MATH 1250 Lecture 4 & 5

Elimination and Row Echelon Form (in Section 3.3)

Elementary row operations on a matrix:

1. Multiply one row by a non-zero number. (aR_i, $a \neq 0$)
2. Interchange two rows. ($R_i \updownarrow R_j$)
3. Add a multiple of one row to another row. ($R_i + bR_j$)

Def^n. A matrix is in row echelon form (REF) if

1). The first nonzero entry in each row is 1, called "leading 1".
2). Leading 1's move to right as you go down.
3). All completely zero rows occur at the bottom if there are any.

Ex. $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 5 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 5 & -8 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

are in row echelon form.

To solve a system of linear equations with augmented matrix $[A|\vec{b}]$, we first do some row operations to get the REF of the augmented matrix.

Case 1. If a leading 1 appears in the last column, the system has no solution.

Case 2. Assume that no leading 1 in the last column.

subcase 1. If # of leading 1's = # of variables, the system has a unique solution.

subcase 2. If # of leading 1's < # of variables, the system has infinitely many solutions. The variables not corresponding to leading 1's are free variables, they can choose any values in $\mathbb{R}$.

Ex. Solve $\begin{cases} x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 3 \\ 2x_1 + 3x_2 + 3x_3 = 9 \end{cases}$

Solution. $[A|\vec{b}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 9 \end{array}\right] \xrightarrow{R_1 \updownarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 9 \end{array}\right]$

$\xrightarrow{R_3+(-2)R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_3+(-1)R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$ (in REF)

Since a leading 1 appears in the last column, the system has no solution.

Note. The last row in the REF represents an equation

$0x_1 + 0x_2 + 0x_3 = 1 \iff 0 = 1$ (impossible).

Ex. Solve $\begin{cases} x_1+2x_2+3x_3=4 \\ 2x_1+3x_2+4x_3=5 \\ x_1+x_2+x_3=1 \end{cases}$

Solution. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3+(-1)R_1]{R_2+(-2)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \end{array}\right]$

$\xrightarrow[(-1)R_2]{R_3+(-1)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$ in REF. The system has infinitely many solutions. (x_3 is a free variable.)

Apply back substitution to get solutions:

From 2nd row we have $x_2+2x_3=3 \Longrightarrow x_2=3-2x_3$.

From 1st row we have $x_1+2x_2+3x_3=4 \Longrightarrow$

$x_1=4-2x_2-3x_3=4-2(3-2x_3)-3x_3=-2+x_3$.

The general solution is

$$\begin{aligned} x_1 &= -2+x_3 \\ x_2 &= 3-2x_3 \\ x_3 &= \quad\ \ x_3 \end{aligned} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \ x_3 \in \mathbb{R}.$$

(vector form)

Ex. The augmented matrix of a system is

$$[A \mid \vec{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & a-1 & a & 3 \\ 0 & 0 & b & -3 \end{array}\right].$$

Determine all values of a, b for which the system has

1). no solution;

2). a unique solution;

3). infinitely many solutions.

<u>Solution</u>. i). If $a \neq 1$ and $b \neq 0$, then

$$[A|\vec{b}] \xrightarrow[\frac{1}{b}R_3]{\frac{1}{a-1}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 1 & \frac{a}{a-1} & \frac{3}{a-1} \\ 0 & 0 & 1 & \frac{-3}{b} \end{array}\right],$$

the system has a unique solution.

ii). If $b=0$ and $a \in \mathbb{R}$, the system has no solution.

(Row 3 means $0x_1 + 0x_2 + 0x_3 = -3 \Longleftrightarrow 0 = -3$.)

iii). If $a=1$, then $[A|\vec{b}] \xrightarrow{R_3+(-b)R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3-3b \end{array}\right]$.

If $a=1$ and $b \neq -1$, then the system has no solution.

If $a=1$ and $b=-1$, then the system has infinitely many solutions.

Def! A matrix is in <u>reduced row echelon form</u> (RREF) if

1. it is in row echelon form, and
2. each leading 1 is the only nonzero entry in its column.

Ex $\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

are in RREF.

Def^n. The rank of a matrix A is the number of leading 1's in a REF of A, denoted by rank A.

Ex $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}$, find the RREF of A and rank A.

Solution

$$A \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3]{(-1)R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 + (-2)R_3} \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ in RREF. rank $A = 3$.

Note. If we use the RREF of the augmented matrix to solve a system of eqns, we do not need to use the back substitution.

Ex. Solve
$$\begin{cases} \quad\;\; -4x_2 + 12x_3 = -8 \\ x_1 + 3x_2 - 2x_3 = 5 \\ 2x_1 + 4x_2 + 3x_3 = 8 \end{cases}$$

Solution
$$\left[\begin{array}{ccc|c} 0 & -4 & 12 & -8 \\ 1 & 3 & -2 & 5 \\ 2 & 4 & 3 & 8 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ (in REF, we can see from here the system has a unique solution.)

$$\xrightarrow[R_2 + 3R_3]{R_1 + 2R_3} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + (-3)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$ in RREF

So
$$\begin{aligned} x_1 &= -15 \\ x_2 &= 8 \\ x_3 &= 2 \end{aligned} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15 \\ 8 \\ 2 \end{bmatrix}$$

Ex.
$$\begin{cases} x_1 + 4x_2 + x_3 + x_4 + 6x_5 = 3 \\ \qquad\qquad\quad x_3 + 2x_4 + 5x_5 = -1 \\ x_1 + 4x_2 + x_3 + 2x_4 + 8x_5 = 4 \end{cases}$$

Solution

$$[A|\vec{b}] = \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 & 5 & -1 \\ 1 & 4 & 1 & 2 & 8 & 4 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 & 5 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 + (-2)R_3]{R_1 + (-1)R_3} \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_1 - R_2} \begin{array}{c} \begin{array}{cccccc} & x_2 & & & x_5 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 4 & 0 & 0 & 3 & 5 \\ 0 & 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right] \end{array} \text{ RREF}$$

From row 1: $x_1 + 4x_2 + 3x_5 = 5 \Longrightarrow x_1 = 5 - 4x_2 - 3x_5$

From row 2: $x_3 + x_5 = -3 \Longrightarrow x_3 = -3 - x_5$

From row 3: $x_4 + 2x_5 = 1 \Longrightarrow x_4 = 1 - 2x_5$,

x_2 and x_5 are free variables.

$$\begin{aligned} x_1 &= 5 - 4x_2 - 3x_5 \\ x_2 &= x_2 \\ x_3 &= -3 - x_5 \\ x_4 &= 1 - 2x_5 \\ x_5 &= x_5 \end{aligned} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \underbrace{x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}}_{\vec{x}_n}$$

$x_2, x_5 \in \mathbb{R}$.

Homogeneous Systems

A system $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b} = \vec{0}$.

A homogeneous system $A\vec{x} = \vec{0}$ always has a solution $\vec{x} = \vec{0}$, but it may have a unique solution or infinitely many solutions.

Ex. Solve the homogeneous system

$$\begin{cases} 2x_1+2x_2+2x_3+3x_4=0 \\ 2x_1+2x_2+3x_3+7x_4=0 \\ 4x_1+4x_2+5x_3+10x_4=0 \end{cases}$$

Solution. For a homogeneous system, we can consider the coefficient matrix A instead of $[A|\vec{0}]$.

$$A=\begin{bmatrix} 2&2&2&3\\2&2&3&7\\4&4&5&10\end{bmatrix} \xrightarrow[R_3-2R_1]{R_2-R_1} \begin{bmatrix} 2&2&2&3\\0&0&1&4\\0&0&1&4\end{bmatrix}$$

$$\xrightarrow[R_3-2R_2]{R_1-2R_2} \begin{bmatrix} 2&2&0&-5\\0&0&1&4\\0&0&0&0\end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1&1&0&-\frac{5}{2}\\0&0&1&4\\0&0&0&0\end{bmatrix} \text{ RREF}$$

$$\begin{aligned} x_1&=-x_2+\tfrac{5}{2}x_4\\ x_2&=\ \ x_2\\ x_3&=\quad\ \ -4x_4\\ x_4&=\qquad\ \ x_4 \end{aligned} \Longrightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=x_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}5/2\\0\\-4\\1\end{bmatrix}$$

$x_2, x_4 \in \mathbb{R}$.

Note. The general solution (or complete solution) to $A\vec{x}=\vec{b}$ can be expressed as $\vec{x}=\vec{x}_p+\vec{x}_n$, where $\vec{x}_p$ is a particular solution to $A\vec{x}=\vec{b}$ and $\vec{x}_n$ is the general solution to the associated homogeneous system $A\vec{x}=\vec{0}$. (see the Ex in page 7)