Homework 1

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Introduction to Analysis II

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Problem 1.

(a) Must a contraction on any metric space have a fixed point? Prove it or give a counterexample.

(b) Let$f : X \to X$, where X is a complete metric space satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ such that$x \neq y$. Must f have a fixed point? Prove it or give a counterexample.

(c) Let $f : X \to X$, where X is a complete compact metric space satisfies $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$such that$x \neq y$. Must f have a fixed point?

Proof.

(a) No, the contraction map need not have a fix point.
Let $X = \{x \in \mathbb{Q} \mid x > 0\}$, and define metric $d(p, q) = |p - q|$, where $p, q \in X$

1. We first show that (X, d) is metric space. Given $p, q, r \in X$
   (1) $d(p, q) = |p - q|, \quad d(p, p) = 0.$
   (2) $d(p, q) = d(q, p) = |p - q|$
   (3) $d(p, r) \leq d(p, q) + d(q, r)$
2. Let $T(x) = x - \frac{x^2-2}{x+2} = \frac{2p+2}{p+2}$, Take $p, q \in X$
   (1) $T(p) = \frac{2p+2}{p+2} \in X$
   (2)
   $$T(p) - T(q) = 2\left(\frac{p+1}{p+2} - \frac{q+1}{q+2}\right) = \frac{2(p-q)}{(p+2)(q+2)} \leq \frac{1}{2}(p-q)$$
   Hence T is a contraction in X.
3. Pick an arbitrary $x_1 \in X$ and $x_{n+1} = T(x_n)$
   $$|x_{n+1}^2 - 2| = 2\left|\frac{x_n^2 - 2}{(x_n + 2)^2}\right| \leq |x_n^2 - 2|$$
   $$\Rightarrow |x_{n+1}^2 - 2| \leq (\frac{1}{2})^n |x_1^2 - 2| \Rightarrow \limsup_{n\to\infty} |x_n^2 - 2| = 0$$
   Since $x_n > 0$ for all $n \in \mathbb{N}$,
   $$\lim_{n\to\infty} x_n = \sqrt{2}$$
   , but$\sqrt{2} \notin X$

(b) Consider $f(x) = x + e^{-x}$ and metric space $X = \mathbb{R}$ with regular metric $d(x, y) = |x - y|$

By MVT,

$$\frac{|f(x) - f(y)|}{|x - y|} = 1 - e^{\xi} < 1 \Rightarrow |f(x) - f(y)| < |x - y|$$

, where $x < \xi < y$

But $f(x) > x, \forall x \in X$, hence f has no fix point in X

(c) Following from the hint, consider $g(x) = d(x, f(x))$

Given $\epsilon > 0$, choose $\delta = \frac{\epsilon}{2}$, such that if $d(x, y) < \delta$,

$$|g(x) - g(y)| = |d(x, f(x) - d(y, f(y)))| \leq d(x, y) + d(f(x), f(y)) < 2d(x, y) < \epsilon$$

Hence g(x) is uniformly continuous on X.

Since X is compact, by EVT, $\inf_{x \in X} g(x) = g(x_m)$, for some $x_m \in X$.

Suppose $g(x_m) > 0$, then $g(f(x_m)) < g(x_m) \rightarrow\leftarrow$

Therefore, $g(x_m) = 0 \Rightarrow x_m = f(x_m)$. Hence $f(x)$ must have a fix point.

$\square$

Problem 2. Consider the initial value problem,

$$\begin{cases} 5x'(t) = sin(tx(t)) \\ x(0) = 0 \end{cases}$$

(a) Use the Contraction Mapping Principle to solve the IVP in $X = C^1[0, \pi/2]$

(b) Write out the first three terms of the Picard iteration scheme for $x(t)$. And find the solution of the IVP.

Proof. (a) By FCT1, since $x(t)$ is continuous.

$$x(t) = \frac{1}{5} \int_0^t sin(ux(u))\, du$$

Let $M = (C^1[0, \pi/2], ||.||_\infty)$ and $\phi(x) = \frac{1}{5} \int_0^t sin(ux(u))\, du$.

1. If $x(t) \in C^1[0, \pi/2]$, then by FCT1 and continuity of $sin(x)$ and $x \Rightarrow \phi(x) \in C^1[0, \pi/2]$. Then $\phi(x) : C^1[0, \pi/2] \rightarrow C^1[0, \pi/2]$.
2.

$$\begin{aligned} ||\phi(x) - \phi(y)|| &= \frac{1}{5} \sup_{t \in [0, \pi/2]} |\int_0^t sin(ux(u)) - sin(uy(y)) du| \\ &\leq \frac{1}{5} \int_0^{\pi/2} ||sin(ux) - sin(uy)|| du \leq \frac{1}{5} \int_0^{\pi/2} ||ux - uy|| du \leq \frac{1}{5} (\frac{\pi}{2})^2 ||x - y|| \end{aligned}$$

And since $\frac{\pi^2}{20} < 1$, then $\phi(x)$ is a contraction map.

By Banach Fixed point theorem, there is a unique solution.

(b) Let $x_0 = 0$, then

$$x_1 = \int_0^t sin(x_0 t)dt = 0, \ \ x_2 = \int_0^t sin(x_1 t)dt = 0, \ \ x_3 = \int_0^t sin(x_2 t)dt = 0$$

$\square$

Problem 3.

(a) (Weierstass Approximation theorem) If $f : [a, b] \to \mathbb{R}$ is a continuous function, then there exist a sequence of polynomials $p_n : [a, b] \to \mathbb{R}$ such that $p_n \rightrightarrows f(x)$ on $[a, b]$

(b) Let $f : [-1, 1] \to \mathbb{R}$, be an even continuous function. Show that there exist a sequence of even polynomials $p_n : [-1, 1] \to \mathbb{R}$ such that $p_n \rightrightarrows f(x)$ on $[-1, 1]$

Proof. (a) Without loss of generality, Suppose $f : [0, 1] \to \mathbb{R}$, we can do this by defining a function $g(x) = f(a + (b - a))$, which is a continuous function on [0, 1], outside of this interval $g(x) = 0$.

Define kernel function $k_n(x) = c_n(1 - x^2)^n$, where $c_n \int_{-1}^{1} (1 - x^2)^n dx = 1$, when $x \in [-1, 1]$ and outside of this interval $k_n(x) = 0$.

1.
$$\int_{-1}^{1} (1 - x^2)dx = 2\int_0^1 (1 - x^2)^n dx \geq 2\int_0^1 (1 - x)^n dx = \frac{2}{n + 1} \Rightarrow c_n \leq \frac{n + 1}{2}$$

2.
$$\int_{1\geq|x|\geq\delta} c_n(1 - x^2)^n dx = 2\int_\delta^1 c_n(1 - x^2)^n dx \leq \int_\delta^1 c_n(1 - \delta^2)^n dx \leq (n + 1)(1 - \delta^2)^n(1 - \delta)$$

$$\Rightarrow \lim_{n\to\infty} \int_{1\geq|x|\geq\delta} c_n(1 - x^2)^n dx = 0$$

3. $(f * k_n)(x) = \int_{-1}^{1} f(x + t)k_n(t)dt = \int_0^1 f(t)k_n(x - t)dt = \int_0^1 f(t)[c_n(1 - (x - t)^2)^n]dt$,which is a polynomial of at most $2n$ degree.
Let $\sup_{x\in[0,1]} |f(x)| = M$.
Given $\epsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that :
(1) $|f(x) - f(y)| < \epsilon$, if $|x - t| < \delta$
(2) $\int_{1\geq|x|\geq\delta} c_n(1 - x^2)^n dx < \frac{\epsilon}{2M}$, if $n \geq N$
Then
$$\Big|(f * k_n)(x) - f(x)\Big| = \Big|\int_{-1}^{1} [f(x + t) - f(x)]k_n(t)dt\Big|$$

$$\leq \int_{1\geq|x|\geq\delta} |f(x + t) - f(t)|k_n(t)dt + \int_{-\delta}^{\delta} |f(x + t) - f(x)|k_n(t)dt \leq 2M \int_{1\leq|x|\leq\delta} k_n(t) + \epsilon \leq 2\epsilon$$
It follows that $(f * k_n)(x) \rightrightarrows f(x)$ on [0, 1], letting $p_n(x) = (f * k_n)(x)$, we are done.

(b) Method 1: Do expansion on $(f * k_n)(x)$ we get:

$$(f * k_n)(x) = \int_{-1}^{1} f(t)[1-(x-t)^2]^n dt = \int_{-1}^{1} \sum_{k=0}^{n} \binom{n}{k} f(t)(-1)^k (t-x)^2 k dt$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{m=0}^{2k} \binom{2k}{m} (-x)^{2k-m} \int_{-1}^{1} f(t) t^m dt$$

If m is odd, then since f(t) is even, $\int_{-1}^{1} f(t)t^m dt = 0$. Hence $p_n(x)$ is the sum of even polynomials, therefore p_n is even.

Method 2: By Weierstass approximation theorem, $\exists\, p_n$ is a sequence of polynomial s.t. $p_n(x) \rightrightarrows f(x)$ as $n \to \infty$, put $q_n(x) = 1/2(p_n(x) + p_n(-x))$, then $q_n(x) \rightrightarrows 1/2(f(x) + f(-x)) = f(x)$ as $n \to \infty$. Since q_n is even functions, we are done.

□

Problem 4. Consider the initial value problem,

$$(E) \begin{cases} x'(t) = x^2(t) + t^2 \\ x(0) = 0 \end{cases}$$

On the region $R = \{(x,t) : |x| \leq 1, |t| \leq 1\}$. Note that $M = \max_R\{x^2 + t^2\} = 2$ and $\alpha = \frac{1}{2}$

(a) Construct the ϵ approximate solution $\phi(t)$ of (E) on $|t| \leq \alpha$ such that $\phi(0) = 0$ for $\epsilon = 1$.

(b) Does the ϵ approximate solution $\phi(t)$ of (E) unique? Explain your answer.

Proof. (a) Let $P = \{x_0 = -\alpha < x_1 < ... < x_n = \alpha\}$ be a partition on $[-\alpha, \alpha]$, such that $mesh(P) < 1/8$. By Euler's method of broken line, define $\phi(t)$ be a continuous function on $[-\alpha, \alpha]$ such that : (1) $x_n(0) = 0$ (2) $x_n'(t) = x_n^2(t_i) + t_i^2 \quad (t_i < t < t_{i+1})$

Given $\epsilon = 1$, Since R is compact, then $f(x,t) = x^2 + t^2$ is uniformly continuous, hence $|f(x,t) - f(y,s)| = |x^2 - y^2 + s^2 - t^2| \leq |x^2 - y^2| + |t^2 - s^2| \leq 2|x-y| + 2|t-s| < 1$, if $|x-y|, |t-s| < \frac{1}{4}$.

Since $\phi'(t) = \phi^2(t_i) + t_i^2 \leq 2$ $(t_i \leq t < t_{i+1})$,then $|\phi(t) - \phi(s)| \leq 2|t-s|$, where $t, s \in [-\alpha, \alpha]$

And since $|t_i - t_{i+1}| < 1/8 < 1/4$, $|\phi(t_i) - \phi(t)| < 1/4$, then

$$|\phi'(t) - f(\phi(t), t)| = |f(\phi(t_i), t_i) - f(\phi(t), t))| < 1$$

, when $t \in [t_i, t_{i+1}]$, .

(b) No. Consider refinement P' of the partition P, then the ϵ approximate solution is distinct from the ϵ approximate solution constructed above.

□