Geometric Measure Theory

Problem Set 1

2.1 - Measures and Measurable Sets

Problem 1. (Numerical Summations)

(a) Let A be a finite subset of $\mathbf{R}$. Prove that there exists one and only one summation operator $\sum_A$ on nonnegative functions such that the following hold:

(i) $\sum_\emptyset f = 0$,

(ii) If $a \in A$, $f(a) \geq 0$, and $f(x) = 0$ whenever $a \neq x \in A$, then $\sum_A f = f(a)$.

(iii) If $f(x) \geq 0$ and $g(x) \geq 0$ whenever $x \in A$, then

$$\sum_A (f+g) = \sum_A f + \sum_A g.$$

Hint: Suppose there exists another summation operator, say $\widetilde{\sum}_A$ on nonnegative functions that satisfies conditions (i)-(iii). Show that (through induction on card A) for all nonnegative functions f that we have $\sum_A f = \widetilde{\sum}_A f$.

(b) Show that if $\sum_A f \in \overline{\mathbf{R}}$ and $h : A \to Y$, then

$$\sum_A f = \sum_{y \in Y} \sum_{h^{-1}\{y\}} f.$$

(c) Use the results of (b) to conclude:

(i) If $0 \neq c \in \mathbf{R}$, then $\sum_A cf = c \sum_A f$,

(ii) If $\sum_A f + \sum_A g \in \overline{\mathbf{R}}$, then $\sum_A (f+g) = \sum_A f + \sum_A g$.

(iii) If $\sum_A f \in \overline{\mathbf{R}}$ and $A = U \times V$, then

$$\sum_A f = \sum_{u \in U} \sum_{v \in V} f(u,v) = \sum_{u \in V} \sum_{v \in U} f(u,v).$$

Problem 2. (Examples of Measures)

(a) Let ϕ be any measure over a set X. For any set $Y \subset X$, show that the function

$$\phi \llcorner Y : \mathbf{2}^X \to [0, \infty] \qquad (\phi \llcorner Y)(A) = \phi(Y \cap A) \text{ for } A \subset X$$

defines a measure over X.

(b) Let $f : X \to Y$ be a function. Show that for any measure ϕ over X, the function

$$f_{\#}(\phi) : \mathbf{2}^X \to [0, \infty] \qquad (f_{\#}\phi)(B) = \phi(f^{-1}B) \text{ for } B \subset Y$$

defines a measure over Y.

(c) Verify that $f^{-1}(B)$ is ϕ measurable if and only if B is $f_{\#}(\phi \llcorner A)$ measurable for every $A \subset X$.

(d) Verify that all subsets of a set X are measurable with respect to the counting measure.

Problem 3. Suppose ϕ measures X. Prove that if A is a ϕ measurable set and $B \subset X$, then

$$\phi(A) + \phi(B) = \phi(A \cap B) + \phi(A \cup B).$$

Problem 4. Show that, if $\phi(X) < \infty$ is a regular measure, $f : X \to Y$ and C is an $f_{\#}\ \phi$ measurable set, then $f^{-1}(C)$ is ϕ measurable.

Problem 5. (Building Regular Measures) Let ϕ be an arbitrary measure over X.

(a) Show that the measure γ defined by the formula

$$\gamma(A) = \inf\{\phi(B) : A \subset B \text{ and } B \text{ is } \phi \text{ measurable}\}$$

for $A \subset X$ is regular.

(b) If A is ϕ measurable, then A is γ measurable and $\phi(A) = \gamma(A)$.

(c) If A is γ measurable and $\gamma(A) < \infty$, then A is ϕ measurable.

Problem 6*. (Ulam Numbers)

(a) Show that $\aleph_0$ is an Ulam number.

(b) Show that the class of all Ulam numbers is an initial segment in the well ordered class of all cardinal numbers.

(c) Show that if there exist any cardinal numbers which are not Ulam numbers, the smallest such number cannot be accessible.

Additional Exercises

Problem 7*. Define functions $\mu_1, \dots \mu_6$ on $\mathbf{2}^X$ by

$$\mu_1(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ 1 & \text{if } A \text{ is nonempty,} \end{cases}$$

$$\mu_2(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ +\infty & \text{if } A \text{ is nonempty,} \end{cases}$$

$$\mu_3(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded,} \end{cases}$$

$$\mu_4(A) = \begin{cases} 0 & \text{if } A \text{ is bounded,} \\ 1 & \text{if } A \text{ is unbounded,} \end{cases}$$

$$\mu_5(A) = \begin{cases} 0 & \text{if } A \text{ is empty,} \\ 1 & \text{if } A \text{ is nonempty and bounded,} \\ +\infty & \text{if } A \text{ is unbounded} \end{cases}$$

$$\mu_6(A) = \begin{cases} 0 & \text{if } A \text{ is countable, and,} \\ +\infty & \text{if } A \text{ is uncountable,} \end{cases}$$

Which of the above functions define measures? For each one that defines a measure, what are the respective measurable subsets of $\mathbf{R}$?

Problem 8. (The 1-Dimensional Lebesgue Measure) For each subset A of $\mathbf{R}$, define the **1-Dimensional Lebesgue Measure**

$$\mathcal{L}^1 : \mathbf{2}^X \to [0, \infty]$$

by

$$\mathcal{L}^1(A) = \inf \sum_i (b_i - a_i)$$

where the infimum is taken over all collections $\mathcal{C}_A = \{(a_i, b_i)\}$ of open intervals whose union $\bigcup_i (a_i, b_i)$ covers A. I.e., $A \subseteq \bigcup_i (a_i, b_i)$. Show that $\mathcal{L}^1(C) = 0$ for every countable subset C of $\mathbf{R}$.