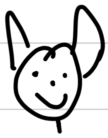


Desem Micu™

Seminar 2

2.2. Decide if the given points are collinear:

1) $P(3,-5)$, $Q(-1,2)$, $R(-5,3)$

2) $P(1,0,-1)$, $Q(0,-1,2)$, $R(-1,-2,5)$

1) P, Q, R - collinear $\Rightarrow$! Only 2D !

$$\Rightarrow \begin{vmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 3 & -5 & 1 \\ -1 & 2 & 1 \\ -5 & 3 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 6 - 3 + 25 - (-10 + 27 + 5) = 0$$

$$\Leftrightarrow 22 - 22 = 0$$

$$0 = 0 \text{ TRUE} \Rightarrow$$

$\Rightarrow P, Q, R$ - collinear

2) $\vec{PQ} = \vec{r}_Q - \vec{r}_P$

$$= (0,-1,2) - (1,0,-1) = (-1,-1,1)$$

$$\vec{PR} = \vec{r}_R - \vec{r}_P = (-1,-2,5) - (1,0,-1)$$

$$= (-2, -2, 6) =$$

$$= 2 \cdot (-1, -1, 3) = 2 \cdot \overrightarrow{PQ} \Rightarrow$$

$\Rightarrow$ P-Q-R collinear

Why the 1st approach works

$$\begin{vmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{vmatrix} \begin{matrix} L_2 \leftarrow L_2 - L_1 \\ = \\ L_3 \leftarrow L_3 - L_1 \end{matrix}$$

$$= \begin{vmatrix} x_P & y_P & 1 \\ x_Q - x_P & y_Q - y_P & 0 \\ x_R - x_P & y_R - y_P & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x_{\overrightarrow{PQ}} & y_{\overrightarrow{PQ}} \\ x_{\overrightarrow{PR}} & y_{\overrightarrow{PR}} \end{vmatrix} = 0 \Leftrightarrow \overrightarrow{PR} = 2\overrightarrow{PQ}$$

Changing reference frames (Euclidean space)

↳ "space where we do geometry"

V - associated v.s.

$$V = \underline{\mathbb{E} \times \mathbb{E}} \atop \sim$$

$\rightsquigarrow$ Basically says: "Taking $\longrightarrow$ is the same as taking $\longrightarrow$"

V - k-v.s.

$$\left.\begin{aligned} b &= (v_1, \dots, v_n) \\ b' &= (v_1', \dots, v_n') \end{aligned}\right\} \text{bases of } V$$

$$M_{b'b} = [id]_{bb'} = \left([v_1]_{b'} \,\middle|\, \dots \,\middle|\, [v_n]_{b'} \right)$$

$$= \text{Base change matrix } b \to b'$$

$$\boxed{\begin{aligned} &\forall\, v \in V: \\ &\qquad [v]_{b'} = M_{b'b} \cdot [v]_b \end{aligned}}$$

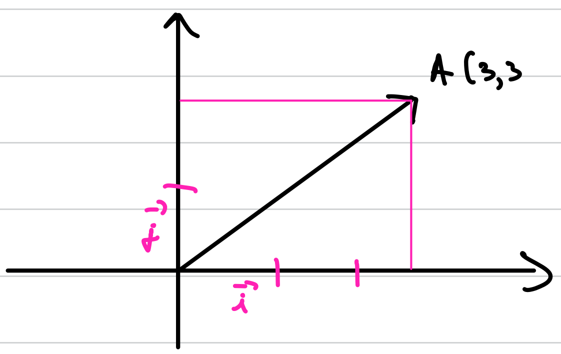
Reference frame = a pair (O, b) where:
- $O \in \mathbb{E}$ - point
- b - base of V/

$\mathcal{K} = (O, b)$ - reference frame

$\forall\, P \in \mathbb{E}$

$$[P]_{\mathcal{K}} := [\overrightarrow{OP}]_b$$

Ex:



$\overrightarrow{OA} = 3\vec{i} + 2\vec{j}$

$\mathcal{K} = (O, b)$
$\mathcal{K}' = (O', b')$

$$\boxed{\begin{aligned}[P]_{\mathcal{K}'} &= [\overrightarrow{O'P}]_{b'} = M_{b'b} \cdot [\overrightarrow{O'P}]_{b} \\ &= M_{b'b} \cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_{b} = \\ &= M_{b'b} \cdot \left([\overrightarrow{OP}]_{b} - [\overrightarrow{OO'}]_{b} \right) \\ &= M_{b'b} \cdot \left([P]_{\mathcal{K}} - [O']_{\mathcal{K}} \right)\end{aligned}}$$

$\Downarrow$

$M_{\mathcal{K}'\mathcal{K}}$ (we will use it more)

2.6. $\left.\begin{aligned}\mathcal{K} &= (O, \mathcal{B}) \\ \mathcal{K}' &= (O', \mathcal{B}')\end{aligned}\right\}$ Reference frames in $\mathbb{E}^{2}$ $\rightarrow$ given dim., NOT $\mathbb{E} \times \mathbb{E}$

$\mathcal{B} = (\vec{i}, \vec{j})$
$\mathcal{B}' = (\vec{i}', \vec{j}')$

$[O']_{\mathcal{K}} = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$

$$[i']_{\mathcal{B}} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$[j']_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$M_{\mathcal{K}'\mathcal{K}} = ?$$

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coordinates of $A, B, C \in \mathbb{E}^2$:

$$[A]_{\mathcal{K}} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$[B]_{\mathcal{K}} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$[C]_{\mathcal{K}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$M_{\mathcal{K}\mathcal{K}'} = M_{\mathcal{B}\mathcal{B}'} = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} -3 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \underset{L_1 \leftrightarrow L_1}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ -3 & 1 & 1 & 0 \end{array}\right)$$

$$\underset{L_2 \leftarrow L_2 + 3L_1}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 3 \end{array}\right)$$

$$\underset{L_2 \leftarrow \frac{L_2}{4}}{\sim} \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$\underset{L_1 \leftarrow L_1 - L_2}{\sim} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & \frac{1}{4} & \frac{3}{4} \end{array}\right)$$

$$\Rightarrow (\mathcal{M}_{BB'})^{-1} = \mathcal{M}_{B'B} = \mathcal{M}_{\mathcal{K}'\mathcal{K}} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Let $P \in \mathbb{E}^2$, $[P]_{\mathcal{K}'} = [\overrightarrow{O'P}]_{B'} = \mathcal{M}_{B'B} \cdot [O'P]_{B} =$

$$= \mathcal{M}_{B'B} \cdot ([\overrightarrow{OP} - \overrightarrow{OO'}]_{B})$$

$$= \mathcal{M}_{B'B} \cdot ([\overrightarrow{OP}]_{B} - [OO']_{B})$$

$$= \mathcal{M}_{\mathcal{K}'\mathcal{K}} \cdot ([P]_{\mathcal{K}} - [O']_{\mathcal{K}}) =$$

$$\hookrightarrow \begin{pmatrix} 6 \\ 0 \end{pmatrix}$$

We can do:

$$[ABC]_{\mathcal{K}'} = \mathcal{M}_{\mathcal{K}'\mathcal{K}} ([ABC]_{\mathcal{K}} - [O'O'O']_{\mathcal{K}})$$

The same thing as doing it 3 times

$$[ABC]_{\mathcal{K}'} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 6 & 3 \\ 2 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

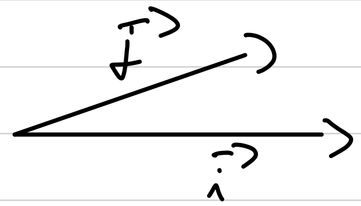
$$= \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \cdot \begin{bmatrix} -6 & 0 & -3 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & 3 & 0 \end{bmatrix} \Rightarrow [A]_{\mathcal{K}'} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[B]_{\mathcal{K}'} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$[C]_{\mathcal{K}'} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\mathbb{E}^2$



$(O, \vec{i}, \vec{j})$ - **right oriented** if we can rotate $\vec{i}$ by a positive angle in $[0, 2\pi)$ to overlap it with $\vec{j}$

left oriented - || - .... $(-2\pi, 0]$ ...

$$\boxed{\text{2 ref. frames } \mathcal{K}, \mathcal{K}' \text{ have the same orientation iff. } \det M_{\mathcal{K}\mathcal{K}'} > 0}$$

2.5. With respect to the basis $\mathcal{B} = (\vec{i}, \vec{j}, \vec{k})$:

$$\vec{u} = \vec{i} + \vec{j}$$
$$\vec{v} = \vec{j} + \vec{k}$$
$$\vec{w} = \vec{i} + \vec{k}$$

check that $\mathcal{B}' = (\vec{u}, \vec{v}, \vec{w})$ is a basis and decide if it's left / right oriented

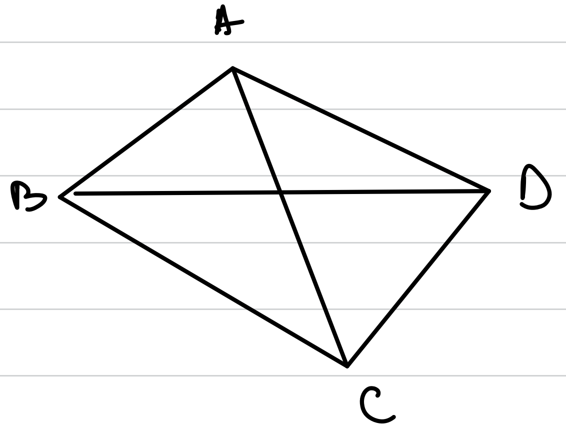
$$\vec{u} = \vec{i} + \vec{j} \Rightarrow [\vec{u}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$
$$\vec{v} = \vec{j} + \vec{k} \Rightarrow [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\vec{w} = \vec{i} + \vec{k} \Rightarrow [\vec{w}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$M_{\mathcal{B}'\mathcal{B}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\det(M_{\mathcal{B}'\mathcal{B}}) = 1+1 = 2 \neq 0 \Rightarrow \mathcal{B}'$ - basis

$\det(M_{\mathcal{B}'\mathcal{B}}) > 0 \Rightarrow$ right - oriented

2.7. $ABCD$ - tetrahedron



Consider the reference frames:

$\mathcal{K}_A = (A, \overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}),\ \mathcal{A} = (\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})$

$\mathcal{K}_B = (B, \overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{BD})$

$\mathcal{K}'_A = (A, \overrightarrow{BA}, \overrightarrow{BD}, \overrightarrow{BC})$

Determine: a) Coords of A, B, C, D in all 3 $\mathcal{K}$'s

b) $M_{\mathcal{K}_A \mathcal{K}'_A}$, $M_{\mathcal{K}_B \mathcal{K}_A}$

c) The orientation of the 3 frames w/ respect to each other

a) $$[ABCD]_{K_A} = \begin{array}{c} \begin{array}{cccc} \vec{AA} & \vec{AB} & \vec{AC} & \vec{AD} \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$[ABCD]_{K_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[ABCD]_{K'_A} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

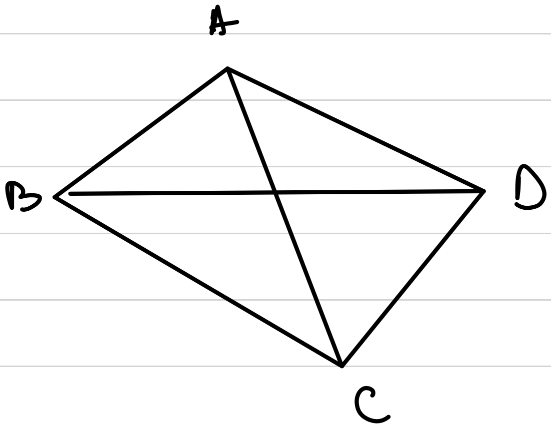
b) $$M_{K_A K'} = \left([BA]_{K_A} \quad [BD]_{K_A} \quad [BC]_{K_A} \right)$$
$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\det M_{K_A K'} = 1 > 0 \Rightarrow K_A, K'_A$ have the same orientation

$$M_{K_B K_A} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det M_{K_B K_A} = -1 < 0 \to K_A, K_B$ - opposite orientation

2.a. What is the # of cartesian frames one could construct w/ the vertices of a tetrahedron



$$(P, \vec{v}_1, \vec{v}_2, \vec{v}_3)$$

$P \in \{A, B, C, D\}$

$\vec{v}_i \in \{\vec{xy} \mid x \neq y,\ x, y \in \{A, B, C, D\}\}$

$\vec{v}_{1,2,3}$ lin. indep.

$$\underbrace{4}_{\text{\# of choices 4 the origin}} \cdot \underbrace{\left(4 \cdot 2 \cdot C_6^2\right)}_{\text{\# of choices for } (\vec{v}_1, \vec{v}_2)} \cdot \underbrace{(2 \cdot 3)}_{\text{\# of choices for } v_3}$$

↳ sides

$= 4 \cdot 8 \cdot 15 \cdot 6 = 2880$