Section 7.1: Integration by Parts

We know that

a formula/rule/property on differentiation $\longleftrightarrow$ a formula/rule/property on integration.

E.g.,

the Chain Rule $\longleftrightarrow$ the Substitution Rule.

Note that the Quotient Rule can be obtained from the Product Rule and the Chain Rule. Then we wonder

the Product Rule $\longleftrightarrow$ **?**

The answer is the Integration by Parts.

The Product Rule also has the form

$$f(x)\,g'(x) \;=\; \big(f(x)g(x)\big)' - f'(x)\,g(x).$$

Taking general antiderivative/indefinite integral to the both sides, we have

Integration by Parts for Indefinite Integral. Suppose f' and g' are continuous. Then

$$\boxed{\int f(x)\,g'(x)\,dx \;=\; f(x)\,g(x) - \int f'(x)\,g(x)\,dx}$$

If we let $u = f(x)$ and $v = g(x)$, then the above formula can be written as

$$\boxed{\int u\,dv \;=\; u\,v - \int v\,du}$$

Combining the above IBP and the FTC-2, we have

Integration by Parts for Definite Integral. Suppose f' and g' are continuous. Then

$$\boxed{\int_a^b f(x)\,g'(x)\,dx \;=\; \Big[f(x)\,g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)\,dx}$$

Remark 1. Like the Substitution Rule, after applying the IBP, we get another integral and we hope it is easier than the original one so that we can solve it. If it is still not solvable, we can try other approaches.

Remark 2. When the integrand is the product of two functions, we can try the substitution Rule first if one factor is related to the derivative of the other factor (e.g., try the GPR). For other cases, we can try the IBP.

Question. Then which factor should be taken as f and which one should be taken as g'?

Principle for Choosing f and g' in applying the IBP. For the two factors in the integrand, the factor with "simpler" derivative should be taken as $f(x)$, and the factor whose antiderivative can be obtained easily should be taken as $g'(x)$.

Example 1. Find the following integrals.

(1) $\int x \sin x \, dx$.

$$\int x \sin x \, dx = x(-\cos x) - \int (-\cos x) \, dx = -x \cos x + \sin x + C.$$

(2) $\int x^{99} \ln x \, dx$.

$$\int x^{99} \ln x \, dx = (\ln x) \frac{1}{100} x^{100} - \int \frac{1}{x} \frac{1}{100} x^{100} \, dx$$

$$= \frac{1}{100} x^{100} \ln x - \frac{1}{100} \int x^{99} \, dx = \frac{1}{100} x^{100} \ln x - \frac{1}{10000} x^{100} + C.$$

(3) $\int \ln x \, dx$.

$$\int \ln x \, dx = \int (\ln x) \, 1 \, dx = (\ln x) \, x - \int \frac{1}{x} \, x \, dx = x \ln x - x + C.$$

(4) $\int \frac{1}{x} \ln x \, dx$.

$$\int \frac{1}{x} \ln x \, dx = \int (\ln x) \frac{1}{x} \, dx = \frac{1}{2} (\ln x)^2 + C.$$

(5) $\int t^2 e^t\,dt$. (**Twice IBP**)

$$\int t^2 e^t\,dt \;=\; t^2 e^t \;-\; \int 2te^t\,dt \;=\; t^2 e^t \;-\; \Big(2te^t \;-\; \int 2e^t\,dt\Big) \;=\; t^2 e^t \;-\; 2te^t \;+\; 2e^t \;+\; C.$$

(6) $\int e^x \sin x\,dx$. (**Twice IBP to get original integral back**)

$$\int e^x \sin x\,dx \;=\; e^x(-\cos x) \;-\; \int e^x(-\cos x)\,dx \;=\; -e^x\cos x \;+\; \int e^x \cos x\,dx$$

$$=\; -e^x\cos x \;+\; e^x \sin x \;-\; \int e^x \sin x\,dx.$$

Therefore, we have

$$2\int e^x \sin x\,dx \;=\; -e^x\cos x \;+\; e^x\sin x \;+\; C;$$

that is,

$$\int e^x \sin x\,dx \;=\; \frac{1}{2}e^x(\sin x - \cos x) \;+\; C.$$

(7) $\int x^5 e^{-x^2}\,dx$. (**SR + twice IBP**)

$$u \;=\; -x^2 \qquad\Longrightarrow\qquad du \;=\; -2x\,dx \qquad\Longrightarrow\qquad x^5\,dx \;=\; -\frac{1}{2}u^2\,du.$$

Therefore, we have

$$\int x^5 e^{-x^2}\,dx \;=\; -\frac{1}{2}\int u^2 e^u\,du \;=\; -\frac{1}{2}\Big(u^2e^u \;-\; 2ue^u \;+\; 2e^u\Big) \;+\; C \quad (\text{by } \mathbf{(5)})$$

$$=\; -\frac{1}{2}\Big(x^4e^{-x^2} \;+\; 2x^2e^{-x^2} \;+\; 2e^{-x^2}\Big) \;+\; C.$$

(8) $\int_0^1 \tan^{-1}x\,dx$. (**IBP + GPR**)

$$\int_0^1 \tan^{-1}x\,dx \;=\; \int_0^1 (\tan^{-1}x)\,1\,dx \;=\; \Big[(\tan^{-1}x)\,x\Big]_0^1 \;-\; \int_0^1 \frac{x}{1+x^2}\,dx$$

$$=\; \frac{\pi}{4} \;-\; \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx \;=\; \frac{\pi}{4} \;-\; \frac{1}{2}\Big[\ln(1+x^2)\Big]_0^1 \;=\; \frac{\pi}{4} \;-\; \frac{1}{2}\ln 2.$$

Tabular Method for Applying IBP Twice

sign	keep taking derivative	keep taking antiderivative
$+$	f	g
$-$	f'	$g^{[1]}$
$+$	f''	$g^{[2]}$
	$f''' = 0$	$g^{[3]}$

In this case, we have

$$\boxed{\int f(x)\,g(x)\,dx \;=\; f(x)\,g^{[1]}(x) \;-\; f'(x)\,g^{[2]}(x) \;+\; f''(x)\,g^{[3]}(x) \;+\; C\,.}$$

This method can be used for $\displaystyle\int t^2 e^t\,dt$, $\displaystyle\int x^2\sin(x)\,dx$, $\displaystyle\int (3x^2-2x+5)\cos(2x)\,dx$, etc.

For example,

$$\int x^2\sin(x)\,dx \;=\; x^2(-\cos x) \;-\; 2x(-\sin x) \;+\; 2\cos x \;+\; C$$

$$= \; -x^2\cos x \;+\; 2x\sin x \;+\; 2\cos x \;+\; C.$$

Section 7.2: Trigonometric Integrals

In this section, we consider three types of trigonometric integrals.

I. Integral of the type $\int \sin^m x\, \cos^n x\, dx$ ($m,\, n \geq 0$ are integers).

The basic identities used in this part include

$$\sin^2 x + \cos^2 x \;=\; 1.$$

$$(\sin x)' \;=\; \cos x, \qquad (\cos x)' \;=\; -\sin x.$$

$$\sin^2 x \;=\; \frac{1}{2}\big(1-\cos(2x)\big), \qquad \cos^2 x \;=\; \frac{1}{2}\big(1+\cos(2x)\big).$$

(a) $\int \sin^{2k+1}x\, \cos^n x\, dx \;=\; \int (1-\cos^2 x)^k\, \cos^n x\; \sin x\, dx \;=\; -\int (1-u^2)^k u^n\, du$ with $u = \cos x$.

(b) $\int \sin^m x\, \cos^{2k+1}x\, dx \;=\; \int \sin^m x (1-\sin^2 x)^k\, \cos x\, dx \;=\; \int u^m (1-u^2)^k\, du$ with $u = \sin x$.

(c) $\int \sin^{2k}x\, \cos^{2\ell}x\, dx \;=\; \int \left(\frac{1}{2}(1-\cos 2x)\right)^k \left(\frac{1}{2}(1+\cos 2x)\right)^\ell dx$ (reduce the power).

Example 2. Find the integrals $\int \sin^2 x\, \cos^3 x\, dx$ and $\int \sin^5 x\, \cos^2 x\, dx$.

Solution. we have

$$\int \sin^2 x\, \cos^3 x\, dx \;=\; \int \sin^2 x\,(1-\sin^2 x)\, \cos x\, dx \;=\; \int u^2(1-u^2)\, du \quad \text{with } u = \sin x$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C \;=\; \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

and

$$\int \sin^5 x\, \cos^2 x\, dx \;=\; \int (1-\cos^2 x)^2\, \cos^2 x\, \sin x\, dx \;=\; -\int (1-u^2)^2\, u^2\, du \quad \text{with } u = \cos x$$

$$= -\int (u^6 - 2u^4 + u^2)\, du \;=\; -\Big(\frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3\Big) + C \;=\; -\Big(\frac{1}{7}\cos^7 x - \frac{2}{5}\cos^5 x + \frac{1}{3}\cos^3 x\Big) + C. \quad \blacksquare$$

Example 3. Find the integral $\int \sin^4 x\, dx$.

Solution. we have

$$\begin{aligned}
\int \sin^4 x\, dx &= \int \Big(\frac{1}{2}(1-\cos 2x)\Big)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{4}\int \Big(1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\Big)\, dx \\
&= \frac{3}{8}\int 1\, dx - \frac{1}{2}\int \cos 2x\, dx + \frac{1}{8}\int \cos 4x\, dx \\
&= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C. \qquad \blacksquare
\end{aligned}$$

II. Integral of the type $\int \tan^m x \sec^n x\, dx$ (m, $n \geq 0$ are integers).

The basic identities used in this part include

$$\sec^2 x = 1 + \tan^2 x.$$

$$(\tan x)' = \sec^2 x, \qquad (\sec x)' = \sec x \tan x.$$

$$\int \tan x\, dx = \ln|\sec x| + C.$$

(a) $\int \tan^{2k+1} x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x\, (\sec x \tan x)\, dx = \int (u^2-1)^k u^{n-1}\, du$ with $u = \sec x$.

(b) $\int \tan^m x \sec^{2k} x\, dx = \int \tan^m x \sec^{2k-2} x \sec^2 x\, dx = \int u^m (1+u^2)^{k-1}\, du$ with $u = \tan x$.

(c) $\int \tan^{2k} x \sec^{2\ell+1} x\, dx = \int (\sec^2 x - 1)^k \sec^{2\ell+1} x\, dx$ (write integrand as odd powers of $\sec x$).

Example 4. Find the integrals $\int \tan^5 x \sec^{19} x\, dx$ and $\int \tan^3 x\, dx$.

Solution. we have

$$\int \tan^5 x \sec^{19} x\, dx = \int \tan^4 x \sec^{18} x\,(\sec x \tan x)\, dx = \int (u^2-1)^2 u^{18}\, du \quad \text{with } u = \sec x$$

$$= \int (u^{22} - 2u^{20} + u^{18})\, du = \frac{1}{23}\sec^{23} x - \frac{2}{21}\sec^{21} x + \frac{1}{19}\sec^{19} x + C$$

and

$$\int \tan^3 x\, dx = \int \tan x(\sec^2 x - 1)\, dx$$

$$= \int \sec x(\tan x \sec x)\, dx - \int \tan x\, dx = \frac{1}{2}\sec^2 x - \ln|\sec x| + C. \qquad \blacksquare$$

Example 5. Find the integral $\int \tan^6 x \sec^4 x\, dx$.

Solution. we have

$$\int \tan^6 x \sec^4 x\, dx = \int \tan^6 x\,(1 + \tan^2 x)\sec^2 x\, dx = \int u^6\,(1 + u^2)\, du \quad \text{with } u = \tan x$$

$$= \frac{1}{7}u^7 + \frac{1}{9}u^9 + C = \frac{1}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C. \qquad \blacksquare$$

In case (c), we need consider integrals of odd powers of $\sec x$. Here we only consider $\int \sec x$. Recall that

$$(\sec x)' = \sec x \tan x \quad \text{and} \quad (\tan x)' = \sec^2 x.$$

Adding these two equalities, we have $(\sec x + \tan x)' = \sec x\,(\sec x + \tan x)$; that is,

$$\sec x = \frac{(\sec x + \tan x)'}{\sec x + \tan x}.$$

Example 6. We have $\int \sec x\, dx = \int \frac{(\sec x + \tan x)'}{\sec x + \tan x}\, dx = \ln|\sec x + \tan x| + C.$

III. Integral of the type $\int \sin mx\, \cos nx\, dx$, $\int \sin mx\, \sin nx\, dx$, $\int \cos mx\, \cos nx\, dx$.

The basic identities used in this part include

$$\sin A \cos B = \frac{1}{2}\Big(\sin(A-B) + \sin(A+B)\Big).$$

$$\sin A \sin B = \frac{1}{2}\Big(\cos(A-B) - \cos(A+B)\Big).$$

$$\cos A \cos B = \frac{1}{2}\Big(\cos(A-B) + \cos(A+B)\Big).$$

Example 7. Find the integrals $\int \sin 4x\, \cos 5x\, dx$ and $\int \cos 9x\, \cos 7x\, dx$.

Solution. we have

$$\int \sin 4x\, \cos 5x\, dx = \frac{1}{2}\int \big(\sin(-x) + \sin 9x\big)\, dx = \frac{1}{2}\Big(\cos x - \frac{1}{9}\cos 9x\Big) + C$$

and

$$\int \cos 9x\, \cos 7x\, dx = \frac{1}{2}\int \big(\cos 2x + \cos 16x\big)\, dx = \frac{1}{2}\Big(\frac{1}{2}\sin 2x + \frac{1}{16}\sin 16x\Big) + C. \quad \blacksquare$$

For other type of trigonometric integrals, we can try to change them to one of the three types discussed above.

Example 8. We have

$$\int \sin^2 x\, \tan x\, dx = \int \sin x\, \frac{1-\cos^2 x}{\cos x}\, dx = \int \Big(\frac{\sin x}{\cos x} - \sin x\, \cos x\Big)\, dx$$

$$= \ln|\sec x| - \frac{1}{2}\sin^2 x + C. \qquad \blacksquare$$

Section 7.3: Trigonometric Substitution

In this section, we consider integral with integrand containing $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 - a^2}$.

If the integrand also contains term x, x^3, x^5, etc., then we can try the Substitution Rule. E.g., by GPR, we have

$$\int x\sqrt{1-x^2}\,dx \;=\; -\frac{1}{2}\int (1-x^2)^{\frac{1}{2}}\,(-2x)\,dx \;=\; -\frac{1}{2}\,\frac{2}{3}(1-x^2)^{\frac{3}{2}} + C \;=\; -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C.$$

<u>**Question**</u>. What should we do if the integrand contains x^2, x^4, x^6, etc.?

<u>**Answer**</u>. In this case, we can try the Trigonometric Substitution to get rid of $\sqrt{\quad}$.

Instead of letting $u = g(x)$ as in the Substitution Rule $\int f(g(x))\,g'(x)\,dx \;=\; \int f(u)\,du$, now we let $x = g(\theta)$ for some trigonometric function g and get

$$\boxed{\int f(x)\,dx \;=\; \int f(g(\theta))\,g'(\theta)\,d\theta}$$

This kind of substitution is also called <u>inverse substitution</u>.

In the following, a denotes a positive number.

<u>**I. Trigonometric Substitution for $\sqrt{a^2 - x^2}$.**</u>

We let $x \;=\; a\sin\theta$ $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$. Then $x^2 \;=\; a^2\sin^2\theta$ and $\sqrt{a^2 - x^2} \;=\; a\cos\theta$.

<u>**II. Trigonometric Substitution for $\sqrt{a^2 + x^2}$.**</u>

We let $x \;=\; a\tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Then $x^2 \;=\; a^2\tan^2\theta$ and $\sqrt{a^2 + x^2} \;=\; a\sec\theta$.

<u>**III. Trigonometric Substitution for $\sqrt{x^2 - a^2}$.**</u>

We let $x \;=\; a\sec\theta$ $(0 \le \theta < \frac{\pi}{2})$. Then $x^2 \;=\; a^2\sec^2\theta$ and $\sqrt{x^2 - a^2} \;=\; a\tan\theta$.

Example 9. Find $\displaystyle\int \frac{\sqrt{9-x^2}}{x^2}\,dx$.

Solution. Let $x = 3\sin\theta$ $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$. Then

$$\sqrt{9-x^2} = 3\cos\theta, \quad x^2 = 9\sin^2\theta, \quad dx = 3\cos\theta\,d\theta.$$

Hence, we have

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = \int \frac{3\cos\theta}{9\sin^2\theta}\,3\cos\theta\,d\theta = \int(\csc^2\theta - 1)\,d\theta = -\cot\theta - \theta + C.$$

Now $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sqrt{9-x^2}}{x}$. Also,

$$x = 3\sin\theta \implies \theta = \sin^{-1}\left(\frac{x}{3}\right).$$

Therefore,

$$\int \frac{\sqrt{9-x^2}}{x^2}\,dx = -\cot\theta - \theta + C = -\frac{\sqrt{9-x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C. \quad \blacksquare$$

Example 10. Find $\displaystyle\int_0^2 \sqrt{4-x^2}\,dx$.

Solution. Let $x = 2\sin\theta$ $(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$. Then

$$\sqrt{4-x^2} = 2\cos\theta \quad \text{and} \quad dx = 2\cos\theta\,d\theta.$$

Also,

$$x = 0 \implies \theta = 0; \qquad x = 2 \implies \theta = \frac{\pi}{2}.$$

Therefore, we have

$$\int_0^2 \sqrt{4-x^2}\,dx = 4\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 2\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta = 2\Big[\theta + \frac{1}{2}\sin 2\theta\Big]_0^{\frac{\pi}{2}} = \pi. \quad \blacksquare$$

Example 11. Find $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

Solution. Let $x = 2\tan\theta$ $(-\frac{\pi}{2} < \theta < \frac{\pi}{2})$. Then

$$\sqrt{x^2+4} = 2\sec\theta, \quad x^2 = 4\tan^2\theta, \quad dx = 2\sec^2\theta\,d\theta.$$

Hence, we have

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = \int \frac{1}{4\tan^2\theta\,2\sec\theta}\,2\sec^2\theta\,d\theta = \frac{1}{4}\int (\sin\theta)^{-2}\cos\theta\,d\theta = -\frac{1}{4}\big(\sin\theta)^{-1} + C.$$

Since $\sin\theta = \dfrac{x}{\sqrt{x^2+4}}$, we have

$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx = -\frac{1}{4}\big(\sin\theta)^{-1} + C = -\frac{\sqrt{x^2+4}}{4x} + C. \qquad \blacksquare$$

Example 12. Find $\displaystyle\int \frac{1}{\sqrt{x^2-4}}\,dx$.

Solution. Let $x = 2\sec\theta$ $(0 \le \theta < \frac{\pi}{2})$. Then

$$\sqrt{x^2-4} = 2\tan\theta, \quad dx = 2\sec\theta\,\tan\theta\,d\theta.$$

Hence, we have

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \int \frac{1}{2\tan\theta}\,2\sec\theta\,\tan\theta\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C.$$

Now $\sec\theta = \dfrac{x}{2}$ and $\tan\theta = \dfrac{\sqrt{x^2-4}}{2}$. Therefore, we have

$$\int \frac{1}{\sqrt{x^2-4}}\,dx = \ln|\sec\theta + \tan\theta| + C = \ln\left|x + \sqrt{x^2-4}\right| + C. \qquad \blacksquare$$

Example 13. Find $\displaystyle\int \frac{1}{\sqrt{4x^2+9}}\,dx$.

Solution. We write

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx \;=\; \frac{1}{2}\int \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx.$$

Now let $x = \dfrac{3}{2}\tan\theta \; \left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$. Then

$$\sqrt{x^2+\left(\frac{3}{2}\right)^2} \;=\; \frac{3}{2}\sec\theta, \qquad dx \;=\; \frac{3}{2}\sec^2\theta\,d\theta.$$

Hence, we have

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx \;=\; \frac{1}{2}\int \frac{1}{\sqrt{x^2+\left(\frac{3}{2}\right)^2}}\,dx \;=\; \frac{1}{2}\int \frac{1}{\frac{3}{2}\sec\theta}\,\frac{3}{2}\,\sec^2\theta\,d\theta$$

$$=\; \frac{1}{2}\int \sec\theta\,d\theta \;=\; \frac{1}{2}\ln|\sec\theta+\tan\theta| \;+\; C.$$

Here, $\tan\theta \;=\; \dfrac{2x}{3}$ and $\sec\theta \;=\; \sqrt{1+\tan^2\theta} \;=\; \dfrac{1}{3}\sqrt{4x^2+9}$. Therefore, we have

$$\int \frac{1}{\sqrt{4x^2+9}}\,dx \;=\; \frac{1}{2}\ln\left|\frac{1}{3}\sqrt{4x^2+9} \;+\; \frac{2x}{3}\right| \;+\; C \;=\; \frac{1}{2}\ln\left|\sqrt{4x^2+9} \;+\; 2x\right| \;+\; C. \qquad \blacksquare$$