



Seminar 2

2.2. Decide if the given points are collinear:

$$1) P(3, -5), Q(-1, 2), R(-5, 5)$$

$$2) P(1, 0, -1), Q(0, -1, 2), R(-1, 2, 5)$$

1) P, Q, R - collinear \Rightarrow ! Only 2D!

$$\Rightarrow \begin{vmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} 3 & -5 & 1 \\ -1 & 2 & 1 \\ -5 & 5 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow 6 - 25 - (-10 + 24 + 5) = 0$$

$$\Leftrightarrow 22 - 22 = 0$$

0 = 0 TRUE \Rightarrow

$\Rightarrow P, Q, R$ - collinear

$$2) \overrightarrow{PQ} = \overrightarrow{R_Q} - \overrightarrow{R_P}$$

$$= (0, -1, 2) - (1, 0, -1) = (-1, -1, 1)$$

$$\overrightarrow{PR} = \overrightarrow{R_R} - \overrightarrow{R_P} = (-1, -2, 5) - (1, 0, -1)$$

$$= (-2, -2, 6) =$$

$$= 2 \cdot (-1, -1, 3) = 2 \cdot \vec{PQ} \Rightarrow$$

$\Rightarrow P-Q-R$ collinear

why the 1st approach works

$$\begin{vmatrix} x_P & y_P & 1 \\ x_Q & y_Q & 1 \\ x_R & y_R & 1 \end{vmatrix} = L_2 \leftarrow L_2 - L_1$$

$$L_3 \leftarrow L_3 - L_1$$

$$= \begin{vmatrix} x_P & y_P & 1 \\ x_Q - x_P & y_Q - y_P & 0 \\ x_R - x_P & y_R - y_P & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{x}_{PQ} & \vec{y}_{PQ} \\ \vec{x}_{PR} & \vec{y}_{PR} \end{vmatrix} = 0 \Leftrightarrow \vec{PR} = 2 \vec{PQ}$$

Changing reference frames

(Euclidean space)

\hookrightarrow "space where we do geometry"

V - associated v.s.

$$V = \frac{\mathbb{E} \times \mathbb{E}}{\sim}$$

Basically says:
"Taking

is the same as taking "

$$\begin{aligned} V &= k\text{-v.s.} \\ b &= (v_1, \dots, v_n) \\ b' &= (v'_1, \dots, v'_n) \end{aligned} \quad \left. \right\} \text{bases of } V$$

$$M_{b'b} = [id]_{b'b'} = \left(\{v_i\}_{b'} \mid \dots \mid \{v_n\}_{b'} \right)$$

= Basis change matrix $b \rightarrow b'$

$\forall v \in V:$

$$\{v\}_{b'} = M_{b'b} \cdot \{v\}_b$$

Reference frame = a pair (O, b) where :

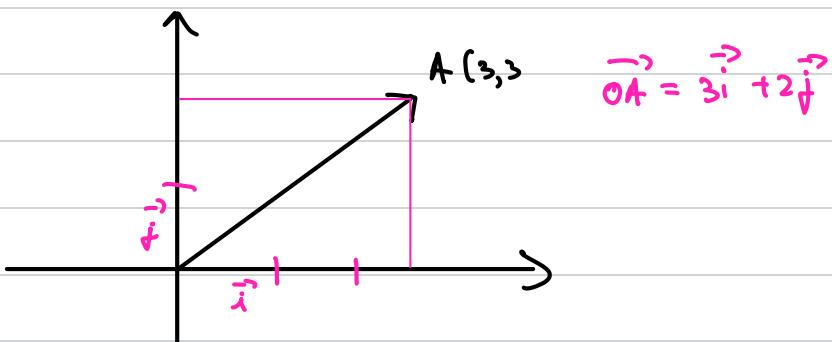
- $O \in \mathbb{E}$ - point
- b - base of V

$K = (O, b)$ - reference frame

$\forall p \in \mathbb{E}$

$$[p]_K := \overrightarrow{OP}_b$$

Ex:



$$\mathcal{K} = (0, b)$$

$$\mathcal{K}' = (0', b')$$

$$\begin{aligned}
 [\mathbf{P}]_{\mathcal{K}'} &= [\overrightarrow{0'P}]_{\mathcal{B}'} = M_{b' \cdot b} \cdot [\overrightarrow{0'P}]_B \\
 &= M_{b' \cdot b} \cdot [\overrightarrow{OP} - \overrightarrow{OO'}]_B = \\
 &= M_{b' \cdot b} \cdot ([\overrightarrow{OP}]_B - [\overrightarrow{OO'}]_B) \\
 &= M_{b' \cdot b} \cdot ([\mathbf{P}]_{\mathcal{K}} - [0']_{\mathcal{K}})
 \end{aligned}$$



$M_{\mathcal{K}' \mathcal{K}}$ (we will use it more)

$$\left. \begin{array}{l} \mathcal{K} = (0, \mathcal{B}) \\ \mathcal{K}' = (0', \mathcal{B}') \end{array} \right\} \text{Reference frames in } \mathbb{E}^2 \xrightarrow{\text{given dim.}, \text{Not } 1 \times 1 \text{E}}$$

$$\mathcal{B} = (\vec{i}, \vec{j})$$

$$\mathcal{B}' = (\vec{i}', \vec{j}')$$

$$[0']_{\mathcal{K}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\{i'\}_B = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\{j'\}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

+

$$M_{KK'} = ?$$

coordinates of A, B, C in E^2 :

$$\{A\}_{K'} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$\{B\}_{K'} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\{C\}_{K'} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$M_{KK'} = M_{BB'} = \begin{pmatrix} \rightarrow & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \xrightarrow[L_1 \leftrightarrow L_1]{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ -3 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[L_2 \leftarrow L_2 + 3L_1]{\sim} \begin{pmatrix} 1 & 1 & | & 0 & 1 \\ 0 & 4 & | & 1 & 3 \end{pmatrix}$$

$$\xrightarrow[L_2 \leftarrow \frac{L_2}{4}]{\sim} \begin{pmatrix} 1 & 1 & | & 0 & 1 \\ 0 & 1 & | & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\xrightarrow[L_2 \leftarrow L_1 - L_2]{\sim} \begin{pmatrix} 1 & 0 & | & -\frac{1}{4} & \frac{1}{4} \\ 0 & 1 & | & \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\Rightarrow (\mathcal{M}_{B'B})^{-1} = \mathcal{M}_{B'B} = \mathcal{M}_{K'K} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$\begin{aligned}
 \text{Let } P \in \mathbb{E}^2, \quad [P]_{K'} &= [\vec{OP}]_{B'} = \mathcal{M}_{B'B} \cdot [O'P]_B = \\
 &= \mathcal{M}_{B'B} \cdot (\vec{OP} - \vec{OO'})_B \\
 &= \mathcal{M}_{B'B} \cdot ([\vec{OP}]_B - [O']_B) \\
 &= \mathcal{M}_{K'K} \cdot ([P]_K - [O']_K) \\
 &\hookrightarrow \begin{pmatrix} 6 \\ 0 \end{pmatrix}
 \end{aligned}$$

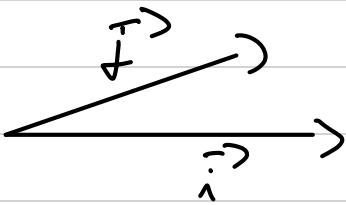
We can do:

$$[ABC]_{K'} = \mathcal{M}_{K'K} ([ABC]_K - [O'O'O']_K)$$

The same thing as doing it 3 times

$$\begin{aligned}
 [ABC]_{K'} &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 6 & 3 \\ 2 & 4 & 1 \end{pmatrix} - \begin{pmatrix} 6 & 6 & 6 \\ 0 & 0 & 0 \end{pmatrix} \right) \\
 &= \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix} \cdot \begin{pmatrix} -6 & 0 & -3 \\ 2 & 4 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix} \Rightarrow \begin{aligned} [A]_{K'} &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ [B]_{K'} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} \\ [C]_{K'} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}
 \end{aligned}$$

IE²



$(\vec{0}, \vec{i}, \vec{j})$ - right oriented if we can rotate \vec{i} by a positive angle in $[0, 2\pi)$ to overlap it with \vec{j} .

left oriented - || - ... $(-2\pi, 0) \dots$

2 ref. frames K, K' have the same orientation if & f.

$\det M_{KK'} > 0$

2.5. With respect to the basis $\beta = (\vec{i}, \vec{j}, \vec{k})$:

$$\begin{aligned}\vec{u} &= \vec{j} + \vec{i} \\ \vec{v} &= \vec{j} + \vec{k} \\ \vec{w} &= \vec{j} + \vec{k}\end{aligned}$$

check that $\beta' = (\vec{u}, \vec{v}, \vec{w})$ is a basis and decide if it's left/right oriented

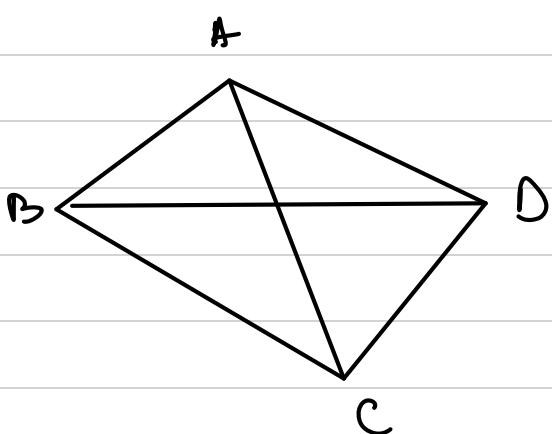
$$\begin{aligned}\vec{u} &= \vec{j} + \vec{i} \Rightarrow \{\vec{u}\}_{\beta'} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\ \vec{v} &= \vec{j} + \vec{k} \Rightarrow \{\vec{v}\}_{\beta'} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ \vec{w} &= \vec{j} + \vec{k} \Rightarrow \{\vec{w}\}_{\beta'} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\end{aligned}$$

$$M_{B'B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\det(M_{B'B}) = 1+1=2 \neq 0 \Rightarrow B' - \text{basis}$$

$\det(M_{B'B}) > 0 \rightarrow$ right-oriented

2.4. ABCD - tetrahedron



Consider the reference frame:

$$K_A = (A, \vec{AB}, \vec{AC}, \vec{AD}), A = (\vec{AB}, \vec{AC}, \vec{AD})$$

$$K_B = (B, \vec{BA}, \vec{BC}, \vec{BD})$$

$$K'_A = (A, \vec{BA}, \vec{BD}, \vec{BC})$$

Derive: a) Coords of A, B, C, D in all 3 K'

b) $M_{K_A K'_A}$, $M_{K_B K_A}$

c) The orientation of the 3 frames
wrt respect to each other

$$a) [ABCD]_{K_A} = \begin{pmatrix} \vec{AA} & \vec{AB} & \vec{AC} & \vec{AD} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[ABCD]_{K_B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[ABCD]_{K_C} = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$b) M_{K_A K_C} = \begin{pmatrix} [BA]_{K_A} & [BD]_{K_C} & [BC]_{K_C} \end{pmatrix}$$

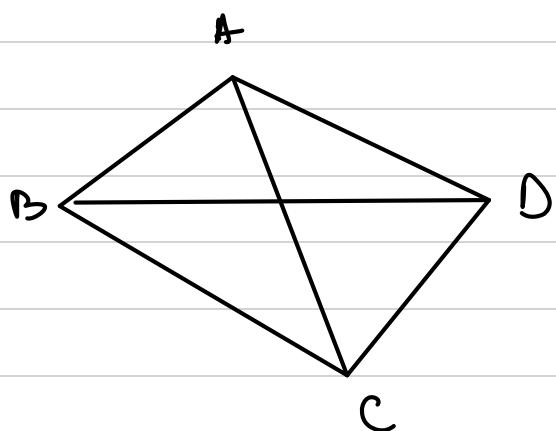
$$= \begin{pmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\det M_{K_A K_C} = 120 \rightarrow K_A, K_C$ have the same orientation

$$M_{K_B K_A} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\det M_{K_B K_A} = -120 \rightarrow K_A, K_B$ - opposite orientation

2.9. What is the # of cartesian frames one could construct w/ the vertices of a tetrahedron.



$$(P, \vec{v}_1, \vec{v}_2, \vec{v}_3)$$

PG $\{A, B, C, D\}$

$\vec{v}_i \in \{\vec{x}\} \mid x \neq y, x, y \in \{A, B, C, D\}$

$\vec{v}_{1,2,3}$ lin. indep.

$$4 \cdot \left(4 \cdot 2 \cdot \binom{C_6^2}{\text{2 sides}} \right) \cdot (2 \cdot 3)$$

of choices at the origin
↓
of choices for \vec{v}_3

↓
of choices for (\vec{v}_1, \vec{v}_2)

$$= 4 \cdot 8 \cdot 15 \cdot 6 = 2880$$

