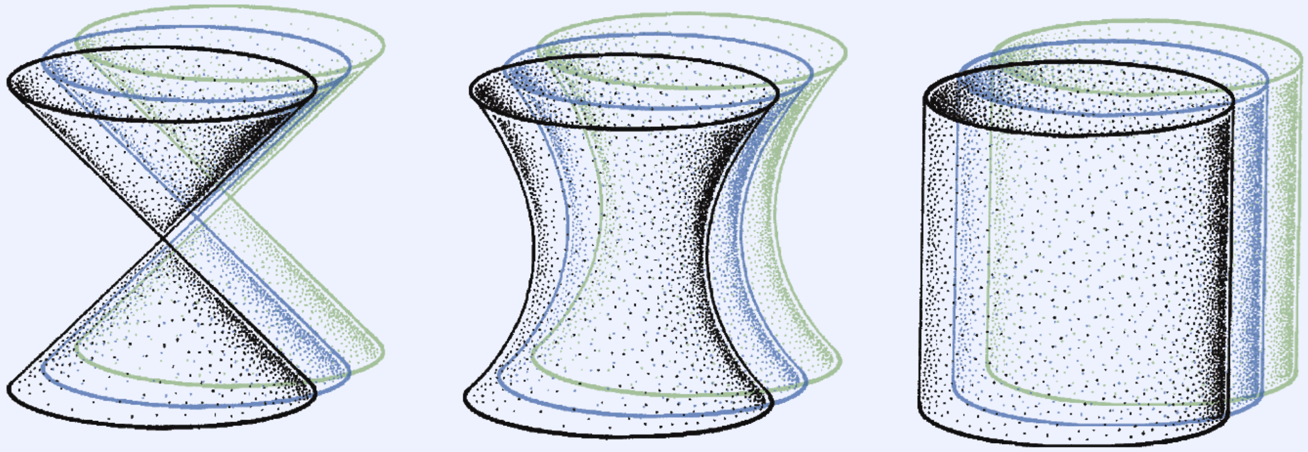
Geometric Measure Theory

An Introduction to the Theory of Inconvenient Surfaces

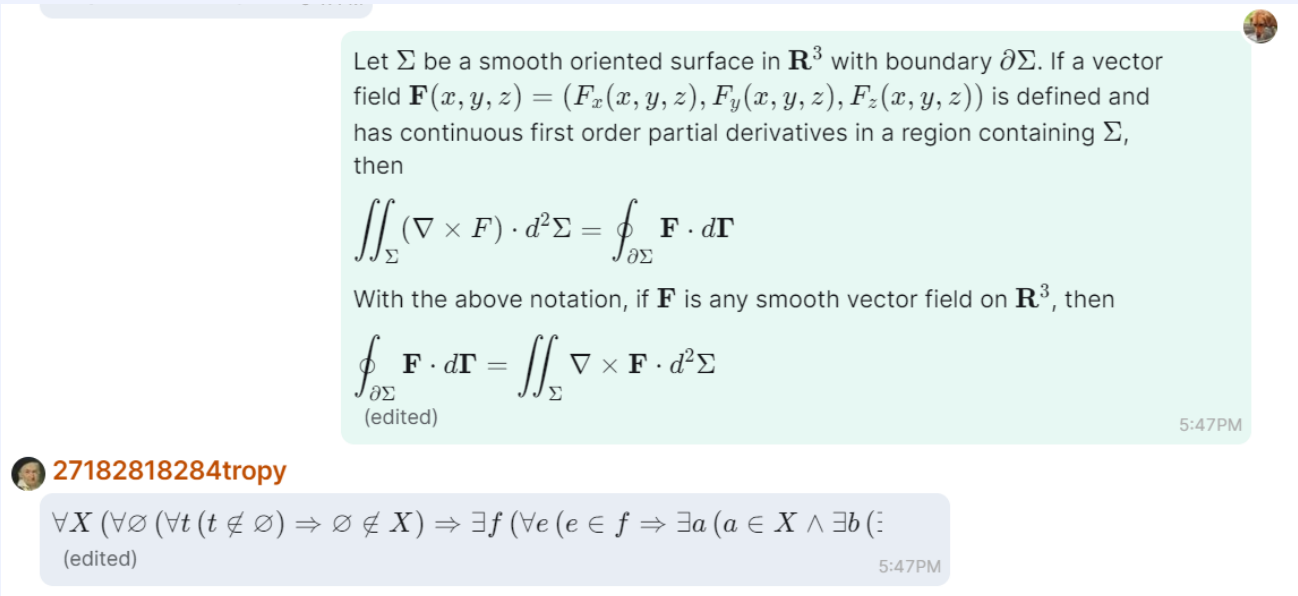
Saturday, December 10, 2022



Course Website

largoscv.github.io

Element.io LaTeX Integration



To Enable LaTeX Rendering:

1. Navigate to:
   - Linux:
     cd ~/.config/Element/
   - Windows:
     C:/Users/user/Appdata/Roaming/Element
   - MacOS:
     ~/Library/Application Support/Element

2. Add/replace config.json with contents in

   https://pastebin.com/8wV9nGR3

3. Restart Element (Task Manager). Then go to All Settings > Labs > “Render LaTeX Maths”

Basic Outline

GMT Part 1: 10 December 2022 to 21 January 2023

- 6 week course in general measure theory a la Carathéorory
- Concepts from geometric measure theory & applications (minimal surfaces) taught in lecture
- 6 homework assignments, first HW due 17 December (3:30pm EST)
- Pace: One section of Federer a week (~10-15 pgs/wk)

Saturday

10

December

- Introduction to GMT
- Relations & Orders
- Transfinite Induction
- Ordinals and Cardinals

Wednesday

14

December

- Study Session

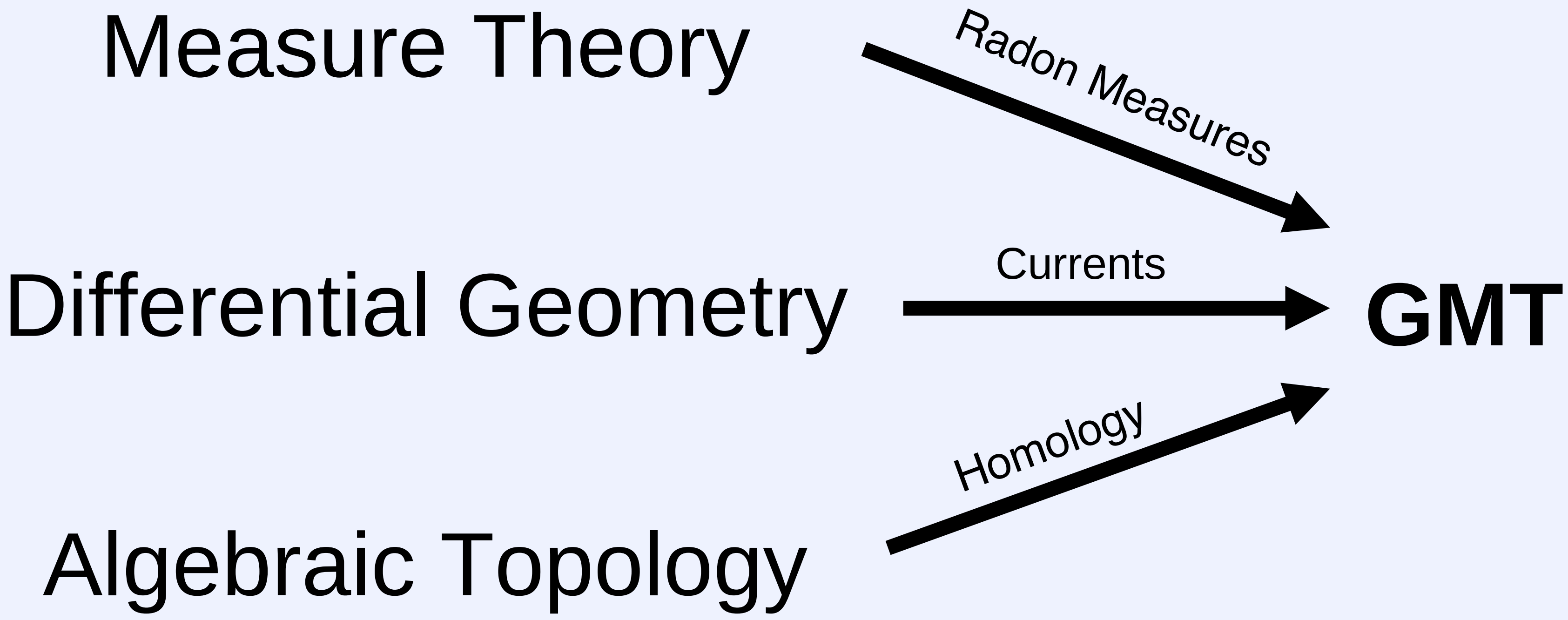
Saturday

17

December

- Hausdorff Measure
- Densities
- Approximate limits
- Approximate continuity

Geometric Measure Theory


Measure Theory
Radon Measures
Differential Geometry
Currents
GMT
Algebraic Topology
Homology

Joseph-Louis Lagrange (1736-1813)

- Given a smooth closed curve γ in $\mathbf{R}^3$, does there exist a surface S of least area such that
$$\partial S = \mathrm{graph}(\gamma)?$$

- Such minimal surfaces may be modeled using soap films, as done by Joseph Plateau (1801-1883). Hence, this problem is known as **Plateau's Problem**.

- Solved by Douglas & Rado (1930) under topological restrictions, Federer & Fleming (1960) used the theory of currents to solve the problem without topological restrictions (orientable version).

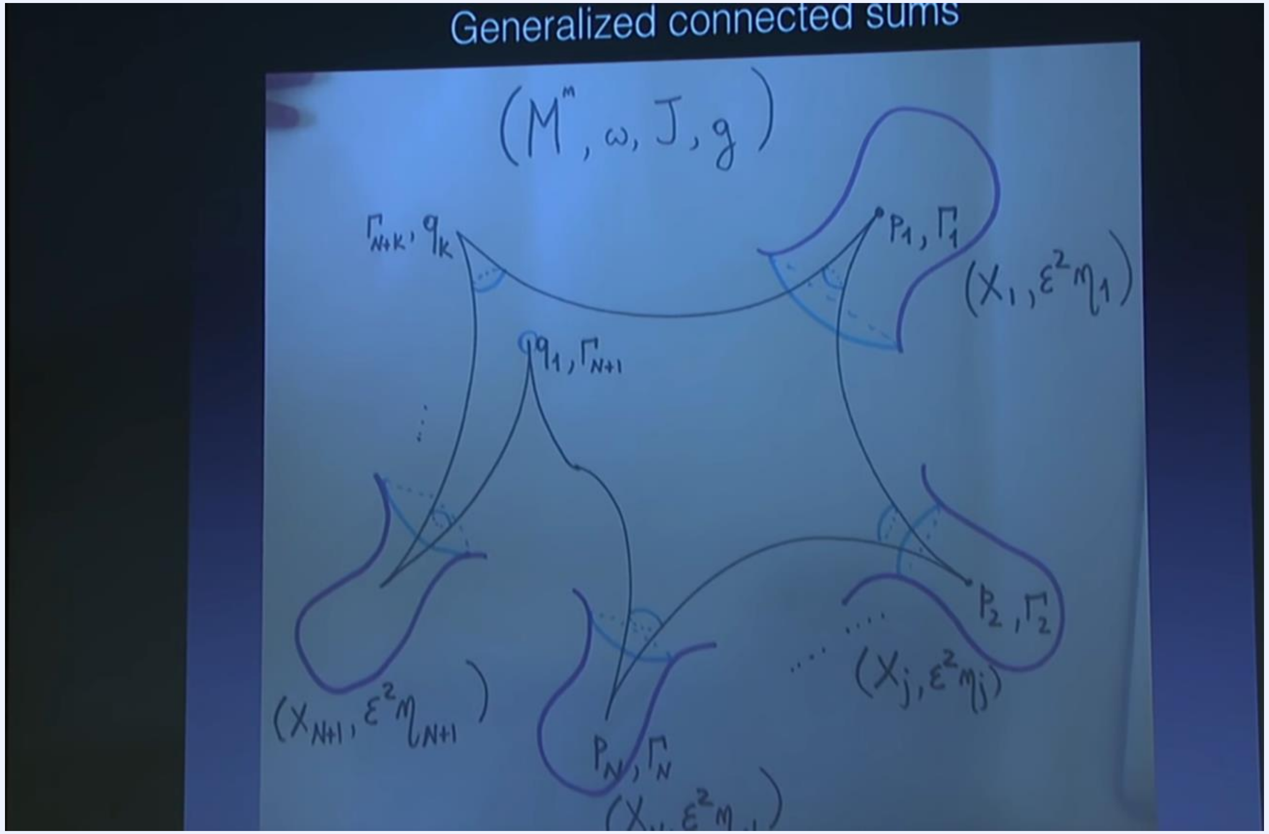
Henry Segerman, 2015

Plateau’s Problem in Higher Dimensions

- Any solution of Plateau’s Problem in $\mathbf{R}^3$ is a “convenient surface” (no corners, tangent plane that exists everywhere).

- In higher dimensions, i.e., $\mathbf{R}^n$, there are solutions of Plateau’s Problems that have singularities (points with no tangent space)

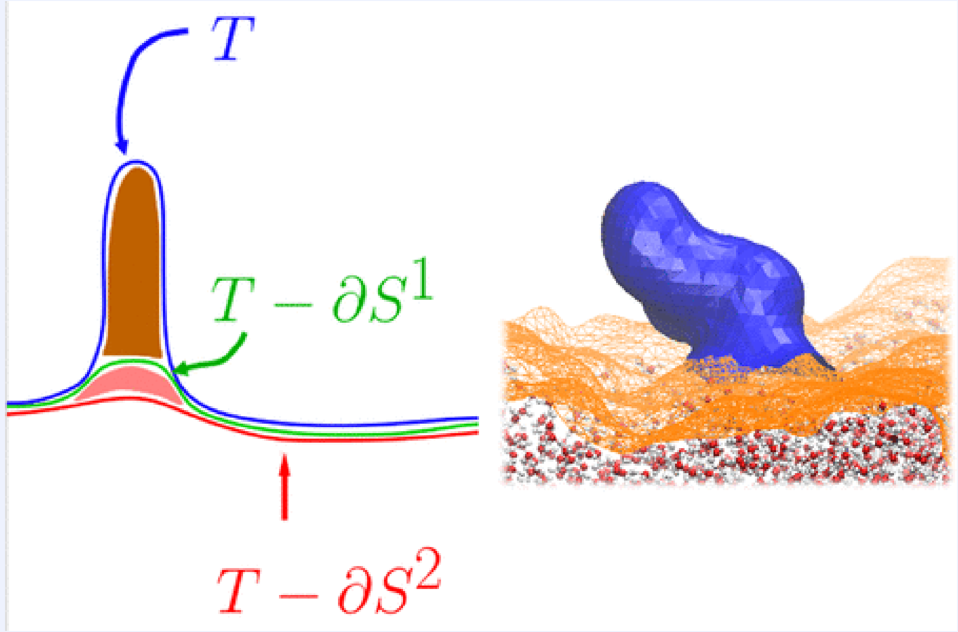
- To study Plateau’s Problem in $\mathbf{R}^n$, we need:
    - Theory of Caccioppoli Sets (De Giorgi)
    - Theory of Rectifiable Currents (Federer & Fleming)



Claudio Arezzo, 2019

GMT Sans Minimal Surfaces

- Partial Differential Equations (Evans)
- Several Complex Variables (Krantz)
- Calculus of Variations
- Differential & Riemannian Geometry



- Continuum Mechanics, Cauchy's Stress Theory (Falach, 2013)
- Analysis of Soft Matter Surfaces (Alvarado et. al., 2020)

Background on ZFC Set Theory

Binary Relations

Given sets X and Y, the Cartesian product $X \times Y$ is the set

$$\{(x, y): x \in X \text{ and } y \in Y\}$$

A **binary relation** over X and Y is a subset $R \subseteq X \times Y$.

If $(x, y) \in R$, we may write xRy.

Let A be a non-empty set and $\preccurlyeq, \prec$ be binary relations on A.

The relation $\preccurlyeq$ is…

Reflexive if $(\forall a \in A)(a \preccurlyeq a)$

Transitive if $(\forall a, b, c \in A)$,

$$(a \preccurlyeq b) \wedge (b \preccurlyeq c) \Rightarrow (a \prec c)$$

Weakly antisymmetric if $(\forall a, b \in A)$,

$$((a \preccurlyeq b) \wedge (b \preccurlyeq a) \Rightarrow (a = b)$$

A **pre-order** on A satisfies the first two properties. A **partial order** satisfies all three.

The relation $\prec$ is…

Irreflexive if $(\forall a \in A)(\neg(a \prec a))$

Antisymmetric if $(\forall a, b \in A)$,

$$(a \prec b) \Rightarrow (\neg(b \prec a))$$

A **strict partial order** on A is an irreflexive, transitive, and antisymmetric relation.

Examples

Total Orders

A partial order $\preccurlyeq$ on a set A is called **total** (or a linear order) if it satisfies the additional property

$$(\forall a, b \in A)(a \preccurlyeq b \ \vee \ b \preccurlyeq a)$$

A strict partial order $\prec$ on a set A is a (strict) total order if the associated partial ordering $\preccurlyeq$ is total.

Well Orders

A **well-order** on A is a total order $\prec$ on A such that every non-empty subset of A has a $\prec$-least element. That is,

$$(\forall B \subseteq A \text{ nonempty})(\exists x \in B)(\forall b \in B)$$

$$x \prec b$$

Well-Ordering Theorem:

Every set can be equipped with a well-order.

(Equivalent to the Axiom of Choice)

Initial Segments

Given a set A equipped with a well order $\prec$, a set $I \subseteq A$ is…

- An **initial segment** of A if

$$(\forall i \in I)(\forall a \in A)(a \prec i \Rightarrow a \in I)$$

- A **proper initial segment** if I is an initial segment and $I \neq A$.

Induction on Well Orderings

Theorem. Given a set A equipped with a strict well order $\prec$, and $\Psi(x)$ a property defined for all $x \in A$. If for all $a \in A$, we have that

$$(\forall b \prec a)\big(\Psi(b)\big) \Rightarrow \Psi(a)$$

then $(\forall a \in A)\big(\Psi(a)\big)$ holds true.

Proof. Straightforward.

The Regularity Axiom:

$$(\forall x)[x \neq \emptyset \Rightarrow (\exists y \in x)(x \cap y = \emptyset)]$$

(Every non-empty set contains an element that is disjoint from it)

Ordinals

A set α is called an **ordinal** if

1. α is transitive, i.e., $(\forall\beta \in \alpha)(\beta \subseteq \alpha)$, and

2. $(\forall\beta, \gamma \in \alpha)(\beta = \gamma \ \vee \ \beta \in \gamma \ \vee \ \gamma \in \beta)$

Theorem. A set α is an ordinal if and only if α is a transitive set and $(\alpha, \in)$ constitutes a well-ordering.

Proof. Regularity axiom.

Properties of Ordinals I

Properties of Ordinals II

For ordinals α, β, we write $\alpha < \beta$ whenever $\alpha \in \beta$.

Natural numbers

Naturals are finite ordinals defined by the recursive rule

$$0 = \emptyset, \qquad n + 1 = n \cup \{n\}$$

The first infinite ordinal is called ω_0 and it is

$$\omega_0 = \{0, 1, 2, 3, 4, \dots\}$$

The first uncountable ordinal is called ω_1, and the first uncountable ordinal which is not equinumerous with ω_1 is called ω_2.

In terms of cardinality, we call those sets $\aleph_0$, $\aleph_1$, and $\aleph_2$ respectively.

Ordinal Arithmetic

The **successor ordinal** β of an ordinal α is given by

$$\beta = \alpha \cup \{\alpha\} = \alpha + 1$$

An ordinal β is called a **limit ordinal** if $\beta \neq 0$ and β is not a successor.

Example

Transfinite Induction

Theorem. Let $\Psi(x)$ be a property defined for all ordinals α. If we have that, for every ordinal α,

$$\Psi(\beta) \text{ is true for all } \beta < \alpha \Rightarrow \Psi(\alpha) \text{ is true}$$

Then Ψ holds for all ordinals.

How to Induct Transfinitely

To prove a property Ψ on ordinals:

1. Base case: Prove that $\Psi(0)$ holds.

2. Successor case: Prove that for any successor ordinal $\alpha + 1$,

$$\Psi(\alpha) \Rightarrow \Psi(\alpha + 1)$$

3. Limit case: Prove that for any limit ordinal β, $\Psi(\beta)$ follows from $\Psi(\alpha)$ for all $\alpha < \beta$.