4.1 Warm-Up Exercises

In the following exercises, all coordinates and components are given with respect to an orthonormal frame $\mathcal{K} = (O, \mathcal{B})$. For the 2-dimensional cases $\mathcal{B} = (\mathbf{i}, \mathbf{j})$ and for the 3-dimensional cases $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$.

4.1. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

4.2. Consider a line ℓ. Show that

a) if $\mathbf{v}(v_1, v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2, -v_1)$ is a normal vector for ℓ,

b) if $\mathbf{n}(n_1, n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2, -n_1)$ is a direction vector for ℓ.

How does this relate to the $\mathbf{J}$-operator?

4.3. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.

4.4. Determine the angle between the lines $\ell_1 : y = 2x + 1$ and $\ell_2 : y = -x + 2$.

4.5. Determine Cartesian equations for the lines situated at distance 4 from the line $12x - 5y - 15 = 0$.

4.6. Determine the values k for which the distance from the point $(2,3)$ to the line $8x + 15y + k = 0$ equals 5.

4.7. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

4.8. Determine a point on the line $5x - 4y - 4 = 0$ for which lies at equal distance from the points $A(1,0)$ and $B(-2,1)$.

4.9. Consider the vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$. Determine the angles between the diagonals of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

4.10. Consider the vectors $\mathbf{q} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p}, \mathbf{q})$ is $5/12$.

4.11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

4.12. Show that a parallelepiped with faces in the planes $2x + y - 2z + 6 = 0$, $2x - 2y + z - 8 = 0$ and $x + 2y + 2z + 1 = 0$ is rectangular.

4.13. Determine the distance between the planes $x - 2y - 2z + 7 = 0$ and $2x - 4y - 4z + 17 = 0$.

4.14. Determine the angles between the plane $x - \sqrt{2}y + z - 1 = 0$ and the plane $x + \sqrt{2}y - z + 3 = 0$.

4.15. Determine the coordinates of a point A on the line $\ell : \frac{x-1}{2} = \frac{y}{3} = \frac{z+1}{1}$ which lies at distance $\sqrt{3}$ from the plane $x + y + z + 3 = 0$.

4.16. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.

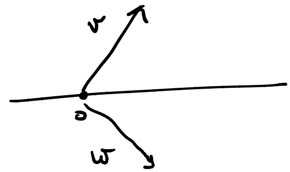
4.1. Determine a Cartesian equations for the line ℓ in the following cases:

a) ℓ contains the point $A(-2,3)$ and has an angle of $60°$ with the Ox-axis,

b) ℓ contains the point $B(1,7)$ and is orthogonal to $\mathbf{n}(4,3)$.

a) there are two unit vectors forming a $60°$ angle with i

$$v\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \qquad w\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$$



$$\Rightarrow \quad \ell: \frac{x+2}{1} = \frac{y-3}{\sqrt{3}} \quad \text{or} \quad \ell: \frac{x+2}{1} = \frac{y-3}{-\sqrt{3}}$$

or, if you prefer, use $\ell: y - y_A = \tan\alpha\,(x - x_A)$

$$y - 3 = \pm\sqrt{3}\,(x+2)$$

b.) $\ell: n_x(x - x_B) + n_y(y - y_B) = 0$

so $\ell: 4(x-1) + 3(y-7) = 0 \iff \ell: 4x + 3y - 25 = 0$

4.2. Consider a line ℓ. Show that

a) if $\mathbf{v}(v_1,v_2)$ is a direction vector for ℓ then $\mathbf{n}(v_2,-v_1)$ is a normal vector for ℓ,

b) if $\mathbf{n}(n_1,n_2)$ is a normal vector for ℓ then $\mathbf{v}(n_2,-n_1)$ is a direction vector for ℓ.

How does this relate to the **J**-operator?

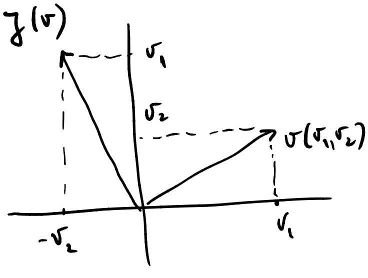
a.) in dimension 2

any vector orthogonal to a direction vector is a normal vector

$$\langle v, n\rangle = v_1 \cdot v_2 - v_2 v_1 = 0 \Rightarrow n \perp v$$

$\Rightarrow n$ is a normal vector

here $n = J(v)$



b.) in dimension 2

any vector orthogonal to a normal vector is a direction vector

$$\langle n, v\rangle = n_1 n_2 - n_2 n_1 = 0 \Rightarrow v \perp n$$

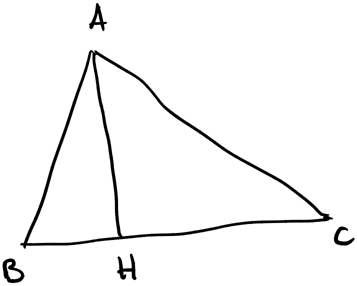
$\Rightarrow v$ is a direction vector

here $v = -J(v)$

4.3. Let $A(1,3)$, $B(-4,3)$ and $C(2,9)$ be the vertices of a triangle. Determine

a) the length of the altitude from A,

b) the line containing the altitude from A.



a) $|AH| = d(A, BC) = \dfrac{|1-3+7|}{\sqrt{1+1}} = \dfrac{5}{\sqrt{2}}$

$\downarrow$

$\overrightarrow{BC}(6,6)$

$\downarrow$

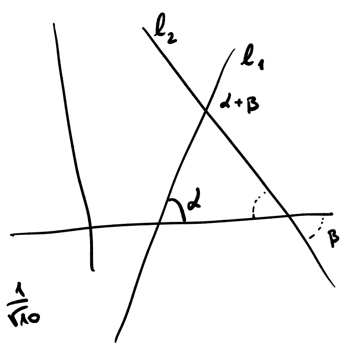
$BC: \dfrac{x+4}{1} = \dfrac{y-3}{1} \Leftrightarrow x-y+7=0$

b) $(1,1) \parallel \overrightarrow{BC}$, $\overrightarrow{BC} \perp \overrightarrow{AH} \Rightarrow$ $(1,1)$ is a normal vector for AH

$\Rightarrow AH: 1\cdot(x-1) + 1(y-3) = 0$

4.4. Determine the angle between the lines $\ell_1 : y = 2x+1$ and $\ell_2 : y = -x+2$.

$\boxed{\text{Sol 1}}$ $\tan\alpha = 2$ $\tan\beta = -1 < 0$



$\Rightarrow$ one angle is $\alpha+\beta$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1+\tan\alpha\tan\beta} = \dfrac{2-1}{1+2} = \dfrac{1}{3} \Rightarrow \alpha+\beta \cong 18.43°$

$\boxed{\text{Sol 2}}$ $\left.\begin{array}{l} n_1(2,-1) \text{ normal vector for } \ell_1 \\ n_2(1,1) \text{ normal vector for } \ell_2 \end{array}\right\} \Rightarrow \cos\measuredangle(n_1,n_2) = \dfrac{\langle n_1,n_2\rangle}{|n_1|\cdot|n_2|} = \dfrac{1}{\sqrt{10}}$

$\Rightarrow \measuredangle(n_1,n_2) \cong 71.57°$

4.5. Determine Cartesian equations for the lines situated at distance 4 from the line $\underbrace{12x-5y-15=0}_{\ell}$.

For a point $P(x_0,y_0)$ we have $4 = d(P,\ell) = \dfrac{|12x_0-5y_0-15|}{\sqrt{144+25}=13}$

$\Rightarrow$ the two lines are

$\ell_{1,2}: 12x-5y = 15 \pm 52$

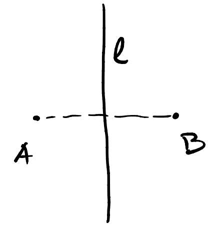
4.6. Determine the values k for which the distance from the point $\underbrace{(2,3)}_{P}$ to the line $\underbrace{8x+15y+k}_{l}=0$ equals 5.

$$5 = d(P,l) = \frac{|16+45+k|}{\sqrt{64+225} = 17} \iff 85 = \pm(61+k)$$

$$\Rightarrow k=-146 \text{ or } k = 24$$

4.7. Consider the points $A(3,-1)$, $B(9,1)$ and $C(-5,5)$. For each pair of these three points, determine the line which is equidistant from them.

- These lines are the perpendicular bisectors of the sides of the triangle



$\boxed{\text{Sol 1}}$ For a point $P(x_0,y_0)$ we have

$$d(A,P) = d(B,P)$$

$$\iff d(A,P)^2 = d(B,P)^2$$

$$\iff (3-x_0)^2+(-1-y_0)^2 = (9-x_0)^2+(1-y_0)^2$$

$$\iff 9-6x_0+\cancel{x_0^2}+1+2y_0+\cancel{y_0^2} = 81-18x_0+\cancel{x_0^2}+1-2y_0+\cancel{y_0^2}$$

$$\iff 12x_0+4y_0-72=0$$

So $l: 3x_0+y_0-18=0$

$\boxed{\text{Sol 2}}$ $\vec{AB}(6,2)$ is a normal vector for l

$M(6,0)$ is the midpoint of $[AB] \Rightarrow M \in l$

$\Rightarrow l: 6(x-6)+2y=0 \iff l: 3x+y-18=0$

4.8. Determine a point on the line $\underbrace{5x-4y-4=0}_{l}$ which lies at equal distance from the points $A(1,0)$ and $B(-2,1)$.

$\boxed{\text{Sol 1}}$ $n(5,-4)$ normal vect for $l \Rightarrow \left.\begin{matrix} v(4,5) \parallel l \\ P(0,-1) \in l \end{matrix}\right\} \Rightarrow l: \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4t \\ 5t-1 \end{bmatrix} = P_t$

$d(P_t,A) = d(P_t,B) \iff \ldots$ write out eq. and solve in t ... $t = \frac{-3}{7}$

$\boxed{\text{Sol}_2}$ The point P which we seek is $l \cap l'$ where l' is the perp. bisector of segment $[AB]$

$\vec{AB}(-3,1)$, $(-\frac{1}{2},\frac{1}{2})$ mid point of $[AB]$

$\Rightarrow \ell' : -3(x+\frac{1}{2}) + 1\cdot(y-\frac{1}{2}) = 0$

$\Rightarrow \ell' : -3x + y - 2 = 0$

$\ell \cap \ell' : \begin{cases} 5x - 4y - 4 = 0 \\ 3x - y + 2 = 0 \end{cases} \Rightarrow y = 3x+2 \Big\} \Rightarrow 5x - 12x - 8 - 4 = 0$

$-7x - 12 = 0$

$\Rightarrow x = \frac{-12}{7}$

$\Rightarrow y = -\frac{22}{7}$

4.9. Consider the vectors $\mathbf{a}(2,1,0)$ and $\mathbf{b}(0,-2,1)$. Determine the angles between the diagonals of a parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

$d_1 = a+b = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad d_2 = a-b = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \cos\sphericalangle(d_1,d_2) = \frac{\langle d_1, d_2\rangle}{|d_1|\cdot|d_2|} = \frac{4-3-1}{\dots} = 0$

$\Rightarrow \sphericalangle(d_1,d_2) = 90^\circ$

4.10. Consider the vectors $\mathbf{q} = 3\mathbf{i} + \mathbf{j}$ and $\mathbf{p} = \mathbf{i} + 2\mathbf{j} + \lambda\mathbf{k}$ with $\lambda \in \mathbb{R}$. Determine λ such that the cosine of the angle $\measuredangle(\mathbf{p},\mathbf{q})$ is $5/12$.

$q = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad p = \begin{bmatrix} 1 \\ 2 \\ \lambda \end{bmatrix} \quad \cos\sphericalangle(p,q) = \frac{3+2+\lambda\cdot 0}{\sqrt{10}\cdot\sqrt{5+\lambda^2}} = \frac{5}{12}$

$\Leftrightarrow \sqrt{10}\cdot\sqrt{5+\lambda^2} = 12 \Leftrightarrow 10(5+\lambda^2) = 144 \Leftrightarrow \lambda^2 = \frac{94}{10}$

$\Rightarrow \lambda = \pm\sqrt{9.4}$

4.11. In an orthonormal basis, consider the vectors $\mathbf{v}_1(0,1,0)$, $\mathbf{v}_2(2,1,0)$ and $\mathbf{v}_3(-1,0,1)$. Use the Gram-Schmidt process to find an orthonormal basis containing $\mathbf{v}_1$.

$w_1 = v_1$

$w_2 = v_2 - \frac{\langle w_1, v_2\rangle}{\langle w_1, w_1\rangle} w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\rangle}{1^2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$w_3 = v_3 - \frac{\langle w_1, v_3\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle w_2, v_3\rangle}{\langle w_2, w_2\rangle} w_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{1^2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{\left\langle \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\rangle}{4} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} =$

$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 0 + \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow w_1, w_2, w_3$ is an orthogonal basis

$\Rightarrow$ an orthonormal basis is $\frac{w_1}{|w_1|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \frac{w_2}{|w_2|} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \frac{w_3}{|w_3|} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

4.12. Show that a parallelepiped with faces in the planes $2x+y-2z+6=0$, $2x-2y+z-8=0$ and $x+2y+2z+1=0$ is rectangular.

- A parallelepiped is rectangular if its faces are orthogonal to each other $\Leftrightarrow$ the planes containing the faces have $90°$ dihedral angles
- normal vectors of the given planes are $n_1(2,1,-2)$, $n_2(2,-2,1)$, $n_3(1,2,2)$

and one checks that $\langle n_1,n_2\rangle=0$, $\langle n_1,n_3\rangle=0$, $\langle n_2,n_3\rangle=0$

4.13. Determine the distance between the planes $\underbrace{x-2y-2z+7=0}_{\pi_1}$ and $\underbrace{2x-4y-4z+17=0}_{\pi_2}$.

$$\left.\begin{array}{l}\text{a normal vector for } \pi_1 \text{ is } n_1(1,-2,-2)\\ \text{a normal vector for } \pi_2 \text{ is } n_2(2,-4,-4)\end{array}\right\} \Rightarrow n_1 \parallel n_2 \Rightarrow \pi_1 \parallel \pi_2$$

$\Rightarrow d(\pi_1,\pi_2)=d(\pi_1,P)\ \forall P\in\pi_2$

choos $P=(-\frac{17}{2},0,0)$ for example. Then $d(\pi_1,\pi_2)=\dfrac{\left|-\frac{17}{2}-2\cdot0-2\cdot0+7\right|}{\sqrt{1+4+4}}=\dfrac{\frac{3}{2}}{\sqrt{9}}=\dfrac{1}{2}$

4.14. Determine the angles between the plane $\underbrace{x-\sqrt{2}y+z-1=0}_{\pi_1}$ and the plane $\underbrace{x+\sqrt{2}y-z+3=0}_{\pi_2}$.

$n_1(1,-\sqrt{2},1)$ is a N.V for π_1

$n_2(1,\sqrt{2},-1)$ — " — π_2

$$\cos\sphericalangle(n_1,n_2)=\frac{\langle n_1\cdot n_2\rangle}{|n_1|\cdot|n_2|}=\frac{1-2-1}{\sqrt{4}\sqrt{4}}=-\frac{2}{4}=\frac{-1}{2}\Rightarrow\sphericalangle(n_1,n_2)=120°$$

$\Rightarrow$ the other angle is $60°$

4.15. Determine the coordinates of a point A on the line $\ell: \frac{x-1}{2}=\frac{y}{3}=\frac{z+1}{1}$ which lies at distance $\sqrt{3}$ from the plane $\underbrace{x+y+z+3=0}_{\pi}$.

the points on ℓ are $P_t=(2t+1,3t,t-1)$ for $t\in\mathbb{R}$

$\Rightarrow \sqrt{3}=d(P,\pi)=\dfrac{|2t+1+3t+t-1|}{\sqrt{3}} \Rightarrow 3=|6t| \Rightarrow t=\pm\frac{1}{2}$

$\Rightarrow$ the two points are $(2,\frac{3}{2},\frac{1}{2})$ and $(0,-\frac{3}{2},-\frac{3}{2})$

4.16. The vertices of a tetrahedron are $A(-1,-3,1)$, $B(5,3,8)$, $C(-1,-3,5)$ and $D(2,1,-4)$. Determine the height of the tetrahedron relative to the face ABC.

$\perp = d(D, ABC)$

$$\overrightarrow{AB} = \begin{bmatrix} 6 \\ 6 \\ 7 \end{bmatrix} \quad \overrightarrow{AC} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \Rightarrow ABC: \begin{vmatrix} x+1 & y+3 & z-1 \\ 6 & 6 & 7 \\ 0 & 0 & 4 \end{vmatrix} = 0$$

$$24(x+1) - 24(y+3) = 0$$

$$\Rightarrow ABC: x - y - 2 = 0$$

$$\Rightarrow d(D, ABC) = \frac{|2-1-2|}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$