Homework 8

Hsin-Wei, Chen
B12902132 CSIE
Introduction to Analysis II

June 27, 2024

Problem 1.
Given a open ball $B_r((0,0))$, let $f(x,y) = x + y + e^{-x}cos(y) : B_r((0,0)) \to \mathbb{R}$, find a polynomial $P(x,y)$ such that

$$|f(x,y) - P(x,y)| \leq C(x^2 + y^2)^{\frac{3}{2}}$$

for all $(x,y) \in B_r((0,0))$.

Proof.
Since $B_r((0,0))$ is open, by Taylor's theorem,

$$f(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial x \partial y}(0,0)y^2) + \frac{1}{3!}(\frac{\partial^3 f}{\partial^3 x}(0,0)x^3 + \frac{\partial^3 f}{\partial^2 x \partial y}(0,0)3x^2y + \frac{\partial^3 f}{\partial x \partial^2 y}(0,0)3xy^2 + \frac{\partial^3 f}{\partial^3 y}(0,0)y^3) + R_4(x,y) = P(x,y) + R_4(x,y)$$

We note that since

$$\frac{|R_4(x,y)|}{(x^2+y^2)^{\frac{3}{2}}} \leq C_1$$

for some $C_1 \in \mathbb{R}$, if $\sqrt{x^2+y^2} \leq \min\{\delta, r\}$, for some $\delta > 0$. And also since $\frac{|R_4(x,y)|}{(x^2+y^2)^{\frac{3}{2}}}$ is continuous on $B_r(0,0) \setminus B_\epsilon(0,0)$, then

$$\frac{R_4(x,y)}{(x^2+y^2)^{\frac{3}{2}}} \leq C_2$$

for some $C_2 \in \mathbb{R}$. Take $C = \min\{C_1, C_2\}$, then

$$|f(x,y) - P(x,y)| = |R_4(x,y)| \leq C(x^2+y^2)^{\frac{3}{2}}$$

Hence

$$P(x,y) = f(0,0) + \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y + \frac{1}{2!}(\frac{\partial^2 f}{\partial x^2}(0,0)x^2 + 2\frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial x \partial y}(0,0)y^2) + \frac{1}{3!}(\frac{\partial^3 f}{\partial^3 x}(0,0)x^3 + \frac{\partial^3 f}{\partial^2 x \partial y}(0,0)3x^2y + \frac{\partial^3 f}{\partial x \partial^2 y}(0,0)3xy^2 + \frac{\partial^3 f}{\partial^3 y}(0,0)y^3) = 1 + y + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}xy^2$$

□

Problem 2. Given $f(x,y) = cos(x^2+y^2)$. Let $T_m((x,y),(x_0,y_0))$ is the Taylor expansion around (x_0,y_0) of degree m (the combination of linear maps $D^r f(x_0,y_0)$, $0 \leq r \leq m$). Compute $T_2((x,y),(0,0))$ and estimate the approximation error for $(x,y) \in [0,0.1]^2$.

Proof.
We first note that f is C^∞; hence, by Clairaut's theorem, every partial derivative is exchangeable. The Taylor expansion around $(0,0)$ of degree 2 is:

$$\begin{aligned}T_2((x,y),(0,0)) &= f(0,0)+\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)x^2+\frac{\partial^2 f}{\partial x\partial y}(0,0)xy+\frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,0)y^2\\ &= 1\end{aligned}$$

For $(x,y)\in[0,0.1]^2$, (we can deduce some calculation by symmetry)

$$\begin{aligned}\Big|\frac{\partial^3 f}{\partial x^3}(x,y)\Big| &= \Big|-12x\cos(x^2+y^2)+8x^3\sin(x^2+y^2)\Big| \le 1.2+16(0.1)^5\\ \Big|\frac{\partial^3 f}{\partial y^3}(x,y)\Big| &= \Big|-12y\cos(x^2+y^2)+8y^3\sin(x^2+y^2)\Big| \le 1.2+16(0.1)^5\\ \Big|\frac{\partial^3 f}{\partial x\partial y^2}(x,y)\Big| &= \Big|-4y\cos(x^2+y^2)+8x^2y\sin(x^2+y^2)\Big| \le 0.4+16(0.1)^5\\ \Big|\frac{\partial^3 f}{\partial x\partial y^2}(x,y)\Big| &= \Big|-4y\cos(x^2+y^2)+8x^2y\sin(x^2+y^2)\Big| \le 0.4+16(0.1)^5\end{aligned}$$

The remainder term is :

$$\begin{aligned}|R_2((0,0),(x,y))| = \frac{1}{3!}\Big|\sum_{i=0}^{3}\binom{3}{i}\frac{\partial^3 f}{\partial^i x\partial^{3-i}y}(c_1,c_2)(c_1)^i(c_2)^{3-i}\Big| &\le (0.8+\frac{32}{3}(0.1)^5)(0.1)^3+(0.4+16(0.1)^5)(0.1)^3\\ &\le 1.2(0.1)^3+\frac{80}{3}(0.1)^8\end{aligned}$$

Hence the error will be less than $1.2(0.1)^3+\frac{80}{3}(0.1)^8$ □

Problem 3.
Let $f:\mathbb{R}\to\mathbb{R}$ be given by

$$f(x)=\begin{cases}\frac{1}{2}x^2+2x^2\sin(\frac{1}{x}), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Mr.Munchlax says the inverse function f^{-1} exists on the neighborhood of $x=0$. Explain whether his statement holds true or not.

Proof.
Since

$$f'(x)=\begin{cases}\frac{1}{2}, & \text{if } x=0\\ \frac{1}{2}+4x\sin(\frac{1}{x})-2\cos(\frac{1}{x}), & \text{if } x\neq 0\end{cases}$$
$$f''(x)=4\sin(\frac{1}{x})-4\frac{1}{x}\cos(\frac{1}{x})-\frac{2}{x^2}\sin(\frac{1}{x}),\ \text{if } x\neq 0$$

We note that since f' is not continuous at $x=0$, therefore it does not satisfy the condition of the inverse function theorem.
Let $t_n=\frac{1}{2n\pi}$ and $r_n=\frac{1}{2n\pi+\pi}$, then

$$\begin{aligned}f'(t_n) &= \frac{1}{2}+4t_n\sin(\frac{1}{t_n})-2\cos(\frac{1}{t_n})=\frac{1}{2}-2<0\\ f'(r_n) &= \frac{1}{2}+4r_n\sin(\frac{1}{r_n})-2\cos(\frac{1}{r_n})=\frac{1}{2}+2>0\end{aligned}$$

By Intermediate value theorem, since $[r_n, t_n]$ is compact and $f'(x)$ is continuous at $x \neq 0$, there exist $x_n \in [r_n, t_n]$ such that $f'(x_n) = 0$. Since

$$f'(x_n) = \frac{1}{2} + 4x \sin(\frac{1}{x_n}) - 2\cos(\frac{1}{x_n}) = 0$$

$$\begin{aligned} f''(x_n) &= 4\sin(\frac{1}{x_n}) - 4\frac{1}{x_n}\cos(\frac{1}{x_n}) - \frac{2}{x^2}\sin(\frac{1}{x_n}) \\ &= 4\sin(\frac{1}{x_n}) - \frac{2}{x_n}(\frac{1}{2} + 4x_n \sin(\frac{1}{x_n})) - \frac{2}{x_n^2}\sin(\frac{1}{x_n}) \\ &= -\frac{1}{x_n^2} - 4\sin(\frac{1}{x_n}) - \frac{2}{x_n^2}\sin(\frac{1}{x_n}) < 0 \end{aligned}$$

, given that $\sin(\frac{1}{x_n}) > 0$. Hence x_n is a local maximum, $\exists \delta_n > 0$ such that $f(x_n) > f(x)$ for all $x \in N_{\delta_n}(x_n)$ and since f is continuous at x_n, take $y \in [x_n - \delta_n, x_n]$ and $z \in [x_n, x_n + \delta_n]$, then by IVT, there exist $r \in [x_n - \delta_n, x_n]$, $s \in [x_n, \delta_n + x_n]$ such that $f(r) = f(s) = \frac{1}{2}(f(x_n) + \max\{f(y), f(z)\})$, hence f is not one to one on $[x_n - \delta_n, x_n + \delta_n]$. Since $x_n \to 0$, given that $t_n \to 0$. It follows that we can always find such x_n in any neighborhood of 0 and hence there is no neighborhood of 0 such that f is invertible on this neighborhood. □

Problem 4.
Let $f(x, y) = (e^x cos(y), e^x sin(y))$

(a) Show that f is C^1 and $Df(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^2$

(b) Show that f is not one-to-one.

(c) Why doesn't this contradict the Inverse Function Theorem?

Proof.

(a) Since $f(x, y)$ is C^1 in its all component, f is differentiable, then $Df(x, y)$ exist and

$$Df(x, y) = \begin{pmatrix} e^x \cos(y) & -e^x \sin(y) \\ e^x \sin(y) & e^x \cos(y) \end{pmatrix}$$

Since every component of $Df(x, y)$ is C^1, it follows that $Df(x, y)$ is C^1.

$$J_f = det(D(f)(x, y)) = e^{2x} > 0$$

for all $(x, y) \in \mathbb{R}^2$. Therefore, $Df(x, y)$ is invertible.

(b) Since $f(0, 2n\pi) = (1, 0)$ for all $n \in \mathbb{N}$, then $f(x, y)$ is not one-to-one on $\mathbb{R}^2$.

(c) This does not contradict the inverse function since although $f(x, y)$ is not one-to-one on $\mathbb{R}^2$ and f satisfy the condition of the inverse function theorem, given a point (x, y) there still exist a neighborhood U such that f is one-to-one on U.

□