Homework 6

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Introduction to Analysis II

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Problem 1. Let $\{w_1, w_2, ..., w_k\} \in [a, b]$, $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$ and continuous on $[a, b] \setminus \{w_1, w_2, ..., w_k\}$. Show that f is Riemann integrable.

Proof.
Since $[a, b]$ is compact, there exists $\delta_1 > 0$, such that

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$$

, if $|x - y| < \delta_1$, for $x, y \in [a, b] \setminus W$.

Let $M = \sup\limits_{x \in [a,b]} f(x)$ and $W = \{w_1, w_2, ..., w_k\}$. Given $\epsilon > 0$, choose partition $P = \{a = x_0 < x_1 < ... < x_n = b\}$ on $[a, b]$ such that $||P|| < \min\{\frac{\epsilon}{4kM}, \delta_1\}$ and $W \cap P = \emptyset$.
Define $U(P, f) = \sum_{i=1}^n M_i \Delta x_i$ and $L(P, f) = \sum_{i=1}^n m_i \Delta x_i$, where $M_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$, $m_i = \sup\limits_{x \in [x_{i-1}, x_i]} f(x)$ and $\Delta x_i = x_i - x_{i-1}$. Then

$$\begin{aligned}
U(P, f) - L(P, f) = \sum_{i=1}^n (M_i - m_i)\Delta x_i &= \sum_{[x_{i-1}, x_i] \cap W = \emptyset} (M_i - m_i)\Delta x_i + \sum_{[x_{i-1}, x_i] \cap W \neq \emptyset} (M_i - m_i)\Delta x_i \\
&\leq \sum_{[x_{i-1}, x_i] \cap W = \emptyset} \frac{\epsilon}{2(b-a)} \cdot \Delta x_i + \sum_{[x_{i-1}, x_i] \cap W \neq \emptyset} 2M \cdot \frac{\epsilon}{4kM} \\
&< \frac{\epsilon}{2(b-a)} \cdot (b-a) + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

□

Problem 2.

(a) Suppose $f : [a, b] \to \mathbb{R}$ is Riemann integrable with $m \leq f(x) \leq M$, for all $x \in [a, b]$, and $\phi : [m, M] \to \mathbb{R}$ is continuous. Show that $\phi \circ f$ is Riemann integrable on $[a, b]$.

(b) Show that f is Riemann integrable on $[a, b]$ implies $|f|$ is Riemann integrable on $[a, b]$.

(c) Given an example that $|f|$ is Riemann integrable on $[a, b]$ but f is not Riemann integrable on $[a, b]$.

Proof. (a) Given $\epsilon > 0$, since $[m, M]$ is compact, there exists $\delta < \epsilon$ such that

$$|\phi(x) - \phi(y)| < \epsilon$$

, if $|x-y|<\delta$ and $x, y \in [m, M]$. Since $f \in \mathscr{R}[a,b]$, there exists partition $P = \{a = x_0 < x_1 < ... < x_n\}$ such that

$$U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \delta^2$$

where $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$, $m_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$ and $\Delta x_i = x_i - x_{i-1}$ and let $M_i^* = \sup\limits_{x\in[x_{i-1},x_i]} \phi(f(x))$, $m_i^* = \sup\limits_{x\in[x_{i-1},x_i]} \phi(f(x))$. Divide the number $i = 1, 2, 3, ..., n$ into two class $A = \{i \mid M_i - m_i < \delta\}$ and $B = \{i \mid M_i - m_i \geq \delta\}$.
For $i \in A$, $\sum_{i\in A}(M_i^* - m_i^*)\Delta x_i < \epsilon \cdot (b-a)$.
For $i \in B$,

$$\sum_{i\in B}\delta\Delta x_i \leq \sum_{i\in B}(M_i - m_i)\Delta x_i < \delta^2$$

. so that $\sum\limits_{i\in B}\Delta x_i < \delta$. Let $M^* = \sup_{x\in[m,M]}\phi(x)$. Hence,

$$\begin{aligned} U(P,\phi\circ f) - L(P,\phi\circ f) = \sum_{i\in A}(M_i^* - m_i^*)\Delta x_i + \sum_{i\in A}(M_i^* - m_i^*)\Delta x_i &\leq \epsilon\cdot(b-a) + 2M^*\cdot\delta \\ &< \epsilon\cdot(b-a+2M^*) \end{aligned}$$

Since ϵ is arbitrary, $\phi \circ f$ is Riemann integrable on $[a,b]$.

(b) Since $\phi(x) = |x|$ is a continuous function and $f \in \mathscr{R}[a,b]$, by (a), $|f| \in \mathscr{R}[a,b]$.

(c) Consider the Dirichlet function f on $[a,b]$,

$$f(x) = \begin{cases} 1,\ x \in \mathbb{Q} \\ -1,\ x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

Then $U(P,f) = (b-a)$ and $L(P,f) = (a-b)$, for any partition P on $[a,b]$, where as $U(P,|f|) = L(P,|f|) = (b-a)$, for all partition P on $[a,b]$. And Therefore, $|f|$ is Riemann integrable on $[a,b]$ but f is not Riemann integrable on $[a,b]$.

□

Problem 3.
Let $f : [0,1] \to \mathbb{R}$ be Riemann integrable and suppose for every $0 \leq a < b \leq 1$, there exist $c \in (a,b)$ such that $f(c) = 0$. Prove that $\int_0^1 f(x)dx = 0$.

Proof. Given $\epsilon > 0$, let $g = |f|$, Since $f \in \mathscr{R}[0,1]$, then $g \in \mathscr{R}[0,1]$, i.e. there exist partition P such that

$$U(P,g) - L(P,g) < \epsilon$$

. Because zero point of f is dense in $[0,1]$, therefore zero point of g is dense in $[0,1]$. Since the zero point of g is dense in $[0,1]$, then $L(P,g) = 0$ for any partition P on $[0,1]$. Then

$$\left|\int_0^1 f(x)dx\right| \leq \int_0^1 |f(x)|dx \leq U(P,g) < \epsilon$$

Since ϵ is arbitrary, $\int_0^1 f(x)dx = 0$. □

Problem 4. (Density Argument)Show that for all $f \in C[0,1]$ the identity

$$\lim_{n\to\infty}(n+1)\int_0^1 f(x)x^n dx = f(1)$$

holds.

Proof. (Method 1) Given $\epsilon > 0$, there exist $\delta > 0$, such that

$$|f(x) - f(1)| < \epsilon/2$$

, if $|x-1| < \delta$, where $x \in [0,1]$. Since

$$\lim_{n\to\infty}(n+1)(1-\delta)^{n+1} = 0$$

, then there exists $N \in \mathbb{N}$ such that if $n > N$,

$$(n+1)(1-\delta)^{n+1} < \frac{\epsilon}{4M}$$

, where $M = \sup_{x\in[0,1]} f(x)$. Then

$$\begin{aligned}
&\left|(n+1)\int_0^1 f(x)x^n dx - f(1)\right| \\
&\leq \left|(n+1)\int_0^1 [f(x)-f(1)]x^n dx\right| \\
&\leq (n+1)\int_0^{1-\delta} |f(x)-f(1)|(1-\delta)^n dx + (n+1)\int_{1-\delta}^1 \frac{\epsilon}{2}x^n dx \\
&\leq 2M(n+1)(1-\delta)^{n+1} + \frac{\epsilon}{2}[1-(1-\delta)^{n+1}] \\
&< 2M\cdot\frac{\epsilon}{4M} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

if $n > N$. Hence

$$\lim_{n\to\infty}(n+1)\int_0^1 f(x)x^n dx = f(1)$$

for all $f \in C[0,1]$.

(Method 2) Let $p(x) = \sum_{k=0}^m a_k x^k$,

$$\begin{aligned}
&\lim_{n\to\infty}(n+1)\int_0^1 p(x)x^n dx \\
&= \lim_{n\to\infty}(n+1)\int_0^1 \sum_{k=0}^m a_k x^{k+n} dx = \lim_{n\to\infty}(n+1)\sum_{k=0}^m \int_0^1 a_k x^{k+n} dx \\
&= \lim_{n\to\infty}(n+1)\sum_{k=0}^m a_k \frac{1}{n+k+1} dx = \sum_{k=0}^m a_k = p(1)
\end{aligned}$$

Given $\epsilon > 0$, by Weierstrass approximation theorem, there exist $p(x)$ polynomial such that $|f(x)-p(x)| < \epsilon/3$, for any $x \in [0,1]$. And by above limit, we see that

$$\left|(n+1)\int_0^1 p(x)x^n dx - p(1)\right| < \epsilon/3$$

, if $n > M$. Then

$$\begin{aligned}
&\left|(n+1)\int_0^1 f(x)x^n dx - f(1)\right| \\
&\leq \left|(n+1)\int_0^1 [f(x)-p(x)]x^n dx\right| + \left|(n+1)\int_0^1 p(x)x^n dx - p(1)\right| + |p(1)-f(1)| \\
&< (n+1)\int_0^1 \frac{\epsilon}{3}\cdot x^n dx + \epsilon/3 + \epsilon/3 = \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon
\end{aligned}$$

if $n > M$. Hence

$$\lim_{n\to\infty}(n+1)\int_0^1 f(x)x^n dx = f(1)$$

for all $f \in C[0,1]$. $\square$