Set Theory Cheat-Sheet

Winter 2022

Contents

1 Partial Orders

For any two sets X and Y, we may form their Cartesian product:

$$X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}.$$

A ***(binary) relation*** on X and Y is a subset $R \subseteq X \times Y$. We use the notation

$$(x, y) \in X \times Y \text{ is written as } xRy.$$

A relation R on a single set X is simply a subset $R \subseteq X \times X$.

Definition 1 (Partial Orders)**.** Let A be a non-empty set and $\preceq, \prec$ be binary relations on A. Then,

(1) The relation $\preceq$ is...

- ***reflexive*** if it satisfies $(\forall a \in A)(a \preceq a)$.
- ***transitive*** if it satisfies $(\forall a, b, c \in A)((a \preceq b) \wedge (b \preceq c) \implies (a \preceq c))$.
- ***weakly antisymmetric*** if it satisfies

$$(\forall a, b \in A)((a \preceq b) \wedge (b \preceq a) \implies (a = b))$$

(2) The relation $\prec$ is...

- ***irreflexive*** if it satisfies $(\forall a \in A)(\neg(a \prec a))$.
- ***antisymmetric*** if it satisfies $(\forall a, b \in A)(a \prec b \implies \neg(b \prec a))$.

A ***pre-order*** on A is a relation on A that is...

- Reflexive
- Transitive

A ***partial order*** on A is a relation on A that is...

- Reflexive
- Transitive
- Weakly antisymmetric

A ***strict partial order*** on A is a relation on A that is...

- Irreflexive
- Transitive
- Antisymmetric

If $\preceq$ is a partial order on a set A, then the relation $\prec$ on A is defined by:

$$a \prec b \iff (a \preceq b) \wedge (a \neq b).$$

If $\prec$ is a strict partial order on a set A, then the relation $\preceq$ on A is defined by

$$a \preceq b \iff (a \prec b) \vee (a = b).$$

For example, assume $B \neq 0$ and let $A = \mathbf{2}^B$, the set of all subsets of B. Consider the relation $\subseteq$ on A, where

$$S_1 \subseteq S_2 \iff (\forall x \in S_1)\, x \in S_2.$$

Then $\subseteq$ is a partial order.

Result 1.

(1) If $\prec$ is a strict partial order on A, then $\preceq$ is a partial order on A.

(2) If $\preceq$ is a partial order on A, then $\prec$ is a strict partial order on A.

Definition 2 (Total/Linear Orders)**.** A partial order $\preceq$ on a set A is called a ***total order*** (or a ***linear order***) if it satisfies the additional property:

$$(\forall a, b \in A)(a \preceq b \text{ or } b \preceq a).$$

A strict partial order $\prec$ on a set A is a ***(strict) total order*** if the associated partial ordering $\preceq$ is total.

Definition 3 (Embeddings and Isomorphisms)**.** Let $(A, \preceq)$ and $(B, \sqsubseteq)$ be partial orders and let $f : A \to B$ be one-to-one (i.e., injective). We say that...

(1) f is an ***embedding*** if for all $a, a' \in A$, we have

$$a \preceq a' \iff f(a) \sqsubseteq f(a')$$

(that is, f is order-preserving!)

(2) f is an ***isomorphism*** if it is a bijective embedding.

Definition 4 (Example of an Ordering)**.** Let ω^ω be the set of all sequences of non-negative integers. We define two binary relations $\leq^+$ and $\leq^*$ on ω^ω. For $f, g \in \omega^\omega$, we let

$$f \leq^+ g \iff (\forall n \in \omega)(f(n) \leq g(n))$$
$$f \leq^\star g \iff (\exists N \in \omega)(\forall n \geq N)(f(n) \leq g(n))$$

where $\leq$ is the natural ordering of ω.

Result 2 (Property of the Example)**.** Suppose that $F \subseteq \omega^\omega$ is countable, i.e., there exists a bijection from ω (set of ***non-negative*** integers) onto F. Then

$$(\exists g \in \omega^\omega)(\forall f \in F)(f \leq^\star g).$$

Proof. Since F is countable, let $F = \{f_0, f_1, f_2, \ldots\}$. Define $g \in \omega^\omega$ by

$$g(n) = 1 + f_0(n) + f_1(n) + \cdots + f_n(n)$$

for $n \in \omega$. Notice that, for all $m, n \in \omega$, we have $f_n(m) \leq g(m)$, since

$$g(m) - f_n(m) = 1 + f_0(m) + \cdots + f_{n-1}(m) + f_{n+1}(m) + \cdots + f_m(m) > 0.$$

Hence $(\forall f \in F)(f \leq^\star g)$. $\square$

1.1 Exercises

Problem 2.1

(1) Let B be a non-empty set. Is $\subseteq$ a total order on B? Why or why not?

(2) Show that $\leq^{+}$ is a partial order on ω^{ω}. Is it total?

(3) Show that $\leq^{*}$ is a pre-order on ω^{ω}. Is it partial?

Problem 2.2 Prove **Result 1**.

Problem 2.3 Let $\mathbf{Q}$ be equipped with the natural inequality $\leq$, and let $(X, \sqsubseteq)$ be a countable linear order. Show that there exists an embedding $f : X \to \mathbf{Q}$.

Problem 2.4 Assume $(X, \subseteq)$ is a pre-order. Define a binary relation on X by

$$a \sim b \iff a \sqsubseteq b \text{ and } b \sqsubseteq a.$$

(a) Show that $\sim$ is an equivalence relation (i.e. transitive, symmetric, and reflexive).

(b) For $A, B \in X/\sim$ (i.e., A and B are equivalence classes of $\sim$) define the relation

$$A \preceq B \iff (\exists a \in A)(\exists b \in B) \text{ such that } a \sqsubseteq b.$$

Show that

(i) $A \preceq B$ if and only if $(\forall a \in A)(\forall b \in B)(a \sqsubseteq b)$, and

(ii) $\preceq$ is a partial order on $X/\sim$.

2 Well Orderings, Initial Segments, and Induction

Definition 5 (Well-founded Relations)**.** Let A be a non-empty set. A relation $R \subseteq A \times A$ is ***well-founded*** provided there is ***no*** infinite sequence $\{a_n : n \in \omega\}$ such that

$$(\forall n \in \omega)(a_n \in A \text{ and } a_{n+1} R a_n).$$

The Regularity Axiom. In ZFC set theory, we assume the following ***Regularity Axiom*** (abbreviated **RA**) holds:

$$(\forall x)[x \neq \emptyset \implies (\exists y \in x)(x \cap y = \emptyset)].$$

Result 3 (Well-founded Relations)**.**

(1) If a relation R is well founded then for every a, b, c:

- $(a, a) \notin R$ and
- $(a, b), (b, c) \in R$ implies $(c, a) \notin R$ (and, more generally, there are no cycles in R).

(2) The following relations are well-founded

- $<$ on ω,
- $\geq$ on $\{-n : n \in \omega\}$, and
- the divisiblity relation on $\omega \backslash \{0, 1\}$:

$$a|b \iff (\exists c \in \omega \backslash \{1\})(a \cdot c = b)$$

Theorem 1 (Regularity Axiom and Well-Foundedness of Membership are Equivalent)**.** Well foundedness of the membership relation $\in$ and **RA** are equivalent.

Definition 6 (Well Orders)**.** A ***well-ordering*** is a well-founded strict linear order. A linear order $(P, \preceq)$ is a ***well-ordering*** if the associated strict linear order $(P, \prec)$ is well-founded.

Result 4 (Characterization of Well-Founded Orders)**.** Let $(P, \leq)$ be a linear order and let $<$ be the associated strict order. Then the relation $<$ on P is well founded if and only if every non-empty subset of P has a smallest element (with respect to the $\leq$ order).

Definition 7 (Initial Segments)**.** Let $(A, \prec)$ be a well-ordering. A set $I \subseteq A$ is an ***initial segment*** of A if

$$(\forall i \in I)(\forall a \in A)(a \prec i \implies a \in I).$$

An initial segment I is ***proper*** if $I \neq A$.

Result 5 (Properties of Initial Segments)**.** Assume $(A, \prec)$ is a well-ordering.

(a) If I is a proper initial segment of A, then for some $b \in A$ we have

$$I = \{a \in A : a \prec b\}.$$

(b) If I, J are initial segments of A, then either $I \subseteq J$ or $J \subseteq I$.

Proof. Exercise. □

Theorem 2 (Transfinite Induction Theorem)**.** Let $(A, \prec)$ be a well-order and $\Psi(x)$ be a formula with domain that includes A. Suppose that for all $a \in A$ we have

$$(\forall b \prec a)(\Psi(b)) \implies \Psi(a).$$

Then $(\forall a \in A)(\Psi(a))$ holds true.

Proof. Suppose for contradiction that the conclusion fails. I.e., suppose for contradiction that there is an $a \in A$ such that $\Psi(a)$ is not true. Then the set $Z = \{b \in A : \Psi(b) \text{ fails}\}$ is not empty. Take the first $\prec$-element b_0 of the set Z (by the characterization of well-founded orders!). By this choice, the statement

$$(\forall b \prec b_0)(\Psi(b))$$

is true, so by our assumption on Ψ we conclude that $\Psi(b_0)$ is true. But, this contradicts $b_0 \in Z$. □

Theorem 3 (Transfinite Recursion Theorem)**.** Suppose that $(A, \prec)$ is a well order, X is a non-empty set, and $\mathcal{X}$ is the set of all functions f such that $\text{dom}(f) \subseteq A$ and $\text{range}(f) \subseteq X$. Let $\mathbb{G} : \mathcal{X} \to X$. Then, there is a unique function $F : A \to X$ such that

$$(\forall a \in A)(F(a) = \mathbb{G}(F|_{\{b \in A : b \prec a\}})).$$

Transfinite recursion is similar to transfinite induction; however, instead of proving that something holds for all ”things” (numbers, ordinals, etc.), we construct a sequence of objects, one for each ”thing” (number, ordinal, etc.).

Definition 8. Two sets X and Y are ***equinumerous*** if there exists a bijection f from X onto Y.

Axiom of Choice (AC). Every set can be well-ordered. Moreover, for every set X there is a binary relation $\prec$ on X such that

- $(X, \prec)$ is a well-ordering, and
- no proper initial segment $(X, \prec)$ is equinumerous with X.

Result 6. A well-ordering is not isomorphic to any of its proper initial segments.

Proof. Suppose that $(A, \prec)$ is a well ordering, I is a proper initial segment of A, and $(I, \prec)$ and $(A, \prec)$ are isomorphic. Let $\pi : I \to A$ be a bijection such that $(\forall a, b \in I)(a \prec b \iff \pi(a) \prec \pi(b))$. Let $B = \{x \in I : \pi(x) \neq x\}$. Note that π cannot be the identity mapping on I since $I \not\subseteq A$, so B is non-empty. Thus, B has a $\prec$-least element b. Then,

(i) $\pi(b) \neq b$, and

(ii) if $x \prec b$, then $x \in I \setminus B$ so $\pi(x) = x$.

If $\pi(b) \prec b$, then by (ii) we have $\pi(\pi(b)) = \pi(b)$. But, since $b \neq \pi(b)$ by (i), there arises a contradiction from the fact that π is one-to-one. Hence, $b \prec \pi(b)$. Since π is onto A, for some $c \in I$ we have $b = \pi(c)$ and thus $\pi(c) = b \prec \pi(b)$. Since π is an order isomorphism we conclude that $c \prec b$. This implies that, by (ii), $\pi(c) = c$ - a contradiction. This completes the proof that a well ordering is not isomorphic to any of its proper initial segments. □

Result 7. Suppose $(A_0, \prec_0)$ and $(A_1, \prec_1)$ are well-orderings. Then ***exactly ONE*** of the following cases holds.

(i) The orders $(A_0, \prec_0)$ and $(A_1, \prec_1)$ are isomorphic.

(ii) The order $(A_0, \prec_0)$ is isomorphic with a unique proper initial segment of $(A_1, \prec_1)$.

(iii) The order $(A_1, \prec_1)$ is isomorphic with a unique proper initial segment of $(A_0, \prec_0)$.

2.1 Exercises

Problem 2.5 Suppose that $A \subseteq \mathbf{R}$ is uncountable. Demonstrate that there exist $a_0, a_1, a_2, \dots \in A$ such that $a_{n+1} < a_n$ for all $n \in \omega$. That is, $(A, <)$ is not well founded.
Problem 2.6 Suppose that a set X can be well ordered. Show that then there is a well ordering $\prec$ of X such that no proper initial segment $(X, \prec)$ is equinumerous with X.
Problem 2.7 Let V be a vector space over $\mathbf{R}$. Recall that a basis for V is a set $B \subseteq B$ such that

- B is linearly independent, i.e., no nontrivial linear combination of finitely many elements of B is the zero element 0, and
- the linear span of B is all of V, i.e., every vector in V is a finite linear combination of the elements of B.

Use the transfinite recursion theorem to show that there exists a base for V.
Problem 2.8 Show that there exists a function $f : \mathbf{R} \to \mathbf{R}$ such that

- $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$, but
- f is not continuous anywhere.

3 Ordinals and Cardinals

Here, ”canonical well-orderings” are introduced, AKA ***ordinals***. Every well-ordering is isomorphic with (exactly one) ordinal.

Definition 9. A set α is called an ***ordinal*** if

(1) α is transitive, i.e., $(\forall \beta \in \alpha)(\beta \subseteq \alpha)$, and

(2) $(\forall \beta, \gamma \in \alpha)(\beta = \gamma$ or $\beta \in \gamma$ or $\gamma \in \beta)$.

Result 8 (Characterization of Ordinals)**.** A set α is an ordinal if and only if α is a transitive set and $(\alpha, \in)$ is a well-ordering.

Proof. Assume α is an ordinal. By property (1) we know that α is transitive. So, we will show that $(\alpha, \in)$ is a well-ordering.

First, suppose $\beta, \gamma, \delta \in \alpha$ are such that $\beta \in \gamma$ and $\gamma \in \delta$. Since the membership relation $\in$ is well founded, we cannot have $\beta = \delta$ nor $\delta \in \beta$. Therefore, it follows from Property (2) of ordinals that $\beta \in \delta$. Hence, the relation $\in$ is transitive on α (This also shows that every member of α is a transitive set.)

We know that $\in$ is antisymmetric and irreflexive as a relation because of **RA**. Therefore, $(\alpha, \in)$ is a strict partial order. It is a total ordering on α by virtue of Property (2) of ordinals. And, $\in$ is well founded by **RA** and **Theorem 1** (equivalence of well-foundedness of $\in$ and **RA**).

Conversely, suppose that α is a transitive set and $(\alpha, \in)$ is a well ordering. Then, the first property of ordinals holds by our assumption and the second follows from the totality of the order. $\square$

Result 9 (Properties of Ordinals)**.**

(1) $\emptyset$ is an ordinal.

(2) If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.

(3) If I is an initial segment of an ordinal α, then $I \in \alpha$ or $I = \alpha$.

(4) If α, β are ordinals and $\beta \not\subseteq \alpha$, then $\beta \in \alpha$.

(5) If α, β are ordinals then either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.

(6) If A is a set of ordinals, then $\bigcup A$, called $\sup(A)$, is an ordinal.

(7) IF α is an ordinal then $\alpha \cup \{\alpha\}$ ***called*** $\alpha + 1$, is an ordinal.

Definition 10 (Ordinal Ordering)**.** For ordinals α, β we write $\alpha < \beta$ whenever $\alpha \in \beta$.

Remark.

(1) For ordinals α, β we have that
$$\alpha < \beta \iff \alpha \not\subseteq \beta.$$

(2) The binary relation $<$ is a well founded linear order on ordinals *except* that the ordinals do not constitute a set. (For every *set* of ordinals there is an ordinal not belonging to it, $\sup(A) \notin A$.)

(3) Ordinals represent all possible well orderings. To be precise: If $(X, \prec)$is a well ordering, then there exists a unique ordinal α such that the well orderings $(X, \prec)$ and $(\alpha, \in)$ are isomorphic.

(4) Using the above result we define ***the cardinality of a set*** X as the minimal ordinal equinumerous with X. Ordinals which are not equinumerous with smaller ordinals are called ***cardinals*** or ***initial ordinals***. Thus the cardinality of X is the unique cardinal equinumerous with X. The cardinality of the real number line $\mathbf{R}$ is typically denoted $\mathfrak{c}$ (so this is the smallest ordinal equinumerous with $\mathbf{R}$.)

Definition 11 (Whole numbers)**.** Whole numbers are finite ordinals defined by the recursive rule

$$0 = \emptyset, \qquad n + 1 = n \cup \{n\}.$$

- The ***first infinite ordinal*** is called ω_0 and it is

  $$\omega_0 = \{0, 1, 2, 3, 4, \dots\}.$$

  If we are interested in cardinality, we call this set $\aleph_0$.

- The first uncountable ordinal is called ω_1, in terms of cardinality, this is $\aleph_1$. The ***continuum hypothesis*** states that $\aleph_1 = \mathfrak{c}$.

- The first uncountable ordinal which is not equinumerous with ω_1 is called ω_2. In terms of cardinality, this is $\aleph_2$.

Definition 12 (Successor and Limit Ordinals)**.**

(1) A ***successor ordinal*** is an ordinal β such that $\beta = \alpha + 1$.

(2) An ordinal β is called a ***limit ordinal*** if $\beta \neq 0$ and β is not a successor ordinal.

Theorem 4 (Stefan Mazurkiewicz)**.** There exists a set $B \subseteq \mathbf{R}^2$ which intersects every straight line in exactly two points.

Proof. Fix a list $\{\ell_\alpha : \alpha < \mathfrak{c}\}$ of all straight lines in the plane.

By induction on $\alpha < \mathfrak{c}$, we construct sets $S_\alpha \subseteq \mathbf{R}^2$ so that the following inductive demands are satisfied:

$(*)_\alpha$ the union $\bigcup_{\beta \le \alpha} S_\beta$ contains exactly two points from the line ℓ_α

$(**)_\alpha$ the union $\bigcup_{\beta \le \alpha} S_\beta$ contains no three colinear points, and

$(***)_\alpha$ S_α has at most two elements.

Suppose that we have defined S_β for $\beta < \alpha$ so that the demands $(*)_\beta - (***)_\beta$ are satisfied. Let $A = \bigcup_{\beta < \alpha} S_\beta$ and note that

$(\otimes)_1$ the set A contains no three colinear points and

$(\otimes)_2$ A of cardinality smaller than the continuum $\mathfrak{c}$, so

$(\otimes)_3$ the family $\mathcal{L}$ of all straight lines passing through two points of A is of size smaller than $\mathfrak{c}$.

If the intersection $\ell_\alpha \cap A$ has exactly two points, then we set $S_\alpha = 0$. If this intersection has one point, then we may use $(\otimes)_3$ to choose a point $x \in \ell_\alpha$ such that $x \notin \ell$ for $\ell \in \mathcal{L}$. We set $S_\alpha = \{x\}$. Finally, if $\ell_\alpha \cap A = \emptyset$, then we use $(\otimes)_3$ to choose two distinct points $x, y \in \ell_\alpha$ such that $x, y \notin \ell$ for all $\ell \in \mathcal{L}$ and we set $S_\alpha = \{x, y\}$. One easily verifies that in each case the demands $(*)_\alpha - (***)_\alpha$ are satisfied.

After the above construction is carried out, we let $B = \bigcup_{\alpha < \mathfrak{c}} S_\alpha$. It follows from $(**)_\alpha$ (for $\alpha < \mathfrak{c}$) that the set B contains no three colinear points. By $(*)_\alpha$ (for $\alpha < \mathfrak{c}$) it intersects each line at exactly two points. $\square$

3.1 Exercises

Problem 2.9 Expand the definition of $\omega_0, \omega_1, \omega_2$ to define $\omega_3, \omega_4, \ldots, \omega_\omega$ and generally ω_α for every ordinal α.

Problem 2.10 We have defined $\alpha + 1$ for every ordinal α in this section. Use induction to expand this definition and introduce $\alpha + \beta$ for any ordinals α, β.

Problem 2.11 Show that is $\alpha = \beta + 1$ then there is no ordinal γ satisfying $\beta < \gamma < \alpha$.

Problem 2.12 For a set $B \subseteq \mathbf{R}^2$ and $x, y \in \mathbf{R}$, define

$$B_x = \{z \in \mathbf{R} : (x, z) \in B\} \qquad \text{and} \qquad B^y = \{z \in \mathbf{R} : (z, y) \in B.\}$$

Show that there exists a set $B \subseteq \mathbf{R}^2$ such that for each $x, y \in \mathbf{R}$:

- the set B_x has exactly one element, and
- the set B^y intersects every non-trivial open interval.

4 Cofinality and Regular and Singular Ordinals

This section is to help you understand topics in Zariski's paper, referenced on page 59 of Federer and kindly translated into English by etropy. This paper justifies the claim Federer makes that the exclusion of inaccessible cardinals is consistent with ZFC, and so we can assume all cardinalities in GMT are Ulam numbers.

Definition 13 (Cofinal subsets, cofinality)**.** Let $(A, \leq)$ be a pre-ordered set.

- A subset $B \subseteq A$ is called ***cofinal*** in A if for every $a \in A$, there exists a $b \in B$ such that $a \leq b$.
- If $\leq$ is a partial order, then the ***cofinality*** $\mathrm{cf}(A)$ of A is the least of the cardinalities of the cofinal subsets of A.

Result 10 (Properties of Cofinal Subsets)**.**

(1) Any partially ordered set (poset) is cofinal in itself.

(2) If B is a cofinal subset of a poset A, and C is a cofinal subset of B (with the partial ordering of A applied to B), then C is also a cofinal subset of A.

(3) If A is a poset with a ***maximal element*** (i.e., there exists $m \in A$ such that there is no $a \in A$ such that $m < a$), then every cofinal subset of A has a maximal element.

(4) For a partially ordered set with a ***greatest element*** (i.e., there exists $m \in A$ such that for all $a \in A$, $a \leq m$, note since the relation is only partial, this is different from the maximal element), a subset of A is cofinal if and only if it contains a greatest element.

Proof. Exercise. □

We have some cofinal subsets of $\mathbf{R}$. For any $-\infty < x < \infty$, the interval (x, ∞) is easily verified to be a cofinal subset of $(\mathbf{R}, \leq)$ but it is ***not*** a cofinal subset of $(\mathbf{R}, \geq)$. The set of naturals $\mathbf{N}$ is a cofinal subset of $(\mathbf{R}, \leq)$, but this is not true of the set of negative integers. You can easily formulate the dual result for $(-\infty, y)$ for $-\infty < y < \infty$.

Definition 14 (Regular and Singular Cardinals)**.** A ***regular cardinal*** is a cardinal number that is equal to its own cofinality. That is, α is a regular cardinal if and only if every unbounded subset $C \subseteq \alpha$ has cardinality α. An infinite well-ordered cardinal that is not a regular cardinal is called a ***singular cardinal***.

Result 11. Assuming **AC**, any cardinal number can be well-ordered, and then the following are equivalent for a cardinal α:

(1) α is a regular cardinal.

(2) If $\alpha = \sum_{i \in I} \beta_i$ and $\beta_i < \alpha$ for all i, then $\mathrm{card}\, I \geq \alpha$.

(3) If $S = \bigcup_{i \in I} S_i$ and if $\mathrm{card}\, I < \alpha$ and $\mathrm{card}\, S_i < \alpha$ for all i, then $\mathrm{card}\, S < \alpha$.

(4) α is a ***regular ordinal***: That is, any infinite ordinal α is a ***regular ordinal*** if it is a limit ordinal that is not the limit of a set of smaller ordinals that have a set order isomorphic to a set less than α.

Remarks.

(1) The ordinals less than ω (set of whole numbers) are finite. A finite sequence of finite ordinals always has a finite maximum, so ω cannot be the limit of any sequence of type less than ω whose elments are ordinals less than ω, and is therefore a regular ordinal.

(2) $\aleph_0$ is a regular cardinal because it is the cardinal sum of a finite number of finite cardinal numbers (and so is, itself finite).

(3) The next ordinal number greater than ω, $\omega + 1$, is singular (since it is not a limit ordinal).

(4) $\omega + \omega$ is the next limit ordinal after ω. It can be written as the limit of the sequence $\omega, \omega + 1, \omega + 2, \ldots$.

(5) $\aleph_1$ is regular, and is the next cardinal number greater than $\aleph_0$, so cardinals less than $\aleph_1$ are countable (finite or denumerable). Assuming **AC**, the union of a countable set of countable sets is itself countable. So, $\aleph_1$ cannot be written as the sum of a countable set of countable cardinal numbers, and so it is regular.

(6) **Important**. Uncountable (weak) ***limit cardinals*** that are also regular are known as (weakly) ***inaccessible cardinals***. They cannot be proved to exist within ZFC, though their existence is not known to be inconsistent with ZFC.

(7) Every successor cardinal is regular.

As an exercise, you could verify (7) above.