Homework 11

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Introduction to Analysis II

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Problem 1. Let c be a simple zero of the polynomial $p = \sum_{k=0}^{n} a_k x^k$ i.e. $p(x) = (x-c)q(x)$ for some polynomial q with $q(c) \neq 0$. Show that c is a C^∞ function locally of the coefficient $a_0, ..., a_n$ of the polynomial p.

Proof. Let $F(c; a_0, a_1, ..., a_n) = p(c)$, denote $a = (a_0, a_1, ..., a_n)$, and note that

$$p'(x) = q(x) + (x-c)q'(x) \Rightarrow p'(c) = q(c) \neq 0$$

also since $p(c)$ is a polynomial, $p(c) \in C^\infty$. By the implicit function theorem, given a fix $a^* \in \mathbb{R}^{n+1}$ corresponds $c^* \in \mathbb{R}$, such that $F(c^*; a_1^*, ..., a_n^*) = F(c^*; a^*) = 0$, there exists an open set A_0 contains the point a^*, and a smooth one-to-one function $c(a) : A_0 \to \mathbb{R}$, such that $F(c(a); a) = p(c(a))$. Then since a^* is arbitrary, we are done. □

Problem 2. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be given by

$$F(x,y,z,w) = (G(x,y,z,w), H(x,y,z,w)) = (y^2 + w^2 - 2xz, y^3 + w^3 + x^3 - z^3)$$

and let $p = (1, -1, 1, 1)$ such that $F(p) = (0, 0)$

(a) Show that we can solve $F(x,y,z,w) = (0,0)$ for (x,z) in terms of (y,w) near $(-1,1)$.

(b) If $(x,z) = \Phi(y,w)$ is a solution of part (a), Show that $D\Phi(-1,1)$ is given by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof.

1. Since $det[D_j f_i((xz); (y,w)] = \begin{vmatrix} -2z & -2x \\ 3x^2 & -3z^2 \end{vmatrix} = 6(x^3 + z^3)$.
   Then $det(D_j f_i((1,1); (-1,1))) = 12 \neq 0$.
   By the Implicit function theorem, there exists an open set $T_0 \subset \mathbb{R}^2$ that contains $(-1,1)$ and a one-to-one C^1 function $\Phi(y,w) : T_0 \to \mathbb{R}^2$ such that
   (a) $g(-1,1) = (1,1)$
   (b) $F(g(y,w); (y,w)) = 0$ on T_0
   (c) $g \in C^1$

   Then $\Phi : T_0 \to \mathbb{R}^2$ is a solution.

2.

$$\nabla F(g(y,w);(y,w)) = \begin{pmatrix} G_x & G_z \\ H_x & H_z \end{pmatrix} D\Phi(y,w) + \begin{pmatrix} G_y & G_w \\ H_y & G_w \end{pmatrix} = O$$
$$\Rightarrow \nabla F(g(-1,1);(-1,1)) = \begin{pmatrix} -2 & -2 \\ 3 & -3 \end{pmatrix} D\Phi(-1,1) + \begin{pmatrix} -2 & 2 \\ 3 & 3 \end{pmatrix} = O$$
$$\Rightarrow D\Phi(-1,1) = -\begin{pmatrix} -2 & -2 \\ 3 & -3 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

□

Problem 3. Show that there exists $p, q > 0$ such that there are unique function $u(x), v(x) : (-1-p, -1+p) \to (1-q, 1+q)$ for which

$$xe^u + ue^v = 0 = xe^v + ve^u = 0$$

for all $x \in (-1-p, -1+p)$ and $u(-1) = v(-1) = 1$.

Proof. Define function $F : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$F(y,z;x) = \begin{pmatrix} xe^u + ue^v \\ xe^v + ve^u \end{pmatrix}$$

Note that $F(1,1,-1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that $det[D_j f_i(y,z;x)] = \begin{vmatrix} xe^u + e^v & ue^v \\ ve^u & xe^v + e^u \end{vmatrix}$.
Then $det[D_i f_j(1,1;-1)] = \begin{vmatrix} 0 & e \\ e & 0 \end{vmatrix} \neq 0$. Therefore, by implicit function theorem, there exists an open set X that contains -1 and an open set Y that contains $(1,1)$ and a one-to-one C^1 function, denoted as$g : X \to Y$, $g(x) = (u(x), v(x))$, such that $F(g(x);x) = 0$. Consider an open interval $(-1-p, -1+p) \subset X$ for some $p > 0$ and some open ball $B_q(1,1)/$ that covers $g((-1-p, -1+p))$ on which g is defined. Then the component u, v of $g : (-1-p, -1+p) \to B_q(1,1)$ is the desired solution. □

Problem 4. (Hadamard inequality) Let $\Delta = det[x_{ij}]$ and $X_i = (x_{i1}, ..., x_{in})$, and set $d_i \geq 0$ such that $||X_i||^2 = d_i^2 (i = 1, ..., n)$. Prove the Hadamard inequality:

$$|\Delta| \leq d_1 ... d_n$$

.

Proof. (Method 1) Without loss of generality, suppose $d_I = 1$, we can always do this since if we normalize all rows of the determinant, we get $|\Delta'| \leq 1$ from the statement, where Δ' is the determinant whose rows are all normalized.
It is suffice to prove $|\Delta'| \leq 1$.
Let $Y = X^* X$, where $X = [x_{ij}]$. Since Y is Hermitian, all eigenvalues are positive and real numbers. Let $\lambda_1, ..., \lambda_n$ be the eignvalues of the matrix Y.

$$|det(X)|^2 = |det(Y)| = \lambda_1 ... \lambda_n \leq (\frac{\lambda_1 + ... + \lambda_n}{n})^n = (\frac{tr(Y)}{n})^n \leq 1$$

It follows from the inequality that $|det(X)| \leq 1$.

(Method 2) Consider constrain $G_i = x_{i1}^2 + ... + x_{in}^2 - d_i = 0$ for $i = 1, ..., n$. Let $F(x_{11}, ..., x_{nn}) = \Delta$, $F : \mathbb{R}^{n^2} \to \mathbb{R}$. By method of Lagrange multipliers,

$$\frac{\partial F}{\partial x_{ij}} = X_{ij} = 2\lambda_i x_{ij}$$

$$\Rightarrow \Delta = \sum_{j=1}^{n} X_{ij} x_{ij} = 2\lambda_i d_{ij}^2$$

$$\Rightarrow x_{ij}\Delta = X_{ij} d_i^2$$

$$\Rightarrow [\Delta X]_{ij} = \begin{pmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{pmatrix} [adj(X)^t]_{ij}$$

$$\Rightarrow \Delta^{n+1} = det[\Delta X] = \begin{vmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{vmatrix} det[adj(X)^t] = \begin{vmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{vmatrix} \Delta^{n-1}$$

$$\Rightarrow \Delta^2 = \begin{vmatrix} d_1^2 & & & \\ & d_2^2 & & \\ & & \ddots & \\ & & & d_n^2 \end{vmatrix} = (d_1 d_2 ... d_n)^2 \Rightarrow |\Delta| = d_1 ... d_n$$

Note that for matrix that are not invertible, the determinat is 0, for instance, say that the first column is all 0, under this condition other $n^2 - n$ points can still be on the constrain, hence $|\Delta| \leq d_1 ... d_n$.

$\square$