Correlated Monte Carlo Simulation using Cholesky Decomposition

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Executive Summary

We outline the steps necessary to peform Monte Carlo simulation on multiple correlated assets.

- Geometric Brownian Motion (GBM)
- Euler Discretization
- Monte Carlo Simulation (MC)
- Multi-Dimensional Monte Carlo
- Asset Correlation
- Cholesky Decomposition
- Correlated Random Variables (RVs)
- Correlation Lower Diagonal Matrices

Geometric Brownian Motion (GBM)

A stock process S(t) could be modelled as a lognormal GBM process growing with drift r and having diffusion or volatility σ,

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{1}$$

Applying Itô's Lemma gives,

$$S(t) = S(0)\ exp\left(\left(r - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)\right) \tag{2}$$

where $W(t) \sim N(0, t)$

Euler Discretization

Euler discretization over the interval $[t_{i+1}, t_i]$ gives,

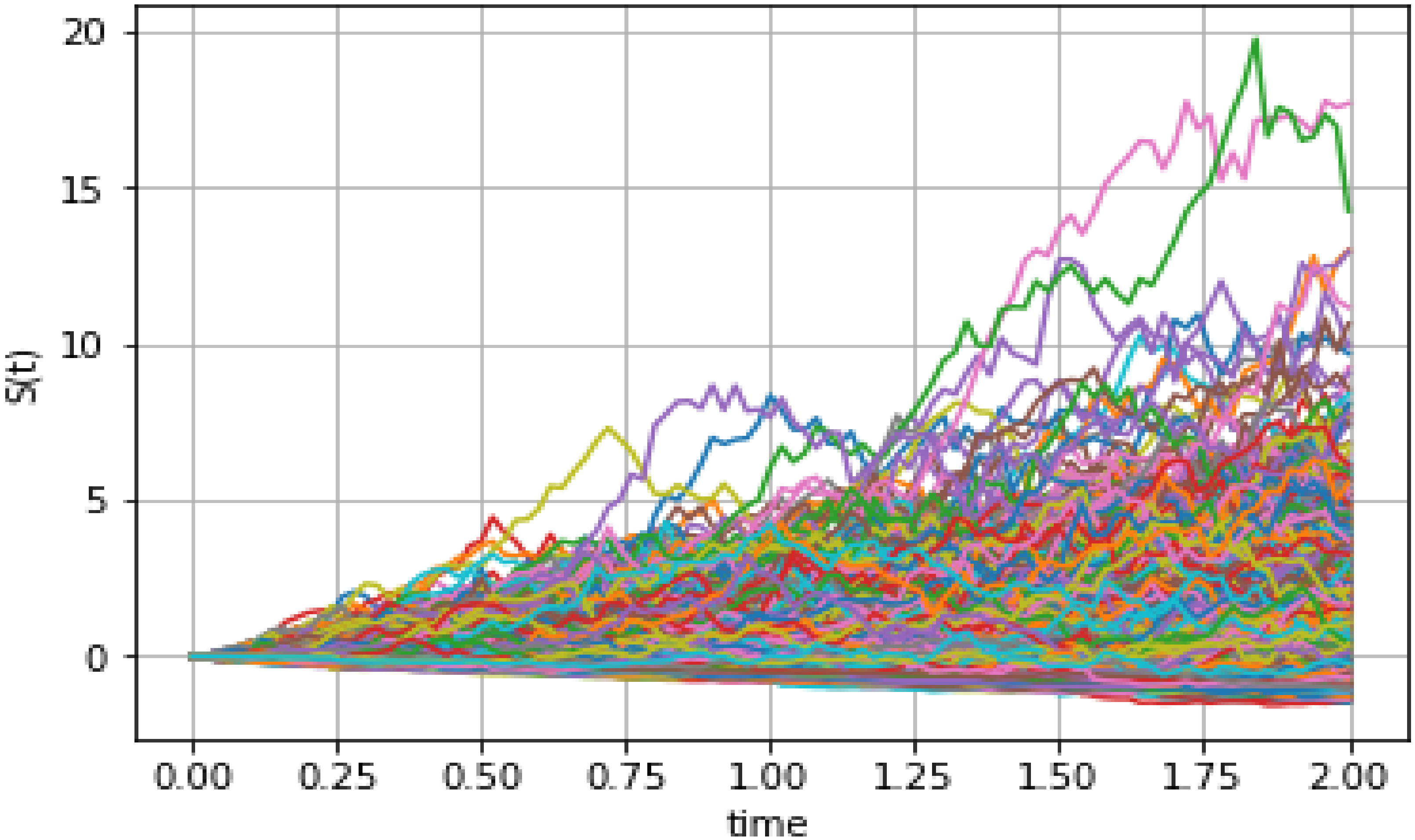
$$S(t_{i+1}) = S(t_i)\ exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(W(t_{i+1}) - W(t_i)\right)\right) \tag{3}$$

as $W(t) \sim N(0,1)$ and given a normal random variate Z we have,

$$S(t_{i+1}) = S(t_i)\ exp\left(\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma Z\sqrt{\Delta t}\right) \tag{4}$$

Monte Carlo Simulation

Performing MC simulation with Euler discretization gives,



Multi-Dimensional Asset Price Processes

For two stocks the process becomes,

$$
\begin{aligned}
dS_1(t) &= r_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t) \\
dS_2(t) &= r_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t)
\end{aligned} \tag{5}
$$

and again applying Itô's Lemma we have,

$$
\begin{aligned}
S_1(t) &= S_1(0)\ exp\left(\left(r_1 - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dW_1(t)\right) \\
S_2(t) &= S_2(0)\ exp\left(\left(r_2 - \frac{1}{2}\sigma_2^2\right)dt + \sigma_2 dW_2(t)\right)
\end{aligned} \tag{6}
$$

where Brownian motions $W_1(t)$ and $W_2(t)$ are correlated with $dW_1(t)dW_2(t) = \rho dt$

Multi-Dimensional Euler Discretization

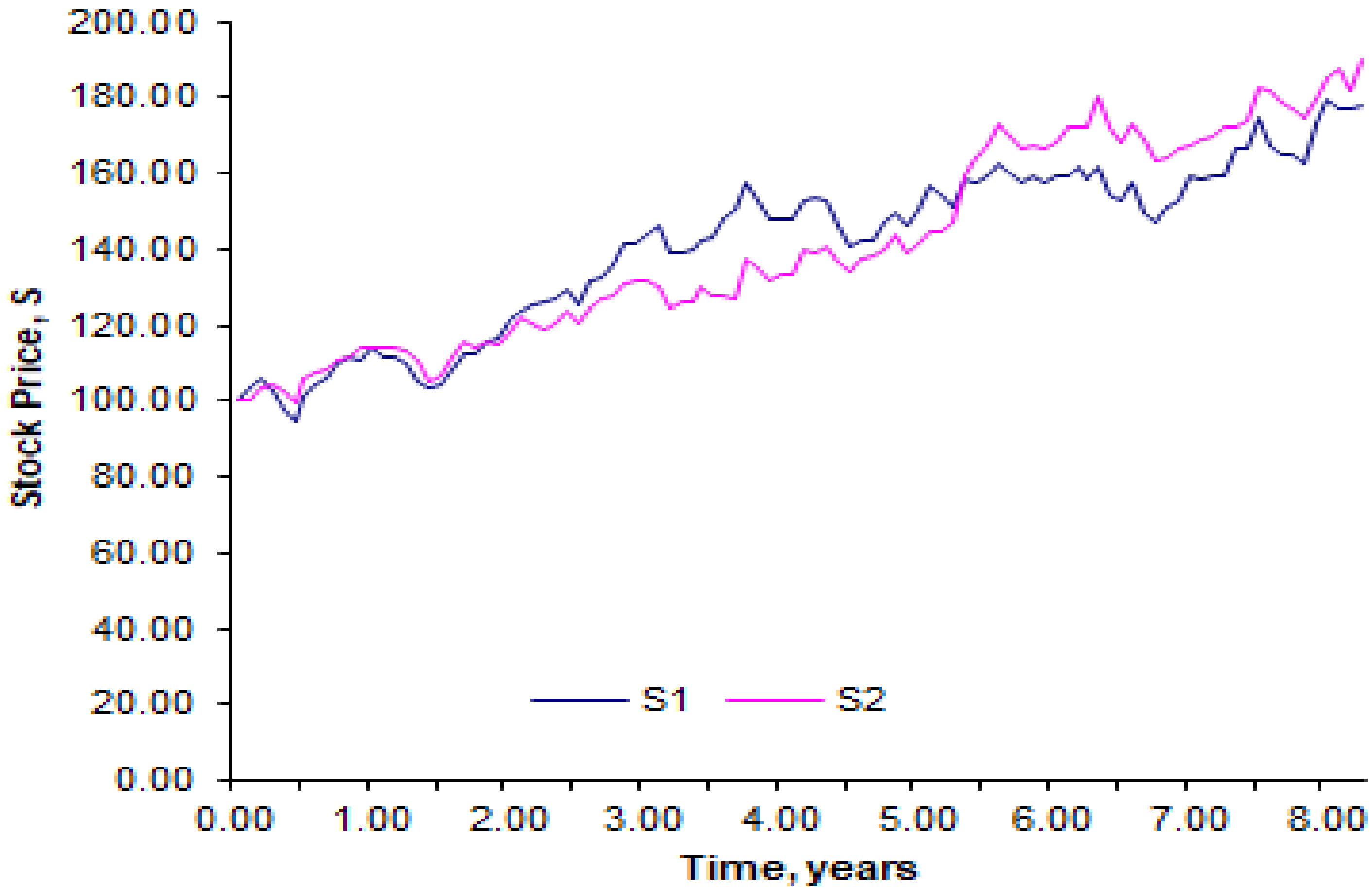
Monte Carlo simulation with Euler discretization for two assets with correlation ρ becomes,

$$
\begin{aligned}
S_1(t) &= S_1(0)\ exp\left(\left(r_1 - \frac{1}{2}\sigma_1^2\right)\Delta t + \sigma_1 \widetilde{Z}_1 \sqrt{\Delta t}\right) \\
S_2(t) &= S_2(0)\ exp\left(\left(r_2 - \frac{1}{2}\sigma_2^2\right)\Delta t + \sigma_2 \widetilde{Z}_2 \sqrt{\Delta t}\right)
\end{aligned}
\tag{7}
$$

where Z_1 and Z_2 denote independent normal random variates and correlated random normal variates are given by $\widetilde{Z}_1 = Z_1$ and $\widetilde{Z}_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2$

Multi-Dimensional Monte Carlo

MC simulation for two assets with correlation $\rho = 0.75$ for a single path gives,



Correlated Assets and Correlated Random Variables

So how do we introduce asset correlation?

- If Brownian motions $W_1(t)$ and $W_2(t)$ are correlated
- then given $W(t) \sim N(0, t)$ the Central Limit Theorem (CLT) for standard normal variables Z states,

$$Z = \frac{W(t) - \mu}{\sigma} = \frac{W(t)}{\sqrt{t}}$$

$$=> W(t) = Z\sqrt{t}$$

- We can adjust the independent random variables Z_1 and Z_2 from $W_1(t)$ and $W_2(t)$ respectively to incorporate correlation
- Cholesky Decomposition can be used to generate correlated random variables $\widetilde{Z_1}$ and $\widetilde{Z_2}$, see next slides.

Cholesky Decomposition

Given a symmetrical positive definite (invertible) correlation matrix,

$$C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{8}$$

We can decompose the correlation matrix into its lower and upper triangular parts. This is called Cholesky decomposition.

$$C = \left(LL^T\right) = \underbrace{\begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix}}_{L^T} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{9}$$

Correlated Processes

Correlated random variables can be generated using the Cholesky Decomposition identity $\boxed{Z = LX}$

As X comprises of standard normal random variables with mean zero and variance one, the covariance of X is $E[X.X^T]$ as shown below,

$$Cov(X.X^T) = E[X.X^T] - E[X]E[X^T] = E[X.X^T] \tag{10}$$

The covariance of Z is the correlation matrix C, which confirms the Cholesky result,

$$\begin{aligned} Cov(Z) = Cov(Z.Z^T) = E[(LX).(LX)^T] = E[(LX).X^T L^T] \\ = LE[X.X^T]L^T = LIL^T = L.L^T = C \end{aligned} \tag{11}$$

Correlated Brownian Motions I

Given a vector of independent Brownian motions $X = (W_1(t), W_2(t))^T$ Cholesky Decomposition creates a new vector of correlated Brownian motions Z as follows,

$$Z = L.X \tag{12}$$

where L is the correlation lower triangular matrix from (9) giving,

$$Z = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} W_1(t) \\ W_2(t) \end{pmatrix} = \begin{pmatrix} W_1(t) \\ \rho W_1(t) + \sqrt{1-\rho^2}\, W_2(t) \end{pmatrix} \tag{13}$$

Correlated Brownian Motions II - Covariance

Furthermore when applying Cholesky Decomposition the covariance between $W_1(t) \sim N(0,t)$ and $W_2(t) \sim N(0,t)$ is ρt as expected,

$$
\begin{aligned}
Cov(W_1(t)W_2(t)) &= \mathbb{E}[W_1(t)W_2(t)] - \underbrace{\mathbb{E}[W_1(t)]\mathbb{E}[W_2(t)]}_{\text{By definition } = 0} \\
&= \mathbb{E}[W_1(t)(\rho W_1(t) + \sqrt{1-\rho}W_2(t))] \\
&= \rho\mathbb{E}[W_1(t)^2] + \sqrt{1-\rho^2}\,\underbrace{\mathbb{E}[W_1(t)]\mathbb{E}[W_2(t)]}_{\text{By definition } = 0} \\
&= \rho Var(W_1(t)) \\
&= \rho t
\end{aligned}
$$

Correlated Random Variables

Given a vector of standard normal variates $X = (Z_1(t), Z_2(t))^T$ Cholesky Decomposition creates a new vector of correlated standard normal variates Z as follows,

$$Z = L.X \tag{14}$$

where L is the correlation lower triangular matrix from (9) giving,

$$Z = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{pmatrix} \tag{15}$$

$Z = (Z_1, \rho Z_1 + \sqrt{1-\rho^2} Z_2)^T$ confirming the result from (7)

Correlation Lower-Diagonal Matrix Notation

Given a correlation matrix C with elements indexed (i, k) and noting the matrix is symmetric with $c_{i,j} = c_{j,i}$,

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix} \tag{16}$$

We need to compute the lower diagonal matrix L, whose elements are indexed (j, k) in order to apply Cholesky decomposition,

$$L = \begin{pmatrix} l_{1,1} & 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 \\ l_{3,1} & l_{3,2} & l_{3,3} & 0 \\ l_{4,1} & l_{4,2} & l_{4,3} & l_{4,4} \end{pmatrix} \tag{17}$$

Correlation Lower-Diagonal Matrix Formulae

To calculate the elements (j,k) of the lower triangular matrix L we use the elements (i,k) of the correlation matrix C and the below formulae, working from top to bottom and left to right,

For **non-diagonal** elements of L when $k \neq i$

$$l_{i,k} = \frac{\left(c_{i,k} - \sum_{j=1}^{i-1} l_{i,k} l_{j,k}\right)}{l_{i,i}} \tag{18}$$

For **diagonal** elements of L when $k = i$

$$l_{i,k} = \sqrt{c_{i,k} - \sum_{j=1}^{k-1} l_{j,k}^2} \tag{19}$$

2x2 Correlation Matrix

Given a symmetric correlation matrix C,

$$C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \tag{20}$$

applying equations (18) and (19) gives,

$$L = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \tag{21}$$

Cholesky Decomposition $\boxed{Z = LX}$ gives correlated rv's,

$$\widetilde{Z} = (Z_1, \rho Z_1 + \sqrt{1-\rho^2} Z_2)^T \tag{22}$$

3x3 Correlation Matrix I

Given a correlation matrix C whose elements have their index notation adjusted to reflect entries are symmetric,

$$C = \begin{pmatrix} 1 & \rho_{1,2} & \rho_{1,3} \\ \rho_{1,2} & 1 & \rho_{2,3} \\ \rho_{1,3} & \rho_{2,3} & 1 \end{pmatrix} \tag{23}$$

applying equations (18) and (19) gives,

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \rho_{1,2} & \sqrt{1-\rho_{1,2}^2} & 0 \\ \rho_{1,3} & \left(\frac{\rho_{2,3}-\rho_{1,2}\rho_{1,3}}{\sqrt{1-\rho_{1,2}^2}}\right) & \sqrt{1-\rho_{1,3}^2-\left(\frac{\rho_{2,3}-\rho_{1,2}\rho_{1,3}}{\sqrt{1-\rho_{1,2}^2}}\right)^2} \end{pmatrix} \tag{24}$$

3x3 Correlation Matrix II

Cholesky Decomposition $\boxed{Z = LX}$ generates correlated random variables as follows,

$$Z = (\omega_1, \omega_2, \omega_3)^T \tag{25}$$

with

$$\omega_1 = Z_1 \tag{26}$$

$$\omega_2 = \rho_{1,2} Z_1 + \sqrt{1 - \rho_{1,2}^2} Z_2 \tag{27}$$

$$\omega_3 = \rho_{1,3} Z_1 + \left(\frac{\rho_{2,3} - \rho_{1,2}\rho_{1,3}}{\sqrt{1 - \rho_{1,2}^2}} \right) Z_2 + \sqrt{1 - \rho_{1,3}^2 - \left(\frac{\rho_{2,3} - \rho_{1,2}\rho_{1,3}}{\sqrt{1 - \rho_{1,2}^2}} \right)^2} Z_3 \tag{28}$$

Conclusion

- Monte Carlo simulation with multiple assets requires that asset simulations are correlated
- Given a symmetric asset correlation matrix with $C = LL^T$.

$$C = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

- We compute the correlation lower diagonal matrix, L
- Cholesky decomposition adjusts independent random variables X by the lower diagonal matrix L to generate correlated random variables Z as follows,

$$Z = LX$$

- All the necessary steps were outlined above. Have Fun!