## 7.1 Warm-Up Exercises

In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K}$.

**7.1.** Starting from the matrix form of the projections and reflections described in this chapter, deduce the matrices of

a) the orthogonal projections on the coordinate axes and on the coordinate hyperplanes of $\mathcal{K}$.

b) the orthogonal reflections in coordinate axes and in coordinate hyperplanes of $\mathcal{K}$.

**7.2.** Determine the orthogonal projection of the point $A(2, 11, -5)$ on the plane $\pi : x + 4y - 3z + 7 = 0$ by determining the matrix form of the projection.

**7.3.** Determine the orthogonal reflection of the point $P(6, -5, 5)$ in the plane $\pi : 2x - 3y + z - 4 = 0$ by determining the matrix form of the reflection.

**7.4.** Project the point $P(4, 1, -2)$ on the $z$-axis parallel to the plane $\pi : x + y + z = 0$.

**7.5.** Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

**7.1.** Starting from the matrix form of the projections and reflections described in this chapter, deduce the matrices of

a) the orthogonal projections on the coordinate axes and on the coordinate hyperplanes of $\mathcal{K}$.

b) the orthogonal reflections in coordinate axes and in coordinate hyperplanes of $\mathcal{K}$.

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = \left(I_n - \frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}\right)\cdot P - \frac{a_{n+1}}{\mathbf{v}^T\cdot\mathbf{a}}\mathbf{v}$$

where $H: a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$

$a(a_1, \dots, a_n)$

$\mathcal{K} = (O, B)$ $B = (e_1, \dots, e_n)$

consider coord hyperplane $\perp e_1 \iff H: x_1 = 0$

then $a = v = (1, 0 \dots 0)$ and $a_n = 0$

so $\mathrm{Pr}_{H,v}(P) = \left(I_n - \frac{v \cdot v^T}{v^T \cdot v}\right) P$

$$v \cdot v^T = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} (1 0 \dots 0) = \begin{pmatrix} 1 & 0 & \\ 0 & 0 & 0 \\ & 0 & \ddots \end{pmatrix}$$

$v^T \cdot v = 1$

$$\Rightarrow \mathrm{Pr}_{H,v}(P) = \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix} \cdot P$$

$$\mathrm{Ref}_{H,\mathbf{v}}(P) = \left(I_n - 2\frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}\right)\cdot P - 2\frac{a_{n+1}}{\mathbf{v}^T\cdot\mathbf{a}}\mathbf{v}. = \left(I_n - 2\, v \cdot v^T\right) \cdot P$$

$$= \begin{pmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix} P$$

$$\mathrm{Pr}_{\ell,\mathbb{W}}(P) = \frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}P + \left(I_n - \frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}\right)Q.$$

where $\mathcal{H}: a_1x_1 + \dots + a_nx_n + a_{n+1} = 0$

$v =$ dir vect of $\ell$

$Q =$ point on $\ell$

$\ell = x_1$ - axis

$\Rightarrow H: x_1 = 0 \quad v = (1,0\dots,0) \quad Q = O(0,\dots,0)$

$$\Rightarrow \quad \mathrm{Pr}_{\ell,W}(P) = \frac{V\cdot V^T}{V^T\cdot V}P = \begin{pmatrix} 1 & & 0 \\ & 0 & \\ 0 & & 0 \end{pmatrix} P = P_1$$

$$\mathrm{Ref}_{\ell,\mathbb{W}}(P) = \left(2\frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}} - I_n\right)P + 2\left(I_n - \frac{\mathbf{v}\cdot\mathbf{a}^T}{\mathbf{v}^T\cdot\mathbf{a}}\right)Q. = \left(2\cdot v\cdot v^T - I_n\right)P$$

$$= \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} P$$

**7.2.** Determine the orthogonal projection of the point $A(2,11,-5)$ on the plane $\pi : x + 4y - 3z + 7 = 0$ by determining the matrix form of the projection.

Let $\pi$ be the plane

A N.V for $\pi$ is $n(1,4,-3)$

So the orthogonal projection is a projection along lines parallel to $n$

Let $P(x_0,y_0,z_0) \in \mathbb{E}^3$ and consider the line $\ell: \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} x_0+t \\ y_0+4t \\ z_0-3t \end{bmatrix}$

The orthogonal projection of $P$ on $\pi$ is $P' = \pi \cap \ell$

$\pi \cap \ell$: $(x_0+t)+4(y_0+4t)-3(z_0-3t)+7=0$

$$x_0+t+4y_0+16t-3z_0+9t+7=0$$

$$\Rightarrow \quad t = \frac{1}{26}(-x_0-4y_0+3z_0-7)$$

one can also apply the formula to get this matrix

$$\Rightarrow P' = \begin{bmatrix} x_0 + \frac{-x_0-4y_0+3z_0-7}{26} \\ y_0 + 4\frac{-x_0-4y_0+3z_0-7}{26} \\ z_0 - 3\frac{-x_0-4y_0+3z_0-7}{26} \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 25 & -4 & 3 \\ -4 & 10 & 12 \\ 3 & 12 & 17 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - \frac{7}{26}\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

in particular

$$Pr_\pi^\perp(A) = \frac{1}{26}\begin{bmatrix} 25 & -4 & 3 \\ -4 & 10 & 12 \\ 3 & 12 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix} - \frac{7}{26}\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} -9 \\ 42 \\ 53 \end{bmatrix} - \frac{7}{26}\begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} = \frac{1}{26}\begin{bmatrix} -16 \\ 14 \\ 74 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -8 \\ 7 \\ 37 \end{bmatrix}$$

**7.3.** Determine the orthogonal reflection of the point $P(6,-5,5)$ in the plane $\pi : 2x-3y+z-4=0$ by determining the matrix form of the reflection.

Let $\pi$ be the plane and let $Q(x_0, y_0, z_0) \in \mathbb{E}^3$

A normal vector for $\pi$ is $n(2,-3,1)$

The orthogonal reflection in $\pi$ is the reflection in $\pi$ along lines parallel to $n$

$$[\mathrm{Ref}^{\perp}_{\pi}(Q)] = [\mathrm{Ref}_{\pi,n}(Q)] = \left(Id_3 - 2\frac{n\cdot a^t}{n^t\cdot a}\right)[Q] - 2\frac{-4}{n^t\cdot a}[n]$$

where $a=n$, so

$$n\cdot a^t = n\cdot n^t = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 2 \\ -6 & 9 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$

$$n^t\cdot a = n^t\cdot n = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 4+9+1 = 14$$

So
$$[\mathrm{Ref}^{\perp}_{\pi}(Q)] = \frac{1}{14}\begin{bmatrix} 6 & 12 & -4 \\ 12 & -4 & 6 \\ -4 & 6 & 12 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + \frac{4}{7}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

In particular $\mathrm{Ref}^{\perp}_{\pi}(P) = \begin{bmatrix} -2 \\ 7 \\ 1 \end{bmatrix}$

**7.4.** Project the point $P(4,1,-2)$ on the $z$-axis parallel to the plane $\pi: x+y+z=0$.

Let $P'$ be the projected point. Then $P' = Oz \cap \pi'$ where $\pi'$ is a plane passing through $P$ and parallel to $\pi$

$$\Rightarrow \pi': (x-4)+(y-1)+(z+2)=0 \iff \pi': x+y+z-3=0$$

So $P': \begin{cases} x+y+z-3=0 \\ x=y=0 \ (z\text{-axis}) \end{cases} \Rightarrow z=3 \Rightarrow P'(0,0,3)$

**7.5.** Write down the vector forms and matrix forms for parallel projections and reflections in $\mathbb{E}^3$.

- We show this for projections in hyperplanes (the other cases are similar)
- the vector forms don't change:

$$\mathrm{Pr}_{H,\mathbf{v}}(P) = P - \frac{\varphi(P)}{\operatorname{lin}\varphi(\mathbf{v})}\mathbf{v}.$$

but $H$ is in this case a plane so it has an equation of the form

$H: ax+by+cz+d=0$

and $v = v(v_x, v_y, v_z)$

- For the matrix form we have

$$[\mathrm{Pr}_{H,\mathbf{v}}(P)]_{\mathcal{K}} = \left(\mathrm{Id}_n - \frac{\mathbf{v}\cdot\mathbf{a}^t}{\mathbf{v}^t\cdot\mathbf{a}}\right)\cdot[P]_{\mathcal{K}} - \frac{a_{n+1}}{\mathbf{v}^t\cdot\mathbf{a}}[\mathbf{v}]_{\mathcal{K}}$$ which in our case becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{av_x+bv_y+cv_z}\begin{pmatrix} v_x a & v_x b & v_x c \\ v_y a & v_y b & v_y c \\ v_z a & v_z b & v_z c \end{pmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$= \frac{1}{av_x+bv_y+cv_z}\left[\begin{matrix} bv_y+cv_z & v_x b & v_x c \\ v_y a & av_x+cv_z & v_y c \\ v_z a & v_z b & av_x+bv_y \end{matrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \frac{d}{av_x+bv_y+cv_z}\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$