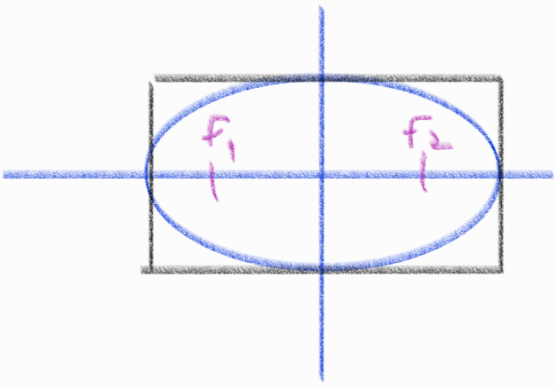
9.1 Determine the semi-minor and semi-major axes as well as the focal points of the ellipse $9x^2+25y^2-225=0$.
Moreover, draw this ellipse.

Sol: $9x^2+25y^2=225 \mid :225$

$$\frac{x^2}{5^2}+\frac{y^2}{3^2}=1 \qquad a=5,\ b=3$$

$c^2=a^2-b^2=16 \Rightarrow c=4$

$F_1(-4,0)$; $F_2(4,0)$



9.2 Determine the position of line $l: 2x+y-10$ relative to the ellipse
$\mathcal{E}: \frac{x^2}{9}+\frac{y^2}{4}-1=0$.

Sol: $l\cap\mathcal{E}: \begin{cases} 2x+y-10=0 \\ \frac{x^2}{9}+\frac{y^2}{4}-1=0 \mid \cdot 36 \end{cases}$

$y=10-2x \Rightarrow \begin{cases} y=10-2x \\ 4x^2+9(10-2x)^2-36=0 \end{cases}$

$4x^2+9(100+4x^2-40x)-36=0$

$40x^2-360x+864=0 \mid :8$

$5x^2-45x+108=0$

$\delta=45^2-4\cdot5\cdot108=5\cdot9(45-48)$

$\Delta<0 \Rightarrow l\cap\mathcal{E}=\emptyset$

9.3 Determine an equation of a line which is orthogonal to $l: 2x-2y-13=0$ and tangent to the ellipse $\mathcal{E}: x^2+4y^2-20=0$.

Sol.: Normal vector to l:

$$(2,-2) \sim \tilde{m}(1,-1)$$

$$\frac{x-x_0}{1} = \frac{y-y_0}{-1} \Rightarrow y = \underbrace{x_0+y_0}_{\alpha} - x$$

$$x^2+4(\alpha-x)^2-20=0$$

$$5x^2-8\alpha x+4\alpha^2-20=0$$

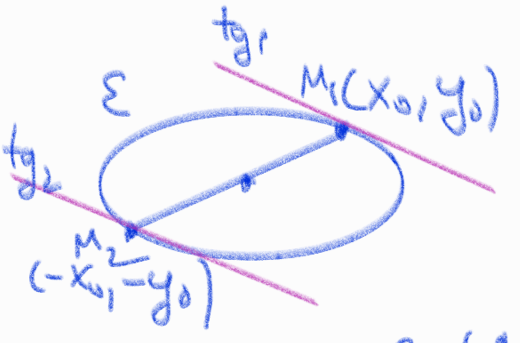
$$\Delta = 64\alpha^2 - 20(4\alpha^2-20)$$
$$= 400-16\alpha^2$$

$\mathcal{E}$ tg to our line $\Leftrightarrow \Delta = 0$
$\Leftrightarrow \alpha^2 = 25$
$\Leftrightarrow \alpha = \pm 5$

Two tangents: $x+y=5$
$x+y=-5$

or: Eq of tg: $xx_0+yy_0-20=0$
at $(x_0,y_0)\in\mathcal{E}$

9.4 A <u>diameter</u> of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.



Sol: $\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$T_{M_1}: \frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

$T_{M_2}: \frac{-xx_0}{a^2}+\frac{-yy_0}{b^2}=1$

normal vectors for both $\left(\frac{x_0}{a^2},\frac{y_0}{b^2}\right) \Rightarrow$ parallel tg

9.5 $\mathcal{E}_a: \frac{x^2}{a^2} + \frac{y^2}{16} = 1.$

For what value $a \in \mathbb{R}$ is $\mathcal{E}_a$ tangent to the line $l: x - \quad + 5 = 0$?

Sol:

$$l \cap \mathcal{E}_a \begin{cases} \frac{x^2}{a^2} + \frac{y^2}{16} = 1 \\ x - y + 5 = 0 \end{cases}$$

$$\frac{x^2}{a^2} + \frac{(x+5)^2}{16} = 1 \quad \Big| \cdot 16a^2$$

$$16x^2 + a^2(x^2 + 10x + 25) = 16a^2$$

$$(16 + a^2)x^2 + 10a^2 x + 9a^2 = 0$$

$$\Delta = 100a^4 - 36a^2(16 + a^2)$$

$$= 64a^4 - 576a^2$$

$\Delta = 0 \Leftrightarrow a = 0$ or $a^2 = \frac{576}{64} = \frac{24^2}{8^2} = 9$

impossible $\quad a = 3$

9.6 Consider the family of lines

$$l_c: \sqrt{5} \cdot x - y + c = 0$$

For what values $c \in \mathbb{R}$ is l_c tangent to the ellipse $\mathcal{E}: x^2 + \frac{y^2}{4} = 1$?

Sol: Another method:

The slope of l_c is $\sqrt{5}$.

The tangents to $\mathcal{E}$ with slope $\sqrt{5}$ are

$$y = \sqrt{5}x \pm \sqrt{5}$$

$$l_c: \sqrt{5}x - y \pm \sqrt{5} = 0 \Rightarrow c = \pm\sqrt{5}$$

9.7 Determine the common tangents to the ellipses
$$\frac{x^2}{45}+\frac{y^2}{9}=1 \text{ and } \frac{x^2}{9}+\frac{y^2}{18}=1.$$

Solution: $x \pm 2y \pm 9 = 0$

Sol: $\ell: Ax+By+C=0$

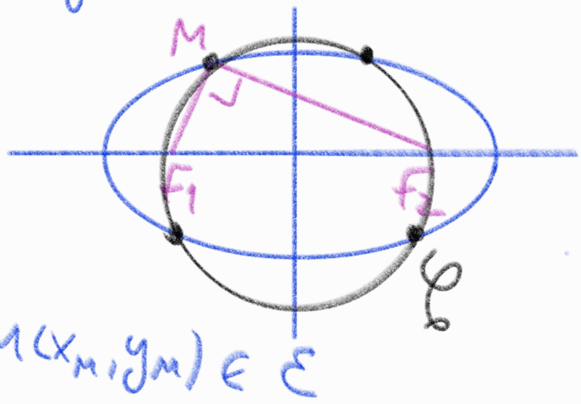
$a^2=45$; $b^2=9$

$$A^2\cdot a^2 + B^2\cdot b^2 = C^2$$

$$\begin{cases} 45A^2+9B^2=C^2 \\ 9A^2+18B^2=C^2 \end{cases} \overset{\ominus}{\Rightarrow} 36A^2=9B^2$$
$\Rightarrow B^2=4A^2$
$\Rightarrow B=\pm 2A$

$$C^2 = 36A^2+45A^2 = 81A^2 \Rightarrow C=\pm 9A$$

9.8 Consider the ellipse $\mathcal{E}: \frac{x^2}{4}+y^2-1=0$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse, for which the angle $\measuredangle F_1MF_2$ is a right angle.



Sol: Method 1:

$$\cos \measuredangle F_1MF_2 = \frac{\overrightarrow{MF_1}\cdot\overrightarrow{MF_2}}{\|\overrightarrow{MF_1}\|\cdot\|\overrightarrow{MF_2}\|}$$

$M(x_M, y_M)\in\mathcal{E}$

(See lecture) $\|MF_1\| = a - \frac{c}{a}x_M$
$\|MF_2\| = a + \frac{c}{a}x_M$

$a=2$; $c=\sqrt{4-1}=\sqrt{3}$

$\overrightarrow{MF_1}(c-x_M, -y_M)$ $\quad \overrightarrow{MF_2}(-c-x_M, -y_M)$

$\cos \measuredangle F_1MF_2 = 0$, computations...

Method 2:

$\measuredangle F_1MF_2 = 90° \Leftrightarrow F_1F_2$ is a diameter in circle $\mathcal{C}$ (so of length $2c$)

In conclusion, the desired points are the intersection of $\mathcal{E}$ with the circle centered at O and radius c.

9.9. Using a rotation of the coordinate system, find the equation of an ellipse centered at origin, with focal points on the line $x=y$ and having semi-major axes equal to 4 and the distance between the focal points equal to $2\sqrt{3}$.

Sol: Supposing that the focal points would be on the x-axis, then: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$2a=4 \Rightarrow a=2$$

$$2c=2\sqrt{3} \Rightarrow c=\sqrt{3} \Rightarrow \sqrt{a^2-b^2}=3 \Rightarrow b=1$$

Eq of the ellipse would be $\frac{x^2}{4}+y^2=1$.

Rotate the ellipse by $45°$ around the origin.

Matrix is $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$.

$$\begin{bmatrix}x\\ y\end{bmatrix} \to \begin{bmatrix}x'\\ y'\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{2}\begin{bmatrix}x-y\\ x+y\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}x'+y'\\ x'-y'\end{bmatrix}$$

The equation becomes $\frac{(x'+y')^2}{8}+\frac{(x'-y')^2}{2}=1$.

9.12 Find an equation for the tangent lines to

a) the hyperbola $\mathcal{H}: \frac{x^2}{3}-\frac{y^2}{5}-1=0$, passing through $P(1,-5)$.

b) the parabola $P: y^2-36x=0$, passing through $P(2,9)$.

Sol:

a) $l_{(x_0,y_0)}: \frac{xx_0}{3}-\frac{yy_0}{5}=1$

$P \in l_{(x_0,y_0)}: \frac{x_0}{3}+y_0=1 \Rightarrow y_0=1-\frac{x_0}{3}$

$(x_0,y_0)\in \mathcal{H}: \frac{x_0^2}{3}-\frac{(1-\frac{x_0}{3})^2}{5}-1=0$

$$x_0=-\frac{3}{14}\pm\frac{3\sqrt{85}}{14} \Rightarrow y_0 \Rightarrow l$$

b) $l_k: y=kx+\frac{f}{2k} \ni P(2,9)$ $\quad y^2=2\cdot 18\cdot x$

$$9=2k+\frac{18}{2k} \Rightarrow 9k=2k^2+9 \Rightarrow k\in\{3,\tfrac{3}{2}\}$$

$k=3$: $y=3x+3$; $\quad k=\frac{3}{2}$ $\quad y=\frac{3}{2}x+6$

9.11 Find an equation for the tangent lines to:

a) the hyperbola $\mathcal{H}: \frac{x^2}{20}-\frac{y^2}{5}-1=0$, orthogonal to the line $l: 4x+3y-7=0$

b) the parabola $P: y^2-8x=0$, parallel to $l: 2x+2y-3=0$

Sol: $l_k: y=kx\pm\sqrt{a^2k^2-b^2}$

$l_k \perp l$: $y=-\frac{4}{3}x+\frac{7}{3}$ $\qquad k=\frac{3}{4}$

$$l_k: y=\frac{3}{4}x\pm\sqrt{20\cdot\frac{9}{16}-5}=\frac{3}{4}x\pm\frac{5}{2}$$

b) $l_k: y=kx+\frac{p}{2k}$ $\qquad p=4$

$l_k \parallel l$: $y=-x+\frac{3}{2}$ $\qquad k=-1$

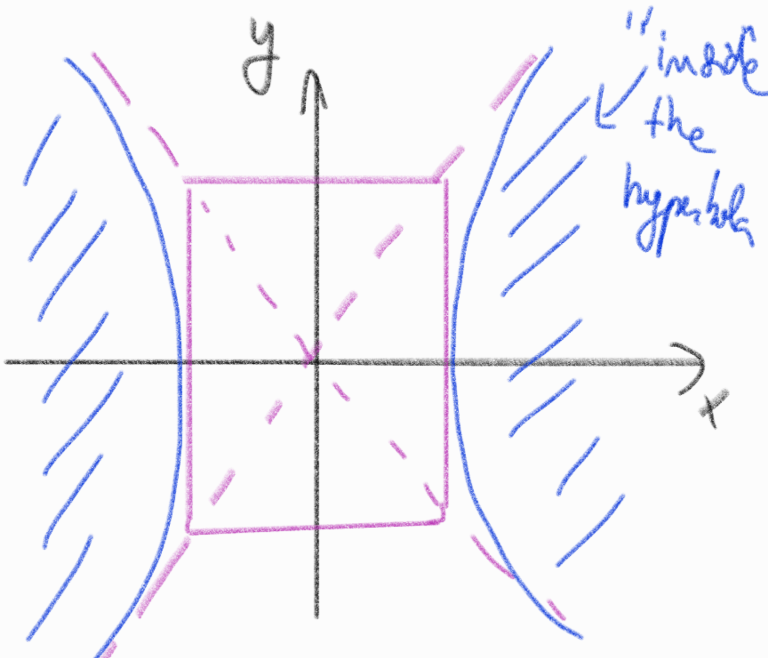
Eq tg: $y=-x-2$

9.10 Draw the hyperbola $9x^2-4y^2=36$, indicate the semi-minor and the semi-major axes, and write down equations for the asymptotes. Moreover, determine the relations between the coordinates (x_p, y_p) of the point P such that P does not belong to any tangent line to the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$.

Sol: $\mathcal{H}: \frac{x^2}{4}-\frac{y^2}{9}=1$. $\qquad a=2$, $b=3$ $\qquad$ asymptotes: $y=\pm\frac{3}{2}x$.

The drawing hints that the desired points are "inside" the hyperbola, described by the eq

$$\frac{x_p^2}{4}-\frac{y_p^2}{9}>1$$



proof: tangent line

$$l_m: y=mx\pm\sqrt{a^2m^2-b^2}$$

for some slope $m\in(-\infty,-\frac{b}{a}]\cup[\frac{b}{a},\infty)$

$$P\in l_m \iff y_p=mx_p\pm\sqrt{a^2m^2-b^2}$$

$$\Rightarrow (y_p-mx_p)^2=a^2m^2-b^2$$

$\Rightarrow m^2(x_p^2 - a^2) - 2m\, y_p x_p + y_p^2 + b^2 = 0$

$P \in l_m \Leftrightarrow \Delta \geq 0 \quad \longrightarrow$ P does not belong to any l_m

$\Leftrightarrow \Delta < 0$

$$\Delta = 4\left(x_p^2 y_p^2 - (x_p^2 - a^2)(y_p^2 + b^2)\right)$$
$$= 4\left(-x_p^2 b^2 + a^2 y_p^2 + a^2 b^2\right)$$

$$\Delta < 0 \Leftrightarrow \frac{x_p^2}{a^2} - \frac{y_p^2}{b^2} - 1 > 0$$