9.1 Warm-Up Exercises

9.1. Find the equation of the circle:

a) of diameter $[A,B]$, with $A(1,2)$ and $B(-3,-1)$,

b) with center $I(2,-3)$ and radius $R = 7$,

c) with center $I(-1,2)$ and passing through $A(2,6)$,

d) centered at the origin and tangent to $\ell : 3x - 4y + 20 = 0$,

e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$,

f) passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$,

g) tangent to both $\ell_1 : 2x + y - 5 = 0$ and $\ell_2 : 2x + y + 15 = 0$ if one tangency point is $M(3,-1)$.

9.2. For a circle $\mathcal{C}$ of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2 - x^2})$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to $\mathcal{C}$.

c) Compare these to the equation of the tangent line $xx_0 + yy_0 = R^2$ where $(x_0, y_0) \in \mathcal{C}$.

9.3. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

9.4. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

9.5. Determine the tangents to the hyperbola $\mathcal{H} : \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell : 4x + 2y - 5 = 0$.

9.6. Determine the tangents to the hyperbola $\mathcal{H} : x^2 - y^2 = 16$ which contain the point $M(-1,7)$.

9.7. Consider the hyperbola $\mathcal{H} : x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that the angle $\measuredangle F_1MF_2$ is a right angle.

9.8. For which value k is the line $y = kx + 2$ tangent to the parabola $\mathcal{P} : y^2 = 4x$?

9.9. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

a) parallel to the line $\ell : 3x - 2y + 30 = 0$;

b) perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

9.10. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2,2)$.

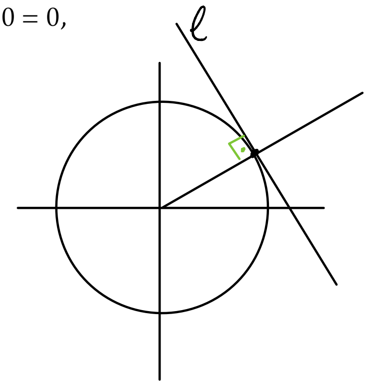
9.1. Find the equation of the circle:

a) of diameter $[A, B]$, with $A(1,2)$ and $B(-3,-1)$,

b) with center $I(2,-3)$ and radius $R = 7$,

c) with center $I(-1,2)$ and passing through $A(2,6)$,

d) centered at the origin and tangent to $\ell : 3x - 4y + 20 = 0$,

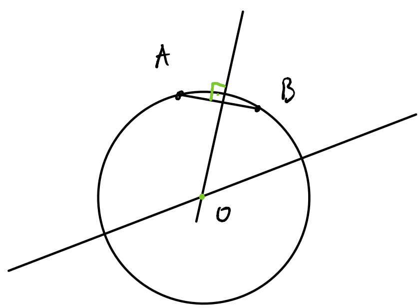


d.) The radius of the circle is

$$d(0, \ell) = \frac{20}{\sqrt{25}} = 4$$

$\Rightarrow \mathcal{C} : x^2 + y^2 = 16$

e) passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$,



Method I

Let O be the center of the circle

Then $O \in \ell$

and $O \in$ perpendicular bisector b of the segment AB

$\Rightarrow$ we can calculate O as $\ell \cap b$ and the radius as $d(O,A) = d(O,B)$

Method II

the equation of a circle of radius r centered at the point $P(x_0, y_0)$

is $\quad \mathcal{C}: \quad (x-x_0)^2 + (y-y_0)^2 = r^2$

since $P \in d: 3x - y - 2 = 0 \Rightarrow \boxed{y_0 = 3x_0 - 2}$ so $P = (x_0, 3x_0 - 2)$ and

$$\mathcal{C}: \quad (x - x_0)^2 + (y - 3x_0 + 2)^2 = r^2$$

since $A(3,1) \in \mathcal{C}$ and $B(-1,3) \in \mathcal{C}$ we have

$$\begin{cases} (3-x_0)^2 + (1 - 3x_0 + 2)^2 = r^2 \quad \text{and} \\ (-1-x_0)^2 + (3 - 3x_0 + 2)^2 = r^2 \end{cases}$$

rearranging the equations we have

$$\begin{cases} 9 - 6x_0 + x_0^2 + 9 - 18x_0 + 9x_0^2 = r^2 \\ 1 + 2x_0 + x_0^2 + 25 - 30x_0 + 9x_0^2 = r^2 \end{cases} \iff \begin{cases} 10x_0^2 - 24x_0 + 18 = r^2 \\ 10x_0^2 - 28x_0 + 26 = r^2 \end{cases} \ominus$$

$$0 + x_0 - 8 = 0$$

$$\Rightarrow \boxed{x_0 = 2} \Rightarrow \boxed{y_0 = 4}$$

$\Rightarrow$ so, the center of $\mathcal{C}$ is $P(2,4)$

and the radius r is the distance from P to A (and to B)

$$r = d(P,A) = \sqrt{(2-3)^2 + (4-1)^2} = \sqrt{1+9} = \sqrt{10}$$

$$\left(\text{check} \quad r = d(P,B) = \sqrt{(2+1)^2 + (4-3)^2} = \sqrt{9+1} = \sqrt{10}\right)$$

f) passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$,

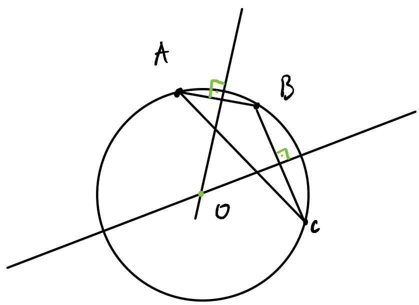
Method I

the circle passing through $A(x_A, y_A)$ $B(x_B, y_B)$ and $C(x_C, y_C)$ has an equation of the form

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_A^2+y_A^2 & x_A & y_A & 1 \\ x_B^2+y_B^2 & x_B & y_B & 1 \\ x_C^2+y_C^2 & x_C & y_C & 1 \end{vmatrix} = 0$$

so, you obtain the equation that is required here if you replace in and expand the determinant

Method II



The center O of the circle lies on the perpendicular bisector b_1 of [AB] and on the perpendicular bisector b_2 of [BC]

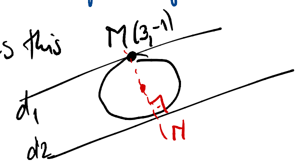
$\Rightarrow$ we can calculate O as $b_1 \cap b_2$ and the radius as $d(O,A)$

g) tangent to both $\ell_1 : 2x+y-5=0$ and $\ell_2 : 2x+y+15=0$ if one tangency point is $M(3,-1)$.

We have a circle which is tangent to two lines $d_1 : 2x+y-5$ and $d_2 : 2x+y+15=0$

What do you notice by looking at the two lines? they are parallel (why?)

So the picture is this



So the midpoint is on the perpendicular line ℓ passing through M, it has equation

$$\ell : -1(x-3)+2(y+1)=0 \quad \text{(why?)}$$

$$\iff \ell : -x+2y+5=0$$

the intersection $N = \ell \cap d_2$ is the solution to $\begin{cases} 2x+y+15=0 \\ -x+2y+5=0 \Rightarrow x = 2y+5 \end{cases} \Rightarrow 5y+25=0$

$$\boxed{\begin{array}{l} \Rightarrow y=-5 \\ \Rightarrow x=-5 \end{array}}$$

$N(-5,-5) \Leftarrow$

the center of $\mathcal{C}$ is the midpoint of $[M,N]$, it is $P(-1,-3)$

the radius of $\mathcal{C}$ is $d(P,M) = \sqrt{(3+1)^2+(-1+3)^2} = \sqrt{16+4} = \sqrt{20}$

$$\Rightarrow \mathcal{C} : (x+1)^2+(y+3)^2 = 20$$

9.2. For a circle C of radius R:

a) Use the parametrization $x \mapsto (x, \pm\sqrt{R^2-x^2})$ to deduce a parametrization of tangent lines to C.

b) Use the parametrization $\theta \mapsto (R\cos(\theta), R\sin(\theta))$ to deduce a parametrization of tangent lines to C.

c) Compare these to the equation of the tangent line $xx_0 + yy_0 = R^2$ where $(x_0, y_0) \in C$.

a) for $f(x) = (x, \sqrt{R^2-x^2})$ $\quad f'(x) = \left(1, -\dfrac{x}{\sqrt{R^2-x^2}}\right)$

So, if ℓ is the tangent line to $\mathcal{C}$ with contact point $f(x_0) \in \mathcal{C}$

then

$$\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ \sqrt{R^2-x_0^2} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\dfrac{x_0}{\sqrt{R^2-x_0^2}} \end{bmatrix} \quad t \in \mathbb{R}$$

Rem a.) notice that the position vector of $f(x_0) \in \mathcal{C}$ is orthogonal to the direction vector $f'(x_0)$

b.) for $f(\theta) = (R\cos\theta, R\sin\theta)$ $f'(\theta) = (-R\sin\theta, R\cos\theta)$

So, if ℓ is the tangent line to $\mathcal{C}$ with contact point $f(\theta_0) \in \mathcal{C}$ then

$$\ell : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} R\cos\theta_0 \\ R\sin\theta_0 \end{bmatrix} + t \begin{bmatrix} -R\sin\theta_0 \\ R\cos\theta_0 \end{bmatrix}$$

Rem b.) notice that the position vector of $f(\theta_0) \in \mathcal{C}$ is orthogonal to the direction vector $f'(\theta_0)$

c.) for $\mathcal{C}: x^2 + y^2 = R^2$ and $(x_0, y_0) \in \mathcal{C}$, the line tangent to $\mathcal{C}$ in (x_0, y_0)

is $\ell : x x_0 + y y_0 = R^2$

$\hookrightarrow$ a normal vector for this line is (x_0, y_0)

$\Rightarrow$ a position vector for this line is $v(-y_0, x_0)$

v is orthogonal to n, which is the position vector of (x_0, y_0)

$\Rightarrow$ if $f(x_0) = (x_0, y_0)$ we get the same line as in a.) and

if $f(\theta_0) = (x_0, y_0)$ we get the same line as in b.)

9.3. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

$$\ell \cap \mathcal{E} : \begin{cases} x^2+3y^2-25=0 \\ x+2y-7=0 \iff x = 7-2y \end{cases} \iff \begin{cases} (7-2y)^2+3y^2-25=0 & (*) \\ x = 7-2y \end{cases}$$

$$(*) \iff 49 - 28y + 4y^2 + 3y^2 - 25 = 0$$

$$7y^2 - 28y + 24 = 0$$

$$\Delta = 4^2\cdot 7^2 - 4\cdot 7\cdot 24 = 4^2\cdot 7(7-6) \Rightarrow y_{1,2} = \frac{4\cdot 7 \pm 4\cdot\sqrt{7}}{2\cdot 7} = 2 \pm \frac{2}{\sqrt{7}}$$

$\Rightarrow$ the two intersection points are $A(7-2y_1, y_1)$ and $B(7-2y_2, y_2)$

So $A\left(3-\frac{4}{\sqrt{7}}, 2+\frac{2}{\sqrt{7}}\right)$ and $B\left(3+\frac{4}{\sqrt{7}}, 2-\frac{2}{\sqrt{7}}\right)$

9.4. Determine the intersection points between the line $\ell : 2x - y - 10 = 0$ and the hyperbola $\mathcal{H} : \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0$.

$$\ell \cap \mathcal{H} : \begin{cases} \frac{x^2}{20} - \frac{y^2}{5} - 1 = 0 \\ 2x - y - 10 = 0 \end{cases} \iff \begin{cases} \frac{x^2}{20} - \frac{(2x-10)^2}{5} - 1 = 0 \\ y = 2x - 10 \end{cases}$$

$$\frac{x^2}{20} - \frac{4(x-5)^2}{5} - 1 = 0 \quad \Big|\cdot 20$$

$$x^2 - 16(x^2 - 10x + 25) - 20 = 0$$

$$-15x^2 + 160x - 420 = 0 \quad \Big| : 5$$

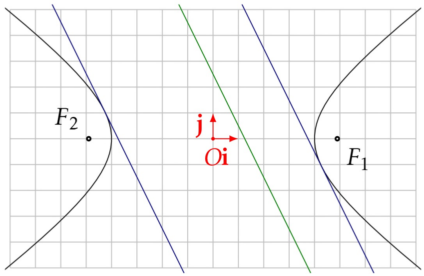
$$-3x^2 + 32x - 84 = 0$$

$$\Delta = \underset{2^{10}}{32^2} - \underset{2^2\cdot 3\cdot 2^2\cdot 3\cdot 7}{4\cdot 3\cdot 84} = 2^4(\underset{64}{2^6} - \underset{63}{3^2\cdot 7}) = 2^4 \Rightarrow x_{1,2} = \frac{-32 \pm 4}{-6} = \begin{cases} \frac{-36}{-6} = 6 \\ \frac{-28}{-6} = \frac{14}{3} \end{cases}$$

$2x-10$

So we have doo intersection points $P_1(\underset{x_1}{6}, 2)$ and $P_2\left(\frac{14}{3}, -\frac{2}{3}\right)$.

9.5. Determine the tangents to the hyperbola $\mathcal{H}: \frac{x^2}{16} - \frac{y^2}{8} - 1 = 0$ which are parallel to the line $\ell: 4x + 2y - 5 = 0$.



The tangents to $\mathcal{H}$ of slope k are

$$\ell_k: \ y = kx \pm \sqrt{a^2k^2 - b^2} \qquad \text{if} \quad k \in \left(-\infty, -\tfrac{b}{a}\right) \cup \left(\tfrac{b}{a}, \infty\right)$$

In our case $a = 4$ and $b = 2\sqrt{2}$ and $k = -2 < -\frac{2\sqrt{2}}{4}$

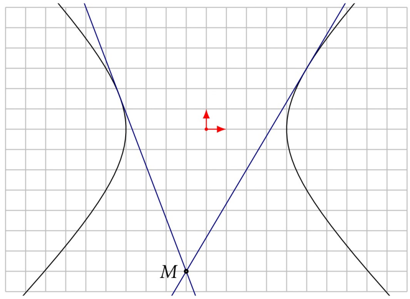
So $\sqrt{a^2k^2 - b^2} = \sqrt{16 \cdot 4 - 8} = 2\sqrt{14}$

and the two tangent lines are

$$\ell_1: \ y = -2x + 2\sqrt{14} \qquad \text{and} \qquad \ell_2: \ y = -2x - 2\sqrt{14}$$

Remark when we deduce the equation of a tangent line for a given slop k. it is clear that $\sqrt{a^2k^2 - b^2}$ has to be a real number in order for the equation to describe a line in the plane $\mathbb{E}^2$

9.6. Determine the tangents to the hyperbola $\mathcal{H}: x^2 - y^2 = 16$ which contain the point $M(-1,7)$.



We search for the possible tangents passing through M in the form

$$\ell : \ y = kx \pm \sqrt{a^2k^2 - b^2} \qquad \text{with } k \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$$

since any tangent line to $\mathcal{H}$ has such an equation, in particular those who pass through M

We know that $M(-1,7) \in \ell$ and that $a = b = 4$

$$7 = -k \pm \sqrt{16k^2 - 16}$$

$$\Rightarrow \quad 7 + k = \pm\sqrt{16k^2 - 16} \quad \Big| ()^2$$

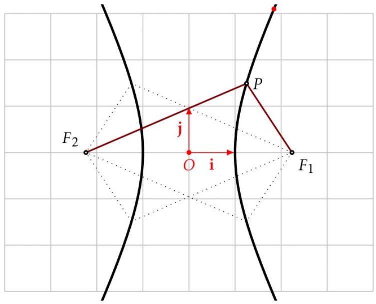
$$\Rightarrow \quad 49 + 14k + k^2 = 16k^2 - 16 \quad \left(\text{since } k^2 \geq \frac{b^2}{a^2}\right)$$

$$\Rightarrow \quad 15k^2 - 14k - 65 = 0$$

$$\Delta = 14^2 + 4 \cdot \underbrace{15 \cdot 65}_{3\cdot 5^2 \cdot 13} = 4(49 + 975) = 4 \cdot \underbrace{(1024)}_{2^{10}} = 2^{12}$$

$$\Rightarrow \quad k_{1,2} = \frac{14 \pm 64}{30} \begin{cases} -\frac{50}{30} = -\frac{5}{3} \\ \frac{78}{30} = \frac{13}{5} \end{cases}$$

9.7. Consider the hyperbola $\mathcal{H}: x^2 - \frac{y^2}{4} - 1 = 0$ with focal points F_1 and F_2. Find the points M situated on the hyperbola such that the angle $\measuredangle F_1MF_2$ is a right angle.



the points are the intersection of $\mathcal{H}$ with a circle centered in the origin and passing through the focal points F_1 and F_2

9.8. For which value k is the line $\underbrace{y = kx + 2}_{\ell_k}$ tangent to the parabola $\underbrace{\mathcal{P}: y^2 = 4x}_{\mathcal{P}_2}$?

$$\ell_k \cap \mathcal{P}_2 : \begin{cases} y^2 = 4x \\ y = kx+2 \end{cases} \iff \begin{cases} (kx+2)^2 = 4x \\ y = kx+2 \end{cases}$$

$$k^2x^2 + 4kx + 4 = 4x$$

$$k^2x^2 + 4(k-1)x + 4$$

$$\Delta = 4^2(k-1)^2 - 4\cdot 4\cdot k^2$$
$$= 4^2(k^2 - 2k - 1 - k^2)$$
$$= 4^2(-2k-1)$$

ℓ_k is tangent to $\mathcal{P}_2$ if $\Delta = 0 \iff k = -\frac{1}{2}$

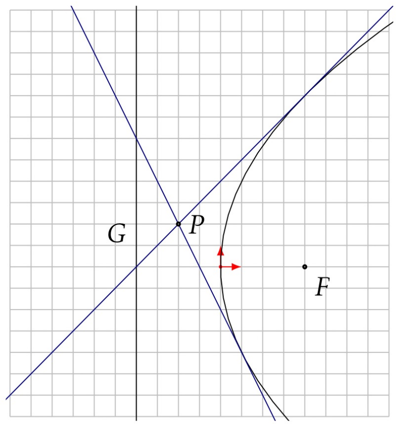
9.9. Consider the parabola $\mathcal{P} : y^2 = 16x$. Determine the tangents to $\mathcal{P}$ which are

a) parallel to the line $\ell : 3x - 2y + 30 = 0$;

b) perpendicular to the line $\ell : 4x + 2y + 7 = 0$.

Hint: use the form $y = kx + \frac{p}{2k}$ for tangents to $\mathcal{P}_p$

9.10. Determine the tangents to the parabola $\mathcal{P} : y^2 = 16x$ which contain the point $P(-2, 2)$.



Hint: use the form $y = kx + \frac{p}{2k}$ for tangents to $\mathcal{P}_p$