Homework 3

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Introduction to Analysis II

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Problem 1. If $\mathbb{T} = \{(x,y) \in \mathbb{R}^2 \ : x^2+y^2=1\}$, then a function $f : \mathbb{T} \to \mathbb{R}$ may be thought of as a 2π-periodic function from $\mathbb{R} \to \mathbb{R}$ by first identifying a number θ with the point $(cos\theta, sin\theta)$ on $\mathbb{T}$.

$$\theta \to (cos\theta, sin\theta) \to f(\theta)$$

Similarly, any 2π-periodic function on $\mathbb{R}$ may be thought of as a function on $\mathbb{T}$. A finite sum of sines and cosines of the sum

$$\frac{a_0}{2} + \sum_{k=1}^{N}(a_k cos(k\theta) + b_k sin(k\theta))$$

is called a **trigonometic polynomial** (N is arbitrary but finite.)

Show that if f is a continuous 2π-periodic function from $\mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$. then there is a trigonometic polynomial q such that $|q(\theta) - f(\theta)| < \epsilon$ for every θ in $\mathbb{R}$.(The number N may depend on f as well as on ϵ.)

Proof.
Let $\mathcal{F} = \{f(x,y) \mid f(x,y) = a_0/2 + \sum_{k=1}^{N}(a_k x^k + b_k y^k),\ (x,y) \in \mathbb{T}\}$.
It is sufficient to show that $\mathcal{F}$ is dense in $C(\mathbb{T})$.

1. Let $f(x,y), g(x,y) \in \mathcal{F}$, then $f(x,y) = a_0/2+\sum_{k=1}^{N}(a_k x^k+b_k y^k)$ and $g(x,y) = \tilde{a}_0/2+\sum_{k=1}^{N}(\tilde{a}_k x^k+\tilde{b}_k y^k)$ and $\alpha \in \mathbb{R}$.

   (1) $f(x,y) + g(x,y) = (a_0 + \tilde{a}_0)/2 + \sum_{k=1}^{N}((a_k + \tilde{a}_k)x^k + (b_k + \tilde{b}_k)y^k) \in \mathcal{F}$

   (2) $\alpha f(x,y) = \frac{\alpha a_0}{2} + \sum_{k=1}^{N}(\alpha a_k x^k + \alpha b_k y^k) \in \mathcal{F}$

   (3) Since $(x,y) \in \mathbb{T}$, $\exists\, \theta \in \mathbb{R}$ such that $(x,y) = (cos\theta, sin\theta)$

   $$f(x,y)g(x,y) = f(sin\theta, cos\theta)g(sin\theta, cos\theta) = \frac{a_0\tilde{a}_0}{4} + \sum_{k=1}^{N}\sum_{l=1}^{N}(a_k cos(k\theta) + b_k sin(k\theta))(\tilde{a}_l cos(l\theta) + \tilde{b}_l sin(l\theta))$$

   $$= \frac{a_0\tilde{a}_0}{4} + \sum_{k=1}^{N}\sum_{l=1}^{N}\Big(\frac{a_k\tilde{a}_l + b_k\tilde{b}_l}{2}\Big)cos(\frac{k+l}{2}\theta) + \Big(\frac{a_k\tilde{a}_l - b_k\tilde{b}_l}{2}\Big)cos(\frac{k-l}{2}\theta)$$

   $$+ \Big(\frac{b_k\tilde{a}_l + a_k\tilde{b}_l}{2}\Big)sin(\frac{k+l}{2}\theta) + \Big(\frac{b_k\tilde{a}_l - a_k\tilde{b}_l}{2}\Big)sin(\frac{k-l}{2}\theta) = h(x,y) \in \mathcal{F}$$

   for some $h(x,y) = h(cos\theta, sin\theta) \in \mathcal{F}$

   Hence $\mathcal{F}$ forms an algebra.

2. $1 \in \mathcal{F}$.

3. let $(x_1, y_1), (x_2, y_2) \in \mathbb{T}$, and $(x_1, y_1) \neq (x_2, y_2)$, then take $r(x, y) = (x - x_1)^2 + (y - y_1)^2 \in \mathcal{F}$, $r(x_1, y_1) = 0 \neq r(x_2, y_2)$. Hence $\mathcal{F}$ separates points on $\mathbb{T}$.

By Stone-Weierstass Theorem, since $\mathbb{T}$ is compact, $\mathcal{F}$ is dense in $C(\mathbb{T})$, For any 2π-periodic function $f(\theta)$ we can write $f(\theta) = F(g(\theta))$, where $g(\theta) = (cos\theta, sin\theta)$, $g : \mathbb{R} \to \mathbb{R}^2$, and $F \in \mathcal{F}$. Given $\epsilon > 0$, $\exists\, p(x, y) \in \mathcal{F}$ such that $|F(x, y) - p(x, y)| < \epsilon$, $\forall (x, y) \in \mathbb{T}$. Let $q(\theta) = p(g(\theta))$, then $|f(\theta) - q(\theta)| = |F(g(\theta)) - p(g(\theta))| < \epsilon$. Note that since $q(\theta)$ is a trigonometric polynomial, we are done. □

Problem 2. Let $f : [2, 7] \to \mathbb{R}$ be continuous. Given $\epsilon > 0$, show that there exists a polynomial p such that $p(2) = f(2)$, $p'(2) = 0$ and $\sup_{x\in[2,7]} |f(x) - p(x)| < \epsilon$.

Proof.
Let $A = \{p(x) : [2, 7] \to \mathbb{R} \mid p(x) = \sum_{k=0}^{N} a_k x^k, a_k \in \mathbb{R}, N \in \mathbb{N}, p'(2) = 0\}$.

1. Let $p(x), q(x) \in A$ and $\alpha \in \mathbb{R}$,

   (1) $(p + q)'(2) = p'(2) + q'(2) = 0 \Rightarrow p + q \in A$

   (2) $(\alpha p)'(2) = \alpha p'(2) = 0 \Rightarrow \alpha p \in A$

   (3) $(pq)'(2) = p'(2)q(2) + p(2)q'(2) = 0 \Rightarrow pq \in A$

   Hence A forms an algebra.

2. $1 \in A$, since $(1)' = 0$,for all $x \in [2, 7]$.

3. Consider $r(x) = (x - 2)^2 \in A$, then for $x_1, x_2 \in [2, 7]$ and $x_1 \neq x_2 \Rightarrow r(x_1) \neq r(x_2)$.
   Hence A separates points in $[2, 7]$.

By Stone-Weierstass theorem, since $[2, 7]$ is compact, A is dense in $C([2, 7])$, therefore $\exists\, p_n(x) \in A$, given $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\sup_{x\in[2,7]} |f(x) - p_n(x)| < \epsilon/2$ for $n > N$. Define $a_n = f(2) - p_n(2)$, $n = 1, 2, 3, ...$, let $q_n(x) = p_n(x) + a_n$, then $q_n(2) = f(2)$ and since $|a_n| = |f(2) - p_n(2)| < \epsilon/2$ for $n > N$, therefore

$$\sup_{x\in[2,7]} |q_n(x) - f(x)| \leq \sup_{x\in[2,7]} |p_n(x) - f(x)| + |a_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

if $n > N$. Take $q_{N+1}(x) = p(x)$, $\sup_{x\in[2,7]} |f(x) - p(x)| < \epsilon$. □

Problem 3. Show the following series converges uniformly on R.

(a) $\sum\limits_{n=1}^{\infty} \sin(\frac{x}{n^2})$, $R = [-M, M]$

(b) $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{1+n^8x^3}$, $R = [0, 1]$

(c) $\sum\limits_{n=1}^{\infty} \frac{x^n \cos(nx)}{n}$, $R = [0, -1]$

(d) $\sum\limits_{n=1}^{\infty} \tan(\frac{x}{n}) - \sin(\frac{x}{n})$, $R = [-1, 1]$

Proof.

(a) Let $f(x) = \sum\limits_{n=1}^{\infty} \sin(\frac{x}{n^2})$. Since $sin(x) < x$ if $x > 0$, then

$$|\sin(\frac{x}{n^2})| \leq \frac{x}{n^2} \leq \frac{M}{n^2}, \qquad \forall x \in [0, M]$$

, and since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges, By Weierstass M-test, $f(x)$ converges uniformly on $[0, M]$. Since $f(x)$ is odd, $f(x)$ converges uniformly on $[-M, M]$

(b) Since $\sin(x) < x$ if $x > 0 \Rightarrow \frac{\sin(nx)}{1+n^8x^3} \leq \frac{nx}{1+n^8x^3}$.
Let $g_n(x) = \frac{nx}{1+n^8x^3} \geq 0$,

$$g_n'(x) = 0 \Rightarrow \frac{n(1+n^8x^3) - 3n^9x^3}{(1+n^8x^3)^2} = \frac{n(1-2n^8x^3)}{(1+n^8x^3)^2} = 0 \Rightarrow x = (\frac{1}{2n^8})^{\frac{1}{3}}$$

Note that

$$g_n(1) = \frac{n}{1+n^8} \leq \frac{1}{n^7} = (\frac{1}{n^{\frac{16}{3}}})(\frac{1}{n^{\frac{5}{3}}}) \leq 2/3n(\frac{1}{2n^8})^{\frac{1}{3}}, \qquad \forall n > 1$$

.
Then $g_n(0) = 0 \leq g_n(x) \leq g_n((\frac{1}{2n^8})^{\frac{1}{3}}) = 2/3n(\frac{1}{2n^8})^{\frac{1}{3}} \leq 2/3\frac{1}{n^{5/3}}$.
Therefore $|\frac{\sin(nx)}{1+n^8x^3}| \leq 2/3\frac{1}{n^{5/3}}$, if $n > 1$. By Weierstass M-test, $\sum\limits_{n=1}^{\infty} \frac{\sin(nx)}{1+n^8x^3}$ converges uniformly on $[0, 1]$.

(c) We prove a stronger result that $\sum\limits_{n=1}^{\infty} \frac{x^n \cos(nx)}{n}$ converges uniformly on $[-1, 1]$.

$$\begin{aligned}
|\sum_{n=1}^{m} x^n cos(nx)| &= \left| Re\Big\{\sum_{n=1}^{m} x^n e^{inx}\Big\} \right| \\
&= \left| Re\Big\{\frac{xe^{ix}(x^m e^{imx} - 1)}{xe^{ix} - 1}\Big\} \right| \\
&= \left| Re\Big\{\frac{xe^{ix}(xe^{imx} - 1)(xe^{-ix} - 1)}{(xe^{ix} - 1)(xe^{-ix} - 1)}\Big\} \right| \\
&= \left| Re\Big\{\frac{xe^{ix}(x^m e^{imx} - 1)(xe^{-ix} - 1)}{1 + x^2 + 2xcos(x)}\Big\} \right| \\
&= \left| Re\Big\{\frac{x^{m+2}e^{imx} - x^{m+1}e^{i(m+1)x} - x^2 + xe^{ix}}{1 + x^2 - 2xcos(x)}\Big\} \right| \\
&= \left| \frac{x^{m+2}cos(mx) - x^{m+1}cos((m+1)x) - x^2 + xcos(x)}{1 + x^2 - 2xcos(x)} \right| \\
&\leq \frac{4}{|(x - cos(x))^2 + sin^2(x)|}
\end{aligned}$$

Note that if $(x + cos(x))^2 + sin^2(x) \geq 0$ and if $sin(x) = 0,\ x \in [-1, 1] \Rightarrow x = 0$, however $(0 - cos(0))^2 = 1 \neq 0$, therefore $(x + cos(x))^2 + sin^2(x) > 0$.
Since $[-1, 1]$ is compact, By Extreme Value theorem, let $c = \min_{x \in [-1,1]}((x + cos(x))^2 + sin^2(x)) > 0$, therefore $|\sum_{n=1}^{m} x^n cos(nx)| \leq 4/c$ is bounded. By Dirichlet test, since $\{1/n\}_{n=1}^{\infty}$ is monotonic decreasing sequence and $1/n \to 0$ as $n \to \infty$, $\sum\limits_{n=1}^{\infty} \frac{x^n \cos(nx)}{n}$ converges uniformly on $[-1, 1]$, and hence on [0, 1].

(d) Method 1: Consider $f(x) = \sum\limits_{n=1}^{\infty} \tan(\frac{x}{n}) - \sin(\frac{x}{n})$ on $[0, 1]$

$$\begin{aligned} tan(\frac{x}{n}) - sin(\frac{x}{n}) &= sin(\frac{x}{n})(sec(\frac{x}{n}) - 1) \\ &\leq sin(\frac{x}{n})\frac{x}{n}(sec(\xi)tan(\xi)), \ \ by\ MVT,\ \xi \in (0, \frac{x}{n}) \\ &\leq \sin(\frac{x}{n})\frac{x}{n}(sec(\frac{x}{n})tan(\frac{x}{n})) \\ &= \frac{x}{n}tan^2(\frac{x}{n}) \leq tan^2(1)(\frac{x}{n})^3 \leq tan^2(1)(\frac{1}{n})^3 \end{aligned}$$

By Weierstrass M-test, since $\sum_{k=1}^{\infty} \frac{1}{n^3}$ converges, $\sum\limits_{n=1}^{\infty} \tan(\frac{x}{n}) - \sin(\frac{x}{n})$ converges uniformly on $[0, 1]$, since $f(x)$ is odd, $\sum\limits_{n=1}^{\infty} \tan(\frac{x}{n}) - \sin(\frac{x}{n})$ converges uniformly on $[-1, 1]$.

Method 2:

$$tan(\frac{x}{n}) - sin(\frac{x}{n}) = tan(\frac{x}{n})(1 - cos(\frac{x}{n})) = 2tan(\frac{x}{n})sin^2(\frac{x}{2n})$$

Since $sin^2(\frac{x}{2n}) \leq \frac{1}{4n^2}$, by Weierstass M-test, $\sum_{n=1}^{\infty} sin^2(\frac{x}{2n})$ converges uniformly on $[-1, 1]$, and $tan(x/n)$ monotonically decreasing and $|tan(x/n)| \leq tan(1),\ \forall x \in [-1, 1]$ is bounded, by Abel's test, $\sum\limits_{n=1}^{\infty} \tan(\frac{x}{n}) - \sin(\frac{x}{n})$ converges uniformly on $[-1, 1]$.

$\mathscr{P}$

$\square$