Homework 5

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Introduction to Analysis II

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Problem 1. Show that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}$$

Proof.

1. Since

$$\begin{aligned} arctan(x) &= \int_0^x \frac{1}{1+t^2} \\ &= \int_0^x \sum_{k=0}^{\infty} (-t^2)^k \quad , |x| < 1 \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, |x| < 1 \end{aligned}$$

We can switch the order of integral and the summation since the partial sum converges uniformly on the radius of convergence, in this case, $\frac{1}{R} = \limsup_{n\to\infty} \sqrt[2n]{|(-1)^n|} = 1$.

2. Also, the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$ converges, by the alternating series test.

Therefore, by Abel theorem,

$$\lim_{x\to 1^-} arctan(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = \frac{\pi}{4}$$

. □

Problem 2. Let function $f, g : \mathbb{R} \to \mathbb{R}$ and f, g are differentiable on $\mathbb{R}$. Use the definition to show the following statement.

(a) $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$

(b) $[f(g(x))]' = f'(g(x))g'(x)$

Proof. (a)

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} &= \lim_{h\to 0} \frac{f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]}{h} \\ &= \left\{\lim_{h\to 0} f(x+h)\right\}\left\{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right\} + g(x)\left\{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right\} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

(b) Fix an x. Rephrasing the definition,

$$f(t) - f(g(x)) = (t - g(x))(f'(g(x)) + u(t))$$
$$g(s) - g(x) = (s - x)(g'(x) + v(s))$$

where $u(t) \to 0$ as $t \to g(x)$ and $v(s) \to 0$ as $s \to x$, then following from this, $u(g(x)) = v(x) = 0$, and thus is continuous function at x. Then

$$\begin{aligned} f(g(s)) - f(g(x)) &= [g(s) - g(x)] \cdot [f'(g(x)) + u(g(s))] \\ &= [f'(g(x)) + u(g(s))] \cdot [g'(x) + v(s)] \cdot (s - x) \end{aligned}$$

If $s \neq x$,

$$\lim_{s \to x} \frac{f(g(s)) - f(g(x))}{s - x} = \lim_{s \to x} [f'(g(x)) + u(g(s))] \cdot [g'(x) + v(s)] = f'(g(x))g'(x)$$

, since $u(t)$ is continuous at $t = g(x)$ and $g(x)$ is continuous at $x = s$.

□

Problem 3. (Thomae's function) Let $f : [0, 1] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational and } 0,\ 1 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \end{cases}$$

where $p, q \geq 0$ with no common factor. Show that f is integrable and compute $\int_0^1 f(x)dx$.

Proof. Let $S_N = \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, ..., \frac{N-1}{N}\}$ and $m = |S_N|$. Given $\epsilon > 0$, choose $N + 1 > \frac{2}{\epsilon}$. Let $P = \{x_0 < x_1 < .. < x_n\}$ be a partition on $[0, 1]$ such that $n > m$, and $||P|| < \epsilon/2m$ and let $\Delta x_i = x_i - x_{i-1}$. There are at most m interval such that $[x_{i-1}, x_i] \cap S_N \neq \emptyset$. Let $M_i = \sup_{t \in [x_{i-1}, x_i]} f(t)$, then

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{[x_{i-1}, x_i] \cap S_N = \emptyset} M_i \Delta x_i + \sum_{[x_{i-1}, x_i] \cap S_N \neq \emptyset} M_i \Delta x_i < 1 \cdot m \cdot \frac{\epsilon}{2m} + \frac{1}{1 + N} \cdot 1 \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since irrational number is dense in $[0, 1]$, $L(P, f) = 0$. Then $U(P, f) - L(P, f) < \epsilon$. Hence f is integrable on $[0, 1]$. And since

$$0 = L(P, f) \leq \underline{\int_0^1} f(x)\, dx \leq \overline{\int_0^1} f(x)\, dx \leq U(P, f) < \epsilon$$

for all $\epsilon > 0$. Therefore,

$$\int_0^1 f(x)dx = \underline{\int_0^1} f(x)\, dx = \overline{\int_0^1} f(x) = 0$$

□

Problem 4. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Proof. By way of contradiction, suppose $f(x_0) > 0$ for some $x_0 \in [a, b]$. Then since $f(x)$ is continuous, there exist a neighborhood $V = N_\delta(x_0)$ such that $f(x) > 0$ for all $x \in V$. Let P be a partition on $[a, b]$ such that $||P|| < \delta$, and let $m = \inf_{x \in N_{\delta/2}(x_0)} f(x)$ then $\int_a^b f(x)dx \geq L(P, f) \geq \delta \cdot m > 0$. $\to\leftarrow$ □

Problem 5. Suppose $f : [0,1] \to \mathbb{R}$ is continuous, f is differentiable on $(0,1)$, and $f(0) = 0$. Assume $|f'(x)| \le |f(x)|$, $0 < x < 1$. Prove $f(x) = 0$ for all $x \in [0,1]$.

Proof. Since $[0,1]$ is compact and f is continuous, let $M = \sup_{x\in[0,1]} |f(x)| = |f(x_0)|$, for some $x_0 \in [0,1]$, Fix $x \in (0,1)$,

$$|f(x) - f(0)| = |f'(z)||x| \le |f(z)||x| \le Mx$$

. In particular, $|f(x_0)| \le |f(x_0)|x_0$, for $x_0 < 1$, then $|f(x_0)| = 0$. Therefore it follows that $f(x) = 0$ for $x \in [0,1]$. □

Problem 6. Let $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a constant M such that $||\mathbf{g}(\mathbf{x})|| \le M||\mathbf{x}||^2$. Let $\mathbf{f}(x) = \mathbf{L}(x) + \mathbf{g}(x)$, where $\mathbf{L}$ is a linear map of $\mathbb{R}^n \to \mathbb{R}^m$. Prove that $\mathbf{f}$ is differentiable at $\mathbf{0}$ and $\boldsymbol{D}\mathbf{f}(\mathbf{0}) = \mathbf{L}$.

Proof. Since $||\mathbf{g}(\mathbf{0})|| \le M||\mathbf{0}|| = 0$, then $\mathbf{g}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{L}$ is a linear map, it follows that $\mathbf{L}(\mathbf{0}) = \mathbf{0}$. Then if $\mathbf{x} \neq \mathbf{0}$,

$$\frac{||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{0}) - \mathbf{L}(\mathbf{x})||}{||\mathbf{x} - \mathbf{0}||} = \frac{||\mathbf{f}(\mathbf{x}) - \mathbf{L}(\mathbf{x})||}{||\mathbf{x}||} = \frac{||\mathbf{g}(\mathbf{x})||}{||\mathbf{x}||} \le M||\mathbf{x}||$$

By squeeze theorem,

$$\lim_{\mathbf{x}\to\mathbf{0}} \frac{||\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{0}) - \mathbf{L}(\mathbf{x})||}{||\mathbf{x} - \mathbf{0}||} = 0$$

Therefore, $\mathbf{f}$ is differentiable at $\mathbf{0}$ and $\boldsymbol{D}\mathbf{f}(\mathbf{0}) = \mathbf{L}$. □

Problem 7. Define $f(0,0) = 0$ and

$$f(x,y) = \frac{x^3}{x^2+y^2} \qquad \text{if } (x,y) \neq (0,0)$$

(a) Prove that $\partial_x f$ and $\partial_y f$ are bounded functions on $\mathbb{R}^2$.(Hence f is continuous).

(b) Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^2$. Show that the directional derivative $(D_{\mathbf{u}} f)(0,0)$ exists, and that its absolute value is at most 1.

(c) Let γ be a differential mapping of $\mathbb{R}^1$ into $\mathbb{R}^2$(in other words, γ is a differentiable curve in $\mathbb{R}^2$), with $\gamma(0) = (0,0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is a differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in C^1$, prove that $g \in C^1$.

(d) In spite of this, prove that f is not differentiable at $(0,0)$.

Proof.

(a)

$$\partial_x f(x,y) = \begin{cases} \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2}, \text{if } (x,y) \neq (0,0) \\ 1, \text{if } (x,y) = (0,0) \end{cases} \qquad \partial_y f(x,y) = \begin{cases} \frac{-2yx^3}{(x^2+y^2)^2}, \text{if } (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$$

Since

$$\frac{x^2(x^2+3y^2)}{(x^2+y^2)} = \frac{x^2}{x^2+y^2} \cdot \frac{x^2+3y^2}{x^2+y^2} \le 1 \cdot \frac{3x^2+3y^2}{x^2+y^2} = 3$$

and

$$\left| \frac{-2yx^3}{(x^2+y^2)^2} \right| = \frac{2|x||y|}{x^2+y^2} \cdot \frac{x^2}{x^2+y^2} \le 1$$

Then both $\partial_x f$ and $\partial_y f$ are bounded functions on $\mathbb{R}^2$.

(b) Let $\mathbf{u} = (a, b)$ and $||\mathbf{u}||_2 = a^2 + b^2 = 1$.

$$(D_{\mathbf{u}}f)(0,0) = \lim_{h\to 0} \frac{f(ah, bh) - f(0,0)}{h} = \lim_{h\to 0} \frac{a^3h^3}{(a^2h^2 + b^2h^2)h} = \frac{a^3}{a^2+b^2} = a^3 \leq 1$$

(c) Since γ is a differentiable map, $\gamma(t) = (x(t), y(t))$, where $x(t)$ and $y(t)$ is differentiable function on $\mathbb{R}$. Since $\partial_x f(x,y)$ and $\partial_y f(x,y)$ is continuous except at $(x,y) = (0,0)$, $g(t)' = \nabla f(\gamma(t)) \cdot \gamma'(t)$ for $\gamma(t) \neq (0,0)$ ... (1). Let $Z(\gamma') = \{t \in \mathbb{R} \mid \gamma'(t) = (0,0)\}$ and $Z(\gamma) = \{t \in \mathbb{R} \mid \gamma(t) = (0,0)\}$.

If $t^* \in Z(\gamma') \cap Z(\gamma)$, then $\gamma'(t^*) = (x'(t^*), y'(t^*)) = (0,0)$. Then

$$\left|\frac{x^3(t)}{(x^2(t) + y^2(t))(t - t^*)}\right| \leq \left|\frac{x(t) - x(t^*)}{t - t^*}\right| ...(2)$$

by letting $t \to t^*$, since $x'(t^*) = 0$, by squeeze theorem, $g'(t^*) = f(\gamma(t^*))' = 0$.

If $t^* \in \overline{Z(\gamma')} \cap Z(\gamma)$, since $|\gamma(t^*)| > 0$, either $x'(t^*)$ of $y'(t^*)$ is non zero, then there exist a neighborhood $V = N_\delta(t^*)$ such that $\gamma(t) \neq (0,0)$ if $x \in V$, then if $t \neq t^*$,

$$\frac{g(t) - g(t^*)}{t - t^*} = \frac{f(\gamma(t)) - f(\gamma(t^*))}{t - t^*} = \frac{x^3(t)}{(x^2(t) + y^2(t))(t - t^*)} = \frac{(\frac{x(t)-x(t^*)}{t-t^*})^3}{(\frac{x(t)-x(t^*)}{t-t^*})^2 + (\frac{y(t)-y(t^*)}{t-t^*})^2} ...(3)$$

by letting $t \to t^*$, we conclude that $g(t^*)' = \frac{x'(t^*)^3}{x'(t^*)^2 + y'(t^*)^2}$.
Hence g is a differentiable for every $t \in \mathbb{R}^1$.

(d)

$$\frac{|f(x,y) - f(0,0) - 1 \cdot x - 0 \cdot y|}{\sqrt{x^2 + y^2}} = \frac{x^3}{(x^2 + y^2)^{3/2}}$$

Along $x = 0$ the limit is 0, whereas along $x = y$ the limit is $\frac{1}{2\sqrt{2}}$. Hence $f(x,y)$ is not differentiable at $(0,0)$.

□

Problem 8. Write $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \text{A is invertable}\}$. Identify $GL_n(\mathbb{R})$ with the Euclidean norm. Define $f : GL_n(\mathbb{R}) \to M_n(\mathbb{R})$by $f(A) = A^{-1}$.

(a) Is f continuous at the identity matrix $I \in GL_n(\mathbb{R})$?

(b) Is f differentiable at $I \in GL_n(\mathbb{R})$ If yes, what is $Df(I)$?

Proof.
Define $||A|| = ||(a_{ij})|| = \sqrt{\sum_{i,j} a_{ij}^2}$. We note that if $A, B \in M_n(\mathbb{R})$

$$||AB|| = \sqrt{\sum_{i,j}\left(\sum_{k=0}^{n} a_ikb_kj\right)^2} \leq \sqrt{\sum_{i,j}(\sum_{k=0}^{n} a_{ik}^2)(\sum_{k=0}^{n} b_{kj}^2)} \leq \sqrt{(\sum_{i=0}^{n}\sum_{k=0}^{n} a_{ik}^2)(\sum_{j=0}^{n}\sum_{k=0}^{n} b_{kj}^2)} \leq ||A|| \cdot ||B||$$

Define $\mathbb{I} = \begin{pmatrix} 1 & 1 & \cdots \\ \vdots & \ddots & \\ 1 & & 1 \end{pmatrix} \in M_n(\mathbb{R})$

(a) Given $\epsilon > 0$, choose $\delta = \min\{\frac{\epsilon}{2}, \frac{1}{2}\}$, if $||H|| < \delta$, by the triangle inequality,

$$||(I+H)^{-1} - I|| = || - H + H^2 - H^3 + ...|| \leq \sum_{k=1}^{\infty} ||H^k|| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon$$

Hence $f(A)$ is continuous at I.

(b) Given $\epsilon > 0$, choose $\delta = \min\{\frac{\epsilon}{2}, \frac{1}{2}\}$, if $||H|| < \delta$,

$$\begin{aligned}\frac{||(I+H)^{-1} - I + \mathbb{I} \cdot H||}{||H||} &= \frac{1}{||H||} \cdot ||H^2 - H^3 + H^4 + ...|| \leq \frac{1}{||H||}||H|| \cdot ||H - H^2 + H^3 - ...|| \\ &= ||H - H^2 + H^3 - ..|| < \epsilon\end{aligned}$$

The last part of the inequality follow directly from (a). Hence

$$\lim_{||H|| \to 0} \frac{||(I+H)^{-1} - I + \mathbb{I} \cdot H||}{||H||} = 0$$

, which follows that $f(A)$ is differentiable at I and $Df(I) = \mathbb{I} = \begin{pmatrix} 1 & 1 & \dots \\ \vdots & \ddots & \\ 1 & & 1 \end{pmatrix} \in M_n(\mathbb{R})$.

$\square$