

Week 3+ : Warm-up exercises

! Week 3-6/7 : Sunt CELE MAI DE BAZĂ !

- ↳ ec. drepturi
- ↳ ec. plomuri

+ noi pe la final chestii cu cercuri, chestii care se legăt de analiza

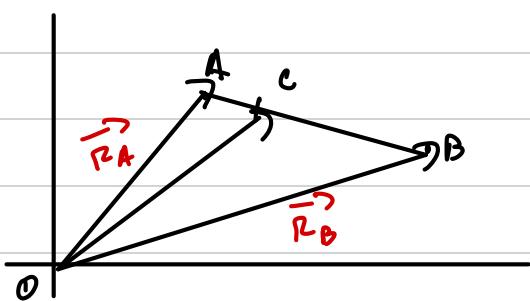
- nu mai e limită organizare de puncte neapărat (1 pt max la mată finală)
- recuperare semințor 0 - online, prob. săpt. astăzi

1.1. Let $\lambda = \frac{CA}{CB}$ - relation in which $C \notin [AB]$ divides it.

$$\text{Prin: } \overrightarrow{OC} = \frac{1}{1+\lambda} \overrightarrow{OA} + \frac{\lambda}{1+\lambda} \overrightarrow{OB}$$



$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$



$$\lambda = \frac{AC}{CB} \Rightarrow AC = \lambda CB \Rightarrow \overrightarrow{AC} = \lambda \cdot \overrightarrow{CB}$$

$$\begin{aligned} \Rightarrow (\lambda + 1) \overrightarrow{AC} &= (\overrightarrow{CB} + \overrightarrow{AC}) \\ &= (\lambda + 1) \cdot \overrightarrow{AB} \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \overrightarrow{OC} &= \overrightarrow{OA} + \lambda \overrightarrow{AB} = \\
 &= \frac{\lambda}{\lambda+1} (\overrightarrow{OA} + \overrightarrow{OB}) + \overrightarrow{OA} \\
 &= \frac{\lambda}{\lambda+1} \overrightarrow{OB} + \left(1 - \frac{\lambda}{\lambda+1}\right) \overrightarrow{OA} \\
 &= \frac{\lambda}{\lambda+1} \overrightarrow{OB} + \frac{1}{\lambda+1} \overrightarrow{OA}
 \end{aligned}$$

Alternatively : (with pos. vectors)

$$\overrightarrow{AC} = \lambda \overrightarrow{CB}$$

$$\overrightarrow{R_C} - \overrightarrow{R_A} = \lambda (\overrightarrow{R_B} - \overrightarrow{R_C})$$

$$(\lambda+1)\overrightarrow{RC} = \lambda \overrightarrow{R_A} + \lambda \overrightarrow{R_B}$$

$$\overrightarrow{R_C} = \frac{\overrightarrow{R_A} + \lambda \overrightarrow{R_B}}{1+\lambda}$$

Nic's way - Trick 4 memory



!

$$\begin{aligned}
 \overrightarrow{R_C} &= \frac{\overrightarrow{R_A} + \lambda \overrightarrow{R_B}}{\lambda+1} = (\lambda+1)\overrightarrow{R_C} \\
 \overrightarrow{R_C} &= \frac{\overrightarrow{R_A} + \lambda \overrightarrow{R_B}}{\lambda+1}
 \end{aligned}$$

$E_x: C \in \{A, B\}$

$$\frac{\overrightarrow{AB}}{BC} = \frac{13}{6}$$



$$\overrightarrow{R_C} = \frac{4 \overrightarrow{R_B} + 6 \overrightarrow{R_A}}{13}$$

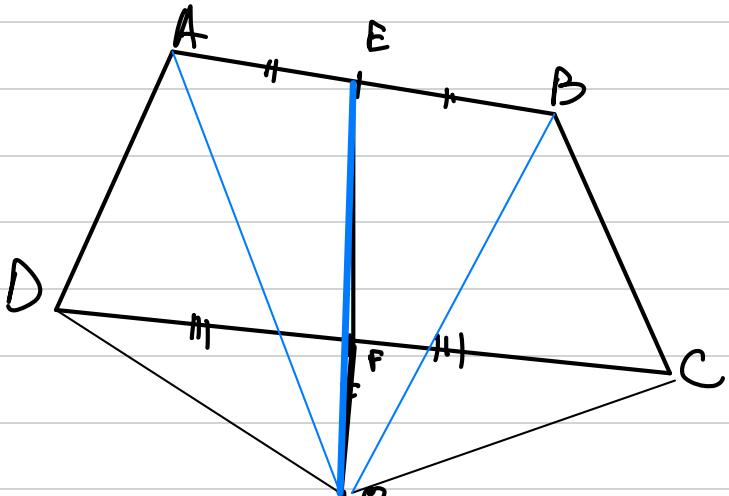
Reversing the scales

1.3. ABCD quadrilateral

E mid point of $\{AB\}$

F mid point of $\{CD\}$

$$\overrightarrow{EF} = \frac{1}{2} (\overrightarrow{AD} + \overrightarrow{BC})$$



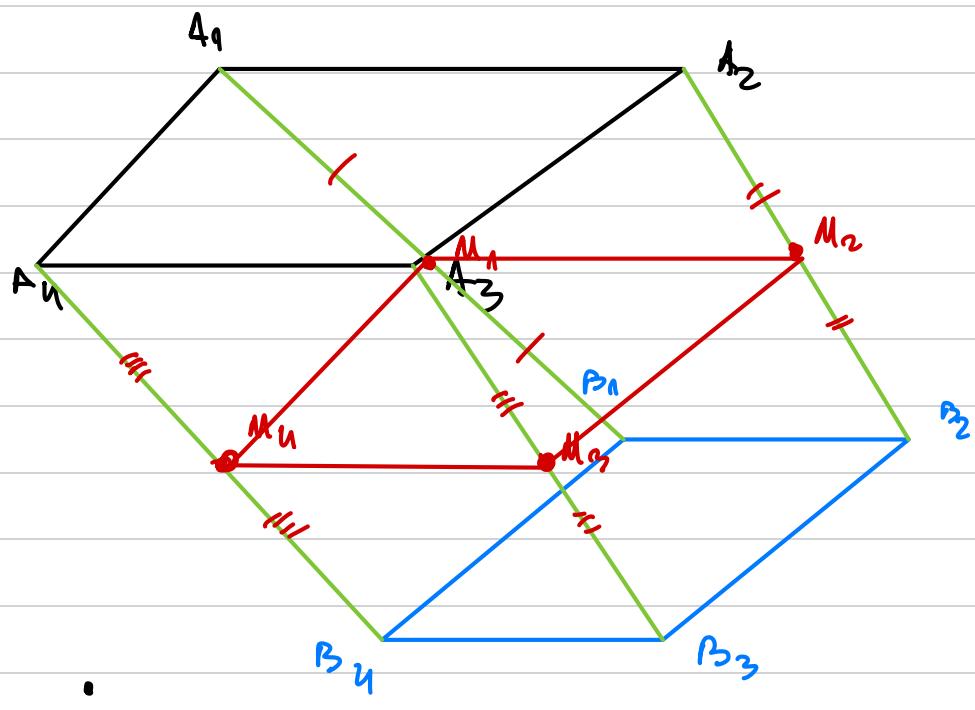
$$\begin{aligned}\overrightarrow{EF} &= \overrightarrow{R_D} - \overrightarrow{R_F} \\ \overrightarrow{R_F} &= \frac{\overrightarrow{R_D} + \overrightarrow{R_C}}{2} \\ \overrightarrow{R_E} &= \frac{\overrightarrow{R_B} + \overrightarrow{R_A}}{2}\end{aligned}$$

$$\begin{aligned}
 \vec{EP} &= \frac{\vec{AD} + \vec{BC}}{2} - \frac{\vec{AB} + \vec{DC}}{2} \\
 &= \frac{\vec{AD} - \vec{DC}}{2} + \frac{\vec{BC} - \vec{AB}}{2} \\
 &= \frac{1}{2} (\vec{AD} + \vec{BC})
 \end{aligned}$$

1.5. $A_1, A_2, A_3, A_n \quad \left. \begin{array}{c} \\ \end{array} \right\}$ Parallelog.
 $B_1, B_2, B_3, B_n \quad \left. \begin{array}{c} \\ \end{array} \right\}$

Show that the midpoints M_1, M_2, M_3, M_n of $\{A_1B_1\}$
 $\{A_2B_2\}$
 $\{A_3B_3\}$
 $\{A_nB_n\}$

are vertices of a parallelogram



$$\begin{aligned}
 \vec{R}_{M_1} &= \frac{\vec{R}_{A_1} + \vec{R}_{B_1}}{2} \\
 \vec{R}_{M_2} &= \frac{\vec{R}_{A_2} + \vec{R}_{B_2}}{2} \\
 \vec{R}_{M_3} &= \frac{\vec{R}_{A_3} + \vec{R}_{B_3}}{2} \\
 \vec{R}_{M_4} &= \frac{\vec{R}_{A_4} + \vec{R}_{B_4}}{2}
 \end{aligned}$$

$$\overrightarrow{M_1 M_3} = \overrightarrow{M_2} - \overrightarrow{M_1} = \frac{1}{2} (\overrightarrow{A_1 A_3} + \overrightarrow{B_1 B_3})$$

$$A_1 A_2 A_3 A_4 - \text{Parallel} \Rightarrow \overrightarrow{A_1 A_3} = \overrightarrow{A_1 A_2}$$

$$B_1 B_2 B_3 B_4 - \text{Parallel} \Rightarrow \overrightarrow{B_1 B_3} = \overrightarrow{B_1 B_2}$$

$$\overrightarrow{M_1 M_3} = \frac{1}{2} (\overrightarrow{A_1 A_2} + \overrightarrow{B_1 B_2}) = \overrightarrow{M_1 M_2}$$

↳ same line
↳ same direction

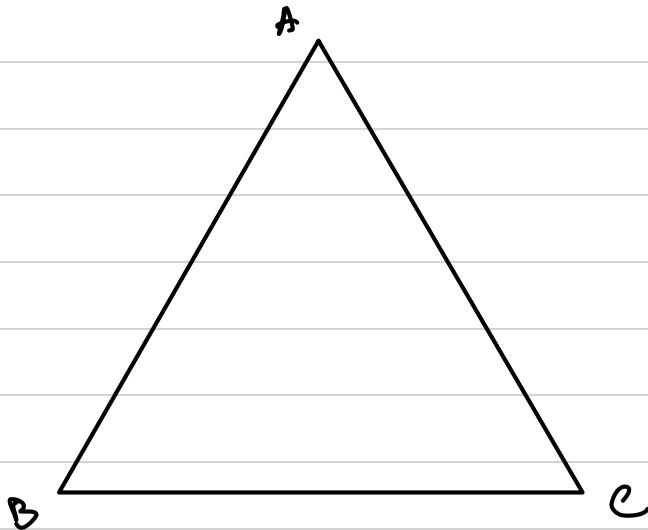
∴

$$\Rightarrow \overrightarrow{M_1 M_3} \stackrel{\parallel}{=} \overrightarrow{M_1 M_2} \Rightarrow M_1 M_2 M_3 M_4 - \text{Parallel}$$

↳ || AND =

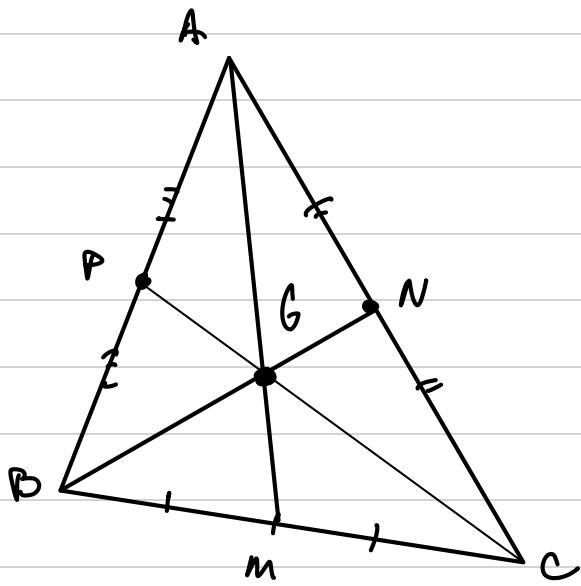
INTERLUDE

1.6. Starting from Hilbert's axioms, prove that the base angles of an isosceles triangle are congruent.



$$\left. \begin{array}{l} AB = AC \\ AC = AB \\ \widehat{BAC} = \widehat{CAB} \end{array} \right\} \xrightarrow{\text{Ax. III.6}} \widehat{ABC} = \widehat{ACB}$$

1.4. Using vectors show that the medians in a \triangle are concurrent and deduce the ratio in which the common intersection point divides the medians.



The vector eq. of a line through points A and B

$\Rightarrow \vec{r}_T = r\vec{r}_A + (1-r)\vec{r}_B$

$$\vec{r}_T = r\vec{r}_A + (1-r)\vec{r}_B$$

Let M, N, P - midpoints of the sides

Let $\{G\} = AM \cap BN \cap CP$

We will find \vec{r}_G and prove $G \in CP$

$$\vec{BA} + \vec{BC} = 2\vec{BN}$$

$$\vec{AC} + \vec{AB} = 2\vec{AM}$$

$$\vec{BA} + \vec{BC} + \vec{AC} + \vec{AB} = 2(\vec{BN} + \vec{AM})$$

$$\vec{BC} + \vec{AC} = 2(\vec{BN} + \vec{AM}) \quad | \cdot (-1)$$

$$\vec{CB} + \vec{CA} = 2(\vec{NB} + \vec{MA})$$

$$2\vec{CP} = 2(\vec{NB} + \vec{MA})$$

$$\vec{CP} = \vec{NB} + \vec{MA}$$

$$\vec{CD} = \vec{NG} + \vec{GB} + \vec{MG} + \vec{G4}$$

Blocoj la tolka + mentol

$$GG AM \Rightarrow \vec{n}_G = \alpha \vec{n}_A + (1-\alpha) \vec{n}_M \\ = \alpha \vec{n}_A + \frac{1-\alpha}{2} \vec{n}_B + \frac{1-\alpha}{2} \vec{n}_C$$

$$GG BN \Rightarrow \vec{n}_G = \mu \vec{n}_B + (1-\mu) \vec{n}_N \\ = \mu \vec{n}_B + \frac{1-\mu}{2} \vec{n}_A + \frac{1-\mu}{2} \vec{n}_C$$

$$\left(\alpha - \frac{1-\mu}{2}\right) \vec{n}_A + \left(\frac{1-\alpha}{2} - \mu\right) \vec{n}_B + \\ + \left(\frac{1-\alpha}{2} - \frac{1-\mu}{2}\right) \vec{n}_C = 0$$

Fix the origin in A:

$$\vec{n}_A = \vec{0}, \vec{n}_B = \vec{AB}, \vec{n}_C = \vec{AC}$$

lin. independent

So :

$$\left(\frac{1-\lambda}{2} - \mu \right) \vec{AB} + \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2} \right) \vec{AC} = 0$$

ΔABC is non-degenerate $\Rightarrow \vec{AB}, \vec{AC}$ lin. indep. \Rightarrow

Recap Base Changes from Lin. Alg.

$$\Rightarrow \begin{cases} \frac{1-\lambda}{2} - \mu = 0 \\ \frac{1-\lambda}{2} - \frac{1-\mu}{2} = 0 \end{cases} \Rightarrow \mu = \lambda$$

$$\Rightarrow 1-\lambda = 2\lambda \Rightarrow \lambda = \frac{1}{3}$$

So:

$$\begin{aligned} \vec{R}_G &= \frac{1}{3} \vec{R}_A + \frac{1}{3} \vec{R}_B + \frac{1}{3} \vec{R}_C = \\ &= \frac{\vec{R}_A + \vec{R}_B + \vec{R}_C}{3} \end{aligned}$$

$$\vec{CG} = \vec{R}_G - \vec{R}_C = \frac{\vec{R}_A + \vec{R}_B - 2\vec{R}_C}{3}$$

$$\vec{CP} = \vec{R}_P - \vec{R}_C = \frac{\vec{R}_A + \vec{R}_B}{2} - \vec{R}_C$$

$$\overrightarrow{CP} = \frac{3}{2} \Rightarrow \overline{CG} \Rightarrow C, P, G - \text{collinear} \Rightarrow$$

$$\Rightarrow G = AM \cap BN \cap CP \text{ and } CP = \frac{3}{2} CG$$

1.8. G, H, U collinear

