Model adaptation (Big Cryptocurrencies)

1 Detail

For the following model there are two types of investors:

- Unrestricted investors (U).
- Restricted investors (R).

In this scenario, the restricted investors (R) exclusively invest in cryptocurrencies that hold a dominant position in terms of popularity and market capitalization.

In traditional CAPM the unrestricted investor fully consumes terminal wealth, with w_U being terminal wealth of the unrestricted investor.

$$\max_{\mathbf{n_U}} \mathrm{E}\left[U(w_U)\right] \tag{1}$$

Subject to:

$$w_U = \boldsymbol{n}_U^{\intercal}\boldsymbol{x} + n_U^f \tag{2}$$

$$\bar{w}_U = \boldsymbol{n}_U^{\intercal}\boldsymbol{P} + n_U^f P_f \tag{3}$$

Where:

- w_U : terminal wealth for the unrestricted investor.
- $\boldsymbol{n}_U$: vector representing the number of shares the unrestricted investor purchases in each N cryptocurrencies.
- $\boldsymbol{x}$: vector of payoffs per share in each of N cryptocurrencies.
- n_U^f: number of risk-free discount bonds with unit payoff purchased by the unrestricted investor.
- $\boldsymbol{P}$: vector of cryptocurrency prices.
- P_f : price of the discount bond.
- $\bar{w}_U$: initial wealth of the unrestricted investor.

From (3) we can conclude the following,

$$n_U^f = \frac{1}{P_f}\left(\bar{w}_U - \boldsymbol{n}_{\boldsymbol{U}}^{\intercal}\boldsymbol{P}\right) \ .$$

Then, substituting the expression in (2) we have the following.

$$w_U = \boldsymbol{n}_U^{\intercal}\boldsymbol{x} + \bar{w}_U\frac{1}{P_f} - \boldsymbol{n}_U^{\intercal}\underbrace{\frac{\boldsymbol{P}}{P_f}}_{\boldsymbol{p}} = \frac{\bar{w}_U}{P_f} + \boldsymbol{n}_U^{\intercal}(\boldsymbol{x}-\boldsymbol{p}) \tag{4}$$

Substituting (4) in (1), and computing the derivative.

$$\frac{d\mathrm{E}}{dw_U} = \mathrm{E}\left[U'(w_U)(\boldsymbol{x}-\boldsymbol{p})\right] = 0 \ .$$

Which corresponds to the first order condition. Now, taking into account that $\boldsymbol{x} \sim \mathcal{N}(\bar{\boldsymbol{x}}, \boldsymbol{\Sigma})$, applying the definition of covariance we can conclude the following expression,

$$\begin{aligned}
\mathrm{E}\left[U'(w_U)(\boldsymbol{x}-\boldsymbol{p})\right] &= E\left[(U'(w_U)-E\left[U'(w_U)\right])(\boldsymbol{x}-\boldsymbol{p}-E\left[\boldsymbol{x}-\boldsymbol{p}\right])\right]+E\left[U'(w_U)\right]E\left[\boldsymbol{x}-\boldsymbol{p}\right]\ , \\
&= E\left[(U'(w_U)-E[U'(w_U)])(\boldsymbol{x}-\bar{\boldsymbol{x}})\right]+E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})\ , \\
&= E\left[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})-E[U'(w_U)](\boldsymbol{x}-\bar{\boldsymbol{x}})\right]+E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})\ , \\
&= E\left[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})\right]-E[U'(w_U)]E\left[\boldsymbol{x}-\bar{\boldsymbol{x}}\right]+E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})\ , \\
\mathrm{E}\left[U'(w_U)(\boldsymbol{x}-\boldsymbol{p})\right] &= E\left[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})\right]+E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})=0\ .
\end{aligned}$$

Then we can define the following equality,

$$-E\left[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})\right]=E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})$$

Stein's lemma application. We have that $\boldsymbol{x} \sim \mathcal{N}(\bar{\boldsymbol{x}}, \boldsymbol{\Sigma})$,

$$\mathrm{E}[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})]=\boldsymbol{\Sigma}\mathrm{E}\left[\frac{\partial U'(w_U)}{\partial \boldsymbol{x}}\right]$$

Taking $w_u=\boldsymbol{x}^{\intercal}\boldsymbol{n}_U+n_U^f\boldsymbol{1} \Rightarrow \frac{\partial w_U}{\partial \boldsymbol{x}}=\boldsymbol{n}_U$, then,

$$\mathrm{E}[U'(w_U)(\boldsymbol{x}-\bar{\boldsymbol{x}})]=\boldsymbol{\Sigma}\mathrm{E}\left[U''(w_u)\boldsymbol{n}_U\right]=\mathrm{E}\left[U''(w_u)\right]\boldsymbol{\Sigma}\boldsymbol{n}_U$$

$\square$

Substituting the lemma application,

$$\begin{aligned}
-E\left[U''(w_U)\right]\boldsymbol{\Sigma}\boldsymbol{n}_U &= E\left[U'(w_U)\right](\bar{\boldsymbol{x}}-\boldsymbol{p})\ , \\
\bar{\boldsymbol{x}}-\boldsymbol{p} &= \frac{-E\left[U''(w_U)\right]}{E\left[U'(w_U)\right]}\boldsymbol{\Sigma}\boldsymbol{n}_U\ , \\
\bar{\boldsymbol{x}}-\boldsymbol{p} &= \theta_U\boldsymbol{\Sigma}\boldsymbol{n}_U\ .
\end{aligned} \tag{5}$$

Where θ_U is akin to the absolute risk aversion, which depends on the initial wealth of investor U and other model. $\boldsymbol{\Sigma}$ is the covariance matrix for risky asset payoffs and $\bar{\boldsymbol{x}}$ the expected payoffs of risky assets.

For investor type R the problem is of similar nature but they invest exclusively in cryptocurrencies that hold a dominant position in terms of popularity and market capitalization.

$$\max_{\mathbf{n_R}} \mathrm{E}\left[U(w_R)\right] \tag{6}$$

Subject to:

$$w_R=\boldsymbol{n}_R^{\intercal}\boldsymbol{x}+n_R^f \tag{7}$$

$$\bar{w}_R=\boldsymbol{n}_R^{\intercal}\boldsymbol{P}+n_R^fP_f \tag{8}$$

Where $\boldsymbol{n}_R$ is the vector of the shares of cryptocurrencies that investor R purchases that comply with their preferences. Then, following the same procedure as before.

$$\theta_R\boldsymbol{\Sigma}_P\boldsymbol{n}_R=\bar{\boldsymbol{x}}_P-\boldsymbol{p}_P \tag{9}$$

Where the matrix of asset payoff covariances is partitioned into popular (P) and non-popular (N) cryptocurrencies.

$$\boldsymbol{\Sigma}=\begin{bmatrix}\boldsymbol{\Sigma}_P & \boldsymbol{\Sigma}_{PN}\\ \boldsymbol{\Sigma}_{NP} & \boldsymbol{\Sigma}_N\end{bmatrix} \tag{10}$$

Where $\boldsymbol{\Sigma}_N$ represents the payoff covariance of all cryptocurrencies that are "non-popular" or have small market capitalization, and $\bar{\boldsymbol{x}}_N$ and $\boldsymbol{p}_N$ are the vectors of mean payoffs and prices, respectively, of the "non-popular" cryptocurrencies.

Assuming q_U investors of type U and q_R investors of type R, the demand for cryptocurrencies may be obtained and set equal to the exogenous supply of cryptocurrencies $\bar{\boldsymbol{n}} = (\bar{\boldsymbol{n}}_N, \bar{\boldsymbol{n}}_P)^\intercal$, and to zero for the risk-free asset, yielding the conditions for market equilibrium.

$$\bar{\boldsymbol{n}} = q_U \boldsymbol{n}_U + q_R \boldsymbol{n}_R, \quad 0 = q_U n_U^f + q_R n_R^f \tag{11}$$

Reorganizing equations (9) and (5) yields the following,

$$\boldsymbol{n}_U = (\theta_U \boldsymbol{\Sigma})^{-1} (\bar{\boldsymbol{x}} - \boldsymbol{p}), \quad \boldsymbol{n}_R = (\theta_R \boldsymbol{\Sigma}_P)^{-1} (\bar{\boldsymbol{x}}_P - \boldsymbol{p}_P) \ .$$

Note that $\boldsymbol{n}_R$ can be represented in the following form,

$$\boldsymbol{n}_R = \theta_R^{-1} \begin{bmatrix} \boldsymbol{\Sigma}_P^{-1} & 0 \\ 0 & 0 \end{bmatrix} (\bar{\boldsymbol{x}} - \boldsymbol{p}) = \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} (\boldsymbol{\Sigma}_P \theta_R)^{-1} \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} (\bar{\boldsymbol{x}} - \boldsymbol{p}) \ .$$

Substituting in (11) yields the following,

$$\bar{\boldsymbol{n}} = \left((\boldsymbol{\Sigma} \theta_U / q_U)^{-1} + \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} (\boldsymbol{\Sigma}_P \theta_R / q_R)^{-1} \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} \right) (\bar{\boldsymbol{x}} - \boldsymbol{p}) \ . \tag{12}$$

From where we want to isolate the expression $\bar{\boldsymbol{x}} - \boldsymbol{p}$, then is necessary to compute the inverse of the expression in parenthesis. The latter can be done using an identity that says the following, given matrices $\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3$ y $\boldsymbol{X}_4$, with $\boldsymbol{X}_1$, $\boldsymbol{X}_4$ having an inverse, the following equality is satisfied.

$$\left(\boldsymbol{X}_1^{-1} + \boldsymbol{X}_2 \boldsymbol{X}_4^{-1} \boldsymbol{X}_3 \right)^{-1} = \boldsymbol{X}_1 + \boldsymbol{X}_1 \boldsymbol{X}_2 \left(\boldsymbol{X}_4 + \boldsymbol{X}_3 \boldsymbol{X}_1 \boldsymbol{X}_2 \right)^{-1} \boldsymbol{X}_3 \boldsymbol{X}_1 \ . \tag{13}$$

Substituting the terms in (13), we have that,

$$\begin{aligned}
&\left((\boldsymbol{\Sigma} \theta_U / q_U)^{-1} + \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} (\boldsymbol{\Sigma}_P \theta_R / q_R)^{-1} \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} \right)^{-1} \\
&= \boldsymbol{\Sigma} \theta_U / q_U - \boldsymbol{\Sigma} \theta_U / q_U \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} \left(\boldsymbol{\Sigma}_P \theta_R / q_R + \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} \boldsymbol{\Sigma} \theta_U / q_U \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} \boldsymbol{\Sigma} \theta_U / q_U \ , \\
&= \boldsymbol{\Sigma} \theta_U / q_U - \boldsymbol{\Sigma} \theta_U / q_U \begin{bmatrix} \boldsymbol{I} \\ 0 \end{bmatrix} (\boldsymbol{\Sigma}_P \theta_R / q_R + \boldsymbol{\Sigma}_P \theta_U / q_U)^{-1} \begin{bmatrix} \boldsymbol{I} & 0 \end{bmatrix} \boldsymbol{\Sigma} \theta_U / q_U \ , \\
&= \theta_U / q_U \left(\boldsymbol{\Sigma} - \frac{\theta_U / q_U}{\theta_U / q_U + \theta_R / q_R} \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{\Sigma}_P^{-1} & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\Sigma} \right) \ .
\end{aligned}$$

Then, substituting the expression in (12) yields the following,

$$
\begin{aligned}
(\bar{\boldsymbol{x}}-\boldsymbol{p}) &= \theta_U/q_U\left(\boldsymbol{\Sigma}-\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\begin{bmatrix}\boldsymbol{\Sigma}_P^{-1} & 0\\ 0 & 0\end{bmatrix}\boldsymbol{\Sigma}\right)\bar{\boldsymbol{n}}\ ,\\
&= \theta_U/q_U\left(\boldsymbol{\Sigma}-\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\begin{bmatrix}\boldsymbol{I} & \boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\\ 0 & 0\end{bmatrix}\right)\bar{\boldsymbol{n}}\ ,\\
&= \theta_U/q_U\left(\boldsymbol{\Sigma}\bar{\boldsymbol{n}}-\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\begin{bmatrix}\bar{\boldsymbol{n}}_N+\boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\bar{\boldsymbol{n}}_P\\ 0\end{bmatrix}\right)\ ,\\
&= \theta_U/q_U\left(\boldsymbol{\Sigma}\bar{\boldsymbol{n}}-\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\bar{\boldsymbol{n}}+\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\begin{bmatrix}-\boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\bar{\boldsymbol{n}}_P\\ \bar{\boldsymbol{n}}_P\end{bmatrix}\right)\ ,\\
&= \theta_U/q_U\left(\frac{\theta_R/q_R}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\bar{\boldsymbol{n}}+\frac{\theta_U/q_U}{\theta_U/q_U+\theta_R/q_R}\boldsymbol{\Sigma}\begin{bmatrix}-\boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\bar{\boldsymbol{n}}_P\\ \bar{\boldsymbol{n}}_P\end{bmatrix}\right)\ ,\\
&= \left(\frac{1}{q_U/\theta_U+q_R/\theta_R}\boldsymbol{\Sigma}\bar{\boldsymbol{n}}+\frac{1}{q_U/\theta_U+q_R/\theta_R}\frac{q_R/\theta_R}{q_U/\theta_U}\boldsymbol{\Sigma}\begin{bmatrix}-\boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\bar{\boldsymbol{n}}_P\\ \bar{\boldsymbol{n}}_P\end{bmatrix}\right)\ ,\\
&= \frac{1}{q_U\bar{w}_U/\rho_U+q_R\bar{w}_R/\rho_R}\boldsymbol{\Sigma}\bar{\boldsymbol{n}}+\frac{1}{q_U\bar{w}_U/\rho_U+q_R\bar{w}_R/\rho_R}\frac{q_R\bar{w}_R/\rho_R}{q_U\bar{w}_U/\rho_U}\boldsymbol{\Sigma}\begin{bmatrix}-\boldsymbol{\Sigma}_P^{-1}\boldsymbol{\Sigma}_{PN}\bar{\boldsymbol{n}}_P\\ \bar{\boldsymbol{n}}_P\end{bmatrix}\ ,\\
&= \gamma\boldsymbol{\Sigma}\bar{\boldsymbol{n}}+\delta\boldsymbol{\Sigma}\bar{\boldsymbol{n}}_K\ .
\end{aligned}
\tag{14}
$$

Where $\bar{\boldsymbol{n}}_K$ represents the known cryptocurrency portfolio. Now, we have to convert (14) in an expression for expected returns rather than expected net payoffs. Given that $P_f = 1/(1+r_f)$ we can define the following,

$$
(1+r_i^s)=\frac{x_i}{P_i}\Leftrightarrow x_i-\frac{P_i}{P_f}=P_i(1+r_i^s)-P_i(1+r_f)=P_i(r_i^s-r_f)\ .
$$

Then, defining the excess return as $r_i = r_i^s - r_f$, and given that in (14) the expression to the left of the equality is represented as an average, it yields that $\mu_i = \mu_i^s - r_f$. In addition, cause $1 + r_i^s = x_i/P_i$, the covariance matrix for the payoffs of the cryptocurrencies $\boldsymbol{\Sigma}$ can be represented in terms of the returns $\sigma_{ij} = \Sigma_{ij}/P_iP_j$, such that, for a specific element of (14) we have that,

$$
\begin{aligned}
P_i\mu_i &= \gamma\Sigma_{im}+\delta\Sigma_{ip}\\
\mu_i &= \gamma P_m\sigma_{im}+\delta P_p\sigma_{ip}
\end{aligned}
\tag{15}
$$

Where m represents the market, $P_m = q_m\bar{w}_M = q_U\bar{w}_U + q_R\bar{w}_R$ is the cost of the market portfolio, and P_p is the cost of the popular portfolio. Now, given (15) we can define μ_m y μ_p, which correspond to the mean returns of the market and popular portfolios, respectively.

$$
\mu_m=\gamma P_m\sigma_m^2+\delta P_p\sigma_{mp}\quad ;\quad \mu_p=\gamma P_m\sigma_{mp}+\delta P_p\sigma_p^2\ .
$$

Solving the system of equations for γP_m y δP_p yields the following,

$$
\delta P_p=\frac{\sigma_{mp}\mu_m-\sigma_m^2\mu_p}{\sigma_{mp}^2-\sigma_p^2\sigma_m^2}\quad ;\quad \gamma P_m=\frac{\sigma_{mp}\mu_p-\sigma_p^2\mu_m}{\sigma_{mp}^2-\sigma_p^2\sigma_m^2}\ .
$$

Substituting in (15) yields,

$$
\begin{aligned}
\mu_i &= \frac{\sigma_m^2\sigma_{ip}-\sigma_{mp}\sigma_{im}}{\sigma_p^2\sigma_m^2-\sigma_{mp}^2}\mu_p+\frac{\sigma_p^2\sigma_{im}-\sigma_{mp}\sigma_{ip}}{\sigma_p^2\sigma_m^2-\sigma_{mp}^2}\mu_m\ ,\\
&= \beta_{ip}\mu_p+\beta_{im}\mu_m\ .
\end{aligned}
\tag{16}
$$

Where β_{ib} and β_{ip} are the population values of the slope estimates for a linear regression of the return of asset i on the market portfolio return and the popular portfolio return.

2 Appendix

2.1 Stein's Lemma

Let X be a random variable that follows a normal distribution with mean μ and variance σ^2. Let g be a function for which $\mathrm{E}\left(g(X)(X-\mu)\right)$ and $\mathrm{E}\left(g'(X)\right)$ exist. Then,

$$\mathrm{E}\left(g(X)(X-\mu)\right) = \sigma^2 \mathrm{E}\left(g'(X)\right).$$

In general, assuming that X and Y have a joint probability distribution, then,

$$\mathrm{Cov}\left(g(X), Y\right) = \mathrm{Cov}(X, Y)\mathrm{E}(g'(X)).$$

For a multivariate normal vector $(X_1, \ldots, X_n) \sim \mathcal{N}(\boldsymbol{u}, \boldsymbol{\Sigma})$, it holds that,

$$\mathrm{E}\left(g(\boldsymbol{X})(\boldsymbol{X}-\boldsymbol{\mu})\right) = \boldsymbol{\Sigma} \cdot \mathrm{E}\left(\nabla g(\boldsymbol{X})\right) \tag{17}$$