



UC San Diego

Econometrics 120A: Discussion Section

Week 5

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Chapter 4: Two Random Variables

Chapter 4: Two Random Variables

Joint Probability Distribution

The Joint Probability Distribution represents the probability of two events, X and Y , occurring simultaneously.

Discrete Bivariate Random Variables:

⇒ The joint probability mass function (PMF) is given by:

$$P(X = x, Y = y) = p(x, y)$$

⇒ Properties:

$$\sum_x \sum_y P(X = x, Y = y) = 1$$

Continuous Bivariate Random Variables:

⇒ The joint probability density function (PDF) is represented as:

$$P(a < X < b, c < Y < d) = \int_a^b \int_c^d p(x, y) dy dx$$

⇒ Properties:

$$\iint_{-\infty}^{\infty} p(x, y) dy dx = 1$$

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Marginal Distribution

A marginal distribution captures the probabilities associated with one variable alone, without considering the dependency on other variables.

Discrete Bivariate Random Variables:

- ⇒ The marginal distribution of Y is obtained by summing the joint probabilities over all values of X .

$$P(Y = y) = \sum_x P(X = x, Y = y)$$

Continuous Bivariate Random Variables:

- ⇒ The marginal distribution of Y is obtained by summing the joint probabilities over all values of X .

$$P(Y = y) = \int_{-\infty}^{\infty} p(x, y) dx$$

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Independence

Independence means knowing the value of one variable provides no information about the other.

For two independent random variables X and Y :

- ⇒ The **joint probability distribution** is equal to the product of the marginals:

$$P(X, Y) = P(X) \cdot P(Y)$$

- ⇒ The **variance** of their sum is the sum of their variances (covariance is 0):

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$$

- ⇒ The **expected value** of the product of X and Y can be calculated simply as the product of their individual expected values:

$$E[XY] = E[X] \cdot E[Y]$$

- ⇒ The **conditional expectation** of one variable given the other is simply the expectation of that variable:

$$E[X|Y] = E[X] \quad \text{and} \quad E[Y|X] = E[Y]$$

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Conditional probability distribution

A conditional probability distribution describes the probability of a random variable X given that another random variable Y has taken on specific values. It is denoted as $P(Y|X)$ and is defined as $P(Y = y|X = x)$

Bayes' Rule allows us to update our beliefs about X given new evidence Y .

$$P(X = x | Y = y) = \frac{P(Y = y, X = x) \cdot P(X = x)}{P(Y = y)}$$

If Y is a random variable and X is another random variable, the **expected value** of Y given X is defined as:

$$E[Y|X] = \sum_y y \cdot P(Y = y|X) \quad (\text{for discrete random variables})$$

$$E[Y|X] = \int_{-\infty}^{\infty} y \cdot p(y|x) dy \quad (\text{for continuous random variables})$$

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Covariance and independence

Covariance measures the degree to which two random variables change together. Specifically, it is defined as:

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Independence Implies Zero Covariance: If X and Y are independent random variables, then their covariance is zero:

⇒ Due to independence, we have:

$$E[XY] = E[X] \cdot E[Y]$$

⇒ Therefore, the covariance becomes:

However, the converse is not necessarily true: **Zero Covariance Does Not Imply Independence**

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Properties expectation and variance

The **expectation** of a linear combination of variables is the linear combination of their expectations.

$$E[aX + bY] = aE[X] + bE[Y]$$

For two variables, their combined **variance** accounts for individual variances and their covariance.

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y)$$

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Kind of PS3 Questions 1-2

Suppose the following table represents the joint distribution of two variables: **Wage (W)** and **Education Level (E)** in a small economy. Here, W represents an individual's income category (low, medium, high), and E represents their education level (high school, bachelor's, master's).

W \ E	High School	Bachelor's	Master's
Low	0.10	0.05	0.02
Medium	0.15	0.20	0.08
High	0.05	0.10	0.25

- (a) Are **Wage** and **Education Level** independent? Justify your answer by calculating the marginal distributions and checking if the product of marginals equals the joint probabilities.
- (b) Calculate the conditional distribution of **Wage** given **Education Level = Bachelor's**. Is this the same as the marginal distribution for **Wage**? Is this consistent with your answer in the previous question?

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Kind of PS3 Questions 1-2

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W \ E	High School	Bachelor's	Master's
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- (b) Calculate the conditional distribution of **Wage** given **Education Level = Bachelor's**. Is this the same as the marginal distribution for **Wage**? Is this consistent with your answer in the previous question?

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Kind of PS3 Question 3

Suppose we have the following joint distribution for two economic variables: *Employment Status* (X) and *Industry* (Y). The variable X represents employment status, where:

$X = 0$ denotes "Unemployed"

$X = 1$ denotes "Employed in a Traditional Job"

$X = 2$ denotes "Self-Employed"

The variable Y represents industry type, where:

$Y = 0$ denotes "Agriculture"

$Y = 1$ denotes "Finance"

The joint probabilities for these variables are given in the table below:

$X \setminus Y$	0 (Agriculture)	1 (Finance)
0 (Unemployed)	0.05	0.15
1 (Employed)	0.20	0.25
2 (Self-employed)	0.10	0.25

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Kind of PS3 Question 3

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Using this table, answer the following questions:

- (a) Verify that this is indeed a probability distribution.
- (b) What are the marginal distributions for X and Y ?
- (c) What are the means of X and Y ?
- (d) What is the conditional distribution of X given $Y = 0$? Also for X given $Y = 1$.
- (e) What do you notice about this result? Does this show that X and Y are independent?
- (f) How does the conditional mean of X depend on Y ?
- (g) Does the result in (f) indicate if the random variables are independent? Be clear.

Kind of PS3 Question 3

Marginal Distribution

The joint probabilities for these variables are given in the table below:

$X \setminus Y$	0 (Agriculture)	1 (Finance)
0 (Unemployed)	0.05	0.15
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(a) Verify that this is indeed a probability distribution.

Kind of PS3 Question 3

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(b) What are the marginal distributions for X and Y ?

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Kind of PS3 Question 3

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(c) What are the means of X and Y ?

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(d) What is the conditional distribution of X given $Y = 0$? Also for X given $Y = 1$.

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- (e) What do you notice about this result? Does this show that X and Y are independent?

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1 (Employed)	0.20	0.25
2 (Self-employed)	0.10	0.25

(g) Does the result in (f) indicate if the random variables are independent? Be clear.

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Kind of PS3 Question 7

A random variable X has mean $E[X] = 20$ and variance of 25; a second random variable Y has mean $E[Y] = 30$ and variance of 55.

- (a) What is the mean and variance of $X + Y$ assuming they are independent?
- (b) What is the mean and variance of $X + Y$ if they have correlation $\rho = 0.5$?

Chapter 5: Random Sampling

Big Picture

⇒ We started from a Bernoulli Random Variable

$$X_i = \begin{cases} 0 & \text{(not cured)} & p \\ 1 & \text{(cured)} & (1-p) \end{cases}$$

⇒ Standard Assumption: X_1, \dots, X_n independent.

⇒ We define the random variable $S_n = \sum_{i=1}^n X_i$ with distribution $S_n \sim \text{Binomial}(n, p)$

⇒ Now we are interested in the distribution of $\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$

Moments of Sample Mean

⇒ We can easily compute the expected value of the sample mean:

$$E[\bar{X}_n] =$$

⇒ We can compute the variance of the sample mean as well:

$$\text{Var}(\bar{X}_n) =$$

Central Limit Theorem

⇒ If $\{X_1, \dots, X_n\}$ are a VSRS with mean μ and variance σ^2 then

$$\bar{X}_n \stackrel{a}{\sim} \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

⇒ Example: if we consider the Bernoulli random variables introduced before we have $\bar{X}_n \stackrel{a}{\sim} \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$

⇒ Given $Y \sim \mathcal{N}(\mu, \sigma^2)$, if we define $Z = aY + b$ by properties of expectation and variance we obtain:

$$Z = aY + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$$

⇒ $n\bar{X}_n \stackrel{a}{\sim} \mathcal{N}(np, np(1-p))$ but we know that $n\bar{X}_n = S_n \sim \text{Binomial}(n, p)$. Is there something wrong?

Central Limit Theorem

- ⇒ **Finite Sample:** When dealing with a finite sample, we cannot rely on the Central Limit Theorem (CLT) to perfectly approximate the distribution of the sample mean as normal, since the CLT only applies asymptotically (as $n \rightarrow \infty$).
- ⇒ **Asymptotic:** As n becomes very large, the CLT approximation improves, and the sample mean \bar{X}_n behaves increasingly like a normal distribution. However, this is an approximation that holds in the limit, meaning it is not exact for finite samples.
- ⇒ For the **Bernoulli random variables** with mean p and variance $p(1 - p)$, the random variable $n\bar{X}_n$ is approximated as follows according to the CLT:

$$n\bar{X}_n \stackrel{d}{\sim} \mathcal{N}(np, np(1 - p))$$

However, for a finite sample size, the actual distribution of $n\bar{X}_n = \sum_{i=1}^n X_i$ is binomial (not normal), and the normal distribution is only an approximation, which may not hold well for small n .