# Homework 3

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Introduction to Analysis II

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**Problem 1.** Find the radius and interval of convergence of the following power series.

1. $\sum_{n=3}^{\infty} \frac{x^n}{log(n)}$
2. $\sum_{n=1}^{\infty} \frac{n!x^n}{n^n}$
3. $\sum_{n=0}^{\infty} n!x^n$

Let the convergence radius be $R$.

1. $\lim_{n\to\infty} \frac{log(n+1)}{log(n)} = 1 \Rightarrow R = 1$
2. $\lim_{n\to\infty} \frac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = 1/e \Rightarrow R = e$
3. $\lim_{n\to\infty} \frac{(n+1)!}{n!} = +\infty \Rightarrow R = 0$

**Problem 2.**

(1) Show that if $\sum_{n=1}^{\infty} a_n = A\ (C,1)$ and $\sum_{n=1}^{\infty} n^2 a_n < \infty$ exist, then the series converges to $A$ as usual sense.

(2) Show that if $\sum_{n=1}^{\infty} a_n = A\ (C,1)$ and $\sum_{n=1}^{\infty} n a_n^2 < \infty$ exist, then the series converges to $A$ as usual sense.

*Proof.* (1) Let $s_n$ be the nth partial sum of the series, and let $\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k$, and since $\sum_{k=1}^{n} a_k$ is Cesaro summable, suppose $\sigma_n \to A$ as $n \to \infty$. Since $\sum_{n=1}^{\infty} n^2 a_n < \infty \Rightarrow (n-1)a_n \to 0$ as $n \to \infty$. Given $\epsilon > 0$, $na_n < \epsilon/2$, for some $N \in \mathbb{N}$. Choose $N' > \max\{\frac{2}{3}(N+1), \frac{2}{\epsilon}\sum_{k=1}^{N-1}(k-1)|a_k|\}$ Since

$$\begin{aligned}
|s_n - \sigma_n| = \left|\frac{\sum_{k=1}^{n}(s_n - s_k)}{n}\right| &= \Big|\frac{1}{n}\sum_{k=1}^{n}(k-1)a_k\Big| \le \sum_{k=1}^{N-1}\frac{k-1}{n}|a_k| + \sum_{k=N}^{n}\frac{k-1}{n}|a_k| \\
&\le \sum_{k=1}^{N-1}\frac{k-1}{n}|a_k| + \frac{(N-n+1)}{n}\frac{\epsilon}{2} \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

if $n > N'$. Hence $s_n = (s_n - \sigma_n) + \sigma_n \to 0 + A = A$.

(2) Let $\sigma_n \to A$ as $n \to \infty$, and $\sum_{k=1}^{\infty} ka_k^2 = M$.

$$s_n - \sigma_n = \frac{\sum_{k=1}^{n}(s_n - s_k)}{n} = \frac{1}{n}\sum_{k=1}^{n}(k-1)a_k = \frac{1}{n}\sum_{k=1}^{n} ka_k - \frac{1}{n}s_n$$
$$\Rightarrow (1+\frac{1}{n})s_n - \sigma_n = \sum_{k=1}^{n}\frac{k}{n}a_k$$
$$\Rightarrow |(1+\frac{1}{n})s_n - \sigma_n|^2 = (\sum_{k=1}^{n}\frac{k}{n}a_k)^2 \le (\sum_{k=1}^{n}\frac{1}{n^2})(\sum_{k=1}^{n}k^2a_k^2) = \frac{1}{n}\sum_{k=1}^{n}k^2a_k^2 \qquad ...(1)$$

Let $s'_n = \sum\limits_{k=1}^{n} ka_k^2$ and $\sigma'_n = \frac{1}{n}\sum\limits_{k=1}^{n} s'_k$

$$\sigma'_n = \frac{1}{n}\sum_{k=1}^{n}(n+1-k)ka_k^2 = (1+\frac{1}{n})\sum_{k=1}^{n}ka_k^2 - \frac{1}{n}\sum_{k=1}^{n}k^2a_k^2 = (1+\frac{1}{n})s'_n - \frac{1}{n}\sum_{k=1}^{n}k^2a_k^2$$
$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}k^2a_k^2 = \lim_{n\to\infty}(1+\frac{1}{n})s'_n - \sigma'_n = M - M = 0$$

Hence $\lim\limits_{n\to\infty}|(1+\frac{1}{n})s_n - \sigma_n|^2 = 0 \Rightarrow \lim\limits_{n\to\infty}(1+\frac{1}{n})s_n - \sigma_n = 0 \Rightarrow \lim\limits_{n\to\infty} s_n = A$

□

**Problem 3.** Let $a_n = \begin{cases} 1 & \text{if } n = 3k+1 \\ 0 & \text{if } n = 3k+2 \\ -1 & \text{if } n = 3k \end{cases}$. Find the $(C,1)$ and the Abel sum of $\sum\limits_{n=1}^{\infty} a_n$.

*Proof.* 1. Let $s_n$ be the partial sum of the sequence $\{a_n\}$, then $s_n = \begin{cases} 1 & \text{if } n \neq 3k \\ 0 & \text{if } n = 3k \end{cases}$,

$$\sum_{n=0}^{m} s_n = \begin{cases} 2k+1 & \text{if } m = 3k+1 \\ 2k+2 & \text{if } m = 3k+2 \\ 2k & \text{if } m = 3k \end{cases}$$

. Since $\limsup\limits_{m\to\infty}\frac{1}{m}\sum\limits_{n=0}^{m} s_n = \lim\limits_{k\to\infty}\frac{2k+2}{3k+2} = \frac{2}{3}$ and that $\liminf\limits_{m\to\infty}\frac{1}{m}\sum\limits_{n=0}^{m} s_n = \lim\limits_{k\to\infty}\frac{2k}{3k} = \frac{2}{3}$, then $\lim\limits_{m\to\infty}\frac{1}{m}\sum\limits_{n=0}^{m} s_n = \frac{2}{3}$. Hence $\sum_{n=1}^{\infty} a_n = \frac{2}{3}$ $(C,1)$.

2.
$$\sum_{n=1}^{\infty} a_n x^n = \sum_{k=0}^{\infty} x^{3k+1} - x^{3k+3} = \frac{x}{1-x^3} - \frac{x^3}{1-x^3} = \frac{x(1+x)}{1+x+x^2}, \qquad x \in [0,1)$$

then $\lim\limits_{x\to 1^-}\frac{x(1+x)}{1+x+x^2} = \frac{2}{3}$. Hence Abel sum is $\frac{2}{3}$.

□

**Problem 4.** (Blancmange function)
Construct the function $g(x)$ by letting $g(x) = |x|$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and extending $g$ so that it becomes periodic on $\mathbb{R}$. Define

$$f(x) = \sum_{k=0}^{\infty}\frac{g(4^k x)}{4^k}$$

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(a) Use Weierstrass-M test to show that $f(x)$ is continuous on $\mathbb{R}$.

(b) Show that $f$ is differentiable at no point.

*Proof.*

(a) Since $|\frac{g(4^k x)}{4^k}| \leq \frac{1}{2 \cdot 4^k}$, and $\sum\limits_{k=0}^{\infty} \frac{1}{2 \cdot 4^k}$ converges, by Weierstass M-test, $\sum_{k=1}^{n} \frac{g(4^k x)}{4^k}$ converges uniformly to $f(x)$ and since $\frac{g(4^k x)}{4^k}$ is continuous on $\mathbb{R}$, $f(x)$ is continuous on $\mathbb{R}$.

(b) Fix a real number $x$ and a positive number $m$.
Put
$$\delta_m = \pm \frac{1}{4^{m+1}}$$
.The sign of $\delta_m$ is chosen so that there is no integer contained in $4^m(x + \delta_m)$ and $4^m x$.
Define
$$\gamma_k = \frac{g(4^k(x + \delta_m)) - g(4^k x)}{\delta_m}$$
Then for $k > m$, $4^k \delta_m$ is an integer, therefore $\gamma_k = 0$. When $0 \leq k \leq m$, $|\gamma_k| = 4^k$, then
$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| (mod\ 2) \equiv \left| \sum_{k=0}^{m} (\frac{1}{4^k}) \gamma_k \right| (mod\ 2) \equiv (m+1)\ (mod\ 2)$$
Hence $\frac{f(x+\delta_m)-f(x)}{\delta_m}$ oscillates between odd numbers and even numbers as $m \to \infty$. Therefore $\lim\limits_{m \to \infty} \frac{f(x+\delta_m)-f(x)}{\delta_m}$ does not exist, and since $x$ is arbitrary, we conclude that $f$ is differentiable at no point.

□