Seminar 9 - Conics

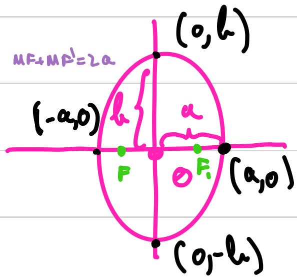
$\mathcal{C}: f(x,y) = 0$

$f \in \mathbb{R}[x,y]$, $\deg f \leq 2$

3 types:

↳ ellipses

"Locus of points where $MF+MF'=2a$"



inner, $b < a$

$\mathcal{E}: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

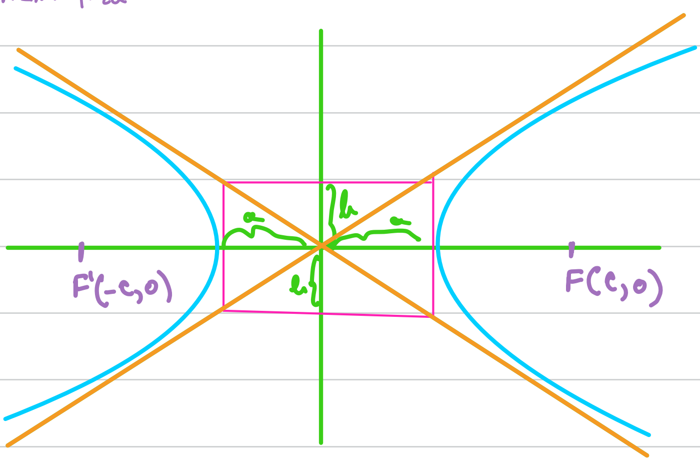
$c = \sqrt{a^2 - b^2}$

$F(-c,0)$, $F(c,0)$ focal points (foci)

"Locus of points where $|MF - MF'| = 2a$"

↳ hyperbolas

$\mathcal{H}: \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

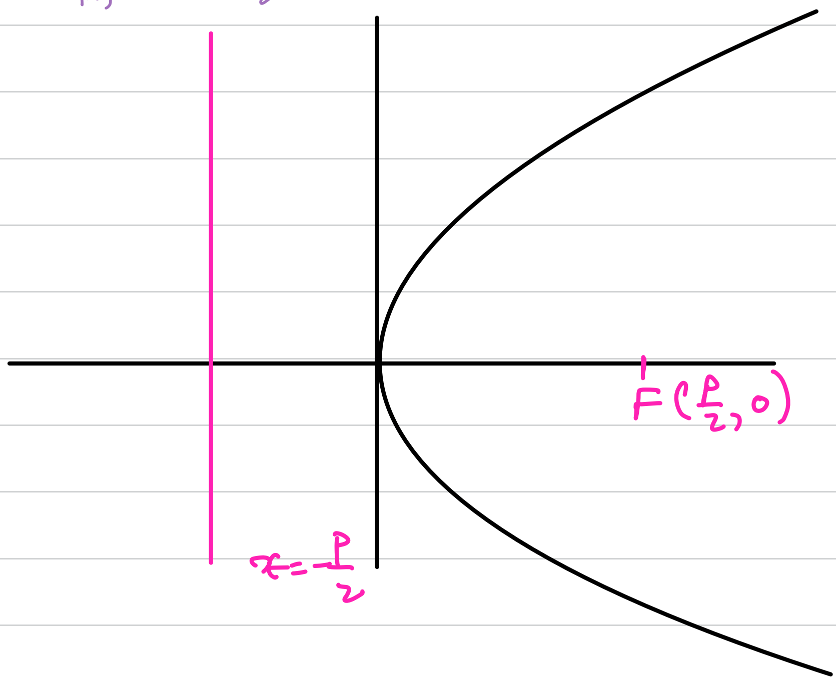


$c = \sqrt{a^2 + b^2}$

$y = \pm \dfrac{b}{a} x$ - asymptotes of H

↱ "Locus of points M where $dist(M,d) = MF = \frac{p}{2}$"

↳ parabolas



$\mathcal{P} : y^2 = 2px$

$\mathcal{E} : f(x, y) = 0$

$T_{(x_0,y_0)}\mathcal{E} : \frac{\partial f}{\partial x}(x_0, y_0) \cdot (x - x_0) +$

↳ the tangent $\frac{\partial f}{\partial y}(x_0, y_0) \cdot (y - y_0)$

$T_{(x_0,y_0)}\mathcal{E} : \frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 1$

$T_{(x_0,y_0)}\mathcal{H} : \frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$

$T_{(x_0,y_0)}\mathcal{P} : y y_0 = p(x + x_0)$

The tangents to a conic $\mathcal{C}$ with slope m are:

- Ellipse: $y = mx \pm \sqrt{m^2a^2+b^2}$

- Hyperbola: $y = mx \pm \sqrt{m^2a^2-b^2}$

9.1. Determine the semi-minor and semi-major axis, as well as the focal points of the ellipse:

$$9x^2 + 25y^2 - 225 = 0$$

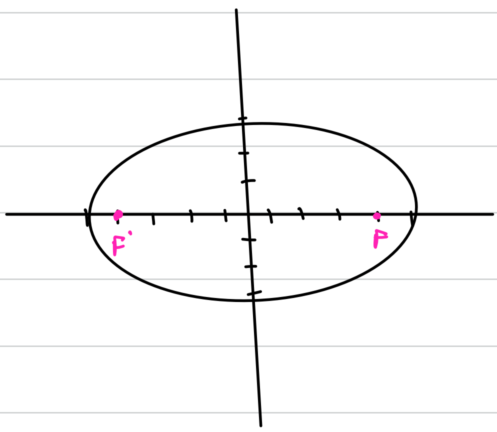
Moreover, draw the ellipse

$$\begin{cases} 9x^2 + 25y^2 = 225 \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} 9x^2 + 25y^2 = 225 \\ b^2x^2 + a^2y^2 = a^2b^2 \end{cases}$$

$$\Rightarrow b = 3$$
$$a = 5$$

$$e = \sqrt{25-9} = 4 \Rightarrow F(4,0)$$
$$F'(0,4)$$



5.3. !!! Determine the equation of a line which is orthogonal to $\ell: 2x-2y-13=0$

and tangent to the ellipse

$\mathcal{E}: x^2+4y^2-20=0$

$\ell: 2x-2y-13=0$

$\ell: 2y=2x-13$

$y=x-\frac{13}{2} \Rightarrow m_\ell=1$

$m_{\ell'}=-1$

$\mathcal{E}: x^2+4y^2-20=0$

$\mathcal{E}: \frac{x^2}{20}+\frac{y^2}{5}=1$

$a^2=20$
$b^2=5$

$\ell': y=mx\pm\sqrt{m^2a^2+b^2}$

$\ell': y=-x\pm\sqrt{20+5}$

$\ell': y=-x\pm 5$

5.6. Consider the family of lines

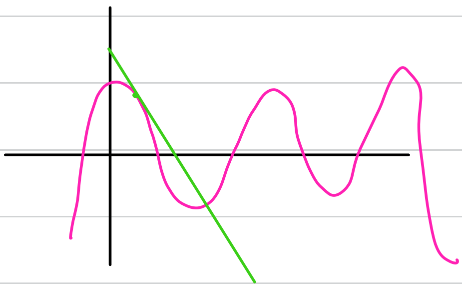
$\ell_c: \sqrt{5}x-y+c=0$

For what values $c\in\mathbb{R}$ is ℓ_c tangent to the ellipse

$\mathcal{E}: x^2+\frac{y^2}{4}=1$

A line ℓ is tangent to a curve $\mathcal{C}$ $\Leftrightarrow$ $|\ell \cap \mathcal{C}| = 1$

Counterexample :



$\mathcal{E}: x^2 - \frac{y^2}{4} = 1 \Rightarrow a^2 = 1$

$b^2 = 4$

$\ell_c: \sqrt{5}\, x - y + c = 0$

$\ell_c \cap \mathcal{E}: \begin{cases} y = \sqrt{5}x + c \\ 4x^2 + y^2 = 4 \end{cases}$

$4x^2 + (\sqrt{5}x + c)^2 = 4$

$4x^2 + 5x^2 + c^2 + 2\sqrt{5}x \cdot c = 4$

$9x^2 + 2\sqrt{5}x \cdot c + c^2 - 4 = 0$

$|\ell_c \cap \mathcal{E}| = 1 \Leftrightarrow$

$\Leftrightarrow \Delta = 0$

$\Delta = (2\sqrt{5} \cdot c)^2 - 4 \cdot 9 \cdot (c^2 - 4)$

$= 20 \cdot c^2 - 36 c^2 + 144$

$\Leftrightarrow -16c^2 = -144$

$\Rightarrow c^2 = 9 \Rightarrow c = \pm 3$

$\Rightarrow \sqrt{5}\, x - y = 3$

$\sqrt{5}\, x - y = -3$

3.10. Draw the hyperbola

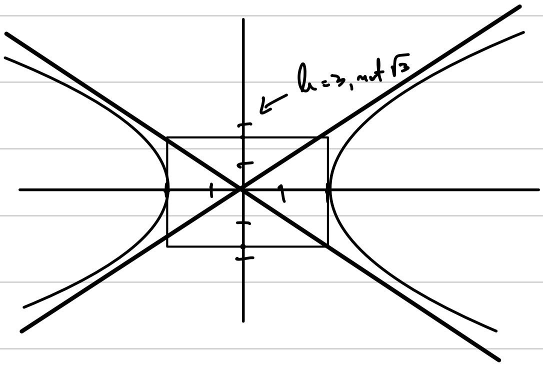
$$\mathcal{H}: 9x^2 - 4y^2 = 36$$

Indicate the semi-major and semi-minor axes and write down equations for

Moreover determine the relation between the coordinates (x_P, y_P) of a point P : $P \notin$ to any tangent to $\mathcal{H}$

$$\mathcal{H}: 9x^2 - 4y^2 = 36$$

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \iff \begin{array}{l} a^2 = 4 \Rightarrow a = 2 \\ b^2 = 9 \Rightarrow b = 3 \end{array}$$



$y = \pm \frac{\sqrt{3}}{2}$ - the asymptotes

$$L_m: y = mx \pm \sqrt{m^2a^2 - b^2}$$

$$L_m: y = mx \pm \sqrt{m^2 4 - 9}$$

$P(x_0, y_0) \in L_m$ iff.

$$(y_P - mx_P)^2 = 4m^2 - 9$$

$$y_P^2 - 2y_P m x_P + m^2 x_P^2 = 4m^2 - 9$$

[...] We reach an equation and we fix $\Delta < 0$ (aka no solutions)

9.12. a) Find the equation of the tangent lines to the parabola

$\mathcal{P}: y^2 - 36x = 0$

that contain $P(2,9)$

$T_{(x_0,y_0)}\mathcal{P}: \quad y y_0 = p(x + x_0)$

$y y_0 = 18(x + x_0)$

$P \in T_{(x_0,y_0)}\mathcal{P} \Rightarrow 9 y_0 = 18(2 + x_0)$

$y_0 = 2(2 + x_0)$

$$\begin{cases} y_0 = 2(2+x_0) \\ y_0^2 = 36 x_0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \quad 4x_0^2 - 20x_0 + 16 = 0$

$x_0^2 - 5x_0 + 4 = 0$

$(x_0)_{1,2} = \dfrac{5 \pm \sqrt{25-16}}{2} =$

$= \dfrac{5 \pm 3}{2} = \dots$

$\Rightarrow x_0 \begin{cases} 4 \\ 1 \end{cases}$

$y_0 \begin{cases} 12 \\ 6 \end{cases}$

So the tangents are

$T_{(4,12)}\mathcal{P}: \quad 12y = 18(x+4)$

$2y = \dots$

$T_{(1,6)}\mathcal{P}: \quad 6y = 18(x+1)$