

Universitatea Babeş-Bolyai, Facultatea de Matematică și Informatică

Sectia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2019

## Linear difference equations. Discrete Dynamical Systems<sup>1</sup>

- 1.** a) Find all the sequences  $(x_k)_{k \geq 0}$  that satisfy  $x_{k+3} = k3^k + 5k - 2$ ,  $k \geq 0$ .  
b) Find all the sequences  $(x_k)_{k \in \mathbb{Z}}$  that satisfy  $x_{k+3} = k3^k + 5k - 2$ ,  $k \in \mathbb{Z}$ .

- 2.** Let  $\eta \in \mathbb{R}$  be a fixed parameter. Find the solution  $(x_k)_{k \geq 0}$  of the initial value problem

$$x_{k+1} = 2x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence?  
Discuss with respect to  $\eta$ .

- 3.** Let  $\lambda \in \mathbb{R}^*$  and  $\eta \in \mathbb{R}$  be fixed parameters. Find the solution  $(x_k)_{k \geq 0}$  of the initial value problem

$$x_{k+1} = \lambda x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence?  
Discuss with respect to  $\lambda$  and  $\eta$ .

- 4.** Find the solution  $(x_k)_{k \in \mathbb{Z}}$  of the initial value problem

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

Check the solution you obtained. What is the long term behavior of this sequence?

- 5.** Find solutions of the form  $x_k = a 3^k$  of the difference equation  $x_{k+1} = 2x_k + 3^k$ ,  $k \geq 0$ . Here we look for  $a \in \mathbb{R}$ .

- 6.** Find solutions of the form  $x_k = ak + b$  of the difference equation  $x_{k+1} = 2x_k - k$ ,  $k \geq 0$ . Here we look for  $a, b \in \mathbb{R}$ .

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For each of the following difference equations find a particular solution of the indicated form. If no form is indicated (and the equation is nonhomogeneous), try a constant solution. Find its general solution using the fundamental theorems for linear difference equations and the characteristic equation method.

**7.**  $x_{k+1} = \frac{1}{3}x_k$ ;    **8.**  $x_{k+1} = \frac{1}{2}x_k + 2$ ;    **9.**  $x_{k+1} = 2x_k + \frac{1}{2}$ ;    **10.**  $x_{k+1} = -3x_k$ ;

**11.**  $x_{k+1} = 4x_k + 3^{k+1}$ ,  $x_k^p = a \cdot 3^k$ ;    **12.**  $x_{k+1} = 1/3x_k + 2^k$ ,  $x_k^p = a \cdot 2^k$ ;

**13.**  $x_{k+2} - 6x_{k+1} + 9x_k = 0$ ;    **14.**  $x_{k+2} + x_{k+1} + x_k = 0$ .

**15.** Find the general solution of the linear planar system

$$x_{k+1} = 1/2x_k + y_k, \quad y_{k+1} = -1/5y_k$$

by reducing it to a second order difference equation in  $x$ .

**16.** Find the general solution of the linear planar system

$$x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k$$

by reducing it to a second order difference equation.

**17.** Write the Euler numerical formula with stepsize  $h > 0$  for the IVP  $x' = -120x$ ,  $x(0) = 1$  and solve the difference equation you obtained in two cases:  $h = 0.1$  and  $h = 0.001$ . Discuss on the long term behavior of the solution of the differential equation and, also, of the solution of the difference equation.

**18.** Write the Euler numerical formula with stepsize  $h > 0$  for the IVP  $x' = 2x(1 - x)$ ,  $x(0) = 1$ .

**19.** Find the fixed points of the map  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x - \frac{1}{4}(x^2 - 2)$  and study their stability.

**20.** Represent the stair-step diagram of the function from the previous exercise and try to discuss the long term behavior of the solution of  $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$ ,  $k \geq 0$  with respect to an arbitrary  $x_0 \in \mathbb{R}$ .

**21.** Prove that the map  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = m + \varepsilon \sin x$ , where  $m > 0$  and  $0 < \varepsilon < 1$ , has a unique fixed point which is an attractor. The equation  $x = m + \varepsilon \sin x$  is known as *Kepler equation* and arises in the study of planetary motion.

**22.** Study the stability of the fixed point  $(0, 0)$  of the following planar linear difference system:

- a)  $x_{k+1} = 3/5 x_k + 1/5 y_k, \quad y_{k+1} = 1/5 x_k + 3/5 y_k;$
- b)  $x_{k+1} = 1/2 x_k + y_k, \quad y_{k+1} = -1/5 y_k;$
- c)  $x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k.$

**23.** Find the fixed points and the 2-periodic points of the maps  $f, g : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = -x^3$  and  $g(x) = x^2 - 1$ .

**24.** In order to study the stability of the fixed point  $0$  of the maps  $f, g, h : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x + x^3$ ,  $g(x) = x - x^3$ ,  $h(x) = x + x^2$ , note that the linearization method can not be applied. Study the stability of the fixed point  $0$  using the stair-step diagram.

**25.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x^2 - 1$ .

(a) Find the fixed points and the 2-periodic points of  $f$  and study their stability using the linearization method.

(b) Represent the graph of  $f$  and find geometrically the fixed points of  $f$ . Also, depict the 2-periodic orbit using the stair-step diagram.

(c) Find directly  $f^k(0)$  for any  $k \geq 0$ . Depict this orbit using the stair-step diagram.

(d) Let  $\eta = 2$ , and, respectively,  $\eta = -1/4$ . Using the stair-step diagram describe the long-term behavior of the orbit that starts at  $\eta$  (in other notation, of the sequence defined by  $x_{k+1} = x_k^2 - 1$ ,  $x_0 = \eta$ ).

**26.** Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \rightarrow \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x}.$$

**27.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = 2x(1 - x)$ .

a) Find its fixed points and study their stability.

b) Let  $I_1 = (-\infty, 0)$ ,  $I_2 = (0, 1)$  and  $I_3 = (1, \infty)$ . Find  $f(I_1)$ ,  $f(I_2)$  and  $f(I_3)$ .

c) Find the orbits corresponding to the initial states  $\eta = 0$  and, respectively,  $\eta = 1$ .

d) Using the stair-step diagram, describe the long-term behavior of the orbits

corresponding to the initial states:  $\eta = 1/8$ ,  $\eta = 7/8$ ,  $\eta = -1/8$  and, respectively,  $\eta = 9/8$ .

- e) Estimate the basin of attraction of the attractor fixed point of  $f$ .

**28.** Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k, \quad x_0 = 0, \quad x_1 = 1.$$

**29\*.** Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

*Hint:* look for  $a, b \in \mathbb{R}$  such that  $(x_k)_p = ak + b$  is a particular solution of the difference equation.

**30\*.** Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $f(x, y) = (-x^2 + y/5, x)$ .

- (a) Find the fixed points of  $f$  and study their stability.
- (b) In case that you found an attracting fixed point, write the consequence of this fact for the sequence  $(f^k(\eta))_{k \geq 0}$  where  $\eta \in \mathbb{R}^2$  is properly chosen. As usual,  $f^k$  denotes the  $k$ -th iterate of  $f$ .

**31.** We consider the map

$$f : \mathbb{R} \rightarrow \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

- (a) Find its fixed points and study their stability.
- (b) Using the stair-step diagram estimate the basin of attraction of the attractor fixed point.
- (c) If  $(x_k)_{k \geq 0}$  represent the number of fish in some lake at month  $k$  and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case  $\eta = 80$  and also in the case  $\eta = 10$ .

**32.** We consider the IVP  $y' = -200y$ ,  $y(0) = 1$ , where the unknown is the function  $y(t)$ .

- a) Find the solution and its limit as  $t \rightarrow \infty$ .
- b) Write the Euler's numerical formula with constant step-size  $h$ .
- c) For  $h = 0.001$ , and, respectively,  $h = 0.01$  find the solution  $(y_k)_{k \geq 0}$  of the difference equation found at b) and decide if it satisfies  $\lim_{k \rightarrow \infty} y_k = 0$ .
- d) Find a range of values for the step-size  $h$  such that the solution  $(y_k)_{k \geq 0}$  of the difference equation found at b) satisfies  $\lim_{k \rightarrow \infty} y_k = 0$ .

**33.** (a) Write the Euler's numerical formula with stepsize  $h = 0.01$  to approximate the solution of the IVP  $y' = y$ ,  $y(0) = 1$ .

- (b) Using (a) find a rational approximation of the Euler's constant  $e$ .

**34.** Let  $g : I \rightarrow \mathbb{R}$  be a  $C^1$  map such that  $g'(x) \neq 0$  for all  $x$  in the interval  $I$ . Assume that there exists  $r \in I$  such that  $g(r) = 0$ . Prove that for  $\eta \in I$  sufficiently close to  $r$  the unique solution  $(x_k)_{k \geq 0}$  of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k \rightarrow \infty} x_k = r.$$

**35.** Let  $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$  be such that  $a_{12} \neq 0$ . Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

**36.** We consider the map

$$T : [0,1] \rightarrow \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

- a) Represent the graph of  $T$ . Find its fixed points.
- b) Let  $n \in \mathbb{N}$ ,  $n \geq 2$ . Find the orbit of the initial state  $\frac{3}{2^n}$ .

- c) Find the 2-periodic points of  $T$ .
- d) Represent the graphs of  $T^2$  and  $T^3$ . How many fixed points they have?
- e) The map  $T$  has a 2-periodic orbit? Or a 3-periodic orbit?  $T$  has a 2019-periodic orbit?

**37.** We consider the IVP  $y' = 1 + xy^2$ ,  $y(0) = 0$ . Write the Euler numerical formula on the interval  $[0, 1]$  with step-size  $h = 0.02$ . Specify the initial values and the number of steps necessary to find the approximate value of  $\varphi(0.5)$  and, respectively, of  $\varphi(1)$ . Here with  $\varphi$  is denoted the exact solution of the given IVP.

**38.** We consider the difference equation

$$x_{k+2} + x_k = \cos \frac{k\pi}{2}.$$

- a) Find a solution of the form  $(x_k)_p = ak \cos \frac{k\pi}{2}$ , with  $a \in \mathbb{R}$ . (Hint: we recall that  $\cos(t + \pi) = -\cos t$  for any  $t \in \mathbb{R}$ )
- b) Find its general solution.
- c) Find the solution with  $x_0 = x_1 = 0$  and describe its long-term behavior (is it periodic? is it bounded? is it oscillatory around 0?).

**39\*.** We consider the linear difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

- a) Study the stability of this system.
- b) Find its general solution.

**40.** Find the linear homogeneous difference equation with constant coefficients, of minimal order, which has as solutions the two sequences

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \dots$$

and

$$1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \frac{1}{2^4}, -\frac{1}{2^5}, \dots$$

**41.** We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k),$$

whose unknown is the sequence  $(x_k)_{k \geq 0}$ , and where  $\lambda \in (0, 1)$  is a parameter. Find its constant solutions (fixed points) and study their stability. Discuss with respect to the parameter  $\lambda$ .

**42.** We consider the IVP  $x' = -10^3x$ ,  $x(0) = 1$ .

- a) Find the solution and its limit as  $t \rightarrow \infty$ .
- b) Write the Euler's numerical formula with constant step-size  $h$ .
- c) Find a range of values for the step-size  $h$  such that the solution  $(x_k)_{k \geq 0}$  of the difference equation found at b) satisfies  $\lim_{k \rightarrow \infty} x_k = 0$ .

**43.** We consider the map  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = x - \frac{1}{4}(x^2 - 2)$  and, given  $x_0 \in \mathbb{R}$ , consider the sequence  $(x_k)_{k \geq 0}$  satisfying the recurrence

$$x_{k+1} = f(x_k).$$

- a) Find the fixed points of  $f$ , and study their stability.
- b) Find  $(x_k)_{k \geq 0}$  when  $x_0 = \sqrt{2}$ .
- c) There exists an  $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$  such that  $\lim_{k \rightarrow \infty} x_k = \sqrt{2}$ ?
- d) There exists an  $x_0 \in \mathbb{R}$  such that  $\lim_{k \rightarrow \infty} x_k = 2$ ?