Section 4.9: Antiderivatives (Review)

Differentiation:

$$f(x) \quad \longrightarrow \quad f'(x), \quad \text{the } \underline{\text{derivative}} \text{ of } f(x), \text{ which is unique.}$$

Integration:

$$f(x) \quad \longrightarrow \quad F(x), \quad \text{an } \underline{\text{antiderivative}} \text{ of } f(x) \text{ (i.e., } F'(x) = f(x)), \text{ which is not unique.}$$

However, $F(x) + C$ is the family of ALL antiderivatives of $f(x)$.

Definition. If $F(x)$ is an antiderivative of $f(x)$ on an interval, then $F(x) + C$ is called

1) the **general antiderivative** of f;

2) the **indefinite integral** of f, written as $\displaystyle\int f(x)\,dx = F(x) + C$;

3) the **general solution** to the differential equation $y' = f(x)$.

Fact. Every derivative formula/property gives an integral formula/property.

E.g., we know that $\left(\frac{1}{3}x^3\right)' = x^2$. Then

$$f(x) = x^2 \quad \Longrightarrow \quad F(x) = \frac{1}{3}x^3 + C.$$

Simple Power Rule for Antiderivative. If $f(x) = x^n$, then

$$F(x) + C = \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1 \\ \\ \ln|x| + C & \text{if } n = -1 \end{cases},$$

where for $n < 0$, the formula holds on the interval $(-\infty, 0)$ or $(0, \infty)$.

Some Other Formulas.

$$f(x) \;=\; e^x \quad\Longrightarrow\quad F(x) + C \;=\; e^x + C.$$

$$f(x) \;=\; \sin x \quad\Longrightarrow\quad F(x) + C \;=\; -\cos x + C.$$

$$f(x) \;=\; \cos x \quad\Longrightarrow\quad F(x) + C \;=\; \sin x + C.$$

$$f(x) \;=\; \sec^2 x \quad\Longrightarrow\quad F(x) + C \;=\; \tan x + C.$$

$$f(x) \;=\; \sec x \,\tan x \quad\Longrightarrow\quad F(x) + C \;=\; \sec x + C.$$

$$f(x) \;=\; \frac{1}{1+x^2} \quad\Longrightarrow\quad F(x) + C \;=\; \tan^{-1}x + C.$$

$$f(x) \;=\; \frac{1}{\sqrt{1-x^2}} \quad\Longrightarrow\quad F(x) + C \;=\; \sin^{-1}x + C.$$

We can add more formulas to this list.

We know that $\Big(F(x) \pm G(x)\Big)' \;=\; F'(x) \pm G'(x)$ and $\Big(kF(x)\Big)'(x) \;=\; kF'(x)$.

Linearity of Antiderivatives.

$$h(x) \;=\; f(x) \pm g(x) \quad\Longrightarrow\quad H(x) + C \;=\; F(x) \pm G(x) + C.$$

$$h(x) \;=\; kf(x) \quad\Longrightarrow\quad H(x) + C \;=\; kF(x) + C.$$

Examples.

i) If $f(x) = 3x^3 + \sqrt{x} - \sin x$, then the general antiderivative of f on $(0, \infty)$ is

$$F(x) + C = \frac{3}{4}x^4 + \frac{1}{\frac{1}{2}+1}x^{\frac{3}{2}} + \cos x + C = \frac{3}{4}x^4 + \frac{2}{3}x^{\frac{3}{2}} + \cos x + C.$$

ii) If $f(x) = -2e^x + \frac{3}{x}$, then the general antiderivative of f on $(-\infty, 0)$ or $(0, \infty)$ is

$$F(x) + C = -2e^x + 3\ln|x| + C.$$

iii) Let $f(x) = \frac{1+x^2}{x}$. Find an antiderivative F of f on $(0, \infty)$ such that $F(1) = 0$.

Solution. Now $f(x) = \frac{1}{x} + x$. Then the general antiderivative of f on $(0, \infty)$ is

$$F(x) + C = \ln|x| + \frac{1}{2}x^2 + C.$$

Thus

$$F(1) = 0 \implies \frac{1}{2} + C = 0 \implies C = -\frac{1}{2}.$$

Therefore, the antiderivative F of f on $(0, \infty)$ satisfying $F(1) = 0$ is $F(x) = \ln x + \frac{1}{2}x^2 - \frac{1}{2}$.

Sections 5.1 - 5.2: Area Problem and Definite Integral (Review)

Suppose $f \geq 0$ is a function defined on $[a, b]$. If f is "good enough" (e.g., f is continuous), then

$$A = \int_a^b f(x)\, dx;$$

that is, the area of the region under the graph of f is given by the definite integral of f.

In general, we consider a function $f : [a, b] \to (-\infty, \infty)$. Let

$$P:\ a = x_0 < x_1 < \cdots < x_n = b$$

be a partition of $[a, b]$. We write

$$\Delta x_i = x_i - x_{i-1} \ (1 \leq i \leq n) \quad \text{and} \quad \|P\| = \max_{1 \leq i \leq n} \Delta x_i.$$

For each i, pick any sample point $x_i^* \in [x_{i-1}, x_i]$ (say, the right end point, or the left end point, or the middle point, etc.). Then we form the <u>Riemann sum</u> of f under the partition P with the choice $\{x_1^*, \cdots, x_n^*\}$ of sample points by

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n.$$

In particular, for any partition P, we have the <u>right Riemann sum</u> $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x_i$ and the <u>left Riemann sum</u> $\displaystyle\sum_{i=1}^{n} f(x_{i-1})\Delta x_i$.

When $f \geq 0$, each of these n terms in a Riemann sum stands for the area of a small rectangle. In this case, a Riemann sum can be considered as an approximation of the area A of the region under the graph of f.

Of course, under the same partition P, we have many Riemann sums. The definite integral $\int_a^b f(x)\, dx$ will be defined as a kind of limit of the Riemann sums.

Definition. Suppose there is $L \in (-\infty, \infty)$ is such that for any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$, when $\|P\|$ is "small", for any choice $x_1^*, \cdots, x_n^*$ of sample points, the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is "close" to L $\big($i.e., $\Big|\sum_{i=1}^{n} f(x_i^*)\Delta x_i - L\Big|$ is "small"$\big)$. Then we say that f is integrable and write

$$\int_a^b f(x)\,dx = L,$$

which is called the definite integral of f on $[a, b]$.

Here, the integral sign $\int$ is stretched from the letter "S", $f(x)$ is called the integrand, and a and b are called the lower and upper limits of the integral.

Example 1. Let $f : [0, 1] \to (-\infty, \infty)$ be given by $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$

For any partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[0, 1]$, we choose rational sample points $\{x_1^*, \cdots, x_n^*\}$ and irrational sample points $\{z_1^*, \cdots, z_n^*\}$. Then

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} f(z_i^*)\Delta x_i = \sum_{i=1}^{n} \Delta x_i = 1.$$

Now there is no number L such that these two Riemann sums both are close to L. Therefore, the definite integral $\int_a^b f(x)\,dx$ does not exist. ■

Question. Can we determine whether f is integrable without using the definition?

Fact 1. If f is continuous or has only finitely many jump discontinuity on $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ exists.

Fact 2. When f is integrable, we can "calculate" $\int_a^b f(x)dx$ via some special Riemann sums.

We consider below two special cases.

Let n be a positive integer. Partition $[a, b]$ into subintervals of equal length:

$$\Delta x = \frac{b-a}{n}, \quad x_0 = a, \quad x_1 = a + \Delta x, \quad \cdots, \quad x_n = a + n\Delta x.$$

Then the right and left Riemann sums of f are given by

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \frac{(b-a)}{n} \sum_{i=1}^{n} f(a + i\Delta x)$$

and

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \frac{(b-a)}{n} \sum_{i=1}^{n} f(a + (i-1)\Delta x).$$

Fact 3. If f is integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n.$

The above denotes limit of a sequence of real numbers, which is discussed in Chapter 11.

Some Useful Formulas (which can be proved by using Mathematical Induction).

(1) $\displaystyle\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$

(2) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$

(3) $\displaystyle\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$

Example 2. Use right Riemann sums to find the following definite integrals.

i) $\int_0^1 (x^2 + x)\,dx.$

Solution. For any positive integer n, we have

$$R_n = \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i^2}{n^2} + \frac{i}{n}\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^2}\sum_{i=1}^{n} i$$

$$= \frac{1}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) + \frac{1}{2}\left(1 + \frac{1}{n}\right) \longrightarrow \frac{2}{6} + \frac{1}{2} = \frac{5}{6} \quad (\text{as } n \to \infty).$$

Therefore, $\int_0^1 (x^2 + x)\,dx = \frac{5}{6}$. ∎

ii) $\int_0^2 x^3\,dx$.

Solution. For any positive integer n, we have

$$\begin{aligned} R_n &= \frac{2}{n}\sum_{i=1}^{n} f\Big(\frac{2i}{n}\Big) = \frac{2}{n}\sum_{i=1}^{n}\Big(\frac{8i^3}{n^3}\Big) = \frac{16}{n^4}\sum_{i=1}^{n} i^3 \\ &= \frac{16}{n^4}\Big(\frac{n(n+1)}{2}\Big)^2 = 4\Big(1+\frac{1}{n}\Big)^2 \longrightarrow 4 \ \ (\text{as } n \to \infty). \end{aligned}$$

Therefore, $\int_0^2 x^3\,dx = 4$. ■

The properties below on definite integrals follows from Fact 3.

Some Properties of Definite Integrals. Assume that f and g are integrable.

(1) $\int_a^b k\,dx = k(b-a)$.

(2) $\int_a^b \big(f(x) \pm g(x)\big)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

(3) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$.

(4) If $a < c < b$, then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

Example 3. Suppose $\int_0^{10} f(x)\,dx = 5$ and $\int_2^{10} f(x)\,dx = 7$. Find $\int_0^2 f(x)\,dx$.

Solution. We have $\int_0^2 f(x)\,dx = \int_0^{10} f(x)\,dx - \int_2^{10} f(x)\,dx = -2$. ■

If $g(x) \geq f(x) \geq 0$, then clearly the area under the graph of g is greater than or equal to the area under the graph of f.

This can be generalized to the general situation in terms of definite integrals and can be obtained from Fact 3.

Comparison Property of Definite Integrals.

Assume that f and g are integrable and $g(x) \le f(x)$ for all $x \in [a,b]$. Then

$$\int_a^b g(x)\,dx \;\le\; \int_a^b f(x)\,dx.$$

In particular, we have

(1) if $f \ge 0$, then $\int_a^b f(x)\,dx \;\ge\; 0$;

(2) if $m \;\le\; f \;\le\; M$, then $m(b-a) \;\le\; \int_a^b f(x)\,dx \;\le\; M(b-a)$, or we can write

$$m \;\le\; \frac{1}{b-a}\int_a^b f(x)\,dx \;\le\; M.$$

Due to Fact 3, the above number $\frac{1}{b-a}\int_a^b f(x)\,dx$ is called the average value of f on $[a,b]$.

Example 4. Show that $1 \;\le\; \int_0^1 \sqrt{1+x^2}\,dx \;\le\; \int_0^1 \sqrt{1+x}\,dx \;\le\; \sqrt{2}$.

Solution. When $0 \le x \le 1$, we have $0 \;\le\; x^2 \;\le\; x$ and hence $1 \;\le\; 1\,+\,x^2 \;\le 1\,+\,x \;\le\; 2$. Therefore, if $0 \le x \le 1$, then

$$1 \;\le\; \sqrt{1+x^2} \;\le\; \sqrt{1+x} \;\le\; \sqrt{2}.$$

It follows from the Comparison Property that

$$1 \;=\; \int_0^1 1\,dx \;\le\; \int_0^1 \sqrt{1+x^2}\,dx \;\le\; \int_0^1 \sqrt{1+x}\,dx \;\le\; \int_0^1 \sqrt{2}\,dx \;=\; \sqrt{2}. \qquad \blacksquare$$