Homework 10

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Introduction to Analysis II

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Problem 1.

(a) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^1 and $Df(x_0)$ has rank m. Show that there exist a whole neighborhood of $f(x_0)$ lying in the image of f.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is of class C^1 is one-to-one and $Df(x_0)$ is one-to-one. Show that f is one-to-one a neighborhood of x_0.

Proof. (a) Let $f(x_1, x_2, ..., x_n) = \begin{pmatrix} f_1(x_1, x_2, ..., x_n) \\ \vdots \\ f_m(x_1, x_2, ..., x_n) \end{pmatrix}$. Since $rank(Df(x_0)) = m$, we can select m of the column vector of $Df(x_0)$ such that the vector space it forms has dimension m, without loss of generality, suppose the first m column of $Df(x_0)$ are linearly independent. Let $g(x_1, x_2, ..., x_n) = \begin{pmatrix} f_1(x_1, x_2, ..., x_n) \\ \vdots \\ f_m(x_1, x_2, ..., x_n) \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix}$, then $Dg(x_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} & \frac{\partial f_1}{\partial x_{m+1}} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} & \frac{\partial f_m}{\partial x_{m+1}} & \cdots & \frac{\partial f_m}{\partial x_n} \\ 0 & & & 1 & & \\ & 0 & & & 1 & \\ & & \ddots & & & \ddots \\ & & 0 & & & 1 \end{pmatrix}$ then $det(Dg(x_0)) \neq 0$. since the product of the first m column is independent, and since $f(x_1, x_2, ..., x_n)$ is C^1, $g(x_1, x_2, ..., x_n)$ is C^1. By Inverse function theorem, there exist $U = B_\delta(x_0) \subset \mathbb{R}^n$ and $V = B_\epsilon(g(x_0)) \subset \mathbb{R}^n$ such that $f(U) = V$. Choose $y \in B_\epsilon(f(x_0))) \in \mathbb{R}^m$. Define $y' = \begin{pmatrix} y \\ x_{m+1} \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$, since $||f(x_0) - y|| = ||g(x_0) - y'|| < \epsilon$, then there exist $x^* \in U$ such that $g(x^*) = y'$. By comparing the component of the vector, we see that $y = f(x^*)$. Hence $y \in im(f)$. Therefore, f is an open mapping.

(b) Since $Df(x_0)$ is injective, $nullity(Df(x_0)) = 0$, by rank and nullity theorem $rank(Df(x_0)) = n \leq m$, without loss of generality suppose the first n column $Df(x_0)$ is linearly independent. let $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix}$, define $g(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$, then $det(Dg(x_0)) \neq 0$. By inverse function theorem, there exists $U = B_\delta(x_0) \subset \mathbb{R}^n$ and $V = B_\delta(g(x_0))$ such that g is one-to-one on U. Suppose $f(x) = f(y)$ where $x, y \in U$, then since $f(x) = \begin{pmatrix} g(x) \\ f_{n+1}(x) \\ f_{n+2}(x) \\ \vdots \\ f_m(x) \end{pmatrix}$, comparing the component, since $g(x) = g(y)$, it follows that $x = y$. Hence f is one-to-one on U. we are done.

□

Problem 2.
Use Inverse Function Theorem to determine whether the system

$$\begin{aligned} u(x,y,z) &= x + xyz \\ v(x,y,z) &= y + xy \\ w(x,y,z) &= z + 2x + 3z^2 \end{aligned}$$

can be solved for x, y, z in terms of u, v, w near $p = (0,0,0)$

Proof. Since $u(x,y,z), v(x,y,z), w(x,y,z)$ is polynomial of x, y, z then u, v, w is C^1. The derivative of the system $f(x,y,z) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$ is $\begin{pmatrix} 1+yz & xz & xy \\ y & 1+x & 0 \\ 2 & 0 & 1 \end{pmatrix}$, and thus $J_f(0,0,0) = 1 \neq 0$. By inverse function theorem, there exists a open set U contains $(0,0,0)$ and open set V that contains $f(0,0,0) = (0,0,0)$ such that $f : U \to V$ is an isomorphism. Given $y = f(x)$ and since $Df^{-1}(y) = \big[Df(x)\big]^{-1}$,

$$Df^{-1}(x,y,z) = \frac{1}{1+yz-2xy-x} \begin{pmatrix} 1+x & -y & -2(1+x) \\ -xz & 1+yz-2xy & 2xz \\ -xy(1+x) & xy^2 & 1+x+yz \end{pmatrix}$$

Notice that the expression is in terms of x, y, z not u, v, w, and is not possible to do substitution to eliminate other variables thus the system can not be solved for x, y, z in terms of u, v, w. □