Seminar 8

Isometries

$f: \mathbb{E}^n \to \mathbb{E}^n$ affine transformation $\mapsto$ linear transformation + addition

$\forall P \in \mathbb{E}^n: \ f(P) = A \cdot P + b$

↳ w/o b it is a linear map

$A \in \mathcal{M}_n(\mathbb{R})$

$b \in \mathcal{M}_{n,1}(\mathbb{R})$

- A.T. preserve DIRECTIONS

$f: \mathbb{E}^n \to \mathbb{E}^n$ isometry if:

$\forall P, Q \in \mathbb{E}^n, \ dist(f(P), f(Q)) = dist(P, Q)$

isometry $\Leftrightarrow$ affine transformations with $A \in O(n)$ $\mapsto \{A \in \mathcal{M}_n(\mathbb{R}) \mid A \cdot A^T = I_n\}$

(orthogonal)

$A \in O(n) \Rightarrow \det A \cdot \underbrace{\det A^t}_{= \det A} = 1$

$\hookrightarrow (\det A)^2 = 1 \Rightarrow$

$\Rightarrow \det A = \pm 1$

$SO(n) = \{A \in O(n) \mid \det A = 1\}$

Direct Isometries : $A \in SO(n)$ ↳ "rotations"

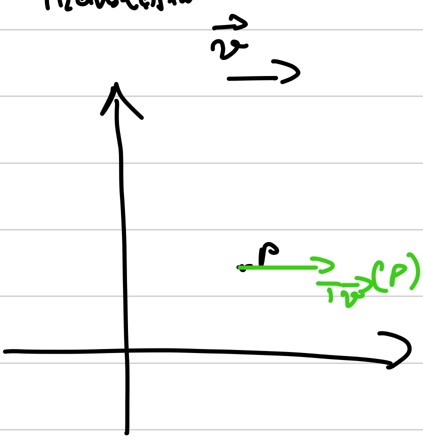
Indirect Isometries : $A \in O(n) \setminus SO(n)$
↳ $\det = -1$
↳ "reflections"

Theorem (Chasles)

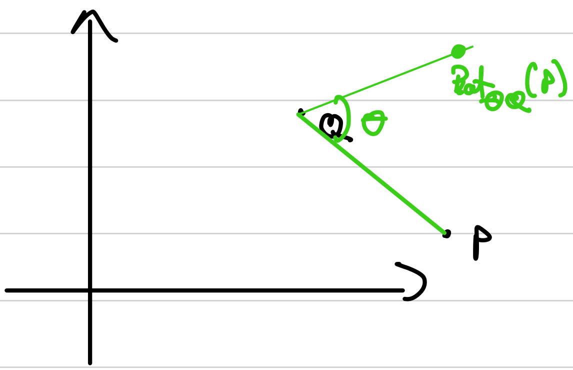
In $\mathbb{E}^2$ the isometries are :

- direct :
  - identity
  - translations
    $T_{\vec{v}} : P \mapsto P + \vec{v}$
  - rotations around a point
    $Rot_{\theta Q}$
- indirect :
  - reflection in a line ℓ
    $Ref_{\ell}^{\perp}$
  - glide-reflections
    $T_{\vec{v}} \circ Ref_{\ell}^{\perp}$, $\vec{v} \in D(\ell)$

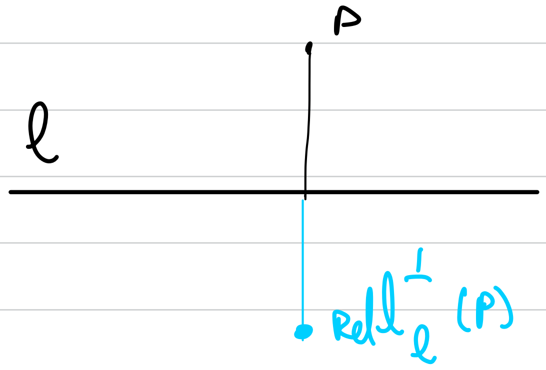
Translation



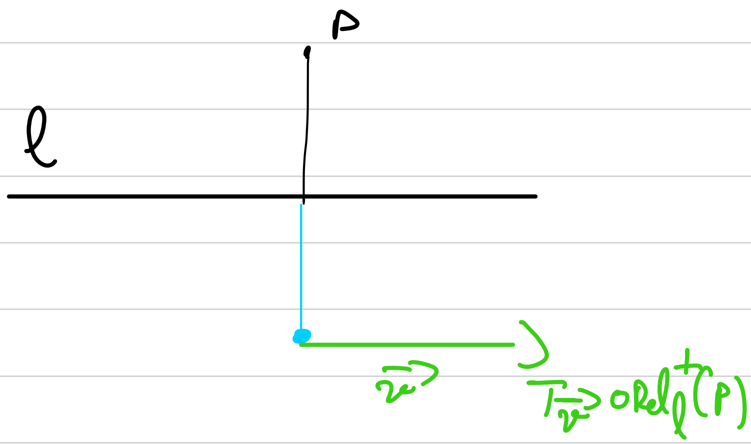
Rotation



Reflection



Glide - reflection



* can be in the exam

8.1. Consider the affine transformation:

$$\varphi(x) = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Explain why it is a rotation.

Calculate the cosine of the rotation angle and determine φ^{-1}

Determine the angle of a rotation

$\cos\theta = \frac{\operatorname{Tr} A}{2}$! in dimension 2

$$A = \begin{pmatrix} \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$$

$$A^T = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$$

$A \cdot A^T = I_2 \Rightarrow \varphi$ is an isometry

$\det A = 1 \Rightarrow$ direct isometry

Fixed point

$f : A \to A$ function

$\operatorname{Fix}(f) = \{x \in A \mid f(x) = x\}$

$A \neq I_2 \Rightarrow \mathcal{E}$ not an identity or a translation $\Rightarrow$

Chasles
$\Rightarrow \mathcal{E}$ is a rotation

$$\cos\theta = \frac{\operatorname{Tr} A}{2} = \frac{\frac{4}{\sqrt{13}}}{2} = \frac{2}{\sqrt{13}}$$

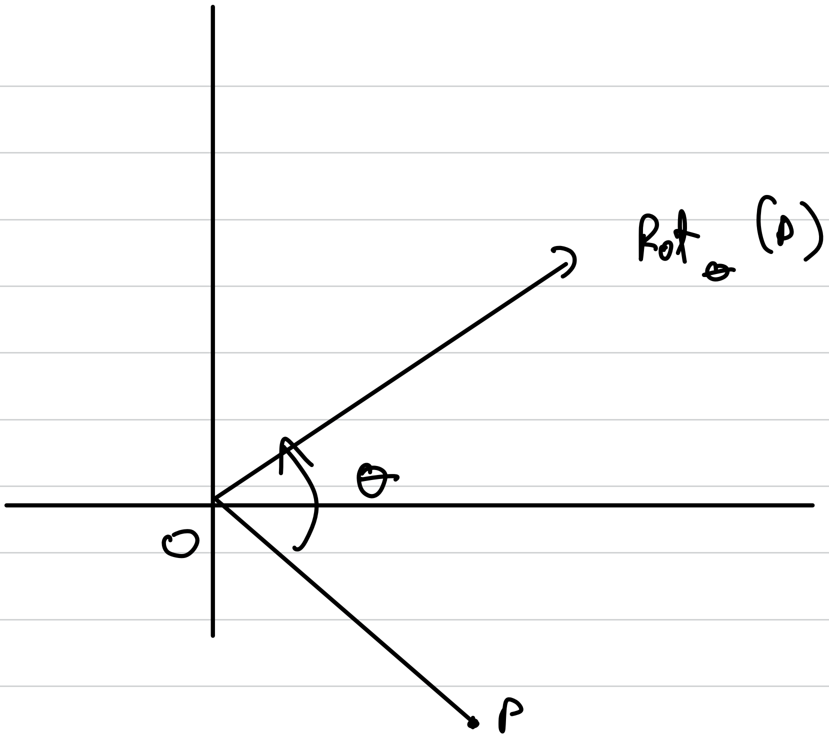
$$\mathcal{E}(x) = y \Leftrightarrow$$

$$\Leftrightarrow y = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$y - \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} x$$

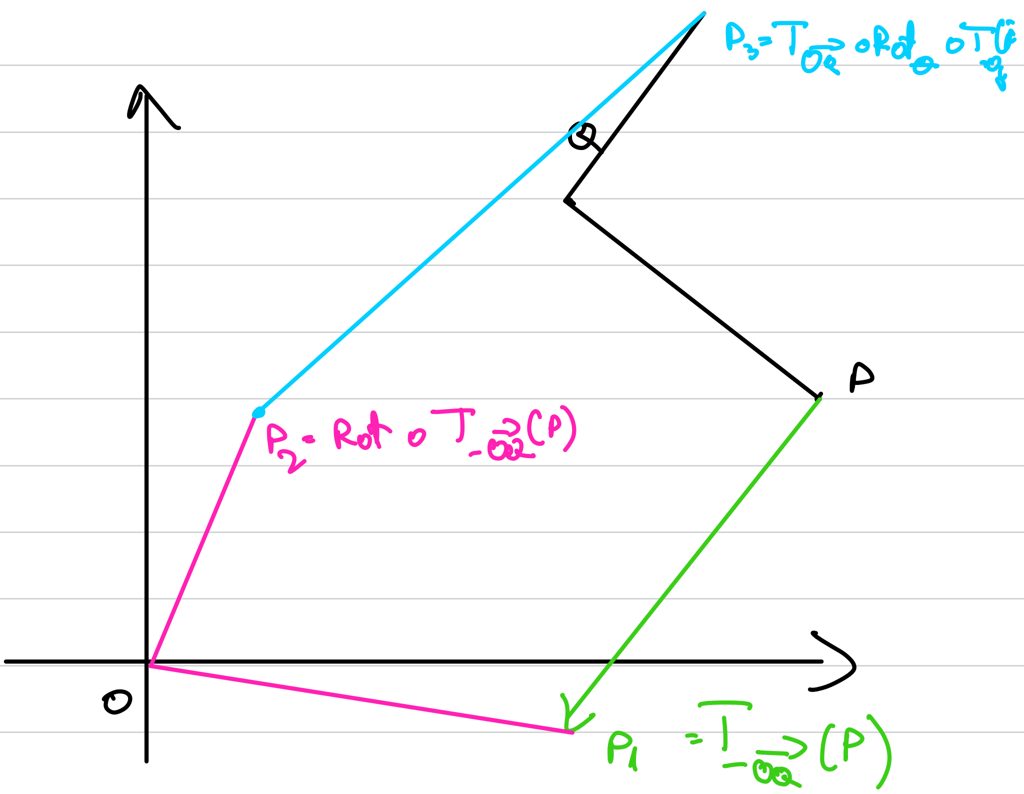
$$x = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \cdot \left(y - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$$

$$\mathcal{E}^{-1}(x) = \frac{1}{\sqrt{13}}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \cdot \left(x - \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right)$$



$$Rot_\theta(P) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

↳ rotation matrix



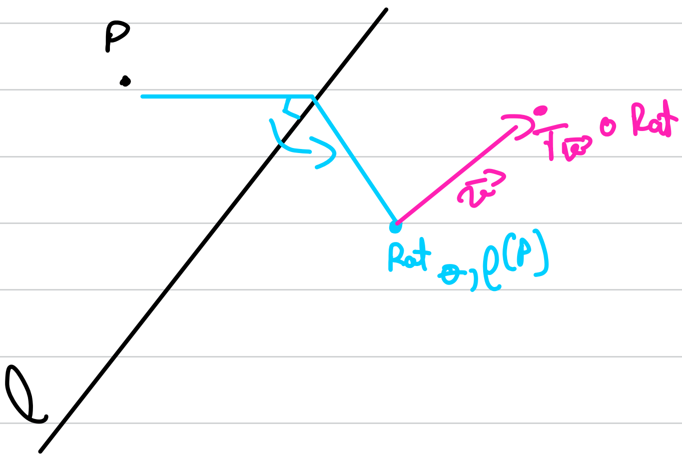
Theorem (Chasles)

In $\mathbb{E}^3$ the isometries are

- direct:
  - identity
  - translations
  - rotations (of a line)
  - glide-rotation (helical displacements)

    $T_{\vec{v}} \circ Rot_{\theta, \ell}$

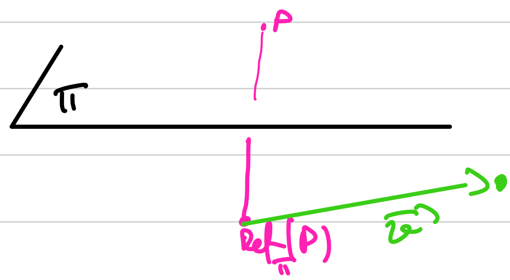
    $\vec{v} \in D(\ell)$



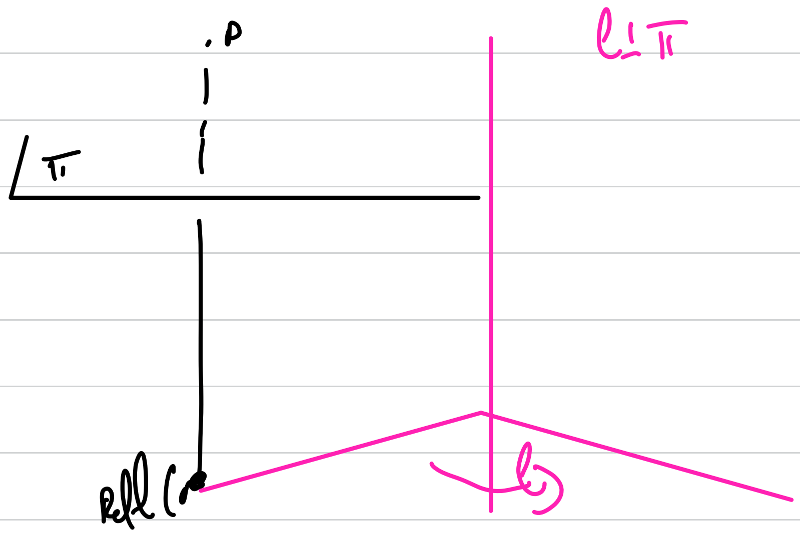
- indirect:
  - reflections (in a plane)

    Ref_{π}
  - glide-reflections

    $T_{\vec{v}} \circ Ref_{\pi}$, $\vec{v} \in D(\ell)$



- rotation reflection



8.3. Consider the affine transformation

$$\varphi(x) = \frac{1}{3} \cdot \begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot x + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

Explain why it's an isometry

Discuss the type of the isometry in terms of the parameter α

$$A = \frac{1}{3} \begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$A^T = \frac{1}{3} \begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = I_3$$

$\Rightarrow$ isometry

$$\det A = \begin{vmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{vmatrix}$$

$$= \frac{1}{3^3}\left(4+4+4-(-8+1-8)\right)$$

$$= \frac{1}{3^3}\cdot(12+15) = 1 \Rightarrow \text{direct isometry}$$

$A \neq I_3 \Rightarrow$ not an identity or translation

Fixed points:

$\varphi(x) = x \quad , \; x = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} \cdot \mathfrak{x} + \begin{pmatrix} \lambda \\ 0 \\ 1 \end{pmatrix} = \mathfrak{x} \quad \rightarrow I_3 \cdot \mathfrak{x} \Rightarrow \text{Ausklammern}$$

$$\left(\frac{1}{3}\begin{pmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{pmatrix} - I_3 \right) x = \begin{pmatrix} -\lambda \\ 0 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} -\frac{4}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\lambda \\ 0 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -\frac{4}{3} & \frac{2}{3} & -\frac{2}{3} & -\lambda \\ -\frac{2}{3} & -\frac{5}{3} & -\frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{1}{3} & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} -4 & 2 & -2 & -3\lambda \\ -2 & -5 & -1 & 0 \\ -2 & 1 & -1 & -3 \end{array}\right)$$

$$\sim \begin{pmatrix} -2 & 1 & -1 & -3 \\ -2 & -5 & -1 & 0 \\ -4 & 2 & -2 & -3\lambda \end{pmatrix}$$

$$\underset{\substack{L_2 - L_1 \\ L_3 - 2L_1}}{\sim} \begin{pmatrix} -2 & 1 & -1 & -3 \\ 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & -3\lambda + 6 \end{pmatrix}$$

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$-3\lambda + 6 = 0 \Rightarrow \lambda = 2 \rightarrow$ system is compatible

$\lambda = 2$

$$\begin{pmatrix} -2 & 1 & -1 & -3 \\ 0 & -6 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} -2a + b - c = -3 \\ -6b = 3 \Rightarrow b = -\frac{1}{2} \\ \end{cases}$$

$$-2a - \frac{1}{2} - c = -3 \Rightarrow -2a - c = -\frac{5}{2}$$

$$c = \frac{5}{2} - 2a \rightarrow$$

$\Rightarrow$ fixed points: $\left\{ \left(a, -\frac{1}{2}, \frac{5}{2} - 2a \right) \middle| a \in \mathbb{R} \right\}$

$$\text{Fixed}(t) = \left\{ \left(0, -\frac{1}{2}, \frac{5}{2}\right) + a \cdot (1, 0, -2) \middle| a \in \mathbb{R} \right\}$$

$$= \left(0, -\frac{1}{2}, \frac{5}{2}\right) + \langle (1, 0, -2) \rangle = \ell$$

$\Rightarrow \mathcal{E} = Rot_{\theta, l}$

In $\mathbb{E}^3$: $\cos \varphi = \frac{\operatorname{Tr} A - 1}{2}$

If $r \neq 2 \Rightarrow$ system is incompatible $\Rightarrow$

$\Rightarrow Fix(\mathcal{E}) = \emptyset \Rightarrow \mathcal{E}$ - glide-rotation