Geometric Measure Theory

Problem Set 2

2.2 - Borel and Suslin Sets

Problem 1. (Borel families)

(a) Show that the collection of all single-point sets in a set X generates the Borel family consisting of all sets that are either at most countable or have at most countable complements.

(b) Show that the Borel family in (a) is strictly smaller than the collection of Borel sets of $\mathbf{R}$ endowed with the standard topology.

(c) Let T be a rotation of $\mathbf{R}^2$ (endowed with the standard topology) about the origin. Show that $A \subset \mathbf{R}^2$ is a Borel set if and only if $T(A)$ is a Borel set.

Problem 2. If ϕ mesaures X, explain why the class of all ϕ measurable sets is a Borel family.

Problem 3. (Support of a measure) Find an example of a measure ϕ over a space with a topology T such that the equation

$$\phi(X \sim \text{spt } \phi) = 0$$

does not hold.

Problem 4. (Borel regularity)

(a) Show that if ϕ is Borel regular and A is a Borel set, then $\phi \llcorner A$ is Borel regular.

(b) If ϕ is any measure over a topological space X such that all Borel subsets of X are ϕ measurable, and if

$$\psi(A) := \inf\{\phi(B) : A \subset B \text{ and } B \text{ is a Borel set}\}$$

whenever $A \subset X$, show that ψ is a Borel regular measure, and that $\psi \equiv \phi$ on the Borel sets of X.

Problem 5. (Theorem 2.2.4, Existence of nonmeasurable subsets) Let ϕ be a Borel regular measure over a complete, separable metric space X, $0 < \phi(A) < \infty$, and $\phi(\{x\}) = 0$ whenever $x \in A$.

(a) Consider the class $\mathit{\Gamma}$ of all closed subsets C of A for which $\phi(C) > 0$. Explain why $\text{card}(C) = 2^{\aleph_0}$, and why $\text{card}(\mathit{\Gamma}) \leq 2^{\aleph_0}$.

(b) Then, prove it follows that there exists a well-ordering of $\mathit{\Gamma}$ such that, for each $C \in \Gamma$, the set $\mathit{\Gamma}_C$ of predecessors of C has cardinal less than $2^{\aleph_0}$.

(c) If A is γ measurable and $\gamma(A) < \infty$, then A is ϕ measurable.

Problem 6*. (Ulam Numbers)

(a) Show that $\aleph_0$ is an Ulam number.

(b) Show that the class of all Ulam numbers is an initial segment in the well ordered class of all cardinal numbers.

(c) Show that if there exist any cardinal numbers which are not Ulam numbers, the smallest such number cannot be accessible.

Additional Exercises

Problem 7*. Define functions $\mu_1, \ldots \mu_6$ on $\mathbf{2}^X$ by

$$\mu_1(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ 1 & \text{if } A \text{ is nonempty,} \end{cases}$$

$$\mu_2(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ +\infty & \text{if } A \text{ is nonempty,} \end{cases}$$

$$\mu_3(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded,} \end{cases}$$

$$\mu_4(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded,} \end{cases}$$

$$\mu_5(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded} \end{cases}$$

$$\mu_6(A) = \begin{cases} 0 & \text{if } A \text{ is countable, and,} \\ +\infty & \text{if } A \text{ is uncountable,} \end{cases}$$

Which of the above functions define measures? For each one that defines a measure, what are the respective measurable subsets of $\mathbf{R}$?

Problem 8. (The 1-Dimensional Lebesgue Measure) For each subset A of $\mathbf{R}$, define the **1-Dimensional Lebesgue Measure**

$$\mathcal{L}^1 : \mathbf{2}^X \to [0, \infty]$$

by

$$\mathcal{L}^1(A) = \inf \sum_i (b_i - a_i)$$

where the infimum is taken over all collections $\mathcal{C}_A = \{(a_i, b_i)\}$ of open intervals whose union $\bigcup_i (a_i, b_i)$ covers A. I.e., $A \subseteq \bigcup_i (a_i, b_i)$. Show that $\mathcal{L}^1(C) = 0$ for every countable subset C of $\mathbf{R}$.