Section 5.3: The Fundamental Theorem of Calculus

Recall that if $F' = f$ (i.e., F is an antiderivative of f) on an interval, then the general antiderivative of f is given by

$$\int f(x)\,dx \;=\; F(x) + C,$$

which is also called the indefinite integral of f. The above equality can also written as

$$\int F'(x)\,dx \;=\; F(x) + C.$$

Question 1. If F' is integrable on $[a,b]$, what is $\int_a^b F'(x)\,dx$?

Answer. $\int_a^b F'(x)\,dx \;=\; F(b) - F(a)$.

This is Part 2 of the FTC, where F' is required to be continuous on $[a,b]$ (and hence F' is integrable on $[a,b]$).

Question 2. When does an integrable function have an antiderivative?

This question is answered in Part 1 of the FTC.

The Fundamental Theorem of Calculus, Part 1 (FTC-1). If f is continuous on $[a,b]$, then the function g given by

$$g(x) \;=\; \int_a^x f(t)\,dt \qquad (a \le x \le b)$$

is differentiable on $[a,b]$ and $g'(x) = f(x)$ for all $x \in [a,b]$.

Idea of the proof. Condisder the case where $h > 0$ and $[x, x+h] \subseteq [a,b]$. Then

$$\frac{g(x+h) - g(x)}{h} \;=\; \frac{1}{h}\Big(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big) \;=\; \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Note that f is continuous on $[x, x+h]$ and $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is an number between the maximum and the minimum of f on the interval $[x, x+h]$ (by the Comparison Property). Then by the Intermediate Value Theorem, there exists c_h in $[x, x+h]$ such that $\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c_h)$. When $h \to 0$, $c_h \to x$ and thus $\frac{g(x+h)-g(x)}{h} = \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c_h) \to f(x) = g'(x)$. ■

Combining FTC-1 with the Chain Rule, we have

Corollary. If f is a continuous function and u and v are differentiable functions, then the function g given by

$$g(x) \;=\; \int_{v(x)}^{u(x)} f(t)\,dt$$

is differentiable and $g'(x) \;=\; f\big(u(x)\big)u'(x) \;-\; f\big(v(x)\big)v'(x)$.

In particular, if $g(x) \;=\; \int_a^{u(x)} f(t)\,dt$, then $g'(x) \;=\; f\big(u(x)\big)u'(x)$.

Example 5. Find $g'(x)$ by using FTC-1.

(i) $g(x) \;=\; \int_0^x \sqrt{1+t^2}\,dt$.

(ii) $g(x) \;=\; \int_1^x \left(e^{t^2} + \sin^2 t + \ln(t^2)\right) dt$.

(iii) $g(x) \;=\; \int_0^{x^3} \sqrt{1+t^2}\,dt$.

Solution. $g'(x) \;=\; \sqrt{1+(x^3)^2}\,3x^2 \;=\; 3x^2\sqrt{1+x^6}$. ■

(iv) $g(x) \;=\; \int_x^{x^2} \sin t\,dt$.

Solution. $g'(x) \;=\; \sin(x^2)\,2x \;-\; \sin x \;=\; 2x\sin(x^2) \;-\sin x$. ■

The following version of Part 2 of the FTC can be obtained from FTC-1.

The Fundamental Theorem of Calculus, Part 2 (FTC-2). If f is continuous on $[a,b]$ and F is an antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)\,dx \;=\; F(b) \;-\; F(a) \;=\; \Big[F(x)\Big]_a^b\,.$$

Proof. Let $g(x) \;=\; \int_a^x f(t)\,dt \;\; (a \le x \le b)$. By FTC-1, we have $g'(x) = f(x)$; i.e., g is also an antiderivative of f. Thus $F(x) \;=\; g(x) + C_0$ for some number C_0. Therefore, we have

$$F(b) \;-\; F(a) \;=\; g(b) \;-\; g(a) \;=\; g(b) \;-\; 0 \;=\; \int_a^b f(t)\,dt;$$

that is, $\int_a^b f(x)\,dx \;=\; F(b) \;-\; F(a)$. ■

Example 6. Find $\int_a^b f(x)\,dx$ by using FTC-2.

(i) $\int_1^3 e^x\,dx \;=\; \Big[\;\mathbf{?}\;\Big]_1^3.$

Solution. $\int_1^3 e^x\,dx \;=\; \Big[\;e^x\;\Big]_1^3 \;=\; e^3 - e.$ ■

(ii) $\int_1^8 \frac{1}{x}\,dx \;=\; \Big[\;\mathbf{?}\;\Big]_1^8.$

Solution. $\int_1^8 \frac{1}{x}\,dx \;=\; \Big[\;\ln|x|\;\Big]_1^8 \;=\; \ln 8.$ ■

(iii) $\int_{-\frac{\pi}{2}}^{2\pi} \cos x\,dx \;=\; \Big[\;\mathbf{?}\;\Big]_{-\frac{\pi}{2}}^{2\pi}.$

Solution. $\int_{-\frac{\pi}{2}}^{2\pi} \cos x\,dx \;=\; \Big[\;\sin x\;\Big]_{-\frac{\pi}{2}}^{2\pi} \;=\; \sin(2\pi) - \sin\Big(-\frac{\pi}{2}\Big) \;=\; 1.$ ■

(iv) $\int_1^2 \frac{3+x^2}{x^3}\,dx \;=\; \Big[\;\mathbf{?}\;\Big]_1^2.$

Solution. $\int_1^2 \frac{3+x^2}{x^3}\,dx \;=\; \int_1^2 \big(3x^{-3} + \frac{1}{x}\big)\,dx \;=\; \Big[\;-\frac{3}{2}x^{-2} + \ln|x|\;\Big]_1^2 \;=\; \frac{9}{8} + \ln 2\,.$ ■

(v) $\int_0^2 \big|x(x-1)\big|\,dx \;=\; \Big[\;\mathbf{?}\;\Big]_0^2.$

Solution. Note that $\int_0^2 \big|x(x-1)\big|\,dx \;\neq\; \Big[\;\big|\frac{1}{3}x^3 - \frac{1}{2}x^2\big|\;\Big]_0^2.$ Now we have

$$\int_0^2 \big|x(x-1)\big|\,dx \;=\; \int_0^1 -x(x-1)\,dx \;+\; \int_1^2 x(x-1)\,dx$$

$$=\; \int_0^1 (x-x^2)\,dx \;+\; \int_1^2 (x^2-x)\,dx$$

$$=\; \Big[\;\frac{1}{2}x^2 - \frac{1}{3}x^3\;\Big]_0^1 \;+\; \Big[\;\frac{1}{3}x^3 - \frac{1}{2}x^2\;\Big]_1^2 \;=\; 1. \quad ■$$

Example 7. Show that $\int_0^{\frac{\pi}{6}} \cos(x^2)\, dx \ \geq\ \frac{1}{2}$.

Solution. When $0 \leq x \leq \frac{\pi}{6}$, we have $0 \leq x^2 \leq x \leq \frac{\pi}{2}$ and hence $\cos(x^2) \geq \cos x$. Therefore, we have

$$\int_0^{\frac{\pi}{6}} \cos(x^2)\, dx \ \geq\ \int_0^{\frac{\pi}{6}} \cos x\, dx \ =\ \Big[\ \sin x\ \Big]_0^{\frac{\pi}{6}} \ =\ \frac{1}{2}.$$

Remark 1. The FTC-2 shows that we can get the definite integral $\int_a^b f(x)\, dx$ from the indefinite integral $\int f(x)\, dx$. However, we are unable to get $\int f(x)\, dx$ from $\int_a^b f(x)\, dx$, since $\int_a^b f(x)\, dx$ is a number.

Remark 2. As the definite integral $\int_a^b f(x)\, dx$ is a number, the choice of the letter for the integral variable does not change the value of $\int_a^b f(x)\, dx$. However, we have

$$\int f(x)\, dx \ =\ F(x) \,+\, C, \qquad \int f(u)\, du \ =\ F(u) \,+\, C, \quad \text{etc.}$$

This is important in particular when we make an substitution (see Section 5.5).

Section 5.4: Indefinite Integrals and the Net Change Theorem

This section is actually a combination of Sections 4.9 and 5.3. The followings are just the definition of indefinite integral and the FTC-2:

$$\boxed{\int F'(x)\,dx \;=\; F(x) + C.}$$

$$\boxed{\int_a^b F'(x)\,dx \;=\; F(b) - F(a).}$$

Recall the fact that **every derivative formula/rule/property gives an integral formula/rule/property**. Therefore, those formulas on antiderivatives given in Section 4.9 can be given in the form of indefinite integrals.

$$\int x^n\,dx \;=\; \begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1 \\ \\ \ln|x| + C & \text{if } n = -1 \end{cases} \qquad (\textbf{Simple Power Rule})$$

where for $n < 0$, the formula holds on $(-\infty, 0)$ or $(0, \infty)$.

$$\int e^x\,dx \;=\; e^x + C; \qquad \int a^x\,dx \;=\; \frac{1}{\ln a}\,a^x + C \;\; (a > 0).$$

$$\int \sin x\,dx \;=\; -\cos x + C; \qquad \int \cos x\,dx \;=\; \sin x + C.$$

$$\int \sec^2 x\,dx \;=\; \tan x + C; \qquad \int \csc^2 x\,dx \;=\; -\cot x + C.$$

$$\int \sec x \tan x\,dx \;=\; \sec x + C.$$

$$\int \frac{1}{1+x^2}\,dx \;=\; \tan^{-1}x + C; \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx \;=\; \sin^{-1}x + C.$$

We can add more formulas to this list of formulas on indefinite integrals.

Section 5.5: The Substitution Rule

The Substitution Rule is an integration rule corresponding to the Chain Rule on derivative.

Suppose $F'(x) = f(x)$. Then for any differentiable function g, we have

$$\big(F(g(x))\big)' \;=\; F'(g(x))\,g'(x) \;=\; f(g(x))\,g'(x).$$

Hence, we have

$$\int f(g(x))\,g'(x)\,dx \;=\; \int \big(F(g(x))\big)'\,dx \;=\; F(g(x)) \,+\, C.$$

Now we write $u = g(x)$. Then we also have

$$F(g(x)) \,+\, C \;=\; F(u) \,+\, C \;=\; \int f(u)\,du.$$

Therefore, we have

<u>The Substitution Rule</u>. If $f(x)$ is continuous and $u = g(x)$ is differentiable, then

$$\boxed{\int f(g(x))\,g'(x)\,dx \;=\; \int f(u)\,du.}$$

Note that if $u = g(x)$, then $du = g'(x)\,dx$. So, the Substitution Rule can be formally obtained by replacing $g(x)$ by u and replacing $g'(x)\,dx$ by du.

The Substitution Rule can help to simplify an integral and eventually solve it. Note that the solved indefinite integral is still a function of x.

<u>Example 8</u>. Find the following integrals.

i) $\int x^3 \cos(x^4 + 2)\,dx \;=$ ________________________.

$$u \;=\; x^4 + 2 \quad\Longrightarrow\quad du \;=\; 4x^3\,dx \quad\Longrightarrow\quad x^3\,dx \;=\; \frac{1}{4}\,du.$$

Therefore,

$$\int x^3 \cos(x^4 + 2)\,dx \;=\; \frac{1}{4}\int \cos u\,du \;=\; \frac{1}{4}\sin u \,+\, C \;=\; \frac{1}{4}\sin(x^4 + 2) \,+\, C.$$

ii) $\int e^{3x}\,dx \;=\; \underline{\hspace{6cm}}.$

$$u \;=\; 3x \quad\Longrightarrow\quad du \;=\; 3\,dx \quad\Longrightarrow\quad dx \;=\; \frac{1}{3}\,du.$$

Therefore,

$$\int e^{3x}\,dx \;=\; \frac{1}{3}\int e^{u}\,du \;=\; \frac{1}{3}e^{u} \;+\; C \;=\; \frac{1}{3}e^{3x} \;+\; C.$$

iii) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx \;=\; \underline{\hspace{6cm}}.$

$$u \;=\; \tan x \quad\Longrightarrow\quad du \;=\; \sec^2 x\,dx \;=\; \frac{1}{\cos^2 x}\,dx.$$

Therefore,

$$\int \frac{e^{\tan x}}{\cos^2 x}\,dx \;=\; \int e^{u}\,du \;=\; e^{u} \;+\; C \;=\; e^{\tan x} \;+\; C.$$

iv) $\int \frac{\sin x}{1+\cos^2 x}\,dx \;=\; \underline{\hspace{6cm}}.$

$$u \;=\; \cos x \quad\Longrightarrow\quad du \;=\; -\sin x\,dx$$

Therefore,

$$\int \frac{\sin x}{1+\cos^2 x}\,dx \;=\; \int \frac{-1}{1+u^2}\,du \;=\; -\tan^{-1}u \;+\; C \;=\; -\tan^{-1}\big(\cos x\big) \;+\; C.$$

If $a \neq 0$, the simple substitutions $u = ax + b$ and $u = \frac{1}{a}x$ give the following useful formulas.

$$\boxed{\int \frac{1}{ax+b}\,dx \;=\; \frac{1}{a}\ln|ax+b| \;+\; C}$$

$$\boxed{\int \frac{1}{a^2+x^2}\,dx \;=\; \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \;+\; C}$$

$$\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx \;=\; \sin^{-1}\left(\frac{x}{a}\right) \;+\; C \quad (a>0)}$$

We know that if $u(x)$ is a differentiable function, then by the Chain Rule, we have

$$\Big(\frac{1}{n+1}u(x)^{n+1}\Big)' \;=\; u(x)^n\,u'(x) \qquad \text{and} \qquad \Big(\ln|u(x)|\Big)' \;=\; \frac{u'(x)}{u(x)}.$$

Therefore, the Substitution Rule has the following special case.

The General Power Rule (GPR). If $u(x)$ is differentiable, then

$$\boxed{\int u(x)^n\,u'(x)\,dx \;=\; \frac{1}{n+1}u(x)^{n+1} + C \quad (n \neq -1)}$$

and

$$\boxed{\int \frac{u'(x)}{u(x)}\,dx \;=\; \ln|u(x)| + C}$$

Some integrals can be found quickly by using the GPR, where one factor of the integrand is related to the derivative of the other factor.

Example 9. Find the following integrals.

(1) $\int 2x\,(1+x^2)^{2020}dx \;=$ ________________________.

$$\int 2x\,(1+x^2)^{2020}dx \;=\; \int (1+x^2)^{2020}\,2x\,dx \;=\; \frac{1}{2021}(1+x^2)^{2021} + C.$$

(2) $\int \frac{x^2}{\sqrt{1+x^3}}\,dx \;=$ ________________________.

$$\int \frac{x^2}{\sqrt{1+x^3}}\,dx \;=\; \int (1+x^3)^{-\frac{1}{2}}\,x^2\,dx \;=\; \frac{1}{3}\int (1+x^3)^{-\frac{1}{2}}\,3x^2\,dx$$

$$=\; \frac{1}{3}\,\frac{1}{-\frac{1}{2}+1}\,(1+x^3)^{-\frac{1}{2}+1} + C \;=\; \frac{2}{3}\sqrt{1+x^3} + C.$$

(3) $\int (2x+1)^{1720}\,dx \;=$ ________________________.

$$\int (2x+1)^{1720}\,dx \;=\; \frac{1}{2}\int (2x+1)^{1720}\,2\,dx \;=\; \frac{1}{2}\,\frac{1}{1720+1}\,(2x+1)^{1721} + C \;=\; \frac{1}{3442}(2x+1)^{1721} + C.$$

(4) $\int (1+x^2)^2\,dx \;\neq\; \frac{1}{2x}\int (1+x^2)^2\,2x\,dx \;=\; \frac{1}{2x}\,\frac{1}{3}(1+x^2)^3 \;+\; C.$

(5) $\int x\sqrt{1+2x}\,dx \;=\;$ ____________________________.

We need a substitution but not applying the GPR.

$$u \;=\; \sqrt{1+2x} \;\Longrightarrow\; u^2 \;=\; 1+2x \;\Longrightarrow\; x \;=\; \frac{1}{2}(u^2-1) \;\text{ and }\; dx \;=\; u\,du.$$

Therefore,

$$\int x\sqrt{1+2x}\,dx \;=\; \int \frac{1}{2}\,(u^2-1)\,u^2\,du \;=\; \frac{1}{2}\int (u^4-u^2)\,du \;=\; \frac{1}{10}u^5 - \frac{1}{6}u^3 \;+\; C$$
$$=\; \frac{1}{10}(1+2x)^{\frac{5}{2}} - \frac{1}{6}(1+2x)^{\frac{3}{2}} \;+\; C,$$

(6) $\int \frac{x}{\sqrt{1+x}}\,dx \;=\;$ ____________________________.

$$u \;=\; \sqrt{1+x} \;\Longrightarrow\; u^2 \;=\; 1+x \;\Longrightarrow\; x \;=\; u^2-1 \;\text{ and }\; dx \;=\; 2u\,du.$$

Therefore,

$$\int \frac{x}{\sqrt{1+x}}\,dx \;=\; \int \frac{u^2-1}{u}\,2u\,du \;=\; 2\int (u^2-1)\,du \;=\; \frac{2}{3}u^3 - 2u \;+\; C$$
$$=\; \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} \;+\; C.$$

(7) $\int \frac{1}{x\,(\ln x)^3}\,dx \;=\;$ ____________________________.

$$\int \frac{1}{x\left(\ln x\right)^3}\,dx \;=\; \int \left(\ln x\right)^{-3}\frac{1}{x}\,dx \;=\; \frac{1}{-2}\left(\ln x\right)^{-2} \;+\; C \;=\; -\,\frac{1}{2\left(\ln x\right)^2} \;+\; C.$$

(8) $\int \frac{e^x+2}{e^x+2x}\,dx \;=\;$ ____________________________.

$$\int \frac{e^x+2}{e^x+2x}\,dx \;=\; \int \frac{(e^x+2x)'}{e^x+2x}\,dx \;=\; \ln|e^x+2x| \;+\; C.$$

(9) $\int \frac{\tan^6 x}{\cos^2 x}\,dx \;=\;$ ____________________________.

$$\int \frac{\tan^6 x}{\cos^2 x}\,dx \;=\; \int (\tan x)^6\,\sec^2 x\,dx \;=\; \frac{1}{7}\tan^7 x \;+\; C.$$

(10) $\int \tan x \, dx \;=\;$ __________________________.

$$\int \tan x \, dx \;=\; \int \frac{\sin x}{\cos x} \, dx \;=\; -\int \frac{(\cos x)'}{\cos x} \, dx \;=\; -\ln|\cos x| \;+\; C.$$

(11) $\int \frac{2+x}{1+x^2} \, dx \;=\;$ __________________________.

$$\int \frac{2+x}{1+x^2} \, dx \;=\; 2\int \frac{1}{1+x^2} \, dx \;+\; \frac{1}{2}\int \frac{2x}{1+x^2} \, dx \;=\; 2\tan^{-1}x \;+\; \frac{1}{2}\,\ln(1+x^2) \;+\; C.$$

The Substitution Rule for Definite Integral $\int_a^b f(g(x))\, g'(x)\, dx$.

Method 1. Find the indefinite integral $\int f(g(x))\, g'(x)\, dx \;=\; \int f(u)\, du \;=\; F(g(x)) + C$, and then evaluate due to FCT-2

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx \;=\; \Big[F(g(x))\Big]_a^b}$$

Example 12. Find $\int_1^e \frac{\ln x}{x} \, dx$.

Solution. First, we find the indefinite integral

$$\int \frac{\ln x}{x} \, dx \;=\; \int (\ln x)\, \frac{1}{x} \, dx \;=\; \frac{1}{2}\,\big(\ln x\big)^2 \;+\; C.$$

Then we have

$$\int_1^e \frac{\ln x}{x} \, dx \;=\; \Big[\,\frac{1}{2}\,\big(\ln x\big)^2\,\Big]_1^e \;=\; \frac{1}{2}. \qquad \blacksquare$$

Method 2. $\boxed{\int_a^b f(g(x))\, g'(x)\, dx \;=\; \int_{g(a)}^{g(b)} f(u)\, du}$

Example 13. Find $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} \, dx$.

Solution. Let $u = \frac{1}{x}$. Then $du = -\frac{1}{x^2}\, dx$. When $x = 1$, $u = 1$; when $x = 2$, $u = \frac{1}{2}$. Therefore, we have

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} \, dx \;=\; \int_1^{\frac{1}{2}} -e^u \, du \;=\; \Big[\,-e^u\,\Big]_1^{\frac{1}{2}} \;=\; e - \sqrt{e}. \qquad \blacksquare$$

Question. Which method is better?

If you can figure out the indefinite integral $\int f(g(x))\, g'(x)\, dx$ easily (e.g., find it by GPR), then Method 1 is faster.

If your substitution is complicated, Method 2 can simplify your calculation; that is, you continue working on the integral $\int_{g(a)}^{g(b)} f(u)\, du$ with the new variable u and you do not need replace u by $g(x)$ in $F(u) + C$.

WARNING. We are not allowed to have both variables x and u under one $\displaystyle\int$ sign.

An Application of the Substitution Rule. Let $a > 0$ and f be integrable on $[-a, a]$.

(i) If f is even, then $\int_{-a}^{a} f(x)\, dx = 2\int_{0}^{a} f(x)\, dx$.

(ii) If f is odd, then $\int_{-a}^{a} f(x)\, dx = 0$.

Proof. In $\int_{-a}^{0} f(x)\, dx$, let $u = -x$. Then we have

$$\int_{-a}^{0} f(x)\, dx = -\int_{a}^{0} f(-u)\, du = \int_{0}^{a} f(-u)\, du = \int_{0}^{a} f(-x)\, dx.$$

Hence, we get

$$\int_{-a}^{a} f(x)\, dx = \int_{-a}^{0} f(x)\, dx + \int_{0}^{a} f(x)\, dx = \int_{0}^{a} \big(f(x) + f(-x)\big)\, dx.$$

Therefore, (i) and (ii) hold. ■

Example 14. Find $\int_{-1}^{1} \frac{\sin x}{1+x^2+x^4+x^6}\, dx$.

Solution. Since $f(x) = \dfrac{\sin x}{1 + x^2 + x^4 + x^6}$ is odd, we have

$$\int_{-1}^{1} \frac{\sin x}{1 + x^2 + x^4 + x^6}\, dx = 0. \qquad ■$$