Section 7.4: Integration of Rational Functions by Partial Fractions

If $P(x)$ is a polynomial, then we can find $\int P(x)\,dx$ easily by the simple power rule.

Recall that a rational function has the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials. Using the long division, we can write

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where $S(x)$ and $R(x)$ are also polynomials with degree $(R) <$ degree (Q). In this case, $\dfrac{R(x)}{Q(x)}$ is called a proper rational function. Then we have

$$\int \frac{P(x)}{Q(x)}\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx.$$

E.g.,

$$\frac{x^3+x}{x-1} = (x^2+x+2) + \frac{2}{x-1}$$

and hence

$$\int \frac{x^3+x}{x-1}\,dx = \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x + 2\ln|x-1| + C.$$

In this section, we consider integral of proper rational functions $\dfrac{R(x)}{Q(x)}$ by writing it as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^k} \quad \text{or} \quad \frac{Ax+B}{(ax^2+bx+c)^n} \ \text{ with } b^2-4ac < 0,$$

which is possible by the algebra theory.

In the following, we always assume that $\dfrac{R(x)}{Q(x)}$ is a proper rational function.

Case 1. $Q(x) = (a_1x + b_1) \cdots (a_kx + b_k)$ **is a product of distinct linear factors.**

In this case, $\dfrac{R(x)}{Q(x)}$ has the partial fraction decomposition

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \cdots + \frac{A_k}{a_kx + b_k}.$$

Then we can apply the formula $\displaystyle\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.$

Example 13. Find $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

Solution. We have

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x-1)(x+2).$$

Thus the integrand has the partial fraction decomposition

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

Multiplying both sides by $x(2x-1)(x+2)$, we get

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).$$

$$\text{Taking } x = 0: \quad -1 = -2A \implies A = \frac{1}{2}.$$

$$\text{Taking } x = \frac{1}{2}: \quad \frac{1}{4} = \frac{5}{4}B \implies B = \frac{1}{5}.$$

$$\text{Taking } x = -2: \quad -1 = 10C \implies C = -\frac{1}{10}.$$

Therefore, we have

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} - \frac{1}{10}\frac{1}{x+2}\right)dx$$

$$= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C. \quad \blacksquare$$

Case 2. $Q(x) = (a_1x+b_1)^{r_1}\cdots(a_kx+b_k)^{r_k}$ **with** $r_1, \cdots, r_k \geq 1$**.**

In this case, $\dfrac{R(x)}{Q(x)}$ has the partial fraction decomposition

$$\frac{R(x)}{Q(x)} = \left(\frac{A_{1,1}}{a_1x+b_1} + \cdots + \frac{A_{1,r_1}}{(a_1x+b_1)^{r_1}}\right) + \cdots + \left(\frac{A_{k,1}}{a_kx+b_k} + \cdots + \frac{A_{k,r_k}}{(a_kx+b_k)^{r_k}}\right).$$

E.g., $\dfrac{x^3+x^2+5}{x^2(x+1)(x-1)^3}$ has the form

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1} + \frac{E}{(x-1)^2} + \frac{F}{(x-1)^3}.$$

Then we can apply the formulas

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C \quad \text{and} \quad \int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a(1-n)}(ax+b)^{1-n} + C \ \ (n \geq 2).$$

Example 14. Find $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$.

Solution. After taking long division, we have

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1} = (x+1) + \frac{4x}{(x-1)^2(x+1)}.$$

Now $\dfrac{4x}{(x-1)^2(x+1)}$ has the form

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Multiplying both sides by $(x-1)^2(x+1)$, we get

$$4x = A(x+1)(x-1) + B(x+1) + C(x-1)^2.$$

$$\text{Taking } x=1: \quad 4 = 2B \implies B = 2.$$

$$\text{Taking } x=-1: \quad -4 = 4C \implies C = -1.$$

$$\text{Taking } x=0: \quad 0 = -A+2-1 \implies A = 1.$$

Therefore, we have

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx \;=\; \int \left(x + 1 \,+\, \frac{1}{x-1} \,+\, \frac{2}{(x-1)^2} \,-\, \frac{1}{x+1} \right) dx$$

$$= \frac{1}{2}x^2 \,+\, x \,+\, \ln|x-1| \,-\, \frac{2}{x-1} \,-\, \ln|x+1| \,+\, C. \quad \blacksquare$$

Case 3. $Q(x)$ **contains irreducible quadratic form** $ax^2 + bx + c$ **with** $b^2 - 4ac < 0$**.**

In this case, the partial fraction decomposition of $\dfrac{R(x)}{Q(x)}$ contains the term $\dfrac{Ax + B}{ax^2 + bx + c}$.

E.g., $\dfrac{x}{(x-1)(x^2+1)(x^2+3)}$ has the partial fraction decomposition

$$\frac{A}{x-1} \,+\, \frac{Bx + C}{x^2+1} \,+\, \frac{Dx + E}{x^2+3}.$$

Then we can apply the formulas

$$\int \frac{x}{x^2 + a^2}\, dx \;=\; \frac{1}{2}\ln(x^2 + a^2) \,+\, C \quad \text{and} \quad \int \frac{1}{x^2 + a^2}\, dx \;=\; \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \,+\, C.$$

Example 15. Find $\displaystyle\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$.

Solution. We have

$$\frac{2x^2 - x + 4}{x^3 + 4x} \;=\; \frac{2x^2 - x + 4}{x(x^2+4)} \;=\; \frac{A}{x} \,+\, \frac{Bx + C}{x^2+4}.$$

Multiplying both sides by $x(x^2+4)$, we get

$$2x^2 - x + 4 \;=\; A(x^2+4) \,+\, (Bx + C)x \;=\; (A+B)x^2 + Cx + 4A.$$

Thus $\Big(A + B \;=\; 2, \quad C \;=\; -1, \quad 4A \;=\; 4\Big) \quad \Longrightarrow \quad \Big(A \;=\; 1, \quad B \;=\; 1, \quad C \;=\; -1\Big)$.

Therefore, we have

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx \;=\; \int \left(\frac{1}{x} + \frac{x - 1}{x^2+4} \right) dx \;=\; \ln|x| + \frac{1}{2}\ln(x^2{+}4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C. \quad \blacksquare$$

Rationalizing Substitution.

Sometimes, through substitutions, we can change other types of functions into rational functions. Then we can solve the integrals by partial fractions.

Example 16. Find $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$.

Solution. We want to get ride of $\sqrt{\ }$, so let $u=\sqrt{x}$. Then

$$x = u^2, \quad dx = 2u\,du.$$

Therefore, we have

$$\int \frac{1}{x+\sqrt{x}}\,dx = \int \frac{2u}{u^2+u}\,du = 2\int \frac{1}{u+1}\,du = 2\ln|u+1|+C = 2\ln|\sqrt{x}+1|+C. \quad \blacksquare$$

Example 17. Find $\displaystyle\int \frac{\sqrt{x+4}}{x}\,dx$.

Solution. Let $u=\sqrt{x+4}$. Then $x = u^2-4$ and $dx = 2u\,du$. Thus we have

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2\int \frac{u^2}{u^2-4}\,du = 2\int \left(1+\frac{4}{u^2-4}\right)du = 2u+8\int \frac{1}{u^2-4}\,du.$$

Now

$$\frac{1}{u^2-4} = \frac{A}{u-2}+\frac{B}{u+2} \quad\Longrightarrow\quad A=\frac{1}{4},\ B=-\frac{1}{4}.$$

Therefore, we have

$$\int \frac{\sqrt{x+4}}{x}\,dx = 2u+8\int \frac{1}{u^2-4}\,du = 2u+2\int\left(\frac{1}{u-2}-\frac{1}{u+2}\right)du$$

$$= 2u+2\ln|u-2|-2\ln|u+2|+C$$

$$= 2\sqrt{x+4}+2\ln\left|\frac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right|+C. \quad \blacksquare$$

Section 7.8: Improper Integrals

Type I. Improper Integrals over Infinite Intervals.

We define

$$\int_a^\infty f(x)\,dx \;=\; \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{if the limit exists.}$$

In this case, we say that the improper integral $\int_a^\infty f(x)\,dx$ is convergent; otherwise, we say that the improper integral $\int_a^\infty f(x)\,dx$ is divergent.

Similarly, we define

$$\int_{-\infty}^a f(x)\,dx \;=\; \lim_{t\to-\infty}\int_t^a f(x)\,dx \quad \text{if the limit exists.}$$

In this case, we say that the improper integral $\int_{-\infty}^a f(x)\,dx$ is convergent; otherwise, we say that the improper integral $\int_{-\infty}^a f(x)\,dx$ is divergent.

We define

$$\int_{-\infty}^\infty f(x)\,dx \;=\; \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

if both improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent.

Remark. We can have that $\int_{-\infty}^\infty f(x)\,dx$ is divergent though $\lim\limits_{t\to\infty}\int_{-t}^t f(x)\,dx$ exists.

Fact 1. When $f \geq 0$, the convergence of the improper integral shows that the area under the graph of f is finite.

Fact 2. In the definition of $\int_{-\infty}^\infty f(x)\,dx$, a can be any number, say, $a = 0$.

Fact 3. $\int_a^\infty f(x)\,dx$ is convergent $\iff$ $\int_b^\infty f(x)\,dx$ is convergent for any $b > a$;

$\int_{-\infty}^a f(x)\,dx$ is convergent $\iff$ $\int_{-\infty}^c f(x)\,dx$ is convergent for any $c < a$.

Example 18. Determine whether the following improper integrals are convergent.

(1) $\int_0^\infty \sin x\, dx$.

It is divergent since $\displaystyle\int_0^\infty \sin x\, dx \;=\; \lim_{t\to\infty}\int_0^t \sin x\, dx \;=\; \lim_{t\to\infty}(1-\cos t)$ DNE.

(2) $\int_{-\infty}^0 e^x\, dx$.

It is convergent since $\displaystyle\int_{-\infty}^0 e^x\, dx \;=\; \lim_{t\to-\infty}\int_t^0 e^x\, dx \;=\; \lim_{t\to-\infty}(1-e^t) \;=\; 1.$

(3) $\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx$.

Now

$$\int_0^\infty \frac{1}{1+x^2}\, dx \;=\; \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\, dx \;=\; \lim_{t\to\infty}\tan^{-1}t \;=\; \frac{\pi}{2}.$$

Similarly, we have

$$\int_{-\infty}^0 \frac{1}{1+x^2}\, dx \;=\; \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\, dx \;=\; \lim_{t\to-\infty}\Big(-\tan^{-1}t\Big) \;=\; \frac{\pi}{2}.$$

Therefore, $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx$ is convergent and

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\, dx \;=\; \int_{-\infty}^0 \frac{1}{1+x^2}\, dx + \int_0^\infty \frac{1}{1+x^2}\, dx \;=\; \pi.$$

(4) $\int_1^\infty \frac{1}{x^p}\, dx$.

Note that $$\int_1^t \frac{1}{x^p}\, dx \;=\; \begin{cases} \frac{1}{1-p}\,t^{1-p} - \frac{1}{1-p} & \text{if } p\neq 1 \\ \ln t & \text{if } p=1 \end{cases},$$

which is convergent as $t\to\infty$ exactly when $p>1$. In this case, we have

$$\int_1^\infty \frac{1}{x^p}\, dx \;=\; \lim_{t\to\infty}\int_1^t \frac{1}{x^p}\, dx \;=\; \lim_{t\to\infty}\left(\frac{1}{1-p}\,t^{1-p} - \frac{1}{1-p}\right) \;=\; \frac{1}{p-1}.$$

Type II. Improper Integrals with Discontinuous Integrands.

Suppose f is continuous on $(a, b]$ and discontinuous at a. We define

$$\int_a^b f(x)\,dx \;=\; \lim_{t\to a^+}\int_t^b f(x)\,dx \quad \text{if the limit exists.}$$

In this case, we say that the improper integral $\int_a^b f(x)\,dx$ is convergent; otherwise, we say that the improper integral $\int_a^b f(x)\,dx$ is divergent.

Similarly, if f is continuous on $[a, b)$ and discontinuous at b. then we define

$$\int_a^b f(x)\,dx \;=\; \lim_{t\to b^-}\int_a^t f(x)\,dx \quad \text{if the limit exists.}$$

In this case, we say that the improper integral $\int_a^b f(x)\,dx$ is convergent; otherwise, we say that the improper integral $\int_a^b f(x)\,dx$ is divergent.

Suppose $a < c < b$ and f is discontinuous at c, then we define

$$\int_a^b f(x)\,dx \;=\; \int_a^c f(x)\,dx \;+\; \int_c^b f(x)\,dx$$

if both improper integrals $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent.

Fact 4. When $f \geq 0$, the convergence of the improper integral shows that the area under the graph of f is finite.

Example 19. Determine whether the following improper integrals are convergent.

(1) $\int_0^1 \frac{1}{x-1}\,dx$.

The integrand is discontinuous at 1. For $0 < t < 1$, we have

$$\int_0^t \frac{1}{x-1}\,dx \;=\; \Big[\ln|x-1|\Big]_0^t \;=\; \ln|t-1|.$$

When $t \to 1^-$, $|t-1| \to 0^+$ and hence $\ln|t-1| \to -\infty$. Therefore, $\int_0^1 \frac{1}{x-1}\,dx$ is divergent.

(2) $\int_0^{\frac{\pi}{2}} \tan x \, dx$.

The integrand is discontinuous at $\frac{\pi}{2}$. For $0 < t < \frac{\pi}{2}$, we have

$$\int_0^t \tan x \, dx \;=\; \Big[-\ln|\cos x| \Big]_0^t \;=\; -\ln|\cos t|.$$

When $t \to \frac{\pi}{2}^-$, $|\cos t| \to 0^+$ and hence $-\ln|\cos t| \to \infty$. Thus $\int_0^{\frac{\pi}{2}} \tan x \, dx$ is divergent.

(3) $\int_0^1 \ln x \, dx$.

The integrand is discontinuous at 0. For $0 < t < 1$, we have

$$\int_t^1 \ln x \, dx \;=\; \Big[x \ln x \Big]_t^1 \;-\; \int_t^1 1 \, dx \;=\; -t\ln t \;-\; (1-t) \;=\; t \;-\; 1 \;-\; t\ln t.$$

By L'Hôpital's Rule, we have

$$\lim_{t\to 0^+} t\ln t \;=\; \lim_{t\to 0^+} \frac{\ln t}{\frac{1}{t}} \;=\; \lim_{t\to 0^+} \frac{\frac{1}{t}}{-\frac{1}{t^2}} \;=\; \lim_{t\to 0^+} (-t) \;=\; 0.$$

Therefore, we have

$$\int_0^1 \ln x \, dx \;=\; \lim_{t\to 0^+} \int_t^1 \ln x \, dx \;=\; \lim_{t\to 0^+} (t \;-\; 1 \;-\; t\ln t) \;=\; -1.$$

Question. What is the area bounded by the x-axis and $y = \ln x$ between 0 and 1?

Example 20. If one writes $\int_{-1}^2 \frac{1}{x} \, dx \;=\; \Big[\ln|x| \Big]_{-1}^2 \;=\; \ln 2$, then what is wrong?

Note that the integrand is discontinuous at 0, one cannot use the FTC-2 to evaluate this improper integral.

In fact, the improper integral $\int_{-1}^2 \frac{1}{x} \, dx$ is divergent, since

$$\int_0^2 \frac{1}{x} \, dx \;=\; \infty$$

is divergent.

Comparison Theorem. Suppose that $0 \le g(x) \le f(x)$.

(i) If $\int_a^\infty f(x)\,dx < \infty$, then $\int_a^\infty g(x)\,dx < \infty$.

(ii) If $\int_a^\infty g(x)\,dx = \infty$, then $\int_a^\infty f(x)\,dx = \infty$.

Remarks.

(a) The converse of (i) or (ii) is not true.

(b) The Comparison Theorem holds for all types of improper integrals.

Example 21. Determine whether the following improper integrals are convergent.

(1) $\int_0^\infty e^{-x^2}\,dx$.

Solution. Note that when $x \ge 1$, $-x^2 \le -x$ and hence $0 \le e^{-x^2} \le e^{-x}$. Since

$$\int_1^\infty e^{-x}\,dx = \frac{1}{e} < \infty,$$

by the Comparison Theorem, we have

$$\int_1^\infty e^{-x^2}\,dx < \infty.$$

Therefore, we have

$$\int_0^\infty e^{-x^2}\,dx = \int_0^1 e^{-x^2}\,dx + \int_1^\infty e^{-x^2}\,dx < \infty$$

is convergent. ■

(2) $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$.

Solution. Note that when $x \ge 1$, we have $\dfrac{1+e^{-x}}{x} \ge \dfrac{1}{x} > 0$. Since $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \infty$, by the Comparison Theorem, we have

$$\int_1^\infty \frac{1+e^{-x}}{x}\,dx = \infty.$$

Therefore, $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ is divergent, ■

(3) $\int_0^1 \frac{\sin^2 x}{\sqrt{x}}\,dx$.

<u>*Solution*</u>. Note that when $x > 0$, we have $0 \le \dfrac{\sin^2 x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$. Since

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \left[2\sqrt{x}\right]_t^1 = 2 < \infty.$$

By the Comparison Theorem, we have

$$\int_0^1 \frac{\sin^2 x}{\sqrt{x}}\,dx < \infty$$

is convergent. ■