DATA STRUCTURES AND ALGORITHMS

LECTURE 9

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In Lecture 8...

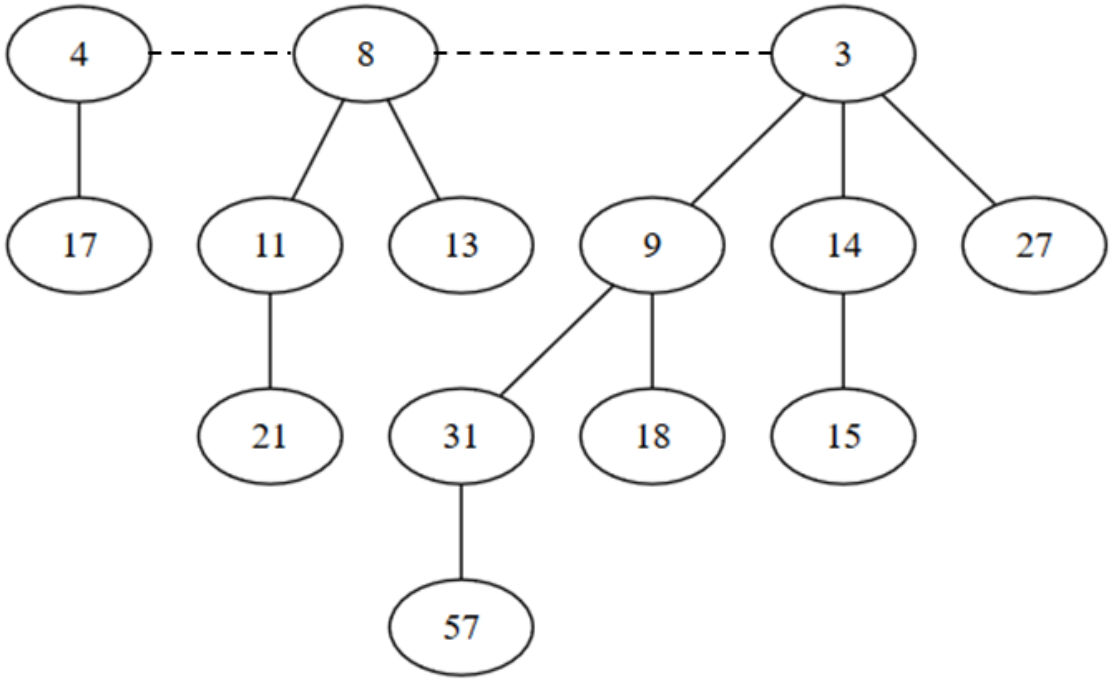
- Binary heap
- Binomial heap

Today

- Binomial heap
- Direct address table
- Hash table
- Collision resolution through separate chaining

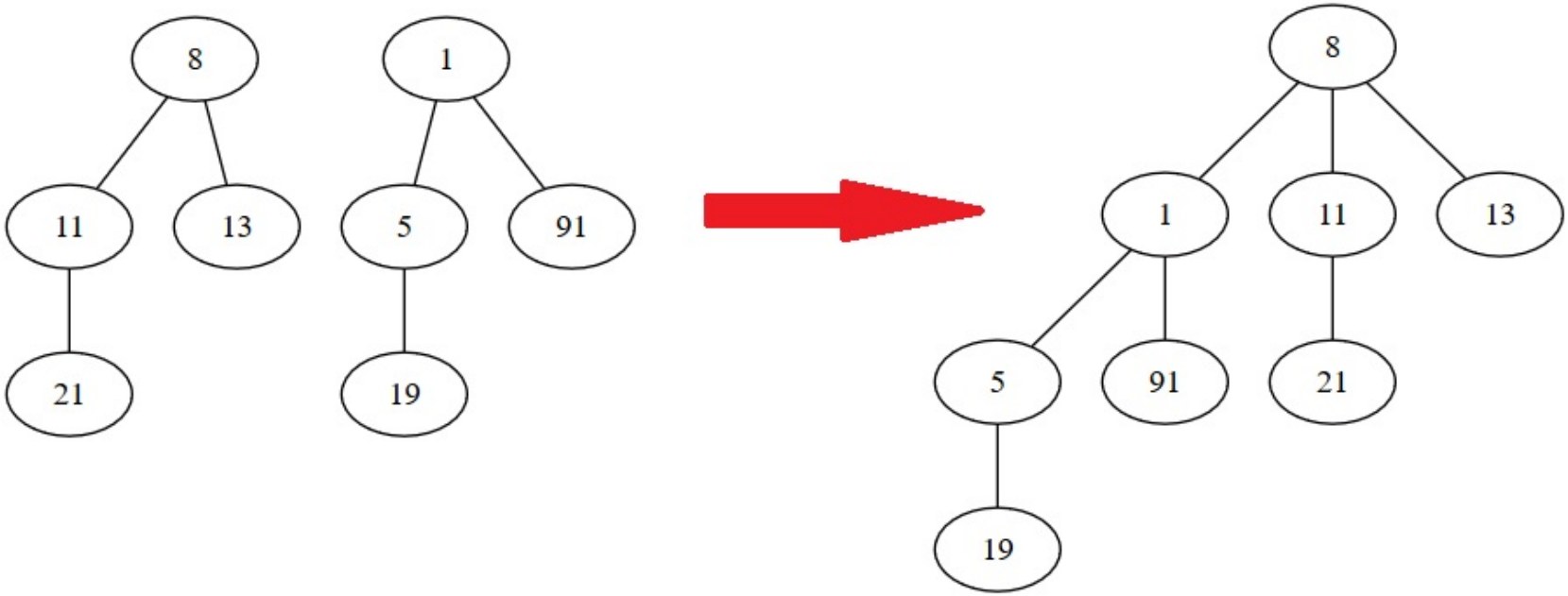
Binomial heap - recap I

- A binomial heap is made up of one or more *binomial trees*, sorted increasingly by their order.



Binomial heap - recap II

- Two binomial trees of the same order, can be merged easily, by making one of them the leftmost child of the other.



Binomial heap - recap III

- Two binomial heaps can be merged by putting all binomial trees together and merging the ones of the same order
- A binomial tree of order k has 2^k nodes and its height is k
- A binomial heap of n elements contains at most $log_2 n$ binomial trees and its height is at most $log_2 n$.

Binomial heap - other operations I

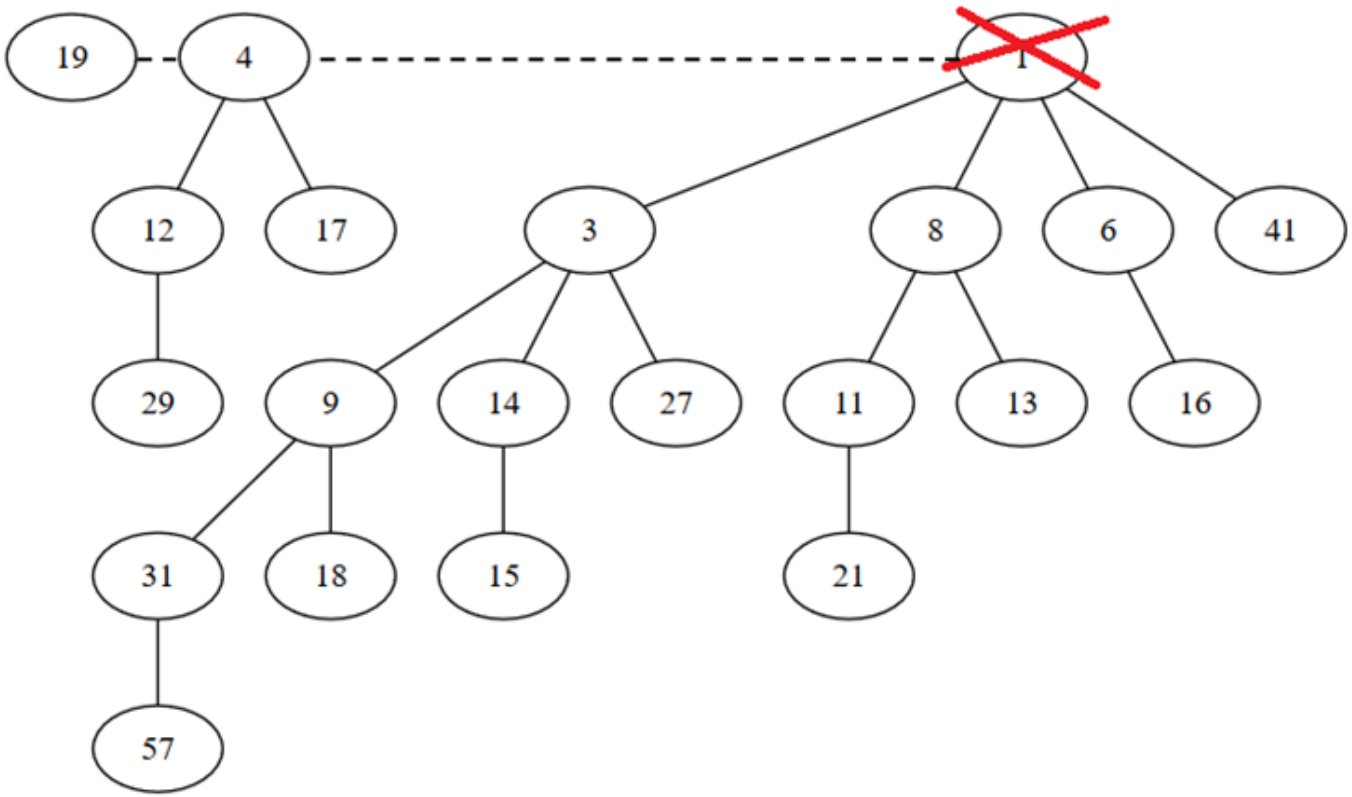
- Most of the other operations that we have for the binomial heap (because we need them for the priority queue) will use the merge operation.
- *Push operation:* Inserting a new element means creating a binomial heap with just that element and merging it with the existing one. Complexity of insert is $\Theta(log_2 n)$ in worst case ($\Theta(1)$ amortized).
- *Top operation:* The minimum element of a binomial heap (the element with the highest priority) is the root of one of the binomial trees. Returning the minimum means checking every root, so it has complexity $O(log_2 n)$.

Binomial heap - other operations II

- *Pop operation:* Removing the minimum element means removing the root of one of the binomial trees. If we delete the root of a binomial tree, we will get a sequence of binomial trees. These trees are transformed into a binomial heap (just reverse their order), and a merge is performed between this new binomial heap and the one formed by the remaining elements of the original binomial heap.

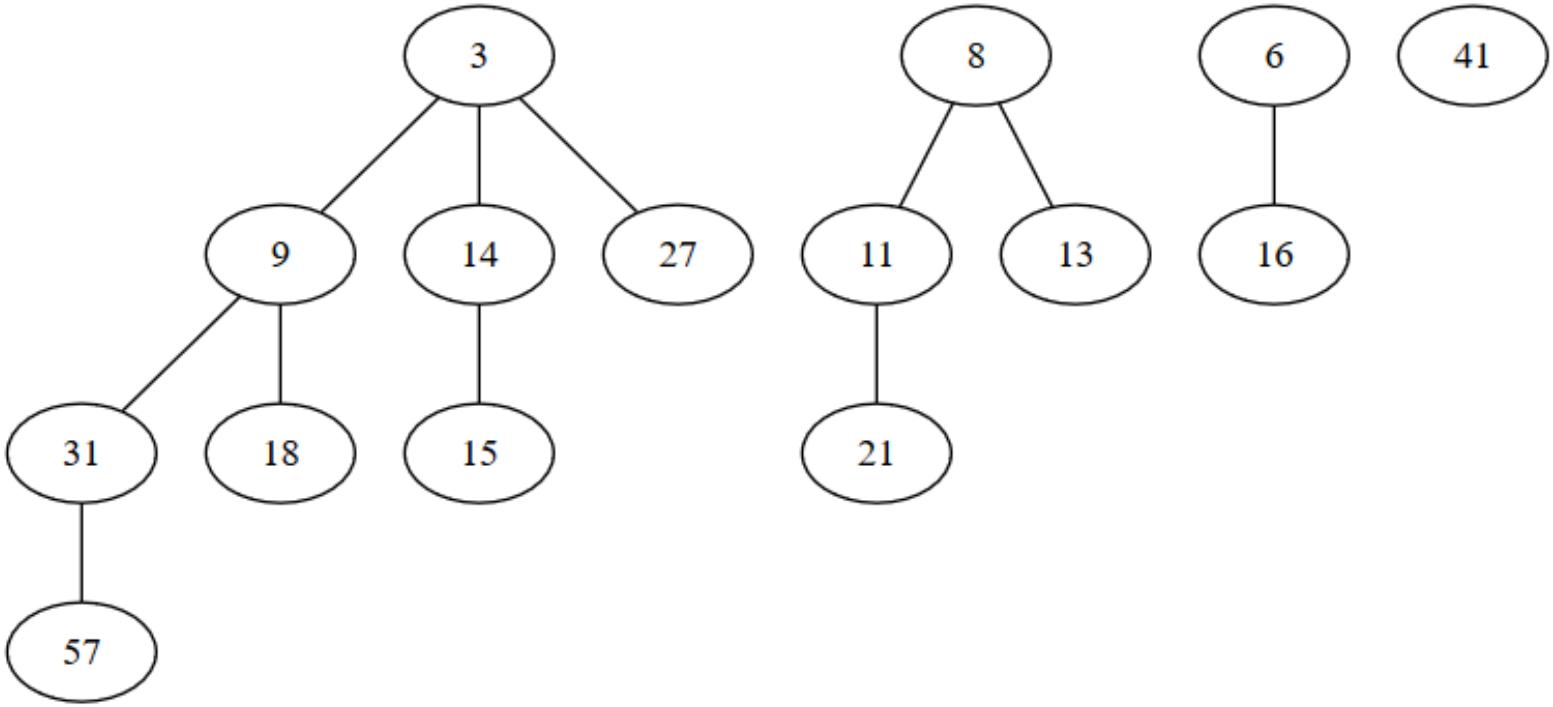
Binomial heap - other operations III

- The minimum is one of the roots.



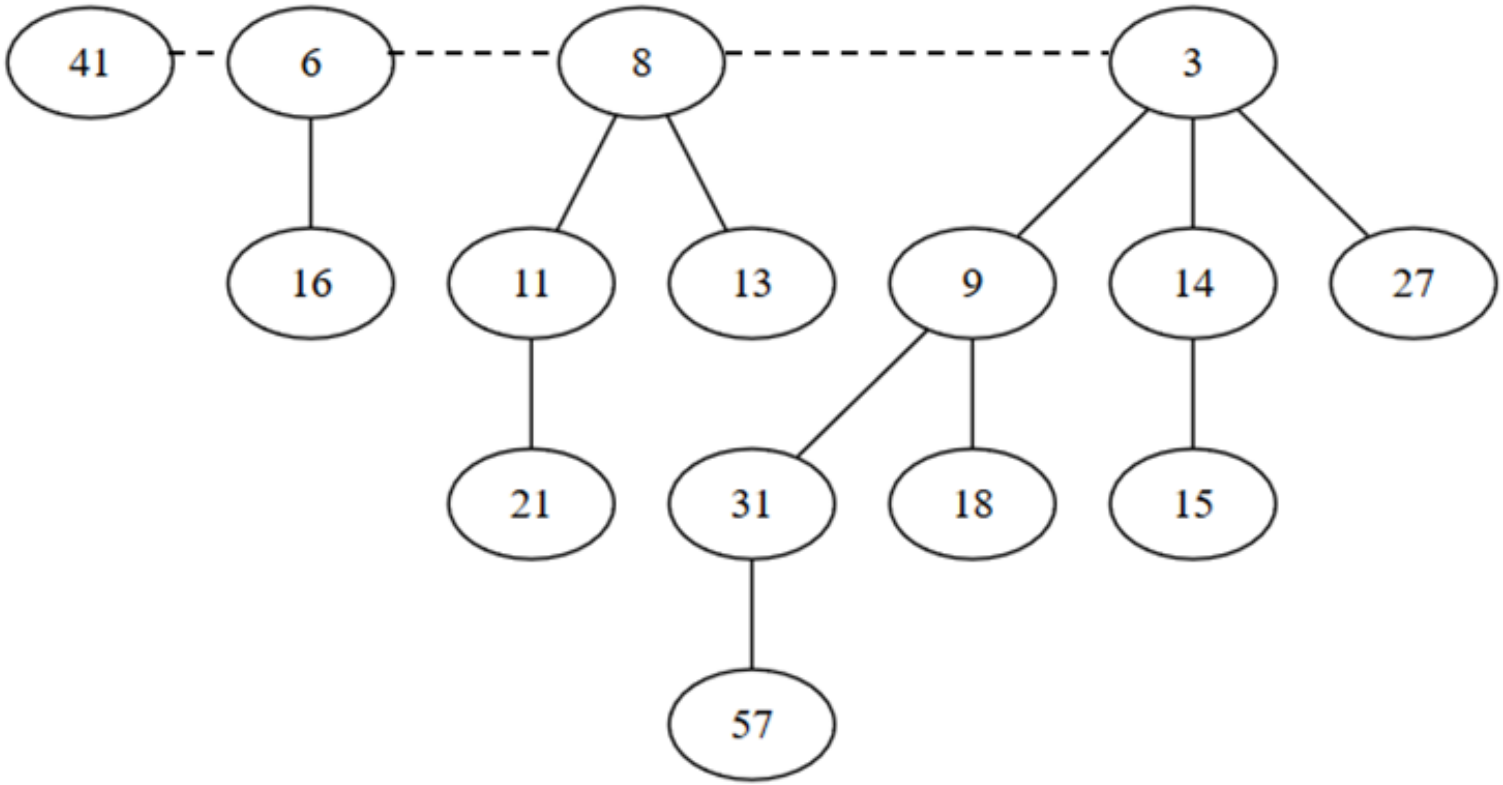
Binomial heap - other operations IV

- Break the corresponding tree into k binomial trees



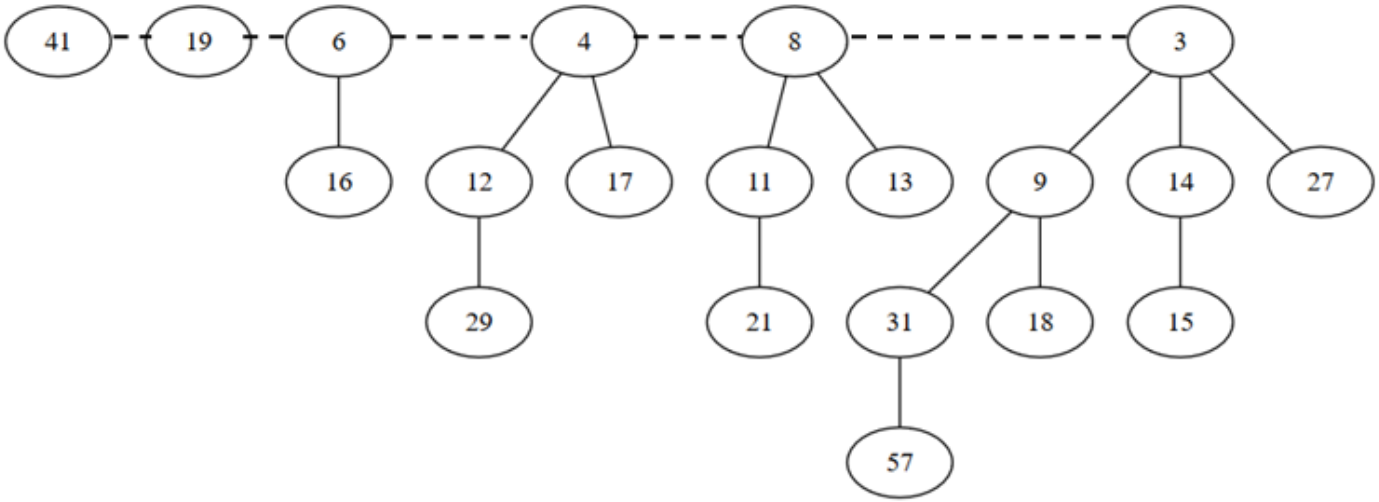
Binomial heap - other operations V

- Create a binomial heap of these trees



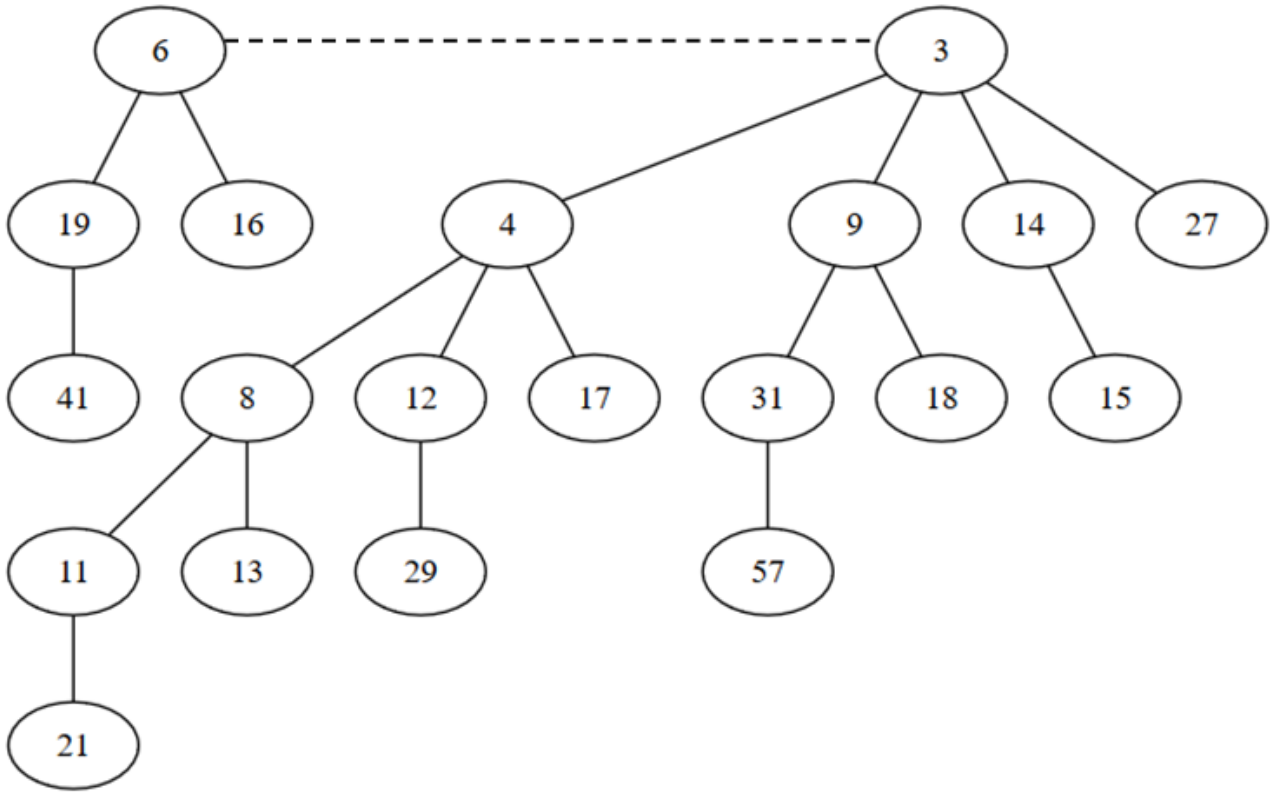
Binomial heap - other operations VI

- Merge it with the existing one (after the merge algorithm)



Binomial heap - other operations VII

- After the transformation



- The complexity of the remove-minimum operation is $O(log_2 n)$

Binomial heap - other operations VIII

- Assuming that we have a pointer to the element whose priority has to be increased (in our figures lower number means higher priority), we can just change the priority and bubble-up the node if its priority is greater than the priority of its parent. Complexity of the operation is: $O(log_2 n)$
- Assuming that we have a pointer to the element that we want to delete, we can first decrease its priority to $-\infty$ (this will move it to the root of the corresponding binomial tree) and remove it. Complexity of the operation is: $O(log_2 n)$

Problems with stacks, queues and priority queues I

- Red-Black Card Game:
  - Statement: Two players each receive $\frac{n}{2}$ cards, where each card can be red or black. The two players take turns; at every turn the current player puts the card from the upper part of his/her deck on the table. If a player puts a red card on the table, the other player has to take all cards from the table and place them at the bottom of his/her deck. The winner is the player that has all the cards.
  - Requirement: Given the number n of cards, simulate the game and determine the winner.
  - Hint: use stack(s) and queue(s)

Problems with stacks, queues and priority queues II

- Robot in a maze:
  - Statement: There is a rectangular maze, composed of occupied cells (X) and free cells (*). There is a robot (R) in this maze and it can move in 4 directions: N, S, E, V.
  - Requirements:
    - Check whether the robot can get out of the maze (get to the first or last line or the first or last column).
    - Find a path that will take the robot out of the maze (if exists).

```
X * * X X X * *
X * X * * * * *
X * * * * * X *
X X X * * * X *
* X * * R X X *
* * * X X X X *
* * * * * * * X
X X X X X X X X
```

Problems with stacks, queues and priority queues III

- Hint:
  - Let T be the set of positions where the robot can get from the starting position.
  - Let S be the set of positions to which the robot can get at a given moment and from which it could continue going to other positions.
  - A possible way of determining the sets T and S could be the following:

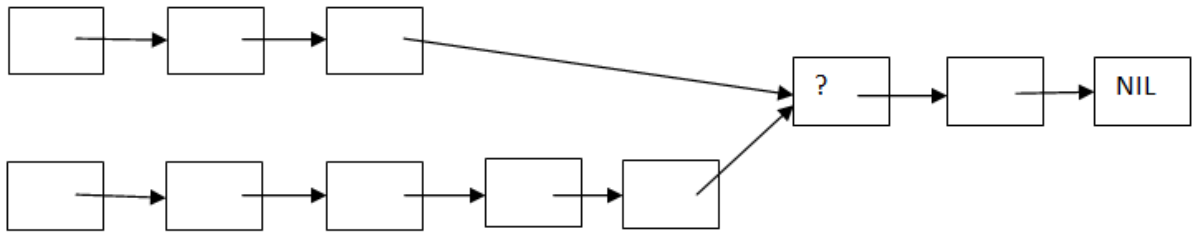
```
T ← {initial position}
S ← {initial position}
while S ≠ ∅ execute
    Let p be one element of S
    S ← S \{p}
    for each valid position q where we can get from p and which is not in T do
        T ← T ∪ {q}
        S ← S ∪ {q}
    end-for
end-while
```

Problems with stacks, queues and priority queues IV

- T can be a list, a vector or a matrix associated to the maze
- S can be a stack or a queue (or even a priority queue, depending on what we want)

Think about it - Linked Lists

- Write a non-recursive algorithm to reverse a singly linked list with $\Theta(n)$ time complexity, using constant space/memory.
- Suppose there are two singly linked lists both of which intersect at some point and become a single linked list (see the image below). The number of nodes in the two list before the intersection is not known and may be different in each list. Give an algorithm for finding the merging point (hint - use a Stack)



Think about it - Stacks and Queues I

- How can we implement a Stack using two Queues? What will be the complexity of the operations?
- How can we implement a Queue using two Stacks? What will be the complexity of the operation?
- How can we implement two Stacks using only one array? The stack operations should throw an exception only if the total number of elements in the two Stacks is equal to the size of the array.

Think about it - Stacks and Queues II

- Given a string of lower-case characters, recursively remove adjacent duplicate characters from the string. For example, for the word "mississippi" the result should be "m".

- Given an integer k and a queue of integer numbers, how can we reverse the order of the first k elements from the queue? For example, if k=4 and the queue has the elements [10, 20, 30, 40, 50, 60, 70, 80, 90], the output should be [40, 30, 20, 10, 50, 60, 70, 80, 90].

Think about it - Priority Queues

- How can we implement a stack using a Priority Queue?
- How can we implement a queue using a Priority Queue?

Example

- Assume that you were asked to write an application for Cluj-Napoca's public transportation service.
- In your application the user can select a bus line and the application should display the timetable for that bus-line (and maybe later the application can be extended with other functionalities).
- Your application should be able to return the info for a bus line and we also want to be able to add and remove bus lines (this is going to be done only by the administrators, obviously).
- And since your application is going to be used by several hundred thousand people, we need it to be very very fast.
  - The public transportation service is willing to maybe rename a few bus lines, if this helps you design a fast application.
- How/Where would you store the data?

Direct-address tables I

- If we want to formalize the problem:
  - We have data where every element has a key (a natural number).
  - The universe of keys (the possible values for the keys) is relatively small, $U = \{0, 1, 2, \ldots, m-1\}$
  - No two elements have the same key
  - We have to support the basic dictionary operations: INSERT, DELETE and SEARCH

Direct-address tables II

- Solution:
  - Use an array T with m positions (remember, the keys belong to the $[0, m-1]$ interval)
  - Data about element with key k, will be stored in the *T[k]* slot
  - Slots not corresponding to existing elements will contain the value NIL (or some other special value to show that they are empty)

Operations for a direct-address table

function search(T, k) **is**:

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Direct-address table - Advantages and disadvantages

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Direct-address table - Advantages and disadvantages

- Advantages of direct address-tables:
  - They are simple
  - They are efficient - all operations run in $\Theta(1)$ time.
- Disadvantages of direct address-tables - restrictions:
  - The keys have to be natural numbers
  - The keys have to come from a small universe (interval)
  - The number of actual keys can be a lot less than the cardinal of the universe (storage space is wasted)

Think about it

- Assume that we have a direct address T of length m. How can we find the maximum element of the direct-address table? What is the complexity of the operation?
- How does the operation for finding the maximum change if we have a hash table, instead of a direct-address table (consider collision resolution by separate chaining, coalesced chaining and open addressing)?

Hash tables

- Hash tables are generalizations of direct-address tables and they represent a *time-space trade-off*.
- Searching for an element still takes $\Theta(1)$ time, but as *average case complexity* (worst case complexity is higher)

Hash tables - main idea I

- We will still have a table T of size m (but now m is not the number of possible keys, $|U|$) - *hash table*
- Use a function h that will map a key k to a slot in the table T - *hash function*

$$h : U \to \{0, 1, ..., m-1\}$$

- Remarks:
  - In case of direct-address tables, an element with key k is stored in $T[k]$.
  - In case of hash tables, an element with key k is stored in $T[h(k)]$.

Hash tables - main idea II

- The point of the hash function is to reduce the range of array indexes that need to be handled $=>$ instead of $|U|$ values, we only need to handle m values.
- Consequence:
  - two keys may hash to the same slot $=>$ **a collision**
  - we need techniques for resolving the conflict created by collisions
- The two main points of discussion for hash tables are:
  - How to define the hash function
  - How to resolve collisions

A good hash function I

- A good hash function:
  - can minimize the number of collisions (but cannot eliminate all collisions)
  - is deterministic
  - can be computed in $\Theta(1)$ time

A good hash function II

- satisfies (approximately) the assumption of simple uniform hashing: **each key is equally likely to hash to any of the m slots, independently of where any other key has hashed to**

$$P(h(k) = j) = \frac{1}{m} \ \forall j = 0, ..., m-1 \ \ \forall k \in U$$

Examples of bad hash functions

- $h(k) =$ constant number

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- $h(k)$ = constant number
- $h(k)$ = random number
- assuming that the keys are CNP numbers:
  - a hash function considering just parts of it (first digit, birth year/date, county code, etc.)
  - assume m = 100 and you use the birth day from the CNP (as a number): h(CNP) = birthday % 100

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- m = 16 and h(k) % m can also be problematic
- etc.

Hash function

- The simple uniform hashing theorem is hard to satisfy, especially when we do not know the distribution of data. Data does not always have a uniform distribution
  - dates
  - group numbers at our faculty
  - postal codes
  - first letter of an English word
- In practice we use heuristic techniques to create hash functions that perform well.

- Most hash functions assume that the keys are natural numbers. If this is not true, they have to be interpreted as natural number. In what follows, we assume that the keys are natural numbers.

The division method

The division method

$h(k) = k \mod m$

For example:

$$m = 13$$
$$k = 63 => h(k) = 11$$
$$k = 52 => h(k) = 0$$
$$k = 131 => h(k) = 1$$

- Requires only a division so it is quite fast
- Experiments show that good values for m are primes not too close to exact powers of 2

The division method

- Interestingly, Java uses the division method with a table size which is power of 2 (initially 16).
- They avoid a problem by using a second function for hashing, before applying the mod:

```
/**
 * Applies a supplemental hash function to a given hashCode, which
 * defends against poor quality hash functions.  This is critical
 * because HashMap uses power-of-two length hash tables, that
 * otherwise encounter collisions for hashCodes that do not differ
 * in lower bits. Note: Null keys always map to hash 0, thus index 0.
 */
static int hash(int h) {
    // This function ensures that hashCodes that differ only by
    // constant multiples at each bit position have a bounded
    // number of collisions (approximately 8 at default load factor).
    h ^= (h >>> 20) ^ (h >>> 12);
    return h ^ (h >>> 7) ^ (h >>> 4);
}
```

Mid-square method

- Assume that the table size is 10^r, for example m = 100 (r = 2)
- For getting the hash of a number, multiply it by itself and take the middle r digits.
- For example, h(4567) = middle 2 digits of 4567 * 4567 = middle 2 digits of 20857489 = 57
- Same thing works for $m = 2^r$ and the binary representation of the numbers
- $m = 2^4$, h(1011) = middle 4 digits of 01111001 = 1110

The multiplication method I

The multiplication method

$$h(k) = floor(m * frac(k * A)) \text{ where}$$
$$m \text{ - the hash table size}$$
$$A \text{ - constant in the range } 0 < A < 1$$
$$frac(k * A) \text{ - fractional part of } k * A$$

For example

m = 13 A = 0.6180339887

k=63 => h(k) = floor(13 * frac(63 * A)) = floor(12.16984) = 12

k=52 => h(k) = floor(13 * frac(52 * A)) = floor(1.790976) = 1

k=129=> h(k)= floor(13 * frac(129 * A)) = floor(9.442999) = 9

The multiplication method II

- Advantage: the value of m is not critical, typically $m = 2^p$ for some integer p
- Some values for A work better than others. Knuth suggests $\frac{\sqrt{5}-1}{2} = 0.6180339887$

Universal hashing I

- If we know the exact hash function used by a hash table, we can always generate a set of keys that will hash to the same position (collision). This reduces the performance of the table.
- For example:

$m = 13$
$h(k) = k \mod m$
k = 11, 24, 37, 50, 63, 76, etc.

Universal hashing II

- Instead of having one hash function, we have a collection $\mathcal{H}$ of hash functions that map a given universe U of keys into the range $\{0, 1, \ldots, m-1\}$
- Such a collection is said to be **universal** if for each pair of distinct keys $x, y \in U$ the number of hash functions from $\mathcal{H}$ for which $h(x) = h(y)$ is precisely $\frac{|\mathcal{H}|}{m}$
- In other words, with a hash function randomly chosen from $\mathcal{H}$ the chance of collision between x and y, where $x \neq y$, is exactly $\frac{1}{m}$

Universal hashing III

Example 1

Fix a prime number $p >$ *the maximum possible value for a key from* U.
For every $a \in \{1, \ldots, p-1\}$ and $b \in \{0, \ldots, p-1\}$ we can define a hash function $h_{a,b}(k) = ((a * k + b) \ mod \ p) \ mod \ m$.

- For example:
  - $h_{3,7}(k) = ((3 * k + 7) \ mod \ p) \ mod \ m$
  - $h_{4,1}(k) = ((4 * k + 1) \ mod \ p) \ mod \ m$
  - $h_{8,0}(k) = ((8 * k) \ mod \ p) \ mod \ m$
- There are $p * (p-1)$ possible hash functions that can be chosen.

Universal hashing IV

Example 2

If the key k is an array $< k_1, k_2, \ldots, k_r >$ such that $k_i < m$ (or it can be transformed into such an array, by writing the k as a number in base m).
Let $< x_1, x_2, \ldots, x_r >$ be a fixed sequence of random numbers, such that $x_i \in \{0, \ldots, m-1\}$ (another number in base m with the same length).
$h(k) = \sum_{i=1}^{r} k_i * x_i \mod m$

Universal hashing V

Example 3

Suppose the keys are $u - bits$ long and $m = 2^b$.
Pick a random $b - by - u$ matrix (called h) with 0 and 1 values only.
Pick $h(k) = h * k$ where in the multiplication we do addition mod 2.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Using keys that are not natural numbers I

- The previously presented hash functions assume that keys are natural numbers.
- If this is not true there are two options:
  - Define special hash functions that work with your keys (for example, for real number from the [0,1) interval $h(k) = [k * m]$ can be used)
  - Use a function that transforms the key to a natural number (and use any of the above-mentioned hash functions) - *hashCode* in Java, *hash* in Python

Using keys that are not natural numbers II

- If the key is a string s:
  - we can consider the ASCII codes for every letter
  - we can use 1 for *a*, 2 for *b*, etc.
- Possible implementations for *hashCode*
  - $s[0] + s[1] + ... + s[n-1]$
    - Anagrams have the same sum *SAUCE* and *CAUSE*
    - *DATES* has the same sum (D = C + 1, T = U - 1)
    - Assuming maximum length of 10 for a word (and the second letter representation), *hashCode* values range from 1 (the word *a*) to 260 (*zzzzzzzzzz*). Considering a dictionary of about 50,000 words, we would have on average 192 word for a *hashCode* value.

Using keys that are not natural numbers III

- $s[0] * 26^{n-1} + s[1] * 26^{n-2} + ... + s[n-1]$ where
  - n - the length of the string
  - Generates a much larger interval of *hashCode* values.
  - Instead of 26 (which was chosen since we have 26 letters) we can use a prime number as well (Java uses 31, for example).

Cryptographic hashing

Cryptographic hashing

- Another use of hash functions besides as part of a hash table
- It is a hash function, which can be used to generate a code (the hash value) for any variable size data
- Used for checksums, storing passwords, etc.

Collisions

- When two keys, x and y, have the same value for the hash function $h(x) = h(y)$ we have a *collision*.
- A good hash function can reduce the number of collisions, but it cannot eliminate them at all:
  - Try fitting $m + 1$ keys into a table of size m
- There are different collision resolution methods:
  - Separate chaining
  - Coalesced chaining
  - Open addressing

The birthday paradox

- *How many randomly chosen people are needed in a room, to have a good probability - about 50% - of having two people with the same birthday?*
- It is obvious that if we have 367 people, there will be at least two with the same birthday (there are only 366 possibilities).

The birthday paradox

- *How many randomly chosen people are needed in a room, to have a good probability - about 50% - of having two people with the same birthday?*
- It is obvious that if we have 367 people, there will be at least two with the same birthday (there are only 366 possibilities).
- What might not be obvious, is that approximately 70 people are needed for a 99.9% probability
- 23 people are enough for a 50% probability

Separate chaining

- Collision resolution by separate chaining: each slot from the hash table T contains a linked list, with the elements that hash to that slot
- Dictionary operations become operations on the corresponding linked list:
  - $insert(T, x)$ - insert a new node to the beginning of the list $T[h(key[x])]$
  - $search(T, k)$ - search for an element with key k in the list $T[h(k)]$
  - $delete(T, x)$ - delete x from the list $T[h(key[x])]$

Hash table with separate chaining - representation

- A hash table with separate chaining would be represented in the following way (for simplicity, we will keep only the keys in the nodes).

Node:
key: TKey
next: ↑ Node

HashTable:
T: ↑Node[] *//an array of pointers to nodes*
m: Integer
h: TFunction *//the hash function*

Hash table with separate chaining - search

```
function search(ht, k) is:
//pre: ht is a HashTable, k is a TKey
//post: function returns True if k is in ht, False otherwise
    position ← ht.h(k)
    currentNode ← ht.T[position]
    while currentNode ≠ NIL and [currentNode].key ≠ k execute
        currentNode ← [currentNode].next
    end-while
    if currentNode ≠ NIL then
        search ← True
    else
        search ← False
    end-if
end-function
```

- Usually search returns the info associated with the key k

Analysis of hashing with chaining

- The average performance depends on how well the hash function h can distribute the keys to be stored among the m slots.
- **Simple Uniform Hashing** assumption: each element is equally likely to hash into any of the m slots, independently of where any other elements have hashed to.
- **load factor** α of the table T with m slots containing n elements
  - is n/m
  - represents the average number of elements stored in a chain
  - in case of separate chaining can be less than, equal to, or greater than 1.

Analysis of hashing with chaining - Insert

- The slot where the element is to be added can be:
  - empty - create a new node and add it to the slot
  - occupied - create a new node and add it to the beginning of the list
- In either case worst-case time complexity is: $\Theta(1)$
- If we have to check whether the element already exists in the table, the complexity of searching is added as well.

Analysis of hashing with chaining - Search I

- There are two cases
  - unsuccessful search
  - successful search
- We assume that
  - the hash value can be computed in constant time ($\Theta(1)$)
  - the time required to search an element with key k depends linearly on the length of the list $T[h(k)]$

Analysis of hashing with chaining - Search II

- **Theorem:** In a hash table in which collisions are resolved by separate chaining, an unsuccessful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.
- **Theorem:** In a hash table in which collisions are resolved by chaining, a successful search takes time $\Theta(1+\alpha)$, on the average, under the assumption of simple uniform hashing.
- Proof idea: $\Theta(1)$ is needed to compute the value of the hash function and α is the average time needed to search one of the m lists

Analysis of hashing with chaining - Search III

- If $n = O(m)$ (the number of hash table slots is proportional to the number of elements in the table, if the number of elements grows, the size of the table will grow as well)
  - $\alpha = n/m = O(m)/m = \Theta(1)$
  - searching takes constant time on average
- Worst-case time complexity is $\Theta(n)$
  - When all the nodes are in a single linked-list and we are searching this list
  - In practice hash tables are pretty fast

Analysis of hashing with chaining - Delete

- If the lists are doubly-linked and we know the address of the node: $\Theta(1)$
- If the lists are singly-linked: proportional to the length of the list

- **All dictionary operations can be supported in $\Theta(1)$ time on average.**
- In theory we can keep any number of elements in a hash table with separate chaining, but the complexity is proportional to α. If α is too large $\Rightarrow$ resize and rehash.

Summary

- Today we have talked about:
  - Direct address table
  - Hash table
- Extra reading - the solution for last week's extra reading.