Homework 8

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Introduction to Analysis II

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Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be define by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$$

(a) Show that f is C^1.

(b) Show that $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

(c) Show that $f(x,y)$ is not C^2.

Proof.

(a) We note that $f(x,y) = -f(y,x)$, and hence

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \frac{x^2y+4x^2y^3-y^5}{(x^2+y^2)^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$$

and $\frac{\partial f}{\partial x}(x,y) = -\frac{\partial f}{\partial y}(y,x)$. Given $\epsilon > 0$, choose $\delta = \frac{2\epsilon}{3}$

$$\left| \frac{x^2y+4x^2y^3-y^5}{(x^2+y^2)^2} \right| \leq |y| \left(\frac{x^2+4x^2y^2+y^4}{(x^2+y^2)^2} \right) \leq |y| \left(1 + \frac{2x^2y^2}{(x^2+y^2)^2} \right) \leq |y| \cdot \frac{3}{2} < \delta \cdot \frac{3}{2} \leq \epsilon$$

, if $\sqrt{x^2+y^2} < \delta$. Hence $\frac{\partial f}{\partial x}$ is continuous and thus by symmetry, $\frac{\partial f}{\partial y}$ is continuous. Since $Df(x,y) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is continuous in both component. $Df(x,y)$ is continuous and therefore f is C^1.

(b)

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x}(0,h) - \frac{\partial f}{\partial x}(0,0)}{h} = \lim_{h \to 0} \frac{-h^5}{h^4} = -1$$

By symmetry, $\frac{\partial^2 f}{\partial y \partial x}(0,0) = -\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1$. Hence $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$.

(c) By Clairaut's theorem, since the $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$, then either $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ or $\frac{\partial^2 f}{\partial y \partial x}(0,0)$ is not continuous at $(0,0)$. And hence $H(f)(x,y)$ is not continuous and therefore f is not C^2.

□

Problem 2. Write $H_n(R) = \{X \in M_n(R) : I - X$ is an invertible n by n matrix$\}$. Identify $H_n(R)$ with $\mathbb{R}^{n^2}$ endowed with the Euclidean norm. Define $f : H_n(R) \to M_n(R)$ by $f(X) = (I - X)^{-1}$. Consider the zero matrix $O \in H_n(R)$.

(a) Show that $H_n(\mathbb{R})$ is a open set in $M_n(\mathbb{R})$.

(b) Find $Df(O)(H)$.

(c) Find $D^2f(O)(H_1, H_2)$.

Proof. (a) Define

$$||A|| = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} |A_{ij}|^2 \right\}^{1/2}$$

. Let $g : H_n(\mathbb{R}) \to \mathbb{R}$, $g(X) = det(I - X)$, which is a polynomial function of X. And hence given $H \in H_n(\mathbb{R})$, there exist $\delta > 0$ such that if $||H - X|| < \delta$, $|det(I - X) - det(I - H)| < |det(I - H)|$, and therefore $|det(I - X)| > 0$, if $||H - X|| < \delta$. Hence if $||H - X|| < \delta$, X is invertible. Then $H_n(\mathbb{R})$ is open.

(b) We note that if $A, B \in M_n(\mathbb{R})$

$$||AB|| = \sqrt{\sum_{i,j} \left(\sum_{k=0}^{n} a_i k b_k j \right)^2} \leq \sqrt{\sum_{i,j} (\sum_{k=0}^{n} a_{ik}^2)(\sum_{k=0}^{n} b_{kj}^2)} \leq \sqrt{(\sum_{i=0}^{n} \sum_{k=0}^{n} a_{ik}^2)(\sum_{j=0}^{n} \sum_{k=0}^{n} b_{kj}^2)} \leq ||A|| \cdot ||B||$$

Given $\epsilon > 0$, choose $\delta = \min\{\frac{\epsilon}{2}, \frac{1}{2}\}$, if $||H|| < \delta$,

$$\frac{||(I - H)^{-1} - I - H||}{||H||} = \frac{1}{||H||} \cdot ||H^2 + H^3 + H^4 + ...|| \leq \frac{1}{||H||} ||H|| \cdot ||H + H^2 + H^3 - ...||$$
$$= ||H + H^2 + H^3 - ..|| \leq \sum_{k=1}^{\infty} ||H^k|| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

Hence

$$\lim_{||H|| \to 0} \frac{||(I + H)^{-1} - I - H||}{||H||} = 0$$

, which follows that $f(A)$ is differentiable at I and $Df(O)(H) = H$.

$\square$

Problem 3.
Given a open ball $B_r((0, 0))$, let $f(x, y) = x + y + e^{-x} cos(y) : B_r((0, 0)) \to \mathbb{R}$, find a polynomial $P(x, y)$ such that

$$|f(x, y) - P(x, y)| \leq C(x^2 + y^2)^{\frac{3}{2}}$$

for all $(x, y) \in B_r((0, 0))$.

Proof. Since $B_r((0, 0))$ is open, by Taylor's theorem,

$$f(x, y) = f(0, 0) + \frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y + \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(0, 0)x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0, 0)xy + \frac{\partial^2 f}{\partial x \partial y}(0, 0)y^2) + \frac{1}{3!}(\frac{\partial^3 f}{\partial^3 x}(0, 0)x^3 +$$
$$\frac{\partial^3 f}{\partial^2 x \partial y}(0, 0)3x^2 y + \frac{\partial^3 f}{\partial x \partial^2 y}(0, 0)3xy^2 + \frac{\partial^3 f}{\partial^3 y}(0, 0)y^3) + R_4(x, y) = P(x, y) + R_4(x, y)$$

We note that since

$$\frac{|R_4(x,y)|}{(x^2+y^2)^{\frac{3}{2}}} \leq C_1$$

for some $C_1 \in \mathbb{R}$, if $\sqrt{x^2+y^2} \leq \min\{\delta, r\}$, for some $\delta > 0$. And also since $\frac{|R_4(x,y)|}{(x^2+y^2)^{\frac{3}{2}}}$ is continuous on $B_r(0,0) \setminus B_\epsilon(0,0)$, then

$$\frac{R_4(x,y)}{(x^2+y^2)^{\frac{3}{2}}} \leq C_2$$

for some $C_2 \in \mathbb{R}$. Take $C = \min\{C_1, C_2\}$, then

$$|f(x,y) - P(x,y)| = |R_4(x,y)| \leq C(x^2+y^2)^{\frac{3}{2}}$$

Hence

$$P(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial x \partial y}(0,0)y^2) + \frac{1}{3!}(\frac{\partial^3 f}{\partial^3 x}(0,0)x^3 + \frac{\partial^3 f}{\partial^2 x \partial y}(0,0)3x^2y + \frac{\partial^3 f}{\partial x \partial^2 y}(0,0)3xy^2 + \frac{\partial^3 f}{\partial^3 y}(0,0)y^3) = 1 + y + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}xy^2$$

□

Problem 4. Let $f(x) \in C^2(a,\infty)$, $M_0 = \sup_{x\in(a,\infty)} |f(x)|$, $M_1 = \sup_{x\in(a.\infty)} |f'(x)|$, $M_2 = \sup_{x\in(a,\infty)} |f''(x)|$.

(a) Show that $M_1^2 \leq 4M_0M_2$

(b) Define norms on $C^2((a,\infty);\mathbb{R})$: $||f||_1 = ||f||_\infty + ||f'||_\infty + ||f''||_\infty$ and $||f||_2 = ||f||_\infty + ||f''||_\infty$. Show that $||\cdot||_1$ is equivalent to $||\cdot||_2$. That is to show that there exists C_1, $C_2 > 0$ such that

$$C_1||f||_2 \leq ||f||_1 \leq C_2||f||_2$$

for all $f \in C^2((a,\infty);\mathbb{R})$

(c) Show that the constant 4 in (a) is the best. That is there exist an interval (a_0,∞) and $h \in C^2((a_0,\infty);\mathbb{R})$ such that the equality holds.

Proof.

(a) If $h > 0$ by Taylor's theorem,

$$f(x+2h) = f(x) + f'(x)(2h) + \frac{f''(\xi)}{2}(2h)^2$$
$$\Rightarrow f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - f''(\xi)h$$
$$\Rightarrow |f'(x)| \leq \frac{1}{2h}[|f(x+2h)| + |f(x)|] + |f''(\xi)|h \leq \frac{1}{2h}2M_0 + hM_2 = \frac{1}{h}M_0 + hM_2 \quad \Rightarrow M_1 \leq \frac{1}{h}M_0 + hM_2$$

Take $h = \sqrt{M_0M_2}$, $M_1 \leq 2\sqrt{M_0M_2}$ that is $M_1^2 \leq 4M_0M_2$

(b) Given $f \in C(a,\infty;\mathbb{R})$, take the definition of M_0, M_1 and M_2 as above. $||f||_2 = M_0 + M_2 \leq M_0 + M_1 + M_2 = ||f||_1$, and $||f||_1 = M_0 + M_1 + M_2 \leq M_0 + M_2 + 2\sqrt{M_0M_2} \leq 2(M_0 + M_2)$.
Therefore,

$$||f||_1 \leq ||f||_2 \leq 2||f||_1$$

Hence $||\cdot||_1$ is equivalent to $||\cdot||_2$.

(c) Take $a_0 = -1$ and

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases}$$

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases}$$

$$f(x) = \begin{cases} 4 & (-1 < x < 0) \\ \frac{4(1-x^2)}{(x^2+1)^3} & (0 \leq x < \infty) \end{cases}$$

Hence $M_0 = 1$, $M_1 = 4$ and $M_2 = 4$, and thus $M_1^2 = 4M_0M_2$. Hence the constant 4 is the best.

□

Problem 5. Determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty}(n^{\frac{1}{n}} - 1)$$

Proof. Let $a_n = (1 + \frac{1}{n})^n$. Then

1. a_n is increasing function, since $((1 + \frac{1}{n}) \cdot 1)^{\frac{1}{n+1}} \leq (1 + \frac{1}{n+1})$, by AM-GM inequality

2. $\lim\limits_{n\to\infty} a_n = e$, hence $a_n \leq e$ for all $n \in \mathbb{N}$

Then

$$(1 + \frac{1}{n})^n \leq e \leq n \Rightarrow \frac{1}{n} \leq n^{\frac{1}{n}} - 1$$

if $n \leq 3$. By direct comparison test, $\sum\limits_{n=2}^{\infty}(n^{\frac{1}{n}} - 1)$ diverges.

□