8.1 Warm-Up Exercises

In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K}$.

8.1. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by 90° around C followed by an orthogonal reflection relative to the line AB.

8.2. Verify that the matrices

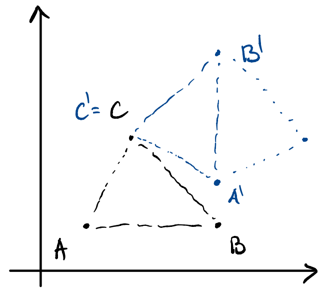
$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to SO(3). Moreover, in each case determine the axis of rotation and the rotation angle.

8.3. Using Euler-Rodrigues formula, deduce the known matrix forms of rotations around the coordinate axes.

8.4. Using rotations around the coordinate axes, give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.
$v=(0,1,1)$

8.1. The vertices of a triangle are $A(1,1)$, $B(4,1)$ and $C(2,3)$. Determine the image of the triangle ABC under a rotation by $90°$ around C followed by an orthogonal reflection relative to the line AB.



$$[Rot_{0,90°}] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$Rot_{C,90°} = T_{\vec{OC}} \circ Rot_{0,90°} \circ T_{-\vec{OC}}$$

$$Rot_{C,90°}\begin{bmatrix} x \\ y \end{bmatrix} = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)\right) + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x-2 \\ y-3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3-y \\ x-2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5-y \\ x+1 \end{bmatrix}$$

in particular $Rot_{C,90°}(A) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = A'$ $\quad Rot_{C,90°}(B) = \begin{bmatrix} 4 \\ 5 \end{bmatrix} = B'$

So the line AB moves to $A'B'$ and a reflection in $A'B'$ can be read-off from the picture

$$\underset{x=4}{Ref^{\perp}_{A'B'}}\begin{bmatrix} x \\ y \end{bmatrix} = T_{(4,0)} \circ Ref^{\perp}_{Oy} \circ T_{(-4,0)}$$

$$= \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\left(\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -4 \\ 0 \end{bmatrix}\right)\right) + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x-4 \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-x \\ y \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 8-x \\ y \end{bmatrix}$$

in particular $Ref^{\perp}_{A'B'}(A') = A'$, $Ref^{\perp}_{A'B'}(B') = B'$, $Ref^{\perp}_{A'B'}(C') = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = C''$

8.2. Verify that the matrices

$$A = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{11}\begin{bmatrix} -9 & -2 & 6 \\ 6 & -6 & 7 \\ 2 & 9 & 6 \end{bmatrix}$$

belong to SO(3). Moreover, in each case determine the axis of rotation and the rotation angle.

- $A \in SO(3) \iff AA^t = I_3$ & $\det A = 1$

$$A^T = \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -1 & -2 & -2 \\ 2 & -2 & 1 \\ -2 & -1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

On checks that $\det A = 1$ so, yeah, $A \in SO(3)$

- the axis of rotation is the line passing through the origin in the direction of the eigenvectors for the eigenvalue 1

this is obtained by solving $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \iff \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$\iff \begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & -2 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ -2 & -5 & -1 \\ -2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow y = 0 \Big\} \Rightarrow 2x + z = 0$$

$$\Rightarrow z = -2x$$

$\Rightarrow$ eigenspace for $\lambda = 1$ is $V_1 = \{(t, 0, -2t) \cdot t \in \mathbb{R}\}$ $\leftarrow$ this is a line passing through the origin, it is the rotation axis

the eigenvectors are the non-zero vectors in V_1

• the angle of rotation θ is determined by

$$\operatorname{tr}(A) = 1 + 2\cos\theta \iff \cos\theta = \frac{\operatorname{tr}A - 1}{2} = \frac{-\frac{1}{3} - 1}{2} = -\frac{2}{3}$$

so $\theta = \arccos\left(-\frac{2}{3}\right)$

the calculation for B is similar.

8.3. Using Euler-Rodrigues formula, deduce the known matrix forms of rotations around the coordinate axes.

$$\mathrm{Rot}_{v,\theta}(x) = \cos\theta\, x + \sin\theta\,(v \times x) + (1-\cos\theta)\underbrace{\langle v, x\rangle v}_{\underbrace{v \cdot \langle v, x\rangle}_{\underbrace{v\cdot(v^T \cdot x)}_{v \otimes v}}}$$ for axis $\mathbb{R}v$

$$= \left[\cos\theta I_n + \sin\theta\,[v\times\_] + (1-\cos\theta)\, v\otimes v\right]\cdot x$$

for a rotation around Ox we choose $v = i$

$$[i\times\_] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad i \otimes i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

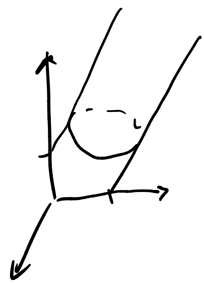
$$\Rightarrow \mathrm{Rot}_{i,\theta}(x) = \left(\begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\sin\theta \\ 0 & \sin\theta & 0 \end{bmatrix} + \begin{bmatrix} 1-\cos\theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) x$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} x$$

Similar for Oy and Oz

8.4. Using rotations around the coordinate axes, give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.

$v = (0,1,1)$



The cylinder contains the line

$$\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and has axis $\ell' : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

We may construct a cylinder $\tilde{\mathcal{C}}$ of diam. $\sqrt{2}$ and axis Oz and then rotate it in the right position

$$\tilde{\mathcal{C}} \ni \tilde{\ell} : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{\mathcal{C}} = Rot_{k,\theta}(\tilde{\ell}) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sqrt{2} \\ t \end{bmatrix} = \begin{bmatrix} -\sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \\ t \end{bmatrix}$$

$$\Rightarrow \mathcal{C} = Rot_{i,45^\circ}(\tilde{\mathcal{C}}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} -\sqrt{2}\cos\theta \\ \sqrt{2}\sin\theta \\ t \end{bmatrix} = \ldots$$