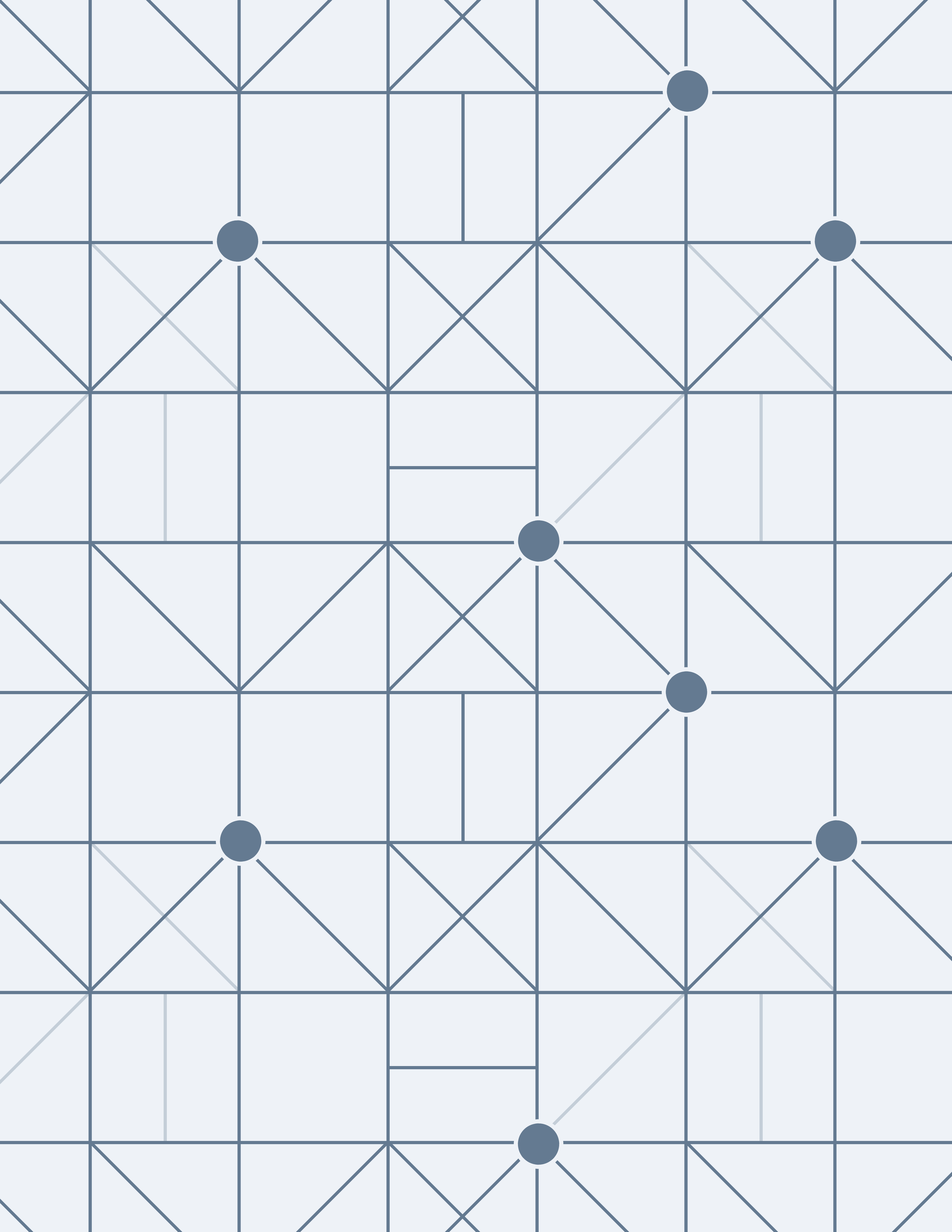
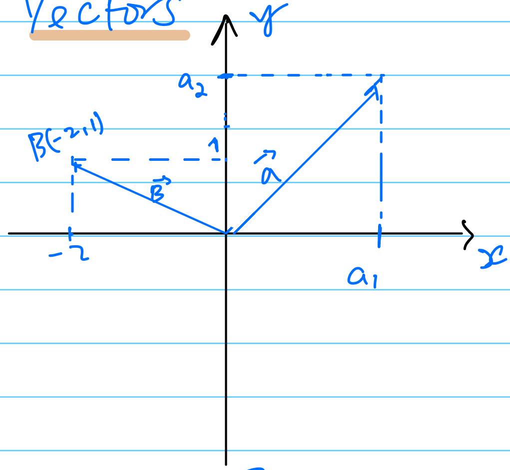


May 7th 2024

Linear algebra

Vectors



$$\vec{a} = \vec{OA} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ or } (a_1, a_2)$$

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} / a_1, a_2 \in \mathbb{R} \right\}$$

Addition: if $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$; $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

then $\vec{a} + \vec{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$

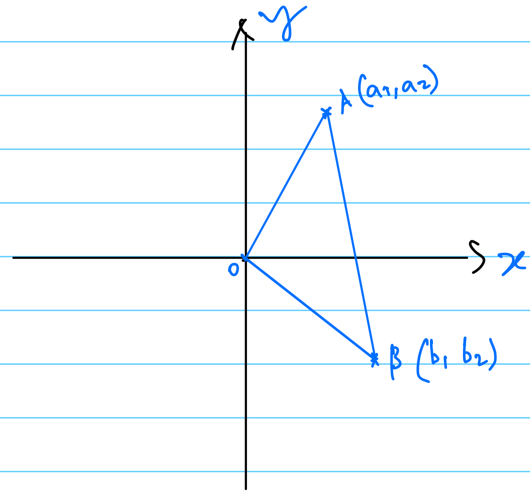
scalar multiple: $k\vec{a} = \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix}$; $k \in \mathbb{R}$

Ex: ⊛ $\begin{bmatrix} -3 \\ 5 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} -3+7 \\ 5+3 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

⊛ $2\begin{bmatrix} 5 \\ -1 \end{bmatrix} + -1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} 10 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 10-3+0 \\ -2-4+6 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$

Linear combination



$$\vec{AB} = \vec{OB} - \vec{OA} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\vec{OA} + \vec{AB} = \vec{OB}$$

Ex: For points $A(2,5), B(3,2)$

we have $\vec{AB} = \vec{OB} - \vec{OA}$

$$= \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Similarly, vectors in $\mathbb{R}^3$: $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} ; \quad c \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ ca_3 \end{bmatrix}$$

Ex: $2 \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 8 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ -6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \ a_1, a_2, \dots a_n \in \mathbb{R} \right\}$$

Ex: $\begin{bmatrix} 2 \\ 5 \\ -7 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ -4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -5 \\ 7 \end{bmatrix}$

Operations are made with vectors of the same space

$$3 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \\ 15 \end{bmatrix}$$

May 9th 2024

Ex ① if $3\vec{u} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ find $\vec{u}$

$$3\vec{u} = \begin{bmatrix} 7 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$3\vec{u} = \begin{bmatrix} 7-2 \\ -1-5 \end{bmatrix} \Rightarrow \vec{u} = \frac{1}{3}\begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 5/3 \\ -2 \end{bmatrix}$$

② if $\vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; $\vec{u} - 3\vec{v} = \begin{bmatrix} -4 \\ 7 \\ -7 \end{bmatrix}$; find $\vec{u}, \vec{v}$

Solution

We need to eliminate $\vec{u}$

$$(\cancel{\vec{u}} + 2\vec{v}) - (\cancel{\vec{u}} - 3\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 \\ 7 \\ -7 \end{bmatrix}$$

$$\Rightarrow 2\vec{v} + 3\vec{v} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix} \Rightarrow 5\vec{v} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix} \Rightarrow \vec{v} = \frac{1}{5}\begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Now we use $\vec{v}$ values to find $\vec{u}$ by substituting the values in first equation:

$$\vec{u} + 2\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} + 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} + \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$$

3) What combination $c\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}-2\\1\end{bmatrix}$ produce $\begin{bmatrix}-8\\-3\end{bmatrix}$?

Sol

$$c\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}-2\\1\end{bmatrix}=\begin{bmatrix}-8\\-3\end{bmatrix}$$

$$\begin{bmatrix}c-2d\\3c+d\end{bmatrix}=\begin{bmatrix}-8\\-3\end{bmatrix}$$

$$\begin{cases}c-2d=-8\\3c+d=-3\end{cases} \Rightarrow d=-3-3c$$

then
$$c-2d=-8$$
$$c-2(-3-3c)=-8$$
$$c=-8+2(-3-3c)$$
$$c=-8-6-6c$$
$$7c=-14$$
$$c=-2$$

If $c=-2$ then
$$3c+d=-3$$
$$3(-2)+d=-3$$
$$-6+d=-3$$
$$d=-3+6$$
$$d=3$$

OR
$$d=-3-3c$$
$$d=-3-3(-2)$$
$$d=-3+6$$
$$d=3$$

1.2 Dot product

$\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$ is also written as $\vec{v}=(v_1,v_2,\dots,v_n)$

Def: Dot product of $\vec{u}=(u_1,u_2,\dots u_n)$ and $\vec{v}=(v_1,v_2,\dots v_n)$

is the number $\vec{u}\cdot\vec{v}=u_1\cdot v_1+u_2\cdot v_2+\dots+u_nv_n$

Ex: $\vec{u}=(1,2,3)$, $\vec{v}=(-2,3,1)$ then $\vec{u}\cdot\vec{v}=1(-2)+2(3)+3(1)=7$

Length of a vector

Def: the length of $\vec{v} = (v_1, v_2, \dots v_n)$ is

$$\|\vec{v}\| = \sqrt{v \cdot v} = (v_1^2 + v_2^2 + \dots + v_n^2)^{1/2}$$

If $\|\vec{v}\| = 1$, then $\vec{v}$ is a unit vector.

Note If $\vec{u} \neq \vec{0}$, then $\frac{1}{\|\vec{u}\|} \vec{u}$ is a unit vector in the direction of $\vec{u}$

$$\frac{1}{\|\vec{u}\|} \vec{u} = \frac{1}{3} \vec{u} \quad \text{if } \|\vec{u}\| = 3$$

Ex: find the length of $\vec{v} = (1, -1, 2, 3)$ and find a unit vector in the direction of $\vec{v}$

Solution $\|\vec{v}\| = \sqrt{1^2 + (-1)^2 + 2^2 + 3^2} = \sqrt{15}$

$$\text{unit vector} = \frac{1}{\|\vec{v}\|} \vec{v} = \frac{1}{\sqrt{15}} (1, -1, 2, 3)$$

Angle between two vectors

The angle θ between $\vec{u}, \vec{v}$ is defined by

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} \ ; \quad 0 \le \theta \le \pi$$

Ex If $\vec{u} = (1, -3)$, $\vec{v} = (2, -1)$ then $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$

$$\cos\theta = \frac{2 + 3}{\sqrt{1+9}\,\sqrt{4+1}} = \frac{5}{\sqrt{50}} = \frac{5}{\sqrt{25 \cdot 2}} = \frac{\cancel{5}}{\cancel{5}\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$ or $\theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Parallel vectors

If $\vec{a} \parallel \vec{b}$; then $\vec{b} = k\vec{a}$ (i.e. If $\vec{a}$ is parallel to $\vec{b}$, then $\vec{a}$ is a multiple of $\vec{b} \rightarrow \vec{b} = k\vec{a}$)

#3 (a) $\vec{a} = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $\vec{b} = \begin{pmatrix}3\\6\\9\end{pmatrix} = 3\begin{pmatrix}1\\2\\3\end{pmatrix} = 3\vec{a}$

$\parallel$ means parallel

$c_1\vec{a} + c_2\vec{b} = c_1\vec{a} + c_2 3\vec{a}$
$= (c_1 + 3c_2)\vec{a}$

* If $\vec{a} \nparallel \vec{b}$, then all combinations $c_1\vec{a} + c_2\vec{b}$ form a plane.

* If $\vec{a} \parallel \vec{b}$, then we get just a line.

Problem set 1.1 (from 5th edition)

(4) $v = \begin{pmatrix}2\\1\end{pmatrix}$, $w = \begin{pmatrix}1\\2\end{pmatrix}$

→ find $3v + w = 3\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}6+1\\3+2\end{bmatrix} = \begin{bmatrix}7\\5\end{bmatrix}$

→ find $cv + dw = c\begin{bmatrix}2\\1\end{bmatrix} + d\begin{bmatrix}1\\2\end{bmatrix}$
$= \begin{bmatrix}2c+d\\c+2d\end{bmatrix}$

(5) $\vec{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $\vec{v} = \begin{bmatrix}-3\\1\\-2\end{bmatrix}$, $w = \begin{bmatrix}2\\-3\\-1\end{bmatrix}$

$u + v + w = \begin{bmatrix}1-3+2\\2+1-3\\3-2-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ it's a zero vector ($\vec{0}$)

$2w + 2v + w = \begin{bmatrix}2-6+2\\4+2-3\\6-2-1\end{bmatrix} = \begin{bmatrix}-6\\3\\3\end{bmatrix}$

Since all combinations $c\vec{u}+k\vec{v}$ form a plane, we have $\vec{w}$ is on the plane.

Problem set 1.2

(3) $\vec{u} = \begin{bmatrix}-6\\8\end{bmatrix}$, $\vec{v} = \begin{bmatrix}4\\3\end{bmatrix}$, $w = \begin{bmatrix}1\\2\end{bmatrix}$

$\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$;

$\vec{v}\cdot\vec{w} = 4+6 = 10$

$\|\vec{v}\| = \sqrt{4^2+3^2} = \sqrt{16+9} = 5$

$\|\vec{w}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\cos\theta = \dfrac{10}{5\sqrt{5}} = \dfrac{2}{\sqrt{5}} \iff \cos\theta = \dfrac{2}{\sqrt{5}}$

for a iff $\vec{u}\cdot\vec{v} = 0 \iff \theta = \frac{\pi}{2}$

!!!

We can choose $\vec{b} = \begin{bmatrix}2\\-1\end{bmatrix}$, then

$\vec{b}\cdot\vec{w} = 1(2)+2(-1) = 0$, so $\theta = 90^\circ$

$\vec{c} = -\vec{w}$ or $\vec{c} = k\vec{w}$; $k<0$

Unit vector

A unit vector u is a vector whose length equals one

Then $u\cdot u = 1$ Also, $u = v/\|v\|$ is a unit vector in the same direction as v

Problem set 1.2

(19) $u \cdot (v+w) = u \cdot v + u \cdot w$

$u = v+w$

prove $\|v+w\|^2 = v \cdot v + 2v \cdot w + w \cdot w$

$\|\vec{v}+\vec{w}\|^2 = \cancel{\sqrt{(\vec{v}+\vec{w}) \cdot (\vec{v}+\vec{w})}^{\,2}}$

$= (\vec{v}+\vec{w}) \cdot (\vec{v}+\vec{w})$

$= \vec{v} \cdot \vec{v} + 2\vec{v}\,\vec{w} + \vec{w}\,\vec{w}$

(2) $u = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$; $v = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\|\vec{u}\| = \sqrt{(-6)^2 + 8^2} = \sqrt{36+64} = \sqrt{100} = 10$

$\|\vec{v}\| = \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5$

$\|\vec{w}\| = \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5}$

$|\vec{u} \cdot \vec{v}| = |-24+24| = 0$

$|\vec{v} \cdot \vec{w}| = |4+6| = 10$

(*) $|\vec{u} \cdot \vec{v}| \leq \|u\| \|v\|$

$0 \leq 50$ True

* $|v \cdot w| \leq \|v\| \cdot \|w\|$

$10 \leq 5\sqrt{5}$

(7) (a) $v = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ $\quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \cdot \|\vec{u}\|}$$

$$= \frac{1 + \sqrt{3}}{(1^2+3^2) \cdot (1^2+0^2)} = \frac{1+\sqrt{3}}{6}.$$

$$\cos\theta = \frac{1+\sqrt{3}}{6} \Rightarrow \theta = \cos^{-1}\left(\frac{1+\sqrt{3}}{6}\right)$$

May 16th 2024

Exercise 1 $\vec{u} = (1, 3t, 2)$, $\vec{v} = (3, 2, -5)$

Find the value of t such that $\vec{u} \perp \vec{v}$

Solution we know that if $\vec{u} \perp \vec{v}$ it means

$$\vec{u} \cdot \vec{v} = 0$$

$\vec{u} \cdot \vec{v} = 1(3) + (3t)2 + 2(-5) \Leftrightarrow 3 + 6t - 10 \Leftrightarrow 6t - 7$

$6t - 7 = 0 \Leftrightarrow t = 7/6$

Ex 2 For $\vec{u} = (1, 1, -2)$, $\vec{v} = (4, 1, -2)$

find c such that $(\vec{v} - c\vec{u}) \perp \vec{v}$

Sol $\vec{u} \cdot (\vec{v} - c\vec{u}) = \vec{u} \cdot \vec{v} - c(\vec{u} \cdot \vec{u})$

We need $\vec{u} \cdot \vec{v} - c(\vec{u} \cdot \vec{u}) = 0$

$\Rightarrow \vec{u} \cdot \vec{v} = c(\vec{u} \cdot \vec{u})$

$\Rightarrow \; c\,(\vec{u}\cdot\vec{u}) = \vec{u}\cdot\vec{v}$

$$c = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}} = \frac{4+1+4}{1+1+4} = \frac{9}{6} = \frac{3}{2}$$

$c = 3/2$

P properties

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

2. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$

3. $(c\,\vec{u})\cdot\vec{v} = c\,(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\,\vec{v})$

Exercise 3 If $\|\vec{u}\| = 2$, $\|\vec{v}\| = 3$; $\vec{u}\cdot\vec{v} = 1$

Find $\|\vec{u}-\vec{v}\|$, where $\vec{u}, \vec{v} \in \mathbb{R}^n$

Solution $\|\vec{u}-\vec{v}\| = \sqrt{(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})}$

$\Rightarrow \sqrt{\vec{u}\cdot\vec{u} - 2\vec{u}\vec{v} + \vec{v}\cdot\vec{v}}$

$\Rightarrow \sqrt{\|\vec{u}\|^2 - 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2}$

i.e. $\Rightarrow \|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$
$\Rightarrow \|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$
therefore $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$
$\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

$\Rightarrow \sqrt{2^2 - 2(1) + 3^2} \Rightarrow \sqrt{4-2+9}$

$\Rightarrow \sqrt{11}$

Ex 4 Find a, b such that $\vec{u} = (a, 1, b)$ is perpendicular to both $\vec{v} = (1,2,1)$ and $\vec{w} = (1,-3,-1)$

Solution Since $\vec{u} \perp \vec{v}$ and $\vec{u}\cdot\vec{w}$; we need

$\vec{u}\cdot\vec{v} = 0$ and $\vec{u}\cdot\vec{w} = 0$

$\Rightarrow \vec{u}\cdot\vec{v} = a+2+b=0 \; ; \; \vec{u}\cdot\vec{w} = a-3-b=0$

$$\begin{cases} a+2+b=0 \\ a-3-b=0 \end{cases} \Rightarrow + \begin{cases} a+\cancel{b} = -2 \\ a-\cancel{b} = 3 \end{cases} \qquad - \begin{cases} \cancel{a}+b = -2 \\ \cancel{a}-b = 3 \end{cases}$$

$$2a = 1 \qquad\qquad 2b = -5$$

$$a = \frac{1}{2} \qquad\qquad b = \frac{-5}{2}$$

$$a = 1/2 \; ; \; b = -5/2$$

Definition of distance between two vectors

The distance between $\vec{u}$ and $\vec{v}$ is

$$\|\vec{u}-\vec{v}\| \quad \text{or} \quad \|\vec{v}-\vec{u}\|$$

Ex: Find the distance between $\vec{u} = (\sqrt{2}, 1, -1)$; $\vec{v} = (0, 2, -2)$

Sol: $\|\vec{u}-\vec{v}\| \Rightarrow \|(\sqrt{2}, -1, 1)\| \Rightarrow \sqrt{(\sqrt{2})^2+(-1)^2+1^2}$

$\Rightarrow \sqrt{2+1+1} = 2$

2.1 in 6th E (2.2 in 5th Ed)

Ex1:
$$\begin{cases} x_1 - 2x_2 - x_3 = 0 & (1) \\ x_1 - x_2 + 2x_3 = 1 & (2) \\ \quad 2x_2 + 7x_3 = 1 & (3) \end{cases}$$

May 17^{th} 2024

Ex 3:
$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 4 \\ 3x_2 + 6x_3 = 9 \end{cases} \Rightarrow \text{eqn } 2 - \text{eqn } 1 \text{ to cancel } x_1 \text{ in eqn } 2$$

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \cancel{-3}x_2 + 2x_3 = 3 \\ \cancel{3x_2} + 6x_3 = 9 \end{cases} \Rightarrow \text{eqn } 3 - 3 \text{ eqn } 2$$

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 3 \\ 0 = 0 \end{cases} \uparrow \text{ back substitute}$$

from eqn 2, we get $= \; x_2 = 3 - 2x_3$

eqn 1 ; $x_1 = 1 - x_2 - x_3$
$= 1 - (3 - 2x_3) - x$
$= -2 + x_3$

The system has many solutions

$$(x_1, x_2, x_3) = (-2 + x_3, \; 3 - 2x_3, \; x_3)$$

where $x_3 \in \mathbb{R}$ (free variable)

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_2 x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Can be represented by the augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right] = [A \mid \vec{b}]$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ (coefficient matrix)

$$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_m \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -3 & -1 & 1 \end{bmatrix}$

$\xrightarrow{R_2 + (-2)R_1}$ $R_1 \to \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -7 & -7 & -7 \end{bmatrix}$ we keep 1st row

$\xrightarrow{-\frac{1}{7}R_2}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ (this is in REF)

Example of a matrix in Row echelon form (RE

$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 4 & 5 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. A, B and C are in REF

May 21st 2024

To solve a system, first write the augmented matrix $[A|\vec{b}]$ then ~~do~~ row operation to get a REF

Case 1: If a leading 1 appears in the last column then the system has no solution.

Case 2: No leading 1 in the last column

(a) If # leading is = # variables, the system has a unique solution.

(b) If # leading is < # variables, the system has ∞ many solution.

Ex: Solve
$$\begin{cases} x_2 + x_3 = 2 \\ x_1 + x_2 + x_3 = 3 \\ 2x_1 + 3x_2 + 3x_3 = 9 \end{cases}$$

Solution:

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 3 & 9 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 2 & 3 & 3 & 9 \end{array}\right]$$

$$\xrightarrow{R_3 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

in R.E.F

So the system has no solution.

Note: The row 3 means $0x_1 + 0x_2 + 0x_3 = 1$
$\Rightarrow 0 = 1$ impossible.

Ex2 Solve $\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + 3x_2 + 4x_3 = 5 \\ x_1 + x_2 + x_3 = 1 \end{cases}$

Solution: $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow[R_3 + (-1)R_1]{R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -1 & -2 & -3 \end{array}\right]$

$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$

The system has ∞ many solution

Use back-substitution.

From row 2: $x_2 + 2x_3 = 3 \iff x_2 = 3 - 2x_3$

From row 1: $x_1 + 2x_2 + 3x_3 = 4$, then

$$x_1 = 4 - 2x_2 - 3x_3 = 4 - 2(3 - 2x_3) - 3x_3$$
$$= 4 - 6 + 4x_3 - 3x_3 = -2 + x_3$$

The general solution is

$$\begin{cases} x_1 = -2 + x_3 \\ x_2 = 3 - 2x_3 \\ x_3 = x_3 \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

vector form

$x_3 \in \mathbb{R}$ is a free variable.

Ex. The system has $[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & a-1 & 9 & 3 \\ 0 & 0 & b & -3 \end{array}\right]$

Determine a, b such that the system has
1) no solution
2) unique solution
3) ∞ many solution

Solution i) if $a-1 \neq 0$ & $b \neq 0$, then

$$[A|\vec{b}] \xrightarrow[\frac{1}{b} R_3]{\frac{1}{a-1} R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 1 & \frac{a}{a-1} & 3/a-1 \\ 0 & 0 & 1 & -3/b \end{array}\right] (REF)$$

The system has a unique solution
(ii) If $b=0$ and $a \in \mathbb{R}$, then row 3 means

$0x_1 + 0x_2 + 0x_3 = -3 \Rightarrow 0 = -3$ impossible

(iii) If $a-1=0$; that is $a=1$, then

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & b & -3 \end{array}\right] \xrightarrow{R_3 + (-b) R_2} \left[\begin{array}{ccc|c} 1 & -2 & 4 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3-3b \end{array}\right]$$

If $a=1$ & $-3-3b \neq 0$ or ($b \neq -1$); no solution

If $a=1$ and $-3-3b=0$ ($b=1$), then ∞ many solutions

Reduced row echelon form (RREF)

Definitions A matrix is in reduced row echelon form (RREF) if:
→ It is in row echelon form, and
→ each leading 1 is the only nonzero entry in its column

Ex: $\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ are in RREF

Rank of a matrix

Def The rank of a matrix A is the number of leading 1s is in REF of A, denoted by Rank A.

Ex: $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}$, find the RREF of A and rank A.

Solution

$$A \xrightarrow{R_2 + (-2)R_1} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 2 & 1 \end{bmatrix} \xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}R_3]{(-1)R_2} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ in (REF) we need to put it RREF

$$\xrightarrow{R_1 + (-2)R_3} \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$ RREF of A

Rank $A = 3$

Note: If we use the RREF of the augmented matrix to solve a system of eqns, we do not need to use the back substitution.

Ex Solve
$$\begin{cases} -4x_2 + 12x_3 = -8 \\ x_1 + 3x_2 - 2x_3 = 5 \\ 2x_1 + 4x_2 + 3x_3 = 8 \end{cases}$$

Solution
$$\left[\begin{array}{ccc|c} 0 & -4 & 12 & -8 \\ 1 & 3 & -2 & 5 \\ 2 & 4 & 3 & 8 \end{array}\right] \xrightarrow{R_1 \updownarrow R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

$$\xrightarrow{R_3 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{(in REF)}$$

nº leading 1 = nº of variables
Therefore, the system has unique solution

$$\xrightarrow[R_2 + 3R_3]{R_3 + 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 + (-3)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \text{in RREF}$$

So: $\begin{cases} x_1 = -15 \\ x_2 = 8 \\ x_3 = 2 \end{cases} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -15 \\ 8 \\ 2 \end{bmatrix}$

(2)
$$\begin{cases} x_1 + 4x_2 + x_3 + x_4 + 6x_5 = 3 \\ x_3 + 2x_4 + 5x_3 = -1 \\ x_1 + 4x_2 + x_3 + 2x_4 + 8x_5 = 4 \end{cases}$$

Solution
$$[A|\vec{b}] = \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 & 5 & -1 \\ 1 & 4 & 1 & 2 & 8 & 4 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 & 5 & -1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right]$$
in REF

$$\xrightarrow[R_2-2R_3]{R_1-R_3} \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \begin{array}{c} \begin{array}{ccccc} & x_2 & & & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} \underline{1} & 4 & 0 & 0 & 3 & 5 \\ 0 & 0 & \underline{1} & 0 & 1 & -3 \\ 0 & 0 & 0 & \underline{1} & 2 & 1 \end{array}\right] \end{array} \rightarrow \text{in R.R.E.F}$$

The system has ∞ many solutions because we have 3 leading 1_s and 5 variables.

Note x_2 and x_5 are free variables because they don't have leading 1_s.

From row 1: $x_1 + 4x_2 + 3x_5 = 5$ then $x_1 = 5 - 4x_2 - 3x_5$

From row 2: $x_3 + x_5 = -3$ then $x_3 = -3 - x_5$

From row 3: $x_4 + 2x_5 = 1$ then $x_4 = 1 - 2x_5$

$$\left.\begin{aligned} x_1 &= 5 - 4x_2 - 3x_5 \\ x_2 &= x_2 \\ x_3 &= -3 \quad - x_5 \\ x_4 &= 1 \quad - 2x_5 \\ x_5 &= \quad x_5 \end{aligned}\right\} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -1 \\ -2 \\ 1 \end{bmatrix}$$

$x_2, x_5 \in \mathbb{R}$

May 28th 2024

Homogenous systems

A system with $[A \mid \vec{b}]$ is homogenous if $\vec{b} = 0$

Ex:
$$\begin{cases} 2x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \\ 2x_1 + 2x_2 + 3x_3 + 7x_4 = 0 \\ 4x_1 + 4x_2 + 5x_3 + 10x_4 = 0 \end{cases}$$

Solution

$$A = \begin{bmatrix} 2 & 2 & 2 & 3 \\ 2 & 2 & 3 & 7 \\ 4 & 4 & 5 & 10 \end{bmatrix} \xrightarrow[R_3+(-2)R_1]{R_2 - R_1} \begin{bmatrix} 2 & 2 & 2 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_2]{R_1 - 2R_2} \begin{bmatrix} 2 & 2 & 0 & -5 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ in RREF}$$

From row 1: $x_1 + x_2 - 5/2\, x_4 = 0$, then $x_1 = -x_2 + 5/2\, x_4$

From row 2: $x_3 + 4x_4 = 0$, then $x_3 = -4x_4$

$$\begin{aligned} x_1 &= x_2 + 5/2\, x_4 \\ x_2 &= x_2 \\ x_3 &= -4x_4 \\ x_4 &= x_4 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5/2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$$

Section 2.4: Rules for Matrix Operations.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & & & & a_{nm} \end{bmatrix} = [a_{ij}]_{m\times n}$$

Size $m\times n$

Addition (sum): $[a_{ij}]_{m\times n} + [b_{ij}]_{m\times n} = [a_{ij} + b_{ij}]_{m\times n}$

Scalar multiple: $c\,[a_{ij}]_{m\times n} = [c\,a_{ij}]_{m\times n}$

Ex: $2\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -1 \end{bmatrix} + (-3)\begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 4 & 6 \\ -4 & 0 & -2 \end{bmatrix} + \begin{bmatrix} 3 & -9 & -6 \\ -3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 2+3 & 4-9 & 6-6 \\ -4-3 & 0-0 & -2+9 \end{bmatrix} = \begin{bmatrix} 5 & -5 & 0 \\ -7 & 0 & 7 \end{bmatrix}$$

Matrix multiplication (RO x 6)

If $A=[a_{ij}]_{m\times n}$; $B=[b_{ij}]_{n\times p}$

The number of rows (n) of A must be equal to the number of columns (n) of B in order to perform the matrix multiplication.

The answer is in the form of $[C_{ij}]_{m\times p}$

$\boxed{AB=[C_{ij}]_{m\times p}}$, where $C_{ij}=\underbrace{(a_{i1},a_{i2},\dots a_{in})}_{\text{row (i) of A}}\cdot\underbrace{(b_{1j},b_{2j},\dots b_{nj})}_{\text{column (j) of B}}$

Ex:

$$\underset{2\times 2}{\begin{bmatrix}1 & 2\\ 1 & -1\end{bmatrix}}\underset{2\times 3}{\begin{bmatrix}4 & 1 & 4\\ 0 & -1 & 3\end{bmatrix}}$$

$$\begin{bmatrix}(1,2)\cdot(4,0) & (1,2)\cdot(1,-1) & (1,2)\,(4,3)\\ (1,-1)\cdot(4,0) & (1,-1)\cdot(1,-1) & (1,-1)\,(4,3)\end{bmatrix}=\begin{bmatrix}4 & -1 & 10\\ 4 & 2 & 1\end{bmatrix}$$

Ex

$$\underset{2\times 3}{\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix}}\underset{3\times 2}{\begin{bmatrix}2 & 1\\ 1 & -2\\ -1 & 0\end{bmatrix}}=\begin{bmatrix}1 & -3\\ 7 & -6\end{bmatrix}$$

$\boxed{\textbf{Note}: AB\neq BA}$

Ex1:

$$\underset{3\times 1}{\begin{bmatrix}4\\ 5\\ -6\end{bmatrix}}\underset{1\times 3}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}=\begin{bmatrix}4 & 8 & 12\\ 5 & 10 & 15\\ -6 & -12 & -18\end{bmatrix}$$

$$\underset{1\times 3}{\begin{bmatrix}1 & 2 & 3\end{bmatrix}}\underset{3\times 1}{\begin{bmatrix}4\\ 5\\ -6\end{bmatrix}}=\begin{bmatrix}-4\end{bmatrix}$$

Therefore $AB\neq BA$

Ex2:

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5 & 4\\ 15 & 10\end{bmatrix}$$

$$\begin{bmatrix}5 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}11 & 18\\ 3 & 4\end{bmatrix}$$

$AB\neq BA$

May 31^{st}
2024

2.7 Transpose

A^T exchange rows and columns

Properties:

1. $(A+B)^T = A^T + B^T$
2. $(aA)^T = aA^T$
3. $(AB)^T = B^T A^T$
4. $(A^T)^T = A$
5. $(A^{-1})^T = (A^T)^{-1}$

Definition A is symmetric if $A^T = A$

Note If A is symmetric, then A is a square matrix.

Ex: $A = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix} \Leftrightarrow A^T = \begin{bmatrix} 2 & 6 \\ 6 & 3 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 5 & -7 \\ 5 & -2 & 0 \\ -7 & 0 & 3 \end{bmatrix}. \Leftrightarrow B^T = \begin{bmatrix} 1 & 5 & -7 \\ 5 & -2 & 0 \\ -7 & 0 & 3 \end{bmatrix}$

A, B are symmetric.

3.1 Spaces of vectors

$\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{R}\} \overset{\text{or}}{=} \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}$

A subset S of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$, if it satisfies

1) $\vec{0}\ (0,0,\dots,0) \in S$

2) if $\vec{u},\vec{v} \in S$, then $\vec{u}+\vec{v} \in S$

3) if $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\,\vec{u} \in S$

Exercise Determine whether

$S = \{ (x_1, x_2, x_3) \mid x_3 = x_1 + 2x_2 \}$ is a subspace of $\mathbb{R}^3$.

Solution

(1) $\vec{0} = (0,0,0) \in S$ because $\overset{x_3}{0} = \overset{x_1}{0} + 2(\overset{x_2}{0})$

(2) if $\vec{x} = (x_1, x_2, x_3) \in S$ and $\vec{y} = (y_1, y_2, y_3) \in S$, then

$x_3 = x_1 + 2x_2$ and $y_3 = y_1 + 2y_2$.

$\vec{x}+\vec{y} = (x_1+y_1,\ x_2+y_2,\ x_3+y_3)$ and

$x_3 + y_3 = (x_1 + 2x_2) + (y_1 + 2y_2) \Rightarrow (x_1+y_1) + 2(x_2+y_2)$

So $\vec{x}+\vec{y} \in S$.

(3) If $\vec{x} = (x_1, x_2, x_3) \in S$ and $c \in \mathbb{R}$ then $x_3 = x_1 + 2x_2$

$c\vec{x} = (cx_1, cx_2, cx_3)$ and $cx_3 = cx_1 + 2cx_2$

So $c\vec{x} \in S$

Therefore S is a subspace of $\mathbb{R}^3$.

June 6^{th} 2024

Span

$B = \{\vec{v_1}, \vec{v_2}, ----, \vec{v_k}\} \subset \mathbb{R}^n$

$\text{Span } B = \{c_1\vec{v_1} + c_2\vec{v_2} + ---- + c_k\vec{v_k} \mid c_1, ---, c_k \in \mathbb{R}\}$

Definition Given a $m \times n$ matrix A, the column space of A is the subspace of $\mathbb{R}^m$ spanned the column of A, denoted by col (A) or C(A). The row space of A is the subspace of $\mathbb{R}^n$ spanned by the rows of A, denoted by Row(A)

Note: $\text{Row}(A) = \text{Col}(A^T)$

Ex: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then

$\text{Col } A = \text{span}\{(1,4), (2,5), (3,6)\}$

$\text{Row } A = \text{Span}\{(1,2,3), (4,5,6)\}$

Remark 1: $\vec{b} \in \text{Col } A \underset{\text{iff}}{\iff} A\vec{x} = \vec{b}$ has solution(s)

Remark 2: $\vec{b}$ is in span $\{\vec{v_1}, \vec{v_2}, ----, \vec{v_k}\}$

$\underset{\text{iff}}{\iff} A\vec{x} = \vec{b}$ has solution (s), where $A = [\vec{v_1}, \vec{v_2}, ---, \vec{v_k}]$

Ex Find the condition on $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, such that $\vec{b} \in \text{Col}(A)$

where $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix}$

Solution:

$$[A|\vec{b}] = \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2-R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & -1 & b_3+2b_1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_1+b_2+b_3 \end{array}\right]$$

$\vec{b} \in col(A)$ iff $\Leftrightarrow$ $A\vec{x}=b$ has a solution iff $\Leftrightarrow$ $b_1+b_2+b_3=0$

Ex: Determine whether

$\vec{b} = \begin{bmatrix} -4 \\ -2 \\ 2 \\ -6 \end{bmatrix}$ is in span $\left\{ \vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}; \vec{v_2} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}$

If so, express $\vec{b}$ as a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

Sol: $[A|\vec{b}] = \begin{array}{c} \begin{array}{ccc} \vec{v_1} & \vec{v_2} & \vec{v_3} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 1 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \\ 1 & 2 & 1 & -6 \end{array}\right] \end{array} \xrightarrow[\substack{-R_3 \\ R_4-R_1}]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -4 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$

$$\xrightarrow[R_4-R_3]{\substack{R_1+R_2 \\ -\frac{1}{2}R_2}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{RREF}$$

The system $A\vec{x} = \vec{b}$ has a solution $\vec{x} = \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$

so $\vec{b}$ is in the span and and

$$\vec{b} = 2\vec{v_1} - 3\vec{v_2} - 2\vec{v_3}$$

3.2 The nullspace of A: Solving $Ax=0$ and $Rx=0$

Theorem. Let A be a $m \times n$ matrix. The set of all solutions to a homogenous system $A\vec{x}=\vec{0}$ is a subspace of A, denoted by $Null(A)$ or $N(A)$

$$Null(A) = \{ x \in \mathbb{R}^n \mid A\vec{x} = 0$$

Ex: Find the nullspace of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix}$

Sol.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}}_{REF} \xrightarrow{R_1 - 2R_2} \underbrace{\begin{array}{c} \begin{array}{ccc} & & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}}_{R.R.E.F} \quad x_3 \text{ is free}$$

from 1^{st} row: $x_1 - x_3 = 0 \Rightarrow x_1 = x_3$

2^{nd} row: $x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$

$\Rightarrow x_3 = x_3$

then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad x_3 \in \mathbb{R}$

So $Null(A) = \left\{ x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} / x_3 \in \mathbb{R} \right\}$

or $Span\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$

Ex Construct a matrix whose nullspace consists of all linear combination of $(2,2,1,0)$ and $(3,1,0,1)$

Solution We need to find a matrix A such that the general solution of $A\vec{x}=\vec{0}$ is

$$\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \underbrace{s\begin{bmatrix}2\\2\\1\\0\end{bmatrix} + t\begin{bmatrix}3\\1\\0\\1\end{bmatrix}}_{s,t\in\mathbb{R}} \iff \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \begin{bmatrix}2s+3t\\2s+t\\s\\t\end{bmatrix}$$

$$\begin{aligned} x_1 &= 2s+3t\\ x_2 &= 2s+t\\ x_3 &= s\\ x_4 &= t\end{aligned} \quad \text{Then} \quad \begin{aligned} x_1 &= 2x_3+3x_4\\ x_2 &= 2x_3+x_4\\ x_3 &= s\\ x_4 &= t\end{aligned} \quad \text{since}$$

$$\iff \begin{aligned} x_1-2x_3-3x_4 &= 0\\ x_2-2x_3-x_4 &= 0\end{aligned}$$

We can choose $A = \begin{bmatrix}1 & 0 & -2 & -3\\ 0 & 1 & -2 & -1\end{bmatrix}$ or $\begin{bmatrix}1 & 0 & -2 & -3\\ 0 & 1 & -2 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$

(many answers)

Inverse Matrices

Remark: If A is invertible, then the system $A\vec{x}=\vec{b}$ has a unique solution $\vec{x}=A^{-1}\vec{b}$

Note $A_{n\times n}$ is invertible $\xLeftrightarrow{\text{iff}}$ $\operatorname{rank} A = n$
$\xLeftrightarrow{\text{iff}}$ I_n is the RREF of A.

Gauss-Jordan Method for computing A^{-1}

$$[A \mid I_n] \xrightarrow[\text{if } \operatorname{rank}A = n]{\text{row operations}} [I_n \mid A^{-1}]$$

If $\operatorname{rank} A < n$, then A is not invertible
(A is singular)

Ex: Find the inverse of $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 5 \\ 2 & 2 & 3 \end{bmatrix}$

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 5 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -1 & -3 & 1 & 0 \\ 0 & 4 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 4 & -1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

Since rank $A < 3$; then A is not invertible.

Properties

① $(A^{-1})^{-1} = A$

② $(AB)^{-1} = B^{-1}A^{-1}$

③ $(A^{k})^{-1} = (A^{-1})^{k}$

④ $(aA)^{-1} = \frac{1}{a}A^{-1}$; if $a \neq 0$

Remark: The 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\Longleftrightarrow$ $ad - bc \neq 0$, in this case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

June 11th 2024

3.3 The complete solution to $A\vec{x}=\vec{b}$

Remark: The complete (or general) solution to $A\vec{x}=\vec{b}$ can be expressed as $\vec{x}=\vec{x}_p+\vec{x}_n$, where $\vec{x}_p$ is a particular solution to $A\vec{x}=\vec{b}$, $\vec{x}_n$ is the general solution to $A\vec{x}=0$. ($\vec{x}_n$ are vectors in Null(A)).

Ex: Let $A=\begin{bmatrix}1&2&3&5\\2&4&8&12\\3&6&7&13\end{bmatrix}$, $\vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

1) Find the condition on b_1, b_2, b_3 such that $A\vec{x}=\vec{b}$ has solution(s). (or such that $\vec{b}\in Col(A)$)

(2) Find the complete solution to $A\vec{x}=b$ for $\vec{b}=\begin{bmatrix}1\\4\\1\end{bmatrix}$

3) What is the Nullspace of A?

4) What is the column space of A.

Solution

1) $[A \,|\, \vec{b}\,] \xrightarrow{\text{row oper.}} \left[\begin{array}{cccc|c}1&2&3&5&b_1\\0&0&1&1&(b_2-2b_1)/2\\0&0&0&0&b_2+b_3-5b_1\end{array}\right]$

$A\vec{x}=\vec{b}$ has solution(s) $\overset{\text{iff}}{\Longleftrightarrow}$ $\vec{b}\in Col(A)$ $\overset{\text{iff}}{\Longleftrightarrow}$ $b_2+b_3-5b_1=0$

(2) For $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$; $[A \mid \vec{b}] \xrightarrow{\text{row operation}}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

if we substitute $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$

$$\xrightarrow{R_1 - 3R_2} \begin{array}{c} \begin{array}{cccc|c} & x_2 & & x_4 & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

(RREF) where x_2, x_4 are free.

$$\left.\begin{aligned} x_1 &= -2 - 2x_2 - 2x_4 \\ x_2 &= x_2 \\ x_3 &= 1 - x_4 \\ x_4 &= x_4 \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{x_p} + \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{x_n}$$

x_p (particular solution)

x_n (homogenous solution or Nullspace)

Note: Consider $A\vec{x} = \vec{0}$

$$A \xrightarrow{\text{row operations}} \begin{array}{c} \begin{array}{cccc} & x_2 & & x_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(RREF) x_2, x_4 are free

If $A\vec{x} = \vec{0}$, then

$$\left.\begin{aligned} x_1 &= -2x_2 - 2x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned}\right\} \Rightarrow \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{The general solution for } A\vec{x} = \vec{0}}$$

(3) From part (2), we get:

$$\text{Nullspace}(A) = \left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

$\underline{\text{or}}$ $\text{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

(4) From (1), we get:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in \text{col}\, A \overset{\text{iff}}{\iff} b_2 + b_3 - 5b_1 = 0$$

$$\text{Col}(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \;\middle|\; \begin{matrix} b_2 + b_3 - 5b_1 = 0 \\ (\text{a plane in } \mathbb{R}^3) \end{matrix} \right\}$$

3.4 Independence, Basis and Dimensions

Linearly independent (Li)

$$\text{This } \begin{bmatrix} x_1 \vec{v}_1 \\ x_2 \vec{v}_2 \\ \vdots \\ x_n \vec{v}_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ iff } \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

iff coefficients are zero

Linearly dependent (L.d)

iff there exists coefficients $t_1, t_2, \dots, t_k$ are not all zero such that $t_1\vec{v_1} + t_2\vec{v_2} + \dots + t_k\vec{v_k} = 0$

Remark $\{v_1, v_2, \dots, \vec{v_k}\}$ is L.i $\overset{iff}{\Longleftrightarrow}$ rank $A = k$

$\{\vec{v_1}, \vec{v_2}, \dots \vec{v_k}\}$ is L.d $\overset{iff}{\Longleftrightarrow}$ rank $A < k$

where $A = [v_1 \quad v_2 \quad \dots \quad v_k]$

Ex1: Determine all the values of t such that the set $\{v_1 = (1,-1,3,1) ; v_2 = (-1,1,2,1) ; v_3 = (-1,1,t,5)\}$ is L.i

Sol: Let
$$\begin{array}{cccc} & \vec{v_1} & \vec{v_2} & \vec{v_3} \\ A = & \left[\begin{array}{c} 1 \\ -1 \\ 3 \\ 1 \end{array}\right. & \begin{array}{c} -1 \\ 1 \\ 2 \\ 1 \end{array} & \left.\begin{array}{c} -1 \\ 1 \\ t \\ 5 \end{array}\right] \end{array} \xrightarrow{\text{row operations}} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & t-12 \\ 0 & 0 & 0 \end{bmatrix}$$

The set is linearly independent $\overset{iff}{\Longleftrightarrow}$ rank $A = 3$

$\Longleftrightarrow t-12 \neq 0$

$\Longleftrightarrow t \neq 12$

because we can do row operations to find the leading 1.

4.1 Orthogonality of the four subspaces.

Def: Two subspace V and W of $\mathbb{R}^n$ are orthogonal if $\vec{v}\cdot\vec{w}=0$ for all $\vec{v}\in V$ and $\vec{w}\in W$, denoted by $V\perp W$.

Note: If the span $\{\vec{v}_1, \vec{v}_2, \cdots\cdots, \vec{v}_k\}$, and $W=\text{span}(\vec{w}_1,\vec{w}_2,\cdots,\vec{w}_k)$ in $\mathbb{R}^n$, then $V\perp W \overset{\text{iff}}{\iff} \vec{v}_i\cdot\vec{w}_j=0$ for all i,j.

Example $V=\text{span}\{\vec{v}_1=(1,1,1)\}$

$W=\text{span}\{\vec{w}_1=(1,2,-3),\ \vec{w}_2=(2,1,-3)\}$

$\vec{v}_1\cdot\vec{w}_1 = 1+2-3=0$, $\vec{v}_1\cdot w_2 = 2+1-3=0$

So $V\perp W$ V is perpendicular/ Orthogonal to W

Note For any $m\times n$ matrix A, we have $\text{Row}(A)\perp \text{Null}(A)$

Proof: $\text{Row}(A)=\text{span}\{\text{rows of } A\}$

$\text{Null}(A)=\{\vec{x} \mid A\vec{x}=\vec{0}\}$

$\vec{w}\in \text{Null}(A) \iff A\vec{w}=\vec{0} \iff$ $\begin{bmatrix}(\text{row } 1 \text{ of } A)\cdot\vec{w}\\(\text{row } 2 \text{ of } A)\cdot\vec{w}\\ \vdots \\(\text{row } m \text{ of } A)\cdot\vec{w}\end{bmatrix}=\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}$

$\iff (\text{row } i \text{ of } A)\cdot\vec{w}=0;\quad i=1,2,3,\cdots,m$

So $\text{Null}(A)\perp \text{Row}(A)$

Similarly, $\text{Col}(A)=\text{Row}(A^T)\perp \text{Null}(A^T)$

Orthogonal complement of a subspace V

V of $\mathbb{R}^n$

$V^\perp=\{\vec{u}\in\mathbb{R}^n \mid \vec{u}\cdot\vec{v}=0 \text{ for all } \vec{v}\in V\}$

The fundamental theorem of linear Algebra

Theorem $(\text{Row}(A))^\perp=\text{Null}(A)$

$(\text{Col}(A))^\perp=\text{Null}(A)$

Remark If $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$, then
$$V^{\perp} = \{\vec{u} \in \mathbb{R}^n \mid \vec{u}\cdot\vec{v}_i = 0,\ i = 1,2,3,\dots,k\}$$

Ex: Let $V = \text{Span}\{\vec{v}_1 = (1,0,1,-1), \vec{v}_2 = (1,1,2,-2)\}$. Find $V^{\perp}$.

Solution 1: We need to find all $\vec{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that
$\vec{x}\cdot\vec{v}_1 = 0$ and $\vec{x}\,\vec{v}_2 = 0$

Then $\begin{cases} x_1 + x_3 - x_4 = 0 \\ x_1 + x_2 + 2x_3 - 2x_4 = 0 \end{cases}$ We need to solve this system.

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & -2 \end{bmatrix} \xrightarrow[R_2 - R_1]{\text{row operations}} \begin{array}{c} \begin{array}{cccc} & & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \end{array}$$ in R.R.E.F.
x_3, x_4 are free.

$$\left.\begin{aligned} x_1 &= -x_3 + x_4 \\ x_2 &= -x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}\right\} \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad x_3, x_4 \in \mathbb{R}$$

So $V^{\perp} = \text{Span}\{(-1,-1,1,0), (1,1,0,1)\}$

Solution 2 Let $A = \begin{bmatrix} \vec{v}_1 \\ \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & 2 & -2 \end{bmatrix}$

Then $V = \text{Span}\{\vec{v}_1, \vec{v}_2\} = \text{Row}(A)$

Then $V^{\perp} = (\text{Row}(A))^{\perp} = \text{Null}(A)$
by the theorem.

So we need to solve $A\vec{x} = \vec{0}$

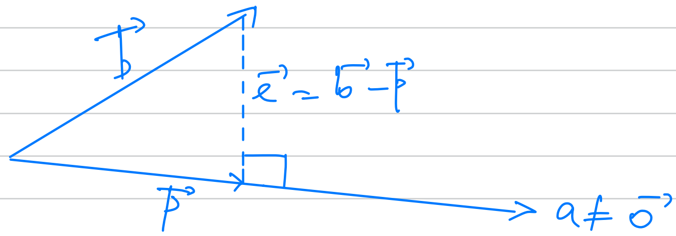
$$A \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$ in R.R.E.F

$$\left.\begin{aligned} x_1 &= -x_3 + x_4 \\ x_2 &= -x_3 + x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}\right\} \Longrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_3, x_4 \in \mathbb{R}.$$

So $V^{\perp} = \text{Null}(A) = \text{Span}\{(-1,-1,1,0)\,(1,1,0,1)\}$

Note If V is a subspace of $\mathbb{R}^n$ then
$\dim(V^{\perp}) = n - \dim(V)$.

Section 4.2 : Projections



We want to find the vector $\vec{p}$ such that $\vec{p} \parallel \vec{a}$ and $\vec{b} - \vec{p} \perp a$

Let $\vec{p} = k\vec{a}$ then $\vec{b} - k\vec{a} \perp a$. Then $\vec{a} \cdot (\vec{b} - k\vec{a}) = 0$

$$\vec{a} \cdot \vec{b} - k(\vec{a} \cdot \vec{a}) = 0$$

$$k = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}}$$

So $\vec{p} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}$, it is called the projection of $\vec{b}$ onto $\vec{a}$, denoted by $\text{proj}_{\vec{a}}(\vec{b})$. $\vec{b} - \vec{p}$ is called the projection of $\vec{b}$ perpendicular to $\vec{a}$, denoted by $\text{perp}_{\vec{a}}(\vec{b})$, or called error $\vec{e}$.

$$\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a}, \quad \vec{e} = \vec{b} - \vec{p} = \text{perp}_{\vec{a}}(\vec{b})$$

The projection matrix P onto $\vec{a}$ is $P = \frac{1}{\vec{a}\,\vec{a}} \vec{a}\vec{a}^T$, where $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ Note that P is the matrix such that

$$\text{proj}_{\vec{a}}(\vec{b}) = P\vec{b} \text{ for all } \vec{b} \in \mathbb{R}^n.$$

Ex 1 If $\vec{a} = (4, 3, -1)$; $\vec{b} = (-2, 5, 3)$ find $\text{proj}_{\vec{a}}(\vec{b})$, $\vec{e}$ and the matrix P.

Solution

$$\vec{p} = \text{proj}_{\vec{a}}(\vec{b}) = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} \vec{a} = \frac{4}{26} \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8/13 \\ 6/13 \\ -2/13 \end{bmatrix}$$

$$\vec{e} = \vec{b} - \text{proj}_{\vec{a}}(\vec{b}) = \begin{bmatrix} -34/13 \\ 59/13 \\ 41/13 \end{bmatrix}, \quad P = \frac{1}{\vec{a} \cdot \vec{a}} \begin{bmatrix} 4 \\ 3 \\ -1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -1 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 16 & 12 & -4 \\ 12 & 9 & -3 \\ -4 & -3 & 1 \end{bmatrix}$$

Check $P\vec{b} = \text{proj}_{\vec{a}}(\vec{b})$

$$\begin{bmatrix} 16 & 12 & -4 \\ 12 & 9 & -3 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2, & 5 & 3 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 8/13 \\ 6/13 \\ -2/13 \end{bmatrix}$$

Tuesday
07-02-24

4.4 Orthonormal Bases and Gram-Schmidt

A set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \dots \vec{v_n}\}$ in $\mathbb{R}^n$ is:

→ orthogonal if $v_i \cdot v_j = 0$ whenever $i \neq j$.

→ Orthonormal if: $\vec{v_i} \, \vec{v_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

(it is orthogonal and $\|\vec{v_i}\| = 1$ for all i)

Ex The set $\{\vec{v_1} = (1,1,1,1), \ \vec{v_2} = (1,-1,1,-1), \vec{v_3} = (1,0,-1,0)\}$

is orthogonal because $\vec{v_1} \cdot \vec{v_2} = 1-1+1-1 = 0$;

$\vec{v_1} \cdot \vec{v_3} = 1+0-1+0 = 0$;

$\vec{v_2} \cdot \vec{v_3} = 1+0-1+0 = 0$;

The set $\left\{ \frac{\vec{v_1}}{\|\vec{v_1}\|}, \frac{\vec{v_2}}{\|\vec{v_2}\|}, \frac{\vec{v_3}}{\|\vec{v_3}\|} \right\} = \left\{ \frac{1}{2}(1,1,1,1), \frac{1}{2}(1,-1,1,-1), \frac{1}{\sqrt{2}}(1,0,-1,0) \right\}$

is orthonormal

Note! $\{\vec{e_1} = (1,0,0,\dots,0); \vec{e_2} = (0,1,0,0 \dots 0), \vec{e_n} = (0,0,\dots 0,1)\}$ is an orthonormal basis of $\mathbb{R}^n$.

Theorem: Any orthogonal set of non-zero vectors in $\mathbb{R}^n$ is linearly independent.

Def: A $n \times n$ matrix Q is orthogonal if $Q^{-1} = Q^T$

or $Q^T Q = I$ or $Q Q^T = I$

Theorem: A $n \times n$ matrix is orthogonal

→ iff the columns of Q form an orthonormal basis of $\mathbb{R}^n$

→ iff the rows of Q form an orthonormal basis of $\mathbb{R}^n$

Def Suppose U is a subspace of $\mathbb{R}^n$ and $\{u_1, u_2, u_3, \dots, u_k\}$ is an orthonormal basis of U. For $\vec{v} \in \mathbb{R}^n$, the projection of $\vec{v}$ onto U is

$$\text{proj}_U(\vec{v}) = (\vec{v} \cdot \vec{u}_1)\vec{u}_1 + (\vec{v} \cdot \vec{u}_2)\vec{u}_2 + \dots + (\vec{v} \cdot \vec{u}_k)\vec{u}_k$$

Gram - Schmidt Procedure:

Suppose $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ is a basis of U. We want to find an orthogonal basis $\{\vec{U}_1, \vec{U}_2, \vec{U}_3, \dots, \vec{U}_k\}$ of U.

Let $\vec{u}_1 = \vec{v}_1$

$$\vec{u}_2 = \vec{v}_2 - \text{proj}_{\vec{u}_1}(\vec{v}_2) = \vec{v}_2 - \frac{\vec{v}_2 \cdot u_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1$$

$$u_3 = \vec{v}_3 - \text{proj}_{\vec{u}_1}(\vec{v}_3) - \text{proj}_{\vec{u}_2}(\vec{v}_3) = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{v}_2}\vec{v}_2$$

$\vdots$

$$\vec{u}_k = \vec{v}_k - \text{proj}_{\vec{v}_1}(\vec{v}_k) - \text{proj}_{\vec{v}_2}(\vec{v}_k) - \dots - \text{proj}_{u_{k-1}}(\vec{v}_k)$$

Theorem: $\text{proj}_U(\vec{v})$ is the unique vector in U which is closest to $\vec{v}$, that is $\|\vec{v} - \text{proj}_U(\vec{v})\| \leq \|\vec{v} - \vec{u}\|$ for all $\vec{u} \in U$

Section 5.1 and 5.2: Determinant by Cofactors, Properties

Def: Let $A = [a_{ij}]_{n\times n}$. Let M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting row i and column j. The **cofactor** of a_{ij} is $C_{ij} = (-1)^{i+j}\det(M_{ij})$

if * $n = 1$; $\det A = \det[a_{11}] = a_{11}$

* $n > 1$; $\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$

Ex: Compute $\det A$ for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Sol: $M_{11} = [a_{22}]$; $C_{11} = (-1)^{1+1}\det M_{11} = (-1)^2 a_{22} = a_{22}$

$M_{12} = [a_{21}]$; $C_{12} = (-1)^{1+2}\det M_{21} = (-1)^3 a_{21} = -a_{21}$

$$\det A = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} + a_{12}(-a_{21})$$
$$= a_{11}a_{22} - a_{12}a_{21}$$

$$\boxed{\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \overset{\text{or}}{=} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}}$$

Ex: $\det\begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 3(2) - (-1)5 = 11$

Theorem: If $A = [a_{ij}]_{n\times n}$, then

$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3} + \cdots + a_{in}C_{in}$ (Use row i)

or

$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (Use row j)

Ex: Compute $\det A$ for $A = \begin{bmatrix} 3 & -2 & 4 \\ 3 & 0 & 0 \\ 0 & 8 & -3 \end{bmatrix}$

Sol: * Use first row

$$\det A = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 3(-1)^2 \begin{vmatrix} 0 & 0 \\ 8 & -3 \end{vmatrix} + (-2)(-1)^3 \begin{vmatrix} 3 & 0 \\ 0 & -3 \end{vmatrix} + 4 \begin{vmatrix} 3 & 0 \\ 0 & 8 \end{vmatrix}$$

$$= 3(0) + 2(-9) + 4(24) = 78.$$

* Use second row

$$\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= 3C_{21} + 0C_{22} + 0C_{23} = 3C_{21} + 0 + 0 = 3C_{21}$$

$$= 3(-1)^3 \begin{vmatrix} -2 & 4 \\ 8 & -3 \end{vmatrix} \Rightarrow -3(6-32) = -18 + 96 = 78$$

Use the fourth row

Ex: $\begin{vmatrix} 2 & 3 & -3 & 1 \\ -3 & 0 & 0 & 5 \\ 1 & 2 & 1 & -2 \\ 3 & 0 & 0 & 0 \end{vmatrix} = a_{41}C_{41} + a_{42}C_{42} + a_{43}C_{43} + a_{44}C_{44}$

$$= 3C_{41} + 0C_{42} + 0C_{43} + 0C_{44}$$

$$= 3(-1)^5 \begin{vmatrix} 3 & -3 & 1 \\ 0 & 0 & 5 \\ 2 & 1 & -2 \end{vmatrix}$$

$$= -3(0C_{21} + 0C_{22} + 5C_{23})$$ (Use of the second row)

$$= -3(5C_{23})$$

$$= 15 \begin{vmatrix} 3 & -3 \\ 2 & 1 \end{vmatrix} = 15(3+6) = 15(9) = \underline{\underline{135}}$$

Column space

Feedback

Here we are given $A = \begin{pmatrix} -1 & 4 & 1 & -5 \\ 2 & 9 & 7 & -12 \\ 1 & 4 & -5 & 2 \\ 4 & 1 & 5 & -2 \end{pmatrix}$. To find a basis for the column space of a matrix, we first row reduce the original matrix, obtaining $B = \begin{pmatrix} 1 & 0 & 0 & \frac{23}{28} \\ 0 & 1 & 0 & \frac{-23}{28} \\ 0 & 0 & 1 & \frac{-25}{28} \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The columns in A that correspond to columns in B that contain leading 1's form a basis for the column space of A. So in this case, $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ -5 \\ 5 \end{pmatrix} \right\}$ forms a basis for the column space.

Orthonormal set of vectors

Is the set of vectors $\begin{pmatrix} \frac{11}{15} \\ \frac{-2}{3} \\ \frac{-2}{15} \end{pmatrix}, \begin{pmatrix} \frac{-2}{15} \\ \frac{-1}{3} \\ \frac{14}{15} \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ orthonormal in $\mathbb{R}^3$?

Your response	Correct response
No	Yes

Auto graded **Grade:** 0/1.0

Total grade: 0.0×1/1 = 0%

Feedback:

Let $\vec{a} = \begin{pmatrix} \frac{11}{15} \\ \frac{-2}{3} \\ \frac{-2}{15} \end{pmatrix}, \vec{b} = \begin{pmatrix} \frac{-2}{15} \\ \frac{-1}{3} \\ \frac{14}{15} \end{pmatrix}, \vec{c} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$.

Then we have:

$\vec{a} \cdot \vec{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3 = 0$

$\vec{a} \cdot \vec{c} = a_1 \cdot c_1 + a_2 \cdot c_2 + a_3 \cdot c_3 = 0$

$\vec{b} \cdot \vec{c} = b_1 \cdot c_1 + b_2 \cdot c_2 + b_3 \cdot c_3 = 0$

Therefore the set is orthogonal.

If the set is orthogonal and the lengths of all vectors are 1, then the set is orthonormal.

$\left\| \vec{a} \right\| = \sqrt{a_1{}^2 + a_2{}^2 + a_3{}^2} = 1$

$\left\| \vec{b} \right\| = \sqrt{b_1{}^2 + b_2{}^2 + b_3{}^2} = 1$

$\left\| \vec{c} \right\| = \sqrt{c_1{}^2 + c_2{}^2 + c_3{}^2} = 1$

Therefore Yes , these are orthonormal.