Q&A SD June 8

Sunday, June 8, 2025 4:58 PM

$$\begin{cases}\dot{x} = -y - x(x^2+y^2)\\ \dot{y} = x - y(x^2+y^2)\end{cases}$$

$\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ $\exists!$ $(\rho,\theta) \in (0,\infty)\times[0,2\pi)$ s.t. $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$.

$$\Rightarrow \begin{cases}\rho^2 = x^2+y^2\\ tg\,\theta = \dfrac{y}{x}\end{cases}$$

$$\begin{cases}\dot{\rho} = f_1(\rho,\theta)\\ \dot{\theta} = f_2(\rho,\theta)\end{cases}$$

$$\begin{cases}\rho^2(t) = x^2(t)+y^2(t)\\ \tan\theta(t) = \dfrac{y(t)}{x(t)}\end{cases}$$

$\Rightarrow$ take the derivative w.r.t. t

$$\begin{cases}2\rho\dot{\rho} = 2x\dot{x} + 2y\dot{y} \quad |:2\\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2}\end{cases}$$

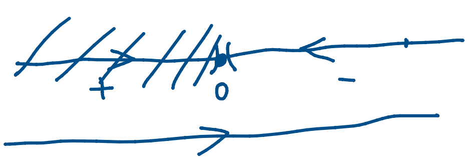
$$\rho\dot{\rho} = -\cancel{xy} - x^2(x^2+y^2) + \cancel{xy} - y^2(x^2+y^2)$$

$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{x^2 - \cancel{xy(x^2+y^2)} + y^2 + \cancel{xy(x^2+y^2)}}{x^2}$$

$$\Rightarrow \rho\dot{\rho} = -(x^2+y^2)^2 \Rightarrow \rho\dot{\rho} = -\rho^4 \Rightarrow \dot{\rho} = -\rho^3$$

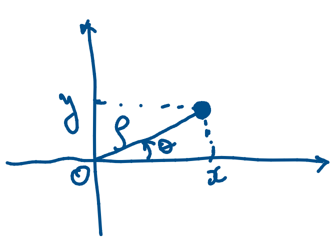
$$\frac{\dot{\theta}}{\cos^2\theta} = \frac{x^2+y^2}{x^2} \Rightarrow \frac{\dot{\theta}}{\cos^2\theta} = \frac{\rho^2}{\rho^2\cos^2\theta} \Rightarrow \dot{\theta} = 1$$

$$\Rightarrow \begin{cases}\dot{\rho} = -\rho^3\\ \dot{\theta} = 1\end{cases}$$



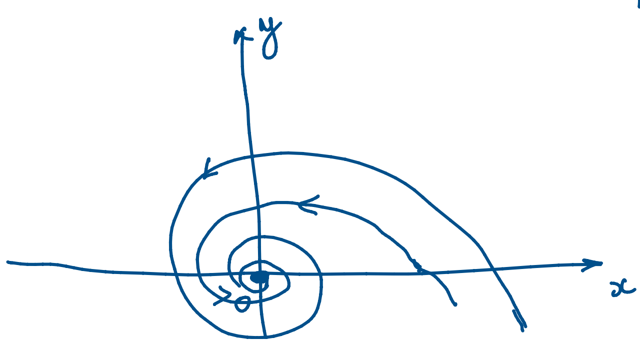
$\dot{\rho} < 0 \Rightarrow \rho$ is strictly decreasing along each orbit

$\dot{\theta} > 0 \Rightarrow \theta$ is strictly increasing along each orbit $\Rightarrow$

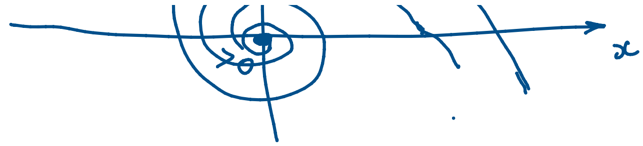


$\Rightarrow$ the orbit rotates around the origin in the trigonometric sense

When a point moves along each orbit, it approaches the origin



$(0,0)$ equilibrium point



$$\begin{cases} \dot{\rho} = -\rho^3 \\ \rho(0) = \rho_0 \end{cases}$$ we have $\lim_{t\to\infty} \rho(t) = 0 \Rightarrow \lim_{t\to\infty} \sqrt{x^2(t)+y^2(t)} = 0 \Rightarrow \lim_{t\to\infty} \|\varphi(t)\| = 0$

for each starting point.

$\Rightarrow (0,0)$ is a global attractor

$\eta > 0$ suff. small $\qquad f(x) = x^2 - 3 \qquad x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)}, \ k \geq 0$

$x_0 = \eta + \sqrt{3}$

$g(x) = x - \dfrac{f(x)}{f'(x)} \qquad x_{k+1} = g(x_k), \ k \geq 0 \Rightarrow x_k = g^k(x_0), \ k \geq 0$

$$g(x) = x - \frac{x^2-3}{2x} = \frac{2x^2 - x^2 + 3}{2x} = \frac{x^2+3}{2x}$$

find the fixed points of g, i.e. solve the eq. $g(x) = x \Leftrightarrow \dfrac{x^2+3}{2x} = x \Leftrightarrow$

$\Leftrightarrow x^2 + 3 = 2x^2 \Leftrightarrow x^2 = 3 \Leftrightarrow x_1 = +\sqrt{3}, \ x_2 = -\sqrt{3}$

for the linearization method, compute $g'(x) = \dfrac{2x \cdot 2x - (x^2+3)\cdot 2}{4x^2} = \dfrac{4x^2 - 2x^2 - 6}{4x^2} = \dfrac{2(x^2-3)}{4x^2}$

$|g'(\sqrt{3})| = 0 < 1 \Rightarrow \sqrt{3}$ is a (super) attractor

$|g'(-\sqrt{3})| = 0 < 1 \Rightarrow (-\sqrt{3})$ — // —

$\sqrt{3}$ is attractor $\Rightarrow$ (by def) $\exists\, r > 0$ s.t. for $|x_0 - \sqrt{3}| < r$ we have $\lim_{k\to\infty} g^k(x_0) = \sqrt{3}$.

Since $\eta > 0$ is suff. small (hyp) we can consider $\eta < r$.

and, since $x_0 = \eta + \sqrt{3}$ we have $|x_0 - \sqrt{3}| < r$.

Thus, $x_k = g^k(x_0)$ is convergent to $\sqrt{3}$.

$f(x) = x^{1/3} \qquad g(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{x^{1/3}}{\frac{1}{3}x^{-\frac{2}{3}}} = x - 3 \cdot x^{1/3} \cdot x^{\frac{2}{3}} = x - 3x =$

$= -2x$

$f'(x)$ $\quad \frac{1}{3}x^{-\frac{2}{3}}$ $\quad = -2x$

$$\begin{cases} x_{k+1} = -2x_k \\ x_0 = \eta \end{cases}$$

$x_{k+1} + 2x_k = 0$

$r + 2 = 0 \qquad r = -2 \longmapsto (-2)^k \qquad \Rightarrow x_k = c\cdot(-2)^k, \ c \in \mathbb{R}$

$$\left.\begin{array}{l} x_0 = c \\ x_0 = \eta \end{array}\right\} \Rightarrow c = \eta$$

$x_k = \eta(-2)^k, \ k \geq 0.$

$x = 0$ is the only fixed point of g

$\eta, -2\eta, 4\eta, -8\eta, 16\eta, \ldots$
is not convergent, is unbounded

$\ddot{\theta} + \omega^2\theta = 0 \qquad \omega > 0$

$$\begin{cases} x = \theta \\ y = \dot{\theta} \end{cases} \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x \end{cases}$$

Try to find a first integral $\qquad \dfrac{dy}{dx} = \dfrac{-\omega^2 x}{y}$

Separate the variables $\qquad y\,dy = -\omega^2 x\,dx$

integrate $\qquad \int y\,dy = -\omega^2 \int x\,dx \Leftrightarrow \dfrac{y^2}{2} = -\omega^2\cdot\dfrac{x^2}{2} + c \ \Big|\cdot 2 \Leftrightarrow$

$\Leftrightarrow \quad \underline{\omega^2x^2 + y^2} = c, \quad c \in \mathbb{R}$

We take $\quad H(x,y) = \omega^2x^2 + y^2 \qquad H: \mathbb{R}^2 \to \mathbb{R} \quad C^1$

H is a f.i. in $\mathbb{R}^2$ iff $\qquad y\cdot\dfrac{\partial H}{\partial x} + (-\omega^2 x)\dfrac{\partial H}{\partial y} = 0$ in $\mathbb{R}^2$ iff

$y\cdot 2\omega^2 x + (-\omega^2 x)\cdot 2y = 0$ in $\mathbb{R}^2$ TRUE.

Conclusion: H is a first integral in $\mathbb{R}^2$.

We represent the level curves of H, i.e. the planar curves of (implicit) eq. $\quad \omega^2x^2 + y^2 = c, \ c \in \mathbb{R}$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a, b > 0.$$

$c = 1 \qquad \omega^2x^2 + y^2 = 1 \Leftrightarrow \dfrac{x^2}{\left(\frac{1}{\omega}\right)^2} + \dfrac{y^2}{1^2} = 1$ ellipse

$c = 2 \qquad \omega^2x^2 + y^2 = 2 \Leftrightarrow \quad x^2 \quad y \quad 1$ ellipse.

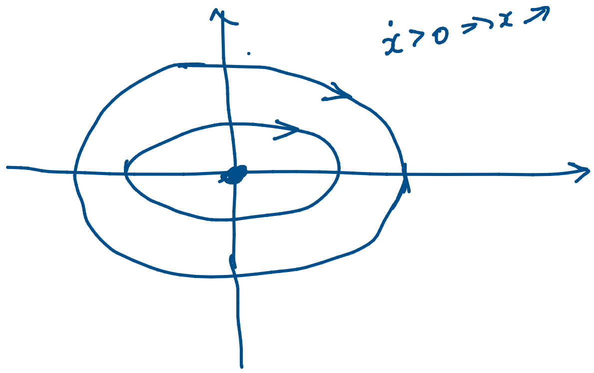
(ω)

$c=2 \quad \omega^2x^2+y^2=2 \iff \dfrac{x^2}{\left(\frac{\sqrt{2}}{\omega}\right)^2}+\dfrac{y^2}{(\sqrt{2})^2}=1$ ellipse.

$c=0 \quad \omega^2x^2+y^2=0$

$\Rightarrow x=y=0$

$\dot{x}>0 \Rightarrow x \nearrow$



the level curves of H are the orbits

$\dot{x}=y$

$x''+tx'=1$ second order LnHDE with non-constant coeff.

$y=x' \qquad y(t)=x'(t) \Rightarrow x''=y'$ first order LnHDE with non-constant coeff.

$\Rightarrow \quad y'+ty=1$

$\mu(t)=e^{\frac{t^2}{2}}$

$y'\cdot e^{t^2/2}+t\cdot e^{t^2/2}\cdot y=e^{t^2/2}$

$\left(y\cdot e^{\frac{t^2}{2}}\right)'=e^{\frac{t^2}{2}} \qquad y\cdot e^{\frac{t^2}{2}}=\int_0^t e^{\frac{s^2}{2}}ds+c, \quad c\in\mathbb{R}$

$y=e^{-t^2/2}\int_0^t e^{\frac{s^2}{2}}ds+c\cdot e^{-\frac{t^2}{2}}$

$x(0)=x'(0)=0$

$x'=e^{-t^2/2}\int_0^t e^{s^2/2}ds+c\cdot e^{-t^2/2}$

$x'(0)=c \quad , x'(0)=0 \Rightarrow c=0$

$x'=e^{-t^2/2}\int_0^t e^{\frac{s^2}{2}}ds \Rightarrow x(t)=\int_0^t e^{-\frac{u^2}{2}}\int_0^u e^{\frac{s^2}{2}}ds\,du+c$

$x(0)=c=0$

$x(t)=\int^t e^{-\frac{u^2}{2}}\int^u e^{\frac{s^2}{2}}ds\,du$

$$\boxed{x(t) = \int_0^t e^{-\frac{u^2}{2}} \int_0^u e^{\frac{s^2}{2}} ds\, du}$$

$\ddot{\theta} + w^2 \sin\theta = 0$

$\begin{cases} x = \theta \\ y = \dot{\theta} \end{cases}$ $\begin{cases} \dot{x} = y \\ \dot{y} = -w^2 \sin x \end{cases}$ $\frac{dy}{dx} = \frac{-w^2 \sin x}{y}$

$y\,dy = -w^2 \sin x\,dx$ $\int y\,dy = -w^2 \int \sin x\,dx$

$\frac{y^2}{2} = +w^2 \cos x + c$ $y^2 = 2w^2 \cos x + 2c$

$H(x,y) = y^2 - 2w^2 \cos x$, $H : \mathbb{R}^2 \to \mathbb{R}$ C^1

H is a f.i. in $\mathbb{R}^2$ iff $y \frac{\partial H}{\partial x} + (-w^2 \sin x)\frac{\partial H}{\partial y} = 0$ in $\mathbb{R}^2$

TRUE.

$\dot{x} = x - 4x^2$

$A = \begin{pmatrix} 14 & 0 \\ -4 & -14 \end{pmatrix}$ $\dot{X} = AX$ $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{cases} \dot{x} = 14x \\ \dot{y} = -4x - 14y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$ $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$.

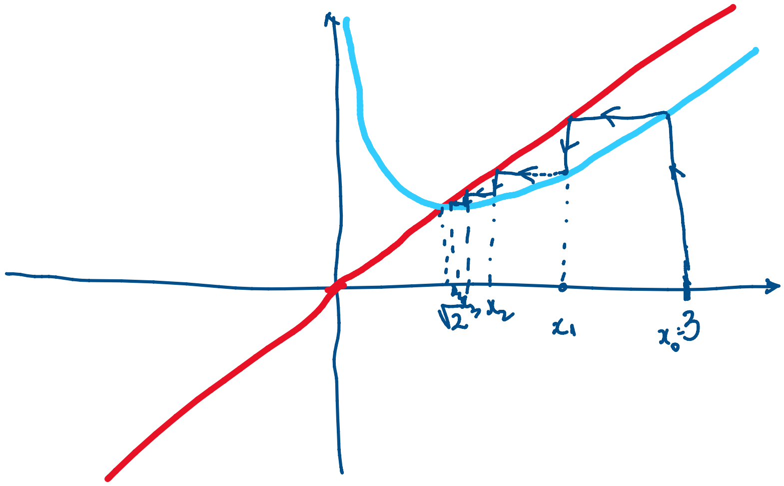
$\varphi(t, \eta_1, \eta_2)$

$g(x) = x^2 - 2$ $f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2-2}{2x} = \frac{x^2+2}{2x}$

$x_{..} = f(x_0)$ $k \in \mathbb{N}$

$g'(x)$

$x_{k+1} = f(x_k), \quad k \in \mathbb{N}$



$x_0 = 1 \quad x_1 = f(1) = \frac{3}{2} \qquad x_2 = f(x_1) = f\left(\frac{3}{2}\right) = \dots$