these elements. Use (c) to show that S_n has at most $n!$ automorphisms and conclude that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ for $n \neq 6$.

Proof. (a) Let $\sigma \in \mathrm{Aut}(G)$. Let $x \in \mathcal{K}$, then for each $y \in \mathcal{K}$, $y = gxg^{-1}$ for some $g \in G$. Let $\mathcal{K}'$ be a conjugacy class containing $\sigma(x)$, then

$$\sigma(y) = \sigma(gxg^{-1}) = \sigma(g)\sigma(x)\sigma(g)^{-1} \in \mathcal{K}'$$

Let $y' \in \mathcal{K}'$, then $y' = g\sigma(x)g^{-1}$ for some $g \in G$, since σ is bijective, $g = \sigma(h)$ for some $h \in G$, then

$$y' = g\sigma(x)g^{-1} = \sigma(h)\sigma(x)\sigma(h)^{-1} = \sigma(hxh^{-1}) \in \sigma(\mathcal{K})$$

then $\sigma(\mathcal{K})$ is a conjugacy class of G.

(b) Suppose $\mathcal{K}'$ be conjugacy class of any element whose cycle type is k 2-cycle ($k \geq 2$). Suppose $|\mathcal{K}'| = |\mathcal{K}|$, then since $n \geq 2k$

$$\frac{n!}{2(n-2)!} = \frac{n!}{k!2^k(n-2k)!} \implies k!2^{k-1} = \frac{(n-2)!}{(n-2k)!} \geq (2k-2)! = 2^{k-1}(k-1)!(2k-3)!!$$

then

$$k \geq (2k-3)!! \geq (2k-3) \Rightarrow 3 \geq k$$

We now check all possibility of k:

- For $k = 2$,

$$2!2^{2-1} = 4 = (n-2)(n-3) \Rightarrow n^2 - 5n + 2 = 0$$

since $(-5)^2 - 4 \cdot 1 \cdot 2 = 17$ is not a perfect square, then there is no integer solution to this case.

- For $k = 3$,

$$3!2^{3-1} = 24 = \frac{(n-2)!}{(n-6)!}$$

and $\frac{(n-2)!}{(n-6)!} \geq 5! = 120$ for $n \geq 7$, it is left to check $n = 6$ case. Since

$$\frac{(6-2)!}{0!} = 4! = 24$$

, then $(n, k) = (6, 3)$ is the only solution.

Since $n \neq 6$, it follows that $|\mathcal{K}'| \neq |\mathcal{K}|$.

(c) Suppose $\sigma((1\,i)) = (a_i\,b_i)$, given that every transposition is mapped to transposition by automorphism for $n \neq 6$. Let $\sigma((1\,j)) = (a_j\,b_j)$, for $j \neq i$, then

$$\sigma((1\,j\,i)) = \sigma((1\,i)(1\,j)) = \sigma((1\,i))\sigma((1\,j)) = (a_i\,b_i)(a_j\,b_j)$$

Let $m = |\{a_i, b_i\} \cap \{a_j, b_j\}|$. If $m = 2$, then $(a_i\,b_i)(a_j\,b_j) = 1$, but $(1\,i\,j)$ have order 3. If $m = 0$, then $(a_i\,b_i)(a_j\,b_j)$ has order 2, but $(1\,i\,j)$ have order 3. Then $\{a_i, b_i\}$ and $\{a_j, b_j\}$ can only have one common element. Relabelling variables so that a_i is the common element is a, then

$$\sigma : (1\,2) \mapsto (a\,b_2), \quad \sigma : (1\,3) \mapsto (a\,b_3), \quad \ldots, \quad \sigma : (1\,n) \mapsto (a\,b_n)$$

(d) Since $(a\,b) = (1\,a)(1\,b)(1\,a)$ and S_n is generated by transpositions, then $S_n = \langle (1\,2), \ldots, (1\,n) \rangle$. By (c), distinct integers $a, b_2, \ldots, b_n$ uniquely determine $\sigma \in \mathrm{Aut}(S_n)$, then $|\mathrm{Aut}(S_n)| \leq n!$. For $n \geq 3$, $Inn(S_n) \simeq S_n$ and $|S_n| = n!$, it follows that $\mathrm{Aut}(S_n) = \mathrm{Inn}(S_n)$ for $n \neq 6$.

□

Problem 19. This exercise shows that $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| \leq 2$ (Exercise 10 in Section 6.3 shows that equality holds by exhibiting an automorphism of S_6 that is not inner).

(a) Let $\mathcal{K}$ be the conjugacy class of transpositions in S_6 and let $\mathcal{K}'$ be the conjugacy class of any element of order 2 in S_6 that is not a transposition. Prove that $|\mathcal{K}| \neq |\mathcal{K}'|$ unless $\mathcal{K}'$ is the conjugacy class of products of three disjoint transpositions. Deduce that $\mathrm{Aut}(S_6)$ has a subgroup of index at most 2 which sends transpositions to transpositions.

(b) Prove that $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| \leq 2$. [Follow the same steps as in (c) and (d) of the preceding exercise to show that any automorphism that sends transpositions to transpositions is inner.]

Proof. (a) Previous problem shows that $|\mathcal{K}| = |\mathcal{K}'|$ if and only if $(n, k) = (6, 3)$ that is $\mathcal{K}'$ is the conjugacy class of product of 3 disjoint transposition. Let

$$H = \{\sigma \in \mathrm{Aut}(S_6) \mid \sigma((i\,j)) = (a_i\,a_j), \forall i, j = 1, ..., n\}$$

then $id_{S_6} \in H$ and $\sigma_1^{-1}\sigma_2 \in H$ for all $\sigma_1, \sigma_2 \in \mathrm{Aut}(S_6)$. Since for all $\sigma \in \mathrm{Aut}(S_6)$, either $\sigma(\mathcal{K}) = \mathcal{K}$ or $\sigma(\mathcal{K}) = \mathcal{K}'$. Let $g_1, g_2 \notin H$, then g_1, g_2 maps transposition to product of 3 transposition, hence

$$g_1^{-1}g_2((a\,b)) = g_1^{-1}((a_1\,b_1)(c_1\,d_1)(e_1\,f_1)) = (a'\,b')$$

for all transposition $(a\,b)$, hence $g_1^{-1}g_2 \in H$. Then $|\mathrm{Aut}(S_6) : H| \leq 2$.

(b) Since $|H| \leq 6!$ following the proof of previous problem, then

$$6! = |\mathrm{Inn}(S_6)| \leq |H| \leq 6!$$

it follows that $H = \mathrm{Inn}(S_6)$, then $|\mathrm{Aut}(S_6) : \mathrm{Inn}(S_6)| \leq 2$.

□

4.5 Sylow's Theorem

Missing exercise number : 46

Problem 1. Prove that if $P \in Syl_p(G)$ and H is a subgroup of G containing P, then $P \in Syl_p(H)$. Give an example to show that, in general, a Sylow p-subgroup of a subgroup of G need not be a Sylow p-subgroup of G.

Proof. Let $P \leq H \leq G$, and $|P| = p^\alpha$ where $|G| = p^\alpha m$ for $p \nmid m$. Since $p^\alpha = |P| \mid |H|$ and $|H| \mid |G| = p^\alpha m$, $|H| = p^\beta m_1$ where $\alpha \leq \beta \leq \alpha$ and $p \nmid m_1$. Then $|H| = p^\alpha m_1$, and hence $P \in Syl_p(H)$.
Consider $G = S_4$ and $H = A_4$. Note that $V_4 \in Syl_2(A_4)$, but since $|V_4| = 4$ and Sylow 2-subgroup in S_4 has order 8, then $V_4 \notin Syl_2(S_4)$. $\square$

Problem 2. Prove that if H is a subgroup of G and $Q \in Syl_p(H)$, then $gQg^{-1} \in Syl_p(gHg^{-1})$ for all $g \in G$.

Proof. Suppose H is a subgroup of G and $Q \in Syl_p(H)$. Take any $g \in G$. Let $|H| = p^\alpha m$ where $p \nmid m$, then $|Q| = |gQg^{-1}| = p^\alpha$ by Proposition 14. Similarly, the subgroup gHg^{-1} has order $|gHg^{-1}| = |H| = p^\alpha m$ and since gQg^{-1} is a subgroup of gHg^{-1} with order p^α , then $gQg^{-1} \in Syl_p(gHg^{-1})$. $\square$

Problem 3. Use Sylow's Theorem to prove Cauchy's Theorem. (Note: We only used Cauchy's Theorem for abelian groups — Proposition 3.21 — in the proof of Sylow's Theorem, so this line of reasoning is not circular.)

Proof. For any finite group G, given prime p dividing $|G|$. By First Sylow Theorem, there exist Sylow p-subgroup $P \in Syl_p(G)$. Since $|P|$ is of prime power, $Z(P)$ is non-trivial by class equation. By Cauchy theorem for abelian group, there exists some element $x \in Z(P)$, such that $|x| = p$. Since $Z(P) \leq P \leq G$, then $x \in G$. $\square$

Problem 4. Exhibit all Sylow 2-subgroups and Sylow 3-subgroups of D_{12} and $S_3 \times S_3$.

Proof. Note that $|D_{12}| = 2^2 \cdot 3$. Observe that $\langle r^2 \rangle \in Syl_3(D_{12})$, since $rr^2r^{-1} = r^2$ and $sr^2s^{-1} = r^4$, then

$$Syl_3(D_{12}) = \{\langle r^2 \rangle\}$$

. Observe that $|\langle r^3, s \rangle| = 4$, then

$$Syl_2(D_{12}) = \{\langle r^3, s \rangle, \langle r^3, sr \rangle, \langle r^3, sr^2 \rangle\}$$

.

Note that $|S_3 \times S_3| = 36 = 2^2 \cdot 3^2$, $n_2 = 1, 3, 9$ and $n_3 = 1, 4$. Since $\langle ((1\,2\,3), 1) \rangle$ and $\langle (1, (1\,2\,3)) \rangle$ are distinct orders of subgroups 3, then $\langle ((1\,2\,3), 1), (1, (1\,2\,3)) \rangle$ is of order 9. Let $Q \in Syl_3(S_3 \times S_3)$. For all nonempty element $x \in Q$, $|x| = 3, 9$ and since $|x| = l.c.m(|\sigma_1|, |\sigma_2|) \leq 6$ where $x = (\sigma_1, \sigma_2)$, then $|\sigma_1| = |\sigma_2| = 3$, so that $x \in \langle ((1\,2\,3), 1), (1, (1\,2\,3)) \rangle$. Then

$$Syl_3(S_3 \times S_3) = \{\langle ((1\,2\,3), 1), (1, (1\,2\,3)) \rangle\}$$

Let $P \in Syl_2(S_3 \times S_3)$, for all nonidentity element $x \in P$, then $|x| = 2, 4$, since $x = (\sigma_1, \sigma_2)$, then it forces $|x| = 2$ since $|x| = 2, 3, 6$. Since $|\langle((1\,2), 1), (1, (1\,2))\rangle| = 4$, then

$$\begin{aligned} Syl_2(S_3 \times S_3) = \{&\langle((1\,2), 1), (1, (1\,2))\rangle, \langle((1\,2), 1), (1, (1\,3)), \langle((1\,2), 1), (1, (2\,3))\rangle \\ &\langle((1\,3), 1), (1, (1\,2))\rangle, \langle((1\,3), 1), (1, (1\,3)), \langle((1\,3), 1), (1, (2\,3))\rangle \\ &\langle((2\,3), 1), (1, (1\,2))\rangle, \langle((2\,3), 1), (1, (1\,3)), \langle((2\,3), 1), (1, (2\,3))\rangle\} \end{aligned}$$

□

Problem 5. Show that a Sylow p-subgroup of D_{2n} is cyclic and normal for every odd prime p.

Proof. Let $P \in Syl_p(D_{2n})$ and $|P| = p^\alpha$, given that $|sr^i| = 2$ for all $0 \leq i < n$, then $sr^i \notin P$. Since $2 \nmid p^\alpha = |P|$ for odd prime p. Then $P \leq \langle r \rangle$, and since $\langle r \rangle$ is cyclic, P is cyclic. $P \operatorname{char} \langle r \rangle \trianglelefteq D_{2n}$, then $P \trianglelefteq D_{2n}$. □

Problem 6. Exhibit all Sylow 3-subgroups of A_4 and all Sylow 3-subgroups of S_4.

Proof. Since $|A_4| = 12 = 2^2 \cdot 3$, direct checking shows that

$$Syl_3(A_4) = \{\langle(1\,2\,3)\rangle, \langle(1\,2\,4)\rangle, \langle(1\,3\,4)\rangle, \langle(2\,3\,4)\rangle\}$$

since there are $4!/3 = 8$ order 3 elements in S_4 and so that there is $8/2$ order 3 group in S_4 and by problem 1,

$$4 = |Syl_3(A_4)| \leq |Syl_3(S_4)| = 4$$

then $Syl_3(A_4) = Syl_3(S_4)$. □

Problem 7. Exhibit all Sylow 2-subgroups of S_4 and find elements of S_4 which conjugate one of these into each of the others.

Proof. Since $D_8 \leq S_4$, then all Sylow 2-subgroup is isomorphic to D_8. Since $|S_4| = 24 = 2^3 \cdot 3$, we have $n_2 = 1, 3$, if $n_2 = 1$, then order 2, and order 4 elements have total 3 elements, and hence $|G| = 3 + 8 < 24$ which is a contradiction, and thus $n_2 = 3$. Then

$$Syl_2(S_4) = \{\langle(1\,2\,3\,4), (1\,3)\rangle, \langle(1\,3\,2\,4), (1\,2)\rangle, \langle(1\,2\,4\,3), (1\,4)\rangle\}$$

Note that

$$(2\,3)\langle(1\,2\,3\,4), (1\,3)\rangle(2\,3)^{-1} = \langle(1\,3\,2\,4), (1\,2)\rangle$$

and that

$$(2\,4)\langle(1\,2\,3\,4), (1\,3)\rangle(2\,4)^{-1} = \langle(1\,2\,4\,3), (1\,4)\rangle$$

□

Problem 8. Exhibit two distinct Sylow 2-subgroups of S_5 and an element of S_5 that conjugates one into the other.

Proof. Since $|S_5| = 120 = 2^3 \cdot 3 \cdot 5$. Let $P \in Syl_2(S_5)$, then for all nonidentity element $x \in P$, $|x| = 2, 4$. Observe that $\langle(1\,2\,3\,4), (1\,3)\rangle$ and $\langle(1\,3\,2\,4), (1\,2)\rangle$ is order 8. Note that

$$(2\,3)\langle(1\,2\,3\,4), (1\,3)\rangle(2\,3)^{-1} = \langle(1\,3\,2\,4), (1\,2)\rangle$$

□

Problem 9. Exhibit all Sylow 3-subgroups of $SL_2(\mathbb{F}_3)$ (cf. Exercise 9, Section 2.1).

Proof. Note that $|SL_2(\mathbb{F}_3)| = \frac{(3^2-1)(3^2-3)}{(3-1)} = 24 = 2^3 \cdot 3$. Since $n_3|4$, then $n_3 \leq 4$. Direct calculation shows that

$$Syl_3(SL_2(\mathbb{F}_3)) = \left\{ \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle \right\}$$

$\square$

Problem 10. Prove that the subgroup of $SL_2(\mathbb{F}_3)$ generated by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the unique Sylow 2-subgroup of $SL_2(\mathbb{F}_3)$ (cf. Exercise 10, Section 2.4).

Proof. By exercise 2.4.10, we have that

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right\rangle \cong Q_8$$

hence the subgroup generated by these two elements is in $Syl_2(SL_2(\mathbb{F}_3))$. Since $n_2 \mid 3$, suppose $n_2 = 3$, then $SL_2(\mathbb{F}_3)$ have at least $4 \cdot 3 + 4 = 16$ order power 2 elements, and the previous problem shows that there are $4 \cdot (3-1) = 8$ order 3 elements, hence all elements have however, $\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ is elements of order 6. Alternative solution can be done by proving the intersection of P_1, P_2 and P_3 of Sylow 2-subgroup has order at least 4, let $\varphi : G \to S_3$ be the permutation presentation afford by this action, since $G/\ker\varphi \hookrightarrow S_3$, we must have $|\ker\varphi| \geq 4$. Since $|N_{SL_2(\mathbb{F}_3)}(P_i)| = \frac{24}{3} = 8$, then we have that $N_{SL_2(\mathbb{F}_3)}(P_i) = P_i$, then

$$|\ker\varphi| = |P_1 \cap P_2 \cap P_3| \geq 4$$

then $|P_1 \cup P_2 \cup P_3| \leq 4 + 3 \cdot 4$ order 2 elements. $\square$

Problem 11. Show that the center of $SL_2(\mathbb{F}_3)$ is the group of order 2 consisting of $\pm I$, where I is the identity matrix. Prove that $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3)) \cong A_4$. (Use facts about groups of order 12.)

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(SL_2(\mathbb{F}_3))$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$$

then we have that $c = 0$, $a = d$ and since

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \Rightarrow \begin{pmatrix} a+b & b \\ a & a \end{pmatrix} = \begin{pmatrix} a & b \\ a & a+b \end{pmatrix}$$

then $b = 0$ and since $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = aI$ commutes with all matrix, then $Z(SL_2(\mathbb{F}_3)) = \{\pm I\}$.
Since

$$\left\langle \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle, \left\langle \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \right\rangle$$

is two distinct order 6 subgroups, then $n_3 = 4$, hence it follows directly that $SL_2(\mathbb{F}_3)/Z(SL_2(\mathbb{F}_3)) \cong A_4$. $\square$

Problem 12. Let $2n = 2^a k$ where k is odd. Prove that the number of Sylow 2-subgroups of D_{2n} is k. (Prove that if $P \in Syl_2(D_{2n})$, then $N_{D_{2n}}(P) = P$.)

Proof. Let $P \in Syl_2(D_{2n})$, then $|P| = 2^a$. Since sr^i for $0 \le i < n$ are order 2 element, then sr^i is in some Sylow 2-subgroup P. If sr^j is also in P, then

$$sr^j sr^i = r^{i-j} \in P$$

thus $\frac{2^{a-1}k}{gcd(i-j,2^{a-1}k)} = |r^{i-j}| \mid |P| = 2^a$, hence $k \mid gcd(i-j, 2^{a-1}k)$, and thus $i - j \equiv 0 \pmod{k}$. Suppose $sr^i \in P$ for $0 \le i < k$, then

$$A \cup B_i = \{1, \dots, r^{(2^{a-1}-1)k}\} \cup \{sr^i, sr^{i+k}, \dots, sr^{i+(2^{a-1}-1)k}\} \subseteq P$$

Since $|A \cup B_i| = 2^{a-1} + 2^{a-1} = 2^a = |P|$, then we have $A \cup B_i = P$. Let $r^l \in N_{D_{2n}}(P)$,

$$r^l sr^i r^{-l} = sr^{i-2l} \in P$$

so that $2l \equiv 0 \pmod{k}$, since $gcd(2,k) = 1$, $l \equiv 0 \pmod{k}$, we have that $r^l \in P$. Let $sr^l \in P$,

$$sr^l sr^i (sr^l)^{-1} = sr^l sr^i r^{-l} s = sr^{2l-i} \in P$$

then $2i - 2l \equiv 0 \pmod{k}$, and then $i - l \equiv 0 \pmod{k}$, and hence $sr^l \in P$. Therefore, $N_{D_{2n}}(P) \subseteq P$, and hence $N_{D_{2n}}(P) = P$. It follows that $n_2 = |D_{2n} : N_{D_{2n}}(P)| = |D_{2n} : P| = k$. ☐

Problem 13. Prove that a group of order 56 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. Let G be a group such that $|G| = 56 = 2^3 \cdot 7$. By Sylow theorem, $n_2 = 1, 7$ and $n_7 = 1, 8$. Suppose both Sylow 2-subgroup and Sylow 7-subgroup are not normal, then $n_2 = 7$ and $n_7 = 8$. Now we have $8 \cdot 6 = 48$ order 7 elements. Let $Syl_2(G) = \{P_1, ..., P_7\}$, then

$$\left| \bigcup_{i=1}^{7} P_i \right| \ge 4 \cdot 7 + 4 = 32$$

and thus G has 32 two power order elements. Then G must have at least $32 + 48 > 56$ distinct elements, which is a contradiction. ☐

Problem 14. Prove that a group of order 312 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. Let G be a group such that $|G| = 312 = 2^3 \cdot 3 \cdot 13$. By Sylow's Theorem, $n_{13} = 1, 14, 27$, since $n_{13} \mid 24$, we must have that $n_{13} = 1$, then let $Q \in Syl_{13}(G)$, $Q \trianglelefteq G$. ☐

Problem 15. Prove that a group of order 351 has a normal Sylow p-subgroup for some prime p dividing its order.

Proof. Let G be a group such that $|G| = 351 = 3^3 \cdot 13$. By Sylow's Theorem, $n_{13} = 1, 27$ and $n_3 = 1, 13$. Suppose that both Sylow 13-subgroup and the Sylow 3-subgroup are not normal, the $n_{13} = 27$ and $n_3 = 13$. Now we have $12 \cdot 27 = 324$ order 13 elements. Let $Syl_3 = \{P_1, ..., P_{13}\}$, then

$$\left| \bigcup_{i=1}^{13} P_i \right| \ge 13 \cdot (27 - 9) + 9 = 242$$

then there are at least 242 order 3 power elements, which is absurd since $242 + 324 > 351$. ☐

Problem 16. Let $|G| = pqr$, where p, q, and r are primes with $p < q < r$. Prove that G has a normal Sylow subgroup for either p, q, or r.

Proof. Suppose Sylow r-subgroup is not normal. Since $n_r \mid pq$ and $n_r = 1 + kr$ for some positive integer k by Sylow's Theorem, then since $r > p, q$, then $n_r = pq$, so that there are $pq(r-1)$ order r elements in G. Since $n_q | rp$ and $n_q = 1 + mq$ for some positive integer q, if $n_q \geq r$, then

$$pqr > pq(r-1) + r(q-1) = pqr - pq + rq - r > pqr - rp + rq - r = pqr + r(q-p-1) \geq pqr$$

, thus $n_q = 1$. Let $Q \in Syl_q(G)$, then G/Q is order pr group; hence, a normal subgroup order r exists. This group corresponds to a subgroup of order rq and again by Sylow theorem, there is a normal subgroup of order r in G, which is a contradiction. $\square$

Problem 17. Prove that if $|G| = 105$, then G has a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.

Proof. Problem 16 has shown that Sylow 7-subgroup is normal. Let $P \in Syl_7(G)$, then G/Q is a group of order 15 and by Sylow's theorem, there exists a subgroup of order 5 in G/Q which is normal. By the Lattice isomorphism theorem, we have that there exists a subgroup of order 35 that is normal in G. Since the subgroup of order 35 is cyclic, then the order 5 subgroup is characteristic and hence normal in G. $\square$

Problem 18. Prove that a group of order 200 has a normal Sylow 5-subgroup.

Proof. Note that $200 = 2^3 \cdot 5^2$, since $n_5(G) \mid 8$ and $n_5(G) = 1 + 5k$ for some integer k. Therefore, $n_5(G) = 1$ and hence, Sylow 5-subgorup in G is normal. $\square$

Problem 19. Prove that if $|G| = 6545$, then G is not simple.

Proof. Note that $6545 = 5 \cdot 7 \cdot 11 \cdot 17$. Suppose G is simple, by Sylow theorem, $n_{17} = 55, 35, 77, 385$ and $n_{17} \equiv 1 \pmod{17}$, then $n_{17} = 35$. By Sylow theorem, $n_{11} \mid 5\dot{7}\dot{1}7$, then $n_{11} = 17, 35, 119, 85, 595$, since $n_{11} \equiv 1 \pmod{11}$, then $n_{11} = 595$. By Sylow theorem, $n_7 = 11, 17, 55, 187, 85, 935$, since $n_7 \equiv 1 \pmod 7$, then $n_7 = 85$. Then this account $35 \cdot 16 + 595 \cdot 10 + 85 \cdot 6 > 6545$ distinct element, which is a contradiction. Therefore, G is not simple. $\square$

Problem 20. Prove that if $|G| = 1365$, then G is not simple.

Proof. Note that $1365 = 3 \cdot 5 \cdot 7 \cdot 13$. Suppose G is a simple subgroup, then by Sylow theorem, direct checking shows that $n_{13} = 105$, $n_7 = 15$ and $n_5 = 21$ or 91, then $|G| > 105 \cdot 12 + 15 \cdot 6 + 21 \cdot 4 = 1260 + 90 + 84 > 1365$ which is a contradiction. Therefore, G is not simple. $\square$

Problem 21. Prove that if $|G| = 2907$, then G is not simple.

Proof. Note that $2907 = 3^2 \cdot 17 \cdot 19$. Suppose G is a simple subgroup, then by Sylow theorem, we have that $n_{19} = 153$, $n_{17} = 171$, then

$$|G| > 153 \cdot 18 + 171 \cdot 16 = 2754 + 2736 > 2907$$

which is a contradiction. Therefore, G is not simple. $\square$

Problem 22. Prove that if $|G| = 132$, then G is not simple.

Proof. Note that $132 = 2^2 \cdot 3 \cdot 11$. Suppose G is a simple subgroup, then by Sylow theorem, we have that $n_{11} = 12$ and $n_3 = 4, 22$ so that there are $(11-1) \cdot 12$ order 11 elements and at least $4 \cdot 2$ order 3 elements, if $n_2 > 1$, then there would be at least 6 elements of order 2 power, which is a contradiction since this would account 134 distinct elements in G. Therefore, G is not simple. $\square$

Problem 23. Prove that if $|G| = 462$, then G is not simple.

Proof. Note that $462 = 2 \cdot 3 \cdot 7 \cdot 11$. Suppose G is a simple subgroup, then by Sylow theorem, we have that $n_{11} \mid 2 \cdot 3 \cdot 7 = 42$, and $n_{11} \equiv 1 \pmod{11}$ then $n_{11} = 1$. Therefore, G is not simple. $\square$

Problem 24. Prove that if G is a group of order 231, then $Z(G)$ contains a Sylow 11-subgroup of G, and a Sylow 7-subgroup is normal in G.

Proof. Note that $|G| = 3 \cdot 7 \cdot 11$. Let $P \in Syl_{11}(G)$ and $Q \in Syl_7(G)$, by problem 16, $P \trianglelefteq G$ (or by direct check on n_{11} show that $n_{11} = 1$). Then by Proposition 13, we have that

$$G/C_G(P) \hookrightarrow \operatorname{Aut}(P) \cong (\mathbb{Z}/11\mathbb{Z})^\times$$

since $|(\mathbb{Z}/11\mathbb{Z})^\times| = 10$ which is coprime with $3, 7$, we have that $C_G(P) = G$, that is, $P \leq Z(G)$.
By Sylow theorem, $n_7 = 1$ and hence $Q \trianglelefteq G$. $\square$

Problem 25. Prove that if G is a group of order 385, then $Z(G)$ contains a Sylow 7-subgroup of G, and a Sylow 11-subgroup is normal in G.

Proof. Note that $|G| = 5 \cdot 7 \cdot 11$. Let $P \in Syl_{11}(G)$ and $Q \in Syl_7(G)$, by problem 16, $P \trianglelefteq G$ (or by direct check on n_{11} show that $n_{11} = 1$). By Sylow theorem, $n_7 = 1$ and hence $Q \trianglelefteq G$. Then by Proposition 13, we have that

$$G/C_G(Q) \hookrightarrow \operatorname{Aut}(P) \cong (\mathbb{Z}/7\mathbb{Z})^\times$$

since $|(\mathbb{Z}/7\mathbb{Z})^\times| = 6$ which is coprime with $5, 11$, we have that $C_G(Q) = G$, that is, $Q \leq Z(G)$.

$\square$

Problem 26. Let G be a group of order 105. Prove that if a Sylow 3-subgroup of G is normal, then G is abelian.

Proof. Note that $105 = 3 \cdot 5 \cdot 7$. Let $P \in Syl_3(G)$. Since $P \trianglelefteq G$, then $N_G(P) = G$ and that

$$G/C_G(P) \hookrightarrow \operatorname{Aut}(P) \cong (\mathbb{Z}/3\mathbb{Z})^\times$$

then $|G/C_G(P)| \mid |\operatorname{Aut}(P)| = 2$, since 105 is coprime with 2, then $C_G(P) = G$, that is $P \leq Z(G)$. We also note that $|G/P| = 35$, and $5 \nmid 7 - 1$, then G/P is cyclic. Let $G/P = \langle aP \langle$, given $g_1, g_2 \in G$, we can write $g_1 = a^i p_1$ and $a^j p_2$ where $p_1, p_2 \in P$ and $i, j \in \mathbb{Z}$, then we have

$$g_1 g_2 = a^i p_1 a^j p_2 = a^{i+j} p_1 p_2 = a^j a^i p_2 p_1 = a^j p_2 a^i p_1 = g_2 g_1$$

so that G is abelian. $\square$

Problem 27. Let G be a group of order 315 that has a normal Sylow 3-subgroup. Prove that $Z(G)$ contains a Sylow 3-subgroup of G and deduce that G is abelian.

Proof. Note that $315 = 3^2 \cdot 5 \cdot 7$. Let $P \in Syl_3(G)$, since $P \trianglelefteq G$, then G/P is a group of order 35 and since $5 \nmid 7-1$, then G/P is cyclic. Since $G = N_G(P)$, we have that

$$G/C_G(P) \hookrightarrow \operatorname{Aut}(P)$$

Since $P \cong \mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, then $|\operatorname{Aut}(P)| = 6$ or $3^2(3-1)(3+1)$, and $P \leq C_G(P)$ given P is abelian, we have that $G = C_G(P)$, that is $P \leq Z(G)$. Since G/P is cyclic, analogous of previous problem shows that G is abelian. $\square$

Problem 28. Let G be a group of order 1575. Prove that if a Sylow 3-subgroup of G is normal, then a Sylow 5-subgroup and a Sylow 7-subgroup are normal. In this situation, prove that G is abelian.

Proof. Note that $1575 = 3^2 \cdot 5^2 \cdot 7$. Suppose $P \in Syl_3(G)$ is normal, then G/P is an order $5^2 \cdot 7$ subgroup. Since $n_7(G/P) = 1$, then there exist a normal subgroup $\overline{H} = H/P$ of order 7 in G/P, thus H is a normal subgroup of order $3^2 \cdot 7$ in G. Since $n_7(H) = 1$, then there exist an order 7 subgroup $S \operatorname{char} H$, and thus $S \trianglelefteq G$. Since $n_5(G/P) = 1$, then there exists a normal subgroup of order 25 in G/P, and hence there exists a normal subgroup N of order $25 \cdot 9$ in G. Since $n_5(N) = 1$, then there exist a order 25 subgroup $T \operatorname{char} N$, then $T \trianglelefteq G$. Since P is abelian, we have $P \leq C_G(P)$ and since

$$G/C_G(P) \hookrightarrow \operatorname{Aut}(P)$$

Since $P \cong \mathbb{Z}/9\mathbb{Z}$ or $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, then $|\operatorname{Aut}(G)| = 6$ or $3^2(3+1)(3-1)$, so that $G = C_G(P)$ and hence $P \leq Z(G)$. Also since T is abelian we have $T \leq C_G(T)$ and that

$$G/C_G(T) \hookrightarrow \operatorname{Aut}(T)$$

Since $P \leq C_G(T)$ by preceding results and $|\operatorname{Aut}(T)| = 20$ or $5^2(5+1)(5-1)$, then $G = C_G(T)$ so that $T \leq Z(G)$, then $|Z(G)| \geq 25 \cdot 9$. It follows that $|G/Z(G)| = 1, 7$ is cyclic, and hence G is abelian. $\square$

Problem 29. If G is a non-abelian simple group of order < 100, prove that $G \cong A_5$. [Eliminate all orders but 60.]

Proof. Supppose G is an non-abelian simple group of order less then 100. Suppose G has prime factor p greater than 7, since G is simple, then $n_p \geq 11$, so that there are at least $n_p \cdot (p-1) \geq n_p \cdot 10 > 100$ distinct order p elements. It follows that $|G|$ has the form $2^a 3^b 5^c 7^d$. Since $2^a 3^b 5^c 7^d = |G| < 100$, then $d \leq 2$, $c \leq 2$, $b \leq 4$ and $a \leq 6$. Suppose $n_7 = 1 + 7k$, then $7(1+7k) < 7^d(1+7k) < 100$, and then $k < 2$, then $k = 1$ or 0, then $n_7 = 1$ or 8 and then the only possibility is $n = 56$, but this account $6 \cdot 8$ order 7 elements and there can be only 1 order 8 subgroup, and then $d = 0$. Again, suppose $n_5 = 1 + 5m$, then $5(1+5m) < 100$, so that $m < 4$, then $n_5 = 1, 6, 11, 16, 21$, and since $n_5 \mid 2^a 3^b$, then $n_5 = 6$ or 16, thus $c \leq 1$. If $n_5 = 16$, then $n = 80$, and there are $16 \cdot 4$ order 5 element so that there is only 1 order 16 subgroup in G. Therefore, $30 \mid n$, hence $n = 30$ or 60. We have proven that $n = 30$ has an order for 15 in the normal group, and then n is forced to be 60. It follows that the order 60 simple group is isomorphic to A_5. If $c = 0$, then $n = 2^a 3^b$, and $n_3 = 1, 4, 16$. If $n_3 = 16$, then $b = 1$, so that $n_2 = 3$, and $|G| > 16 \cdot 3$. Let ϕ be the conjugation action on Sylow 2-subgroups, then

$$G \overset{\phi}{\hookrightarrow} S_3$$

given that G is simple so that $\ker \phi = 1$, hence $|G| \leq 6$ which is a contradiction. Then $n_3 = 4$, but similar arguments on conjugation action on Sylow 3-subgroups show that

$$G \hookrightarrow S_4$$

then $|G| \mid 24$ and $|G| = 12$ or 24, but this means that $G \cong S_4$ or A_4 which is solvable. $\square$

Problem 30. How many elements of order 7 must there be in a simple group of order 168?

Proof. Note that $168 = 2^3 \cdot 3 \cdot 7$. Since $n_7 \mid 24$ and $n_7 \equiv 1 \pmod 7$, then $n_7 = 1, 8$. Because G is simple, $n_7 = 8$ and there are $8 \cdot (7-1) = 48$ order 7 elements. $\square$

Problem 31. For $p = 2, 3,$ and 5, find $n_p(A_5)$ and $n_p(S_5)$. [Note that $A_4 \leq A_5$.]

Proof. According to Sylow theorem, $n_5(A_5) = 1, 6$, but since A_5 is simple, $n_5(A_5) = 6$. Since there are $\frac{5 \cdot 4 \cdot 3}{3} = 20$ order 3 elements in A_5, then there are $20/(3-1) = 10$ order 3 subgroups in A_5 (note that $10 \equiv 1 \pmod 3$) and hence $n_3(A_5) = 10$. Since $V_4 \leq A_4 \leq A_5$, then $V_4 \in Syl_2(A_5)$. Since $V_4 = \langle (1\,2)(3\,4), (1\,3)(2\,4) \rangle$ and any pair of Sylow 2-subgroups is isomorphic, then it is obvious that any four label of 1, ..., 5 determines a subgroup isomorphic to V_4. Then $n_2(A_5) = \binom{5}{4} = 5$ (note that $5 \equiv 1 \pmod 2$). Since $S_5 \geq A_5$, then $n_3(S_4) = 10$ and $n_5(S_4) = 6$. Since $\langle (1\,3), (1\,2\,3\,4) \rangle \cong D_8$ is a subgroup of order 8 in S_4, then $D_8 \in Syl_2(S_5)$. Since any 4 label of $\{1, \ldots, 5\}$ can determine the vertices of D_8 and there are 3 possible label at the diagonal end of any label, then $n_2(S_5) = \binom{5}{4} \cdot 3 = 15$. $\square$

Problem 32. Let P be a Sylow p-subgroup of H and let H be a subgroup of K. If $P \trianglelefteq H$ and $H \trianglelefteq K$, prove that P is normal in K. Deduce that if $P \in Syl_p(G)$ and $H = N_G(P)$, then $N_G(H) = H$ (in words: normalizers of Sylow p-subgroups are self-normalizing).

Proof. Since $P \trianglelefteq H$ and $P \in Syl_p(H)$, then $P \operatorname{char} H$ and since $H \operatorname{char} K$, we have that $P \trianglelefteq K$.
Since $P \trianglelefteq H = N_G(P) \trianglelefteq N_G(H)$, then $P \trianglelefteq N_G(H)$, so that $N_G(H) \leq H$. Since $H \leq N_G(H)$, then we have that $N_G(H) = H$. $\square$

Problem 33. Let P be a normal Sylow p-subgroup of G and let H be any subgroup of G. Prove that $P \cap H$ is the unique Sylow p-subgroup of H.

Proof. Suppose $|H| = p^\beta m$, $p \nmid m$, and $|P| = p^\alpha$. By second isomorphism theorem, since $G = N_G(P)$ we have that

$$HP/P \cong H/(H \cap P)$$

Since $HP \leq G$, $|HP| = p^\alpha m_1$, where $p \nmid m_1$. Then $m_1 = |H/(H \cap P)|$ is co-prime with p and hence $|H \cap P| = p^\beta$. Therefore, $H \cap P \in Syl_p(H)$, and $H \cap P$ is normal in H by the second isomorphism theorem and hence a unique Sylow p-subgroup in H. $\square$

Problem 34. Let $P \in Syl_p(G)$ and assume $N \trianglelefteq G$. Use the conjugacy part of Sylow's Theorem to prove that $P \cap N$ is a Sylow p-subgroup of N. Deduce that PN/N is a Sylow p-subgroup of G/N (note that this may also be done by the Second Isomorphism Theorem — cf. Exercise 9, Section 3.3).

Proof. Let $Q \in Syl_p(N)$, then Q is a p-group in G, by the second part of Sylow theorem, $Q \leq gPg^{-1}$ for some $g \in G$. Then $Q \leq gPg^{-1} \cap N = gP \cap Ng^{-1}$. Since $P \cap N$ is a p-group in N and $|Q| \leq |gP \cap Ng^{-1}| = |P \cap N|$, so that $|Q| = |P \cap N|$, then $P \cap N \in Syl_p(N)$. Since

$$\left|\frac{G/N}{PN/N}\right| = \left|\frac{G/P}{PN/P}\right| = \frac{|G/P|}{|N/N \cap P|}$$

and the numerator and the denominator are co-prime with p, then $|\frac{G/N}{PN/N}|$ is co-prime with p and since $|PN/N| = |P/P \cap N|$ is of order p power, then $PN/N \in Syl_p(G/N)$ $\square$

Problem 35. Let $P \in Syl_p(G)$ and let $H \leq G$. Prove that $gPg^{-1} \cap H$ is a Sylow p-subgroup of H for some $g \in G$. Give an explicit example showing that $hPh^{-1} \cap H$ is not necessarily a Sylow p-subgroup of H for any $h \in H$ (in particular, we cannot always take $g = 1$ in the first part of this problem, as we could when H was normal in G).

Proof. Let $Q \in Syl_p(H)$, since Q is a p-group in G, then $Q \leq gPg^{-1}$ for some $g \in G$, then $Q \leq gPg^{-1} \cap H$. Since $|gPg^{-1} \cap H| = |gP \cap Hg^{-1}| = |P \cap H|$, and since $P \cap H \leq P$, thus $|P \cap H|$ is order of p power, then $gPg^{-1} \cap H$ is a p-group in H and since $|Q| \mid |gPg^{-1} \cap H|$, then $gPg^{-1} \cap H \in Syl_p(H)$.
For example, let $H = \langle (1\,2\,3) \rangle$, and $G = A_4$, then $H \in Syl_3(G)$. Since $n_4 = 4$, let $P \in Syl_3(G) - \{H\}$, then $1P1^{-1} \cap H = \{1\}$ which is not a Sylow 3-subgroup of G. □

Problem 36. Prove that if N is a normal subgroup of G then $n_p(G/N) \leq n_p(G)$.

Proof. Suppose $N \trianglelefteq G$, let

$$\varphi : Syl_p(G) \to Syl_p(G/N), \qquad \varphi(P) = PN/N$$

it is well defined by problem 35. Let $\overline{H} \in Syl_p(G/N)$, and let $|N| = p^{\alpha_1} m_1$ and $|G| = p^{\alpha} m$, where $p \nmid m_1$ and $p \nmid m$, then

$$|G/N| = \frac{p^{\alpha} m}{p^{\alpha_1} m_1}$$

since $\frac{m}{m_1}$ is coprime with p, then $|\overline{H}| = p^{\alpha - \alpha_1}$ and the preimage H has order

$$|H| = |\overline{H}||N| = p^{\alpha-\alpha_1} \cdot p^{\alpha_1} m_1 = p^{\alpha} m_1$$

Let $P \in Syl_p(H)$, note that $|P| = p^{\alpha}$ and $H \leq P$, it follows that $PN/N \leq HN/N = \overline{H}$ and since $|PN/N| = |\overline{H}| = p^{\alpha-\alpha_1}$, hence $\varphi(P) = \overline{H}$ and the map φ is surjective, so that $n_p(G/N) = |Syl_p(G/N)| \leq |Syl_p(G)| = n_p(G)$ □

Problem 37. Let R be a normal p-subgroup of G (not necessarily a Sylow subgroup).

(a) Prove that R is contained in every Sylow p-subgroup of G.

(b) If S is another normal p-subgroup of G, prove that RS is also a normal p-subgroup of G.

(c) The subgroup $O_p(G)$ is defined to be the group generated by all normal p-subgroups of G. Prove that $O_p(G)$ is the unique largest normal p-subgroup of G and $O_p(G)$ equals the intersection of all Sylow p-subgroups of G.

(d) Let $\overline{G} = G/O_p(G)$. Prove that $O_p(\overline{G}) = 1$ (i.e., $\overline{G}$ has no nontrivial normal p-subgroup).

Proof. (a) Let R be a normal p-subgroup in G, let $P \in Syl_p(G)$, by Sylow's theorem, $R \leq gPg^{-1}$, for some $g \in G$. Let $gPg^{-1} = Q \in Syl_p(G)$, for all $S \in Syl_p(G)$, second Sylow theorem gives that $S = hQh^{-1}$ for some $h \in G$, then

$$R = hRh^{-1} \leq hQh^{-1} = S$$

then $R \leq \bigcap_{P \in Syl_p(G)} P$.

(b) Let S be some normal p-subgroup of G, then

$$|RS| = \frac{|R||S|}{|R \cap S|}$$

which is of p power and since

$$gRSg^{-1} = gRg^{-1}gSg^{-1} = RS$$

for all $g \in G$, then $RS \trianglelefteq G$. Hence RS is a normal p-subgroup of G.

(c) Let $O_p(G) = \langle R_1, \dots, R_m \rangle$, note that $m < \binom{n}{p}$. Let $z \in O_p(G)$, write $z = r_1 \dots r_k$ for some finite $k \in \mathbb{N}$ and $r_i \in R_j$ for some j. Since $r_i \in R_j$ and R_j is a normal p-subgroup of G, then $r_i \in R_j \leq \bigcap_{P \in Syl_p(G)} P$ by (a), then $z = r_1 \dots r_k \in \bigcap_{P \in Syl_p(G)} P$, and thus $O_p(G) \leq \bigcap_{P \in Syl_p(G)} P$. Since G acts transitively on $Syl_p(G)$, then

$$g\Big(\bigcap_{P \in Syl_p(G)} P\Big)g^{-1} = \bigcap_{P \in Syl_p(G)} gPg^{-1} \leq \bigcap_{P \in Syl_p(G)} P$$

for all $g \in G$. It follows that the intersection is a normal p-subgroup of G and hence $\bigcap_{P \in Syl_p(G)} P \leq O_p(G)$. Therefore, $O_p(G) = \bigcap_{P \in Syl_p(G)} P$. By construction, $O_p(G)$ is largest in the sense that since $O_p(G)$ is generated by normal p-subgroups of G, then any normal p-subgroups of G is contained in $O_p(G)$. Uniqueness of $O_p(G)$ follows directly from the unique intersection of all Sylow p-subgroup.

(d) Let $\overline{R}$ be a normal p-subgroup of $\overline{G}$, note that the preimage R is a normal subgroup that has order $|R| = |\overline{R}| \cdot |O_p(G)|$ which is of order p power. Because R is a normal p-subgroup of G, then $R \leq O_p(G)$, but since $O_p(G) \leq R$ given by the quotient, then one have $O_p(G) = R$, which implies that $\overline{R} = \overline{1}$. Therefore, $O_p(\overline{G}) = \overline{1}$.

□

Problem 38. Use the method of proof in Sylow's Theorem to show that if n_p is not congruent to $1 \pmod{p^2}$ then there are distinct Sylow p-subgroups P and Q of G such that $|P : P \cap Q| = |Q : P \cap Q| = p$.

Proof. Let $P \in Syl_p(G)$ act on $Syl_p(G)$, let $\mathcal{O}_1, ..., \mathcal{O}_r$ the orbits induced by this action, and the corresponding representative is $P_1, ..., P_r$,

$$n_p = \sum_{i=1}^{r} |P : P_i \cap P|$$

since $n_p \pmod{p} = 1$, then we can write $n_p = 1 + kp$, and since $1 + kp = n_p \not\equiv 1 \pmod{p}^2$, then k can not be 0, so that there is at least 1 Sylow p-subgroup P_i such that $|P : P \cap P_i| = p$ take $P_i = Q$ we are done. Note that since $|P \cap Q| = |P|/p$, we automatically have that $|Q : P \cap Q| = |Q|/(|P|/p) = p$. □

Problem 39. Show that the subgroup of strictly upper triangular matrices in $GL_n(\mathbb{F}_p)$ (cf. Exercise 17, Section 2.1) is a Sylow p-subgroup of this finite group. [Use the order formula in Section 1.4 to find the order of a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.]

Proof. Recall that $|GL_n(\mathbb{F}_p)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$. Factor p power from each brackets, we see that $|GL_n(\mathbb{F}_p)| = p^{\sum_{k=1}^{n-1} k} m = p^{\frac{n(n-1)}{2}} m$, where $p \nmid m$. Let W be the subgroup of all strictly upper triangular matrices, since $|W| = p^{\frac{n(n-1)}{2}}$, then $W \in Syl_p(GL_n(\mathbb{F}_p))$, we are done. □

Problem 40. Prove that the number of Sylow p-subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$. [Exhibit two distinct Sylow p-subgroups.]

Proof. Let $G = GL_2(\mathbb{F}_p)$. Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} ad-bc-\alpha bc & \alpha a^2 \\ -\alpha b^2 & ad-bc+\alpha ab \end{pmatrix}\frac{1}{ad-bc} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

for $a, b, c, d, \alpha \in \mathbb{F}_p$ if and only if $b = 0$ for non-zero α. Let W be the subgroup of all strictly upper triangular matrices, then elements $A \in N_G(W)$ if and only if

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in GL_2(\mathbb{F}_p)$$

Since there are $(p-1)^2$ possibilities for the diagonal entries and p possibilities for c, then $|N_G(W)| = (p-1)^2 p$ and so

$$n_p = |G : N_G(W)| = \frac{(p-1)^2 p(p+1)}{(p-1)^2 p} = p+1$$

□

Problem 41. Prove that $SL_2(\mathbb{F}_4) \cong A_5$ (cf. Exercise 9, Section 2.1 for the definition of $SL_2(\mathbb{F}_4)$).

Proof. It is suffice to show that $|SL_2(\mathbb{F}_4)| = 60$ and that there exists more than one Sylow 5-subgroup. Recall the formula, $|SL_n(\mathbb{F}_p)| = \frac{|GL_n(\mathbb{F}_p)|}{(p-1)}$, then $|SL_2(\mathbb{F}_4)| = \frac{(4^2-1)(4^2-4)}{(4-1)} = 60$.
Note that

$$\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \quad \text{and} \quad \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\rangle$$

are two order 5 subgroup in $SL_2(\mathbb{F}_4)$, then by Proposition 21 and 23, we have that $SL_2(\mathbb{F}_4) \cong A_5$. □

Problem 42. Prove that the group of rigid motions in $\mathbb{R}^3$ of an icosahedron is isomorphic to A_5. [Recall that the order of this group is 60: Exercise 12, Section 1.2.]

Proof. Since there are 20 faces and each face is an equilateral triangle, then there are $3 \cdot 20 = 60$ ridged rotations for this group.
Notice that the 5 triangles form a pentagon from a top-down view of the solid with one vertex on the top level and since there are 6 vertex pairs on the diagonal, then $n_5 > 1$ so that this group is isomorphic to A_5. □

Problem 43. Prove that the group of rigid motions in $\mathbb{R}^3$ of a dodecahedron is isomorphic to A_5. (As with the cube and the tetrahedron, the icosahedron and the dodecahedron are dual solids.) [Recall that the order of this group is 60: Exercise 12, Section 1.2.]

Proof. Since there are 12 faces and each face is a pentagon, there is a $12 \cdot 5 = 60$ ridge motion in this group. Notice that rotation on some fixed face is a subgroup of order 5 of this group. Since there are pairs of 6 faces on the diagonal, then $n_5 > 1$ so that this group is isomorphic to A_5. □

Problem 44. Let p be the smallest prime dividing the order of the finite group G. If $P \in Syl_p(G)$ and P is cyclic, prove that $N_G(P) = C_G(P)$.

Proof. Let p be the smallest prime dividing $|G|$. Assume that $P \in Syl_p(G)$ and P is cyclic. Let $|P| = p^\alpha$. Note that $P \leq C_G(P)$ so that $|N_G(P)/C_G(P)|$ is coprime with p. Since

$$N_G(P)/C_G(P) \hookrightarrow \mathrm{Aut}(P) \cong (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$$

then $|N_G(P)/C_G(P)| \mid p^{\alpha-1}(p-1)$, the minimality of p forces $|N_G(P)/C_G(P)| = 1$. Therefore, $N_G(P) = C_G(P)$. □

Problem 45. Find generators for a Sylow p-subgroup of S_{2p}, where p is an odd prime. Show that this is an abelian group of order p^2.

Proof. Let $P \in Syl_p(S_{2p})$. Note that $|S_{2p}| = (2p)!$. Since p is an odd prime, then the only numbers with divisor p are p and $2p$, then $|P| = p^2$. Since there are no p^2 elements in S_{2p}, then $P \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let I be the set with label 1 to $2p$ and let A be some subset of order p and $B = I - A$. Then

$$P = \langle \sigma, \tau \mid \sigma \in S_A, \tau \in S_B \rangle \cong \langle \sigma \rangle \times \langle \tau \rangle$$

and we are done. □

Problem 46. Find generators for a Sylow p-subgroup of S_{p^2}, where p is a prime. Show that this is a non-abelian group of order p^{p+1}.

Proof. Since p is an □

Problem 47. Write and execute a computer program which

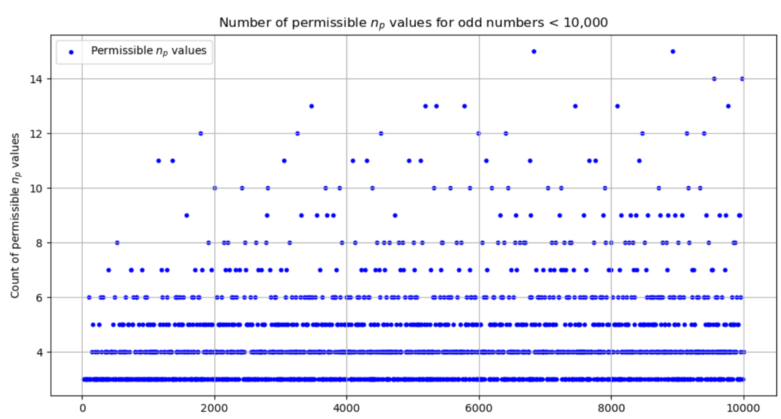
(i) gives each odd number $n < 10,000$ that is not a power of a prime and that has some prime divisor p such that n_p is not forced to be 1 for all groups of order n by the congruence condition of Sylow's Theorem, and

(ii) gives for each n in (i) the factorization of n into prime powers and gives the list of all permissible values of n_p for all primes p dividing n (i.e., those values not ruled out by Part 3 of Sylow's Theorem).

Proof. The code is available from the following link

`https://github.com/Laplacian2004/Dummit-Foote-sol/tree/main/code`

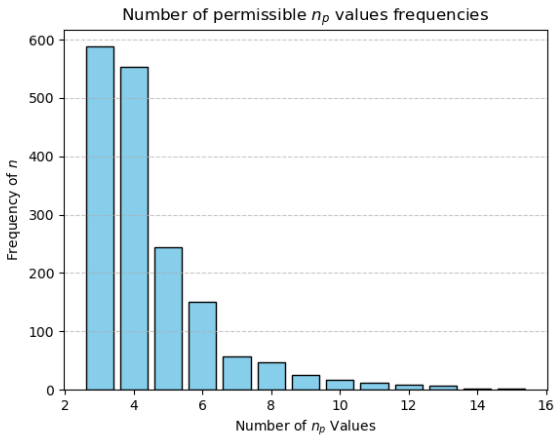
You can run `Sylow.py` to generate some figures and it will output the list in `np_list.out`. Read the `Makefile` for some detailed instructions. Here are some results:

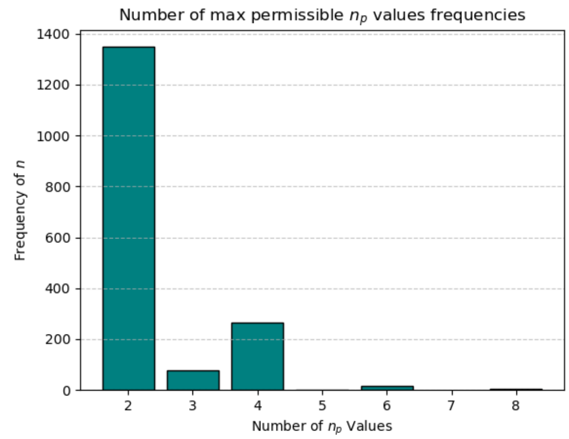


□

Problem 48. Carry out the same process as in the preceding exercise for all even numbers less than 1000. Explain the relative lengths of the lists versus the number of integers tested.

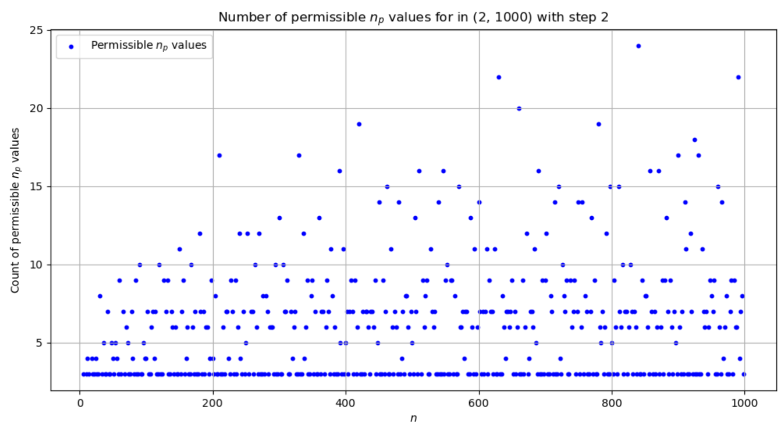
Proof. Similar to **Problem 47.**, you can run `Sylow.py` in

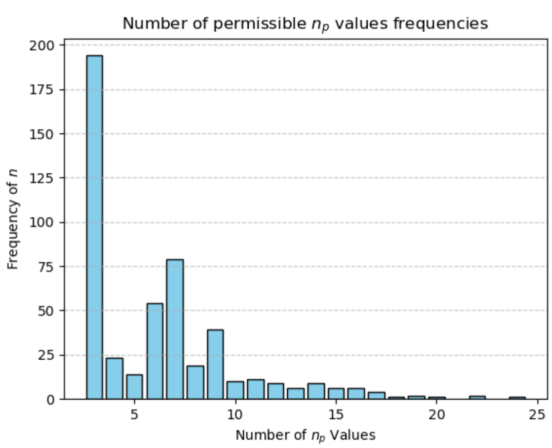


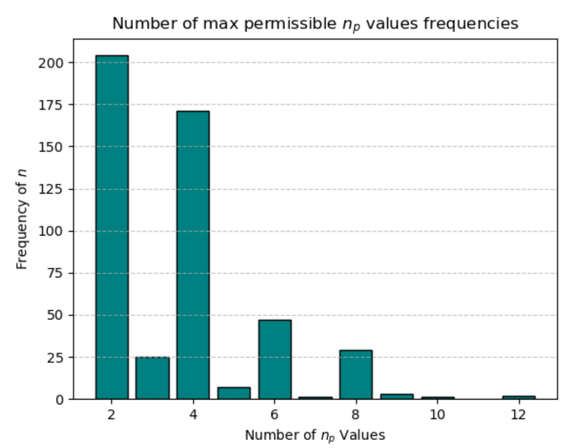


`https://github.com/Laplacian2004/Dummit-Foote-sol/tree/main/code`

with upper bound 100, lower bound 2, and step 2, it will generate some figures and it will output the list in `np_list.out`. Read the `Makefile` for some detailed instructions. Here are some results:







□

Problem 49. Prove that if $|G| = 2^m m$ where m is odd and G has a cyclic Sylow 2-subgroup then G has a normal subgroup of order m. [Use induction and Exercises 11 and 12 in Section 2.]

Proof. We shall prove it by induction on n. When $n = 1$, the statement has been proven in Exercises 11 and 12 in Section 2. Suppose that a normal subgroup of order m exists for n less than k. For n equals k since G has a cyclic Sylow 2-subgroup, there exists element x of order 2^n, let G acts on itself by left multiplication, let π be the permutation presentation afford by this action, then $\pi(x)$ is the product of m 2^n-cycles, which is an odd permutation so that $\phi = \epsilon \circ \pi : G \to \{\pm 1\}$ is a surjective function. The first isomorphism theorem gives that

$$G/\ker(\phi) \cong Z_2$$

then $|\ker(\phi)| = 2^{n-1}m$, so by induction, there exist a order m subgroup of $\ker(\phi)$, which is also a subgroup of G. ☐

Problem 50. Prove that if U and W are normal subsets of a Sylow p-subgroup P of G then U is conjugate to W in G if and only if U is conjugate to W in $N_G(P)$. Deduce that two elements in the center of P are conjugate in G if and only if they are conjugate in $N_G(P)$. (A subset U of P is normal in P if $N_P(U) = P$.)

Proof. Assumes that $P \in Syl_p(G)$ and $P = N_G(U) = N_G(W)$. Suppose $gUg^{-1} = W$ for some $g \in G$. Note that

$$gpg^{-1}W(gpg^{-1})^{-1} = gpg^{-1}gUg^{-1}(gpg^{-1})^{-1} = gUg^{-1} = W$$

for all $p \in P$. Then $gPg^{-1} \leq N_G(W) = P$, and thus $g \in N_G(P)$. THerefore, U is conjugates to W in $N_G(P)$. The other direction is trivial, given that $N_G(P) \leq G$.
Let $x, y \in Z(P)$, let $X = \{x\}$ and $Y = \{y\}$, note that X and Y are normal by definition. Suppose x, y are conjugates in G, i.e. X, Y is congugates in G, apply previous result on X and Y deduce that X and Y are congugates in $N_G(P)$, i.e. x and y are congugates in $N_G(P)$. The other direction is trivial, given that $N_G(P) \leq G$. ☐

Problem 51. Let P be a Sylow p-subgroup of G and let M be any subgroup of G which contains $N_G(P)$. Prove that $|G : M| \equiv 1 \pmod p$.

Proof. Since $P \in Syl_p(G)$, and $P \leq N_G(P) \leq M$, then we have that $P \in Syl_p(M)$. Since

$$|G : M||M : N_G(P)| = |G : N_G(P)| = n_p(G)$$

, and $|M : N_G(P)| = |M : N_G(P) \cap M| = |M : N_M(P)| = n_p(M)$, so that

$$|G : M|n_p(M) = n_p(G)$$

By Sylow's theorem, since $n_p(M) \equiv 1 \pmod p$ and $n_p(G) \equiv 1 \pmod p$, then $|G : M| \equiv 1 \pmod p$. ☐

Problem 52. Suppose G is a finite simple group in which every proper subgroup is abelian. If M and N are distinct maximal subgroups of G, prove $M \cap N = 1$. [See Exercise 23 in Section 3.]

Proof. Assume that G is a finite simple group in which every proper subgroup is abelian. Let N and M be distinct maximal subgroups in G. Suppose $N \cap M$ is non-trivial. Since G is simple, then $N_G(N \cap M) \neq G$. Since M and N are abelian subgroup then $N, M \leq N_G(N \cap M)$, so that the maximality of N and M forces $N_G(N \cap M) = N = M$, contradicts the fact that N and M are distinct. Therefore, $N \cap M = 1$ ☐

Problem 53. Use the preceding exercise to prove that if G is any non-abelian group in which every proper subgroup is abelian, then G is not simple. [Let G be a counterexample to this assertion and use Exercise 24 in Section 3 to show that G has more than one conjugacy class of maximal subgroups. Use the method of Exercise 23 in Section 3 to count the elements which lie in all conjugates of M and N, where M and N are nonconjugate maximal subgroups of G; show that this gives more than $|G|$ elements.]

Proof. By way of contradiction, suppose G be a non-abelian simple group in which every proper subgroup is abelian. We first claim that there exists maximal subgroup such that they are not in the same conjugacy class. Suppose that every maximal subgroup is conjugate to another. Let M be an arbitrary maximal subgroup in G. For any $x \in G$, let M_x be the maximal subgroup containing x, then $M_x = gMg^{-1}$. Since

$$G = \bigcup_{x \in G} M_x \leq \bigcup_{g \in G} gMg^{-1} \leq G$$

then $G = \bigcup_{g \in G} gMg^{-1}$, contradicting Exercise 24 in Section 3. Let M and N be two distinct maximal subgroups in G such that they are not in the same conjugacy class. Since $gMg^{-1} = hMh^{-1}$ if and only if $gM = hM$, then there are exactly $|G : M|$ in the conjugacy class of M. Suppose $gMg^{-1} < H < G$ then $M < g^{-1}Hg < G$, which is not possible, then gMg^{-1} is also maximal. Since the intersection of distinct maximal subgroups is 1, then

$$|G| > |G : M|(|M| - 1) + |G : N|(|N| - 1) \geq \frac{|G|}{2} + \frac{|G|}{2} = |G|$$

which is a contradiction. Therefore, G can not be simple. $\square$

Problem 54. Prove the following classification: if G is a finite group of order $p_1 p_2 \cdots p_r$, where the p_i's are distinct primes such that p_i does not divide $p_j - 1$ for all i and j, then G is cyclic. [By induction, every proper subgroup of G is cyclic, so G is not simple by the preceding exercise. If N is a nontrivial proper normal subgroup, N is cyclic, and G/N acts as automorphisms of N. Use Proposition 16 to show that $N \leq Z(G)$ and use induction to show $G/Z(G)$ is cyclic, hence G is abelian by Exercise 36 of Section 3.1.]

Proof. We show the statement by doing induction on r. For $r = 1$, since $|G| = p_1$, then $G \cong Z_{p_1}$, and is cyclic. Suppose for $r < k$ the statement is true. For $r = k$, since every subgroup of G is cyclic by induction, then G is not simple by the previous exercise. Let N be a nontrivial normal subgroup in G, then G is cyclic by induction. Since N is normal let G acts on N by conjugation, then

$$G/C_G(N) \hookrightarrow \mathrm{Aut}(N) \cong (\mathbb{Z}/n\mathbb{Z})^\times$$

where $n = |N|$. Since $|H| = p_{n_1} \ldots p_{n_m}$ where $n_i \in \{1, \ldots r\}$ and $n_i \neq n_j$ for $i \neq j$, then $|(\mathbb{Z}/n\mathbb{Z})^\times| = (p_{n_1} - 1) \ldots (p_{n_m} - 1)$. Since no p_j divides $p_{n_i} - 1$ where $j \in \{1, \ldots r\} \setminus \{n_1, \ldots, n_m\}$ and $N \leq C_G(N)$ so that there is no factor of p_{n_i} in $|G/C_G(N)|$, then we immediately have $|G/C_G(N)| = 1$, i.e. $G = C_G(N)$ or $N \leq Z(G)$. Since N is non-trivial, $Z(G) \neq 1$ so that $G/Z(G)$ is cyclic by induction; hence, G is abelian. Cauchy theorem gives that there exist elements x_i with order p_i for each $i \in \{1, ..., r\}$. Since $x_1 \cdots x_r$ is element of order $p_1 \ldots p_r$, then G is cyclic. This completes the proof. $\square$

Problem 55. Prove the converse to the preceding exercise: if $n \geq 2$ is an integer such that every group of order n is cyclic, then $n = p_1 p_2 \cdots p_r$ is a product of distinct primes and p_i does not divide $p_j - 1$ for all i, j. [If n is not of this form, construct noncyclic groups of order n using direct products of noncyclic groups of order p^2 and pq, where $p \mid q - 1$.]

Proof. Let G be a cyclic subgroup of order n. By way of contradiction, suppose that n is not a product of distinct prime or $p_i \mid p_j - 1$ for some i, j. If n is not a product of distinct prime i.e there exists $p_i = p_j$ for some $i \neq j$, then construct a group G such that

$$G = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \simeq Z_{p_i} \times Z_{p_i} \times Z_m$$

where $n = mp_i^2$. Note that $|G| = p_i^2 m = n$ and since for every $x \in G$, write $x = a^i b^i c^j$, $x^{pm} = 1$, then there is no order mp_i^2 element in G and hence G is not cyclic, which is a contradiction. Otherwise suppose $n = p_1 \dots p_r$ and $p_i \mid p_j - 1$ for some i, j. Construct a group G such that

$$G = M \times Z_m$$

where $n = mp_ip_j$ and M obtained by first let $P = \langle x \rangle \in Syl_{p_j}(S_{p_j})$ and since $|N_{S_{p_j}}(P)| = p_j(p_j - 1)$, by Cauchy theorem, there exists a subgroup $Q = \langle y \rangle$ of order p_i and QP is a group of order p_ip_j, and take $M = QP$. Note that M is non-abelian since $C_{S_{p_j}}(P) = P$, so that $(x, 1)$ does not commute with $(y, 1)$. Therefore, G is a non-abelian subgroup of order $|G| = p_ip_jm = n$, thus G is not cyclic, which is a contradiction. □

Problem 56. If G is a finite group in which every proper subgroup is abelian, show that G is solvable.

Proof. We shall induct on the order of $|G|$. For $|G| = 1$, the statement is trivial. Suppose G is abelian, then $1 \trianglelefteq G$ is trivially a normal tower so that G is solvable. Suppose G is non-abelian, then **Exercise 53** shows that G is not simple so that there exists proper nontrivial normal subgroup N of G, and N is abelian by assumption. Let $H/N = \overline{H} < G/N$, since the preimage H is abelian, then H/N is also abelian, so that every subgroup of G/N is then abelian. Since $|G/N| < |G|$, induction shows that there exists an abelian normal tower $\{M_i\}$ such that

$$M_1 = \overline{1} \trianglelefteq \cdots \trianglelefteq G/N = M_k$$

of G/N. Let N_i be the preimage of $M_i = N_i/N$ under the canonical homomorphism, and Lattice isomorphism gives that

$$N_1 = N \trianglelefteq \cdots \trianglelefteq G = N_k$$

Since $N_1 = N$ is abelian by assumption, then $\{N_i\}$ is a normal abelian tower in G, and thus G is solvable. □

4.6 The Simplicity of A_n

Missing exercise number : 8

Problem 1. Prove that A_n does not have a proper subgroup of index $< n$ for all $n \geq 5$.

Proof. Suppose that H is a subgroup such that $m = |A_n : H| < n$. Let A_n act on the set of left cosets of H by left multiplication, and φ is the homomorphism afforded by this action. Since A_n is simple for $n \geq 5$ and $\ker \varphi = \cap_{g \in G} gHg^{-1} \leq H$, then we have that $\ker \varphi = 1$. Then

$$A_n \overset{\varphi}{\hookrightarrow} S_m$$

but since $m! \leq (n-1)! = \frac{n!}{n} < n!/2$ since $n \geq 5$, then $\varphi(A_n)$ cannot be a subgroup of S_m, contradict with the fact that φ is injective. □

Problem 2. Find all normal subgroups of S_n for all $n \geq 5$.

Proof. Suppose that N is a non-trivial proper normal subgroup of S_n for $n \geq 5$. Let $M = A_n \cap N$ is normal, since A_n is simple, $M = A_n$ or $M = 1$. If $M = A_n$, then $A_n \leq N$, but since A_n has index 2, one must have $N = A_n$. If $M = 1$, then $N \neq A_n$, and by second isomorphism theorem,

$$S_n/N = NA_n/N \cong A_n/(A_n \cap N)$$

which gives that $|N| = 2$. then $N = \langle \sigma \rangle$ where σ is tranpositions. But since transpositions are conjugates with each other, N is not normal.
It is possible to show since $S_n' = A_n$, and $[S_n, N] \leq N \cap S_n' = M = 1$, then $N \leq Z(S_n) = 1$. Therefore, the only possible normal groups for S_n are $\{1, A_n, S_n\}$ for all $n \geq 5$. □

Problem 3. Prove that A_n is the only proper subgroup of index $< n$ in S_n for all $n \geq 5$.

Proof. Let H be a proper subgroup such that $m = |S_n : H| < n$. Let G act on the set of left cosets of H by conjugation, and let φ be the permutation presentation afforded by this action, then

$$\varphi : S_n \to S_m$$

. Since $n > m$, then $\ker \varphi \neq 1$ and since $\ker \varphi \leq H$, then $\ker \varphi < G$. But previous exercise show that the only normal subgroup of S_n is A_n, and therefore $\ker \phi = A_n$. Since $A_n = \ker \varphi \leq H \leq G$, then $H = A_n$. □

Problem 4. Prove that A_n is generated by the set of all 3-cycles for each $n \geq 3$.

Proof. Let $\sigma \in A_n$. Write $\sigma = \lambda_1 \dots \lambda_{2t}$ where λ is transposition in S_n. It suffices to show that a product of two transpositions can be written as a product of 3-cycles.

1. If two transpositions $\tau = (a\,b), \gamma = (c\,d)$ are disjoint (implictly, $n \geq 4$), then

$$(a\,b)(c\,d) = (a\,b)(b\,c)(b\,c)(c\,d) = (b\,c\,a)(c\,d\,b)$$

2. If two transpositions $\tau = (a\,b), \gamma = (a\,c)$ intersect one element, then

$$(a\,b)(a\,c) = (a\,c\,b)$$

3. If two transpositions are the same, then

$$(a\,b)(a\,b) = 1$$

Therefore, every element in A_n is generated by 3-cycles for $n \geq 3$. □

Problem 5. Prove that if there exists a chain of subgroups $G_1 \leq G_2 \leq \cdots \leq G$ such that $G = \bigcup_{i=1}^{\infty} G_i$ and each G_i is simple, then G is simple.

Proof. When $|G| = 1$ the statement is trivial. Assume that $|G| \neq 1$. Suppose N is a normal subgroup of G, then

$$N = G \cap N = \cup_{i=1}^{\infty}(G_i \cap N)$$

Since $(G_i \cap N) \trianglelefteq G_i$, simplicity of G_i forces $G_i \cap N = 1$ or $G_i \cap N = G_i$. Suppose $G_i \cap N \neq 1$ for some i, then $G_i \leq N$ so that $G_i \leq N \cap G_j$ for $j \geq i$. Since $G_i \neq 1$, then $N \cap G_j \neq 1$ so that $N \cap G_j = G_j$ for $j \geq i$, and it follows that $G \leq N$ and thus $N = G$. Suppose $G_i \cap N = 1$ for all i, then $N = 1$. Therefore, G is simple. □

Problem 6. Let D be the subgroup of S_Ω consisting of permutations which move only a finite number of elements of Ω (described in Exercise 17 in Section 3), and let A be the set of all elements $\sigma \in D$ such that σ acts as an even permutation on the (finite) set of points it moves. Prove that A is an infinite simple group. [Show that every pair of elements of D lie in a finite simple subgroup of D.]

Proof. Let $\tau, \sigma \in A$. Since $\tau\sigma^{-1}$ fix finite number of points and $\tau\sigma^{-1}$ is even, then $\tau\sigma^{-1} \in A$, so A is a group. Suppose N is some non-trivial normal subgroup in A with $\gamma \in N - \{1\}$, and let $\Delta \subseteq \Omega$ be the set move either by τ or γ. If $|\Delta| < 5$ append elements in Ω so that $|\Delta| = 5$. Let H be a subgroup of D such that it fixed all points in $\Omega \setminus \Delta$, then $H \cong A_\Delta$. Since $\gamma, \tau \in H$ and $\gamma \in N$, then $H \cap N \trianglelefteq H$, and simplicity of H forces $H \cap N = H$, given that $1 \neq \gamma \in H \cap N$. So since $\tau \in H$, then $\tau \in N$. Therefore, $A \leq N$ and thus $N = A$. ☐

Problem 7. Under the notation of the preceding exercise, prove that if $H \trianglelefteq S_\Omega$ and $H \neq 1$, then $A \leq H$, i.e., A is the unique (nontrivial) minimal normal subgroup of S_Ω.

Proof. Let $H \trianglelefteq S_\Omega$ and $H \neq 1$. Since $H \cap A \trianglelefteq A$, then simplicity of A forces $H \cap A = 1$ or $H \cap A = A$. For $H \cap A = A$, then $A \leq H$, we are done. For $H \cap A = 1$, note that $A \trianglelefteq S_\Omega$ given by $D \trianglelefteq S_\Omega$ and $A_\Omega \trianglelefteq S_\Omega$, then $[H, A] \leq H \cap A = 1$, then $H \leq C = C_{S_\Omega}(A)$. For all $a \in \Omega$, construct $(a\,b\,c) \in A_a$, note that C commutes with $(a\,b\,c)$ for all $b, c \in \Omega \setminus \{a\}$ and hence have cycles that does not contain a. Therefore, $C = 1$ and hence $H = 1$ which is a contradiction. It follows that $A \leq H$, i.e., A is the unique (nontrivial) minimal normal subgroup of S_Ω. ☐

Problem 8. Under the notation of the preceding two exercises, prove that $|D| = |A| = |\Omega|$. Deduce that

$$\text{if } S_\Omega \cong S_\Delta, \text{ then } |\Omega| = |\Delta|.$$

[Use the fact that D is generated by transpositions. You may assume that countable unions and finite direct products of sets of cardinality $|\Omega|$ also have cardinality $|\Omega|$.]

Proof. ☐