Stochastic Interest Rates and Swaptions

STA2503: Applied Probability for Mathematical Finance

Authors:

Renee So: 1005472338

John Ndolo: 1009482560

Michael Luan: 1004282867

Master of Financial Insurance

DEPARTMENT OF STATISTICAL SCIENCES,
FACULTY OF ARTS & SCIENCE

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Abstract

In this project we investigate several aspects of the stochastic interest rates using the two-factor Vasicek model. We first derive the analytical formula of a T-maturity bond. Then using a Euler-Scheme to discretize the SDEs, we simulate 1,000 risk neutral interest rate paths and visualize the respective quantile paths. Using the simulated paths, we obtain Monte-Carlo estimate of bond yields then compute the yields for maturities ranging from one month to 10 years in steps of one month, and compare with the analytical formula of T-maturity bond derived from the probabilistic approach. Ultimately, we explore the pricing of a swaption and determine the Black implied volatility of a swaption with strike equal to today's swap-rate; then vary the strike and establish the effect it has on the Black implied volatility. Some of notable observations include; the bond yield is inversely proportional to the volatility and the bonds option price decrease linearly with increase in strike.

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1 Introduction

Predicting how interest rates evolve can be difficult. Investors and analysts have several tools at their disposal to help them figure out how interest rates change over time in order to make well-informed decisions about their investments and the economy. The stochastic interest rate models such as the two-factor Vasicek Interest Rate Model are among the models that can be used to help estimate how interest rates will change in the future. This plays an important role in pricing different financial instruments such as bonds, swaps among others. Therefore, in this project we explore the two-factor Vasicek model and how it is applied in modeling interest rates changes in pricing a bond and swaptions. Some of the previous models such as the Black-Scholes model assumes interest rates are constant [1]. However, in reality, the interest rates are stochastic. Hence the adoption of short-rate or stochastic interest models helps in modeling and describing the evolution of interest rates in the future. In this report, we focus on evaluating a two-factor stochastic interest rate model (two factor Vasicek Model). We work under the assumption two Brownian motion of the model are independent.

First, we derive an analytical formula of a T-maturity bond under the risk-neutral measure by solving the distribution of $\int_0^T r_u du$ and drawing the term structure of interest rates with the given base parameters. We then use an Euler-scheme to discretize the SDE (with discretization of 1 step per day, 252 trading days in a year) and generate 1,000 simulations of risk-neutral interest rate paths out to 10 years and plot the quantiles of the paths. We then use the simulated paths to evaluate the Monte Carlo estimate of bond yields (with confidence bands), compute the yields for maturities ranging from one month to 10 years in steps of one month, and compare the results with the analytical formula derived in Subsection (2.3). We further compare the term structures of the interests rates and investigate the role of parameters α, β, σ and η described in Subsection (2.3). Lastly, we determine the Black implied volatility of a swaption with strike equal to today's swap-rate with the black implied volatility defined as an Longstaff-Schwarz (LSM) model that makes the LSM price equal the price you obtained; then investigate how the implied volatility change with variation of strike. In summary, our report is structured into four parts; the introduction, methodology, results & discussions and the conclusion.

2 Methodology

Assume a bank account given as $B = (B_t)_{t\geq 0}$ and represented by SDE;

$$dB_t = r_t B_t + 0 \cdot dW_t, \qquad B_0 = 1$$

where $r = (r_t)_{t\geq 0}$ is a called a short rate process. The solution of the above SDE of the bank account is given as

$$B_t = B_0 e^{\int_0^t r_u du},$$

since r_t is assumed to be stochastic.

There are many kinds of stochastic interest rate models, in this report we focus on the Vasicek interest rate model, especially the two-factor Vasicek model. Besides the two factor models, there exist one factor models such as Ho-Lee models, one factor Vasicek model and Cox, Ingersoll, Ross (CIR) model. The one factor and two factor Vasicek models are described below and used to derive an analytical formula of pricing a T-maturity bond.

2.1 One-factor Vasicek Model

One-factor short rate models work under the assumption that the future evolution of the interest rates is dependent on only one stochastic factor [2]. Although unrealistic, the models provide good estimates of the term structure of interest rates if the different factors that affect the interest rates are highly interrelated. A one-factor Vasicek interest rate Model is a good example of one factor interest models and satisfies the SDE:

$$dr_t = k(\theta - r_t)dt + \sigma dW_t$$

where W_t is a risk-neutral Brownian motion, θ is the long-term mean level (long-term interest rate), k determines the speed of reversion and σ is the instantaneous volatility. The solution of the SDE is given as

$$r_t = \theta + (r_s - \theta)e^{-k(t-s)} + \sigma \int_s^t e^{-k(t-u)} dW_u, \qquad 0 \leq s < t.$$

The one-factor models, however, are insufficient to model interest rates because there are numerous real-world factors that affect interest rates. As a result, two factor models are a consideration.

2.2 Two-factor Vasicek Model

Two-factor Vasicek is also know as Hull-White model. Let $r = (r_t)_{t\geq 0}$ denote the short rate of interest and suppose that $r_t = \phi_t + x_t + y_t$ where x and y satisfy the SDEs;

$$\begin{aligned} dx_t &= -\alpha x_t dt + \sigma dW_t^1, \\ dy_t &= -\beta y_t dt + \eta dW_t^2 \end{aligned}$$

where $W^{1,2} = (W_t^{1,2})_{t\geq 0}$ are independent risk-neutral Brownian motions, and ϕ_t is a deterministic function of time. This is called a two-factor interest rate model.

2.3 Bond pricing

The two-factor Vasicek model can be conveniently used to price a T-maturity bond [3]. We denote the bond's price process by $P(T) = (P_t(T))_{t\in[0,T]}$ and there exists a deterministic functions $A_t(T), B_t(T)$ and $C_t(T)$ such that;

$$P_t(T) = \exp\{A_t(T) - B_t(T)x_t - C_t(T)y_t\}.$$

We determine the functions A, B and C by using probabilistic approach by solving for the distribution of $\int_0^T r_u du$. Assuming , a bond maturity T and by the fundamental theorem of asset pricing (FTAP), the bond price is given as

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right],$$

where $\mathbb{Q}$ is the risk-neutral measure.

Now, let's define
$y_t = e^{\alpha t} x_t$, by Itô's product rule it follows that;

$$\begin{aligned} dy_t &= x_t d\left(e^{\alpha t}\right) + e^{\alpha t} dx_t + d\left[x_t, e^{\alpha t}\right] \\ &= \alpha e^{\alpha t} x_t dt + e^{\alpha t} dx_t + 0 \\ &= \alpha e^{\alpha t} x_t dt + e^{\alpha t}\left[-\alpha x_t dt + \sigma dw_t^1\right] \\ &= \sigma e^{\alpha t} dW_t^1 \end{aligned}$$

By integrating both sides, we get

$$\begin{aligned} \int_t^s dy_u &= \int_t^s \sigma e^{\alpha u} dW_u^1 \\ y_s - y_t &= \int_t^s \sigma e^{\alpha u} dW_u^1 \\ e^{\alpha s} x_s - e^{\alpha t} x_t &= \int_t^s \sigma e^{\alpha u} dW_u^1 \\ x_s &= e^{-\alpha(s-t)} x_t + \sigma \int_t^s e^{-\alpha(s-u)} dW_u^1 \end{aligned}$$

Also, by integrating SDE, $dx_t = -\alpha x_t dt + \sigma dW_t^1$ we have,

$$\begin{aligned} \int_t^T dx_t &= \int_t^T (-\alpha x_s ds + \sigma dW_s^1) \\ &= -\alpha \int_t^T x_s ds + \sigma \int_t^T dW_s^1 \\ x_T - x_t &= -\alpha \int_t^T x_s ds + \sigma \int_t^T dW_s^1 \\ \int_t^T x_s ds &= \frac{1}{\alpha}\left[-(x_T - x_t) + \sigma \int_t^T dW_s^1\right] \end{aligned}$$

But we know

$$x_T = e^{-\alpha(T-t)} x_t + \sigma \int_t^T e^{-\alpha(T-s)} dW_s^1,$$

thus upon substitution in the above equation we have;

$$\begin{aligned} \int_t^T x_s ds &= \frac{1}{\alpha}\left[-(x_T - x_t) + \sigma \int_t^T dW_s^1\right] \\ &= \frac{1}{\alpha}\left[-\left(e^{-\alpha(T-t)} x_t + \sigma \int_t^T e^{-\alpha(T-s)} dW_s^1 - x_t\right) + \sigma \int_t^T dW_s^1\right] \\ &= \frac{1}{\alpha}\left[\left(1 - e^{-\alpha(T-t)}\right) x_t + \sigma \int_t^T \left(1 - e^{-\alpha(x-s)}\right) dW_s^1\right] \end{aligned}$$

$$= l_t x_t + \sigma \int_t^T l_s dW_s^1$$

where $l_t = \frac{\left(1-e^{-\alpha(T-t)}\right)}{\alpha}$ and $l_s = \frac{\left(1-e^{-\alpha(T-s)}\right)}{\alpha}$. Therefore, we the distribution $\int_t^T x_s ds$ is approximated as,

$$\int_t^T x_s ds \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(l_t x_t, \sigma^2 \int_t^T l_s^2 ds\right), \text{ by Ito's isometry}$$

Similarly, using the same idea above, we can derive the distribution of $\int_t^T y_s ds$ under the SDE, $dy_t = -\beta y_t dt + \eta dW_t^2$ such that;

$$\int_t^T y_s ds \overset{\mathbb{Q}}{\sim} \mathcal{N}\left(m_t y_t, \eta^2 \int_t^T m_s^2 ds\right)$$

where $m_t = \frac{\left(1-e^{-\beta(T-t)}\right)}{\beta}$ and $m_s = \frac{\left(1-e^{-\beta(T-s)}\right)}{\beta}$.

Now, we already know, a T-maturity bond is given as;

$$P_t(T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right]$$

and $r_t = \phi_t + x_t + y_t$ thus we can have $P_t(T)$ as;

$$\begin{aligned}
P_t(T) &= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_s ds}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (\phi_s + x_s + y_s) ds}\right] \\
&= e^{-\int_t^T \phi_s ds} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (x_s + y_s) ds}\right] \\
&= e^{-\int_t^T \phi_s ds} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T x_s ds}\right] \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T y_s ds}\right] \qquad \text{(Independence property)}
\end{aligned}$$

By M.g.f technique we have;

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T x_s ds}\right] = e^{-l_t x_t + \frac{\sigma^2}{2} \int_t^T l_s^2 ds}$$

and

$$\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T y_s ds}\right] = e^{-m_t y_t + \frac{\eta^2}{2} \int_t^T m_s^2 ds}.$$

We also know, $\phi = a + b\left(\frac{(1-e^{-\lambda t})}{\lambda t} - e^{-\lambda t}\right)$, thus,

$$e^{-\int_t^T \phi_s ds} = e^{-\int_t^T a + b\left(\frac{(1-e^{-\lambda s}}{\lambda s} - e^{-\lambda s}\right) ds} = e^{-a - b \int_t^T \left(\frac{(1-e^{-\lambda s}}{\lambda s} - e^{-\lambda s}\right) ds}$$

By combining the above equations, we come up with an affine model of the T-maturity bond pricing as;

$$\begin{aligned}
P_t(T) &= e^{-\int_t^T \phi_s ds} \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (x_s + y_s) du}\right] \\
&= e^{-\int_t^T \phi_s ds} \cdot e^{-l_t x_t + \frac{\sigma^2}{2} \int_t^T l_s^2 ds} \cdot e^{-m_t y_t + \frac{\eta^2}{2} \int_t^T m_s^2 ds} \\
&= e^{A_t(T) - B_t(T) x_t - C_t(T) y_t}
\end{aligned}$$

where,

$$A_t(T) = -\int_t^T \phi_s ds - \frac{1}{2}\int_t^T \sigma^2 (l_t)^2 + \eta^2 (m_s)^2 du$$

$$B_t(T) = l_t = \frac{\left(1 - e^{-\alpha(T-t)}\right)}{\alpha}$$

$$C_t(T) = m_t = \frac{\left(1 - e^{-\beta(T-t)}\right)}{\beta}$$

$$\phi = a + b\big(\frac{(1 - e^{-\lambda t})}{\lambda t} - e^{-\lambda t}\big)$$

$$m_s = \frac{\left(1 - e^{-\beta(T-s)}\right)}{\beta}$$

Hence, we have obtained the analytical functions A, B and C hence shown above.

2.4 Euler-Scheme discretization

To obtain the simulations of the bond price process and the Monte Carlo estimates, we first discretize the SDEs using the Eule-scheme. The discretization and simulation is done in the following steps;

1. We first evaluate the values of ϕ_t which is function given as,

$$\phi_t = a + b\Big(\frac{1 - e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\Big)$$

2. We then initialize the value of the intial interest rates at $t = 0$ as $r_t = \phi_o + x_0 + y_0$
3. By Euler-scheme, we discretize the SDEs and obtain the discretized forms of x and y as;

$$x_{t+\Delta_t} = x_t - \alpha x_t \Delta_t + \sigma\sqrt{\Delta_t} Z_1$$

and

$$y_{t+\Delta_t} = y_t - \beta y_t \Delta_t + \eta\sqrt{\Delta_t} Z_2$$

where $Z_1 \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$ and $Z_2 \overset{\mathbb{Q}}{\sim} \mathcal{N}(0,1)$.
4. We then use the discretized values of ϕ_t, x_t and y_t to estimate the interests rates as ;

$$r_{t+1} = \phi_{t+1} + x_{t+1} + y_{t+1}$$

5. We then use the simulated paths of r_t to find an estimate of $\int_0^T r_s ds$ for a T-maturity band where the estimate is give as a summation of r_t over every discretized points ($\sum_0^N r_i \Delta_t$) then evaluate the expectation of the integral under risk-neutarl measure to obtain a monte-Carlo estimate of the bond given as ,

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\Big[e^{-\int_t^T r_s ds}\Big].$$

3 Results and discussions

3.1 Simulation of risk-neutral interest rate paths

Using the derived analytical formula of the T-maturity bond price process, we apply an Euler-scheme to discretize the SDE (with discretization of 1 step per day, 252 trading days in a year) and generate 1,000 simulations of risk-neutral interest rate paths out to 10 years. We then plot the quantiles of the paths and investigate x, y, and r on separate plots as illustrated in the figures below,

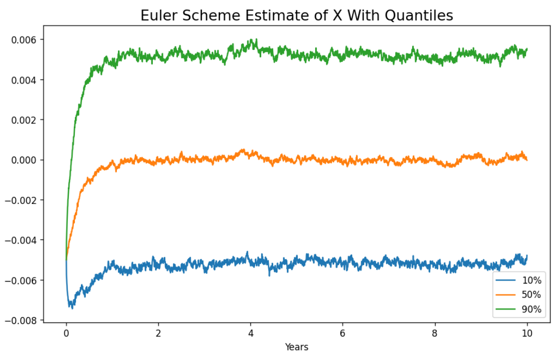

Figure 1: x analysis plot

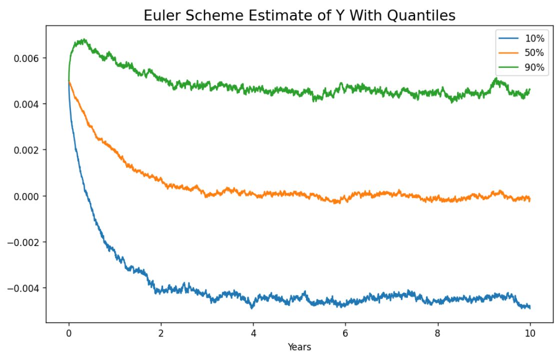

Figure 2: y analysis plot

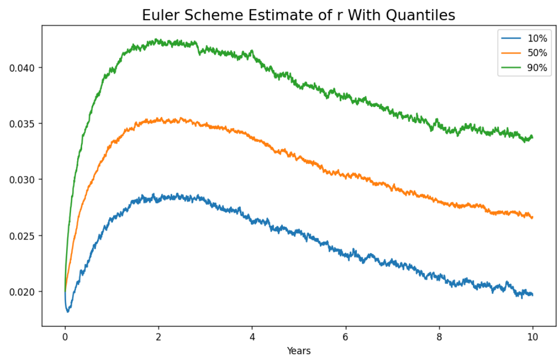

Figure 3: r analysis plot

We observe in Figure (1), the values of x increase sharply in early years then stagnates after sometimes, and decrease slowly. On the other, in Figure (2), we observe that, y decreases with time then stagnates after sometime. Figure (3) represents the interest rates paths which shows an increasing trend at early times of the bonds maturity then starts decreasing at a point.

3.2 Monte Carlo estimates and yield curve

We use 10,000 simulated paths to obtain Monte Carlo estimate of bond yields (with confidence bands). Then compute the yields for maturities ranging from one month to 10 years in steps of one month, and compare with the analytical formula we derived in Subsection (2.3) using the following base parameters;

$$x_0 = -0.5\%, \quad \alpha = 3, \quad \sigma = 1\%, \quad \beta = 1, \quad \eta = 0.5\%,$$

and

$$\phi_t = a + b\left(\frac{1-e^{-\lambda t}}{\lambda t} - e^{-\lambda t}\right), \qquad a = 2\%, \quad b = 5\%, \quad \lambda = 0.75.$$

The figure below illustrates the bonds price and yield curve obtained using the analytical formula and the Monte Carlo estimates;

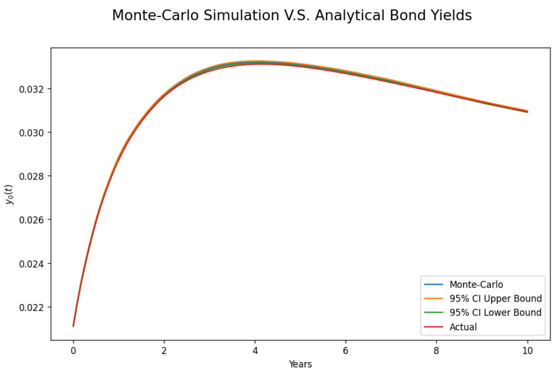

Figure 4: Yield curve using MC simulation and Analytical formula

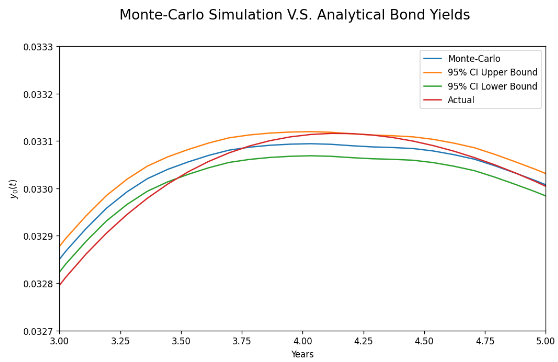

Figure 5: Yield curve using MC simulation and Analytical formula(zoomed)

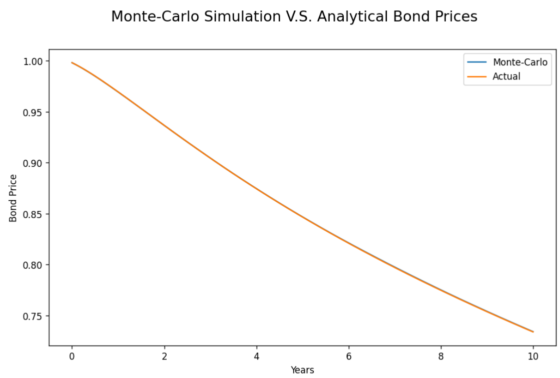

Figure 6: MC simulation V.S. Analytical formula Bonds Price

From figure (4), we see the yield curve with base parameters. We can see that since the short-term mean reversion term is larger than the long-term mean reversion term, and so we see larger yields in the initial years with its peak at around 3.8 years while tapering off as the years progress.
As evident in Figure (6), the Bonds price curve under Monte Carlo simulation overlaps with the one obtained from the analytical formula meaning that the MC estimates are approximately closer to the estimates from analytical formula. In addition the plotted yield curves relays almost same information as evident from the bonds price curve, where Figure (5) is a zoomed version of Figure

(4). The orange and green bands represent the upper and lower 95% confidence interval respectively. We can observe that the confidence intervals have a small width implying that the simulated estimates and analytical formula estimates are approximately close. However, since the simulated yields highly depends on volatility σ, if volatility increases to a higher value, we expect the simulated yields to have more fluctuations while the analytical yield curve is still smooth.

3.3 Role of base parameters on the term structure of interest rates

Moreover, we vary the base parameters α, β, η and σ and investigate their role in determining the term structure of interest rates using the analytical formula. This effects are observed in the figures presented below. We first begin with variations in short-term mean reversion α. As seen in Figure (7), we observe that the the smaller the value is for α, the lower the yield curve . Conversely, the larger the alpha, the higher the resulting yield curve which converges after sometime. Regardless of value, the yield curve still maintains its original shape inline with the x analysis we observed in Figure (1). We therefore, conclude that, by increasing the value of α, the yield increases and thus the shift of the curve upwards at earlier times which is characteristic of short-term mean reversion as the a higher mean reversion term implies faster return to the expected value which suggests if short-term interest rates are expected to fall in the future, the yield curve may become less downward sloping, as investors are willing to accept lower yields on long-term bonds. The vice-versa is true. In addition, we note that, the rate of increase over a short time of maturity is sharp and later on it converges at a point and start decreasing slowly, suggesting increasing the α value further has a minor impact at this point.

We also observe the effect of changes in long-term mean reversion β in Figure (8) where we note that with a smaller β value, the yield curve shifts upward and its maximum value subsequently also increases. This was to be expected as if long-term interest rates are expected to increase in the future, the yield curve may become more upward sloping, as investors demand higher yields on long-term bonds to compensate for the anticipated increase in interest rates. Conversely, larger β values causes the resulting yield curve to decrease with diminishing returns especially as after five years. We can see the difference in α and β as having inverse relationships on the yield curve with respect to each other. This inverse relationship between α and β can be further inferred from their respective term structure plots of observed in Figure (1) and Figure (2). This was to be expected as highly volatile long-term interest rate volatility can lead to increased uncertainly about future interest rates which un tirn flattens the yield curve due to investors being less certrain about the long term returns.

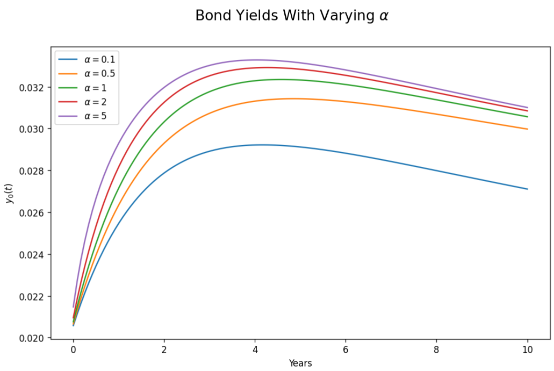

Figure 7: Yield curve with varying α

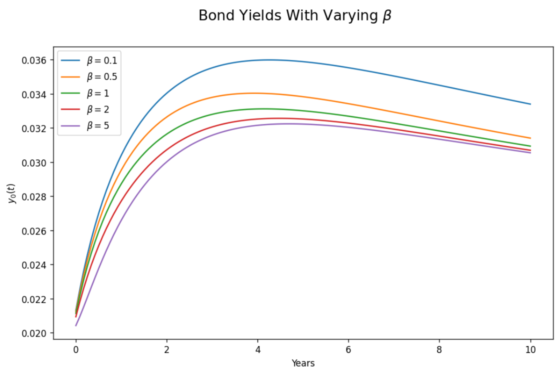

Figure 8: Yield curve with varying β

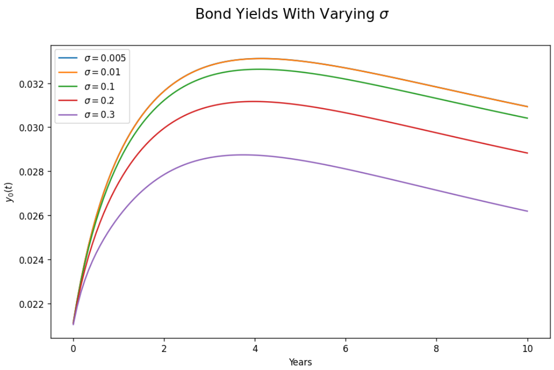

Figure 9: Yield curve with varying σ

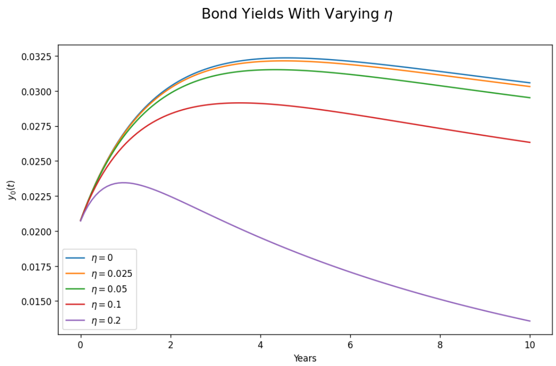

Figure 10: Yield curve with varying η

Next, we observe the effects of varying volatility σ in the short-term interest rate. We can see in Figure (9) that, by decreasing the volatility to a small value ($\sigma = 0.05\%$), the yield curve overlap in earlier years and the variations between the curves is too small to draw conclusions with confidence. Therefore, we vary the volatility more by increasing the values of σ and we note that, the yield curve decreases with increase in volatility which can be attribute to the fact that, the higher the volatility the higher the bond price and vice versa is true. Thus, based on this observations, the bonds yield decrease with increase in volatility and increases with decrease in volatility. This was to be expected as high volatility can lead to increased uncertainty about future interest rates and a flattening of the yield curve due to investors being less certain about the direction of interest rate changes this demanding higher yields on long-term bonds to compensate for the increased risk.

Similarly, looking at Figure (10), we can observe that, η behaves almost the same as the σ. Increasing the value of η leads to increase in Bonds price which consequently decreases the yield. Therefore, the higher the value of η the lower the yield curve as observed in the figure above. As a result, we conclude that, the η value is inversely proportional to the bond yield.

3.4 Interest Rate Swap & Implied Volatility

An Interest Rate Swap (IRS) occurs when there is an exchange of fixed cash flows, the fixed leg, for another set of cash flows valued using floating interest rate, the floating leg.

The fixed leg is calculated by discounting all cash flows to the present time using bond prices:

$$V_t^{\text{fixed}} = N \cdot F \cdot \sum_{k=1}^{n} \Delta_{\tau_k} \cdot P_t(\tau_k)$$

where N is the notional amount, F is the fixed rate, $\tau = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ is the tenure structure with τ_0 as reset date (no payment) , and $P_t(\tau_k)$ is the price of a bond valued at time t with maturity T.

On the other hand, the value of the floating leg is:

$$V_t^{\text{floating}} = N \cdot \sum_{k=1}^n \left(P_t(\tau_{k-1}) - P_t(\tau_k)\right) = N \cdot \left(P_t(\tau_0) - P_t(\tau_n)\right)$$

By equating the value of the two legs, we derive the following swap rate at time t:

$$S_t = \frac{(P_t(\tau_0) - P_t(\tau_n))}{\sum_{k=1}^n \Delta\tau_k P_t(T_k)}$$

There are two forms of swaption: a payer swaption and a receiver swaption. A payer swaption gives the holder the right but not obligation to enter an IRS, in which they pay the fixed rate K and receive the floating rate at time T, whereas the receiver swaption is the opposite.

The value of a payer swaption at maturity T is:

$$V_T = \mathbb{1}\{S_T > K\} \cdot \left(V_T^{\text{floating}} - V_T^{\text{fixed}}\right) = A_T(S_T - K)_+$$

where $A_T = \sum_{k=1}^n \Delta\tau_k P_T(\tau_k)$ is the value of an annuity at time T.

Under the fundamental theorem of asset pricing (FTAP), we can use A_t as the numeraire asset and obtain:

$$\frac{V_t}{A_t} = \mathbb{E}^{\mathbb{Q}_A}\left[\frac{A_T(S_T - K)_+}{A_T}\right] = \mathbb{E}^{\mathbb{Q}_A}\left[(S_T - K)_+\right]$$

Therefore,

$$V_t = A_t \mathbb{E}^{\mathbb{Q}_A}\left[(S_T - K)_+\right]$$

Since $S_t = \frac{P_t(\tau_0) - P_T(\tau_n)}{A_t}$ is the relative price of a numeraire asset, S_t is a $\mathbb{Q}_A$ - martingale and satisfies the following SDE:

$$\frac{dS_t}{S_t} = \sigma_t^S dW_t^{\mathbb{Q}^A}$$

We assume that σ_t^S is a deterministic function such that the swap rate has a lognormal distribution, and can be computed using the Black-Scholes option formula as follows:

$$S_T = \exp\left\{-\frac{1}{2}\int_t^T \sigma_u^2 du + \int_t^T \sigma_u dW_u^{\mathbb{Q}_A}\right\}$$

As a result, the value of the swaption is:

$$V_t = A_t\left[S_t\Phi(d_+) - K\Phi(d_-)\right]$$

where $d_\pm = \frac{\log(S_t/K) \pm \frac{1}{2}\Omega^2}{\Omega}$, and $\quad \Omega = \sqrt{\int_t^T \sigma_u^2 du}.$

The model described above is a Lognormal Swap Rate (LSM) model. Assuming a swaption with strike K equal to today's swap rate S_0, we can determine the black implied volatility such that the LSM price equals the market price. Suppose we hold an IRS with tenure structure $\tau = \{3, 3.25, \ldots, 6\}$ with 3 is the first reset date (no payment) and the first payment is 3.25 after that. Using the formula derived in the methodology on bond pricing, we find that today's swap rate S_0 is 3.28133%, and the Black implied volatility $\Omega/\sqrt{T-t}$ is 0.0011279.

In addition, we investigate the change in Black implied volatility as the strike price increases under the LSM. From Figure (12), we can observe that as the strike price increases, the implied volatility decreases. This can be explained using the relationship between the swaption price and the strike price, as shown in Figure (11). As the strike approaches today's swap rate, the swaption value decreases, suggesting a lower volatility.

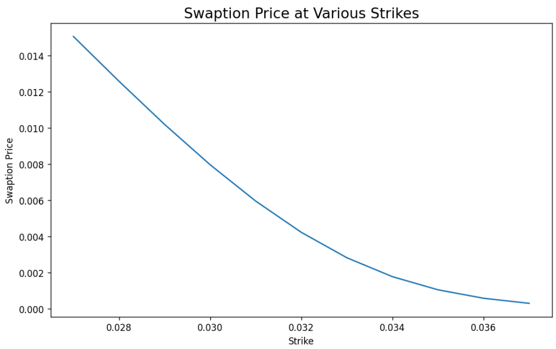

Figure 11: Swaption Price at Various Strikes

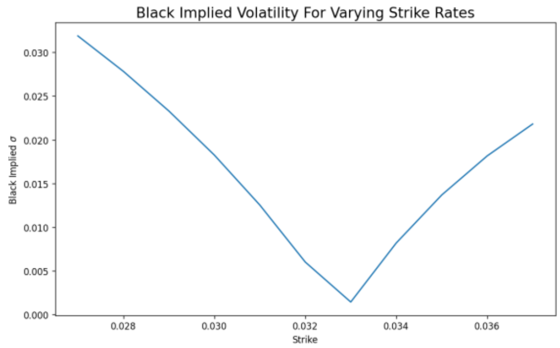

Figure 12: Black Implied Volatility For Varying Strikes Rates

4 Conclusion

In this project, we assume a two-factor Vasicek model to derive the analytical formula of pricing a T-maturity bond. We then plot and investigate the term structure of interest rates against the bonds maturity. We observe that, the interest rates are proportional to the time to maturity converging at a point in time. We also generate interest rates paths using Euler-Scheme discretization and Monte Carlo simulations and compute the yields for maturities ranging from one month to 10 years in steps of one month, and compare with the derived analytical formula using the base parameters. we observe that, the yields from both Monte Carlo estimates and the analytical formula overlap and the confidence bands width is small. This implies the is a small variation between the analytical formula estimates and the Monte Carlo estimates. However, we noted the variation depends on the level of volatility used to generate the Monte Carlo estimates. Moreover, we investigate the role of parameters α, β, σ and η by varying them and determine how the affect the term structure using the base parameters. We observe that, the bonds yield decreases with increase in value of σ, as well increases with increase in the value of η although the slop of the curves increments vary slightly at different points of the bonds maturity. we also note that, the bonds yield is inversely proportional to the bonds maturity. Besides, the yield increase with increase in value of α and the inverse is true for the value of β. Lastly, we examine a 3-year interest rate swap (IRS) by investigating its price and the Black implied volatility. We estimate the value of the IRS under the risk-neutral measure. Furthermore, assuming the swap rate follows the Log-normal Swap Rate (LSR) model, we determine the Black implied volatility and study its relationship with the strike price. We notice that as the strike price increases, the Black implied volatility decreases. However, it is known that implied volatility is known to only decrease due to the assumption that the strike price always equals the current day's swap rate. The major source of error for this are errors in programming.

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