The HOUSEHOLD solve the following Max problem (a)

$$\begin{cases} \max\limits_{\{C_t, N_t, B_t, M_t\}} U_t = E_t\left[\sum_{s=0}^{\infty} \beta^{t+s} \left(\frac{X_{t+s}^{1-\gamma} - 1}{1-\gamma} - \chi \frac{N_{t+s}^{1+\varphi}}{1+\varphi} \right) \right] \\ \text{s.t.} \quad P_t C_t + B_t + M_t \leq W_t N_t + Q_{t-1} B_{t-1} + M_{t-1} + P_t (TR_t + PR_t) \end{cases}$$

Where we have $X_t = \left[(1-\theta) C_t^{1-\nu} + \theta \left(\frac{M_t}{P_t} \right)^{1-\nu} \right]^{\frac{1}{1-\nu}}$

LAGRANGEAN : $\mathcal{L} = U_t + E_t \left[\sum_{t=0}^{\infty} \beta^t \lambda_t \left[W_t N_t + Q_{t-1} B_{t-1} + M_{t-1} + P_t (TR_t + PR_t) - P_t C_t - B_t - M_t \right] \right]$

FOC :

$[C_t]$ $\quad \cancel{\beta^t} X_t^{\nu-\gamma} (1-\theta) C_t^{-\nu} = \cancel{\beta^t} \lambda_t P_t$

$[N_t]$ $\quad \chi N_t^{\varphi} = \lambda_t W_t$

$[B_t]$ $\quad E_t [\lambda_{t+1} \beta^{\cancel{t}+1} Q_t] = \lambda_t \cancel{\beta^t}$

$[M_t]$ $\quad \cancel{\beta^t} X_t^{\nu-\gamma} \theta \left(\frac{M_t}{P_t} \right)^{-\nu} \frac{1}{P_t} - \lambda_t \cancel{\beta^t} + E_t [\lambda_{t+1} \beta^{\cancel{t}+1}] = 0$

THEN STATIC LABOR SUPPLY : $\quad X_t^{\nu-\gamma} (1-\theta) C_t^{-\nu} \frac{W_t}{P_t} = \chi N_t^{\varphi}$

DYNAMIC EULER EQUATION : $\quad \frac{X_t^{\nu-\gamma} C_t^{-\nu}}{P_t} = \beta Q_t E_t \left[\frac{X_{t+1}^{\nu-\gamma} C_{t+1}^{-\nu}}{P_{t+1}} \right]$

DYNAMIC FOC WRT MONEY SUPPLY : $\quad \frac{X_t^{\nu-\gamma} C_t^{-\nu}}{P_t} = \underbrace{X_t^{\nu-\gamma} \frac{\theta}{1-\theta} \left(\frac{M_t}{P_t} \right)^{-\nu} \frac{1}{P_t}}_{\text{MARGINAL VALUE OF LIQUIDITY SERVICE}} + \beta E_t \left[\frac{X_{t+1}^{\nu-\gamma} C_{t+1}^{-\nu}}{P_{t+1}} \right]$

(b)

When is consumption independent from money ? Let's consider the government BC :

$B_t + M_t = P_t TR_t + Q_{t-1} B_{t-1} + M_{t-1}$

$\frac{B_t}{P_t} + \frac{M_t}{P_t} = TR_t + \frac{Q_{t-1} B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t}$ IN REAL TERMS

Given the LOCALLY NON SATIATED Utility function, the household is going to exhaust the budget constraints. then

$P_t C_t + B_t + M_t = W_t N_t + Q_{t-1} B_{t-1} + M_{t-1} + P_t (TR_t + PR_t)$

$C_t = \frac{W_t}{P_t} N_t + \frac{Q_{t-1} B_{t-1}}{P_t} + TR_t + PR_t - \frac{M_{t-1}}{P_t} - TR_t - \frac{Q_{t-1} B_{t-1}}{P_t} - \frac{M_{t-1}}{P_t}$

$C_t = \frac{W_t}{P_t} N_t + PR_t = A_t N_t + PR_t \longrightarrow$ but W_t / P_t depends on X_t !

Remember from the static labour supply condition that

$$X_t^{\nu-\gamma}(1-\theta)C_t^{-\nu}\frac{W_t}{P_t} = \chi N_t^{\varphi} \implies A_t = \frac{W_t}{P_t} = \chi N_t^{\varphi}\left(X_t^{\nu-\gamma}(1-\theta)C_t^{-\nu}\right)^{-1}$$

Then if $\nu = \gamma$ the X_t terms goes away and consumption decision would not depend on Money, id est money is NEUTRAL.

STEADY STATE : c

- If $A_t = 1$ then $W^{ss} = P^{ss}$
- Market Clearing $C^{ss} = Y^{ss} = 1 \cdot N^{ss} \implies C^{ss} = N^{ss}$
- Static labor supply $1 = \chi C^{ss\,\varphi} X_t^{\gamma-\nu}(1-\theta)^{-1}C^{ss\,\nu} \implies \chi/(C^{ss})^{\nu+\varphi} = (1-\theta)(X^{ss})^{\nu-\gamma}$
- $C^{ss} = C_t = C_{t+1}$ and knowing that $Q_t\frac{P_t}{P_{t+1}} = R_{t+1}$ with $R_{t+1} = R^{ss}$ then the euler equation becomes :

$$\frac{X_t^{\nu-\gamma}C_t^{-\nu}}{P_t} = \beta Q_t E_t\left[\frac{X_{t+1}^{\nu-\gamma}C_{t+1}^{-\nu}}{P_{t+1}}\right] \implies E_t\left[Q_t\left(\frac{P_t}{P_{t+1}}\right)\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\right]\beta = 1$$

$$E_t[R^{ss}]\beta = 1 \implies R^{ss}\beta = 1$$

- From the MONEY DEMAND EQUATION :

$$1 = \beta E_t\left[\frac{P_t}{P_{t+1}}\left(\frac{X_{t+1}}{X_t}\right)^{\nu-\gamma}\left(\frac{C_{t+1}}{C_t}\right)^{-\nu}\right] + \frac{\theta}{1-\theta}C_t^{\nu}\left(\frac{M_t}{P_t}\right)^{-\nu}$$

$$1 = \frac{1}{Q_t} + \frac{\theta}{1-\theta}C_t^{\nu}\left(\frac{M_t}{P_t}\right)^{-\nu}$$

$$\left(\frac{M_t}{P_t}\right) = \left(1-\frac{1}{Q_t}\right)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}C_t$$ IN STEADY STATE $P_t = P_{t+1} = P^{ss}$ Then $R^{ss} = Q^{ss}$ $R^{ss} = 1/\beta$

$$\left(\frac{M^{ss}}{P^{ss}}\right) = (1-\beta)^{-\frac{1}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{\nu}}C^{ss}$$ $\longrightarrow P_t/P_{t+1} = \Pi_t$ but $\Pi^{ss} = P^{ss}/P^{ss} = 1$

- THEN $$X^{ss} = \left[(1-\theta)C^{ss\,1-\nu} + \theta(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}C^{ss}\right]^{\frac{1}{1-\nu}}$$

$$X^{ss} = \left[(1-\theta) + \theta(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{1}{1-\nu}}C^{ss}$$

- BUT $$(1-\theta)X^{ss\,\nu-\gamma} = \chi C^{ss\,\varphi+\nu}$$

$$(1-\theta)\left[(1-\theta) + \theta(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{\nu-\gamma}{1-\nu}}(C^{ss})^{\nu-\gamma} = \chi C^{ss\,\varphi+\nu}$$

$$C^{ss} = \left(\chi^{-1}(1-\theta)\left[(1-\theta) + \theta(1-\beta)^{-\frac{1-\nu}{\nu}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1-\nu}{\nu}}\right]^{\frac{1}{1-\nu}}\right)^{\frac{\nu-\gamma}{\varphi+\nu}}$$ THEN HAVING C^{ss} everything is in terms of parameters

(d) ALGORITHM : given $\{\chi, \beta, \theta, v, \gamma, \varphi\}$ I can obtain C^{ss}. Given β I can obtain R^{ss} and Q^{ss}. From C^{ss} I get X^{ss}, $(M/P)^{ss}$ and N^{ss}.

(e) Since $X_t = \left[(1-\theta)C_t^{1-v} + \theta\left(\frac{M_t}{P_t}\right)^{1-v}\right]^{\frac{1}{1-v}}$ θ = relative weights to consumption compared to real money demand, I would try to calibrate θ to reflect this share. If I have the data for C_t and the value v, I would solve for the value of θ given my equilibrium conditions evaluated at the steady state.

(f) $\left(\frac{M^{ss}}{P^{ss}}\right) = (1-\beta)^{-\frac{1}{v}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{v}} C^{ss}$ Knowing this relationship in steady state, I would

express $P^{ss} = M^{ss}\left[(1-\beta)^{-\frac{1}{v}}\left(\frac{\theta}{1-\theta}\right)^{\frac{1}{v}} C^{ss}\right]^{-1}$ s.t. $P^{ss} = f(M^{ss}; v, \beta, C^{ss}, \theta)$

Set P^{ss} and solve for the value of M^{ss} s.t. $P^{ss} = 1$ in steady state.

(g) LOG - LINEARIZATION : given a generic $\hat{\omega} \equiv \log \Omega_t - \log \Omega^{ss}$

[PR] $Y_t = A_t N_t$ by log diff : $\hat{y}_t = \hat{a}_t + \hat{n}_t$

[MC] $Y_t = C_t$ then : $\hat{y}_t = \hat{c}_t$

[MPL] $W_t / P_t = A_t$ then trivially $\hat{w}_t - \hat{p}_t = \hat{a}_t$

[LS] $X_t^{v-\gamma}(1-\theta)C_t^{-v}\frac{W_t}{P_t} = \chi N_t^{\varphi}$ BY LOG DIF : $\varphi \hat{n}_t = \hat{w}_t - \hat{p}_t + (v-\gamma)\hat{x}_t - v\hat{c}_t$

[MD] $M_t/P_t = C_t\left(\frac{1-\theta}{\theta}\left(1-\frac{1}{Q_t}\right)\right)^{-1/v}$

$M_t = g(z)$

$g'_z z \hat{z}_t = f'_p(\cdot) P \hat{p}_t + f'_c(\cdot) C \hat{c}_t + f'_q(\cdot) Q \hat{q}_t$

$M\hat{m}_t = C\left(\frac{\theta}{1-\theta}\right)^{1/v}\left(1-\frac{1}{Q}\right)^{-1/v} P\hat{p}_t + C\left(\frac{\theta}{1-\theta}\right)^{1/v}\left(1-\frac{1}{Q}\right)^{-1/v} C\hat{c}_t - \frac{1}{v}\frac{M}{(1-\frac{1}{Q})Q} Q_t$

$M\hat{m}_t = M\hat{p}_t + M\hat{c}_t - \frac{1}{v}\frac{M}{(Q-1)}\hat{q}_t$

$\hat{m}_t - \hat{p}_t = \hat{c}_t - \frac{1}{v}\frac{\beta}{\beta-1} q_t$ in steady state $1 = \beta Q$

[EE] $C_t^{-v} = \beta\left[Q_t \frac{P_t}{P_{t+1}}\left(\frac{X_{t+1}}{X_t}\right)^{v-\gamma} C_{t+1}^{-v}\right]$

simply log diff : $v(\hat{c}_{t+1} - \hat{c}_t) = \hat{q}_t + (\hat{p}_t - \hat{p}_{t+1}) + (v-\gamma)(\hat{x}_{t+1} - \hat{x}_t)$

[X_t] $\hat{x}_t = (1-\theta)\left(\frac{C}{X}\right)^{1-v}\hat{c}_t + \theta\left(\frac{M/P}{X}\right)^{1-v}(\hat{M}_t - \hat{P}_t)$

ENDOGENOUS $\{\hat m_t\}$; EXOGENOUS $\{\hat a_t \, ; \, \hat m_t = \varphi m_{t-1} + \varepsilon_t\}$

$$\hat y_t = \hat a_t + \hat m_t$$
$$\hat\omega_t - \hat p_t = \hat a_t$$
$$\hat\omega_t - \hat p_t = \varphi \hat m_t + v \hat c_t - (v-\gamma)\hat x_t$$
$$\hat y_t = \hat c_t$$
$$\hat m_t - \hat p_t = \hat c_t - \frac{1}{v}\frac{\beta}{1-\beta}\hat q_t \quad \star$$
$$v(\hat c_{t+1} - \hat c_t) = \hat q_t + (\hat p_t - \hat p_{t+1}) + (v-\gamma)(\hat x_{t+1} - \hat x_t)$$
$$\hat x_t = \underbrace{(1-\theta)\left(\frac{c}{x}\right)^{1-v}}_{B} \hat c_t + \underbrace{\theta\left(\frac{M/P}{x}\right)^{1-v}}_{A} (\hat M_t - \hat p_t) \quad \star$$
$$m_t = \varphi m_{t-1} + \varepsilon_t$$

- 1 $\hat y_t = \hat a_t + \hat m_t$
- 2 $\hat\omega_t - \hat p_t = \hat a_t$
- 3 $\hat c_t = y_t$

$$\hat\omega_t - \hat p_t = \varphi \hat m_t + v \hat c_t - (v-\gamma)\hat x_t$$

- 4 $\hat x_t = \frac{1}{v-\gamma}\left(\varphi \hat m_t + v\hat c_t - a_t\right)$
- 5 $\star$ $\hat p_t = \hat m_t + A^{-1}B c_t - A^{-1}\hat x_t$
- 6 $\star$ $\hat q_t = \left[\frac{v(1-\beta)}{\beta}\right]\left(\hat c_t - \hat m_t + \hat p_t\right)$

FINAL, JUST TO CHECK

- $v(\hat c_{t+1} - \hat c_t) = \hat q_t + (\hat p_t - \hat p_{t+1}) + (v-\gamma)(\hat x_{t+1} - \hat x_t)$

SINCE $\varepsilon_t = e1$

$$\overbrace{\begin{vmatrix} \hat y_t = \hat a_t + \hat m_t \\ \hat\omega_t - \hat p_t = \hat a_t \\ \hat c_t = y_t \\ \hat x_t = \frac{1}{v-\gamma}\left(\varphi \hat m_t + v\hat c_t - a_t\right) \\ \hat m_t = \varphi m_{t-1} + \varepsilon_t \\ \hat p_t = \hat m_t + A^{-1}B c_t - A^{-1}\hat x_t \\ \hat q_t = \left[\frac{v(1-\beta)}{\beta}\right]\left(\hat c_t - \hat m_t + \hat p_t\right) \end{vmatrix}}^{H_t(Y,a)=0} = 0$$

$$H_U = H_Y \frac{\partial Y}{\partial U} \qquad Y = \begin{vmatrix} y \\ \omega_t - p \\ c \\ x \\ m \\ p \\ q \end{vmatrix}$$

H_Y FROM BOND DEMAND

$$\underbrace{v(\hat c_{t+1} - \hat c_t)} - q_t + (\hat p_{t+1} - \hat p_t) - (v-\gamma)(\hat x_{t+1} - \hat x_t) = 0$$

$$\Phi_{BD,c} = v \cdot \begin{vmatrix} 1 & -1 & & \\ 0 & 1 & -1 & \\ & & \ddots & \\ & & & 1 \end{vmatrix}_{T\times T}$$

$$H_y = \begin{vmatrix} \Phi_{BD,y} & \Phi_{BD\,w-p} & \Phi_{BD\,c} & \Phi_{BD\,x} & \Phi_{BD\,p} & \Phi_{BD\,q} \end{vmatrix}$$

$$\frac{\partial Y}{\partial u} = \begin{vmatrix} \Phi_{y,m} \\ \Phi_{wp,m} \\ \Phi_{c,m} \\ \Phi_{x,m} \\ \Phi_{p,m} \\ \Phi_{q,m} \end{vmatrix} \qquad \frac{\partial Y}{\partial Z} = \begin{vmatrix} \Phi_{y,a} & \Phi_{y,m} \\ \Phi_{wp,a} & \Phi_{wp,m} \\ \Phi_{c,a} & \Phi_{c,m} \\ \Phi_{x,a} & \Phi_{x,m} \\ \Phi_{p,a} & \Phi_{p,m} \\ \Phi_{q,a} & \Phi_{q,m} \end{vmatrix}$$

$$dU = -H_U^{-1} H_Z \, dZ$$

NOW LET'S GROUP ALL VARIABLE OF INTEREST inside $Y_t = M(U,Z)$ and let's compute

$$dY = \Theta_{\cdot,j,\cdot} \equiv \frac{\partial M}{\partial U} dU + \frac{\partial M}{\partial Z} dZ = M_U[-H_U^{-1} H_Z \, dZ] + M_Z \, dZ$$

$$\boxed{\Theta_{\cdot,j,\cdot} = M_U[-H_U^{-1} H_Z e_1] + M_Z e_1 \qquad \text{SVMA}(\infty)}$$

$$\underbrace{\nu(\hat{c}_{t+1} - \hat{c}_t)} - q_t + (\hat{\rho}_{t+1} - \hat{\rho}_t) - (\nu - \gamma)(\hat{x}_{t+1} - \hat{x}_t) = 0$$

$$\Phi_{BD,c} = \nu \cdot \begin{vmatrix} 1 & -1 & & \\ 0 & 1 & -1 & \\ & & \ddots & \\ & & & 1 \end{vmatrix}_{T\times T}$$

$$H_y = \begin{vmatrix} \Phi_{BD,y} & \Phi_{BD\,w-p} & \Phi_{BD\,c} & \Phi_{BD\,x} & \Phi_{BD\,\rho} & \Phi_{BD\,q} \end{vmatrix}$$

$$H_y = \begin{vmatrix} 0_T & 0_T & \nu\Delta_T & -(\nu-\gamma)\Delta_T & \Delta_T & -I_T \end{vmatrix} \quad (T \times TN)$$

$$\left| \begin{array}{l} \overbrace{\hspace{6cm}}^{H_t(Y,a)=0} \\ \hat{y}_t = \hat{a}_t + \hat{m}_t \\ \hat{w}_t - \hat{p}_t = \hat{a}_t \\ \hat{c}_t = y_t \\ \hat{x}_t = \frac{1}{\nu-\gamma}\left(\varphi \hat{m}_t + \nu \hat{c}_t - a_t\right) \\ \hat{\rho}_t = \hat{m}_t + A^{-1} B c_t - A^{-1} \hat{x}_t \\ \hat{q}_t = \underbrace{\left[\frac{\nu(1-\beta)}{\beta}\right]}_{\Delta} \left(\hat{c}_t - \hat{m}_t + \hat{\rho}_t\right) \end{array} \right| = 0$$

$$\frac{\partial Y}{\partial u} = \begin{vmatrix} \Phi_{y,m} \\ \Phi_{wp,m} \\ \Phi_{c,m} \\ \Phi_{x,m} \\ \Phi_{\rho,m} \\ \Phi_{q,m} \end{vmatrix} \qquad \frac{\partial Y}{\partial Z} = \begin{vmatrix} \Phi_{y,a} & \Phi_{y,m} \\ \Phi_{wp,a} & \Phi_{wp,m} \\ \Phi_{c,a} & \Phi_{c,m} \\ \Phi_{x,a} & \Phi_{x,m} \\ \Phi_{\rho,a} & \Phi_{\rho,m} \\ \Phi_{q,a} & \Phi_{q,m} \end{vmatrix}$$

$$\frac{\partial Y}{\partial u} = \begin{vmatrix} \Phi_{y,m} \\ \Phi_{wp,m} \\ \Phi_{c,m} \\ \Phi_{x,m} \\ \Phi_{\rho,m} \\ \Phi_{q,m} \end{vmatrix} = \begin{vmatrix} I_T \\ 0_T \\ I_T \\ [(\nu-\gamma)^{-1}\varphi + 1] I_T \\ A^{-1} B I_T - A^{-1} \Phi_{xm} \\ \Delta I_T + \Delta \Phi_{\rho,m} \end{vmatrix}$$

$(TN \times T)$

$$\frac{\partial Y}{\partial Z} = \begin{bmatrix} \Phi_{y,a} & \Phi_{y,m} \\ \Phi_{wp,a} & \Phi_{wp,m} \\ \Phi_{c,a} & \Phi_{c,m} \\ \phi_{x,a} & \phi_{x,m} \\ \Phi_{p,a} & \Phi_{p,m} \\ \Phi_{q,a} & \Phi_{q,m} \end{bmatrix} = \begin{bmatrix} I_T & 0_T \\ I_T & 0_T \\ I_T & 0_T \\ \left[(\nu-1)/(\nu-\gamma)\right] I_T & 0_T \\ A^{-1}B\, I_T - A^{-1}\Phi_{xa} & I_T \\ \Delta I_T + \Delta \Phi_{pa} & 0_T \end{bmatrix} \quad (TN \times 2T)$$

$\hat{m}_t$ and $\hat{p}_t$ cancel out.

j) Money supply increase consumption only when $\nu > 1$, so we can rule out $\nu = \{0.25, 0.5, 1\}$

THE CHARTS ARE ON GITHUB. In the code I explained my DAG which is different from the others