In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K} = (O, \mathcal{B})$ where $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$.

5.1. Consider the vectors $\mathbf{a}(3,-1,-2)$ and $\mathbf{b}(1,2,-1)$. Calculate

$$\mathbf{a} \times \mathbf{b}, \quad (2\mathbf{a}+\mathbf{b}) \times \mathbf{b}, \quad (2\mathbf{a}+\mathbf{b}) \times (2\mathbf{a}-\mathbf{b}).$$

5.2. Consider the points $A(1,2,0)$, $B(3,0,-3)$, $C(5,2,6)$, $D(1,0,1)$. Determine

a) the area of the triangle ABC and the distance from C to AB.

b) the volume of the tetrahedron $ABCD$ and the distance from D to ABC.

c) the common perpendicular line of the lines AB and CD.

5.3. Determine the surface area and the volume of a regular tetrahedron.

5.4. Consider two lines ℓ_1 and ℓ_2 in $\mathbb{E}^3$. Suppose that the common perpendicular line is

$$\ell : \begin{cases} x = 1 + t \\ y = 2 - t \\ z = t \end{cases},$$

that $P_1(1,0,1) \in \ell_1$ and that $P_2(-1,1,0) \in \ell_2$. Determine the two lines.

5.5. (Law of Sines) Let ABC be a triangle and let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, $\mathbf{w} = \overrightarrow{CA}$. Show that

$$\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{w} = \mathbf{w} \times \mathbf{u}.$$

and deduce the law of sines in a triangle.

5.6. Fix $\mathbf{v} \in \mathbb{V}^3$ and let $\phi : \mathbb{V}^3 \to \mathbb{V}^3$ be the map $\phi(\mathbf{w}) = \mathbf{v} \times \mathbf{w}$. Is the map linear? Explain why. Give the matrix of ϕ relative to a right oriented orthonormal basis. What changes if we define ϕ by $\phi(\mathbf{w}) = \mathbf{w} \times \mathbf{v}$?

5.7. Prove the following identities:

a) the Jacobi identity,

b) the Lagrange identity:

$$\langle \mathbf{a} \times \mathbf{b}, \mathbf{c} \times \mathbf{d} \rangle = \langle \mathbf{a}, \mathbf{c} \rangle \cdot \langle \mathbf{b}, \mathbf{d} \rangle - \langle \mathbf{b}, \mathbf{c} \rangle \cdot \langle \mathbf{a}, \mathbf{d} \rangle = \begin{vmatrix} \langle \mathbf{a}, \mathbf{c} \rangle & \langle \mathbf{a}, \mathbf{d} \rangle \\ \langle \mathbf{b}, \mathbf{c} \rangle & \langle \mathbf{b}, \mathbf{d} \rangle \end{vmatrix} \quad \forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{V}^3,$$

c) the formula for the cross product of two cross products:

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{b} \cdot [\mathbf{a}, \mathbf{c}, \mathbf{d}] - \mathbf{a} \cdot [\mathbf{b}, \mathbf{c}, \mathbf{d}] = \mathbf{c} \cdot [\mathbf{a}, \mathbf{b}, \mathbf{d}] - \mathbf{d} \cdot [\mathbf{a}, \mathbf{b}, \mathbf{c}] \quad \forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{V}^3.$$

From c) deduce a geometric interpretation of Cramer's rule in dimension 3.