3.1 Warm-Up Exercises

3.1. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

3.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

3.3. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

d) π contains the point $M(1,7,1)$ and is parallel to the coordinate plane Oyz,

e) π contains the points $M_1(5,3,4)$, $M_2(1,0,1)$ and is parallel to the vector $\mathbf{a}(1,3,-3)$,

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

3.4. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2+3u-4v,\ y = 4-v,\ z = 2+3u$ with $u,v \in \mathbb{R}$,

b) $\pi : x = u+v,\ y = u-v,\ z = 5+6u-4v$ with $u,v \in \mathbb{R}$.

3.5. Determine parametric equations for the plane π in the following cases:

a) $\pi : 3x-6y+z=0$,

b) $\pi : 2x-y-z-3=0$.

3.6. Show that the points $A(1,0,-1)$, $B(0,2,3)$, $C(-2,1,1)$ and $D(4,2,3)$ are coplanar.

3.7. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $M_0(2,0,3)$ and is parallel to the vector $\mathbf{a}(3,-2,-2)$,

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

c) ℓ contains the points $M_1(1,2,3)$ and $M_2(4,4,4)$.

3.8. Give Cartesian equations for the lines ℓ in the previous exercise.

3.9. Determine parametric equations for the line contained in the planes $x+y+2z-3=0$ and $x-y+z-1=0$.

3.10. In each of the following, find a Cartesian equation for the plane containing the point Q and the line ℓ.

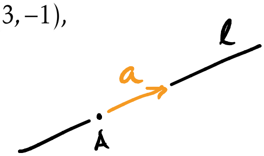
a) $Q=(3,3,1)$, $\ell: x=2+3t, y=5+t, z=1+7t$ with $t \in \mathbb{R}$,

b) $Q=(2,1,0)$, $\ell: x-y+1=0, 3x+5z-7=0$.

Solutions

3.1. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $A(1,2)$ and is parallel to the vector $\mathbf{a}(3,-1)$,



$$\ell : \begin{cases} x = 1+3t \\ y = 2-t \end{cases} \quad t \in \mathbb{R}$$

$$\Leftrightarrow \ \ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} t \quad t \in \mathbb{R}$$

$$\Leftrightarrow \ \ell : x = 1+3t, \ y = 2-t, \ t \in \mathbb{R}$$

b) ℓ contains the origin and is parallel to $\mathbf{b}(4,5)$,

$$\ell : \begin{cases} x = 4s \\ y = 5s \end{cases} \quad s \in \mathbb{R}$$

c) ℓ contains the point $M(1,7)$ and is parallel to Oy,

$\Downarrow$

ℓ has the same direction vectors as Oy
The basis of $\mathbb{V}^2$ is (i, j)
so j is a direction vector for ℓ
The components of j are $(0,1)$

$$\ell : \begin{cases} x = 1 \\ y = 7+t \end{cases} \quad t \in \mathbb{R}$$

d) ℓ contains the points $M(2,4)$ and $N(2,-5)$.

$\hookrightarrow$ $\overrightarrow{MN}(0,-9)$ is a direction vector for ℓ

$$\ell : \begin{cases} x = 2 \\ y = 4-9s \end{cases} \quad s \in \mathbb{R}$$

3.2. For the lines ℓ in the previous exercise

a) determine a Cartesian equation for ℓ,

b) describe all direction vectors for ℓ.

3.1.a)

$$\ell : \begin{cases} x = 1+3t \\ y = 2-t \end{cases} \quad t \in \mathbb{R} \quad \begin{matrix} \Rightarrow t = \frac{x-1}{3} \\ \Rightarrow t = \frac{y-2}{-1} \end{matrix} \Bigg\} \Rightarrow \ell : \frac{x-1}{3} = \frac{y-2}{-1}$$

$\Rightarrow \ell: \; x + 3y - 7 = 0$

since $a(3,-1)$ is a direction vector all other direction vectors are $\{ \lambda \cdot a = (3\lambda, -\lambda) : \lambda \in \mathbb{R} \}$

3.1.b) $\ell: \begin{cases} x = 4s \\ y = 5s \end{cases} \quad s \in \mathbb{R} \quad \begin{matrix} \Rightarrow s = \frac{x}{4} \\ \Rightarrow s = \frac{y}{5} \end{matrix} \Bigg\} \Rightarrow \ell: \frac{x}{4} = \frac{y}{5}$

$\Rightarrow \ell: \; 5x - 4y = 0$

the direction vectors are $\{ \lambda \cdot b(4,5) = (4\lambda, 5\lambda) : \lambda \in \mathbb{R} \}$

3.1.c) $\ell: \begin{cases} x = 1 \\ y = 7 + t \end{cases} \quad t \in \mathbb{R}$

$\Rightarrow \ell: \; x = 1$ (y is arbitrary)

the direction vectors are $\{ \lambda \cdot j = (0, \lambda) : \lambda \in \mathbb{R} \}$

3.1.d) $\ell: \begin{cases} x = 2 \\ y = 4 - 9s \end{cases} \quad s \in \mathbb{R}$

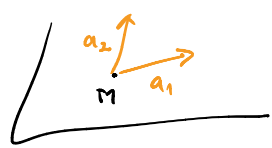
$\Rightarrow \ell: \; x = 2$

since $\overrightarrow{MN} = 9 \cdot j$ ℓ has direction vectors

3.3. Determine parametric equations for the plane π in the following cases:

a) π contains the point $M(1,0,2)$ and is parallel to the vectors $\mathbf{a}_1(3,-1,1)$ and $\mathbf{a}_2(0,3,1)$,

$\pi: \begin{cases} x = 1 + 3t \\ y = -t + 3s \\ z = 2 + t + s \end{cases} \quad t, s \in \mathbb{R}$

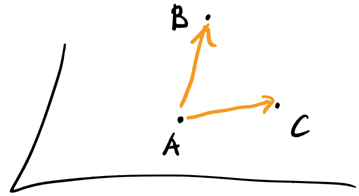


$\Leftrightarrow \pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} s \quad t, s \in \mathbb{R}$

$\Leftrightarrow \pi: \; x = 1 + 3t, \; y = -t + 3s, \; z = 2 + t + s, \; t, s \in \mathbb{R}$

b) π contains the points $A(-2,1,1)$, $B(0,2,3)$ and $C(1,0,-1)$,

$\vec{AB} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \quad \vec{AC} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$



- $\vec{AB}, \vec{AC}$ are vectors represented with points in π
- $\vec{AB}, \vec{AC}$ are linearly independent so they form a basis of the direction space $D(\pi)$

$$\pi : \begin{cases} x = -2 + 2t + 3s \\ y = 1 + t - s \\ z = 2 + 2t - 2s \end{cases} \quad t, s \in \mathbb{R}$$

$$\Leftrightarrow \quad \pi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} s \quad t, s \in \mathbb{R}$$

c) π contains the point $A(1,2,1)$ and is parallel to $\mathbf{i}$ and $\mathbf{j}$,

$i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ since by convention (i, j, k) is a basis of V^3

$$\pi : \begin{cases} x = 1 + t \\ y = 2 + s \\ z = 1 \end{cases} \quad t, s \in \mathbb{R} \quad \Leftrightarrow \quad \pi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad t, s \in \mathbb{R}$$

d) π contains the point $M(1,7,1)$ and is parallel to the coordinate plane Oyz,

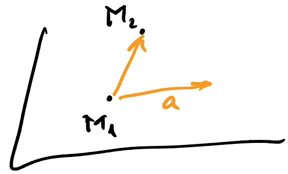
a basis for $D(Oyz)$ is (j, k) $\quad j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\pi : \begin{cases} x = 1 \\ y = 7 + t \\ z = 1 + s \end{cases} \quad t, s \in \mathbb{R} \quad \Leftrightarrow \quad \pi : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad t, s \in \mathbb{R}$$

e) π contains the points $M_1(5,3,4)$, $M_2(1,0,1)$ and is parallel to the vector $\mathbf{a}(1,3,-3)$,

$\overrightarrow{M_2M_1}(4,3,3)$ is a vector represented with points in π so it is parallel to π

$\overrightarrow{M_2M_1}$ and a are linearly indep. vectors
$\Rightarrow$ they form a basis of $D(\pi)$



$$\pi: \begin{cases} x = 5 + 4t + s \\ y = 3 + 3t + 3s \\ z = 4 + 3t - 3s \end{cases} \quad t,s \in \mathbb{R} \iff \pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix} \quad t,s \in \mathbb{R}$$

f) π contains the point $A(1,5,7)$ and the coordinate axis Ox.

$Ox \subseteq \pi \Rightarrow O \in \pi \Rightarrow \overrightarrow{OA}(1,5,7)$ is parallel to π
$\Rightarrow i(1,0,0)$ is parallel to π
since $\overrightarrow{OA}$ and i are linearly independent, they form a basis of $D(\pi)$

$$\pi: \begin{cases} x = t + s \\ y = 5t \\ z = 7t \end{cases} \quad t,s \in \mathbb{R} \iff \pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad t,s \in \mathbb{R}$$

3.4. Determine Cartesian equations for the plane π in the following cases:

a) $\pi : x = 2 + 3u - 4v,\ y = 4 - v,\ z = 2 + 3u$ with $u, v \in \mathbb{R}$,

b) $\pi : x = u + v,\ y = u - v,\ z = 5 + 6u - 4v$ with $u, v \in \mathbb{R}$.

a) $$\pi: \begin{cases} x = 2 + 3u - 4v \\ y = 4 - v \\ z = 2 + 3u \end{cases} \quad u,v \in \mathbb{R} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} + u \cdot 3 \cdot \underset{a}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}} - v \underset{b}{\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}} \quad u,v \in \mathbb{R}$$

$$\pi: \begin{array}{r} \\ a \rightarrow \\ b \rightarrow \end{array} \begin{vmatrix} x-2 & y-4 & z-2 \\ 1 & 0 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 0 \iff (x-2)\cdot(-1) - (y-4)\cdot(-4) + (z-2) = 0$$

$$\Rightarrow \pi: -x + 4y + z - 16 = 0$$

b.) $\pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} + u \begin{bmatrix} 1 \\ 1 \\ 6 \end{bmatrix} + v \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix} \quad u, v \in \mathbb{R}$

$$\Rightarrow \pi: \begin{vmatrix} x & y & z-5 \\ 1 & 1 & 6 \\ 1 & -1 & -4 \end{vmatrix} = 0 \iff x \underbrace{\begin{vmatrix} 1 & 6 \\ -1 & -4 \end{vmatrix}}_{2} - y \underbrace{\begin{vmatrix} 1 & 6 \\ 1 & -4 \end{vmatrix}}_{-10} + (z-5) \underbrace{\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}}_{-2}$$

$$\Rightarrow \pi: x + 5y - z + 5 = 0$$

3.5. Determine parametric equations for the plane π in the following cases:

a) $\pi : 3x - 6y + z = 0$,

b) $\pi : 2x - y - z - 3 = 0$.

a.) choose x and y as parameters $\begin{cases} x = x \\ y = y \\ z = -3x + 6y \end{cases}$

$$\Rightarrow \pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 6 \end{bmatrix} \quad t, s \in \mathbb{R}$$

we relabel x and y

b.) choose $x = t$ and $z = s$ as parameters, then $\begin{cases} x = t \\ y = 2x - z - 3 = 2t - s - 3 \\ z = s \end{cases}$

$$\Rightarrow \pi: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

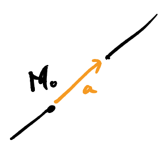
3.6. Show that the points $A(1, 0, -1)$, $B(0, 2, 3)$, $C(-2, 1, 1)$ and $D(4, 2, 3)$ are coplanar.

A, B, C, D coplanar $\iff$ $\vec{AB}, \vec{AC}, \vec{AD}$ linearly dependent, true since

$$\vec{AB} = \begin{vmatrix} -1 \\ 2 \\ 4 \end{vmatrix} \quad \vec{AC} = \begin{vmatrix} -3 \\ 1 \\ 2 \end{vmatrix} \quad \vec{AD} = \begin{vmatrix} 3 \\ 2 \\ 4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} -1 & -3 & 3 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{vmatrix} = -4 + 12 - 24 - 12 + 24 + 4 = 0$$

3.7. Determine parametric equations for the line ℓ in the following cases:

a) ℓ contains the point $M_0(2, 0, 3)$ and is parallel to the vector $\mathbf{a}(3, -2, -2)$,



$$\ell: \begin{cases} x = 2 + 3t \\ y = -2t \\ z = 3 - 2t \end{cases} \quad t \in \mathbb{R} \iff \ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}$$

b) ℓ contains the point $A(1,2,3)$ and is parallel to the Oz-axis,

$\hookrightarrow$ k is a direction vector, $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\ell : \begin{cases} x = 1 \\ y = 2 \\ z = 3 + t \end{cases} \quad t \in \mathbb{R} \iff \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

3.8. Give Cartesian equations for the lines ℓ in the previous exercise.

3.7.a) $$\ell : \begin{cases} x = 2 + 3t \Rightarrow t = \frac{x-2}{3} \\ y = -2t \Rightarrow t = \frac{y}{-2} \\ z = 3 - 2t \Rightarrow t = \frac{z-3}{-2} \end{cases} \Rightarrow \frac{x-2}{3} = \frac{y}{-2} = \frac{z-3}{-2}$$

so we can choose the following two equations

$$\ell : \begin{cases} \frac{x-2}{3} = \frac{y}{-2} \\ \frac{y}{-2} = \frac{z-3}{-2} \end{cases} \iff \ell : \begin{cases} 2x + 3y - 4 = 0 \\ 2y - 2z + 6 = 0 \end{cases}$$

3.7.b) $$\ell : \begin{cases} x = 1 \\ y = 2 \\ z = 3 + t \end{cases}$$ Notice that z is arbitrary

so $$\ell : \begin{cases} x = 1 \\ y = 2 \end{cases} \iff \ell : \begin{cases} x - 1 = 0 \\ y - 2 = 0 \end{cases}$$

3.9. Determine parametric equations for the line contained in the planes $x + y + 2z - 3 = 0$ and $x - y + z - 1 = 0$.

$$\ell : \begin{cases} x + y + 2z - 3 = 0 \\ x - y + z - 1 = 0 \end{cases} \quad (*)$$

$$\begin{pmatrix} 1 & 1 & 2 & -3 \\ 1 & -1 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & -2 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 1 & 1/2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3/2 & -2 \\ 0 & 1 & 1/2 & -1 \end{pmatrix}$$

so system $(*)$ is equivalent to

$$\begin{cases} x + \frac{3z}{2} - 2 = 0 \\ y + \frac{z}{2} - 1 = 0 \end{cases}$$

if we choose the parameter $t = z/2$ then

$$\begin{cases} x = 2-3t \\ y = 1-t \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

3.10. In each of the following, find a Cartesian equation for the plane containing the point Q and the line ℓ.

a) $Q = (3,3,1)$, $\ell : x = 2+3t, y = 5+t, z = 1+7t$ with $t \in \mathbb{R}$,

b) $Q = (2,1,0)$, $\ell : x - y + 1 = 0, 3x + 5z - 7 = 0$.

a) $Q = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$ and $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}}_{= A} + t \underbrace{\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}}_{= v}$ $t \in \mathbb{R}$

so the plane $\begin{cases} \text{contains the points } Q \text{ and } A \\ \text{is parallel to } v \end{cases}$

proceed as for Exercise 3.3.e)

b) $\ell : \begin{cases} x - y + 1 = 0 \\ 3x + 5z - 7 = 0 \end{cases}$

transform these equations into parametric equations for ℓ

then, proceed as for point a)