Homework 2

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Introduction to Analysis II

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Problem 1. Define the following set:

$$\mathbb{P}_{even}(I) = \{p : I \to \mathbb{R},\ p(x) = \sum_{k=0}^{n} a_k x^{2k}\ | a_k \in \mathbb{R}, n \in \mathbb{N}\}$$

$$\mathbb{P}(I^2) = \{p(x,y)\ is\ polynomial\ on\ I \times I\}$$

$$C_s(A \times B) = \{h(x,y) = \sum_{k=0}^{n} f_k(x) g_k(y) \mid n \in \mathbb{N}, f_k \in C(A), g_k \in C(B)\}$$

$$\mathcal{F}_\sigma = \{g(x) = \sum_{k=0}^{n} a_k \sigma(b_k x + c_k) : [0,100] \to \mathbb{R} \mid a_k, b_k, c_k \in \mathbb{R}, n \in \mathbb{N}\}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$

(a) Is $\mathbb{P}_{even}([0,1])$ is dense in $C([0,1])$?

(b) Is $\mathbb{P}_{even}([-1,1])$ is dense in $C([-1,1])$?

(c) Is $\mathbb{P}([0,1]^2)$ is dense in $C([0,1]^2)$?

(d) Is $C_s([a,b] \times [c,d]))$ is dense in $C([a,b] \times [c,d])$?

(e) Is $\mathcal{F}_\sigma$ is dense in $C([0,100])$?

Proof.

(a) Yes. Let $p(x),\ q(x) \in \mathbb{P}_{even}([0,1])$, $\alpha \in \mathbb{R}$ then we can write $p(x) = \sum_{k=0}^{n} a_k x^{2k}$, and $q(x) = \sum_{k=0}^{n} b_k x^{2k}$ (if the degrees of p and q are not the same we put 0 coefficient so that they are the same degree)

  (1) (i) $p(x) + q(x) = \sum_{k=0}^{n} (a_k + b_k) x^{2k} \in \mathbb{P}_{even}([0,1])$
  (ii) $(\alpha p)(x) = \sum_{k=0}^{n} (\alpha a_k) x^{2k} \in \mathbb{P}_{even}([0,1])$
  (iii) $p(x)q(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + ... + a_n b_n x^n \in \mathbb{P}_{even}([0,1])$
  Hence $\mathbb{P}_{even}([0,1])$ is an algebra.

  (2) Constant function $1 \in \mathbb{P}_{even}([0,1])$

  (3) Let $x_1,\ x_2 \in [0,1]$ and without loss of generality , suppose $x_1 > x_2$, consider $r(x) = x^2 \in \mathbb{P}_{even}([0,1])$ then $r(x_1) = x_1^2 > x_2^2 = r(x_2)$, hence $r(x_1) \neq r(x_2)$
  Hence $\mathbb{P}_{even}([0,1])$ separates points in $[0,1]$

Since [0, 1] is compact, by Stone-Weierstass theorem, $\mathbb{P}_{even}([0,1])$ is dense in $C([0,1])$

(b) No. By way of contradiction, suppose $\mathbb{P}_{even}([-1,1])$ is dense in $C([-1,1])$, then since $x \in C([-1,1])$, $\exists p_n$ sequence of even polynomial on $[-1,1]$ such that $p_n(x) \rightrightarrows x$ as $n \to \infty$.
Given $\epsilon > 0$ there exist N such that $|p_n(x) - x| < \epsilon/2$, if $n > N$, $\forall x \in [-1,1]$.
Then $|x - (-x)| \le |x - p_n(x)| + |p_n(x) + x| = |x - p_n(x)| + |p_n(-x) - (-x)| < \epsilon$ if $n > N$.
Therefore, since ϵ is arbitrary, $2|x| = 0$, $\forall x \in [-1,1]$ $-\!\!\times\!\!-$

(c) Yes. Let $p(x,y), q(x,y) \in \mathbb{P}([0,1]^2)$, $\alpha \in \mathbb{R}$

(1) (i) $p(x,y) + q(x,y) \in \mathbb{P}([0,1]^2)$
(ii) $(\alpha p)(x,y) \in \mathbb{P}_{even}([0,1])$
(iii) $p(x,y)q(x,y) \in \mathbb{P}([0,1]^2)$
Hence $\mathbb{P}([0,1]^2)$ is an algebra.

(2) Constant $1 \in \mathbb{P}([0,1]^2)$

(3) Given $(x_1,y_1),(x_2,y_2) \in [0,1]^2$, and $(x_1,y_1) \neq (x_2,y_2)$.
Consider the function $r(x,y) = (x-x_1)^2 + (y-y_1)^2$, then $r(x_1,y_1) \neq r(x_2,y_2)$.
Hence $\mathbb{P}([0,1]^2)$ separates points in $[0,1]^2$.

Then since $[0,1]^2$ is compact, by Stone-Weierstass theorem, $\mathbb{P}([0,1]^2)$ is dense in $C([0,1]^2)$.

(d) Yes. Let $h(x,y), \tilde{h}(x,y) \in C_s([a,b]\times[c,d])$, then $h(x,y) = \sum_{k=0}^{n} f_k(x)g_k(y)$ and $\tilde{h}(x,y) = \sum_{k=0}^{n} \tilde{f}_k(x)\tilde{g}_k(y)$.
(We can make the number of terms equal by considering $f_k(x)$ to be 0)

(1) (i) $h(x,y) + \tilde{h}(x,y) \in C_s([a,b]\times[c,d])$, by re-index $\tilde{f}_k = f_{k+n+1}$ and $\tilde{g}_k = g_{k+n+1}$.
(ii) $(\alpha h)(x,y) = \sum_{k=0}^{n} \alpha f_k(x)g_k(y) \in C_s([a,b]\times[c,d])$
(iii) $h(x,y)\tilde{h}(x,y) \in C_s([a,b]\times[c,d])$, since product of variable-seperable function is also variable-seperable
Hence $C_s([a,b]\times[c,d])$ is an algebra.

(2) Constant $1 \in C_s([a,b]\times[c,d])$.

(3) Given $(x_1,y_1),(x_2,y_2) \in [a,b]\times[c,d]$, and $(x_1,y_1) \neq (x_2,y_2)$.
Consider the function $r(x,y) = (x-x_1)^2 + (y-y_1)^2$, then $r(x_1,y_1) \neq r(x_2,y_2)$.
Hence $C_s([a,b]\times[c,d])$ separates points in $[a,b]\times[c,d]$

Then since $[a,b]\times[c,d]$ is compact, by Stone-Weierstass theorem, $C_s([a,b]\times[c,d]))$ is dense in $C([a,b]\times[c,d])$.

(e) Yes. Let $g(x), \tilde{g}(x) \in \mathcal{F}_\sigma$, then $g(x) = \sum_{k=0}^{n} a_k cos(b_k x + c_k)$ and $\tilde{g}(x) = \sum_{k=0}^{n} \tilde{a}_k cos(\tilde{b}_k x + \tilde{c}_k)$.
(We can make the number of terms equal by considering the coefficient to be 0)

(1) (i) $g(x) + \tilde{g}(x) \in \mathcal{F}_\sigma$, by re-index $\tilde{a}_k = a_{k+n+1}$, $\tilde{b}_k = b_{k+n+1}$, and $\tilde{c}_k = c_{k+n+1}$.
(ii) $(\alpha g)(x) = \sum_{k=0}^{n} \alpha a_k cos(b_k x + c_k) \in \mathcal{F}_\sigma$
(iii) $g(x)\tilde{g}(x) \in \mathcal{F}_\sigma$, since the product of cosine functions can be expressed as sum of cosine functions
Hence $\mathcal{F}_\sigma$ is an algebra.

(2) By considering $a_k = 1, b_k = 0, c_k = 0$, constant $1 \in \mathcal{F}_\sigma$.

(3) Let $x_1, x_2 \in [0,100]$ and $x_1 < x_2$ Take $b_k = \pi/200$,

$$cos(\pi/200x_2) - cos(\pi/200x_1) = sin(\xi)\pi/200(x_1 - x_2) \neq 0$$

, since $x_1 \neq x_2$ and $0 \leq \pi/200x_1 < \xi < \pi/200x_2 \leq \pi/2.$, implies that $sin(\xi) \in (0,1)$.
Then $\mathcal{F}_\sigma$ separates points.

Then since $[0,100]$ is compact, by Stone-Weierstass theorem, $\mathcal{F}_\sigma$ is dense in $C([0,100])$.

$\square$

Problem 2. (a) If $f : [0,1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x)x^n dx = 0$$

for each integer number$n \in \mathbb{R}$, prove that $f(x) = 0$ for all $x \in [0,1]$

(b) If $f : [0,1] \to \mathbb{R}$ is a continuous function such that

$$\int_0^1 f(x)x^n dx = 0$$

for each integer number$n > 2024$, prove that $f(x) = 0$ for all $x \in [0,1]$

Proof.

(a) Since $f(x)$ is a continuous function on compact set $[0,1]$, then $\exists\, p_n(x)$ a sequence of polynomial such that $p_n(x) \rightrightarrows f(x)$ as $n \to \infty$.
By the property given above,

$$\int_0^1 f(x)p_n(x)dx = 0, \qquad \forall n \in \mathbb{N}$$

as $n \to \infty$,

$$\Rightarrow \int_0^1 f^2(x)dx = 0$$

Suppose $|f(x_0)| > 0$ for some $x_0 \in [0,1]$, Since$|f(x)|$ is continuous on [0, 1], $\exists\, \delta > 0$ such that $|f(x)| > 0$, if $|x - x_0| < \delta$.
Therefore,

$$\int_0^1 f^2(x)dx \geq \int_{x_0-\delta/2}^{x_0+\delta/2} f^2(x)dx = \delta \inf_{|x-x_0|\leq\delta/2}[f^2(x)] \geq 0 \rightarrow\!\leftarrow$$

Hence $|f(x)| = 0, \ \forall x \in [0,1]$, it follows that $f(x) = 0, \ \forall x \in [0,1]$.

(b) Let $N = 2024$. Consider $g(x) = x^N f(x)$ on $[0,1]$, Then since $g(x)$ is continuous on $[0,1]$ and $\int_0^1 g(x)x^n dx = 0, \ \forall x \in [0,1]$, By Problem 2.(a), $g(x) = 0, \ \ x \in [0,1] \Rightarrow x^N f(x) = 0, \ \ x \in [0,1] \Rightarrow f(x) = 0, \ \ x \in (0,1]$,Since $f(x)$ is continuous, $f(x) = 0, \ \ x \in [0,1]$.

$\square$

Problem 3. Please answer the following questions.

(a) Show that the derivative of $B_{n+1}f$ is

$$(B_{n+1}f)' = \sum_{k=0}^{n} \binom{n}{k} \frac{f(\frac{k+1}{n+1}) - f(\frac{k}{n+1})}{\frac{1}{n+1}} x^k(1-x)^{n-k}$$

(b) If $f :\to [0,1]$ has a continuous first derivative on $[0,1]$,use the Mean Value Theorem and the uniform continuity to show that

$$\lim_{n\to\infty} ||(B_n f)' - f'||_\infty = 0$$

Proof. (a)

$$B_{n+1}f = \sum_{k=0}^{n+1} \binom{n+1}{k} f(\frac{k}{n+1}) x^K(1-x)^{n+1-k}$$

$$\begin{aligned}
\Rightarrow (B_{n+1}f)' &= \sum_{k=0}^{n+1} \binom{n+1}{k} f(\frac{k}{n+1})[kx^{k-1}(1-x)^{n+1-k} - (n+1-k)x^k(1-x)^{n-k}] \\
&= \sum_{k=1}^{n+1} \frac{(n+1)!}{(k-1)!(n+1-k)!} f(\frac{k}{n+1}) x^{k-1}(1-x)^{n+1-k} - \sum_{k=0}^{n} \frac{(n+1)!}{k!(n-k)!} f(\frac{k}{n+1}) x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \frac{(n+1)!}{k!(n-k)!} f(\frac{k+1}{n+1}) x^k (1-x)^{n-k} - \sum_{k=0}^{n} \frac{(n+1)!}{k!(n-k)!} f(\frac{k}{n+1}) x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{f(\frac{k+1}{n+1}) - f(\frac{k}{n+1})}{\frac{1}{n+1}} x^k (1-x)^{n-k}
\end{aligned}$$

(b) Suppose $f' \in C^1[0,1]$, let $M = \sup_{x\in[0,1]} |f'(x)|$, By Mean Value Theorem,

$$\begin{aligned}
(B_{n+1}f)' &= \sum_{k=0}^{n} \binom{n}{k} \frac{f(\frac{k+1}{n+1}) - f(\frac{k}{n+1})}{\frac{1}{n+1}} x^k(1-x)^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} f'(x_k) x^k (1-x)^{n-k}
\end{aligned}$$

where $\frac{k}{n+1} < x_k < \frac{k+1}{n+1}$.
Since $f'(x)$ is uniform continuous on $[0,1]$, given $\epsilon > 0$, $\exists\, \delta > 0$ such that $|f'(x) - f'(y)| < \epsilon/2$, if $|x-y| < \delta$, $\forall x \in [0,1]$, then

$$\begin{aligned}
|(B_n f)' - f'| &= |\sum_{k=0}^{n} \binom{n}{k} [f'(x_k) - f(x)] x^k (1-x)^{n-k}| \\
&\le \sum_{|x_m - x| < \delta} |f(x) - f(x_k)| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{|x_m - x| \ge \delta} |f(x) - f(x_k)| \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \epsilon/2 + \sum_{|x_m - x| \ge \delta} 2M \binom{n}{k} x^k (1-x)^{n-k} \\
&\le \epsilon/2 + 2M \sum_{k=0}^{n+1} \frac{(\frac{k}{n+1} - x)^2}{(\delta - 1/n+1)^2} \binom{n+1}{k} x^k (1-x)^{n+1-k} \frac{1}{(n+1)(1-x)}
\end{aligned}$$

Since $x \geq \delta + x_k \geq \frac{k}{n+1} + \delta$ and $x \leq x_k - \delta \leq \frac{k+1}{n+1} - \delta$ then $|x - \frac{k}{n+1}| \geq \delta - \frac{1}{n+1}$.
Then

$$|(B_n f)' - f'| \leq \epsilon/2 + \frac{2M(n+1)x(1-x)}{(n+1)^2(\delta - \frac{1}{n+1})^2(n+1)(1-x)}$$
$$= \epsilon/2 + \frac{2Mx}{(n+1)^2(\delta - \frac{1}{n+1})^2} \leq \epsilon/2 + \frac{2M}{(n+1)^2(\delta - \frac{1}{2})^2} \leq \epsilon/2 + \epsilon/2 = \epsilon$$

, if $n > \sqrt{\frac{4M}{\epsilon(\delta - 1/2)^2}}$

Therefore, $\lim_{n\to\infty} ||(B_n f)' - f'||_\infty = 0$.

$\square$

Problem 4.. Let $f : \mathbb{R} \to \mathbb{R}$ be uniform limit of a polynomial. Prove that f is a polynomial.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ and $p_n \rightrightarrows f$ as $n \to \infty$ on $\mathbb{R}$, By Cauchy criterion, there exist N, such that $sup_{x\in\mathbb{R}}|p_m(x) - p_n(x)| < \epsilon$,if $m, n > N$, notice that $p_m(x) - p_n(x)$ is a polynomial, therefore if $p_m(x) - p_n(x)$ has degree at least 1, then $|p_m(x) - p_n(x)| \to \infty$ as $x \to \infty$ $\rightarrow\!\leftarrow$.
Hence $p_m(x) - p_n(x) \in \mathbb{R}$. In particular, let $p_{n+1}(x) - p_n(x) = a_n$, then

$$p_n(x) = p_0 + \sum_{k=0}^{n-1} a_k$$

Note that $|p_m(x) - p_n(x)| = |\sum_{k=n}^{m-1} a_k| < \epsilon$, if $m, n > N \Rightarrow \sum_{k=0}^{\infty} a_k$ converges. hence

$$f(x) = \lim_{n\to\infty} p_n(x) = p_0(x) + \sum_{k=0}^{\infty} a_k$$

Then since $p_0(x)$ is a polynomial, it follows that $f(x)$ is a polynomial.

$\square$