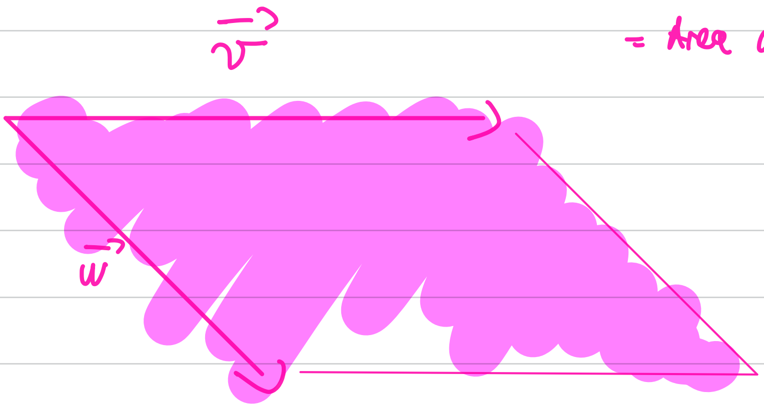
Seminar 5

Cross product

$\vec{v}, \vec{w} \in V^3 \Rightarrow \vec{v} \times \vec{w} \in V^3$

- $|\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\widehat{\vec{v}, \vec{w}})$

= Area of parallelog.



- $\vec{v} \times \vec{w} \perp \vec{v}$, $\vec{v} \times \vec{w} \perp \vec{w}$

- orientation: right-hand rule
(screw rule)

↳ middle finger UP!

Properties of $\cdot \times \cdot$

- $(\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$

- $(\alpha \vec{v}) \times \vec{w} = \alpha (\vec{v} \times \vec{w})$

- $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$

• $\vec{v} \times \vec{v} = 0$

↳ $\sin(v, v) = 0$

If we assume the reference frame to be right-orthonormal and we take $v\,(x_1, y_1, z_1)$

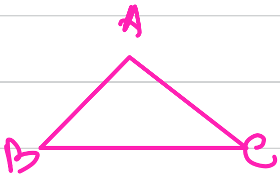
$w\,(x_2, y_2, z_2)$

$$v \times w = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i \cdot \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \cdot j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

Application :

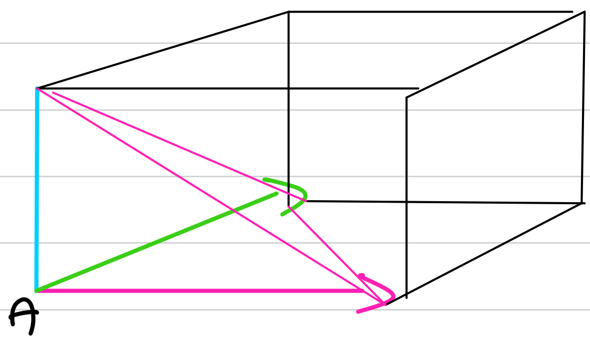
• areas :



$$A_{ABC} = \frac{1}{2} \left| \vec{AB} \times \vec{AC} \right|$$

↳ half the A of ▱

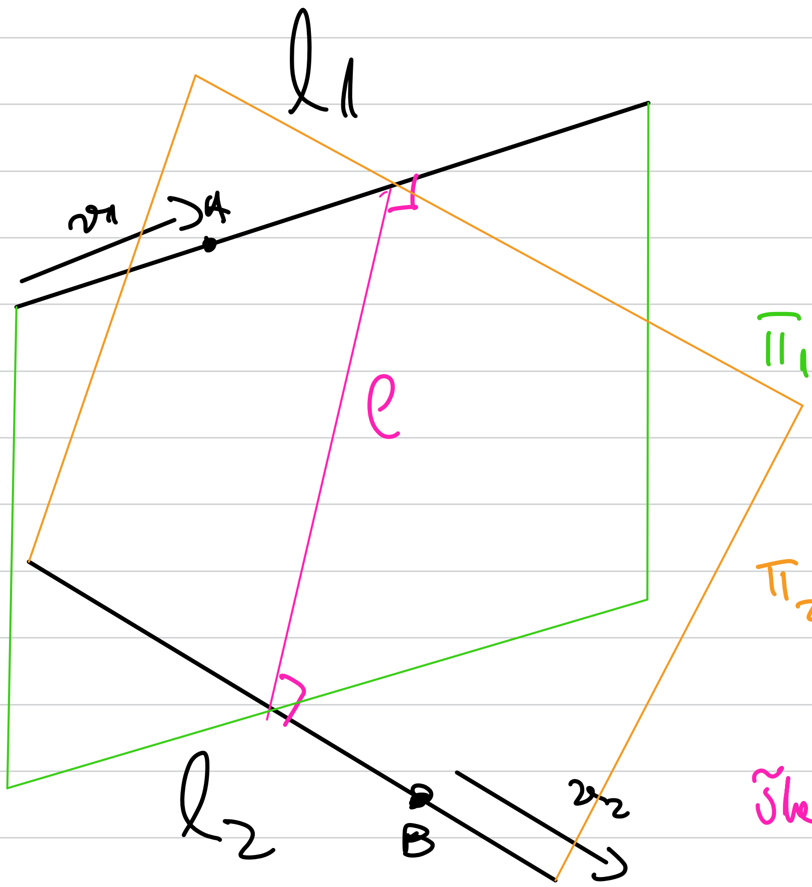
• Volumes :



$$\mathrm{Vol}(ABCDA'B'C'D') = [\vec{AB}, \vec{AC}, \vec{AD}]$$
$$= (\vec{AB} \times \vec{AC}) \vec{AD}$$

$$\text{Vol of a tetrahedron} = \frac{1}{6} [\vec{AB}, \vec{AC}, \vec{AA'}]$$
$$= \mathrm{dist}(A', ABD) \cdot A_{ABD} \cdot \frac{1}{3}$$

The common perpendicular of 2 skew lines



$A \in l_1$
$B \in l_2$
$v_1 \in D(l_1)$
$v_2 \in D(l_2)$

π_1 = plane given by $A, \vec{v_1}, \vec{v_1} \times \vec{v_2}$

π_2 = plane given by $B, \vec{v_2}, \vec{v_1} \times \vec{v_2}$

The common perp.:

$$l : \begin{cases} \pi_1 : \dots \\ \pi_2 : \dots \end{cases}$$

5.1. Consider:

$\vec{a}\ (3, -1, -2)$

$\vec{b}\ (1, 2, 1)$

Calculate:

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = -i - 2j + 6k - (-k - 4i + 3j)$$

$$= 5i - 5j + 7k$$

$$(2\vec{a} + \vec{b}) \times \vec{b} = 2\vec{a} \times \vec{b} + \underbrace{\vec{b} \times \vec{b}}_{0}$$

$$= 2(\vec{a} \times \vec{b})$$

$$= 6i - 10j + 14k$$

$$(2\vec{a} + \vec{b}) \times (2\vec{a} - \vec{b}) =$$

$$= 2\vec{a} \times (2\vec{a} - \vec{b}) + \vec{b} \times (2\vec{a} - \vec{b})$$

$$= -(2\vec{a} - \vec{b}) \times 2\vec{a} - (2\vec{a} - \vec{b}) \times \vec{b}$$

$$= \underbrace{-2\vec{a} \times 2\vec{a}}_{0} + \vec{b} \times 2\vec{a} \underbrace{- 2\vec{a} \times \vec{b}}_{+\vec{b} \times 2\vec{a}} + \underbrace{\vec{b} \times \vec{b}}_{0}$$

$$= 2(\vec{b} \times 2\vec{a}) = 4(\vec{a} \times \vec{b}) =$$

$$= -12i + 20j - 28k$$

Multiple choice ???

5.2. $A(1,2,0)$, $B(3,0,-3)$, $C(5,2,6)$, $D(1,0,1)$

a) Area of ΔABC ; Dist(C, AB)

b) Vol $(ABCD)$, dist (D, ABC)
↳ tetrahedron

c) the common perp. of AB and CD

a) $A_{ABC} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$\vec{AB}(2,-2,-3)$

$\vec{AC}(4,0,6)$

$$|\vec{AB} \times \vec{AC}| = \left| \begin{vmatrix} i & j & k \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} \right|$$

$$= |-12i - 12j - (-8k + 12j)|$$

$$= |-12i - 24j + 8k|$$

$$= |4(-3i - 6j + 2k)|$$

$$= 4\sqrt{(9 + 36 + 4)}$$

$$= 4 \cdot 7 = 28$$

$A_{ABC} = 14$

$$A_{ABC} = \frac{dist(C, AB) \cdot |AB|}{2}$$

$|AB| = \sqrt{17}$

$28 = dist(C, AB) \cdot \sqrt{17}$

$\Rightarrow dist(C, AB) = \frac{28\sqrt{17}}{17}$

b) $Vol(ABCD) = \frac{1}{6} \{\vec{AB}, \vec{AC}, \vec{AD}\}$

$\vec{AB}(2, -2, -3)$

$\vec{AC}(4, 0, 6)$

$\vec{AD}(0, -2, 1)$

$|(\vec{AB} \times \vec{AC}) \cdot \vec{AD}| =$

$= |(-12, -24, 8) \cdot (0, -2, 1)|$

$= 56$

$\Rightarrow V_{ABCD} = \frac{56}{6} = \frac{28}{3}$

$$\frac{28}{3} = \frac{1}{3} \cdot dist(D, ABC) \cdot A_{ABC}$$

$$A_{ABC} = \frac{1}{2} |AB \times AC| = 14$$

$\Rightarrow$

$$\Rightarrow dist(D, ABC) = 2$$

General formula:

$$\Pi : Ax + By + Cz + D = 0$$

$$dist(P, \Pi) = \frac{|Ax_P + By_P + Cz_P|}{\sqrt{A^2 + B^2 + C^2}}$$

c) $l : \begin{cases} \Pi_1 \ldots \\ \Pi_2 \ldots \end{cases}$

5.5. (Law of sines)

$\triangle ABC$.

$\vec{u} = \vec{AB}$, $\vec{v} = \vec{BC}$, $\vec{w} = \vec{CA}$

Show that $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$

and deduce the law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\vec{u} = \left(-\vec{v} - \vec{w}\right)$$

$$\vec{u} \times \vec{v} = (-\vec{v} - \vec{w}) \times \vec{v} = \underbrace{-\vec{v} \times \vec{v}}_{0} - \vec{w} \times \vec{v}$$

$$= \vec{v} \times \vec{w}$$

$$\vec{w} \times \vec{u} = -\vec{u} \times \vec{w} = (\vec{v} + \vec{w}) \times \vec{w}$$

$$= \vec{v} \times \vec{w} + \vec{w} \times \vec{w} = \vec{v} \times \vec{w}$$

$|u \times v| = |v \times w| \Rightarrow$

$\Rightarrow |AB \times BC| = |BC \times CB|$

$|AB|\,|BC| \sin B = |BC|\,|CB| \sin C$

$$\Rightarrow \frac{|AB| \cdot \cancel{|BC|}}{\sin C} = \frac{\cancel{|BC|}\,|CA|}{\sin B}$$

$$\Rightarrow \frac{|AB|}{\sin C} = \frac{|CA|}{\sin B}$$

Proof of the Jacobi's identity

$$(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{c} \times \vec{a}) \times \vec{b} = 0$$

Lemma : $(\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ b \cdot c & a \cdot c \end{vmatrix}$

$$= \langle a \cdot c \rangle \vec{b} - \langle b, c \rangle \vec{a}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ \vec{b} \cdot \vec{a} & \vec{a} \cdot \vec{c} \end{vmatrix}$$

Proof : $a(x_1, y_1, z_1)$

$b(x_2, y_2, z_2)$

$c(x_3, y_3, z_3)$

$$a \times b = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

Just do the computations

(even for $(a \times b) \times c = \begin{vmatrix} i & j & k \\ \dots & \dots & \dots \\ x_c & y_c & z_c \end{vmatrix}$)

Back to Jacobi:

$(\vec{a} \times b) \times c + (b \times c) \times a + (c \times a) \times b$

$= \ldots = 0$