MATH 1250 Lecture 7

Section 2.5 Inverse Matrices

Def^{n}! Let A be a square matrix. A is <u>invertible</u> if there is a matrix A^{-1} such that $AA^{-1} = A^{-1}A = I$. A^{-1} is the <u>inverse</u> of A.

<u>Remark</u>. If A is invertible, then the system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x} = A^{-1}\vec{b}$.

<u>Note</u>. $A_{n\times n}$ is invertible $\iff$ rank $A = n$

$\iff$ I_n is the RREF of A.

<u>Gauss-Jordan Method</u> for ~~find~~ computing A^{-1}:

$$[A \mid I_n] \xrightarrow[\text{if rank } A = n]{\text{row operations}} [I_n \mid A^{-1}]$$

If rank $A < n$, then A is not invertible (A is <u>singular</u>)

Ex. Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

<u>Solution</u>

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 + \frac{1}{2}R_2]{R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[\substack{-\frac{1}{2}R_2 \\ (-1)R_3}]{R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -1 \end{array}\right]$$

$$= [I_3 \mid A^{-1}]$$

Ex. Find the inverse of $A=\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 5 \\ 2 & 2 & 3 \end{bmatrix}$.

Solution $[A \mid I_3]=\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 5 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-3R_1}$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -1 & -3 & 1 & 0 \\ 0 & 4 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

Since $\operatorname{rank} A = 2 < 3$, A is not invertible.

Properties. If A and B are $n \times n$ invertible matrices, then

1). $(A^{-1})^{-1}=A$, 2). $(AB)^{-1}=B^{-1}A^{-1}$,

3). $(A^{k})^{-1}=(A^{-1})^{k}$, 4). $(aA)^{-1}=\frac{1}{a}A^{-1}$ if $a\neq 0$.

Remark. The 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible

$\iff ad-bc\neq 0$, in this case $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Ex. Find A if $\left(A-3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\right)^{-1}=\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution $A-3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}=\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1}=\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$,

$$A=\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}+3\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}=\begin{bmatrix} \frac{7}{2} & -\frac{1}{2} \\ \frac{11}{2} & -\frac{3}{2} \end{bmatrix}.$$

Ex. Find B if $ABA^{-1}=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ and $A=\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$.

Solution $A^{-1}(ABA^{-1})A = A^{-1}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}A$

$$\Longrightarrow B = IBI = (A^{-1}A)B(A^{-1}A) = A^{-1}(ABA^{-1})A$$

$$= A^{-1}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}A = \begin{bmatrix}3 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}$$

$$= \begin{bmatrix}0 & 2\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix} = \begin{bmatrix}4 & 6\\ 1 & 1\end{bmatrix}.$$

Ex. $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{bmatrix}$, find A^{-1}.

Solution $\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 5 & 5 & 1 & 0 & 0 & 1\end{array}\right] \xrightarrow{R_3-5R_1} \left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 0 & -4 & -5 & 0 & 1\end{array}\right]$

$$\xrightarrow[\substack{R_2+\frac{3}{4}R_3\\ -\frac{1}{4}R_3}]{R_1+\frac{1}{4}R_3} \left[\begin{array}{ccc|ccc}1 & 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4}\\ 0 & 2 & 0 & -\frac{15}{4} & 1 & \frac{3}{4}\\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4}\end{array}\right]$$

$$\xrightarrow[\frac{1}{2}R_2]{R_1-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{13}{8} & -\frac{1}{2} & -\frac{1}{8}\\ 0 & 1 & 0 & -\frac{15}{8} & \frac{1}{2} & \frac{3}{8}\\ 0 & 0 & 1 & \frac{5}{4} & 0 & -\frac{1}{4}\end{array}\right] = [I \mid A^{-1}]$$

Section 2.7 Transposes and Permutations

Def. The transpose of A is the matrix obtained by exchanging rows and columns of A, denoted by A^T.

Ex. $A=\begin{bmatrix}1&2\\3&4\\5&6\end{bmatrix}$, $A^T=\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$

Properties. 1. $(A+B)^T=A^T+B^T$, 2. $(aA)^T=aA^T$, 3. $(AB)^T=B^TA^T$, 4. $(A^T)^T=A$, 5. $(A^{-1})^T=(A^T)^{-1}$.

Def. A is symmetric if $A^T=A$.

A is anti-symmetric if $A^T=-A$.

Ex. $\begin{bmatrix}1&5\\5&3\end{bmatrix}$ and $\begin{bmatrix}-2&4&6\\4&3&-5\\6&-5&1\end{bmatrix}$ are symmetric.

$\begin{bmatrix}0&2&-3\\-2&0&4\\3&-4&0\end{bmatrix}$ and $\begin{bmatrix}0&8\\-8&0\end{bmatrix}$ are anti-symmetric.

Ex. If A is a square matrix, then

$(A+A^T)^T=A^T+(A^T)^T=A^T+A=A+A^T$,

$(A-A^T)^T=A^T-(A^T)^T=A^T-A=-(A-A^T)$

So $A+A^T$ is symmetric, $A-A^T$ is anti-symmetric.