LQG Control and Applications to Neural Control of Movement

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Part I: Basic Theory

LQG Control Framework

- **Linear**: Linear dynamics in state and control variables,
- **Quadratic**: Quadratic cost-function in state and control variables,
- **Gaussian**: Assume that the noise variables are normally distributed ($X \sim \mathcal{N}(\mu, \sigma^2)$).

Definitions

Control System:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \xi_k, \\ y_k &= Hx_k + \omega_k, \end{aligned}$$

$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. The initial state is given (x_1).

Cost Function:

$$\begin{aligned} J_k(x_k, u_k) &= x_k^T Q_k x_k + u_k^T R u_k, \qquad Q_k \geq 0, \quad k = 1, 2, \ldots, N-1, \\ J_N(x_N) &= x_N^T Q_N x_N. \qquad\qquad\qquad R > 0. \end{aligned}$$

Noise Parameters:

$$\xi_k \sim \mathcal{N}(0, \Omega_\xi), \quad \omega_k \sim \mathcal{N}(0, \Omega_\omega).$$

Optimal Control Problem

Find a control sequence, $u_1, u_2, \ldots u_{N-1}$, which minimizes:

$$J \quad := \quad E\left[J_N + \sum_{k=1}^{N-1} J_k(x_k, u_k)\right],$$

where $E[.]$ denotes the expected value of the argument.

Theorem: Fully Observable Case. *The optimal solution is a linear state feedback controller, and the cost-to-go at each time step is given by:*

$$v_k(x_k, u_k) = x_k^T S_k x_k + s_k,$$

where S_k are non-negative matrices and s_k are non-negative scalar quantities.

Solution of the Optimal Control Problem

Proof Sketch (Induction).

- The claim is true when $k = N$ with $S_N = Q_N$ and $s_N = 0$.
- Let $1 \leq k < N$. We must solve (Bellman's equation):

$$\begin{aligned} v_k &= \min_{u_k}\Big[J(x_k, u_k) + E\left(v_{k+1}|x_k, u_k\right)\Big]. \\ v_k &= \min_{u_k}\Big[x_k^T Q_k x_k + u_k^T R u_k + E\left(v_{k+1}|x_k, u_k\right)\Big]. \end{aligned}$$

  Expanding the conditional expected value of v_{k+1} given x_k and u_k from the induction hypothesis gives:

$$\begin{aligned} v_k = \min_{u_k}\Big[& x_k^T\left(Q_k + A^T S_{k+1} A\right) x_k + u_k^T\left(R + B^T S_{k+1} B\right) u_k \\ & + 2x_k^T A^T S_{k+1} B u_k + \operatorname{tr}\left(S_{k+1}\Omega_\xi\right) + s_{k+1}\Big]. \end{aligned}$$

Proof (Cont'd).

The previous equation is a quadratic form in u_k, which is minimized when u_k satisfies:

$$\begin{aligned} u_k &= -\left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A x_k, \\ &:= -L_k x_k. \end{aligned}$$

By plugging the expression of the optimal control variable into the expression of v_k we obtain:

$$\begin{aligned} v_k &= x_k^T \left(Q_k + A^T S_{k+1}(A - BL_k)\right) x_k + s_k, \\ &:= x_k^T S_k x_k + s_k. \end{aligned}$$

where $s_k := s_{k+1} + \mathrm{tr}(S_{k+1}\Omega_\xi) > 0$. We found the required expression for v_k, and we must verify that $S_k \geq 0$ to complete the proof. ■

Control: Practical Formulation

The optimal control policy is a linear function of the state. The optimal feedback gains are given by the following backward recursion:

$$\begin{aligned} L_k &= \left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A, \\ S_k &= Q_k + A^T S_{k+1}(A - BL_k), \\ s_k &= s_{k+1} + \mathrm{tr}(S_{k+1}\Omega_\xi), \\ S_N &= Q_N, \quad s_N = 0. \end{aligned}$$

The closed loop control system is described by:

$$x_{k+1} = (A - BL_k)x_k + \xi_k.$$

State Estimation: Predictive Case

The fully observable case assumes perfect knowledge of the state vector. In many cases, this is not a realistic assumption. In biology, the measurement of the state (sensory feedback) can be modelled by a noisy mixture of state variables.

Control System:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \xi_k, \\ y_k &= Hx_k + \omega_k. \end{aligned}$$

We assume a convex combination of prior and feedback:

$$\begin{aligned} \hat{x}_{k+1} &= (1-K) \times \mathrm{prior} + K \times \mathrm{feedback}, \\ \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - H\hat{x}_k). \end{aligned}$$

The estimation error has the following dynamics:

$$e_{k+1} = (A - K_k H)e_k + \xi_k - K_k\omega_k.$$

Predictive Case (Cont'd).

The optimal Kalman gain minimize the estimation error:

$$\begin{aligned} K_k &= \arg\min_K \| e_{k+1} \|^2, \\ &= \arg\min_K \left[\mathrm{tr}\left(E(e_{k+1}e_{k+1}^T)\right)\right]. \end{aligned}$$

From the error dynamics, the terms of the error covariance matrix that depend on K_k give:

$$a(K_k) := \mathrm{tr}\left(-2K_k H\Sigma_k + K_k(H\Sigma_k H^T + \Omega_\omega)K_k^T\right),$$

which is minimized over K_k when:

$$\nabla a(K_k) = 0 \quad \Rightarrow K_k = A\Sigma_k H^T(H\Sigma_k H^T + \Omega_\omega)^{-1}.$$

Estimation: Practical Solution

The optimal state estimates and Kalman gains are obtained in a forward recursion (Σ_1 known):

$$\begin{aligned}\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - H\hat{x}_k),\\ K_k &= A\Sigma_k H^T(H\Sigma_k H^T + \Omega_\omega)^{-1},\\ \Sigma_{k+1} &= \Omega_\xi + (A - K_k H)\Sigma_k A^T.\end{aligned}$$

Certainty Equivalence Principle

The solution to the Bellman equation is the same when the true state x_k is replaced by the estimated state $\hat{x}_k$ (why?). Thus under partial state observation, the optimal control law can be applied to the estimated state (certainty equivalence principle). The full closed loop system can be written as follows:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A - BL_k & BL_k \\ 0 & A - K_k H \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} \xi_k \\ \xi_k - K_k\omega_k \end{bmatrix}.$$

Observe that the stability depends on the eigenvalues of $A - BL$ and $A - KH$ separately, which is known as the ***separation principle***.

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Hands-on tutorial on OFC !!!