MATH 1250 Lecture 6

Section 2.4 Rules for Matrix Operations

Addition: $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij}+b_{ij}]_{m\times n}$

Note. $A+B$ is defined only if they have same size.

Scalar multiple: $c[a_{ij}]_{m\times n} = [c\,a_{ij}]_{m\times n}$.

Ex. $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 2 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 3 \\ 6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 1 \\ 9 & 4 & 2 \end{bmatrix}$

Matrix multiplication:

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{n\times p}$, then $AB=[c_{ij}]_{m\times p}$,

where $c_{ij} = \underbrace{(a_{i1}, a_{i2}, \cdots, a_{in})}_{\text{row } i \text{ of } A} \cdot \underbrace{(b_{1j}, b_{2j}, \cdots, b_{nj})}_{\text{column } j \text{ of } B}$

Note AB is defined only if # of columns of A = # of rows of B.

Ex. $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} (1,2)\cdot(4,0) & (1,2)\cdot(1,-1) & (1,2)\cdot(4,3) \\ (1,-1)\cdot(4,0) & (1,-1)\cdot(1,-1) & (1,-1)\cdot(4,3) \end{bmatrix}$

$= \begin{bmatrix} 4 & -1 & 10 \\ 4 & 2 & 1 \end{bmatrix}$.

Ex. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} (1,2,3)\cdot(2,1,-1) & (1,2,3)\cdot(1,-2,0) \\ (4,5,6)\cdot(2,1,-1) & (4,5,6)\cdot(1,-2,0) \end{bmatrix}$

$= \begin{bmatrix} 2+2-3 & 1-4+0 \\ 8+5-6 & 4-10+0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 7 & -6 \end{bmatrix}$

Ex. $\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ -6 & -12 & -18 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$

Ex. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 15 & 10 \end{bmatrix}$, $\begin{bmatrix} 5 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 3 & 4 \end{bmatrix}$

Note $AB \neq BA$.

zero matrix $O_{m\times n} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m\times n}$.

Identity matrix $I \overset{\text{or}}{=} I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$

Note If A is a $m\times n$ matrix, then $I_m A = A = A I_n$

Ex. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 7-7 & 2-2 \\ -7+7 & -2+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Note $AB = 0$ is possible even if $A \neq 0$ and $B \neq 0$.

If A is a $n\times n$ matrix (square matrix), then

$$A^2 = AA, \quad A^K = \underbrace{AA\cdots A}_{K}.$$

Ex $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$

Properties: $A+B=B+A$, $A+(B+C)=(A+B)+C$,
$A(BC)=(AB)C$, $A(B+C)=AB+AC$,
$(B+C)A=BA+CA$. $\quad$ ($AB \neq BA$ in general)

Note $AB = A[\vec{b_1}\ \vec{b_2} \cdots \vec{b_p}] = [A\vec{b_1}\ A\vec{b_2} \cdots A\vec{b_p}]$,
where $\vec{b_1}, \vec{b_2}, \cdots, \vec{b_p}$ are columns of B.

Section 2.3 Elimination Using Matrices

Def^n. A matrix E is an elementary matrix if it is obtained from the identity matrix I by a single row operation.

Remark. If A is a $m\times n$ matrix and E is a elementary matrix obtained from I_m by a certain row operation, then EA is the matrix obtained from A by the same row operation.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow{R_2+(-4)R_1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \end{bmatrix} = B$

$\xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} = C \xrightarrow{R_1+(-2)R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} = D$

(in RREF)

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{R_2+(-4)R_1} \begin{bmatrix}1&0\\-4&1\end{bmatrix}=E_1, \quad \begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix}1&0\\0&-\frac{1}{3}\end{bmatrix}=E_2,$$

$$\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{R_1+(-2)R_2} \begin{bmatrix}1&-2\\0&1\end{bmatrix}=E_3,$$ (E_1, E_2, E_3 are elementary matrices)

$$E_1A=\begin{bmatrix}1&0\\-4&1\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}1&2&3\\0&-3&-6\end{bmatrix}=B.$$

$$E_2B=\begin{bmatrix}1&0\\0&-\frac{1}{3}\end{bmatrix}\begin{bmatrix}1&2&3\\0&-3&-6\end{bmatrix}=\begin{bmatrix}1&2&3\\0&1&2\end{bmatrix}=C$$

$$E_3C=\begin{bmatrix}1&-2\\0&1\end{bmatrix}\begin{bmatrix}1&2&3\\0&1&2\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&1&2\end{bmatrix}=D$$

So $E_3E_2E_1A=D$.

Ex $A=\begin{bmatrix}0&1&-3&2\\1&3&-2&5\\2&4&3&8\end{bmatrix} \xrightarrow{R_1 \updownarrow R_2} B \xrightarrow{R_3-2R_1} C$, then

$$B=E_1A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}A=\begin{bmatrix}1&3&-2&5\\0&1&-3&2\\2&4&3&8\end{bmatrix}$$

$$C=E_2B=\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}B=\begin{bmatrix}1&3&-2&5\\0&1&-3&2\\0&-2&7&-2\end{bmatrix}$$