Seminar 4

$\vec{v}, \vec{w} \in V$

$$\boxed{\vec{v} \cdot \vec{w} = \begin{cases} 0, & \vec{v} = 0 \text{ or } \vec{w} = 0 \\ |\vec{v}| \cdot |\vec{w}| \cdot \cos(\vec{v}, \vec{w}) \end{cases}}$$

$$\boxed{\begin{array}{l} \vec{v} \perp \vec{w} \Rightarrow \vec{v} \cdot \vec{w} = 0 \\ \vec{v} \cdot \vec{v} = |\vec{v}|^2 \end{array}}$$

Say we work in $\mathbb{E}^n$

From now on, we will work over <u>orthonormal</u> reference frames

orthonormal = orthogonal + normalized

$R = (O, \vec{v}_1, \dots, \vec{v}_n)$

- orthogonal : $\forall i \neq j : \vec{v}_i \perp \vec{v}_j$
- normalized : $|\vec{v}_1| = \dots = |\vec{v}_n| = 1$

Yk $\vec{v}, \vec{w} \in V^n$

$$[v]_R = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$[w]_R = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$\vec{v}\cdot\vec{w} = [v]_B^t \cdot [w]_B^t = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

4.1. $\vec{m}, \vec{n}$ unit vectors

$\measuredangle(\vec{m}, \vec{n}) = 120^0$

Determine: $\measuredangle(\vec{a}, \vec{b})$ if

$\vec{a} = 2\vec{m} + 4\vec{n}$

$\vec{b} = \vec{m} - \vec{n}$

$$\vec{m}\cdot\vec{n} = |\overset{1}{\vec{m}}|\cdot|\overset{1}{\vec{n}}|\cdot\cos(120^\circ) = -\frac{1}{2}$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos(\vec{a}, \vec{b})$$

$$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(2\vec{m}+4\vec{n})^2} = \sqrt{4\vec{m}^2 + 16\vec{n}^2 + 16\vec{m}\cdot\vec{n}} = \sqrt{20-8} = \sqrt{12} = 2\sqrt{3}$$

$$|\vec{b}| = \sqrt{\vec{b}\cdot\vec{b}} = \sqrt{(\vec{m}-\vec{n})^2} = \sqrt{2+1} = \sqrt{3}$$

$$\vec{a}\cdot\vec{b} = (2\vec{m}+4\vec{n})(\vec{m}-\vec{n}) = 2\vec{m}^2 - 2\vec{m}\vec{n} + 4\vec{n}\vec{m} - 4\vec{n}^2 = 2 - 1 - 4 = 2 - 5 = -3$$

$$-3 = \sqrt{3}\cdot 2\sqrt{3}\cdot\cos(\measuredangle a, b) \Rightarrow \cos(a, b) = -\frac{\cancel{3}}{2\cancel{\sqrt{3}}\cdot\cancel{\sqrt{3}}} = -\frac{1}{2}$$

$\widehat{a, b} = 120°$

$\mathbb{E}^n$

Take a hyperplane H in $\mathbb{E}^n$

$H: a_1x_1 + a_2x_2 + \dots + a_nx_n + b = 0$

$D(H): \underbrace{a_1x_1 + \dots + a_nx_n}_{(a_1 \dots a_n) \cdot (x_1 \dots x_n) = 0} = 0$

$D(H)^{\perp} = \{\vec{w} \in V \mid \vec{w} \perp \vec{v}, \forall \vec{v} \in D(\pi)\}$

$= \langle \vec{n_\pi} \rangle \quad \forall \vec{v} \in D(\pi)$

Ex: $\pi: 2x + 6y - 3z + 4 = 0$

$\vec{n_\pi}(2, 6, -3)$

The normal vector is given by the equation. + it gives the orientation

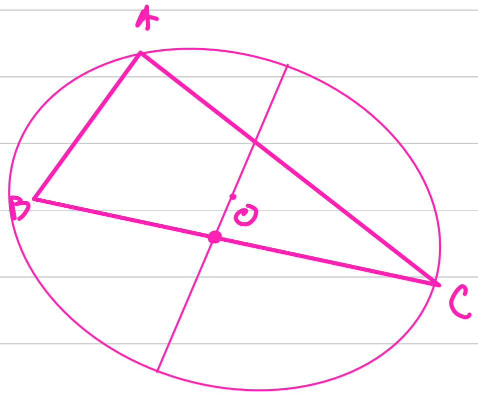
4.2. $A(1,2)$
$B(3,-2)$
$C(5,6)$

a) Calculate the circumcenter of ΔABC

b) Calculate the orthogonal centre of ΔABC

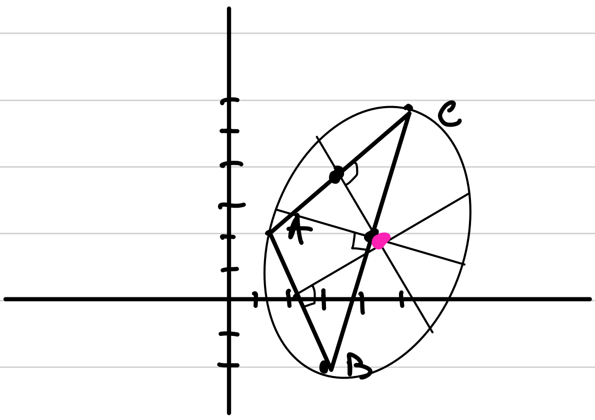
c) Find the eq. of the angle bisector.

circumcentre = $\cap$ of $\perp$ bisectors
↳ mediatoare



orthocenter = $\cap$ of the heights

$A(1,2)$
$B(3,-2)$
$C(5,6)$



M - midpoint of BC $\rightarrow M = (4,2)$

N - midpoint of AC => N(3, 4)

$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{8}{2} = 4 \Rightarrow m_{OM} = -\frac{1}{4}$

$m_{AC} = \frac{y_C - y_A}{x_C - x_A} = \frac{4}{4} = 1 = m_{ON} = -1$

$OM: y - y_M = m(x - x_M)$

$y - 2 = -\frac{1}{4}(x - 4) \Rightarrow y = \frac{-x+12}{4}$

$ON: y - 4 = -1(x - 3)$

$y = -x + 4$

$OM \cap ON = \begin{cases} y = \frac{-x+12}{4} & | \cdot 4 \\ y = -x + 4 \end{cases} \Leftrightarrow -x + 4 = \frac{-x+12}{4}$

$-4x + 28 = -x + 12$

$-3x = -16$

$x = \frac{16}{3}$

$y = \frac{5}{3}$

$\Rightarrow O\left(\frac{16}{3}, \frac{5}{3}\right)$

b) Let $M \in AC$: $\vec{BM} \perp \vec{AC} \Rightarrow m_{\vec{BM}} = -\frac{1}{m_{\vec{AC}}}$

$$m_{\vec{AC}} = \frac{y_C - y_A}{x_C - x_A} = \frac{4}{4} = 1 \Rightarrow m_{\vec{BM}} = -1$$

$N \in BC$ [...] $m_{AN} = -\frac{1}{4}$

AN : $y - y_A = m_{AN}(x - x_A)$

$$y - 2 = -\frac{1}{4}(x - 1)$$

AN : $y = \frac{-x+1}{4} + 2$

BM : $y - y_B = m_{BM}(x - x_B)$

$$y + 2 = -x + 3$$

BM : $y = 1 - x$

$AN \cap BM$ - $\begin{cases} y = \frac{-x+1}{4} + 2 \\ y = 1 - x \end{cases}$

$$\Leftrightarrow 1 - x = \frac{-x+1}{4} + 2$$

$$-1 - x = \frac{-x+1}{4}$$

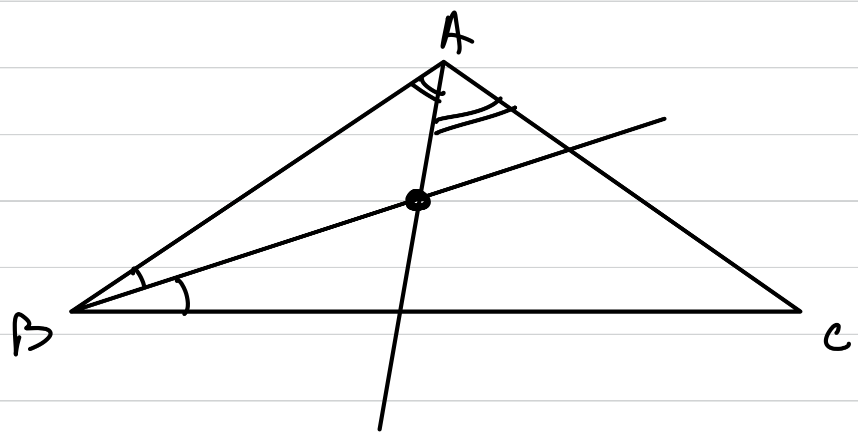
$$-4 - 4x = -x + 1$$

$$-3x = 5$$

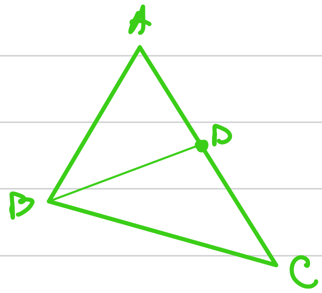
$$x = -\frac{5}{3}$$

$y = \frac{8}{3} \Rightarrow$

$\Rightarrow AN \cap BM = O\left(-\frac{5}{3}, \frac{8}{3}\right)$



First approach: bisector th.



$$\frac{AB}{BC} = \frac{AD}{DC}$$

$AB = \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5}$

$BC = \sqrt{2^2+8^2} = \sqrt{68} = 2\sqrt{17}$

→ th. from 1st week

$$\frac{AD}{DC} = \frac{\sqrt{5}}{\sqrt{17}} \Rightarrow \vec{BD} = \frac{\sqrt{5}\,\vec{BC} + \sqrt{17}\cdot\vec{BA}}{\sqrt{5}+\sqrt{17}}$$

$$= \frac{1}{\sqrt{5}+\sqrt{17}} \cdot \left(\sqrt{5}(2,8) + \sqrt{17}(-2,4)\right)$$

$$BD: \begin{cases} x = 3 + 2 \cdot \frac{2\sqrt{5} - 2\sqrt{17}}{\sqrt{5}+\sqrt{17}} \\ y = -2 \cdot 2 \cdot \frac{8\sqrt{5} - 4\sqrt{17}}{\sqrt{5}+\sqrt{17}} \end{cases}$$

2^{nd} approach: Parallelogram rule

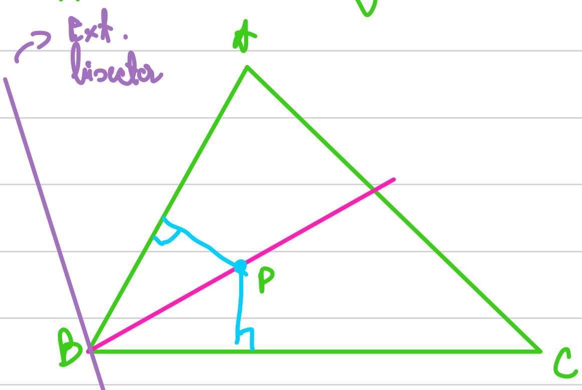
The direction of the bisector is given by:

$$\frac{\vec{BA}}{|\vec{BA}|} + \frac{\vec{BC}}{|\vec{BC}|}$$ ⇒ True bc. $BB_1B_2B_3$ = rhombus

$$\frac{1}{2\sqrt{5}}(-2,4) + \frac{1}{2\sqrt{14}}(2,8)$$

So: $BD = \begin{cases} x = 3 + \lambda\left(-\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{14}}\right) \\ y = -2 + \lambda\left(\frac{2}{\sqrt{5}} + \frac{4}{\sqrt{14}}\right) \end{cases}$

3^{rd} approach: using distances



P ∈ bisector of B ⟺

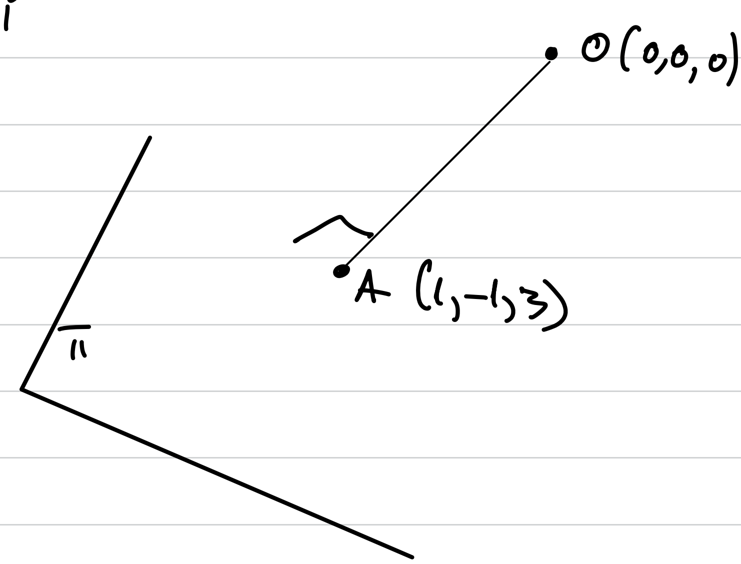
⟺ dist(P, AB) = dist(P, BC)

This gives two lines l_1 and l_2.

To decide which one is the internal bisector we use the half-plane rule

↳ Plugging the coords of A and C and pick the line that separates them

4.6. Determine a Cartesian equation of the plane π if $A(-1,3)$ is the orthogonal projection of the origin on π



$$\pi : x - y + 3z + d = 0$$

$$A \in \pi \Rightarrow 1 + 1 + 9 + d = 0 \Rightarrow d = -11$$

$$\Rightarrow d = -11$$

$$\Rightarrow x - y + 3z - 11 = 0$$

$OA \perp \pi \Rightarrow$

$\Rightarrow \overrightarrow{OA} \in D(\pi)$

$\hookrightarrow \overrightarrow{OA}$ is a normal vector for π

$\Rightarrow$ we can use its coordinates for the eq. of π

4.5. Consider the vector $\vec{v}(x, y, z)$ so that

$\vec{v} \perp \vec{a}(4, -2, -3)$

$\vec{v} \perp b(0, 1, 3)$

$\sphericalangle(v, Ox) < \frac{\pi}{2} \Rightarrow \cos(\widehat{v, Ox}) > 0$

$|\vec{v}| = 26$

$\vec{v} \cdot \vec{a} = 0 \Leftrightarrow 4x - 2y - 3z = 0$

$\vec{v} \cdot \vec{b} = 0 \Leftrightarrow y + 3z = 0$

$\vec{v} \cdot Ox(1, 0, 0) = |\vec{v}| \cdot 1 \cdot \cos(v, \vec{Ox}) > 0$

$\vec{v} \cdot \vec{Ox} > 0 \Rightarrow (x, y, z)(1\ 0\ 0) > 0 \Rightarrow x > 0$

$$v: \begin{cases} 4x - 2y - 3z = 0 \\ y + 3z = 0 \Rightarrow z = -\frac{y}{3} \\ \cos(\widehat{v, Ox}) > 0 \\ \Leftrightarrow x > 0 \end{cases} \Rightarrow 4x - y = 0 \Rightarrow x = \frac{y}{4}$$

$$|\vec{v}| = 26 \Rightarrow \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{x^2+y^2+z^2} = 26$$

$$\Leftrightarrow \begin{cases} 4x - 3y - 6z = 0 \\ \sqrt{x^2+y^2+z^2} = 26 \Rightarrow \\ x > 0 \end{cases}$$

$$\Rightarrow \sqrt{\left(\frac{y}{4}\right)^2 + y^2 + \left(-\frac{y}{3}\right)^2} = 26$$

$$\Leftrightarrow \sqrt{\frac{y^2}{16} + y^2 + \frac{y^2}{9}} = 26$$

$$\Leftrightarrow \sqrt{\frac{9y^2 + 144y^2 + 16y^2}{144}} = 26$$

$$\Leftrightarrow \frac{13|y|}{12} = 26$$

$$|y| = 24$$

$$\Rightarrow |y| = \pm 24$$

$$x = \frac{y}{4} \Rightarrow x = \pm 6$$

$$x > 0 \Rightarrow x = 6,\ y = 24$$

$$z = -\frac{y}{3} \Rightarrow z = -8$$

$v(6, 24, -8)$