

Linear difference equations. Discrete Dynamical Systems¹

1. a) Find all the sequences $(x_k)_{k \geq 0}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \geq 0$.
b) Find all the sequences $(x_k)_{k \in \mathbb{Z}}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \in \mathbb{Z}$.

2. Let $\eta \in \mathbb{R}$ be a fixed parameter. Find the solution $(x_k)_{k \geq 0}$ of the initial value problem

$$x_{k+1} = 2x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to η .

3. Let $\lambda \in \mathbb{R}^*$ and $\eta \in \mathbb{R}$ be fixed parameters. Find the solution $(x_k)_{k \geq 0}$ of the initial value problem

$$x_{k+1} = \lambda x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to λ and η .

4. Find the solution $(x_k)_{k \in \mathbb{Z}}$ of the initial value problem

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

Check the solution you obtained. What is the long term behavior of this sequence?

5. Find solutions of the form $x_k = a3^k$ of the difference equation $x_{k+1} = 2x_k + 3^k$, $k \geq 0$. Here we look for $a \in \mathbb{R}$.

6. Find solutions of the form $x_k = ak + b$ of the difference equation $x_{k+1} = 2x_k - k$, $k \geq 0$. Here we look for $a, b \in \mathbb{R}$.

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For each of the following difference equations find a particular solution of the indicated form. If no form is indicated (and the equation is nonhomogeneous), try a constant solution. Find its general solution using the fundamental theorems for linear difference equations and the characteristic equation method.

7. $x_{k+1} = \frac{1}{3} x_k$; **8.** $x_{k+1} = \frac{1}{2} x_k + 2$; **9.** $x_{k+1} = 2 x_k + \frac{1}{2}$; **10.** $x_{k+1} = -3x_k$;

11. $x_{k+1} = 4 x_k + 3^{k+1}$, $x_k^p = a \cdot 3^k$; **12.** $x_{k+1} = 1/3 x_k + 2^k$, $x_k^p = a \cdot 2^k$;

13. $x_{k+2} - 6x_{k+1} + 9x_k = 0$; **14.** $x_{k+2} + x_{k+1} + x_k = 0$.

15. Find the general solution of the linear planar system

$$x_{k+1} = 1/2 x_k + y_k, \quad y_{k+1} = -1/5 y_k$$

by reducing it to a second order difference equation in x .

16. Find the general solution of the linear planar system

$$x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k$$

by reducing it to a second order difference equation.

17. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = -120x$, $x(0) = 1$ and solve the difference equation you obtained in two cases: $h = 0.1$ and $h = 0.001$. Discuss on the long term behavior of the solution of the differential equation and, also, of the solution of the difference equation.

18. Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = 2x(1 - x)$, $x(0) = 1$.

19. Find the fixed points of the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and study their stability.

20. Represent the stair-step diagram of the function from the previous exercise and try to discuss the long term behavior of the solution of $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$, $k \geq 0$ with respect to an arbitrary $x_0 \in \mathbb{R}$.

21. Prove that the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = m + \varepsilon \sin x$, where $m > 0$ and $0 < \varepsilon < 1$, has a unique fixed point which is an attractor. The equation $x = m + \varepsilon \sin x$ is known as *Kepler equation* and arises in the study of planetary motion.

22. Study the stability of the fixed point $(0, 0)$ of the following planar linear difference system:

a) $x_{k+1} = 3/5 x_k + 1/5 y_k, \quad y_{k+1} = 1/5 x_k + 3/5 y_k;$

b) $x_{k+1} = 1/2 x_k + y_k, \quad y_{k+1} = -1/5 y_k;$

c) $x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k.$

23. Find the fixed points and the 2-periodic points of the maps $f, g : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = -x^3$ and $g(x) = x^2 - 1$.

24. In order to study the stability of the fixed point 0 of the maps $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x + x^3$, $g(x) = x - x^3$, $h(x) = x + x^2$, note that the linearization method can not be applied. Study the stability of the fixed point 0 using the stair-step diagram.

25. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points and the 2-periodic points of f and study their stability using the linearization method.

(b) Represent the graph of f and find geometrically the fixed points of f . Also, depict the 2-periodic orbit using the stair-step diagram.

(c) Find directly $f^k(0)$ for any $k \geq 0$. Depict this orbit using the stair-step diagram.

(d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at η (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1$, $x_0 = \eta$).

26. Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \rightarrow \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x}.$$

27. Let $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x(1 - x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits

corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the attractor fixed point of f .

28. Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k, \quad x_0 = 0, \quad x_1 = 1.$$

29*. Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

Hint: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

30*. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be defined by $f(x, y) = (-x^2 + y/5, x)$.

(a) Find the fixed points of f and study their stability.

(b) In case that you found an attracting fixed point, write the consequence of this fact for the sequence $(f^k(\eta))_{k \geq 0}$ where $\eta \in \mathbb{R}^2$ is properly chosen. As usual, f^k denotes the k -th iterate of f .

31. We consider the map

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the attractor fixed point.

(c) If $(x_k)_{k \geq 0}$ represent the number of fish in some lake at month k and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

- 32.** We consider the IVP $y' = -200y$, $y(0) = 1$, where the unknown is the function $y(t)$.
- a) Find the solution and its limit as $t \rightarrow \infty$.
 - b) Write the Euler's numerical formula with constant step-size h .
 - c) For $h = 0.001$, and, respectively, $h = 0.01$ find the solution $(y_k)_{k \geq 0}$ of the difference equation found at b) and decide if it satisfies $\lim_{k \rightarrow \infty} y_k = 0$.
 - d) Find a range of values for the step-size h such that the solution $(y_k)_{k \geq 0}$ of the difference equation found at b) satisfies $\lim_{k \rightarrow \infty} y_k = 0$.

- 33.** (a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y$, $y(0) = 1$.
- (b) Using (a) find a rational approximation of the Euler's constant e .

- 34.** Let $g : I \rightarrow \mathbb{R}$ be a C^1 map such that $g'(x) \neq 0$ for all x in the interval I . Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to r the unique solution $(x_k)_{k \geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k \rightarrow \infty} x_k = r.$$

- 35.** Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

- 36.** We consider the map

$$T : [0, 1] \rightarrow \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

- a) Represent the graph of T . Find its fixed points.
- b) Let $n \in \mathbb{N}$, $n \geq 2$. Find the orbit of the initial state $\frac{3}{2^n}$.

- c) Find the 2-periodic points of T .
- d) Represent the graphs of T^2 and T^3 . How many fixed points they have?
- e) The map T has a 2-periodic orbit? Or a 3-periodic orbit? T has a 2019-periodic orbit?

37. We consider the IVP $y' = 1 + xy^2$, $y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with φ is denoted the exact solution of the given IVP.

38. We consider the difference equation

$$x_{k+2} + x_k = \cos \frac{k\pi}{2}.$$

- a) Find a solution of the form $(x_k)_p = ak \cos \frac{k\pi}{2}$, with $a \in \mathbb{R}$. (Hint: we recall that $\cos(t + \pi) = -\cos t$ for any $t \in \mathbb{R}$)
- b) Find its general solution.
- c) Find the solution with $x_0 = x_1 = 0$ and describe its long-term behavior (is it periodic? is it bounded? is it oscillatory around 0?).

39*. We consider the linear difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

- a) Study the stability of this system.
- b) Find its general solution.

40. Find the linear homogeneous difference equation with constant coefficients, of minimal order, which has as solutions the two sequences

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \dots$$

and

$$1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \frac{1}{2^4}, -\frac{1}{2^5}, \dots$$

41. We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k),$$

whose unknown is the sequence $(x_k)_{k \geq 0}$, and where $\lambda \in (0, 1)$ is a parameter. Find its constant solutions (fixed points) and study their stability. Discuss with respect to the parameter λ .

42. We consider the IVP $x' = -10^3 x$, $x(0) = 1$.

- a) Find the solution and its limit as $t \rightarrow \infty$.
- b) Write the Euler's numerical formula with constant step-size h .
- c) Find a range of values for the step-size h such that the solution $(x_k)_{k \geq 0}$ of the difference equation found at b) satisfies $\lim_{k \rightarrow \infty} x_k = 0$.

43. We consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and, given $x_0 \in \mathbb{R}$, consider the sequence $(x_k)_{k \geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k) .$$

- a) Find the fixed points of f , and study their stability.
- b) Find $(x_k)_{k \geq 0}$ when $x_0 = \sqrt{2}$.
- c) There exists an $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$ such that $\lim_{k \rightarrow \infty} x_k = \sqrt{2}$?
- d) There exists an $x_0 \in \mathbb{R}$ such that $\lim_{k \rightarrow \infty} x_k = 2$?