Section 6.1: Areas Between Curves

We know that if $f(x) \geq 0$ is continuous on $[a,b]$, then the area of the region under the graph of f and above the x-axis is given by $A = \int_a^b f(x)\,dx$.

Example 1. The area of the region under the graph of $f(x) = xe^x$ and above the x-axis between 0 and 1 is

$$A = \int_0^1 xe^x\,dx = \Big[xe^x - e^x\Big]_0^1 = 1.$$

Here, we use IBP and FTC-2. ■

In general, if f and g are two continuous functions on $[a,b]$, then the area between the graphs of f and g is given by

$$\boxed{A = \int_a^b \big|f(x) - g(x)\big|\,dx\,.}$$

Note that $\big|f(x) - g(x)\big|$ is a piecewise defined function given by

$$\big|f(x) - g(x)\big| = \begin{cases} f(x) - g(x) & \text{if } f(x) \geq g(x) \\ \\ g(x) - f(x) & \text{if } f(x) < g(x) \end{cases}.$$

So, to find the area A, we need find the intersection points of the two graphs.

Example 2. Find the area of the region enclosed by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.

Solution. First, for $0 \leq x \leq \frac{\pi}{2}$, $\sin x = \cos x \implies x = \frac{\pi}{4}$.

Now $\cos x \geq \sin x$ on $[0, \frac{\pi}{4}]$, and $\sin x \geq \cos x$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$. Therefore,

$$A = \int_0^{\frac{\pi}{2}} \big|\sin x - \cos x\big|\,dx = \int_0^{\frac{\pi}{4}} \big(\cos x - \sin x\big)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \big(\sin x - \cos x\big)\,dx$$

$$= \Big[\sin x + \cos x\Big]_0^{\frac{\pi}{4}} + \Big[-\cos x - \sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = 2(\sqrt{2} - 1). \qquad ■$$

Example 3. Find the area of the region enclosed by $y = \sqrt{x-1}$ and $y = x - 1$.

Solution. Note that now $x \geq 1$. For $x \geq 1$,

$$\sqrt{x-1} = x-1 \quad \Longrightarrow \quad x-1 = (x-1)^2 \quad \Longrightarrow \quad x-1 = 0 \;\text{ or }\; x-1 = 1;$$

that is, $x = 1$ or $x = 2$. So, the two curves have exactly two intersection points. Note that $\sqrt{x-1} \geq (x-1)$ on $[1,2]$. Therefore, we have

$$A = \int_1^2 \big(\sqrt{x-1} - (x-1)\big)\,dx = \Big[\frac{2}{3}(x-1)^{\frac{3}{2}} - \frac{1}{2}x^2 + x\Big]_1^2 = \frac{1}{6}. \quad \blacksquare$$

When calculating the area of a region bounded by curves, sometimes it is more convenient to express the equation of a curve by regarding x as a function of y.

Example 4. Find the area of the region enclosed by $x - y - 1 = 0$ and $2x - y^2 + 6 = 0$.

Solution. We write the equations of the two curves as

$$x = y + 1 \qquad \text{and} \qquad x = \frac{1}{2}y^2 - 3.$$

Then

$$y + 1 = \frac{1}{2}y^2 - 3 \quad \Longrightarrow \quad y^2 - 2y - 8 = 0 \quad \Longrightarrow \quad y = -2 \;\text{ or }\; y = 4.$$

So, the two curves have exactly two intersection points. Note that $y + 1 \geq \frac{1}{2}y^2 - 3$ on $[-2, 4]$. Therefore, we have

$$\begin{aligned} A &= \int_{-2}^4 \Big((y+1) - \big(\frac{1}{2}y^2 - 3\big)\Big)\,dy = \int_{-2}^4 \Big(y + 4 - \frac{1}{2}y^2\Big)\,dy \\ &= \Big[\frac{1}{2}y^2 + 4y - \frac{1}{6}y^3\Big]_{-2}^4 = 18. \quad \blacksquare \end{aligned}$$

Remark. Note that $\int_a^b \big|f(x) - g(x)\big|\,dx \neq \Big|\int_a^b \big(f(x) - g(x)\big)\,dx\Big|$. For example,

$$1 = \int_0^2 \big|x - 1\big|\,dx \neq \Big|\int_0^2 (x-1)\,dx\Big| = 0.$$

Section 6.2: Volumes

We know that the volume of a cylinder is given by

$$V = A\,h,$$

where A is the area of the base and h is the height of the cylinder. For example, the volume of a circular cylinder is $V = \pi r^2 h = Ah$, and the volume of a rectangular box is $V = lwh = Ah$.

Let us consider the volume of a more general solid S. For each $a \le x \le b$, let P_x denote the plane perpendicular to the x-axis and passing through x, and let $A(x)$ denote the area of the cross-section of S in P_x. Suppose that the solid S is between P_a and P_b. By the definition of a definite integral, the volume of S is given by

$$\boxed{V = \int_a^b A(x)\,dx.}$$

In this section, we consider the special case, where S is a solid of revolution so that there is an easy way to figure out the area function $A(x)$ of the cross-section.

Suppose $f(x) \ge 0$ is continuous on $[a, b]$. Let R denote the region enclosed by $y = f(x)$, $y = 0$, $x = a$, and $x = b$. Revolving R about the x-axis, we get a solid S of revolution. In this case, for $a \le x \le b$, $A(x) = \pi\big[f(x)\big]^2$. Therefore, the volume of S is given by

$$\boxed{V = \int_a^b \pi\big[f(x)\big]^2\,dx.}$$

Similarly, if $x = g(y) \ge 0$ is continuous on $[c, d]$ and R is the region enclosed by $x = g(y)$, $x = 0$, $y = c$ and $y = d$, then the volume of the solid obtained by revolving R about the y-axis is given by

$$\boxed{V = \int_c^d \pi\big[g(y)\big]^2\,dy.}$$

Example 5. Find the volume of the solid obtained by rotating about the x-axis the region enclosed by $y = x^3$, $y = 0$ and $x = 2$.

Solution. $V = \int_0^2 \pi\left(x^3\right)^2 dx = \int_0^2 \pi x^6 \, dx = \left[\frac{\pi}{7}x^7\right]_0^2 = \frac{128}{7}\pi$. ■

Example 6. Find the volume of the solid obtained by rotating about the y-axis the region enclosed by $y = x^3$, $y = 8$ and $x = 0$.

Solution. $V = \int_0^8 \pi\left(y^{\frac{1}{3}}\right)^2 dy = \int_0^8 \pi y^{\frac{2}{3}} \, dy = \pi\left[\frac{3}{5}y^{\frac{5}{3}}\right]_0^8 = \frac{96}{5}\pi$. ■

More generally, Suppose $f(x) \geq g(x)$ on $[a, b]$ and R is the region enclosed by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$. Then the volume of the solid obtained by revolving R about the x-axis is given by

$$\boxed{V = \int_a^b \pi\left(f(x)^2 - g(x)^2\right) dx.}$$

Similarly, if $p(y) \geq q(y)$ on $[c, d]$ and R is the region enclosed by $x = p(y)$, $x = q(y)$, $y = c$ and $y = d$, then the volume of the solid obtained by revolving R about the y-axis is given by

$$\boxed{V = \int_c^d \pi\left(p(y)^2 - q(y)^2\right) dy.}$$

Example 7. Find the volume of the solid obtained by rotating the region R enclosed by $y = x$ and $y = x^2$ about the x-axis and the y-axis, respectively.

Solution. The volume obtained by rotating the region R about the x-axis is

$$V_1 = \int_0^1 \pi\left(x^2 - (x^2)^2\right) dx = \pi\left[\frac{1}{3}x^3 - \frac{1}{5}x^5\right]_0^1 = \frac{2}{15}\pi.$$

The volume obtained by rotating the region R about the y-axis is

$$V_2 = \int_0^1 \pi\left((\sqrt{y})^2 - y^2\right) dy = \int_0^1 \pi\left(y - y^2\right) dy = \pi\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1 = \frac{1}{6}\pi. \quad ■$$

Section 6.3: Volumes by Cylindrical Shells

Let $f(x) \geq 0$ be continuous on $[a, b]$ and let R be the region enclosed by $y = f(x)$, $y = 0$, $x = a$ and $x = b$. In Section 6.2, we calculate the volume of the solid obtained by rotating R about the x-axis, which is given by $\int_a^b \pi f(x)^2 \, dx$. This method of calculation is call the disk/washer method.

Now we consider the volume V of the shell obtained by rotating the same region R about the y-axis.

First, we note that f is not invertible in general so that we cannot express the curve by regarding x as a function y. Also, it may be very complicated to find the volume by dividing R into some subregions and using the method discussed in Section 6.2. In this section, we will use the method of cylindrical shells to find the volume V by using only the given $f(x)$ and the interval $[a, b]$.

We start with the simplest case where f is a constant function. Then

$$\begin{aligned} V &= \text{volume of the outer cylinder} \; - \; \text{volume of the inner cylinder} \\ &= \pi r_2^2 h \; - \; \pi r_1^2 h \; = \; \pi(r_2 + r_1)(r_2 - r_1)h \; = \; 2\pi\left(\frac{r_2 + r_1}{2}\right)(r_2 - r_1)h \; = \; 2\pi r h \Delta r, \end{aligned}$$

where $r = \dfrac{r_2 + r_1}{2}$ and $\Delta r = r_2 - r_1$.

For the general case, by the definition of a definite integral, we can obtain the volume by

$$\boxed{V = \int_a^b 2\pi x f(x) \, dx.}$$

Similarly, if $g(y) \geq 0$ is continuous on $[c, d]$ and R is the region enclosed by $x = g(y)$, $x = 0$, $y = c$ and $y = d$, then the volume of the shell obtained by rotating R about the x-axis is given by

$$\boxed{V = \int_c^d 2\pi y g(y) \, dy.}$$

Also, if $f(x) \geq g(x)$ on $[a, b]$ and R is the region enclosed by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$, then the volume of the solid obtained by rotating R about the y-axis is given by

$$\boxed{V \;=\; \int_a^b 2\pi x\Big(f(x) - g(x)\Big)\,dx.}$$

Example 8. Find the volume of the solid obtained by rotating about the y-axis the region enclosed by $y = x^2(1 - x)$ and $y = 0$.

Solution. Note that $x^2(1-x) = 0 \implies x = 0$ or $x = 1$. Therefore, we have

$$V \;=\; \int_0^1 2\pi x x^2(1-x)\,dx \;=\; 2\pi\int_0^1 (x^3 - x^4)\,dx \;=\; 2\pi\Big[\frac{1}{4}x^4 - \frac{1}{5}x^5\Big]_0^1 \;=\; \frac{\pi}{10}. \qquad \blacksquare$$

Example 9. Find the volume of the solid obtained by rotating about the y-axis the region enclosed by $y = x$ and $y = x^2$.

Solution. Note that $x = x^2 \implies x = 0$ or $x = 1$. Also, when $0 \leq x \leq 1$, $x \geq x^2$. Therefore, we have

$$V \;=\; \int_0^1 2\pi x(x - x^2)\,dx \;=\; 2\pi\int_0^1 (x^2 - x^3)\,dx \;=\; 2\pi\Big[\frac{1}{3}x^3 - \frac{1}{4}x^4\Big]_0^1 \;=\; \frac{\pi}{6}.$$

Example 10. Find the volume of the solid obtained by rotating about the x-axis the region R enclosed by $y = x^2$ and $y = 9$.

Solution. Note that $x^2 = 9 \implies x = \pm 3$. We can solve this problem in two methods.

Method from Section 6.2 (Disk/Washer Method)

$$V \;=\; \int_{-3}^3 \pi\big(9^2 - (x^2)^2\big)\,dx \;=\; \pi\int_{-3}^3 (81 - x^4)\,dx \;=\; \pi\Big[81x - \frac{1}{5}x^5\Big]_{-3}^3 \;=\; \frac{1944\pi}{5}.$$

Method of Cylindrical Shells Note that the region R is symmetric about the y-axis. When $0 \leq x \leq 3$, the curve can be expressed as $x = \sqrt{y}$. Therefore, by the method of the cylindrical shells, we have

$$V \;=\; 2\int_0^9 2\pi y\sqrt{y}\,dy \;=\; 4\pi\int_0^9 y^{\frac{3}{2}}\,dy \;=\; 4\pi\Big[\frac{2}{5}y^{\frac{5}{2}}\Big]_0^9 \;=\; \frac{1944\pi}{5}. \qquad \blacksquare$$

Section 6.4: Work

Newton's Second Law of Motion says

$$F \quad = \quad m \, \frac{d^2 S}{dt^2} \, ,$$

where F = force, m = mass, $\dfrac{d^2S}{dt^2}$ = acceleration.

<u>Units in Metric System</u>:

m – kilogram (**kg**), t – second (**s**), S – meter (**m**), F – newton (**N**).

If F is a constant, then the <u>work</u> done on a object moving distance d is defined by

$$W \quad = \quad F\,d \qquad (\text{work} \quad = \quad \text{force} \times \text{distance})$$

In the metric system, where the unit for the force F is newton and the unit for the distance d is meter, the unit for the work W is joule (**J**).

More general, suppose a object moves along the x-axis from a to b. At each point x between a and b, a force $f(x)$ acts on the object. We wonder what is the total work done.

Let $P = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[a, b]$. The work done in moving the object from x_{i-1} to x_i is

$$W_i \quad \approx \quad f(x_i)\, \Delta x_i .$$

So, the total work done is

$$W \quad = \quad \sum_{i=1}^{n} W_i \quad \approx \quad \sum_{i=1}^{n} f(x_i)\, \Delta x_i \quad \longrightarrow \quad \int_a^b f(x)\, dx .$$

Therefore, we have

$$\boxed{W \quad = \quad \int_a^b f(x)\, dx}$$

where $f(x)$ is the <u>force function</u>.

Example 11. When a particle is at a distance x meters from the origin, a force of $x^2 + 2x$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 3$?

Solution. We have $W = \int_1^3 (x^2 + 2x)\, dx = \left[\frac{1}{3}x^3 + x^2\right]_1^3 = \frac{50}{3}\,\mathbf{J}.$ ■

Remark. Note that units used in the British system are

$$F - \text{pound } (\mathbf{lb}), \qquad d - \text{foot } (\mathbf{ft}), \qquad W - \mathbf{ft\text{-}lb}\,.$$

We know that $1\ \mathbf{ft\text{-}lb} = 1.36\ \mathbf{J}$.

So, in Example 11, if the distance is measured in foot and the unit for force is pound, then the answer will be $\frac{50}{3}$ **ft-lb**.

Hooke's Law. The force required to hold a spring stretched x units beyond its natural length is

$$f(x) = kx\,,$$

where k is a constant, called the spring constant.

Example 12. A force of 40N is required to hold a spring stretched from its natural length of 10cm to a length 15cm. How much work is done to stretch the spring from 15cm to 18cm?

Solution. First, let us find the spring constant k. Note that

$$x = 15\text{cm} - 10\text{cm} = 5\text{cm} = 0.05\text{m}.$$

Since $f(x) = kx$, we have

$$40 = 0.05\,k \implies k = 800.$$

Therefore, we have

$$W = \int_{0.05}^{0.08} 800\,x\,dx = \left[400\,x^2\right]_{0.05}^{0.08} = 1.56\,\mathbf{J}.$$ ■

Section 6.5: Average Value of a Function

If $y_1, \cdots, y_n$ are some real numbers, then the average value is $y_{\text{ave}} = \frac{1}{n}\big(y_1 + \cdots + y_n\big)$. Now let $f(x)$ be a continuous function on $[a, b]$. Then what is the average value of f?

Recall that $\int_a^b f(x)\,dx$ can be obtained from right Riemann sums:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$$

$$= (b-a) \lim_{n\to\infty} \frac{1}{n} \Big(f(x_1) + \cdots + f(x_n)\Big).$$

Therefore, we define <u>the average value of f on $[a, b]$</u> by

$$\boxed{f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx\,.}$$

<u>**Fact 1**</u>. If $f(x) \geq 0$ on $[a, b]$, then the area A under the graph of f and above the x-axis is given by

$$A = \int_a^b f(x)\,dx = f_{\text{ave}}\,(b-a) = \text{the area of a rectangle.}$$

<u>**Fact 2**</u>. If f is continuous on $[a, b]$ and $m \leq f(x) \leq M$, where m is the absolute minimum of f on $[a, b]$ and M is the absolute maximum of f on $[a, b]$, then by the Comparison Property of definite integrals,

$$m \leq f_{\text{ave}} \leq M.$$

By the Intermediate Value Theorem for a continuous function, there exists c in $[a, b]$ such that $f(c) = f_{\text{ave}}$. Therefore, we have the following

<u>**Mean Value Theorem for Integrals**</u>. If f is continuous on $[a, b]$, then there exists c in $[a, b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Example 13. Let $f(x) = \sin x$. Find f_{ave} on $[0, \pi]$.

Solution. We have $f_{\text{ave}} = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x\, dx = \dfrac{1}{\pi}\Big[-\cos x\Big]_0^{\pi} = \dfrac{2}{\pi}$.

Note that it is not easy to find c in $[0, \pi]$ such that $f(c) = \dfrac{2}{\pi}$ though there are two values of such c. ■

Example 14. Let $f(x) = 1 + x^2$ $(-1 \le x \le 2)$. Find c in $[-1, 2]$ such that $f(c) = f_{\text{ave}}$.

Solution. We have $f_{\text{ave}} = \dfrac{1}{3}\displaystyle\int_{-1}^{2}(1 + x^2)\, dx = \dfrac{1}{3}\Big[x + \dfrac{1}{3}x^3\Big]_{-1}^{2} = 2$. Solving

$$f(c) = f_{\text{ave}} = 2,$$

we have

$$1 + c^2 = 2 \implies c = -1,\ 1. \qquad ■$$

Example 15. Let $f(x) = 3x^2 + 1$. Find the numbers a such that f_{ave} on the interval $[a,\ a+1]$ is equal to 2.

Solution. By the definition, f_{ave} on the interval $[a,\ a+1]$ is

$$f_{\text{ave}} = \int_a^{a+1}(3x^2 + 1)\, dx = \Big[x^3 + x\Big]_a^{a+1} = (a+1)^3 + (a+1) - a^3 - a = 3a^2 + 3a + 2.$$

Solving $f_{\text{ave}} = 3a^2 + 3a + 2 = 2$, we get that $a = 0,\ -1$. ■