Exercise 26 Using linear resolution, prove that $(P \land Q) \to (R \land S)$ follows from $P \to R$ and $R \land P \to S$.

Exercise 27 Convert these axioms to clauses, showing all steps. Then prove $\mathit{Winterstorm} \to \mathit{Miserable}$ by resolution:

$$\begin{array}{cc} \mathit{Rain} \land (\mathit{Windy} \lor \neg\mathit{Umbrella}) \to \mathit{Wet} & \mathit{Winterstorm} \to \mathit{Storm} \land \mathit{Cold} \\ \mathit{Wet} \land \mathit{Cold} \to \mathit{Miserable} & \mathit{Storm} \to \mathit{Rain} \land \mathit{Windy} \end{array}$$

8 Skolem Functions and Herbrand's Theorem

Propositional logic is the basis of many proof methods for first-order logic. Eliminating the quantifiers from a first-order formula reduces it nearly to propositional logic. This section describes how to do so.

8.1 Prenex normal form

The simplest method of eliminating quantifiers from formula involves first moving them to the front.

Definition 11 A formula is in *prenex normal form* if and only if it has the form

$$\underbrace{Q_1 x_1 \; Q_2 x_2 \; \cdots \; Q_n x_n}_{\text{prefix}} \; \underbrace{(A)}_{\text{matrix}},$$

where A is quantifier-free, each Q_i is either $\forall$ or $\exists$, and $n \geq 0$. The string of quantifiers is called the *prefix* and A is called the *matrix*.

Using the equivalences described above, any formula can be put into prenex normal form.

Examples of translation.

The affected subformulæ will be underlined.

Example 16 Start with

$$\neg(\underline{\exists x\, P(x)}) \land (\exists y\, Q(y) \lor \forall z\, P(z))$$

Pull out the $\exists x$:

$$\forall x\, \neg P(x) \land (\underline{\exists y\, Q(y)} \lor \forall z\, P(z))$$

Pull out the $\exists y$:

$$\forall x \neg P(x) \wedge (\underline{\exists y\,(Q(y) \vee \forall z\, P(z))})$$

Pull out the $\exists y$ again:

$$\exists y\,(\forall x \neg P(x) \wedge (Q(y) \vee \underline{\forall z\, P(z)}))$$

Pull out the $\forall z$:

$$\exists y\,(\forall x \neg P(x) \wedge \underline{\forall z\,(Q(y) \vee P(z))})$$

Pull out the $\forall z$ again:

$$\exists y\,\forall z\,(\underline{\forall x \neg P(x)} \wedge (Q(y) \vee P(z)))$$

Pull out the $\forall x$:

$$\exists y\,\forall z\,\forall x\,(\neg P(x) \wedge (Q(y) \vee P(z)))$$

Example 17 Start with

$$\forall x\; P(x) \to \exists y\,\forall z\, R(y,z)$$

Remove the implication:

$$\neg\forall x\; P(x) \vee \exists y\,\forall z\, R(y,z)$$

Pull out the $\forall x$:

$$\exists x\, \neg P(x) \vee \exists y\,\forall z\, R(y,z)$$

Distribute $\exists$ over $\vee$, renaming y to x:[6]

$$\exists x\,(\neg P(x) \vee \forall z\, R(x,z))$$

Finally, pull out the $\forall z$:

$$\exists x\,\forall z\,(\neg P(x) \vee R(x,z))$$

8.2 Removing quantifiers: Skolem form

Now that the quantifiers are at the front, let's eliminate them! We replace every existentially bound variable by a Skolem constant or function. This transformation does not preserve the meaning of a formula; it does preserve *inconsistency*, which is the critical property, since resolution works by detecting contradictions.

[6] Or simply pull out the quantifiers separately. Using the distributive law is marginally better here because it will result in only one Skolem constant instead of two; see the following section.

How to Skolemize a formula

Suppose the formula is in prenex normal form.[7] Starting from the left, if the formula contains an existential quantifier, then it must have the form

$$\forall x_1 \forall x_2 \cdots \forall x_k \exists y\, A$$

where A is a prenex formula, $k \geq 0$, and $\exists y$ is the leftmost existential quantifier. Choose a k-place function symbol not present in A (that is, a *new* function symbol). Delete the $\exists y$ and replace all other occurrences of y by $f(x_1, x_2, \ldots, x_k)$. The result is another prenex formula:

$$\forall x_1 \forall x_2 \cdots \forall x_k\, A[f(x_1, x_2, \ldots, x_k)/y]$$

If $k = 0$ above then the prenex formula is simply $\exists y\, A$, and other occurrences of y are replaced by a new constant symbol c. The resulting formula is $A[c/y]$.

The remaining existential quantifiers, if any, are in A. Repeatedly eliminate all of them, as above. The new symbols are called *Skolem functions* (or Skolem constants).

After Skolemization the formula is just $\forall x_1 \forall x_2 \cdots \forall x_k\, A$ where A is quantifier-free. Since the free variables in a formula are taken to be universally quantified, we can drop these quantifiers, leaving simply A. We are almost back to the propositional case, except the formula typically contains terms. We shall have to handle constants, function symbols, and variables.

Examples of Skolemization

The affected expressions are underlined.

Example 18 Start with

$$\underline{\exists x}\, \forall y\, \exists z\, R(\underline{x}, y, z)$$

Eliminate the $\exists x$ using the Skolem constant a:

$$\forall y\, \underline{\exists z}\, R(a, y, \underline{z})$$

Eliminate the $\exists z$ using the 1-place Skolem function f:

$$\forall y\, R(a, y, f(y))$$

Finally, drop the $\forall y$ and convert the remaining formula to a clause:

$$\{R(a, y, f(y))\}$$

[7] This makes things easier to follow. However, some proof methods merely require the formula to be in negation normal form. The basic idea is the same: remove the outermost existential quantifier, replacing its bound variable by a Skolem term. Pushing quantifiers in as far as possible, instead of pulling them out, yields a better set of clauses.

Example 19 Start with

$$\underline{\exists u}\,\forall v\,\exists w\,\exists x\,\forall y\,\exists z\,((P(h(\underline{u}, v)) \vee Q(w)) \wedge R(x, h(y, z)))$$

Eliminate the $\exists u$ using the Skolem constant c:

$$\forall v\,\underline{\exists w}\,\exists x\,\forall y\,\exists z\,((P(h(c, v)) \vee Q(\underline{w})) \wedge R(x, h(y, z)))$$

Eliminate the $\exists w$ using the 1-place Skolem function f:

$$\forall v\,\underline{\exists x}\,\forall y\,\exists z\,((P(h(c, v)) \vee Q(f(v))) \wedge R(\underline{x}, h(y, z)))$$

Eliminate the $\exists x$ using the 1-place Skolem function g:

$$\forall v\,\forall y\,\underline{\exists z}\,((P(h(c, v)) \vee Q(f(v))) \wedge R(g(v), h(y, \underline{z})))$$

Eliminate the $\exists z$ using the 2-place Skolem function j (note that function h is already used!):

$$\forall v\,\forall y\,((P(h(c, v)) \vee Q(f(v))) \wedge R(g(v), h(y, j(v, y))))$$

Finally drop the universal quantifiers, getting a set of clauses:

$$\{P(h(c, v)), Q(f(v))\} \qquad \{R(g(v), h(y, j(v, y)))\}$$

Correctness of Skolemization

Skolemization does *not* preserve meaning. The version presented above does not even preserve validity! For example,

$$\exists x\,(P(a) \to P(x))$$

is valid. (Why? In any model, the required value of x exists — it is just the value of a in that model.)

Replacing the $\exists x$ by the Skolem constant b gives

$$P(a) \to P(b)$$

This has a different meaning since it refers to a constant b not previously mentioned. And it is not valid! For example, it is false in the interpretation where $P(x)$ means 'x equals 0' and a denotes 0 and b denotes 1.

Our version of Skolemization does preserve *consistency* — and therefore inconsistency. Consider one Skolemization step.