Q&A June 22

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$$\begin{cases} \dot{x} = -y - x(x^2+y^2) \\ \dot{y} = x - y(x^2+y^2) \end{cases}$$ (0,0) is an equil. point

$$f(x,y) = \begin{pmatrix} -y - x(x^2+y^2) \\ x - y(x^2+y^2) \end{pmatrix} = \begin{pmatrix} -y - x^3 - xy^2 \\ x - x^2y - y^3 \end{pmatrix}$$

$f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (0,0)$ is an equil. point

$A = Jf(0,0)$ $\quad Jf(x,y) = \begin{pmatrix} -3x^2 - y^2 & -1-2xy \\ 1-2xy & -x^2-3y^2 \end{pmatrix}$ $\quad A = Jf(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \dot{X} = AX \iff \begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ this is the linearized system around (0,0)

the gen sol: $\ddot{x} = -\dot{y} \Rightarrow \ddot{x} = -x \to \ddot{x} + x = 0 \quad r^2+1=0 \quad r_{1,2} = \pm i$

$$\begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = c_1 \sin t - c_2 \cos t \end{cases} \quad y = -\dot{x}, \quad c_1, c_2 \in \mathbb{R}.$$

find a first integral: $\dfrac{dy}{dx} = \dfrac{x}{-y} \Rightarrow -y\,dy = x\,dx \to -\dfrac{y^2}{2} = \dfrac{x^2}{2} + c$

$\Rightarrow x^2 + y^2 = -2c, \quad c \in \mathbb{R}$

$H(x,y) = x^2+y^2, \quad H: \mathbb{R}^2 \to \mathbb{R} \quad C^1$

H is a f.i in $\mathbb{R}^2 \iff -y\dfrac{\partial H}{\partial x} + x\dfrac{\partial H}{\partial y} = 0$ in $\mathbb{R}^2 \iff$

$\iff -y \cdot 2x + x \cdot 2y = 0$ in $\mathbb{R}^2 \iff 0 = 0$ true.

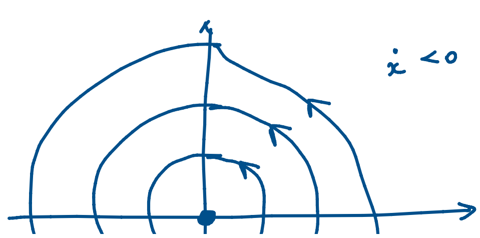
we represent now the level curves of H

$x^2+y^2 = c, \quad c \in \mathbb{R}$

$c=1 \quad x^2+y^2=1$

$c=2 \quad x^2+y^2=2$

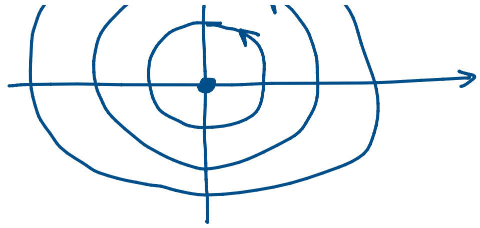
$c=0 \quad x=y=0$



$\dot{x} < 0 \Rightarrow x \searrow$

$c=0 \qquad x=y=0$

$c<0 \qquad \emptyset$



the orbits are the level curves of H.

$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ linear system having $(0,0)$ as the only equil. point

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ we need the eigenvalues

$\det(A - \lambda I) = 0 \iff \lambda^2 - (\operatorname{tr} A)\lambda + \det A = 0 \iff \lambda^2 + 1 = 0 \iff \lambda_{1,2} = \pm i$

This linear system is a center, thus it is stable.

$\begin{cases} \dot{x} = 2x - 3y \\ \dot{y} = x - 2y \end{cases} \qquad A = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}$

$\lambda_1 = 1 \qquad u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad \lambda_2 = -1 \qquad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

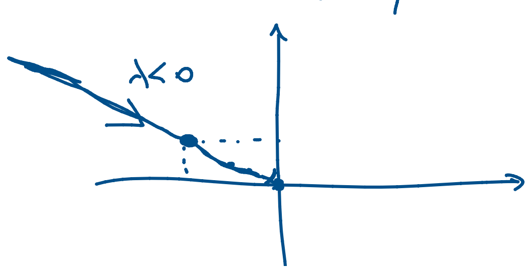
$\varphi(t, u_1) = e^{t\lambda_1} u_1 \quad , \quad \varphi(t, u_2) = e^{t\lambda_2} u_2$

$\dot{X} = AX \qquad \lambda \in \mathbb{R} \, , \quad u \in \mathbb{R}^2$ s.t. $Au = \lambda u$, $u \neq (0,0)$

$\varphi(t,u) = e^{t\lambda} u \iff e^{t\lambda} u$ is the unique sol. of the ivp $\begin{cases} \dot{X} = AX \\ X(0) = u \end{cases}$

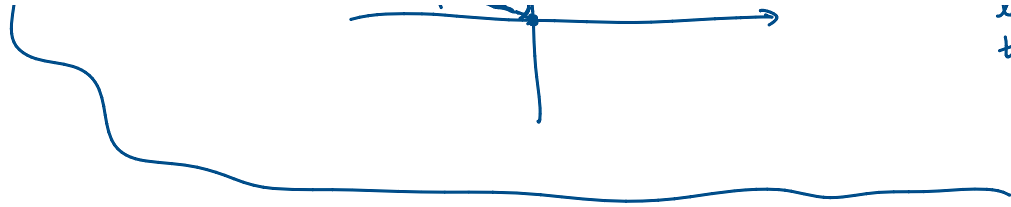
$\iff \begin{cases} \frac{d}{dt}\left(e^{t\lambda} u\right) = \lambda e^{t\lambda} u = e^{t\lambda} \cdot (\lambda u) = e^{t\lambda} A u = A \cdot \left(e^{t\lambda} u\right) \\ e^{t\lambda} u \big|_{t=0} = e^0 u = u \end{cases}$ TRUE.

$\gamma_u = \{ \varphi(t,u) : t \in \mathbb{R} \} = \{ e^{t\lambda} u : t \in \mathbb{R} \} = \{ \alpha u : \alpha > 0 \} \subset \mathbb{R}^2$



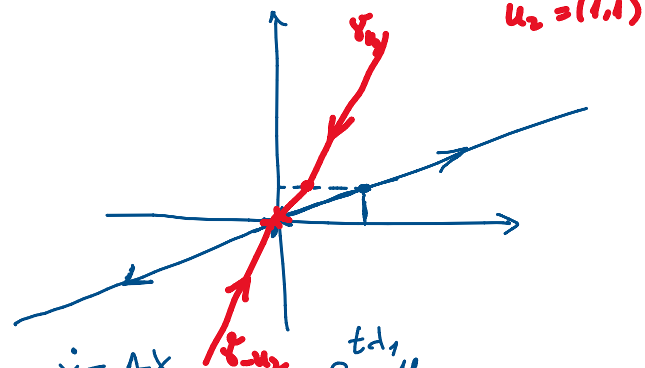
$\lambda < 0 \Rightarrow \lim_{t \to \infty} e^{t\lambda} = 0 \Rightarrow$

$\Rightarrow \lim_{t \to \infty} \varphi(t,u) = (0,0)$



$\varphi(t, u_1) = e^{t} u_1 \quad , \quad u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\gamma_{u_1} = \{ e^{t} u_1 \ : \ t \in \mathbb{R} \}$



$\gamma_{-u_1} = ?$

$\varphi(t, -u_1) = -e^{t\lambda_1} u_1$

$e^{t\lambda_1} u_1$

$\begin{cases} \dot{x} = AX \\ X(0) = -u_1 \end{cases} \qquad \begin{cases} \dot{x} = AX \\ X(0) = u_1 \end{cases}$

$\gamma_{-u_1} = \{ \underbrace{-e}^{t\lambda_1} u_1 \ : \ t \in \mathbb{R} \} \qquad \lambda_1 = 1$

$\dot{x} = f(x) \qquad H : \mathbb{R}^2 \to \mathbb{R}$ first integral $\qquad f : \mathbb{R}^2 \to \mathbb{R}^2$

For $\eta \in \mathbb{R}^2$, we consider the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$ and denote its unique sol. by

$\varphi(t, \eta), \quad t \in I_\eta$.

$H : \mathbb{R}^2 \to \mathbb{R}$ is a f.i of $\dot{x} = f(x) \overset{\text{def}}{\Longleftrightarrow}$ H is not a constant function and

$$H(\varphi(t,\eta)) = H(\eta), \quad \forall t \in I_\eta, \ \forall \eta \in \mathbb{R}^2.$$

Assume, by contradiction, that $\dot{x} = f(x)$ has a global attractor $\Rightarrow$

$\exists \ \eta^* \in \mathbb{R}^2$ an equilibrium point s.t. $\lim_{t \to \infty} \varphi(t, \eta) = \eta^*$, $\forall \eta \in \mathbb{R}^2$.

H is cont and $\lim_{t\to\infty} \varphi(t,\eta) = \eta^* \Rightarrow \lim_{t\to\infty} H(\varphi(t,\eta)) = H(\eta^*)$

H is a f.i. $\Rightarrow H(\varphi(t,\eta)) = H(\eta), \ \forall \eta \in \mathbb{R}^2$ $\Bigg\} \Rightarrow$

$$\lim_{t\to\infty} \text{ H is a f.i.} \Rightarrow \lim_{t\to\infty} H(\varphi(t,\eta)) = H(\eta), \forall \eta \in \mathbb{R}^2$$

$\Rightarrow H(\eta) = H(\eta^*), \forall \eta \in \mathbb{R}^2 \Rightarrow$ H is a constant function in $\mathbb{R}^2$

this contradicts the definition of a f.i.

$g(x) = x^2 - 2 \qquad g : (0,\infty) \to \mathbb{R}$

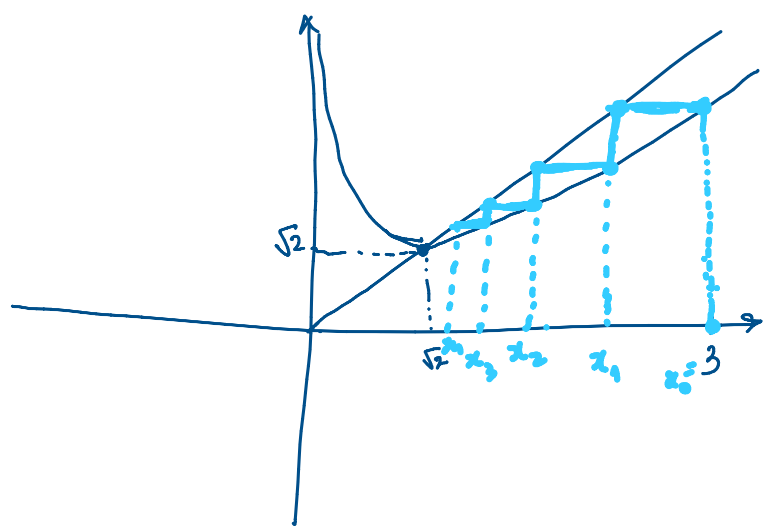
$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2-2}{2x} = \frac{2x^2 - x^2 + 2}{2x} = \frac{x^2+2}{2x}$$

$f : (0,\infty) \to \mathbb{R} \qquad f(x) = \dfrac{x^2+2}{2x}$

$$\begin{cases} x_{k+1} = f(x_k), & k \geq 0 \\ x_0 = \eta \end{cases} \qquad x_0 = 1 \qquad x_1 = f(x_0) = f(1) = \frac{3}{2}$$

$$x_2 = f(x_1) = f\left(\frac{3}{2}\right) = \frac{\frac{9}{4}+2}{3} = \frac{17}{12}$$

~~cw~~ <u>cobweb diagram</u> G_f and the line $y = x$



c) We have to prove that $\sqrt{2}$ is an attractor fixed point of f.
(by the definition of an attractor fixed point).

it is sufficient to prove that $f(\sqrt{2}) = \sqrt{2}$ and $|f'(\sqrt{2})| < 1$.

$$f(x) = \frac{x^2+2}{2x} \qquad f(\sqrt{2}) = \frac{(\sqrt{2})^2+2}{2\sqrt{2}} = \frac{4}{2\sqrt{2}} = \sqrt{2}. \checkmark$$

$$f'(x) = \frac{2x \cdot 2x - 2(x^2+2)}{4x^2} = \frac{2x^2-4}{4x^2} = \frac{2(x^2-2)}{4x^2}$$

$$f'(\sqrt{2}) = 0 \Rightarrow |f'(\sqrt{2})| < 1 \quad \checkmark$$

$x' + tx = 1$ first order Ln-HDE with non-constant coeff.

Find the gen. sol.

$x = x_h + x_p$

x_h is the gen sol. of $x' + tx = 0$

$$\frac{dx}{dt} = -tx \qquad \frac{dx}{x} = -t\,dt \qquad \ln|x| = -\frac{t^2}{2} + c$$

$x = 0$ is a sol. $\qquad |x| = e^c \cdot e^{-t^2/2}$

$x_h = c e^{-t^2/2}, \quad c \in \mathbb{R}$

$x_p = ?$ $\qquad x_p(t) = \varphi(t) e^{-\frac{t^2}{2}} \qquad x_p' + t x_p = 1 \iff$

$$\iff \varphi'(t) e^{-t^2/2} \cancel{- t\varphi(t) e^{-t^2/2}} + \cancel{t\varphi(t) e^{-t^2/2}} = 1, \quad \forall t \in \mathbb{R} \iff$$

$$\varphi'(t) = e^{t^2/2} \Rightarrow \varphi(t) = \int_0^t e^{s^2/2}\,ds$$

$$\Rightarrow x_p = e^{-t^2/2} \int_0^t e^{s^2/2}\,ds$$

$$x = c e^{-t^2/2} + e^{-t^2/2} \int_0^t e^{s^2/2}\,ds, \quad c \in \mathbb{R}.$$

$\eta \qquad f(\eta) = \eta \qquad f(\eta) = \eta \quad \dots$

$\eta \qquad f(\eta) \neq \eta \qquad f(f(\eta)) = \eta \qquad$ 2-cycle

$f^2 = f \circ f \qquad f^2$

$f^2 = f \circ f$ $\qquad$ f^2