Seminar 3

l - line in $\mathbb{E}^2$

$A_0 \in l$, $\vec{v} \parallel l$

(i.e. $\vec{v} \in D(l)$)

↳ direction subspace

$$\boxed{\forall M \in l \quad \exists \lambda \in \mathbb{R}: \ \vec{r}_M = \vec{r}_{A_0} + \lambda \vec{v}}$$

↳ Vector eq. of l

Fix a reference frame:

$$l: \boxed{\begin{cases} x = x_{A_0} + \lambda \cdot x_{\vec{v}} \\ y = y_{A_0} + \lambda \cdot y_{\vec{v}} \end{cases}}$$

PARAMETRIC EQ. OF l

× add as many eq. as dimensions

$$l: \boxed{\frac{x - x_{A_0}}{x_{\vec{v}}} = \frac{y - y_0}{y_{\vec{v}}}}$$

SYMMETRIC EQU.

$$l: \boxed{Ax + By + C = 0}$$

IMPLICIT EQU.

ℓ_1 $\boxed{y = \boxed{m}x + n}$ EXPLICIT EQU

$\hookrightarrow$ slope

! NO slopes in 3D !

Same side $\Leftrightarrow$ Same sign (a line splits a plane into 2 halves)

3.1. $a \in \mathbb{R}$ parameter

$\ell_1: 2x + y = 1$
$\ell_2: x + ay = -1$

Discuss the relative position of the 2 lines in terms of a
Determine $a \in \mathbb{R}$: O, P(-2,-2) lie on the same side of ℓ_2.

$\ell_1: \quad y = 1 - 2x \Rightarrow m_1 = -2$

$\ell_2: \quad y = -\frac{1}{a}(1+x) \Rightarrow \begin{cases} m_2 = -\frac{1}{a}, & a \neq 0 \\ m_2 = -\infty, & a = 0 \end{cases}$

for $a = 0$, $\ell_1 \not\parallel \ell_2$

for $a \neq 0$:

$-\frac{1}{a} = -2 \Rightarrow a = \frac{1}{2} \Rightarrow \ell_1 \parallel \ell_2$

$a \neq \frac{1}{2} \Rightarrow \ell_1 \not\parallel \ell_2$

$f_2(x,y) = x + ay + 1$

$f_2(0,0) = 1$

$f_2(-2,-2) = -2 - 2a + 1$

O, P are on the same side iff:

$$f_2(-2,-2) > 0 \iff$$

$$\iff -2a - 1 > 0$$

$$a < -\frac{1}{2}$$

V - k v.s. (Ambient space)

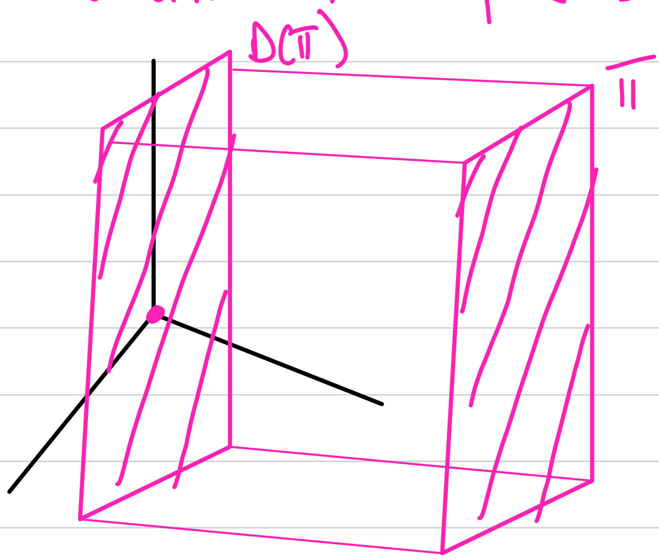
A linear variety is a subset $S \subseteq V$ of the form

$S = a + U = \{a + u \mid u \in U\}$

where $U \leq_k V$

$a \in V$

The direction space of S is U, denoted D(S)



S_1, S_2 linear varieties

$S_1 \parallel S_2 \iff D(S_1) \subseteq D(S_2)$

OR

$D(S_2) \subseteq D(S_1)$

Being parallel means the vector is included in the things of the other.

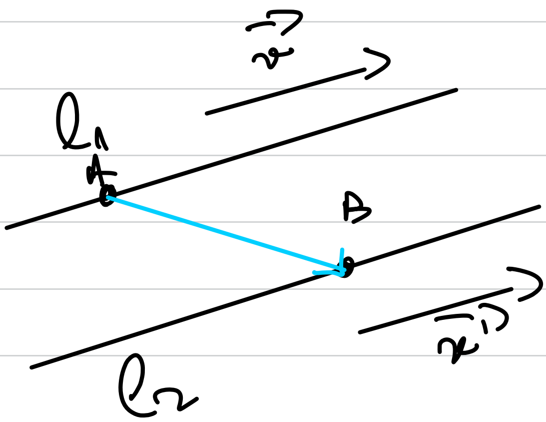
3.2. $\ell_1: x-1 = \frac{y-1}{2} = z-3$

$\ell_2: \frac{x-3}{3} = \frac{y}{6} = \frac{z+2}{3}$

1. Are the lines parallel?
2. If they are, give an eq. for the plane containing them
3. Check if the plane separates the point $P(1,1,1)$ from the origin

The eq. of a plane given by a point A_0 and 2 vectors $\vec{v}, \vec{w}$

$$\begin{vmatrix} x - x_{A_0} & y - y_{A_0} & z - z_{A_0} \\ x_v & y_v & y_w \\ x_w & z_v & z_w \end{vmatrix} = 0 \quad [\dots] = Ax + By + Cz + D = 0$$



1. $\exists A \in l$, $A(1,1,3)$ $\quad \frac{x-1}{1} = \frac{y-1}{2} = \frac{z-3}{1}$

$D(l_1) = \langle(1,2,1)\rangle$

l_2: $B \in l_2$; $B(3,0,-2)$

$D(l_2) = \langle(3,6,3)\rangle$
$= 3\langle(1,2,1)\rangle = D(l_1)$

$\rightarrow l_1 \parallel l_2$

$\vec{AB}(2,-1,-5)$

$$\begin{vmatrix} x-1 & y-1 & z-3 \\ 1 & 2 & 1 \\ 2 & -1 & -5 \end{vmatrix} = 0 \Leftrightarrow [...]$$

$\Leftrightarrow -5x + 7y - 5z + 17 = 0$

$f(x,y,z) = -5x + 7y - 5z + 17$

$$\left.\begin{array}{l} f(0,0,0) = 17 > 0 \\ f(1,1,1) = 10 > 0 \end{array}\right\} \Rightarrow \underline{\text{NO}} \text{ separation}$$

3.3. What is the relative position of the lines:

$$l_1: \begin{cases} x = -3t \\ y = 2+3t \\ z = 1 \end{cases} \quad (t \in \mathbb{R})$$

$$l_2: \begin{cases} x = 1+5s \\ y = 1+13s \\ z = 1+10s \end{cases} \quad (s \in \mathbb{R})$$

$$D(l_1) = \langle(-3,3,0)\rangle$$

$$D(l_2) = \langle(5,13,10)\rangle$$

$(-3,3,0)$ and $(5,13,10)$ are lin. indep. $\Rightarrow l_1 \nparallel l_2$

$$l_1 \cap l_2: \begin{cases} x = -3t \\ y = 2+3t \\ z = 1 \\ x = 1+5s \\ y = 1+13s \\ z = 1+10s \end{cases} \Leftrightarrow \begin{cases} -3t = 1+5s \\ 2+3t = 1+13s \\ 1 = 1+10s \end{cases} \Rightarrow$$

$\Rightarrow s = 0$

$\Rightarrow 2+3t = 1 \Rightarrow t = -\frac{1}{3}$

$-3t = 1+5s \Leftrightarrow -\frac{1}{3} \cdot -3 = 1$

$\Leftrightarrow 1 = 1$ TRUE

$\Rightarrow$

$\Rightarrow l_1, l_2$ concurrent $\Rightarrow$ in $A(1,1,1) = l_1 \cap l_2$

3.4. Determine the values a and d for which the line

$l: \frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$

is contained in the plane

$\Pi: ax + y - 2z + d = 0$

- $D(l) = \langle (3, 2, -2) \rangle$

$t = \frac{x-2}{3} = \frac{y+1}{2} = \frac{z-3}{-2}$

$$\begin{cases} x = 2+3t \\ y = -1+2t \\ z = 3-2t \end{cases}$$

$a(2+3t) + (-1+2t) - 2(3-2t) + d = 0$

$2a + 3at - 1 + 2t - 6 + 4t + d = 0$

$(3a+6)t + (2a-7+d) = 0 \quad \forall t$

$$\begin{cases} 3a+6=0 \Rightarrow a=-2 \\ 2a-7+d=0 \Leftrightarrow -4-7+d=0 \Rightarrow d=11 \end{cases}$$

3.5. Determine the relative positions of the line

$\ell: \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-1}{-4}$

and the plane

$\pi: 2x - y + z - 1 = 0$

See if the line punctures the plane, determine the intersection point

$t = \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-1}{-4}$ $\quad \exists A(x_A, y_A, z_A) \in \ell$

$$\begin{cases} x = t+1 \\ y = -2t+2 \\ z = -4t+1 \end{cases}$$

Let $f(x,y,z) = 2x - y + z - 1 \Rightarrow$

$\Rightarrow f(A) = 2(t+1) - (2-2t) + (1-4t) - 1 =$

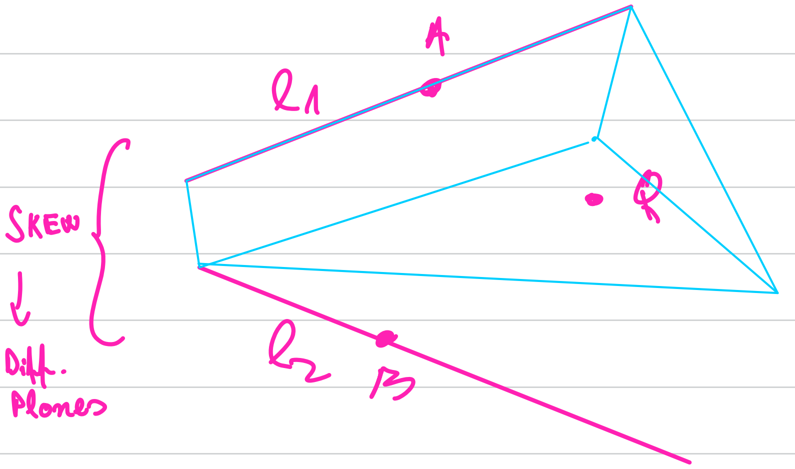
$= 0 \Rightarrow f(A) = 0, \forall A \in \ell \Rightarrow \ell \subset \pi$

3.6. Determine cartesian equations for the line ℓ passing through $Q(1,1,2)$ and coplanar to the lines:

$$\ell' : \begin{cases} 3x-5y+z=-1 \\ 2x-3z=-5 \end{cases}$$

$$\ell'' : \begin{cases} x+5y=3 \\ 2x+2y-2z=-4 \end{cases}$$

idea:



we need to write

$\pi_1 : Q, \vec{v_1} \in D(\ell_1), \vec{QA}$

$\pi_2 : Q, \vec{v_2} \in D(\ell_2), \vec{QB}$

$$\ell : \begin{cases} \pi_1 : \\ \pi_2 : \end{cases}$$

Rezolvare 3.6

3.6. $Q(1,1,2)$

l coplanar cu: $l': \begin{cases} 3x - 5y + z = -1 \\ 2x - 3z = -5 \end{cases}$

$l'': \begin{cases} x + 5y = 3 \\ 2x + 2y - 4z = -4 \end{cases}$

$$l: \begin{cases} 3x - 5y + z + 1 = 0 \\ 2x - 3z + 5 = 0 \end{cases}$$

$$\begin{pmatrix} 3 & -5 & 1 & 1 \\ 2 & 0 & -3 & 5 \end{pmatrix} \underset{\frac{1}{3}L_1}{\sim} \begin{pmatrix} 1 & -\frac{5}{3} & \frac{1}{3} & \frac{1}{3} \\ 2 & 0 & -3 & 5 \end{pmatrix}$$

$$\underset{L_2-2L_1}{\sim} \begin{pmatrix} 1 & -\frac{5}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{10}{3} & -\frac{11}{3} & \frac{25}{3} \end{pmatrix}$$

$$\underset{\substack{L_1 \cdot 3 \\ L_2 \cdot 3}}{\sim} \begin{pmatrix} 3 & -5 & 1 & 1 \\ 0 & 10 & -11 & 25 \end{pmatrix}$$

$$\underset{L_1+\frac{1}{2}L_2}{\sim} \begin{pmatrix} 3 & 0 & -\frac{9}{2} & \frac{27}{2} \\ 0 & 10 & -11 & 25 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} 3x - \frac{3}{2}z + \frac{27}{2} = 0 \\ 10y - 11z + 25 = 0 \end{cases} \Leftrightarrow \begin{cases} 3x = \frac{3}{2}z - \frac{27}{2} \\ 10y = 11z - 25 \end{cases}$$

$$z = t \Leftrightarrow \begin{cases} x = -\frac{9}{2} + \frac{3}{2}t \\ y = \frac{-5}{2} + \frac{11}{10}t \end{cases}$$

$$\Rightarrow d_l = \begin{bmatrix} \frac{3}{2} \\ \frac{11}{10} \\ 1 \end{bmatrix}$$

$$l' : \begin{cases} x + 5y = 3 \\ 2x + 2y - 7z = -7 \end{cases} \Leftrightarrow \begin{cases} x + 5y - 3 = 0 \\ 2x + 2y - 7z + 7 = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} 1 & 5 & 0 & -3 \\ 2 & 2 & -7 & 7 \end{pmatrix} \underset{L_2 - 2L_1}{\sim} \begin{pmatrix} 1 & 5 & 0 & -3 \\ 0 & -8 & -7 & 13 \end{pmatrix}$$

$$\underset{\substack{L_1 \cdot 8 \\ L_2 \cdot 5}}{\sim} \begin{pmatrix} 8 & 40 & 0 & -24 \\ 0 & -40 & -35 & 45 \end{pmatrix}$$

$$\underset{L_1 + L_2}{\sim} \begin{pmatrix} 8 & 0 & -35 & 21 \\ 0 & -40 & -35 & 45 \end{pmatrix}$$

$$\underset{L_2 \cdot \frac{1}{5}}{\sim} \begin{pmatrix} 8 & 0 & -35 & 21 \\ 0 & -8 & -7 & 9 \end{pmatrix}$$

$$\begin{cases} 8x - 35z + 21 = 0 \\ -8y - 7z + 5 = 0 \end{cases} \Leftrightarrow$$

$$t = -7z$$