Group Project 2 - Delta-Gamma Hedging

STA2503H: Applied Probability for Mathematical Finance

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January 13, 2023

Abstract

Delta and delta-gamma hedging schemes are popular techniques used to mitigate risk exposure from the sale of a product which value is of a probabilistic nature. In this scenario, we examine daily and move based delta and delta-gamma hedging techniques in order or cover our exposure of an at-the-money put option written on an underlying asset whose price is modeled with Black-Scholes. We consider in addition to our bank account and held assets; a long position in a call option of the same Black-Scholes-modeled asset for delta-gamma hedging techniques. As for the result of our study, we observe the comparisons in performance of hedging techniques in the form of profits and loss distributions for each hedging method and its accompanying conditional value at risk of 90 percent. We conclude our discussion by comparing the results of having different band sizes on performance.

Contents

1 Introduction

This paper aims to examine the moved-based and the time-based approaches under hedging strategies and take the transaction cost into account. The structure of the rest of the paper is as follows. First, we introduce the methodology of price simulation, basic model setup, the Black-Scholes formula for options, and the evolution of different hedging strategies. Next, we create time-based and move-based Delta hedging and Delta-gamma hedging for shorting put position to explore the distribution of the Profit and Loss (PnL) and transaction costs of each hedging strategy. We discuss the impact of varying rebalancing bandwidth on the Delta hedging strategy. Lastly, we conclude the findings in our analysis and make suggestions for possible future work on this subject.

2 Methodology

2.1 Assumptions and Model Setup

Assume that an asset $S = (S_t)_{t>0}$ follows the Black-Scholes model with

$$S_0 = 100, \sigma = 20\%, \mu = 10\%, r = 2\%$$

the initial stock price is S_0, the variable σ is the volatility of the underlying stock price, μ is the drift of the stock price, r is the constant risk-free rate. And the Bank Account $B = (B_t)_{t>0}$ satisfies:

$$\frac{dB_t}{B_t} = rdt \tag{1}$$

Since the risk-free rate r is constant, we have $B_{t+\delta t} = B_t e^{r\delta t}$.

We have just sold an at-the-money $\frac{1}{4}$ year put written option on this asset S_t with strike price $K = 100$ at time 0. This means we agree to purchase the underlying asset at a strike price of K from the buyer when the buyer exercises the put option. Shorting a put option exposes us to a risk of losing money when the price of the underlying asset is below K at maturity. Note that the maximum profit is the option price we received when shorting the option. However, the loss can be tremendous if the asset price drops dramatically at maturity and afterward.

One of the ways to reduce and hedge this risk of adverse price change of the underlying asset is using dynamic hedging. The purpose of dynamic hedging is to construct a replicating portfolio and update hedge positions regularly as market conditions change. To hedge the short put option, we consider investing in the underlying asset S_t, the bank account B_t, and a call option on the same asset with the same strike K but the maturity of $\frac{1}{2}$ year.

Notice that perfect hedging under the Black-Scholes framework is impossible because of the presence of transaction costs. We account for the transaction cost of trading in every one unit of Equity is 0.005$ and the transaction cost of trading in every unit of Options is 0.01$.

2.2 Call Option and Put Option Pricing Under the Black-Scholes Model

Since the underlying asset S follows the Black-Scholes Model, the put option and the call option written on S also follow the Black-Scholes Model. Under the Black-Scholes formula, the pricing process of call and put options on the asset S with value S_t at time t with strike price K is shown below:

$$f^{call}(S,T) = S_t N(d_+) - Ke^{-r(T-t)} N(d_-) \tag{2}$$

$$f^{put}(S,T) = Ke^{-r(T-t)} N(-d_-) - S_t N(-d_+) \tag{3}$$

where

$$d_+ = \frac{ln(S_0/K) + (r + (1/2)\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_- = \frac{ln(S_0/K) + (r - (1/2)\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

The Black-Scholes model can only be used for a European-style option and assumes no dividends are paid out. The price calculated from the Black-Scholes model assumes there are no transaction costs in buying the option and the volatility of the underlying stock remains constant. The prices of the underlying asset follow a log-normal distribution.

Consider we want to value a call option and a put option at a general time t (not at time zero). Since T is the maturity date, then the time to maturity is $T - t$. The function $N(d)$ is the cumulative probability distribution function for a variable d with a standard normal distribution. The term $N(d_-)$ in both Equations 2 and 3 has a simple interpretation. It is the probability that a call option will be exercised in a risk-neutral world. The final payment of the option is only paid if the stock price is greater than K and the option holder chooses to exercise the option. The call option $f^{call}(S,T)$ is the present value of the expected payoff in a risk-neutral world.

During the hedging process, we assume any number, including non-integer, of call and put options are tradeable. In addition, we assume any amount of options is tradeable immediately with no transition time.

2.3 Delta Hedging

Delta hedging represents the change in option value concerning the change in the price of the underlying asset written on. It involves constructing a cost-free self-financing portfolio with the underlying asset S and the bank account B. The portfolio we hold contains the underlying asset S, the bank account B, and a put option with units $(\alpha_t, \beta_t, -1)_{t>0}$ respectively. Note that α_t is the number of units we invest in the underlying asset S, and β_t is the number of units we invest in the Bank Account. Thus, the portfolio value at time t is

$$V_t = \alpha_t S_t + \beta_t B_t - f_t \tag{4}$$

At time 0, we determine α_0 to achieve an instantaneously risk-free portfolio. According to Ito's Lemma, it can be shown that

$$\Delta put = \partial_s V_t^{put} = N(d_+) - 1 \tag{5}$$

where d_+ and $N(d_+)$ are defined as in Equation 3. The formula gives the delta of a put option, which is the ratio between the change in the put option price and the change in the price of the underlying asset S. This value also tells us how the asset movement affects the option price. For instance, if the put option has $\Delta_{put} = \$0.5$, when the price of S increased by \$1 per share, the value of the put option will rise by \$0.5 while everything else remains the constant.

Notice that $N(*)$ is the CDF of a standard normal distribution, α_t is always non-positive, which means when we short a put option, we always need to short the underlying asset S under delta hedging. The α_t is between 0 to -1.

In Delta hedging, the timing and frequency of rebalancing can have a significant impact on the profit and loss. We divide Delta hedging strategies into time-based and move-based rebalancing based on how we want to determine the number of rebalancing steps.

2.3.1 Time-Based Delta Hedging

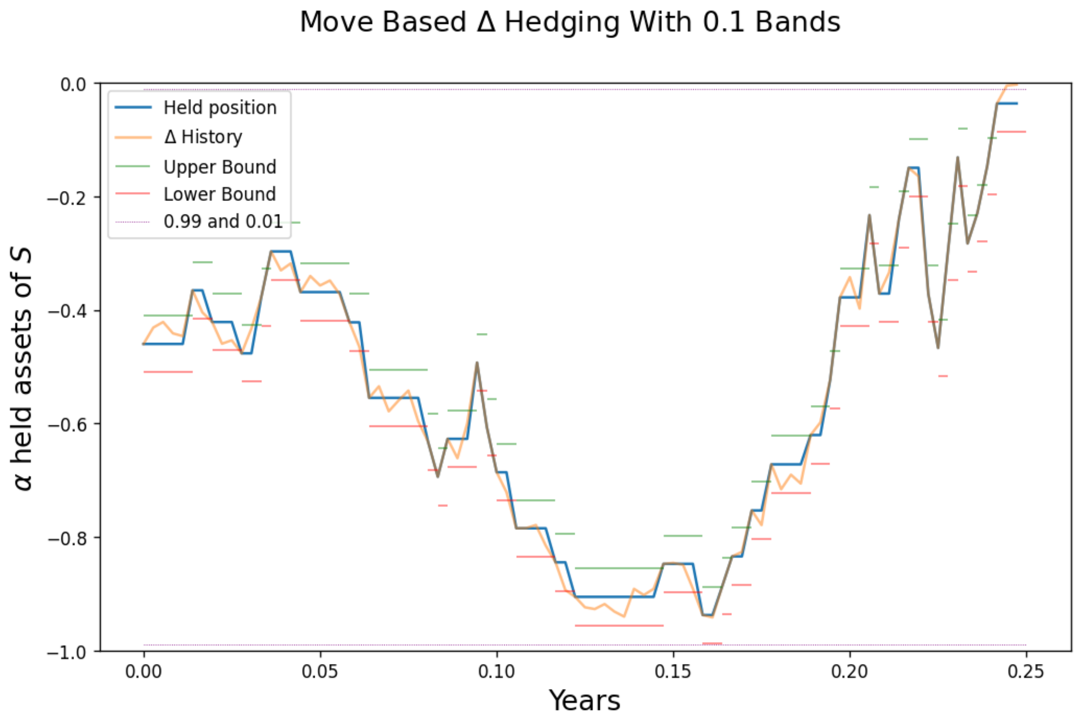
For the time-based Delta hedging, the rebalancing occurs at each time step. For instance, we hedge the sale of the option in equally sized, discrete time steps Δt (where $\Delta t = \frac{T}{N}$). Therefore, the evolution of a time-based delta hedging portfolio is:

Time	Evolution of Delta Hedging Portfolio
$t = t_0$	Sell the Put Option at V_0^{put} Sell $\lvert\alpha_0\rvert$ units of S Bank Account: $B_0 = V_0^{put} - \alpha_0 S_0 - TC_0$ where $TC_0 = 0.01 + 0.005\lvert\alpha_0\rvert$
$t = t_1^-$	Put Option: $V_{t_1^-}^{put}$ Asset holdings value: $\alpha_0 S_{t_1^-}$ Bank Account: $B_{t_1^-} = (V_0^{put} - \alpha_0 S_0 - TC_0)e^{r\Delta t}$
$t = t_1$	Put Option: V_1^{put} Rebalance Asset: Short $\lvert\alpha_0\rvert$ units of S, long α_{t_1} units $\rightarrow$ Cash Flow $= (\alpha_0 - \alpha_{t_1})S_{t_1}$ Rebalance Bank Account: $\rightarrow B_{t_1} = B_0 e^{r\Delta t} + (\alpha_0 - \alpha_{t_1})S_{t_1} - TC_{t_1}$ where $TC_{t_1} = 0.005\lvert\alpha_{t_1} - \alpha_0\rvert$
	...
$t = t_{t_k^-}$	Put Option : V_1^{put} Asset holdings value: $\alpha_{k-1} S_{t_k^-}$ Bank Account: $B_{t_k^-} = B_{t_{k-1}^-} e^{r\Delta t}$
$t = t_k$	Put Option: V_k^{put} Rebalance Asset: Short α_{k-1} units of S, long α_k units $\rightarrow$ Cash Flow $= (\alpha_{k-1} - \alpha_k)S_{t_k}$ Rebalance Bank Account: $\rightarrow B_{t_k} = B_{k-1}e^{r\Delta t} + (\alpha_{k-1} - \alpha_k)S_k - TC_{t_k}$ where $TC_{t_1} = 0.005\lvert\alpha_{k_1} - \alpha_k\rvert$
	...
$t = T$ at maturity	Sell the Put Option at V_1^{put} Asset holdings value: $\alpha_{k-1} S_T$ Bank Account: $B_T = B_{T-1}e^{r\Delta t} + \alpha_{T-1}S_T - 0.005\lvert\alpha_{T-1}\rvert$
PnL	$B_T - (K - S_T)_+$

2.3.2 Moved-Based Delta Hedging

The difference between time-based Delta hedging and move-based Delta hedging is the timing of rebalancing. While the time-based hedging strategy rebalances between each time interval, the move-based hedging strategy rebalances when the delta of the hedged option first hit the band set. We set a default baseband and we only rebalance when the Delta of the put option (Δ^{put}) hit the baseband. When the asset price shifts, we check if α_t hit this band. In our initial assumption, the baseband = 0.1. Figure 1 illustrates the move-based Delta hedging strategy with a 0.1 baseband. We rebalance if the Delta hit either the upper bound or lower bound of the band set. Therefore, when $|\alpha_t - \alpha_{t-1}| < 0.05$ and it excess the edge bands of 0.01 and 0.99 below or above the hedge position, we do not rebalance. Otherwise, we rebalance at the time t follow the exact same rule as in the time-based Delta hedging.

Figure 1: Band of Put Option



Note that when we do not rebalance at time t, there is no cash flow in the asset holding position, and thus no transaction cost. In this case, the Bank Account position is simply $B_t = B_{t-1}e^{r*\Delta t}$.

2.4 Delta-Gamma Hedging

Delta-gamma hedging combines delta hedging and gamma hedging together to mitigate the risk of adverse changes in asset price and delta. Specifically, Delta Hedging only accounts for first-order changes in the asset price S whereas Delta-Gamma Hedging, by its name, incorporates Gamma (Γ^{put}), which is the second-order derivative of the put option with respect to S, written as $\partial_{ss}V_t^{put}$. We shall consider the following second-order Taylor expansion:

$$\begin{aligned} g(t, X_{t+\Delta t}) &= g(t, X_t + \Delta X_t) \\ &= g(t, X_t) + \partial_x g(t, X_t) * \Delta X_t + \frac{1}{2} * \partial_{xx} g(t, X_t) * (\Delta X_t)^2 \end{aligned} \tag{6}$$

As we require net zero Delta for Delta Hedging, intuitively, we want to construct a portfolio with both zero net Delta and Gamma under Delta-Gamma Hedging. By introducing another hedging call option with value $h(t, X_t)$, we are able to obtain Delta-Gamma neutral given $\Delta^S = \partial_s S_t = 1$ and $\Gamma^S = \partial_{ss} S_t = 0$. In order to achieve a delta-gamma neutral portfolio, we hold positions at $(\alpha, \beta, \eta, -1) = (\alpha_t, \beta_t, \eta_t, -1)_{t\geq 0}$ which are investments into the underlying asset S, bank account B, a call option on S with strike = 100 and maturity in 1/2 year, as well as the associated put option respectively. The value of our portfolio at time t is given by:

$$V_t = \alpha_t X_t + B_t B_t + \eta_t h_t - g_t \tag{7}$$

Also, the delta and gamma regarding the portfolio at time t is given as:

$$\Delta_t^V = \alpha_t + \eta_t \Delta_t^h - \Delta_t^g \tag{8}$$

$$\Gamma_t^V = \eta_t \Gamma_t^h - \Gamma_t^g \tag{9}$$

By setting the above two equations to zero for achieving Delta-Gamma neutrality, it tells us the acquired number of units to hold in the underlying asset and the hedging option:

$$\alpha_t = \Delta_t^g - \frac{\Gamma_t^g}{\Gamma_t^h} \Delta_t^g \tag{10}$$

$$\eta_t = \frac{\Gamma_t^g}{\Gamma_t^h} \tag{11}$$

In a theoretical world, we would expect Delta-Gamma hedging to be more precise; however, the impact of transaction costs is significant and unneglectable in real life.

2.4.1 Time-Based Delta-Gamma Hedging

In a similar manner to time-based Delta-Hedging, the rebalancing actions occur in equal time steps. The evolution framework of a time-based Delta-Hedging is the same as the one for the time-based Delta heading, referring to the table provided in the 2.3.1 section. Notice that the only difference we have is that we additionally incorporate the call option into the Delta-Gamma hedging.

Starting from time 0, we need to short the put option and receive V_o^g. Also, we make investments for α_0 units of S and η_0 units of the call option. According to dynamic hedging requirements, we have $V_t = 0$ for all time t. Thereby, for bank account investments, we have the amount as

$$B_0 = V_0^g - \alpha_0 S_0 - \eta_0 V_0^h - TC_0, \text{ and } TC_0 = 0.01 * (1+ \mid \eta_0 \mid) + 0.005* \mid \alpha_0 \mid$$

where g represents put option and h represents call option.

Suppose that we rebalance at the time t-1. Then at the time t, we have $\alpha_t - \alpha_{t-1}$ units of S and $\eta_t - \eta_{t-1}$ units of the call option. Also, our investments for the bank account at the time t is

$$B_t = B_{t-1}e^{r\Delta t} - (\alpha_t - \alpha_{t-1})S_t - (\eta_t - \eta_{t-1})V_t^h - TC_t$$

where

$$TC_t = 0.01* \mid \eta_t - \eta_{t-1} \mid +0.005* \mid \alpha_t - \alpha_{t-1} \mid$$

Upon maturity, the amounts in the bank account turn into

$$B_T = B_{T-1}e^{r\Delta t} + \alpha_{T-1})S_T + \eta_{T-1}V_t^h - TC_T$$

where

$$TC_T = 0.01* \mid \eta_{T1} \mid +0.005* \mid \alpha_{T-1} \mid$$

The profit and loss at maturity would be $B_T - (K - S_T)_+$.

2.4.2 Moved-Based Delta-Gamma Hedging

The same rules apply to move-based Delta-Gamma hedging as they do to move-based Delta hedging. Instead of rebalancing at each rebalancing step, we establish a baseband and only rebalance when the delta of the put option hit this band. Basically, at each t_n, we need to check if the delta of the put option hit the predetermined band which is 0.1. Under the condition that $\mid \Delta_{t_n}^{put} - \Delta_{t_{n-1}}^{put} \mid < 0.05$, there is no need to rebalance, thereby, no transaction costs. However, if $\mid \Delta_{t_n}^{put} - \Delta_{t_{n-1}}^{put} \mid \geq 0.05$, we need to rebalance our portfolio at time t_n following the same rules we adopted in the time-based Delta-Gamma hedging.

3 Analysis

In order to study the various strategies available to hedge a put option, namely time-based and move-based delta and delta-gamma hedging and to do a comparative analysis of these trading strategies, we implement them and perform 10000 simulations for each of the methods.

3.1 Time-Based vs. Move-Based Delta Hedging

3.1.1 Stock Price Change and the unit of Stock Change Over time in Time-based Delta Hedge

Figure 2: Simulated Stock Price Paths

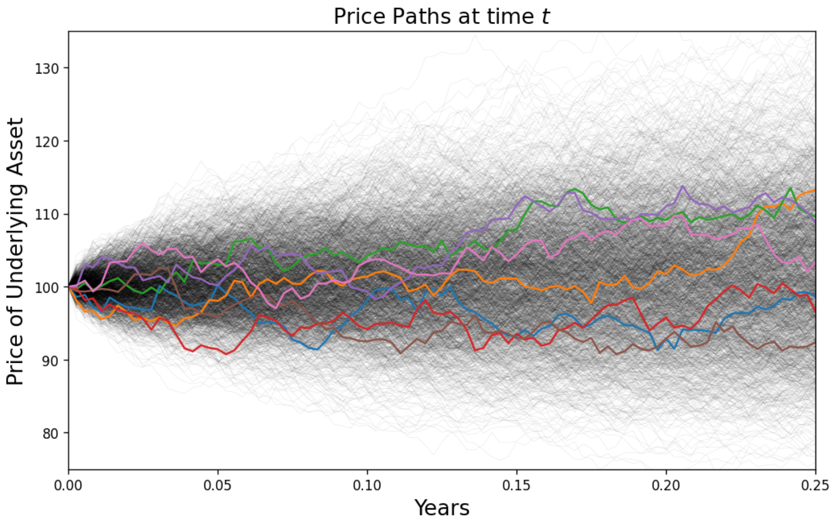


Figure 3: Simulated Unit of Stock in time-based delta hedging

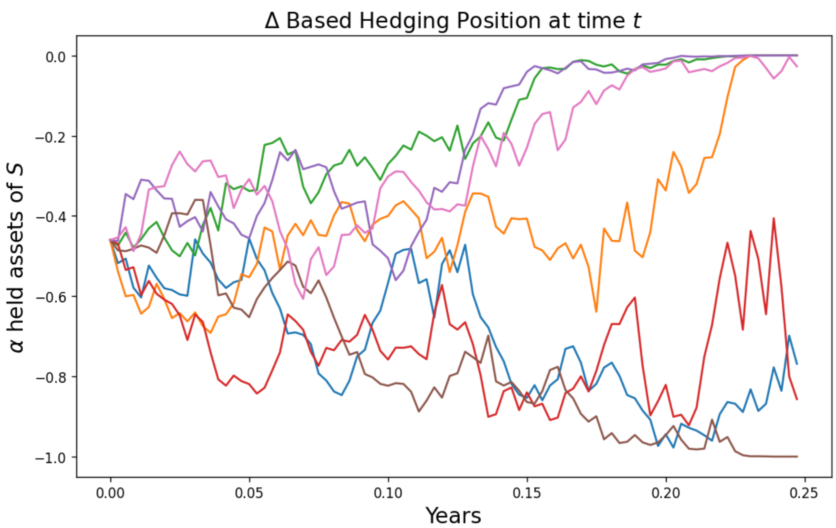
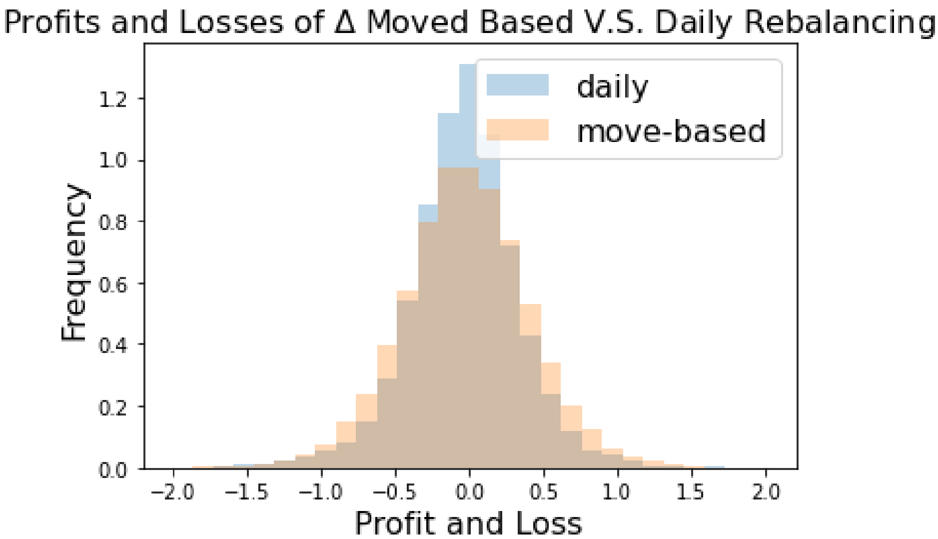


Figure 2 simulates 8 stock price paths over time and Figure 3 shows the unit of these 8 stock paths in time-based delta hedging. Recall that the initial stock price is $S_0 = 100$, so all stock price paths are starting from 100, with 4 paths moving above the strike price most of the time and 3 paths moving below the strike price. At maturity, there are 3 paths lower than the strike price $K = 100$. We found that when the stock price paths are above the strike price at maturity, its corresponding unit of stock gradually approaches 0. On the other hand, when the stock price path is lower than the strike price at maturity, its corresponding unit of stock gradually goes to -1.

3.1.2 Profits and Losses (PnL): Time-based vs. Move-based

In figure 4, the plot illustrates the Profit and Loss (PnL) under two histograms, which are when the move-based strategy with a baseband of 0.1 and when the time-based strategy under 90 times during the $\frac{1}{4}$ year (Daily). The PnL histograms of both the move-based strategy and time-based are roughly symmetric around 0. We can see that the move-based strategy slightly outperforms the time-based (daily) strategy. This is reasonable because the move-based strategy has to rebalance less frequently, which results in a lower transaction cost and a higher PnL.

Figure 4: Histogram of PnL



3.1.3 The Charging Price using Time-Based and Move-Based hedging

We aim to determine the charging price using time-based and move-based hedging so that the Conditional value-at-risk at level 0.9 is no larger than 0.02, which also means that

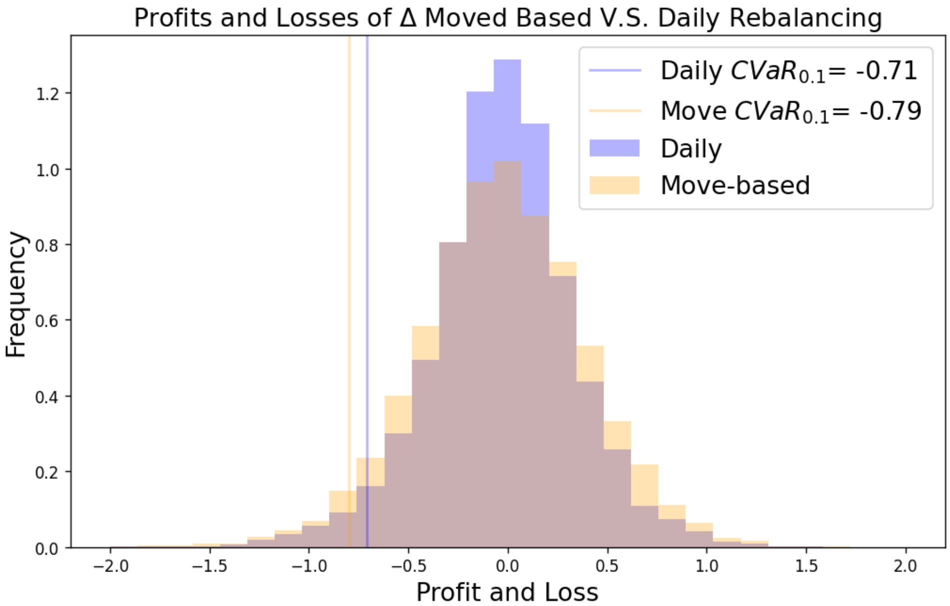
$$CVaR_{0.1} \geq -0.02 \tag{12}$$

In figure 5, the two straight lines show the values of $CVaR_0.1$ when they are time-based strategy and move-based strategy, which are -0.71 and -0.79 respectively. To obtain the optimal charging price, we compute the difference between our absolute CVaR value and -0.02, and discount back to our put price at time 0,

$$OptimalPrice = [|CVaR_{0.1}| - 0.02] * e^{-rT} + PutOption \tag{13}$$

Where the value of the put option for our result is around 3.7334. Therefore, the optimal price under time-based (daily) $CVaR_{0.1}$ is around 4.42, and the optimal price under move-based $CVaR_{0.1}$ is around 4.996.

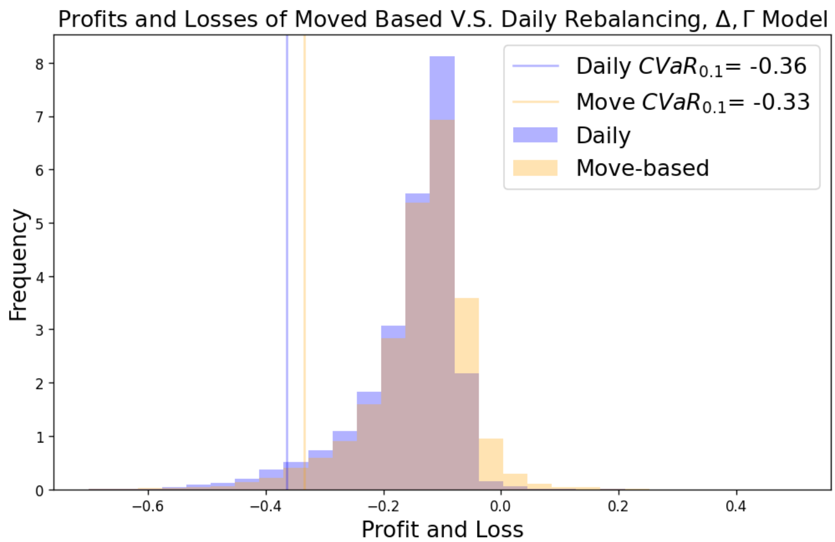
Figure 5: Conditional Value-at-Risk at level 0.1



3.2 Time-Based vs. Move-Based Delta-Gamma Hedging

To compare Delta-Gamma hedging under time-based and move-based techniques, we plot their profits and losses histograms against each other in figure 6, respectively for daily-based and the baseband of 0.1. From the plot, we can observe that both histograms tend to be left-skewed with the bulk of observations around -0.2 to 0 and a few observations that are smaller than -0.2. Additionally, we can see that, for Delta-Gamma Hedging, moved based technique slightly outperforms the time-based (daily) technique due to fewer rebalancing actions acquired which is the same as the summary we derived from Delta hedging. We also see these changes reflected in our relatively lower values at risk where it was effectively cut in half in relation to delta only rebalancing. This suggests that we are able to sell the option for consideraly less for an at-the-money price.

Figure 6: Histogram of PnL for Delta-Gamma Hedging

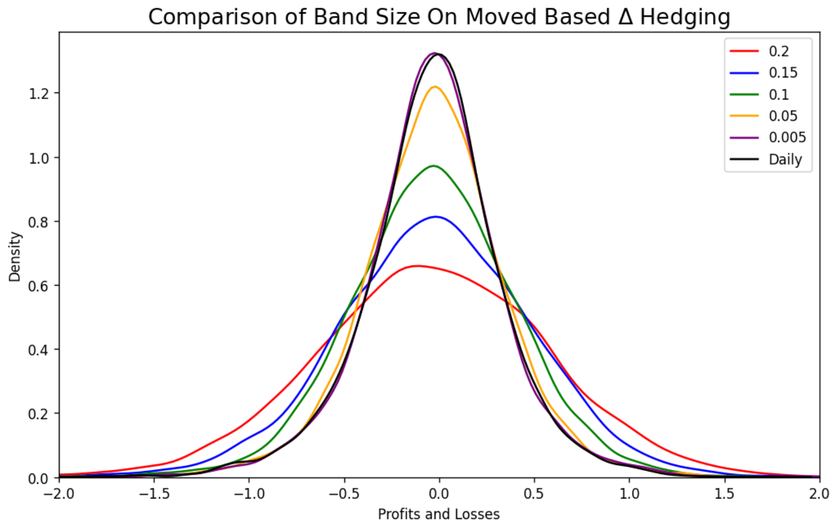


3.3 The impact of Rebalancing-Band on the Delta Hedge

3.3.1 The impact of Changing the Band Width for the Moved-Based Delta Hedging

We want to investigate the effect of rebalancing bands on profits and losses by computing the mean and variance of profits and losses for various bandwidth assumptions. Figure 7 illustrates the profit and loss distributions under delta hedging with varying bandwidths. We can observe that the average loss falls when the rebalancing band widens, owing to fewer rebalancing actions, which saves more transaction costs. Additionally, when bandwidth grows, so does the volatility of earnings and losses. The reasoning is that hedgers do not react to modest changes in the delta, causing the hedging mistake to rise. Special notice to profits and losses of bandwidth 0.005 and daily adjustments; we can see that the purple line and black line are almost fully overlapped. This can be explained by having a relatively small bandwidth leading to frequent rebalancing actions. For risk-averse investors to adopt time-based techniques to hedge, a smaller band is a better choice. A larger bandwidth, on the other hand, gives better rewards to investors with a high-risk tolerance.

Figure 7: The impact of Rebalancing-Band on the Delta Hedge



4 Conclusion and Future Work

In this project, we explore four different types of dynamic discrete time hedging strategies under Black-Scholes model asset pricing assumptions. We have seen the performance of discrete time daily delta hedging, discrete time movement based delta hedging, discrete time daily delta-gamma hedging using a long call option, and using the same long call option; discrete time movement based delta-gamma hedging. It is found that movement based delta hedging of a 0.1 rebalancing band produced a wider profit and loss distribution after 10,000 price path simulations compared to daily rebalancing alongside a lower higher conditional value at risk. This suggests that a rebalancing band of 0.1 is a less desirable strategy compared to daily rebalancing under our transaction costs assumptions as a higher value at risk suggests that our put option pricing would be higher for the client for us to maintain a minimum 10 percent conditional values at risk of 0.02. It is observed that with a large rebalancing band for delta hedging strategy, the corresponding profits and loss distribution seem to widen even further; which in turn would give us a less desirable conditional values-at-risk. Conversely, with a smaller rebalancing bands, the profits and loss distributions converge closer and closer to daily rebalancing which follows our logic as infinitesimally smaller bands would suggest that we are effectively rebalancing every day.

For delta-gamma daily rebalancing, we saw that our profits and loss functions produced a tighter distribution that resulted in lower conditional values-at-risk despite losing out on being symmetrically distributed around zero. Alternatively to delta only rebalancing, we see that the profts and loss function for move based rebalancing with a 0.1 band seems to improve compared to the daily rebalancing strategy by having a similar distribution shape to that of the daily rebalancing strategy but with a larger tail on the right which subsequently leads to a more desirable conditional values-at-risk. Thus from a risk management perspective, delta-gamma rebalancing seems to the more desirable strategy as we are able to hedge the same option by charging the client a smaller at-the-money price.

A major factor we did not observe was the rebalancing frequency distributions of the different types of hedging strategies. In our discussion, we assumed a constant transaction cost which does not change with respect to the underlying cost of the product purchased/sold. In reality, we would like to observe profit and loss distributions under various transaction rates, however these situations are very unique and specific to different clients that it would be unfeasible to observe these distributions in a general manner. Of course in an ideal environment free of transactions we would like to rebalance as much as possible to ensure perfect risk hedging, however the combination of unfeasibility in infinite rebalancing and transaction costs render our discussions to a fixed daily rebalancing and flat transaction costs.