Introduction

May 7th 2024

* Integers: eg: $-1, 2, -2, 3, \ldots$ ($\mathbb{Z}$)
* Rational n°: eg ratio of integer a/b $b \neq 0$ ($\mathbb{Q}$)
* Real number: $1, 2, -2, 1, 2, \ldots$ $\mathbb{R}$
* Complex numbers ($\mathbb{C}$)

Absolute values

* $|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a \leq 0 \end{cases}$ eg: $\begin{cases} |2| = 2 \\ |-2| = 2 \end{cases}$

* $|a+b| \leq |a| + |b|$, with equality when a and b have the same sign.

e.g: $|-1+2| \overset{?}{=} |-1| + |2|$
$\Rightarrow |1| \overset{?}{=} 1 + 2$
$\Rightarrow 1 \neq 3$

statement is equal if they have the same sign.

eg: $|-1+(-2)| \overset{?}{=} |-1| + |-2|$
$\Rightarrow |-1-2| \overset{?}{=} 1+2$
$|-3| \overset{?}{=} 3$
$3 = 3$

OR

$|1+2| = |1| + |2|$
$|3| = 1+2$
$3 = 3$

* $|a \cdot b| = |a| \cdot |b|$

* $|a^n| = |a|^n$

eg $|2^{-1}| = |2|^{-1}$
$\left|\frac{1}{2}\right| = 2^{-1}$
$0.5 = \frac{1}{2}$
$0.5 = 0.5$

* $|a/b| = |a|/|b|$ (this follows the product rule of the second bullet - How?)

i.e: $|a| = \left|b \cdot \frac{a}{b}\right| = |b| \cdot \left|\frac{a}{b}\right|$

$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

$\frac{\cancel{|b|} \cdot \left|\frac{a}{b}\right|}{\cancel{|b|}} = \left|\frac{a}{b}\right|$ hence $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Problems

② $|3+2| \leq |3| + |2| \Rightarrow 3+2 = 5$

$|-3+2| \leq |-3| + |2| \Rightarrow 3+2 = 5$

③ $|-3 \cdot 2| = |-3| \cdot |2| = (-(-3)) \cdot 2 \Rightarrow 3 \cdot 2 = 6$

④ $|-2^3| = |-2|^3 = (-(-2))^3 = 2^3 = 8$

⑤ $|a - b| \leq |a - b|$

i.e $|a| = |a - b + b| \leq |a-b| + |b|$

$\Rightarrow |a| - |b| \leq |a-b| + \cancel{|b|} - \cancel{|b|}$

therefore $|a| - |b| \leq |a - b|$

Laws of exponents

* $a^n = a \cdot a \cdot a, \cdots, a$ (n factors), a is known as the base and n as the exponent.

* For $a \neq 0$; $a^0 = 1$

* For $a \neq 0$; $a^{-n} = 1/a^n$

* $a^m \cdot a^n = a^{m+n}$

* $a^n \cdot b^n = (a \cdot b)^n$. (different bases, but same exponent)

* $(a^m)^n = a^{mn}$

* $\frac{a^m}{a^n} = a^{m-n}$

* $a^n/b^n = (a/b)^n$ (different bases, but same exponent)

Logarithm

Definition

let $b > 0$ but not equal 1 and let x be positive. The logarithm of x to the base b is a real number y such that $b^y = x$

i.e $\log_b x = y$ or $\ln_e x = y$

Properties of logarithms

① $\log_b(xy) = \log_b x + \log_b y$

② $\log_b(x/y) = \log_b x - \log_b y$

③ $\log_b x^n = n \log_b x$

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(4) $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$

Order, Range, inequalities

(1) $x \leq x$ (reflexive law)

(2) $x = y$ iff $x \leq y$ and $y \leq x$ (symmetry law)

(3) if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitive law)

Range

(*) $a \leq x \leq b$ described as $[a, b]$ $(\,(a,b)\,)$

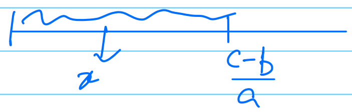
(*) $a < x \leq b$ " $(a, b]$ $(\,[a, b)\,)$

Solving linear inequalities

$ax + b \leq c$

$ax \leq c - b$

$x \leq \dfrac{c-b}{a}$



$\left(-\infty, \dfrac{c-b}{a}\right]$

ex: $a = 2$; $b = 3$; $c = 5$

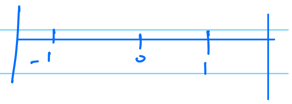
$x \leq \dfrac{5-3}{2}$

$x \leq 1 \Rightarrow (-\infty, 1]$

Floor and ceiling functions

The floor of x is denoted by $\lfloor x \rfloor$

ex : if $x = 3.5$, then $\lfloor 3.5 \rfloor = 3$



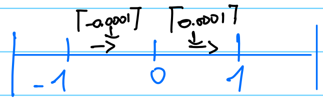
$x = -3.5$ then $\lfloor 3.5 \rfloor = -4$

$x = -3.1$ then $\lfloor -3.1 \rfloor = -4$

Ceiling

its the smallest integer exceeding x denoted by $\lceil x \rceil$

eg $\lceil 0.0001 \rceil = 1$



$\lceil -0.0001 \rceil = 0$

Show $\lfloor x \rfloor \leq \lceil x \rceil$

$$\left.\begin{array}{l}\lfloor x \rfloor \leq x \\ \lceil x \rceil \geq x\end{array}\right\} \quad \lfloor x \rfloor \leq x \leq \lceil x \rceil$$

$\Downarrow$

transitive law

$$\lfloor x \rfloor \leq \lceil x \rceil$$

$$\left.\begin{array}{l}\lfloor 3.5 \rfloor = 3 \leq 3.5 \\ \lceil 3.5 \rceil = 4 \text{ ie } 4 \leq \lceil 3.5 \rceil\end{array}\right\}$$

meaning $\lfloor x \rfloor \leq \lceil x \rceil$ because $\lfloor x \rfloor \leq x \leq \lceil x \rceil$

exercises

If n is an integer show that $\lfloor x + n \rfloor = \lfloor x \rfloor + n$

Propositional Logic

A proposition is a declarative sentence that is either true or false.

They are denoted by single letters. {T, F}

Propositional variables : p, q, r, s, --

Compound propositions

They are constructed from logical connectives and other propositions.

- Negation $\neg$
- Conjunction $\wedge$ (times $\times$)
- Disjunction $\vee$ (plus $+$)
- Implication $\rightarrow$
- Biconditional $\leftrightarrow$

Negation

P	$\neg$ P
T	F
F	T

Conjunction (and or times ($\times$))

P	Q	$P \wedge Q$
T(1)	F(0)	F(0)
F(0)	T(1)	F(0)
F(0)	F(0)	F(0)
T(1)	T(1)	T(1)

Disjunction (OR or + plus)

P	Q	$P \vee Q$

Exclusive OR $\oplus$

For $p \oplus q$ to be true, one of p and q must be true but not both.

Implication

P	Q	P → Q
T	F	F
F	T	T
F	F	T
T	T	T

False iff P is True and Q is False. Otherwise true.

Biconditional

P	Q	P ↔ Q
T	T	T
T	F	F
F	T	F
F	F	T

* True iff both are true or iff both are false

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Construction of Truth tables

for n proposition we get 2^n rows and columns

Precedence (Priority) of logical operators

Operator	Precedence
$\neg$	**1**
$\wedge$	**2**
$\vee$	**3**
$\rightarrow$	**4**
$\leftrightarrow$	**5**

Translating english sentences to propositions

Ex, : "The automated reply cannot be sent when the file system is full"

Sol : * P : "The automated reply can be sent."
* Q : "The file system is full."

Translation : $\neg P \rightarrow Q$

* **Tautology** : is a proposition that is always true.
* **Contradiction** : is " " " is " false.
* **Contigency** " " " " neither a tautology nor a contradiction.
* **Logical equivalent** : p and q are logically equivalent if $p \leftrightarrow q$ is a tautology. (ie we write $p \equiv q$)

De Morgan's law

* $\neg(p \wedge q) \equiv \neg p \vee \neg q$ (law 1)
* $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (law 2)

Key logical Equivalences

Identity law :	$p \wedge T \equiv p$	$p \vee F \equiv p$
Domination law :	$p \vee T \equiv T$	$p \wedge F \equiv F$
Idempotent law:	$p \vee p \equiv p$	$p \wedge p \equiv p$

Double negation law: $-(-p) \equiv p$

Negation law: $p \vee -p \equiv T$ $\qquad$ $p \wedge -p \equiv F$

Commutative law: $p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

Associative law: $p \vee (q \vee r) \equiv (p \vee q) \vee r$

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

Distributive law: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ [! not true in math!]

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Absorption laws: $p \vee (p \wedge q) \equiv p$

$p \wedge (p \vee q) \equiv p$

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Ex

Show that $p \vee q \to p \vee q \equiv -(p \wedge q) \vee (p \vee q)$

$\equiv (-p \vee -q) \vee (p \vee q)$

$\equiv (-p \vee p) \vee (-q \vee q) \equiv T \vee T \equiv T$

ie

$(-p \vee -q) \vee (p \vee q) \equiv ((-p \vee q) \vee p) \vee q \equiv -p \vee (-q \vee p) \vee q$

$(-p \vee (p \vee -q)) \vee q \equiv ((-p \vee p) \vee -q) \vee q \equiv (-p \vee p) \vee (-q \vee q)$

$T \vee T \equiv T$

Ex2

Show that $-(p \to -q) \to -q$ is a tautology

p	q	$-q$	$p \to -q$	$-(p \to -q)$	$-(p \to -q) \to -q$
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T

It is not a tautology

Ex3: Show that $p \to q$ and $-q \to -p$ are logically equivalent.

$$p \to q \equiv -p \vee q$$
$$\equiv -p \vee -(-q)$$
$$\equiv -(-q) \vee -p$$
$$\equiv -q \to -p$$ by using the same logic $p \to q \equiv -p \vee q$

Note: $\boxed{p \to q \equiv -p \vee q}$ By truth table

Consistent system specifications

A list of propositions is consistent if it is possible to assign truth values to the proposition variables so that each proposition is true.

e.g: 1. $p \vee q$

2. $\neg q$

3. $p \to q$

- When p is false and q is true all the statements are true, so the specifications are consistent.

Interval notation

$[a,b] = \{x \mid a \le x \le b\}$

$[a,b) = \{x \mid a \le x < b\}$

$(a,b] = \{x \mid a < x \le b\}$

$(a,b) = \{x \mid a < x < b\}$

Set Equality

A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$

Subset

If $A = \{1,2,3\}$ and $B = \{1,2\}$ then $B \subseteq A$
subset.

$A \subseteq B \equiv \forall x\,(x \in A \to x \in B)$

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for any set S
2. Because $a \in S \to a \in S$, $S \subseteq S$ for any set S.

$$\underbrace{A = B \equiv}_{} \quad \forall x \left[\underbrace{(x \in A \to x \in B)}_{A \subseteq B} \wedge \underbrace{(x \in B \to x \in A)}_{B \subseteq A} \right]$$

Example

If $A = \{1,2,3,4\}$ and $B = \{1,2,3,4,5\}$ Then $A \subseteq B$ but $B \nsubseteq A$

Proper subsets

$A \subset B \equiv A \subseteq B$ but $A \neq B$

eg $A \subset B$ if $A = \{1,2\}$ and $B = \{1,2,3\}$
proper subset

$\forall x\,(x \in A \to x \in B) \wedge \exists x\,(x \in B \wedge x \notin A)$

Set Cardinality

The cardinality of a finite set A denoted by $|A|$ is the number of (distinct) element of A

examples

1. $|\{1,2,3,3,4\}| = 4$
2. $|\{1,2,3,4\}| = 4$
3. $|\emptyset| = 0$
4. $|\{\{1,2,3\},1,2,3\}| = 4$
5. $|\{\emptyset\}| = 1$

Power sets

The set of all subsets of a set A denoted $P(A)$, is called the power set of A.

Example ⊛ $A = \{1,2,3\}$

$P(A) = 2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

⊛ $B = \{a,b\}$

$P^B = 2^B = \{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$

→ If a set has n element, then the cardinality of the power set is $2^n = 2^{|A|}$.

Cartesian Product

$A \times B = \{(a,b) \mid a \in A \wedge b \in B\}$

Example $A = \{1,2\}$; $B = \{3,4\}$

$A \times B = \{(1,3),(1,4),(2,3),(2,4)\}$

The cartesian product of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

Note $|\{1, 1, 2\}| = |\{1, 2\}| = |\{2, 1\}| = 2$

or $|\{\emptyset, \emptyset\}|, |\{\emptyset\}| = 1$

because it's the distinct element

- $A = B$ iff $A \subseteq B \wedge B \subseteq A$
- $|A \times B| = |A|\,|B|$
- $|A \times B \times C| = |A|\,|B|\,|C|$

The cardinality of two or more set multiplied is the number of combinations needed.

Examples:

(i) $A = \{a, b\}$, $B = \{1, 2, 3\}$

$A \times B = \{\underbrace{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)}_{6 \text{ elements}}\}$.

(ii) $P(x) \equiv |x| = 1$

$x \in \{-1, 0, 1\} = D$

$\{x \in D \mid P(x)\} = \{-1, 1\}$

(iii) Suppose A, B, C are sets such that $A \subseteq B$ and $B \subseteq C$. Show $A \subseteq C$.

Solution $A \subseteq B \wedge B \subseteq C$ to show $A \subseteq C$

$\forall x\,(\underbrace{x \in A}_{p} \to \underbrace{x \in B}_{q}) \wedge \forall x\,(\underbrace{x \in B}_{q} \to \underbrace{x \in C}_{r})$

Reminder

$p \to q$

$q \to r$

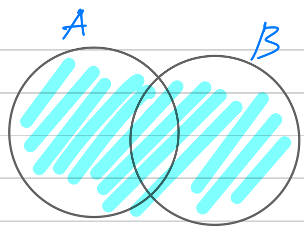
$\therefore p \to r$ (H.S)

$\forall x\,(x \in A \to x \in C) \equiv A \subseteq C$

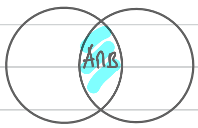
Union

Example ① $A = \{1,2,3\}$; $B = \{3,4,5\}$

$A \cup B = \{1,2,3,4,5\}$

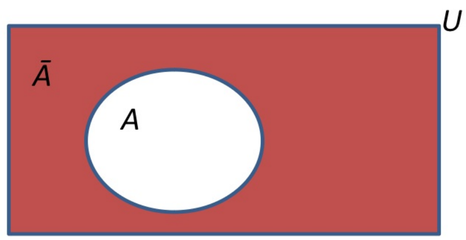


Intersection : When A and B meet



Complement ; $\bar{A} = \{x \in U \mid x \notin A\}$

Venn Diagram for Complement

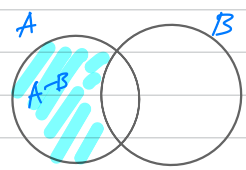


Ex: If $U = \{x \in \mathbb{Z}^+ \mid 1 \leq x < 100\}$

$S = \{x \in U \mid x \geq 70\}$

$\bar{S} = \{x \in U \mid 1 \leq x \leq 70\}$

Difference : $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \bar{B}$



Ex: $U = \{1,2,3,4,5\}$

$\underbrace{\{1,2,3\}}_{A} - \underbrace{\{3,4,5\}}_{B}$

$A - B = \{1,2\}$

Also

$A \cap \bar{B} = \{1,2,3\} \cap \{1,2\}$

$= \{1,2\}$

The cardinality of the union of the sets.

$|A \cup B| = |A| + |B| - |A \cap B|$

if A and B are totally different or apart like this

then $|A \cup B| = |A| + |B|$

Proving set identities

When proving set identities

① $A \equiv B$ iff $A \subseteq B \wedge B \subseteq A$

Ex Let's prove that $\overline{A \cap B} \equiv \bar{A} \cup \bar{B}$ meaning $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B} \wedge \overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

$x \notin A \cap B$

$\neg(x \in A \cap B)$

$\neg(x \in A \wedge x \in B)$

$\neg(x \in A) \vee \neg(x \in B)$

$x \notin A \vee x \notin B$

$x \in \bar{A} \vee x \in \bar{B}$

$x \in \bar{A} \cup \bar{B}$

② Builder notation (i.e. $A = \{x \mid P(x)\}$

Ex: $\overline{A \cap B} = \{x \mid x \notin A \cap B\}$

$= \{x \mid \neg(x \in (A \cap B)\}$

$= \{x \mid \neg(x \in A \wedge x \in B)\}$

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$

$= \{x \mid x \notin A \vee x \notin B\}$

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$

③ Membership tables (Truth tables)

* Union $(A \cup B)$ is considered as addition (+)
* Intersection $(A \cap B)$ is considered as multiplication (*)
* Difference $(A - B)$ " " as Subtraction (−)

Problem

① Let $A = \{1,2,3,4,5\}$ and $B = \{0,3,6\}$

⊛ $A \cup B = \{0,1,2,3,4,5,6\}$; ⊛ $A \cap B = \{3\}$; ⊛ $A - B = \{1,2,4,5\}$

$\{x \mid x \in A \vee x \in B\}$ $\{x \mid x \in A \wedge x \in B\}$ $\{x \mid x \in A \wedge x \in \bar{B}\}$

⊛ $B - A = \{0,6\}$

$\{x \mid x \in B \wedge x \in \bar{A}\}$

② Prove $A \cup B = B \cup A$ (Commutative law) using all the methods

Sol: ⊛ $A \cup B \subseteq B \cup A$ and $B \cup A \subseteq A \cup B$

1. $x \in A \cup B$
2. $x \in A \vee x \in B$ (definition of Union from 1)
3. $x \in B \vee x \in A$ (Commutative law for propositions from 2)
4. $x \in B \cup A$ (definition of union from 3)

⊛ $A \cup B = \{x \mid x \in A \cup B\}$

$= \{x \mid x \in A \vee x \in B\}$ (definition of union)

$= \{x \mid x \in B \vee x \in A\}$ (Commutative law)

$= \{x \mid x \in B \cup A\}$ (definition of union)

Therefore $B \cup A \equiv A \cup B$

⊛

A	B	$A \cup B$ ✓	$B \cup A$ ✓
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

Therefore $A \cup B \equiv B \cup A$

⊛ Show that if A and B are sets then $A - B = A \cap \overline{B}$

Sol: $A - B = A \cap \overline{B}$

$A - B \subseteq A \cap \overline{B}$ and $A \cap \overline{B} \subseteq A - B$

$x \in (A - B)$ (assumption)
$x \in A \wedge x \notin B$ (definition of difference of operations)
$x \in A \wedge x \in \overline{B}$ (def. of negation)
$x \in (A \cap \overline{B})$ (def. of intersection)

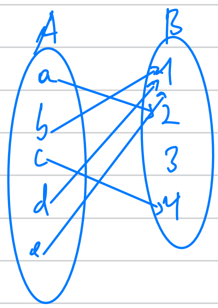
Note: When we get asked:

⊛ Equivalence = Two ways

⊛ Inference = One ways

Note: (*) Not onto if $|B| > |A|$

(*)



$F(A) = \{F(a), F(b), F(c), F(d), F(e)\}$
$= \{2, 1, 4, 1, 1\}$
$= \{1, 2, 4\}$

Problem

$f(x) = x^3, \quad x \in \mathbb{Z}$

Find $f^{-1}(x)$.

Solution

$f(x) = x^3$ Therefore $f^{-1}(x) = \sqrt[3]{x}$

$y = x^3$

$x = \sqrt[3]{y}$

Composition function

Ex $f(x) = 2x+3$; $g(x) = 3x+2$

$$f \circ g(x) = 2(3x+2)+3 = 6x+4+3 = \underline{6x+7}$$

$$g \circ f(x) = 3(2x+3)+2 = 6x+9+2 = \underline{6x+11}$$

Properties of Algorithms

Definiteness

output

C C D E F + I/O

Completeness

Finiteness

Input

Correctness

Effectiveness

Note: **Algorithm**

```
max:= an
for i:= an-1 downto 1
if i > max then
return (max)
end for
```