Project: Interest Rate Risk Measurement

STA 2540: Insurance Risk Management

Songjie Lin (1009289422)
Zhiye Luan (1004282867)
Zhengyu Li (1003939660)

University of Toronto
Department of Statistical Sciences
Master of Financial Insurance
March 14, 2023

1 Executive Summary

The purpose of this report is to analyze the interest rate risk exposure in the setting of a series of fixed cash flows. In particular, figure 1. Our approach involved selecting relevant risk factors, gathering historical data, performing principle component analysis (PCA), and using Monte Carlo simulation to derive a P&L distribution. Our analysis revealed significant interest rate risk exposure, with potential losses of approximately \$16,703 and \$16,625 at the left CL99.5 and CTE99 risk levels respectively. Effective risk management strategies are crucial to mitigate these losses and protect the institution's portfolio. Overall, this report highlights the importance of implementing effective risk management strategies to manage interest rate risk, especially for financial instruments that are exposed to interest rate risk.

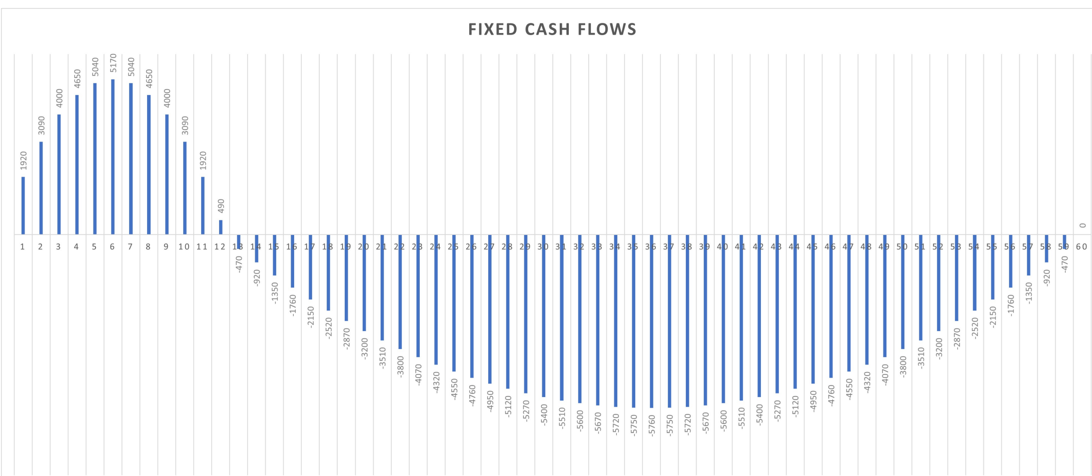


Figure 1: Fixed Cashflow

2 Introduction

Interest rate risk refers to the possibility of incurring losses or gains as a result of changes in interest rates. To manage their exposure to this risk, financial institutions use interest rate risk measurement techniques to quantify the potential risk. These techniques help institutions identify and analyze the potential impact of interest rate movements on their portfolios, enabling them to implement effective risk management strategies.

In this report, we approach the modeling interest rate movement with a principle component model. We first describe the selection of risk factors, modeling time step, number of principal components, and drift assumptions. We then provide an overview of the data sources and the results of our analysis, highlighting the potential losses at different confidence levels by performing PCA analysis, and using Monte Carlo simulation to derive the P&L distribution.

Finally, we discuss the implications of our findings and provide recommendations for mitigating interest rate risk exposure. Our analysis reveals that the financial institution is exposed to significant interest rate risk, with potential losses at the CL99.5 and CTE99 confidence levels, respectively.

3 Parameters & Stochastic Modeling

3.1 Model and Parameters

Risk factors can be constructed using principal component analysis (PCA) by identifying the principal components that explain the majority (but not all) of the variation in a given data set. These principal components can then be combined in a suitable way to develop new variables that reflect the underlying sources of risk in the data. This is done to reduce the model inputs and reduce model dimensionality especially when working with large amounts of historical data.

To generate principal components, we start by calculating a covariance matrix that describes the changes in the yield curve over time. Then, we perform either eigenvalue decomposition or singular value decomposition on this covariance matrix. This allows us to extract the principal components of the yield curve.

The following steps present how the covariance metric works,

1. Generate a set of yield curves (in terms of forward rates) from the structural model. Construct a data matrix of forward rates:

$$\boldsymbol{F} = [F_{ij}],$$

where F_{ij} is the j^{th} forward rate of the i^{th} yield curve.

2. Transform the values in the data matrix as desired.

3. Centre and scale the data matrix if desired:

   1. Subtract the mean of each of the forward rates to obtain data that has a zero mean.

   2. Subtract the mean and then divide by the standard deviation of the forward rates, to obtain data that has zero mean and standard deviation of 1.0.

4. Calculate a matrix of co-variances between the forward rates

$$\boldsymbol{C} = \mathrm{Cov}(\boldsymbol{F}),$$

   where this is interpreted as a matrix of co-variances of pairs of columns. Note that if the data has a mean of zero and a standard deviation of 1.0 this will also be the correlation matrix.

5. Compute the eigenvalue decomposition or singular value decomposition, with eigenvalues or singular values in decreasing order. The eigenvalue decomposition is:

$$\boldsymbol{C} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{\top}$$

   where $\boldsymbol{V}$ is an orthonormal matrix and $\boldsymbol{\Lambda}$ is a diagonal matrix. The singular value decomposition is:

$$\boldsymbol{C} = \boldsymbol{U}\boldsymbol{S}\boldsymbol{V}^{\top}$$

   where $\boldsymbol{S}$ is a diagonal matrix and $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthonormal matrices that will be identical since the covariance matrix is positive semi-definite.

6. The significance, ψ_i, of each principal component is given by:

$$\psi_i = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}, \quad \text{or} \quad \psi_i = \frac{s_i}{\sum_{j=1}^{n} s_j}.$$

In our model, the "drift" term refers to the mean of the components. The "drift" term is a concept in finance that is used to measure the impact of changes in interest rates on the value of assets or liabilities over time. It is a component of mathematical models that aim to predict interest rate behavior and its effects on financial instruments. The drift term is calculated by comparing the current interest rate to the long-term average interest rate and reflects the anticipated direction and magnitude of future interest rate changes and is important for financial modeling especially in our case as it affects the valuation of financial instruments such as bonds, options, and futures contracts. By taking the drift term into account, financial analysts can make better predictions about the future performance of these instruments and make more informed investment decisions. In our case, the drift of the interest rate directly affects our series of cash flows.

In finance, the logarithmic transformation is commonly used in interest rate risk models to transform interest rate data into a form that is easier to analyze and model. Therefore, after the "drift" term and principle components are obtained, our risk factor is the logarithmic transformation of the annual returns. Using logarithmic transformation is advantageous since gives the data symmetry, as positive and negative deviations from the mean have an equal influence on the variance. Additionally, it renders the data time-invariant, which means that the relationship between different interest rates is preserved, even after transformation. This is particularly useful in analyzing the relative changes in interest rates over time. Lastly, the logarithmic transformation makes the data additive, as the log of the product of two interest rates is equal to the sum of their logs. This allows for the use of linear regression models, which assume the additivity of the predictor variables.

Once every component has been verified, our risk factor is going to be described solely by components. These components are linear combinations of the original variables that explain the maximum amount of variance in the data thereby reducing the number of variables in our dataset, rendering it easier to analyze and visualize, while still retaining most of the important information.

It should be noted that if we are modeling the yield curves as 50 forward rates as an example, then the covariance matrix will be 50×50 and there will be 50 PC's. In particular, it can perfectly reproduce all of the yield curves on which the

PCA was performed. Such a model should be called a full statistical model, and can be represented as:

$$F = \mu + \sum_{i=1}^{n} \alpha_i v_i,$$

where n is the number of forward rates, equal to 50 in the present example. This can be expressed in matrix notation as:

$$F = \mu + \boldsymbol{V}\alpha,$$

where $\boldsymbol{V} = [v_1, v_2, \ldots, v_n]$ is a square $n \times n$ matrix who's columns are the PCs (i.e. the eigenvectors of the covariance matrix), and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_n]^\top$ is a vector of magnitudes of the PCs.

PCA simplifies the overly flexible full statistical model while maintaining its ability to reproduce most yield curves produced by the structural model. By reducing the number of PC's used to only the most important ones (as indicated by their eigenvalues), we only retain those PC's that are associated with the largest eigenvalues. The reduced model can be expressed as:

$$F = \mu + \sum_{i=1}^{k} \alpha_i v_i$$

where k is the number of PC's retained. This means that the yield curves are expressed as a linear combination of the k PC's. This can be expressed in matrix notation as:

$$F = \mu + \boldsymbol{V}^{(\boldsymbol{k})}\alpha$$

where $\boldsymbol{V}^{(k)} = [v_1, v_2, \ldots, v_k]$ is a rectangular $n \times k$ matrix whos columns are the k PC's corresponding to the largest k eigenvalues, and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_k]^\top$ is a vector of magnitudes of the PCs.

In a reduced model, the possible yield curves are also reduced. However, if the number of PC's is sufficient, most yield curves used in the analysis can be accurately reproduced. In yield curve modeling, typically two or three PC's are sufficient, but the appropriate number can be determined by analyzing the corresponding eigenvalues.

In our study, we investigate the effects of risk associated with yearly interest rate change. This means our risk factor is $\Delta \ln(y)$. We obtained the month-end yield rates from U.S. Department of the Treasury, Daily Treasury Par Yield Curve Rates) of maturities: 6 months, 1 year, 2 year, 3 year, 5 year, 7 year, 10 year, 20 year, and 30 years from December 2002 to December 2022. We then proceed to reduce this model into a three-component model with each component having a maturity corresponding to the periods mentioned.

3.2 Monte Carlo Sampling

After obtaining our components, we then proceed to simulate 10,000 new yield curves with our components in the following fashion.

$$\hat{y} = ye^{\mu\Delta t+\alpha\epsilon}, \quad \epsilon^T = [N(0,\sigma_1), N(0,\sigma_2), N(0,\sigma_3)]$$

Where:

- $\hat{y}$ is the estimated new yield curve.
- y is the yield curve from December 31, 2022.
- μ is the mean of the components.
- Δt is the length of each time step.
- α_i is the component size of each component.
- σ_i is the component variance.

In our case, we are interested in a 1-year risk horizon. Since our data is using monthly changes in interest rate, our Δt in this study is 12.

4 Pricing Formula, Risk Measure

After the model is determined, the present value can be calculated from our stream of cash flows.

4.1 Pricing Formulas

We calculate the present value of our cash flows as follows:

$$P = \sum_{i=1}^{n} \frac{CS_i}{(1+y_i)^i}$$

Where CS_i is the cashflow at year i, n is the final year of the cash flow, and y_i is the spot rate at year i.

Furthermore, the following formulas would be used to convert par semi-annual yields to spot rates (zero coupon yields) obtained from the OSFI LICAT bootstrapping method. Despite calculating the spot rates semi-annually, we only use the annual spot rates.

$$\mathrm{PV}_{\text{factor },t} = \begin{cases} \frac{1}{1+\frac{1}{2}\text{ Yield }_{\text{par semi,t }}}, & \text{if t} = \frac{1}{2} \\ \frac{1}{(1+\text{ Yield }_{\text{zero coupon },t})^t}, & \text{if t} > \frac{1}{2} \end{cases}$$

$$\mathrm{PV}_{\text{last payment },t} = 100\left(1 - \frac{\text{Yield }_{\text{par semi,t}}}{2}\sum_{n=1}^{t\times 2-1}\mathrm{PV}_{\text{factor },n/2}\right)$$

$$\text{Yield }_{\text{zero coupon },t} = \left[100 \times \left(1 + \frac{\text{Yield }_{\text{par semi,t}}}{2}\right)/\mathrm{PV}_{\text{last payment, },t}\right]^{\frac{1}{t}} - 1$$

Risk-free par yields that are not obtained directly from the data can be inferred using linear interpolation (i.e. for duration years of 4, 6, etc.). The resulting quantities of yield zero coupons, t for $t = 1, 2, \ldots, 20$ as determined above would constitute the risk-free spot rate curve.

4.2 Risk Measures

4.2.1 Confidence Level

CL, or Confidence Level, is a risk measure that calculates the tail end of the profit and loss distribution. It is a popular risk measure in financial risk management, especially in the assessment of credit risk. In this study, we aim to investigate the left tail of the 99.5 percent confidence level.

4.2.2 Conditional Tail Expectation

CTE, or Conditional Tail Expectation, is a risk measure that is used to estimate the expected value of a portfolio's losses beyond a certain threshold. It is also known as expected shortfall or expected tail loss.

CTE is calculated by averaging all of the portfolio's losses that exceed a specific threshold, typically the 95th percentile of the loss distribution. This means that CTE takes into account only the losses that are more severe than the threshold, which is also known as the "tail" of the loss distribution.

CTE is a popular risk measure in financial risk management, particularly for assessing potential losses in extreme market conditions. It is used in combination with other risk measures such as VaR, expected return, and expected volatility to help investors and risk managers make informed decisions about their portfolios. In this study we aim to investigate the 95 percent conditional tail end.

5 Simulation Results

5.1 Principal Components

Before the simulation, let us observe the variable reduction effect from the principle component analysis.

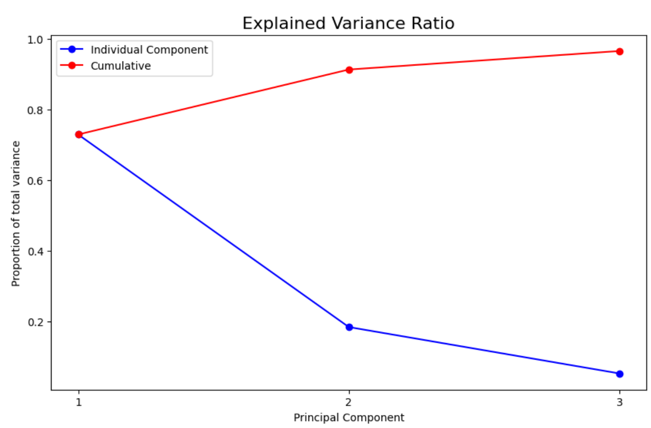
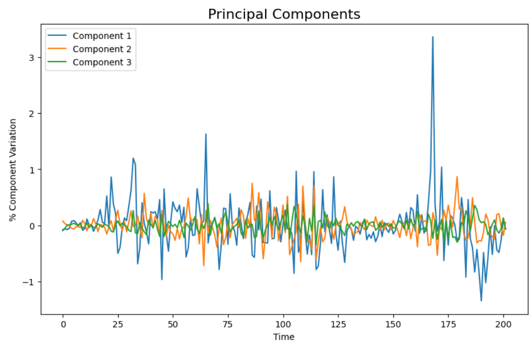
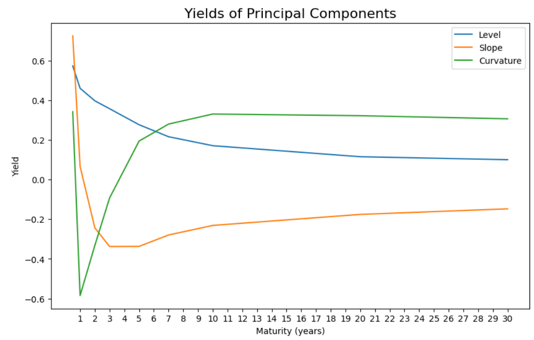


Figure 2: Explained Variance of Each Component

From the results above, we can see that each component contributes a proportion of the total variance exhibited by the data with the first component having an explained variance of 72.5%, the second component having an explained variance of 18.4%, and the final component having an explained variance of around 5.2%. In total, the components account for 96.5% of the total variance exhibited by all of the monthly interest rate movements across 20 years.



(a) Variance of components over time



(b) Yields of Principal Components

Figure 3: Effects of principle components

From above, we can see the explained variance of the principle components over time, As we can see, the three components seem to capture different aspects of the interest rate changes. The figure depicts the behavior of the components over time. Component 1 seems to capture large spikes in interest rate changes. Component 2 seems to capture the mild changes in interest rates. Component 3 seems to reflect the small daily noise in interest rates.

The figure shows the effects of the principal components in one yield curve. It seems like the components are reflecting characteristics of the interest rate curve. The level seems to decrease logarithmically as time goes on, and there seems to eventually converge to a certain level if time goes infinitely. The slope indicates the magnitude of change between the yield-to-maturity on a long-maturity bond and the yield-to-maturity on a shorter-maturity bond. In our case, yield-to-maturity seems to be higher on shorter maturity bonds. Finally, we have the curvature, the relationship between short, intermediate, and long-term yields-to-maturity depicted in a dip for short term bonds which then returns back to a normal level.

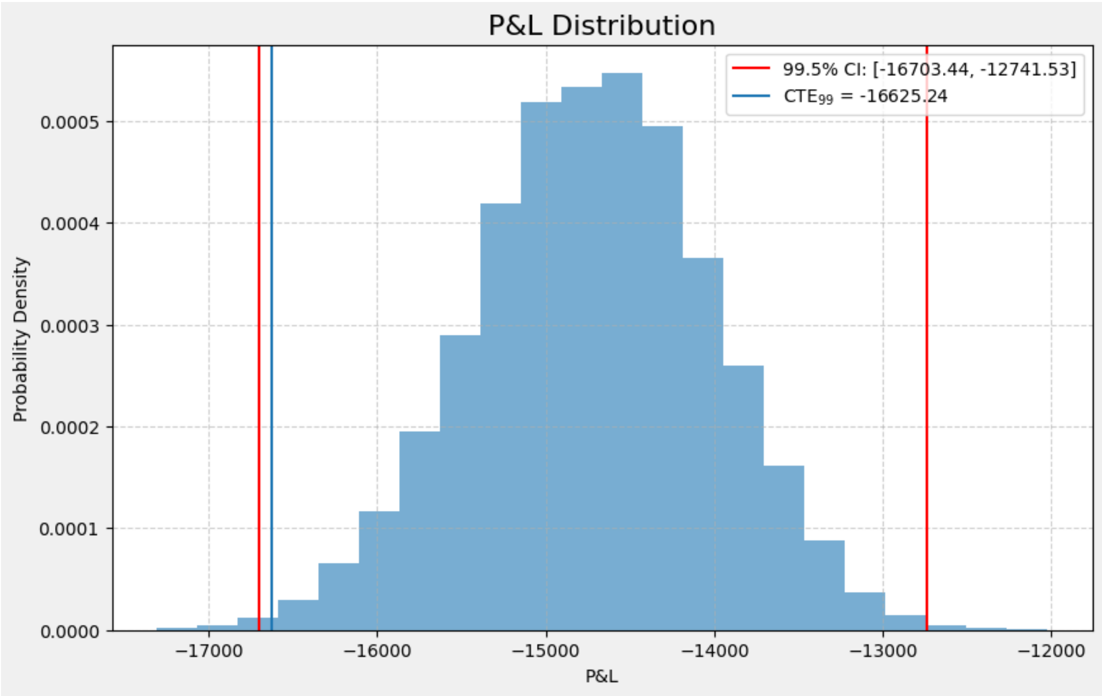


Figure 4: Yields of Principal Components

As we can see from Figure 4. Our conditional tail end at 99% is around -$16,625 and our 99.5% confidence levels are -$16,703 and -$12,741.

6 Conclusion

Based on our analysis, we selected relevant risk factors, gathered historical data, performed principle component analysis, and used Monte Carlo simulation to derive a P&L distribution. First, retaining only the first three PCs might be appropriate because they explain a high proportion of the variance in the data, and retaining more PCs would not provide much additional information. Secondly, our conditional tail end at 99% is around -$16,625 and our 99.5% confidence levels are -$16,703 and -$12,741.