Non Linear Latent Variable Models

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GPRS
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Outline

Nonlinear Latent Variable Models

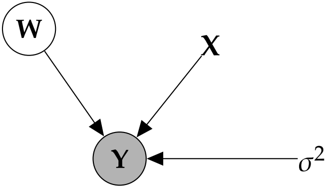
Extensions

Outline

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Define *linear-Gaussian relationship* between latent variables and data.
- **Novel** Latent variable approach:
    - Define Gaussian prior over *parameteters*, $\mathbf{W}$.
    - Integrate out *parameters*.



$$p\left(\mathbf{Y}|\mathbf{X},\mathbf{W}\right) = \prod_{i=1}^{n} \mathcal{N}\left(\mathbf{y}_{i,:}|\mathbf{W}\mathbf{x}_{i,:},\sigma^2\mathbf{I}\right)$$

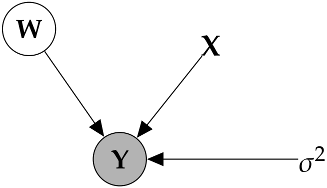
$$p\left(\mathbf{W}\right) = \prod_{i=1}^{p} \mathcal{N}\left(\mathbf{w}_{i,:}|\mathbf{0},\mathbf{I}\right)$$

$$p\left(\mathbf{Y}|\mathbf{X}\right) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0},\mathbf{X}\mathbf{X}^{\top}+\sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...

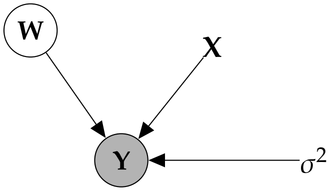


$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{X}\mathbf{X}^{\top} + \sigma^2\mathbf{I}\right)$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = \mathbf{X}\mathbf{X}^\top + \sigma^2\mathbf{I}$$

Non-Linear Latent Variable Model

Dual Probabilistic PCA

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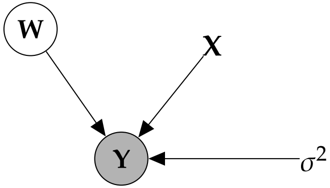
  - The covariance matrix is a covariance function.
  - We recognise it as the 'linear kernel'.



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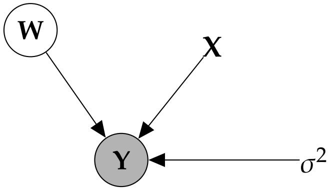
This is a product of Gaussian processes with linear kernels.

Non-Linear Latent Variable Model

Dual Probabilistic PCA

- Inspection of the marginal likelihood shows ...
  - The covariance matrix is a covariance function.
  - We recognise it as the 'linear kernel'.
  - We call this the Gaussian Process Latent Variable model (GP-LVM).



$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

$$\mathbf{K} = ?$$

Replace linear kernel with non-linear kernel for non-linear model.

Non-linear Latent Variable Models

Exponentiated Quadratic (EQ) Covariance

- The EQ covariance has the form $k_{i,j} = k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right)$, where

$$k\left(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}\right) = \alpha \exp\left(-\frac{\left\|\mathbf{x}_{i,:} - \mathbf{x}_{j,:}\right\|_2^2}{2\ell^2}\right).$$

- No longer possible to optimise wrt $\mathbf{X}$ via an eigenvalue problem.
- Instead find gradients with respect to $\mathbf{X}, \alpha, \ell$ and σ^2 and optimise using conjugate gradients.

Outline

Nonlinear Latent Variable Models

Extensions

Other Topics

- Local distance preservation ▸ Details
- Dynamical models ▸ Details
- Hierarchical models ▸ Details
- Bayesian GP-LVM ▸ Details

Back Constraints I

Local Distance Preservation **(Lawrence and Quiñonero Candela, 2006)**

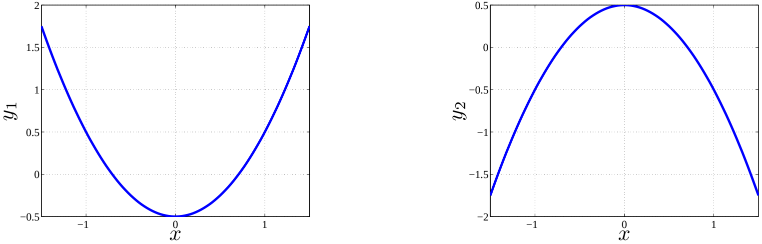
- Most dimensional reduction techniques preserve local distances.
- The GP-LVM does not.
- GP-LVM maps smoothly from latent to data space.
  - Points close in latent space are close in data space.
  - This does not imply points close in data space are close in latent space.
- Kernel PCA maps smoothly from data to latent space.
  - Points close in data space are close in latent space.
  - This does not imply points close in latent space are close in data space.

Back Constraints II

Forward Mapping (`demBackMapping` in oxford toolbox)

- Mapping from 1-D latent space to 2-D data space.

$$y_1 = x^2 - 0.5, \quad y_2 = -x^2 + 0.5$$



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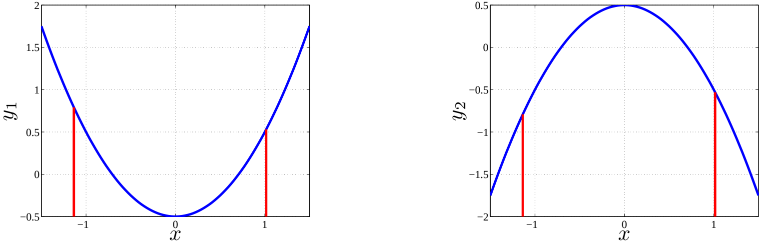

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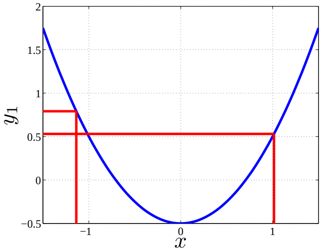
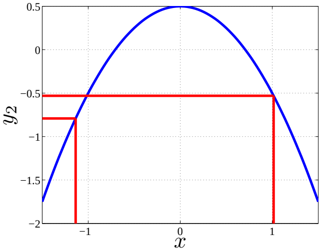


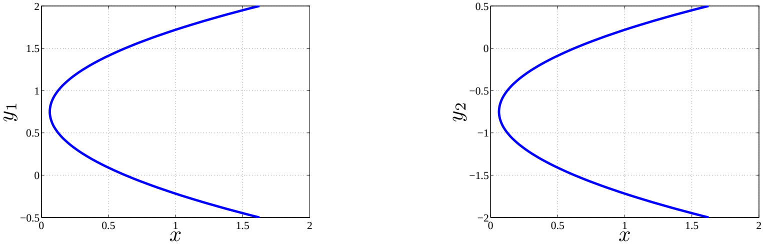


Back Constraints II

Backward Mapping (`demBackMapping` in oxford toolbox)

- Mapping from 2-D data space to 1-D latent.

$$x = 0.5\left(y_1^2 + y_2^2 + 1\right)$$



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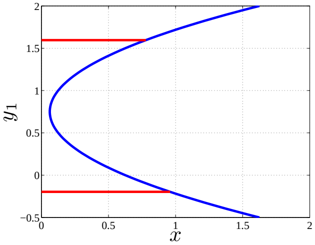

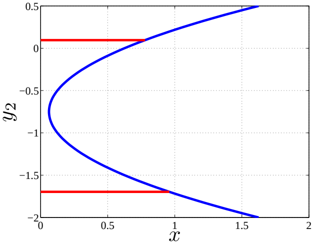

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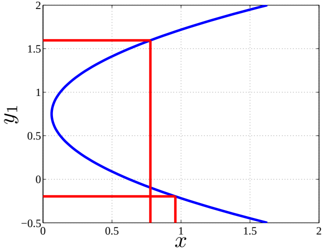


Multi-Dimensional Scaling with a Mapping

- Lowe and Tipping (1997) made latent positions a function of the data.

$$x_{i,j} = f_j\left(\mathbf{y}_{i,:}; \mathbf{v}\right)$$

- Function was either multi-layer perceptron or a radial basis function network.
- Their motivation was different from ours:
  - They wanted to add the advantages of a true mapping to multi-dimensional scaling.

Back Constraints in the GP-LVM

Back Constraints

- We can use the same idea to force the GP-LVM to respect local distances.(Lawrence and Quiñonero Candela, 2006)
  - By constraining each $\mathbf{x}_i$ to be a 'smooth' mapping from $\mathbf{y}_i$ local distances can be respected.
- This works because in the GP-LVM we maximise wrt latent variables, we don't integrate out.
- Can use any 'smooth' function:
  1. Neural network.
  2. RBF Network.
  3. Kernel based mapping.

Optimising BC-GPLVM

Computing Gradients

- GP-LVM normally proceeds by optimising

$$L(\mathbf{X}) = \log p(\mathbf{Y}|\mathbf{X})$$

  with respect to $\mathbf{X}$ using $\frac{\mathrm{d}L}{\mathrm{d}\mathbf{X}}$.
- The back constraints are of the form

$$x_{i,j} = f_j(\mathbf{y}_{i,:}; \mathbf{v})$$

  where $\mathbf{v}$ are parameters.
- We can compute $\frac{\mathrm{d}L}{\mathrm{d}\mathbf{v}}$ via chain rule and optimise parameters of mapping.

Motion Capture Results

`demStick1` **and** `demStick3`

Figure : The latent space for the motion capture data with (*right*) and without (*left*) back constraints.

Motion Capture Results

`demStick1` and `demStick3`

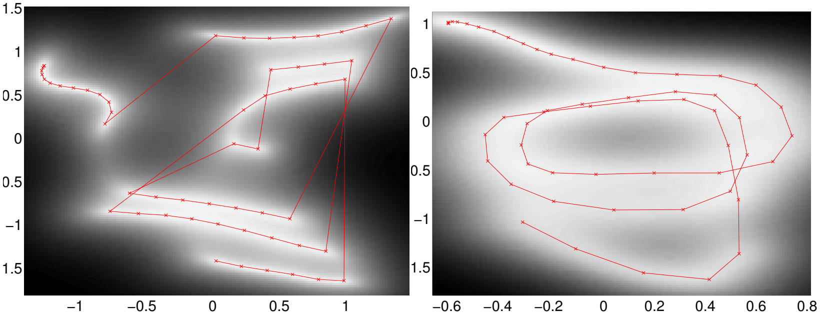
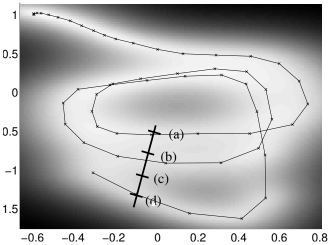


Figure : The latent space for the motion capture data with (*right*) and without (*left*) back constraints.

Stick Man Results

`demStickResults`



(a) (b) (c) (d)

Projection into data space from four points in the latent space. The inclination of the runner changes becoming more upright.

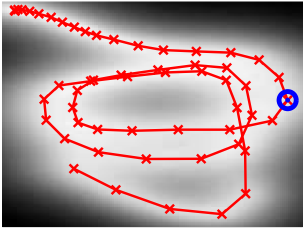
Adding Dynamics

MAP Solutions for Dynamics Models

- Data often has a temporal ordering.
- Markov-based dynamics are often used.
- For the GP-LVM
  - Marginalising such dynamics is intractable.
  - But: MAP solutions are trivial to implement.
- Many choices: Kalman filter, Markov chains *etc*..
- Wang et al. (2006) suggest using a Gaussian Process.

GP-LVM with Dynamics

- Autoregressive Gaussian process mapping in latent space between time points.



t

Gaussian Process Dynamics

GP-LVM with Dynamics

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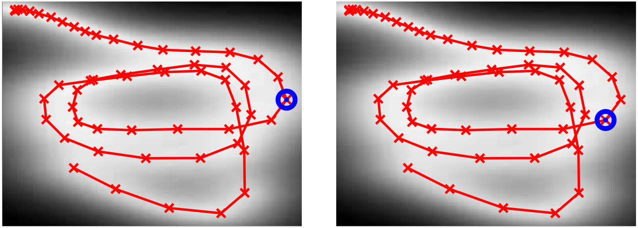

t $\qquad$ $t+1$

Gaussian Process Dynamics

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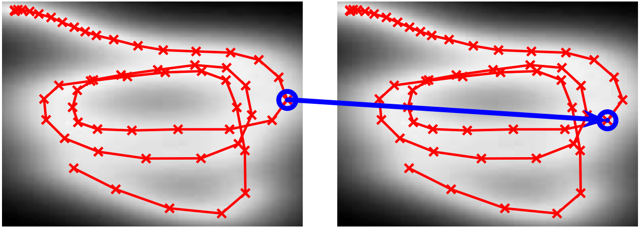

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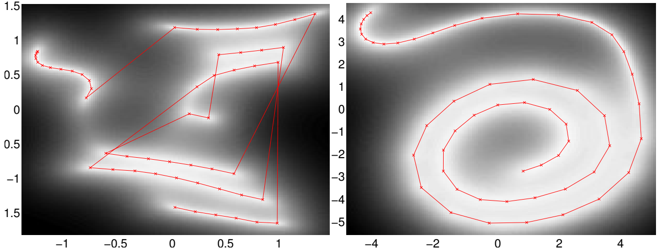
Motion Capture Results

`demStick1` **and** `demStick2`

Figure : The latent space for the motion capture data without dynamics (*left*), with auto-regressive dynamics (*right*) based on an exponentiated quadratic kernel.

Motion Capture Results

`demStick1` and `demStick2`



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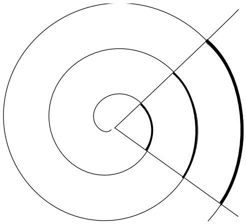
Regressive Dynamics

Inner Groove Distortion

- Autoregressive unimodal dynamics, $p(\mathbf{x}_t|\mathbf{x}_{t-1})$.
- Forces spiral visualisation.
- Poorer model due to inner groove distortion.



Regressive Dynamics

Direct use of Time Variable

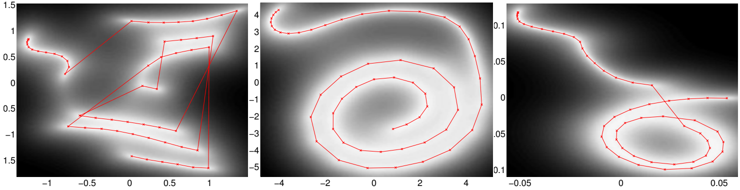
- Instead of auto-regressive dynamics, consider regressive dynamics.
- Take $\mathbf{t}$ as an input, use a prior $p(\mathbf{X}|\mathbf{t})$.
- User a Gaussian process prior for $p(\mathbf{X}|\mathbf{t})$.
- Also allows us to consider variable sample rate data.

`demStick1`, `demStick2` **and** `demStick5`

Figure : The latent space for the motion capture data without dynamics (*left*), with auto-regressive dynamics (*middle*) and with regressive dynamics (*right*) based on an exponentiated quadratic kernel.

Motion Capture Results

`demStick1`, `demStick2` **and** `demStick5`



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Return

Hierarchical GP-LVM

(Lawrence and Moore, 2007)

Stacking Gaussian Processes

- Regressive dynamics provides a simple hierarchy.
  - The input space of the GP is governed by another GP.
- By stacking GPs we can consider more complex hierarchies.
- Ideally we should marginalise latent spaces
  - In practice we seek MAP solutions.

Two Correlated Subjects

(Lawrence and Moore, 2007)

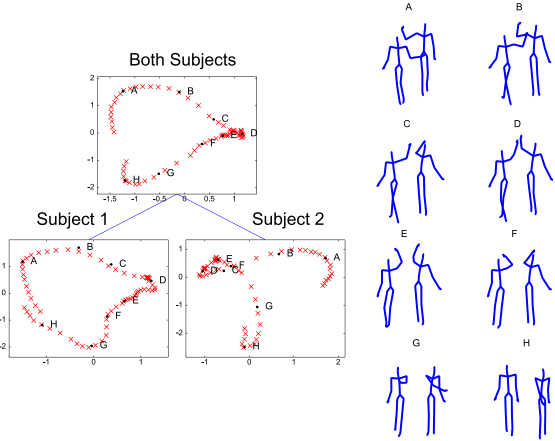


Figure : Hierarchical model of a 'high five'.

Within Subject Hierarchy

(Lawrence and Moore, 2007)

Decomposition of Body

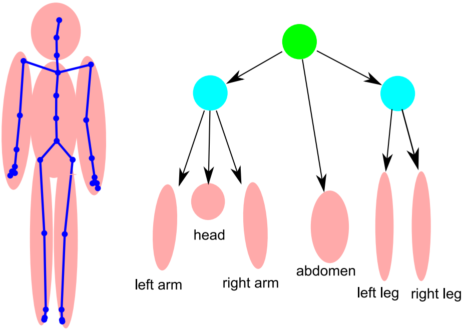


Figure : Decomposition of a subject.

Single Subject Run/Walk

(Lawrence and Moore, 2007)

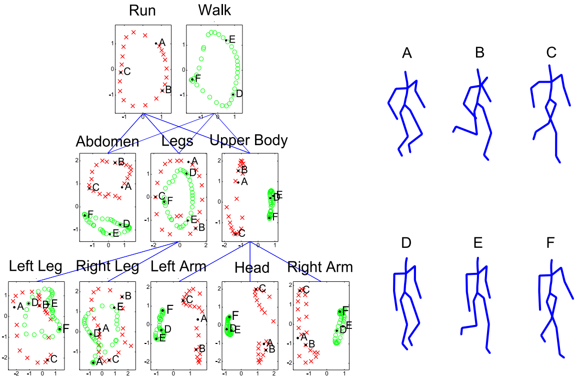


Figure : Hierarchical model of a walk and a run.

Return

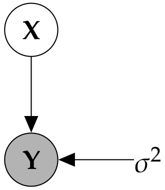
Selecting Data Dimensionality

- GP-LVM Provides probabilistic non-linear dimensionality reduction.
- How to select the dimensionality?
- Need to estimate marginal likelihood.
- In standard GP-LVM it increases with increasing q.

Integrate Mapping Function and Latent Variables

Bayesian GP-LVM

- Start with a standard GP-LVM.



$$p\left(\mathbf{Y}|\mathbf{X}\right) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

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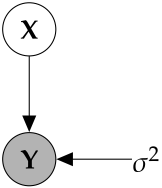
- Apply standard latent variable approach:
  - Define Gaussian prior over *latent space*, $\mathbf{X}$.



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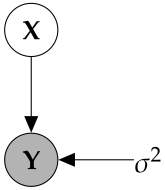

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{j=1}^{p} \mathcal{N}\left(\mathbf{y}_{:,j}|\mathbf{0}, \mathbf{K}\right)$$

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Integrate Mapping Function and Latent Variables

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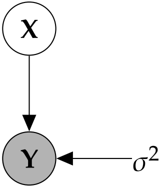
  - Integrate out *latent variables*.
  - Unfortunately integration is intractable.



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$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \alpha_i^{-2}\mathbf{I}\right)$$

$$p(\mathbf{Y}|\boldsymbol{\alpha}) = ??$$

Standard Variational Approach Fails

- Standard variational bound has the form:

$$\mathcal{L} = \left\langle \log p(\mathbf{y}|\mathbf{X}) \right\rangle_{q(\mathbf{X})} + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)$$

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- Requires expectation of $\log p(\mathbf{y}|\mathbf{X})$ under $q(\mathbf{X})$.

$$\log p(\mathbf{y}|\mathbf{X}) = -\frac{1}{2}\mathbf{y}^\top \left(\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right)^{-1} \mathbf{y} - \frac{1}{2}\log\left|\mathbf{K}_{\mathbf{f},\mathbf{f}} + \sigma^2\mathbf{I}\right| - \frac{n}{2}\log 2\pi$$

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- Extremely difficult to compute because $\mathbf{K}_{\mathbf{f},\mathbf{f}}$ is dependent on $\mathbf{X}$ and appears in the inverse.

Variational Bayesian GP-LVM

- Consider collapsed variational bound,

$$p(\mathbf{y}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y}| \left\langle \mathbf{f} \right\rangle, \sigma^2 \mathbf{I}\right) p(\mathbf{u}) \mathrm{d}\mathbf{u}$$

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$$p(\mathbf{y}|\mathbf{X}) \geq \prod_{i=1}^{n} c_i \int \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{u}) \mathrm{d}\mathbf{u}$$

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- Apply variational lower bound to the inner integral.

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- Apply variational lower bound to the inner integral.

$$\begin{aligned}
\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X} \\
\geq & \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})} \\
& + \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f} \right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) \right\rangle_{q(\mathbf{X})} \\
& + \mathrm{KL}\left(q(\mathbf{X}) \,\|\, p(\mathbf{X})\right)
\end{aligned}$$

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- Apply variational lower bound to the inner integral.

$$\int \prod_{i=1}^{n} c_i \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right) p(\mathbf{X})\mathrm{d}\mathbf{X}$$

$$\begin{aligned} &\geq \left\langle \sum_{i=1}^{n} \log c_i \right\rangle_{q(\mathbf{X})} \\ &+ \left\langle \log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{X})}, \sigma^2\mathbf{I}\right)\right\rangle_{q(\mathbf{X})} \\ &+ \mathrm{KL}\left(q(\mathbf{X})\,\|\,p(\mathbf{X})\right) \end{aligned}$$

- Which is analytically tractable for Gaussian $q(\mathbf{X})$ and some covariance functions.

Required Expectations

- Need expectations under $q(\mathbf{X})$ of:

$$\log c_i = \frac{1}{2\sigma^2}\left[k_{i,i} - \mathbf{k}_{i,\mathbf{u}}^\top \mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1} \mathbf{k}_{i,\mathbf{u}}\right]$$

and

$$\log \mathcal{N}\left(\mathbf{y}|\left\langle \mathbf{f}\right\rangle_{p(\mathbf{f}|\mathbf{u},\mathbf{Y})}, \sigma^2\mathbf{I}\right) = -\frac{1}{2}\log 2\pi\sigma^2 - \frac{1}{2\sigma^2}\left(y_i - \mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{u}\right)^2$$

- This requires the expectations

$$\left\langle \mathbf{K}_{\mathbf{f},\mathbf{u}} \right\rangle_{q(\mathbf{X})}$$

and

$$\left\langle \mathbf{K}_{\mathbf{f},\mathbf{u}}\mathbf{K}_{\mathbf{u},\mathbf{u}}^{-1}\mathbf{K}_{\mathbf{u},\mathbf{f}} \right\rangle_{q(\mathbf{X})}$$

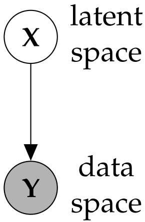
which can be computed analytically for some covariance functions.

Priors for Latent Space

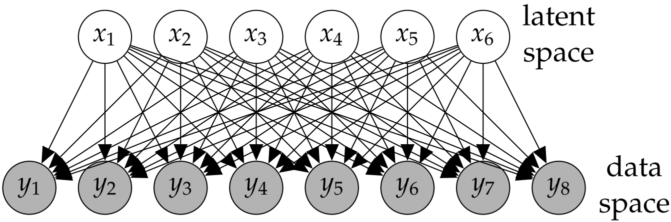
Titsias and Lawrence (2010)

- Variational marginalization of $\mathbf{X}$ allows us to learn parameters of $p(\mathbf{X})$.
- Standard GP-LVM where $\mathbf{X}$ learnt by MAP, this is not possible (see e.g. Wang et al., 2008).
- First example: learn the dimensionality of latent space.

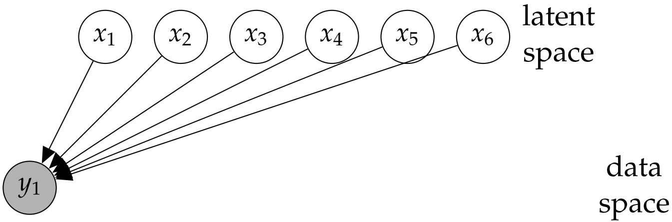
Graphical Representations of GP-LVM



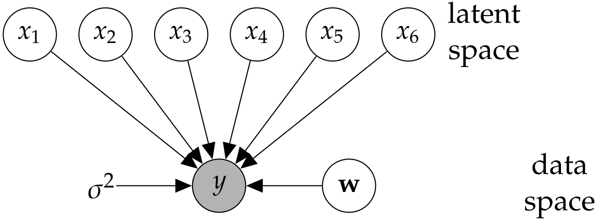
Graphical Representations of GP-LVM



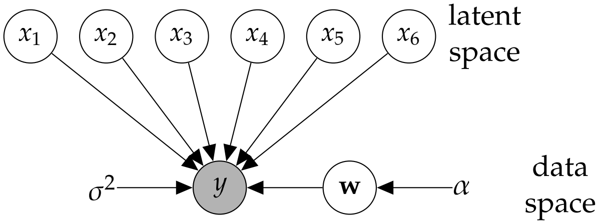
Graphical Representations of GP-LVM



Graphical Representations of GP-LVM



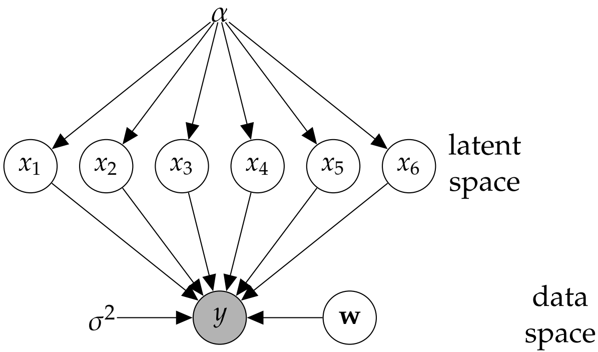
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \alpha\mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad \mathbf{x} \sim \mathcal{N}(0, \alpha\mathbf{I})$$

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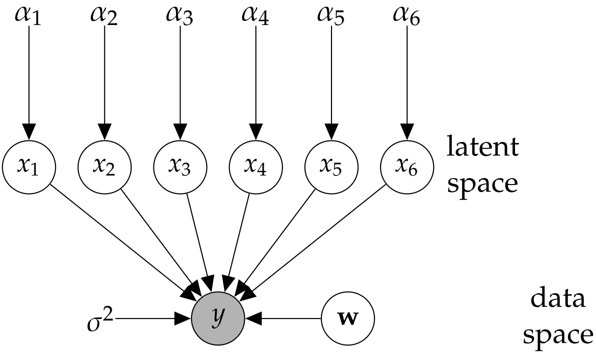
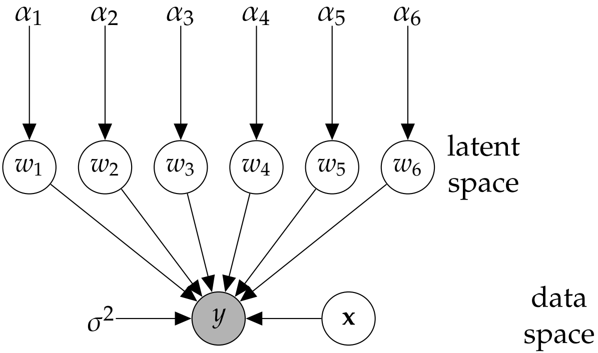
Graphical Representations of GP-LVM



$$\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \quad x_i \sim \mathcal{N}(0, \alpha_i)$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Graphical Representations of GP-LVM



$$w_i \sim \mathcal{N}(0, \alpha_i) \quad \mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$$

$$y \sim \mathcal{N}\left(\mathbf{x}^\top \mathbf{w}, \sigma^2\right)$$

Non-linear $f(\mathbf{x})$

- In linear case equivalence because $f(\mathbf{x}) = \mathbf{w}^\top \mathbf{x}$

$$p(w_i) \sim \mathcal{N}(\mathbf{0}, \alpha_i)$$

- In non linear case, need to scale columns of $\mathbf{X}$ in prior for $f(\mathbf{x})$.
- This implies scaling columns of $\mathbf{X}$ in covariance function

$$k(\mathbf{x}_{i,:}, \mathbf{x}_{j,:}) = \exp\left(-\frac{1}{2}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})^\top \mathbf{A}(\mathbf{x}_{:,i} - \mathbf{x}_{:,j})\right)$$

$\mathbf{A}$ is diagonal with elements α_i^2. Now keep prior spherical

$$p(\mathbf{X}) = \prod_{j=1}^{q} \mathcal{N}\left(\mathbf{x}_{:,j}|\mathbf{0}, \mathbf{I}\right)$$

- Covariance functions of this type are known as ARD (see e.g. Neal, 1996; MacKay, 2003; Rasmussen and Williams, 2006).

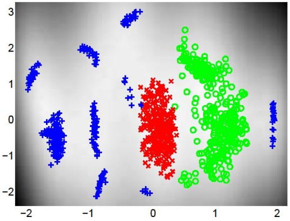
Automatic dimensionality detection

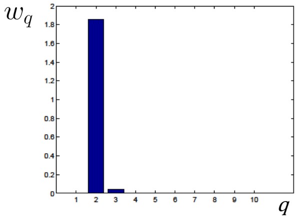
- Achieved by employing an *Automatic Relevance Determination (ARD)* covariance function for the prior on the GP mapping

- $f \sim GP(\mathbf{0}, k_f)$ with

$$k_f\left(\mathbf{x}_i, \mathbf{x}_j\right) = \sigma^2 \exp\left(-\frac{1}{2}\sum_{q=1}^{Q} w_q \left(x_{i,q} - x_{j,q}\right)^2\right)$$

- Example





Applications

Style Based Inverse Kinematics

- Facilitating animation through modeling human motion (Grochow et al., 2004)

Tracking

- Tracking using human motion models (Urtasun et al., 2005, 2006)

Assisted Animation

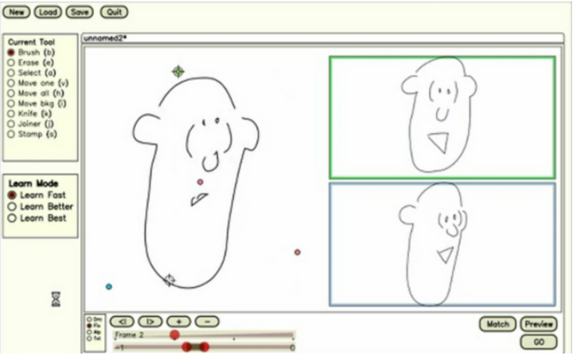
- Generalizing drawings for animation (Baxter and Anjyo, 2006)

Shape Models

- Inferring shape (e.g. pose from silhouette). (Ek et al., 2008b,a; Priacuriu and Reid, 2011a,b)

Example: Latent Doodle Space

(Baxter and Anjyo, 2006)



http://vimeo.com/3235882

Example: Latent Doodle Space

(Baxter and Anjyo, 2006)

Generalization with much less Data than Dimensions

- Powerful uncertainly handling of GPs leads to surprising properties.
- Non-linear models can be used where there are fewer data points than dimensions *without overfitting*.

References I

W. V. Baxter and K.-I. Anjyo. Latent doodle space. In *EUROGRAPHICS*, volume 25, pages 477–485, Vienna, Austria, September 4-8 2006.

C. H. Ek, J. Rihan, P. Torr, G. Rogez, and N. D. Lawrence. Ambiguity modeling in latent spaces. In A. Popescu-Belis and R. Stiefelhagen, editors, *Machine Learning for Multimodal Interaction (MLMI 2008)*, LNCS, pages 62–73. Springer-Verlag, 28–30 June 2008a. [PDF].

C. H. Ek, P. H. Torr, and N. D. Lawrence. Gaussian process latent variable models for human pose estimation. In A. Popescu-Belis, S. Renals, and H. Bourlard, editors, *Machine Learning for Multimodal Interaction (MLMI 2007)*, volume 4892 of *LNCS*, pages 132–143, Brno, Czech Republic, 2008b. Springer-Verlag. [PDF].

K. Grochow, S. L. Martin, A. Hertzmann, and Z. Popovic. Style-based inverse kinematics. In *ACM Transactions on Graphics (SIGGRAPH 2004)*, pages 522–531, 2004.

N. D. Lawrence and A. J. Moore. Hierarchical Gaussian process latent variable models. In Z. Ghahramani, editor, *Proceedings of the International Conference in Machine Learning*, volume 24, pages 481–488. Omnipress, 2007. [Google Books] . [PDF].

N. D. Lawrence and J. Quiñonero Candela. Local distance preservation in the GP-LVM through back constraints. In W. Cohen and A. Moore, editors, *Proceedings of the International Conference in Machine Learning*, volume 23, pages 513–520. Omnipress, 2006. [Google Books] . [PDF].

D. Lowe and M. E. Tipping. Neuroscale: Novel topographic feature extraction with radial basis function networks. In M. C. Mozer, M. I. Jordan, and T. Petsche, editors, *Advances in Neural Information Processing Systems*, volume 9, pages 543–549, Cambridge, MA, 1997. MIT Press.

D. J. C. MacKay. *Information Theory, Inference and Learning Algorithms*. Cambridge University Press, Cambridge, U.K., 2003. [Google Books] .

R. M. Neal. *Bayesian Learning for Neural Networks*. Springer, 1996. Lecture Notes in Statistics 118.

V. Priacuriu and I. D. Reid. Nonlinear shape manifolds as shape priors in level set segmentation and trackign. In *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, 2011a.

V. Priacuriu and I. D. Reid. Shared shape spaces. In *IEEE International Conference on Computer Vision (ICCV)*, 2011b.

C. E. Rasmussen and C. K. I. Williams. *Gaussian Processes for Machine Learning*. MIT Press, Cambridge, MA, 2006. [Google Books] .

References II

M. K. Titsias and N. D. Lawrence. Bayesian Gaussian process latent variable model. In Y. W. Teh and D. M. Titterington, editors, *Proceedings of the Thirteenth International Workshop on Artificial Intelligence and Statistics*, volume 9, pages 844–851, Chia Laguna Resort, Sardinia, Italy, 13-16 May 2010. JMLR W&CP 9. [PDF].

R. Urtasun, D. J. Fleet, and P. Fua. 3D people tracking with Gaussian process dynamical models. In *Proceedings of the IEEE Computer Society Conference on Computer Vision and Pattern Recognition*, pages 238–245, New York, U.S.A., 17–22 Jun. 2006. IEEE Computer Society Press.

R. Urtasun, D. J. Fleet, A. Hertzmann, and P. Fua. Priors for people tracking from small training sets. In *IEEE International Conference on Computer Vision (ICCV)*, pages 403–410, Bejing, China, 17–21 Oct. 2005. IEEE Computer Society Press.

J. M. Wang, D. J. Fleet, and A. Hertzmann. Gaussian process dynamical models. In Y. Weiss, B. Schölkopf, and J. C. Platt, editors, *Advances in Neural Information Processing Systems*, volume 18, Cambridge, MA, 2006. MIT Press.

J. M. Wang, D. J. Fleet, and A. Hertzmann. Gaussian process dynamical models for human motion. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 30(2):283–298, 2008. ISSN 0162-8828. [DOI].