Practice Test
Moving Average of Order 1

Let $\{\varepsilon_t\}_{t=1}^{\infty}$ be an independent and identically distributed sequence of white noise, i.e. $\varepsilon_t \sim iid\ \mathcal{N}(0, \sigma^2)$, consider the following discrete-time stochastic process:

$$y_t = c + \varepsilon_t + \theta\varepsilon_{t-1} \tag{1}$$

also known as moving average of order 1, MA(1), with deterministic intercept c.

Question 1

Suppose that we are interested in the sample average ($\hat{\mu}$) of the observed outcomes (y_t) for $t = 1, \ldots, T$. Can you apply WLLN to study the probability limit of the estimator? Find the probability limit of the estimator $\hat{\mu}$.

Question 2

Can you apply CLT? Under what assumption would you be able to use it?

Question 3

Informally, the CLT can be applied to dependent data if the degree of dependency does not grow with t or grows slowly enough as t increases. However, to study the asymptotic distribution of our estimator, we need to take a step back and study the properties of the stochastic process y_t.

Find the unconditional distribution of y_t, along with its mean and variance. Think carefully: do you need CLT? Hint: ε_t, ε_{t-1} are independent.

Question 4

Find the conditional distribution of y_t given the realization from the previous period y_{t-1}. What does this distribution imply about dependency?

Question 5

Suppose we now have a Vector Moving Average of order 1 defined as

$$Y_t = C + \epsilon_t + \Theta\epsilon_{t-1} \tag{2}$$

$$Y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \quad \epsilon_t = \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \right)$$

Find the distribution of Y_t.

Question 6

Now suppose that we have $Y_t = |y_t \quad y_{t-1}|'$. Find the unconditional distribution of Y_t. Hint: combine the results from question 3 and 5 - you only need to estimate one additional parameter.

Question 7

Let's extend the problem further: suppose you now define $Y_t = |y_t \;\; y_{t-1} \quad \cdots \quad y_{t-h}|'$ for a very large h. Find the unconditional distribution of Y_t. Note that now the dimensions of C, ϵ_t, Θ now depend on the number of lags considered, h.

Question 8

We are almost ready to apply the CLT to our estimator $\hat{\mu}$. We need to verify two conditions:

1. $E[y_t] = k$ where k is a constant, $\forall t$.
2. $\lim_{t \to \infty} \; t \, Var(\hat{\mu}) < \infty$

The second condition holds if the stochastic process satisfies the so-called absolute summability property. Given our stochastic process y_t, define the autocovariance function as:

$$\gamma(h) = Cov(y_t, y_{t-h})$$

where h is the lag time. The stochastic process is absolutely summable if $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$. Verify whether the MA(1) process satisfies the abovementioned conditions.

Question 9

Given $a_t(\hat{\mu} - p)$ where a_t is the appropriate convergence rate and p is the probability limit of $\hat{\mu}$, apply the CLT and compute the variance. Hint: Substituting the MA inside the summation and expanding the sum will cause a pattern to appear.

Question 10

Claim: y_t can be represented as an autoregressive process of order infinity:

$$y_t = \mu + \varepsilon_t + \sum_{i=1}^{\infty} \psi_i \, y_{t-i}$$

Derive the AR(∞) representation of y_t.

How could the covariance $Cov(y_t, y_{t-k})$ be equal to zero? We know that y_{t-k} appears in the infinite sum. Is this a contradiction?