Homework 12

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Introduction to Analysis II

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Problem 1. Prove the following properties of exterior measure.

(a) If $E = E_1 \cup E_2$, then $m_*(E) \leq m_*(E_1) + m_*(E_1)$

(b) If $E_2 = E_1 \cup E_2$ with $d(E_1, E_2) = \inf\{||x-y||_{\mathbb{R}^n} : x \in E_1, y \in E_2\} > 0$. Then $m_*(E) = m(E_1) + m_*(E_2)$.

Proof.

1. Given $\epsilon > 0$, there exist countable open elementary covering $\{A_n\}$ and $\{B_n\}$ of E_1, E_2 such that $A = \bigcup_{i=1}^{\infty} A_n \subset E_1$, $\sum_{n=1}^{\infty} m(A_n) \leq m_*(E_1) + \epsilon/2$ and $B = \bigcup_{i=1}^{\infty} B_n \subset E_2$, $\sum_{n=1}^{\infty} m(B_n) \leq m_*(E_2) + \epsilon/2$. Since $E = E_1 \cup E_2$, then union of both $\{A_n\}$ and $\{B_n\}$ forms countable a open elementary covering of E, then $m_*(E) \leq \sum_{i=1}^{\infty}(m(A_i) + m(B_i)) \leq m_*(E_1) + m_*(E_2) + \epsilon$. Since ϵ is arbitrary, we have $m_*(E) \leq m_*(E_1) + m_*(E_2)$.

2. Given $\epsilon > 0$, there exist disjoint countable elementary covering $\{V_n\}$ of E such that $m_*(E) + \epsilon > \sum_{i=1}^{\infty} m(V_i)$, since $V_i \in \mathscr{E}$, then V_i is finitely unioned disjoint intervals, we can then partition each interval to finite interval such that the diameter of the intervals are less than $d(E_1, E_2)$. Let $V = \bigcup_{i=1}^{\infty} V_i = \bigcup_{i=1}^{\infty} I_i$, where $I_i \cap I_j = \emptyset$ and $diam(I_i) < d(E_1, E_2)$, then for each i, either $I_i \cap E_1 \neq \emptyset$ or $I_i \cap E_2 \neq \emptyset$, we simply discard the case that I_i contains no elements of E. Let $A = \{i \mid I_i \cap E_1 \neq \emptyset\}$ and $B = \{i \mid I_i \cap E_2 \neq \emptyset\}$. Then $V = (\bigcup_{i\in A} I_i) \cup (\bigcup_{i \in B} I_i)$ and since $(\bigcup_{i\in A} I_i)$ is an countable elementary covering of E_i then $m_*(E_1) \leq \sum_{i\in A} m(I_i)$ and $m_*(E_2) \leq \sum_{i \in B} m(I_i)$, then $m_*(E_1) + m_*(E_2) \leq \sum_{i=1}^{\infty} m(I_i) = \sum_{i=1}^{n} m(V_i) \leq m_*(E) + \epsilon$. Since ϵ is arbitrary, then $m_*(E) \geq m_*(E_1) + m_*(E_2)$. Combine with the previous problem, $m_*(E) = m_*(E_1) + m_*(E_2)$.

□

Problem 2. Let $f : D \subset \mathbb{R} \to \mathbb{R}$. Define the graph of f, $G_f = \{(x, f(x)) \in \mathbb{R}^2 : x \in D\}$.

(a) Let $D = [a, b]$, $f \in C(D)$, show that G_f has measure zero in $\mathbb{R}^2$.

(b) Let $f \in C(\mathbb{R})$, show that G_f has measure zero in $\mathbb{R}^2$

Proof.

(a) Given $\epsilon > 0$, since f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$, then there exist some $\delta_1 > 0$, such that $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$, if $|x - y| < \delta_1$, choose $\delta = \min\{\delta_1, (b-a)/2\}$. Let $\{B_\delta(x)\}$

be collection of open balls that center at $x \in [a,b]$ with radius δ. Since $[a,b]$ is compact, there exist a finite subcover $\{B_\delta(x_i)\}$ with some $r \in \mathbb{N}$ such that $[a,b] \subset \bigcup_{i=1}^r B_\delta(x_i)$, let $V_i = B_\delta(x_i) - \bigcup_{j=1}^{i-1} B_\delta(x_j)$, then $[a,b] \subset \bigcup_{i=1}^r V_i$, $V_i \subset B_\delta(x_i)$ thus $diam(V_i) < 2\delta$ and $V_i \cap V_j = \emptyset$, where $i,j = 1,..,r$. Then $(x, f(x)) \in V_i \times [m_i, M_i]$ where $m_i = \inf_{x \in V_i} f(x)$ and $M_i = \sup_{x \in V_i} f(x)$. $m_*(G_f) \leq \sum_{i=1}^r m(V_i \times [m_i, M_i]) \leq (2\delta + b - a) \cdot \frac{\epsilon}{b-a} \leq 2(b-a) \cdot \frac{\epsilon}{2(b-a)} = \epsilon$ Since ϵ is arbitrary, $m_*(G_f) = 0$.

(b) Define $G_{f,i} = \{(x, f(x)) \in \mathbb{R}^2 : x \in [i, i+1]\}$. Then $G_f = \bigcup\limits_{i=-\infty}^{\infty} G_{f,i}$, since $m_*(G_f) \leq \sum_{i=-\infty}^{\infty} m_*(G_{f,i})$, since $m_*(G_{f,i}) = 0$, where $i = 1, ..., n$, then $m_*(G_f) = 0$.

□

Problem 3.

(a) If $A \subset [a,b]$ has measure zero in $\mathbb{R}$, prove that $[a,b]/A$ does not have measure zero in $\mathbb{R}^2$.

(b) Let $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ be obtain from $F_0 = [0,1]$ by removing the middle third. Reapt, obtaining

$$F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$$

In general, F_n is a union of intervals and F_{n+1} is obtained by removing the middle third of these intervals. Let $C = \bigcap_{i=1}^{\infty} F_n$, the Cantor set. Show that the Cantor set has measure zero in $\mathbb{R}$. (Note that the Cantor set is uncountable.)

Proof.

(a) Suppose $m_*([a,b] \setminus A) = 0$. Since $[a,b] \in \mathscr{E}$, we see that $m_*([a,b]) = m([a,b]) = b - a \neq 0$ if $b \neq a$. Since $A \cup [a,b] \setminus A = [a,b]$, then $b - a = m_*([a,b]) \leq m_*(A) + m_*([a,b] \setminus A) = 0 \Rightarrow b - a = 0 \rightarrow\!\leftarrow$.

(b) From construction we see that $m_*(F_n) = m(F_n) = (\frac{2}{3})^n$. Since $C \subset F_n$ for all $n \in \mathbb{N}$. Since $m_*(C) \leq m_*(F_n) = (\frac{2}{3})^n$, taking $n \to \infty$ we see that $m_*(C) = 0$.

□

Problem 4. Determine whether f is integrable on $E = [0,1]^2$. Evaluate $\int_E f$ is exists.

(a) $f(x,y) = \begin{cases} 1, \text{ if } (x,y) = (\frac{p}{2^n}, \frac{q}{2^n}), 0 < q, p < 2^n, p, q, r \in \mathbb{N} \\ 0, \text{otherwise} \end{cases}$

(b) $f(x,y) = \begin{cases} 0, \text{ when } x = 0, \text{ or when } x \text{ or } y \text{ is irrational} \\ \frac{1}{q}, \text{ when } x, y \in \mathbb{Q}, x = \frac{p}{q} \text{ in reduce form} \end{cases}$

(c) $f(x,y) = \begin{cases} 1, \text{ if } x \text{ is rational} \\ 0, \text{ if } x \text{ is irrational} \end{cases}$

Proof.

1. Let $P_1 = \{(x,y) = (\frac{p}{2^n}, \frac{q}{2^n}) \mid p, q, n \in \mathbb{N}\}$, since p, q, n are all integers, P_1 forms a countable set, hence the measure $m_*(P_1) = 0$. Let $S_a = \{(x,y) \mid f(x,y) > a\}$, then $S_a = \emptyset$ for all $a > 1$, $S_a = P$ for all $1 \geq a > 0$ and $S_a = E$ for all $a \leq 0$, since P, E is measurable, f is a measurable function. Since

$$\int_E f = \int_{P_1} f + \int_{E \setminus P_1} f$$

and $m_*(P_1) = 0$, $\int_{P_1} f = 0$. Since $f = 0$ for all $(x, y) \in E \setminus P_1$, $\int_{E \setminus P_1} f = 0$. Therefore,

$$\int_E f = 0 < +\infty$$

2. Let $P_2 = \{(x, y) \mid x, y \in \mathbb{Q}, x = \frac{p}{q} \text{ in reduce form}\}$. Since $\mathbb{Q}$ is countable, P_2 is a countable subset of E, hence $m_*(P_2) = 0$. Let $S_a = \{(x, y) \mid f(x, y) > a\}$ and $W_q = \{(x, y) \mid x, y \in \mathbb{Q}, x = \frac{p}{q}, \text{where } q \text{ is fixed}\}$, then

$$S_a = \begin{cases} \emptyset, & \text{if } a > 1 \\ \bigcup_{i=1}^{q} W_i, & \text{if } \frac{1}{q+1} < a \leq \frac{1}{q}, q \in \mathbb{N} \\ \bigcup_{i=1}^{\infty} W_i, & \text{if } a \leq 0 \end{cases}$$

, since all $W_i \subset P_2$ is measurable set with $m_*(W_i)$, it follows that S_a is a measurable set and hence f is measurable function, and thus $\int_{P_2} f = 0$. Since

$$\int_E f = \int_{P_2} f + \int_{E \setminus P_2} f$$

and $f(x, y) = 0$ for all $(x, y) \in E \setminus P_2$, $\int_{E \setminus P_2} f = 0$. Therefore,

$$\int_E f = 0 < +\infty$$

3. Let $R = \{(x, y) \mid x \in \mathbb{Q}\}$, and let $R_q = \{(q, y) \mid \text{where } q \text{ is fixed rational number}\}$. Then $R = \bigcup_{q \in Q} R_q$. Given $\epsilon > 0$, define $I_{\delta,q} = [q - \delta, q + \delta] \times [0, 1]$, then $I_{\delta,q}$ forms an open covering of R_q, for arbitrary $\delta > 0$, hence $m_*(R_q) < \delta$. Since $\mathbb{Q}$ is a countable set, there exists a function $\phi : \mathbb{N} \to \mathbb{Q}$ such that ϕ is one-to-one. Hence $R = \bigcup_{i=1}^{\infty} R_{\phi(i)}$. Now since $m_*(R_{\phi(i)}) < \frac{\epsilon}{2^{i-1}}$, it follows that $m_*(R) < \epsilon$. Since ϵ is arbitrary, $m_*(R) = 0$. Let $S_a = \{(x, y) \mid f(x, y) > a\}$, then

$$S_a = \begin{cases} \emptyset, & \text{if } a > 1 \\ R, & \text{if } 1 \geq a > 0 \\ E, & \text{if } a \leq 0 \end{cases}$$

Since both R, E is measurable, f is a measurable function. Therefore

$$\int_E f = \int_R f + \int_{E \setminus R} f = \int_{E \setminus R} f = 0 < +\infty$$

The last equality follows directly from that $f = 0$ for all $(x, y) \in E \setminus R$.

$\square$