Translator's note

This is a translation of Alfred Tarski's paper "Über unerreichbare Kardinalzahlen.", originally in German, published 1938 in the journal "Fundamenta Mathematicae", available online under the DOI of "10.4064/FM-30-1-68-89".
I am not a professional translator nor do qualify as one, I translated the paper solely for the purpose of its use in an English reading group of geometric measure theory I took-part in, given I happen to speak both German and English acceptably well.
I tried to stay true to the original paper as much as I was able to without using too obscure terminology due to the time and language difference, though a reader should beware that as pointed out above my translation is far from perfect. I hence appologize to Tarski for partially butchering a translation of his paper.
It might be beneficial to point out the use of the notation

$$\underset{x}{\mathrm{E}}[x \mid \varphi(x)]$$

in the paper, in place of the modern notation $\{x \mid \varphi(x)\}$.
Further $X non \in Y$ is old notation for what is modernly denoted $X \notin Y$, i.e. the negation of that $X \in Y$.

Regarding inaccesible cardinal numbers

Alfred Tarski(Warszawa)

1938

Already for a long time has one in set theoretic investigations encountered the notion of the inaccessible cardinal number (though it was only recently that the naming "*inaccessible cardinal*" was introduced into the specialized literature)[1]). To begin however one viewed the inaccessible numbers rather as a curiosity; so e.g. expresses Hausdorff in his famous work *Grundzüge der Mengenlehre* the opinion, these numbers are of such an "exorbitant" size, that they for the common mathematical purposes of set theory will barely ever come to be regarded[2]). Only later was the significance for foundational questions of the regarded notion recognized[3]). And in the last couple of years it turned out, that the inaccessible numbers also were not insignificant for some important problems of set theory, and even play a key role in some investigations[4]). For these reasons does it appear today worth the effort, to investigate the notion of inaccessible cardinal number more precisely; the following essay should be a contribution to this problem.

§ 1. Definition and characteristic properties of the inaccessible cardinal numbers.

The present executions will be grounded upon the Zermelo – Fraenkel axiomatic system[5]). Should the operating with cardinal numbers be made possible on the basis of said system, then one can proceed as follows. Regard the notion "*the cardinal number* (or *cardinality*) *of the set M*", in symbols "$\overline{\overline{M}}$", as a new fundamental notion and add two new axioms into the system, namely:

Axiom I. For each set M there corresponds a $\mathfrak{m}$ such that $\overline{\overline{M}} = \mathfrak{m}$.

Axiom II. Two arbitrary sets M and N are equinumerous if and only if $\overline{\overline{M}} = \overline{\overline{N}}$.

One could get by freely without these axioms: however one would have to always speak about the sets themselves rather than about their cardinalities.

We define now:

Definition 1. *The cardinal number* $\mathfrak{m}$ *is called in the* broader sense inaccessible, *whenever* $\mathfrak{m}$ *is different from* 0 *and suffices the following conditions:*

$\mathscr{B}_1$. *In case* X *is an arbitrary set of cardinality* $< \mathfrak{m}$ *and to each element* $x \in X$ *a cardinal number* $\mathfrak{n}_x < \mathfrak{m}$ *is associated, so follows*

$$\sum_{x \in X} \mathfrak{n}_x < \mathfrak{m};$$

$\mathscr{B}_2$. *If* $\mathfrak{n} < \mathfrak{m}$, *so there exists a cardinal number* $\mathfrak{p}$ *such that* $\mathfrak{n} < \mathfrak{p} < \mathfrak{m}$.

Definition 2. *The cardinal number* $\mathfrak{m}$ *is called in the* narrower sense inaccessible, *whenever* $\mathfrak{m}$ *is different from* 0 *and suffices the condition* $\mathscr{B}_1$ *as well the following condition:*

$\mathscr{B}_3$. *If* $\mathfrak{n} < \mathfrak{m}$ *and* $\mathfrak{p} < \mathfrak{m}$, *so* $\mathfrak{n}^{\mathfrak{p}} < \mathfrak{m}$ [6]*).*

We now give with a series of theorems, which are partially completely elementary, various characteristic properties of both kinds of inaccessible numbers.

Auxiliary Theorem 1. *The cardinal number* $\mathfrak{m}$ *suffices the condition* $\mathscr{B}_1$ *if and only if* $m \leq 2$ *or there exists an ordinal number* α *such that* $\mathfrak{m} = \aleph_\alpha$ *and with that* ω_α *is a regular initial number.*

Proof. Out of the enlightening formula: $\mathfrak{m} = (\mathfrak{m} - 1) + 1$ one sees immediately, that no finite $\mathfrak{m} > 2$ suffices the condition $\mathscr{B}_1$. If $\mathfrak{m}$ is infinite, so then because of the well-ordering theorem one can regard $\mathfrak{m}$ as an aleph: $\mathfrak{m} = \aleph_\alpha$. One can further choose a sequence of numbers $\mathfrak{n}_\xi$ of type $\omega_{cf(\alpha)}$ in a way such that

$$\aleph_\alpha = \sum_{\xi < \omega_{cf(\alpha)}} \mathfrak{n}_\xi \textit{ and } \mathfrak{n}_\xi < \mathfrak{m} \textit{ for each } \xi < \omega_{cf(\alpha)} \text{ [7]}). \tag{1}$$

If now ω_α were singular, that is $cf(\alpha) < \alpha$, so then we'd have

$$\aleph_{cf(\alpha)} = \overline{\overline{\underset{\xi}{\mathrm{E}}[\xi < \omega_{cf(\alpha)}]}} < \aleph_\alpha,$$

and the formulae (1) would contradict $\mathscr{B}_1$.

So if $\mathfrak{m} = \aleph_\alpha$ were to satisfy this condition, so ω_α must be regular. Equally easily one can show that as well the numbers $0, 1$ and 2 as well any number $\mathfrak{m} = \aleph_\alpha$ with regular initial number ω_α suffice the condition $\mathscr{B}_1$. Auxiliary theorem 1 hence applies in both directions.

Auxiliary Theorem 2. *The cardinal number $\mathfrak{m}$ suffices the condition $\mathscr{B}_2$ if and only if $\mathfrak{m} = 0$ or when there exists an ordinal number α such that $\mathfrak{m} = \aleph_\alpha$ and with that α is 0 or a limit ordinal.*

Proof : obvious when considering the well-ordering theorem

Theorem 3. *The cardinal number $\mathfrak{m}$ is in the broader sense inaccessible if and only if there exists an ordinal α such that $\mathfrak{m} = \aleph_\alpha$ and with that either $\alpha = 0$ or ω_α is a regular initial number with limit index.*

Proof. The theorem follows out of definition 1 as well as the auxiliary theorems 1 and 2. Out of theorem 3 it follows immediately

Theorem 4. *$\aleph_0$ is the smallest in the narrower sense inaccessible cardinal number.*

Auxiliary Theorem 5. *The numbers $0, 2$ and $\aleph_0$ suffice the condition $\mathscr{B}_3$; there exists no finite from 0 and 2 distinct cardinal number $\mathfrak{m}$, which suffices this condition.*

Proof. The condition $\mathscr{B}_3$ is trivially fulfilled by 0. The definition of exponentiation of cardinal numbers gives $0^0 = 1^0 = 1$ and $0^1 = 0$; the number 2 suffices hence the condition $\mathscr{B}_3$, however the number 1 does not. Also no finite number $\mathfrak{m} > 2$ suffices this condition, because e.g. $\mathfrak{n} = 2 < \mathfrak{m}$, $\mathfrak{p} = \mathfrak{m} - 1 < \mathfrak{m}$ and yet $\mathfrak{n}^\mathfrak{p} = 2^{\mathfrak{m}-1} > \mathfrak{m}$. If finally the numbers $\mathfrak{n}$ and $\mathfrak{p}$ are finite, so is $\mathfrak{n}^\mathfrak{p}$, the number $\aleph_0$ thus suffices the condition $\mathscr{B}_3$.

Theorem 6. *2 is the smallest and $\aleph_0$ the next largest in the narrower sense inaccessible cardinal number.*

Proof : by definition 2 as well auxiliary theorems 1 and 5.

It is to note, that various theorems, which initially only were formulated and proven for infinite inaccessible numbers, also prove true for 2 (however no other finite cardinal number) [8]). It hence seems expedient to formulate the notion of in the narrower sense inaccessible cardinal number in a way, such that 2 too also falls under this notion.

Auxiliary Theorem 7. *Every cardinal number $\mathfrak{m}$, which suffices the following condition:*

$\mathscr{B}_4$. *If* $\mathfrak{p} < \mathfrak{m}$ *then so* $2^{\mathfrak{p}} < \mathfrak{m}$,

suffices also condition $\mathscr{B}_2$ *and is hence either* 0 *or infinite.*

If $\mathfrak{m} \neq 2$ *then so the conditions* $\mathfrak{B}_3$ *and* $\mathfrak{B}_4$ *are equivalent.*

Proof. Suppose $\mathfrak{m}$ were to suffice $\mathscr{B}_4$. By Cantor's inequality $\mathfrak{n} < 2^{\mathfrak{n}}$ we see immediately that $\mathfrak{m}$ must suffice the condition $\mathscr{B}_2$; by auxiliary theorem 2 is hence $\mathfrak{m}$ either 0 or infinite.

If $\mathfrak{m} = 0$, then clearly $\mathfrak{m}$ suffices the condition $\mathscr{B}_3$. If however $\mathfrak{m}$ is infinite, then we derive $\mathscr{B}_3$ and $\mathscr{B}_4$ as follows. Let $\mathfrak{n} < \mathfrak{m}$ and $\mathfrak{p} < \mathfrak{m}$; it then holds $\mathfrak{n} \cdot \mathfrak{p} < \mathfrak{m}$ as is well known, after which due to $\mathscr{B}_4$ it holds that $2^{\mathfrak{n} \cdot \mathfrak{p}} < \mathfrak{m}$; since on the other hand $\mathfrak{n}^{\mathfrak{p}} \leq (2^{\mathfrak{n}})^{\mathfrak{p}} = 2^{\mathfrak{n} \cdot \mathfrak{p}}$, then so $\mathfrak{n}^{\mathfrak{p}} < \mathfrak{m}$.

Suffices conversely a number $\mathfrak{m} \neq 2$ the condition $\mathscr{B}_3$, then so by auxiliary theorem 5 $\mathfrak{m} > 2$ (aside from the trivial case $\mathfrak{m} = 0$); $\mathscr{B}_4$ is then as a special-case contained in $\mathscr{B}_3$. With this the proof is finished.

From definition 2 and auxiliary theorem 7 it follows immediately

Theorem 8. *For that* $\mathfrak{m}$ *is a from* 2 *different (that is infinite) in the narrower sense inaccessible number, it is necessary and sufficient, that* $\mathfrak{m}$ *is different from* 0 *and suffices the conditions* $\mathscr{B}_1$ *and* $\mathscr{B}_4$.

Theorem 9. *Every from* 2 *different (that is infinite) in the narrower sense inaccessible cardinal number is also inaccessible in the broader sense.*

If the Cantor aleph-hypothesis:

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for every ordinal number } \alpha$$

correct, then so is conversely every in the broader sense inaccessible cardinal number inaccessible in the narrower sense [6]*).*

Proof. The first part of the theorem follows immediately out of theorems 8 and 9 due to definition 1.

If we now suppose the validity of Cantor's aleph-hypothesis. If the number $\mathfrak{m}$ is inaccessible in the broader sense, so by theorem 3 it follows $\mathfrak{m} = \aleph_\alpha$ where α is either 0 or a limit ordinal. If $\mathfrak{p} < \mathfrak{m}$, so then it follows $2^{\mathfrak{p}} < \mathfrak{m}$ if $\mathfrak{p}$ is finite, if however $\mathfrak{p}$ is infinite, that is $\mathfrak{p} = \aleph_\beta$, so then by Cantor's aleph-hypothesis it follows that $2^{\mathfrak{p}} = 2^{\aleph_\beta} = \aleph_{\beta+1}$ and hence $2^{\mathfrak{p}} < \aleph_\alpha = \mathfrak{m}$. Hence the number $\mathfrak{m}$ suffices the condition $\mathscr{B}_4$ and hence by theorem 8 is in the narrower sense inaccessible, which we desired to prove.

Theorem 10. *The cardinal number $\mathfrak{m}$ is in the narrower sense inaccessible if and only if $\mathfrak{m}$ is distinct from 0 and suffices the following condition:*

$\mathscr{B}_5$. *In case X is an arbitrary set of cardinality $< \mathfrak{m}$ and to each element $x \in X$ a cardinal number $\mathfrak{n}_x < \mathfrak{m}$ is associated, so follows*

$$\prod_{x \in X} \mathfrak{n}_x < \mathfrak{m}.$$

Proof. Let $\mathfrak{m}$ be an in the narrower sense inaccessible cardinal number. Suppose that each element x of a set X of cardinality $\overline{\overline{X}} = \mathfrak{p} < \mathfrak{m}$ has a number $\mathfrak{n}_x$ associated. It follows then by $\mathscr{B}_1$ that

$$\sum_{x \in X} \mathfrak{n}_x < \mathfrak{m}$$

and thus by $\mathscr{B}_3$

$$(\sum_{x \in X} \mathfrak{n}_x)^{\mathfrak{p}} < \mathfrak{m}$$

as on the other hand

$$\prod_{x \in X} \mathfrak{n}_x \leq (\sum_{x \in X} \mathfrak{n}_x)^{\mathfrak{p}}$$

(aside from the trivial case: $X = 0$), so one obtains finally $\prod_{x \in X} \mathfrak{n}_x < \mathfrak{m}$. The number $\mathfrak{m}$ suffices the condition $\mathscr{B}_5$.

If we now conversely suppose, that $\mathfrak{m} \neq 0$ fulfills the condition $\mathscr{B}_5$. Then the condition $\mathscr{B}_3$ is a special case contained in $\mathscr{B}_5$:
one has that

$$\mathfrak{n}^{\mathfrak{p}} = \prod_{x \in X} \mathfrak{n}_x,$$

if $\overline{\overline{X}} = \mathfrak{p}$ and $\mathfrak{n}_x = \mathfrak{n}$ for each $x \in X$. $\mathscr{B}_1$ follows from $\mathscr{B}_5$ due to the following well known inequality:

$$\sum_{x \in X} \mathfrak{n}_x \leq \prod_{x \in X} \mathfrak{n}_x$$

(this inequality does not hold, when among all the other numbers $\mathfrak{n}_x$ the number 0 appears or when all the numbers $\mathfrak{n}_x$ equal 1; these both exceptional cases can however without any difficulties be done). The numbers $\mathfrak{m} \neq 0$ suffices the conditions $\mathscr{B}_1$ and $\mathscr{B}_3$ and is hence in the narrower sense inaccessible, which we desired to prove.

A little deeper than the previous couple theorems lies the further below given theorem 13, which provides seemingly the easiest characterisation of the in the narrower sense inaccessible cardinal numbers. We give prior to that theorem two auxiliary theorems.

Auxiliary Theorem 11. *Suffices the infinite cardinal number* $\mathfrak{m}$ *the condition* $\mathscr{B}_1$*, so then it follows for every* $\mathfrak{p} \neq 0$

$$\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m} \cdot \sum_{\mathfrak{n}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}}.$$

Proof. First of all it holds completely generally

$$\mathfrak{m} \cdot \sum_{\mathfrak{n}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} \leq \mathfrak{m} \cdot \mathfrak{m}^{\mathfrak{p}} \cdot \overline{\overline{\underset{\mathfrak{n}}{\mathrm{E}}[\mathfrak{n}< \mathfrak{m}]}} \leq \mathfrak{m} \cdot \mathfrak{m}^{\mathfrak{p}} \cdot \mathfrak{m}$$

and by this, since $\mathfrak{m}$ is infinite and $\mathfrak{p} > 0$,

$$\mathfrak{p} \cdot \sum_{\mathfrak{n}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} \leq \mathfrak{m}^{\mathfrak{p}}. \tag{1}$$

In the proof of the inverse inequality

$$\mathfrak{m}^{\mathfrak{p}} \leq \mathfrak{p} \cdot \sum_{\mathfrak{n}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} \tag{2}$$

it is to differentiate three cases, depending on whether $\mathfrak{p} \leq \aleph_0$, $\aleph_0 \leq \mathfrak{p} < \mathfrak{m}$ or $\mathfrak{p} \geq \mathfrak{m}$.

If $\mathfrak{p}$ is finite, so then (2) holds surely as one has $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$.

If $\mathfrak{p}$ is infinite and $< \mathfrak{m}$, so then we set

$$\mathfrak{m} = \aleph_\alpha \textit{ and } \mathfrak{p} = \aleph_\gamma. \tag{3}$$

by auxiliary theorem 1 it must be the case that ω_α is a regular initial number, such that

$$\gamma < cf(\alpha) = \alpha. \tag{4}$$

If α is a limit ordinal, then so with the use of a well known theorem [9]) (while considering (3) and (4)):

$$\mathfrak{m}^{\mathfrak{p}} = \aleph_\alpha^{\aleph_\gamma} = \sum_{\xi<\alpha} \aleph_\xi^{\aleph_\alpha} = \sum_{\aleph_0 \leq \mathfrak{n}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} \leq \sum_{\mathfrak{n}<} \mathfrak{n}^{\mathfrak{p}},$$

and from this one immediately obtains (2). Is however α not a limit ordinal, then so α must be of the form $\beta + 1$ (since by (4) one has $\alpha \neq 0$). By the Hausdorff recursion formula [10]) it follows then

$$\mathfrak{m}^{\mathfrak{p}} = \aleph_{\beta+1}^{\aleph_\gamma} = \aleph_{\beta+1} \cdot \aleph_\beta^{\aleph_\gamma} = \mathfrak{m} \cdot \aleph_\beta^{\mathfrak{p}},$$

from where (2) immediately follows.

Let us at last consider the case $\mathfrak{p} \geq \mathfrak{m}$. We then have, as is well known, $\mathfrak{m}^{\mathfrak{p}} = 2^{\mathfrak{p}}$ [10]); from this formula however immediately follows the inequality (2).

The inequalities (1) and (2) result immediately in the equality that was desired to show.

Auxiliary Theorem 12. *Every cardinal number* $\mathfrak{m}$*, which suffices the following condition:*

$\mathscr{B}_6$. *If* $0 < \mathfrak{p} < \mathfrak{m}$*, then so* $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$*,*

also suffices the condition $\mathscr{B}_1$*.*

The converse of this theorem is equivalent to Cantor's aleph-hypothesis.

Proof. Suppose first that $\mathfrak{m}$ suffice the condition $\mathscr{B}_6$. Let X be an arbitrary set of cardinality $\overline{\overline{X}} = \mathfrak{p} < \mathfrak{m}$; further associate to each element $x \in X$ a number $\mathfrak{n}_x < \mathfrak{m}$. Set further also $\mathfrak{m}_x = \mathfrak{m}$ for $x \in X$. Since $\mathfrak{n}_x < \mathfrak{m}_x$ for every $x \in X$, so follows from a well known theorem of Zermelo [11]), that

$$\sum_{x \in X} \mathfrak{n}_x < \prod_{x \in X} \mathfrak{m}_x = \mathfrak{m}^{\mathfrak{p}},$$

and hence due to $\mathscr{B}_6$

$$\sum_{x \in X} \mathfrak{n}_x < \mathfrak{m}.$$

The number $\mathfrak{m}$ thus suffices the condition $\mathscr{B}_1$.

Suppose now the validity of Cantor's aleph-hypothesis and consider a number $\mathfrak{m}$ which fulfills $\mathscr{B}_1$. If $\mathfrak{m}$ is finite, then so by auxiliary theorem 1 it must be one of the numbers $0, 1, 2$; each of these numbers clearly suffices the condition $\mathscr{B}_6$. Let now $\mathfrak{m}$ be infinite and $0 < \mathfrak{p} < \mathfrak{m}$. By auxiliary theorem 11 it then follows

$$\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m} \cdot \sum_{\mathfrak{n} < \mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} \leq \mathfrak{m} \cdot \sum_{\mathfrak{n} < \mathfrak{m}} (2^{\mathfrak{n}})^{\mathfrak{p}} = \mathfrak{m} \cdot \sum_{\mathfrak{n} < \mathfrak{m}} 2^{\mathfrak{n} \cdot \mathfrak{p}}. \tag{1}$$

If $\mathfrak{n} < \mathfrak{m}$ and $\mathfrak{p} < \mathfrak{m}$ then so also $\mathfrak{n} \cdot \mathfrak{p} < \mathfrak{m}$; out of Cantor's aleph-hypothesis it then is easy to conclude $2^{\mathfrak{n} \cdot \mathfrak{p}} < \mathfrak{m}$, from which it follows due to (1) that

$$\mathfrak{m}^{\mathfrak{p}} \leq \mathfrak{m} \cdot \mathfrak{m} \cdot \overline{\overline{\underset{\mathfrak{n}}{\mathrm{E}}[\mathfrak{n} < \mathfrak{m}]}} \leq \mathfrak{m} \cdot \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}.$$

Since the inverse inequality $\mathfrak{m} \leq \mathfrak{m}^{\mathfrak{p}}$ is obvious, one finally obtains that $\mathfrak{m} = \mathfrak{m}^{\mathfrak{p}}$, i.e. that $\mathscr{B}_6$ is satisfied.

It remains still to from the statement:

(2) *Every cardinal number* $\mathfrak{m}$ *that suffices the condition* $\mathscr{B}_1$ *also suffices the condition* $\mathscr{B}_6$

to conclude Cantor's aleph-hypothesis. For this purpose we consider an arbitrary ordinal number α. Since $\omega_{\alpha+1}$ is regular it suffices to show that $\mathfrak{m} = \aleph_{\alpha+1}$ suffices the condition $\mathscr{B}_1$ (compare auxiliary theorem 1). Considering auxiliary theorem (2) it hence suffices to show that $\aleph_{\alpha+1}$ suffices the condition $\mathscr{B}_6$; for $\mathfrak{p} = \aleph_\alpha$ we obtain from here the formula $\aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}$. However it is well known the formula $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ to be equivalent to this [12]). With this the proof is finished.

Theorem 13. *The cardinal number* $\mathfrak{m}$ *is inaccessible in the narrower sense if and only if* $\mathfrak{m}$ *is different from* 0 *and suffices the conditions* $\mathscr{B}_3$ *and* $\mathscr{B}_6$.

Proof. If the number $\mathfrak{m}$ is inaccessible in the narrower sense, then so by definition 2 it suffices conditions $\mathscr{B}_1$ and $\mathscr{B}_3$. By theorem 6 is here $\mathfrak{m}$ either 2 or infinite. The number $\mathfrak{m} = 2$ suffices the condition $\mathscr{B}_6$. If $\mathfrak{m}$ is infinite, then so by auxiliary theorem 11 one obtains the formula

$$\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m} \cdot \sum_{\mathfrak{n} < \mathfrak{m}} \mathfrak{n}^{\mathfrak{p}}$$

for each $\mathfrak{p} > 0$; if here $\mathfrak{p} < \mathfrak{m}$, then so one obtains with the help of $\mathscr{B}_3$

$$\mathfrak{m}^{\mathfrak{p}} \leq \mathfrak{m} \cdot \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}$$

and further $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$. The number $\mathfrak{m}$ hence again suffices the condition $\mathscr{B}_6$.

If conversely $\mathfrak{m} \neq 0$ fulfills conditions $\mathscr{B}_3$ and $\mathscr{B}_6$, so out of auxiliary theorem 12 and definition 2 one sees immediately that $\mathfrak{m}$ is inaccessible in the narrower sense, which we desired to prove.

Theorem 14. *For* $\mathfrak{m}$ *to be a from* 2 *distinct (that is infinite) in the narrower sense inaccessible cardinal number, it is necessary and sufficient that* $\mathfrak{m}$ *is distinct from* 0 *and suffices the conditions* $\mathscr{B}_4$ *and* $\mathscr{B}_6$.

The condition $\mathscr{B}_4$ *can here be replace through the following condition:*

$\mathscr{B}_7$. *There exists no cardinal number* $\mathfrak{p}$, *for which* $2^{\mathfrak{p}} = \mathfrak{m}$.

Proof. According to theorems 8 and 13 fulfills every in the narrower sense inaccessible cardinal number $\mathfrak{m} \neq 2$ the conditions $\mathscr{B}_4$ and $\mathscr{B}_6$; from $\mathscr{B}_4$ follows $\mathscr{B}_7$ immediately (due to the inequality: $\mathfrak{p} < 2^{\mathfrak{p}}$). Suppose now that the number $\mathfrak{m} \neq 0$ were to suffice the conditions $\mathscr{B}_6$ and $\mathscr{B}_7$. By auxiliary theorem 12 it follows that $\mathfrak{m}$ suffices the condition $\mathscr{B}_1$ too. From $\mathscr{B}_7$ one sees immediately, that $\mathfrak{m} > 2$ must hold. So if $\mathfrak{p} < \mathfrak{m}$, then it follows $2^{\mathfrak{p}} \leq \mathfrak{m}^{\mathfrak{p}}$ and hence $2^{\mathfrak{p}} \leq \mathfrak{m}$ because of $\mathscr{B}_6$; since by $\mathscr{B}_7$ $2^{\mathfrak{p}} \neq \mathfrak{m}$ holds, it follows further $2^{\mathfrak{p}} < \mathfrak{m}$, thus the number $\mathfrak{m}$ fulfills $\mathscr{B}_4$. By theorem 8 it hence follows that $\mathfrak{m}$ is a distinct from 2 in the narrower sense inaccessible number. Theorem 14 is thus proven.

Theorem 15. *For* $\mathfrak{m}$ *to be a from* 2 *distinct (that is infinite) in the narrower sense inaccessible cardinal number it is necessary and sufficient that* $\mathfrak{m} \neq 0$ *and to suffice the condition* $\mathscr{B}_4$ *as well the following condition:*

$\mathscr{B}_8$. *If* $\mathfrak{p} \neq 0$ *then it holds that* $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m} \cdot 2^{\mathfrak{p}}$.

Proof. By theorem 14 every infinite in the narrower sense inaccessible number $\mathfrak{m}$ suffices the conditions $\mathscr{B}_4$ and $\mathscr{B}_6$. If hence $0 < \mathfrak{p} < \mathfrak{m}$, then so one obtains $2^{\mathfrak{p}} < \mathfrak{m} = \mathfrak{m}^{\mathfrak{p}}$ and hence

(1) $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m} \cdot 2^{\mathfrak{p}}$.

If however $\mathfrak{p} \geq \mathfrak{m}$, then so as is well known $\mathfrak{m} \leq \mathfrak{m}^{\mathfrak{p}} = 2^{\mathfrak{p}}$ [10]), out of which (1) follows. Thus the formula (1) holds for each $\mathfrak{p} \neq 0$M in other words: $\mathfrak{m}$ suffices the condition $\mathscr{B}_8$.

Suppose now conversely the number $\mathfrak{m} \neq 0$ were to suffice the conditions $\mathscr{B}_4$ and $\mathscr{B}_8$. For every number $\mathfrak{p}$, $0 < \mathfrak{p} < \mathfrak{m}$, one has $2^{\mathfrak{p}} < \mathfrak{p}$ and (1); since here by auxiliary theorem 7 $\mathfrak{m}$ is infinite, one obtains $\mathfrak{m}^{\mathfrak{p}} = \mathfrak{m}$. The number $\mathfrak{m}$ suffices thus the condition $\mathscr{B}_6$. Considering theorem 14 is $\mathfrak{m}$ an infinite in the narrower sense inaccessible number, which we desired to prove.

Auxiliary Theorem 16. *Let* M *be an arbitrary set of cardinality* $\overline{\overline{M}} = \mathfrak{m}$. *For* $\mathfrak{m}$ *to be a from* 2 *distinct and the condition* $\mathscr{B}_6$ *sufficient cardinal number, it is necessary and sufficient that* M *fulfills the following condition:*

$\mathscr{C}_1$. M *is equinumerous with the set system* $E_X[X \subset M$ *and* $\overline{\overline{X}} < \overline{\overline{M}}]$.

Proof. We shall set the trivial case $\mathfrak{m} < \aleph_0$ aside (one sees easily, that every finite number $\mathfrak{m} > 2$ fulfills neither $\mathscr{B}_6$ nor $\mathscr{C}_1$). Let $\mathfrak{m} \geq \aleph_0$. As consequence of a theorem of Sierpiński [12]) has the system $\mathrm{E}_X[X \subset M \textit{ and } \overline{\overline{X}} < \overline{\overline{M}}]$ cardinality $\sum_{\mathfrak{p}<\mathfrak{m}} \mathfrak{m}^{\mathfrak{p}}$ [13]); the condition $\mathscr{C}_1$ is thus equivalent to the formula

(1)
$$\mathfrak{m} = \sum_{\mathfrak{p}<\mathfrak{m}} \mathfrak{m}^{\mathfrak{p}}$$
.

It is however easy to derive this formula from $\mathscr{B}_6$; due to $\mathscr{B}_6$ namely

$$\mathfrak{m} \leq \sum_{\mathfrak{p}<\mathfrak{m}} \mathfrak{m}^{\mathfrak{p}} \leq \mathfrak{m} \cdot \mathfrak{m} = \mathfrak{m}.$$

Equally as easy it is to obtain the reverse implication.
The conditions $\mathscr{B}_6$ and (1), that is $\mathscr{C}_1$, are hence equivalent, which we desired to prove.

Theorem 17. *If M is an arbitrary set of cardinality $\overline{\overline{M}} = \mathfrak{m}$, then is $\mathfrak{m}$ a from 2 different (that is infinite) in the narrower sense inaccessible number if and only if M is non-empty and suffices the condition $\mathscr{C}_1$ as well as the following condition:*

$\mathscr{C}_2$. *There exists no set P, such that M is equinumerous with the set system $E_X[X \subset P]$.*

Proof : by theorem 14 and auxiliary theorem 16 (the condition $\mathscr{C}_2$ expresses essentially the same as $\mathscr{B}_7$).

Auxiliary Theorem 18. *Let α be an arbitrary ordinal number and N an arbitrary set of cardinality $\overline{\overline{N}} < \aleph_\alpha$; associate to every ordinal number $\xi < \omega_\alpha$ a set system N_ξ; and namely in a way such that:*

(i)
$$N_0 = \underset{X}{E}[X \subset N],$$

(ii)
$$N_\xi = \underset{X}{E}[X \subset \sum_{\eta<\xi} N_\eta \textit{ and } \overline{\overline{X}} < \aleph_\alpha] \textit{ for every } \xi \textit{ and } 0 < \xi < \omega_\alpha;$$

let further

(iii)
$$M = \sum_{\xi<\omega_\alpha} N_\xi;$$

Then the following hold:

(I) *M satisfies the condition*

$\mathscr{D}_1$. *If $X \in M$ and $Y \subset X$, then also $Y \in M$.*

(II) *suffices the number $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_4$, so then M suffices the condition*

$\mathscr{D}_2$. *If $X \in M$, then so also $E_Y[Y \subset X] \in M$;*

(III) *suffices the number $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_6$, so then is the set system M of cardinality $\overline{\overline{M}} = \aleph_\alpha$ and also suffices the condition*

$\mathscr{D}_3$. *If $X \subset M$ and $\overline{\overline{X}} < \overline{\overline{M}}$, then so $X \in M$;*

(IV) *if $N = 0$ and suffices the number $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_6$, so then M suffices the condition*

$\mathscr{D}_4$. *$M = E_X[X \subset M$ and $\overline{\overline{X}} < \overline{\overline{M}}]$.*

Proof. From (i) and (ii) we obtain immediately:

(1) If $\xi < \omega_\alpha$, $X \in N_\xi$ and $Y \subset X$, then so also $Y \in N_\xi$;

considering (iii) we obtain from this:

(2) If $X \in M$ and $Y \subset X$, then so also $Y \in M$; in other words M suffices the condition $\mathscr{D}_1$.

Since $\overline{\overline{N}} < \aleph_\alpha$ it follows further from (i)-(iii) that

(3) $\overline{\overline{X}} < \aleph_\alpha$ for every $X \in N_\xi$, $\xi < \omega_\alpha$ and more generally for every $X \in M$.

Let us now suppose that the number $\aleph_\alpha$ were to suffice the condition $\mathscr{B}_4$. If then a set X has cardinality $\overline{\overline{X}} = \mathfrak{p} < \aleph_\alpha$, so then, as is well known, has the system $\mathrm{E}_Y[Y \subset X]$ the cardinality $2^{\mathfrak{p}}$, and so also a cardinality $< \aleph_\alpha$. Considering this and from (1), (3) and (ii):

(4) suffices $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_4$ and if $\xi < \omega_\alpha$ as well $X \in N_\xi$, then so is $\mathrm{E}_Y[Y \subset X] \subset N_\xi \subset \sum_{\eta < \xi+1} N_\eta$, $\overline{\overline{\mathrm{E}_Y[Y \subset X]}} < \aleph_\alpha$ and finally $\mathrm{E}_Y[Y \subset X] \in N_{\xi+1}$.

From (4) and (iii) it follows immediately

(5) suffices $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_4$ and is $X \in M$, so then also $\mathrm{E}_Y[Y \subset X] \in M$; the set M hence suffices the condition $\mathscr{D}_2$.

We now wish to show that

(6) $$\overline{\overline{M}} \geq \aleph_\alpha.$$

Due to (iii) holds (6) surely whenever there exists a number $\xi < \omega_\alpha$, for which

$$\overline{\overline{\sum_{\eta<\xi} N_\eta}} \geq \aleph_\alpha.$$

If however no such number ξ exists, then one concludes from (ii) that

$$N_\xi = \underset{X}{\mathrm{E}}[X \subset \sum_{\eta<\xi} N_\eta]$$

for every ξ for which $0 < \xi < \omega_\alpha$. From here it follows immediately that the set system N_ξ is of a larger cardinality than the sum of all preceding set systems; one therefore has:

(7) $$N_\xi - \sum_{\eta<\xi} N_\eta \neq 0 \; for\ every\ \xi,\ 0 < \xi < \omega_\alpha.$$

From (iii) one also obtains the formula:

(8) $$M = N_0 + \sum_{0<\xi<\omega_\alpha} \left(N_\xi - \sum_{\eta<\xi} N_\eta \right).$$

Considering (7) one sees that (8) determines a decomposition of the set system M into $\aleph_\alpha$ disjoint non-empty subsystems; M hence must fulfill the formula (6).

From (3) and (6) we obtain immediately:

(9) $$If\ X \in M\ then\ \overline{\overline{X}} < \overline{\overline{M}}.$$

We now suppose further that the number $\aleph_\alpha$ is to satisfy $\mathscr{B}_6$. By transfinite induction we wish to show that

(10) $$\overline{\overline{N_\xi}} \leq \aleph_\alpha$$

for every $\xi < \omega_\alpha$. If indeed $\xi = 0$ and if one is to set $\overline{\overline{N}} = \mathfrak{p}$, so one sees from (i) that N_ξ has cardinality $2^{\mathfrak{p}}$; since however $\mathfrak{p} < \aleph_\alpha$, so follows from $\mathscr{B}_6$ that $\overline{\overline{N_\xi}} = 2^{\mathfrak{p}} \leq \aleph_\alpha^{\mathfrak{p}} \leq \aleph_\alpha$.

Let now $0 < \xi < \omega_\alpha$. Suppose that all set systems N_η with $\eta < \xi$ are of a cardinality $\leq \aleph_\alpha$, we obtain

$$\overline{\overline{\sum_{\eta<\xi} N_\eta}} = \mathfrak{n} \leq \aleph_\alpha \cdot \overline{\overline{\underset{\eta}{\mathrm{E}}[\eta < \xi]}} \leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha.$$

If $\mathfrak{n} < \aleph_\alpha$, so we conclude from (ii) that

$$N_\xi = \mathop{\mathrm{E}}_{X}[X \subset \sum_{\eta < \xi} N_\eta]$$

and thus $\overline{\overline{N_\eta}} = 2^{\mathfrak{n}}$; from here it follows (precisely like in the case $\xi = 0$) the formula (10). Is however $\mathfrak{n} = \aleph_\alpha$, so then we set in auxiliary theorem 15:

$$\mathfrak{m} = \aleph_\alpha, \quad M = \sum_{\eta < \xi} N_\eta$$

and obtain immediately (again due to (ii) and $\mathscr{B}_6$) $\overline{\overline{N_\xi}} = \aleph_\alpha$.

Thus (10) is proven for every $\xi < \omega_\alpha$. By (iii) it follows from this that $\overline{\overline{M}} \leq \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$; with (6) together this inequality provides

(11)
$$\overline{\overline{M}} = \aleph_\alpha.$$

Consider now an arbitrary set system X such that $X \subset M$ and $\overline{\overline{X}} < \overline{\overline{M}}$. Considering (iii) and (11) we have

(12)
$$X \subset \sum_{\xi < \omega_\alpha} N_\xi \textit{ and } \overline{\overline{X}} < \aleph_\alpha.$$

We now apply auxiliary theorems 1 and 12. Since $\aleph_\alpha$ suffices the condition $\mathscr{B}_6$, so it follows ω_α must be regular, i.e.

(13)
$$\aleph_\alpha = \aleph_{cf(\alpha)}.$$

Due to a well known theorem [14]) one concludes from (11) and (12) the existence of a number ξ, $0 < \xi < \omega_\alpha$ for which

$$X \subset \sum_{\eta < \xi} N_\eta$$

holds; considering (11), (ii) and (iii) one from this obtains $X \in N_\xi$ and $X \in M$.

We formulate the final result of this observation:

(14) Suffices $\mathfrak{m} = \aleph_\alpha$ the condition $\mathscr{B}_6$, so then is the set system M of cardinality $\overline{\overline{M}} = \aleph_\alpha$ and meanwhile it suffices the condition $\mathscr{D}_3$:
if $X \subset M$ and $\overline{\overline{X}} < \overline{\overline{M}}$, so then $X \in M$.

To finally justify $\mathscr{D}_4$ consider the following implication:

(15) If $X \in N_\xi$ so $X \subset M$.

If $0 < \xi < \omega_\alpha$, so then (15) follows inevitably from (ii) and (iii).
For $\xi = 0$ is (15) wrong in general. However if $N = 0$, so one sees from (i) that N_0 contains the null-set as the sole element; supposing this then so (15) also holds for $\xi = 0$ and hence for every $\xi < \omega_\alpha$. Due to (iii) one then obtains:

(16) If $X \in M$ then $X \subset M$.

Suppose now that $N = 0$ and simultaneously that $\aleph_\alpha$ suffices the condition $\mathscr{B}_6$. There then are implications: (9), (16) as well, when considering (14), the implication $\mathscr{D}_3$; these three implications can be clearly summarised in a formula:

$$M = \underset{X}{\mathrm{E}}[X \subset M \textit{ and } \overline{\overline{X}} < \overline{\overline{M}}].$$

It thus holds:

(17) If $N = 0$ and if $\mathfrak{m} = \aleph_\alpha$ suffices the condition $\mathscr{B}_6$, then so N suffices the formula $\mathscr{D}_4$: $M = \mathrm{E}_X[X \subset M \textit{ and } \overline{\overline{X}} < \overline{\overline{M}}]$.

According to (2), (5), (14) and (17) has the set system M all of the asserted properties.

Auxiliary Theorem 19. *For that a number $\mathfrak{m}$ is a from 2 different and the condition $\mathscr{B}_6$ sufficing cardinal number, it is necessary and sufficient that there exists a set system M of cardinality $\overline{\overline{M}} = \mathfrak{m}$ which fulfills the condition $\mathscr{D}_3$.*
Here the condition $\mathscr{D}_3$ can be replaced with the condition $\mathscr{D}_4$.

Proof. The case $\mathfrak{n} < \aleph_0$ can be done easily: there exist, as is seen immediately, only two finite numbers $\mathfrak{m} \neq 2$ which fulfill $\mathscr{B}_6$ namely $\mathfrak{m} = 0$ and $\mathfrak{m} = 1$, and there too are only two finite set systems which suffice the condition $\mathscr{D}_3$, or also $\mathscr{D}_4$, namely the null-set and the system which contains the null-set as its sole element.
Let now $\mathfrak{m} \geq \aleph_0$ and further $\mathfrak{m} = \aleph_\alpha$. We set $N = 0$ and define means recursion the set systems N_ξ fir $\xi < \omega_\alpha$ by the in auxiliary theorem 18 given formulae (i) and (ii); through the formula (iii) one further determines the set system M (the existence of the systems N_ξ and M one obtains from the axiom of replacement). Let us now suppose that $\mathfrak{m}$ were to suffice to the condition $\mathscr{B}_6$. By the statement

of auxiliary theorem 18 it follows that the system M has cardinality $\overline{\overline{M}} = \mathfrak{m}$; it further fulfills the formula $\mathscr{D}_4$ and all the more the condition $\mathscr{D}_3$. which is an inevitable consequence of $\mathscr{D}_4$.

Let us now now conversely suppose there to be a set system M of cardinality $\overline{\overline{M}} = \mathfrak{m} = \aleph_\alpha$ fulfilling $\mathscr{D}_3$. From $\mathscr{D}_3$ follows that the system

$$N = \underset{X}{\mathrm{E}}[X \subset M \textit{ and } \overline{\overline{X}} < \overline{\overline{M}}]$$

is a subsystem of M. On the other hand is M equinumerous with a subsystem of N, and namely with the system

$$P = \underset{X}{\mathrm{E}}[X \subset M \textit{ and } \overline{\overline{X}} = 1]$$

Following from the $\mathsf{Cantor} - \mathsf{Bernstein}$ equivalence theorem are the set systems M and N equinumerous. Through application of auxiliary theorem 16 we conclude that the number $\mathfrak{m} = \overline{\overline{M}}$ suffices the condition $\mathscr{B}_6$, which we desired to prove.

Considering auxiliary theorem 19 we want to remark that the condition $\mathscr{B}_6$ for $\mathfrak{m} = \aleph_{\alpha+1}$ reduces to the formula

$$\aleph_{\alpha+1}^{\aleph_\alpha} = \aleph_{\alpha+1}, \textit{ or } 2^{\aleph_\alpha} = \aleph_{\alpha+1}$$

(cf. the proof of auxiliary theorem 12). If in particular one sets $\mathfrak{m} = \aleph_1$ so one sees the continuum hypothesis

$$2^{\aleph_0} = \aleph_1$$

equivalent toe the following assertion:

There exists a set M of cardinality $\overline{\overline{M}} = \aleph_1$, which suffices the condition $\mathscr{D}_3$ or $\mathscr{D}_4$.

Theorem 20. *Let $\mathfrak{m}$ be an arbitrary cardinal number and N a set of cardinality $\overline{\overline{N}} = \mathfrak{n}$. The number $\mathfrak{m}$ is a from 2 different (that is infinite) in the narrower sense inaccessible cardinal number $> \mathfrak{n}$ if and only if there exists a set system M of cardinality $\overline{\overline{M}} = \mathfrak{m}$ that suffices the conditions $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ and contains N as an element.*

Proof. Let us suppose the number $\mathfrak{m} \neq 2$ were to be in the narrower sense inaccessible and $> \mathfrak{n}$. Starting with the set N we construct the set system M in the way as is given in auxiliary theorem 18.

Since $\mathfrak{m}$ suffices the conditions $\mathscr{B}_4$ and $\mathscr{B}_6$ (theorem 14), it follows M is of cardinality $\overline{\overline{M}} = \mathfrak{m}$ and fulfills the conditions $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$; from the formulae (i) and (iii) (auxiliary theorem 18) one obtains that $N \in M$.

Let, conversely, a set system M of cardinality $\overline{\overline{M}} = \mathfrak{m}$ be given, such that it fulfills the conditions $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ and contains N as an element. Since M is non-empty it follows $\mathfrak{m} \neq 0$. Let us consider a number $\mathfrak{p} < \mathfrak{m}$. There definitely exists a system $X \subset M$ of cardinality $\overline{\overline{X}} = \mathfrak{p}$; by $\mathscr{D}_3$ it follows $X \in M$. Let us set:

$$U = \mathop{\mathrm{E}}_{Y}[Y \subset X] \text{ and } V = \mathop{\mathrm{E}}_{Y}[Y \subset U].$$

By $\mathscr{D}_2$ is $U \in M$; by $\mathscr{D}_1$ from this follows that $V \subset M$; consequently

$$\overline{\overline{V}} \leq \overline{\overline{M}} = \mathfrak{m}. \tag{1}$$

On the other hand we have

$$\overline{\overline{U}} = 2^{\mathfrak{p}} < \overline{\overline{V}} = 2^{2^{\mathfrak{p}}}; \tag{2}$$

the inequalities (1) and (2) result immediately in that $2^{\mathfrak{p}} < \mathfrak{m}$. The number $\mathfrak{m}$ thus suffices the condition $\mathscr{B}_4$. Due to auxiliary theorem 19 one further obtains from $\mathscr{D}_3$ that the condition $\mathscr{B}_6$ is satisfied. We can now apply theorem 14: it turns out that $\mathfrak{m}$ is an infinite in the narrower sense inaccessible number. Since after-all $N \in M$, by $\mathscr{D}_2$ follows

$$\mathop{\mathrm{E}}_{Y}[Y \subset N] \subset M;$$

one sees from this that $\mathfrak{m} = \overline{\overline{M}}$ is greater than $\mathfrak{n} = \overline{\overline{N}}$. Thus the number $\mathfrak{m}$ has the desired properties and theorem 20 is proven in both directions.

Theorem 21. $\mathfrak{m}$ *is a from* 2 *different (that is infinite) in the narrower sense inaccessible cardinal number if and only if there exists a non-empty set system* M *of cardinality* $\overline{\overline{M}} = \mathfrak{m}$ *which suffices the conditions* $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$.

Here one can replace the conditions $\mathscr{D}_1$ *and* $\mathscr{D}_3$ *by* $\mathscr{D}_4$.

Proof. To substantiate the first part of the theorem it suffices to set $\mathfrak{n} = 0$ in theorem 20. If one however desires to show that one can replace the conditions $\mathscr{D}_1$ and $\mathscr{D}_3$ by $\mathscr{D}_4$, then one needs to make use of auxiliary theorem 18 for $N = 0$: as a consequence of the assertion of this theorem fulfills the system M not only the conditions $\mathscr{D}_1$, $\mathscr{D}_2$, $\mathscr{D}_3$ but also $\mathscr{D}_4$. On the other hand are $\mathscr{D}_1$ and $\mathscr{D}_3$ obvious consequences of $\mathscr{D}_4$; thus if M suffices the conditions $\mathscr{D}_2$ and $\mathscr{D}_4$, then $\mathfrak{m}$ has to (by the first part of this theorem) be a from 2 distinct in the narrower sense inaccessible cardinal number.

§ 2. Axiomatic introduction of the inaccessible numbers.

When considering theorems 4 and 6 one naturally obtains the question, whether there exist in the narrower or at least in the broader sense inaccessible numbers, which are $> \aleph_0$. This problem is until now undecided and can probably not be decided at all. Especially when the matter is about in the narrower sense inaccessible cardinal numbers, so it is possible to very rigorously verify, that their existence cannot be justified on the ground of the Zermelo – Fraeknel system [3]). If one wishes to ensure the existence of arbitrarily large inaccessible cardinal numbers, then one requires to enrich the Zermelo – Fraeknel system with a new "principle of generation", that is, if took formally, to enrich by a further axiom. Considering theorem 20, this axiom can be formulated as the following:

Axiom $\mathscr{A}$. (Axiom of the inaccessible sets). To each set N there exists a set M with the following properties:

$\mathscr{A}_1$. *$N \in M$;*

$\mathscr{A}_2$. *if $X \in M$ and $Y \subset X$ then $Y \in M$;*

$\mathscr{A}_3$. *if $X \in M$ and Z contains all sets $Y \subset X$ and contains no other things as elements, then $Z \in M$;*

$\mathscr{A}_4$. *if $X \subset M$ and if the sets X and M aren't equinumerous, then $X \in M$.*

There are by the way various equivalent restatements of this axiom known. One can e.g. replace the conditions $\mathscr{A}_1 - \mathscr{A}_4$ respectively by the following conditions (and namely independently from each other):

$\mathscr{A}_{1'}$. *$N \subset M$;*

$\mathscr{A}_{2'}$. *if $X \in M$ and $Y \in X$ then $Y \in M$ (in other words: if $X \in M$ then $X \subset M$);*

$\mathscr{A}_{3'}$. *if $X \in M$ then there exists a set $Z \in M$ containing all the sets $Y \subset X$ as elements;*

$\mathscr{A}_{4'}$. *if $X \subset M$ and if M is equinumerous with no subset $Y \subset X$, then $X \in M$.*

The axiom $\mathscr{A}$ is logically so powerful, that its adaption allows various simplifications of the axiomatic system: some of the other axioms become through it provable theorems and thus can be omitted. It can be seen that e.g. the axiom of the powerset can be derived from $\mathscr{A}$, namely from $\mathscr{A}_1$ and $\mathscr{A}_2$ means the axiom of specification (which can, as is well know, be derived from the axiom of replacement [15])); analogous concerns the axiom of infinity; further is the axiom of pairing a consequence of axiom $\mathscr{A}$ and the axiom of replacement. If one is to replace in $\mathscr{A}$ the conditions $\mathscr{A}_1$ and $\mathscr{A}_2$ by the conditions $\mathscr{A}_{1'}$ and $\mathscr{A}_{2'}$, then beyond the aforementioned axioms the axiom of union also becomes derivable.

All of these relations of the consequences are quite trivial. It appears more interesting the fact, that on the ground of the extended axiomatic system the axiom of choice too becomes a provable theorem. The derivation of the axiom of choice is however not very easy. We thus wish to outline the thought-process of the proof here [16]).

As is well known is the axiom of choice a consequence of the well-ordering theorem [17]); it thus suffices to deduce the well-ordering theorem from axiom $\mathscr{A}$ without the use of the axiom of choice. As is well known Zermelo gave two proofs of the theorem which both rely on the axiom of choice [16]). If one analyses the second Zermelo's proof, so then one obtains the following auxiliary theorem, which can be justified already without the axiom of choice:

I. *If M is an arbitrary set and F a function, which associates to each proper subset X of M an element $F(X) \in M - X$ uniquely, then M can be well ordered.*

One can now sharpen this auxiliary theorem as the following:

II. *If M is an arbitrary set and F a function, which associates to each subset X of M not equinumerous with M, an element $F(X) \in M - X$ uniquely, then M can be well-ordered.*

The basic idea of the proof remains here unchanged. The difference consists in the following: if the hypothesis of I is fulfilled, so then one can "effectively" define a binary relation, through which the set M can be well-ordered; in the proof of theorem II one can only "effectively" construct a well-ordered subset $P \subset X$, which one can show to be equinumerous with M (from this one concludes immediately, that the set M too is capable to be well-ordered, however one is not capable to well-order M "effectively).

From II one obtains the following auxiliary theorem:

III. *Every set M that suffices the condition $\mathscr{A}_4$ can be well-oredred.*

To prove this, let us consider the function F given by the formula:

$$F(X) = \mathop{\mathrm{E}}_{Y}[Y \in X \ \mathit{and}\ Y non \in Y]$$

(if the possibility $Y \in Y$ is excluded through a special axiom [15]), then one just sets $F(X) = X$). This function associates to each set X a subset $F(X) \subset X$, one can further show that $F(X) non \in X$ is true [17]). If now $X \subset M$ and $\overline{\overline{X}} \neq \overline{\overline{M}}$, then so also $F(X) \subset M$ and $\overline{\overline{F(X)}} \neq \overline{\overline{M}}$ (here one obviously needs the Cantor – Bernstein equivalence theorem; if one is to avoid that, one requires to replace $\mathscr{A}_4$ by $\mathscr{A}_{4'}$); by $\mathscr{A}_4$ thus $F(X) \in M$ and consequently $F(X) \in M - X$. The function F hence suffices the condition of II, and the set M can be well-ordered.

Due to the axiom $\mathscr{A}$ one can derive the general well-ordering theorem from III:

IV. *Every set N can be well-oredred.*

Let in fact N be an arbitrary set. Following from axiom $\mathscr{A}$ there exists a set M sufficing the conditions $\mathscr{A}_1 - \mathscr{A}_4$ (in this proof by the way we don't come to see $\mathscr{A}_3$; also $\mathscr{A}_2$ is superfluous if one replaces $\mathscr{A}_1$ with $\mathscr{A}_{1'}$). By theorem III can M be well-ordered. On the other hand follows from $\mathscr{A}_1$ and $\mathscr{A}_2$, that N is equinumerous with a subset of M, namely with the set

$$P = \mathop{\mathrm{E}}_{X}[X \subset N \ \mathit{and}\ \overline{\overline{X}} = 1].$$

From here one concludes immediately that N too can be well-ordered.

At last we remark the following. Considering the results in the branch of mathematics that were achieved in the last couple of years, it would be foolish to think that the axiom of the inaccessible sets only plays a role in highly abstract set-theoretic investigations. After-all it is possible to build all of analysis and pure number theory within Zermelo – Fraenkel set theory. One can thus in accordance with the by Gödel developed method construct certain theorems, which are fully formulated in terms of pure number theory and due to Zermelo – Fraenkel system can be neither proven nor disproved; these theorems however become decidable, if one adds the axiom A to this system [18]).

On the other hand it would be naive to believe, that the problem, to axiomatize intuitive Cantor set theory in its full extent, is finally solved or at least substantially promoted by the axiomatic introduction of inaccessible numbers; one can after-all define further kinds of cardinal numbers, whose existence cannot be ensured on the grounds of the extended axiomatic system and can only be granted by the introduction of new "principles of generation". The problem of the axiomatization of the "Cantor absolute" (a problem whose precise formulation itself poses a considerable difficulty) remains still open, if it even can ever be solved.

Remarks.

The author reported about the in this essay contained results on 18. VI. 1937 to the Polish mathematical society, department Warsaw.

[1]) The first time it must have been F. Hausdorff in his work *Grundzüge einer Theorie der geordneten Mengen*, Math. Ann. **65**, 1908, p. 443, to raise the question, whether there exist regular initial numbers ω_α with limit ordinal index α. The with this initial numbers alephs $\aleph_\alpha$ are precisely in the present treatise what we call the in the narrower sense inaccessible cardinals (aside from $\aleph_0$ or ω_0, which are accounted for here but not with Hausdorff; cf. definition 1 and theorems 3 and 4). The expression "*unerreichbare Kardinalzahl*" ("*nombres cardinaux inaccessibles*") was suggested by C. Kuratowski. The expression of the in the narrower sense inaccessible cardinal number was, as far as I know, introduced by me and used the first time in the joint work of W. Sierpiński and myself: *Sur une propriété caractéristique des nombres inaccessibles*, Fund. Math. 15, 1930, defined p. 292.

[2]) S. F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914, p. 131 (one is to consider the previous remark, especially regarding the number $\aleph_0$).

[3]) Cf. for this (near the below [1]) cited treatise of Sierpiński and Tarski): C. Kuratowski, *Sur l'état actuel de l'axiomatique de la théorie des ensembles*, Ann. Soc. Pol. Math. **3**, p. 146 f.; A. *Tarski*, *Communication sur les recherches de la Théorie des Ensembles*, C. R. Soc. Sc. Vars. **19**,, Cl. III, 1926, p. 322 ff.; R. Baer, *Zur Axiomatik der Kardinalzahlen*, Math. Ztschr. **29**, 1929, p. 380 ff., in particular p. 382 f.; E. Zermelo, *Über Grenzzahlen und Mengenbereiche*, Fund. Math. **16**, 1930, p. 29 ff., in particular pp. 43-47 (the by Zermelo considered Grenzzahlen cover themselves with the initial numbers ω_α, as with the in the narrower sense inaccessible alephs $\aleph_\alpha$ accordingly).

[4]) Cf. for this the below [1]) cited treatise of Sierpiński and Tarski in the following works: S. Banach, *Über additive Maßfunktionen in abstrakten Mengen*, Fund. Math. **15**, 1930, p. 97 ff.; A. Koźniewski et A. Lindenbaum, *Sur les opérations d'addition et de multiplication dans les classes d'ensembles* ibid., p. 342 ff.; U. Ulam, *Zur Maßtheorie in der allgemeinen Mengenlehre*, Fund. Math. **16**, 1930, p. 140 ff.; W. Sierpiński, *Sur un théorème de recouvrement dans la théorie générale des ensembles*, Fund. Math. **20**, 1933, p. 214 ff.; S. Ulam, *ÜbergewisseZerlegungenvonMengen*, ibid., p. 221 ff.; *A.* Tarski, *Über additive und Multiplikative Mengenkörper und Mengenfunktionen*, C. R. Soc. Se. Vars. **30**, 1937 p. 151 ff. cf. also the book by W. Sierpiński, *Hypothèse du continu*, Monogr. Matem. **4**, Warsaw-Lwów 1934, in particular p. 107 and p. 152 ff. In connection to this there are also some older investigations of the theory of ordered sets to be considered, namely the under [1]) cited work of Hausdorff, e.g. p. 477 f., as well the treatise of P. Mahlo, *Über lineare transfinite Mengen*, Leipz. Ber. Math.-phys. Kl. **63**, (1911), p. 187 ff.

[5]) Cf. e.g. A. Fraenkel, *Einleitung in die Mengenlehre*, Berlin 1928, p. 268 ff.

[6]) Cf. for this the under [1]) cited tretise of Sierpiński and Tarski, p. 292 ff. (Définition 1, Théoèrme 1 and Théorème 2); we don't presuppose readers to have the knowledge of this work here.

[7]) The symbol "$cf(\alpha)$" refers to the index of the smallest with the number ω_α cofinal initial number. For (1) cf. A. Tarski, *Sur les classes d'ensembles closes par rapport à certains opérations élémentaires*, Fund. Math. **16**, 1930, p. 185, Lemme 2^a.

[8]) This concerns e.g. the theorems 10 and 13 as well the main theorem of the under [1]) cited treatise of Sierpinski and Tarski (p. 300, Théorème 5; cf. for this the under [4]) mentioned work of Koźniewski and Lindenbaum, p. 354 f.).

[9]) S. A. Tarski, *Quelques théorème sur les alephs*, Fund. Math. **7**, 1925, p. 7, (Théorème 7^a).

[10]) S. e.g. A. Schoenflies, *Entwickelung der Mengenlehre und ihrer Anwendungen*, Leipzig. and Berlin 1913, p. 136 f.

[11]) Ibidem, p. 66 f.

[12]) Cf. e.g. my under [7]) mentioned work, p. 194.

[13]) Ibidem, p. 195, Lemme 10^b

[14]) Ibidem, p. 185 f., Lemme 3^e.

[15]) Cf. e.g. the below [3]) cited treatise of Zermelo, p. 31.

[16]) Cf. e.g. the below [2]) cited work of Hausdorff, o, 133 ff. To the following cf. also the section of the well-ordering theorem and the axiom of choice in the work of J. v. Neumann, *Die Axiomatisierung der Mengenlehre*, Math. Ztschr. **27**, 1928, p. 726 ff.

[17]) Cf. E. Zermelo, *Untersuchungen über die Grundlagen der Mengenlehre*, I, Math. Ann. **65**., 1908, p. 264 f.

[18]) Cf, for this K. Gödel, *Über formal unentscheidbare Sätze der Principa Mathematica und verwandter Systeme I*, Monatsh. Math. Phys. **1**, 1931, p. 187 ff., as well A. Tarski, *Der Wahrheitsbegriff in den formalisierten Sprachen*, Stud. Phil. **1**, 1935, in particular p. 397, Rmrk. [106]), and p. 400 ff.