5.1 Warm-Up Exercises

In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K} = (O, \mathcal{B})$ where $\mathcal{B} = (\mathbf{i}, \mathbf{j}, \mathbf{k})$.

5.1. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Calculate $\mathbf{a} \times \mathbf{b}$.

5.2. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6, 0, 1)$ and $\overrightarrow{AC}(1.5, 2, 1)$.

5.3. Consider the vectors $\mathbf{a}(2, -3, 1)$, $\mathbf{b}(-3, 1, 2)$ and $\mathbf{c}(1, 2, 3)$. Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

5.4. The points $A(1, 2, -1)$, $B(0, 1, 5)$, $C(-1, 2, 1)$ and $D(2, 1, 3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

5.5. Consider the tetrahedron with vertices $A(2, -1, 1)$, $B(5, 5, 4)$, $C(3, 2, -1)$ and $D(4, 1, 3)$. Determine

a) the volume of the tetrahedron.

b) the common perpendicular of the sides AB and CD.

5.6. Consider the vectors $\mathbf{a}(2, 3, -1)$ and $\mathbf{b}(1, -1, 3)$. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\langle \mathbf{p}, 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \rangle = 51$.

5.7. Consider the points $A(2, 1, -1)$, $B(3, 0, 1)$, $C(2, -1, 3)$ and $D \in Oy$. Determine the coordinates of D if the volume of a tetrahedron $ABCD$ is 5.

5.8. Let π be a plane with normal vector $(1, 1, 1)$ intersecting the coordinate axes in A, B and C. Give formulas for the volume of the tetrahedron $OABC$ in terms of $h = d(O, \pi)$, $a = |OA|$ and $s = |AB|$ respectively.

5.9. Determine the matrices of the linear maps $\phi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = (2\mathbf{i} - \mathbf{j}) \times \mathbf{v}$.

5.10. Consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

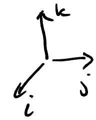
5.1. Consider the vectors $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 7\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$. Calculate $\mathbf{a} \times \mathbf{b}$.

$$a \times b = (i + 2j - 2k) \times (7i + 4j + 6k)$$

$$= 7\underbrace{i \times i}_{=0} + 4\, i \times j + 6\, i \times k +$$

$$+ 14\, j \times i + 8\underbrace{j \times j}_{=0} + 12\, j \times k +$$

$$- 14\, k \times i - 8\, k \times j - 12\underbrace{k \times k}_{=0}$$

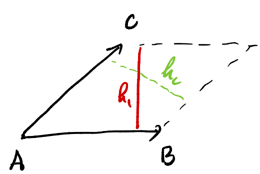


$$= -10\underbrace{i \times j}_{=k} + 20\underbrace{i \times k}_{-j} + 20\underbrace{j \times k}_{=i}$$

$$= 20i - 20j - 10k$$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 2 & -2 \\ 7 & 4 & 6 \end{vmatrix} = 20i - 20j - 10k$$

5.2. Determine the distances between opposite sides of a parallelogram spanned by the vectors $\overrightarrow{AB}(6, 0, 1)$ and $\overrightarrow{AC}(1.5, 2, 1)$.



Let a be the area of the parallelogram

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = a = h_1 \cdot |\overrightarrow{AB}| = h_2 \cdot |\overrightarrow{AC}|$$

$$\Rightarrow h_1 = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB}|} = \frac{\sqrt{673}}{2\sqrt{37}}$$

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \left| \begin{vmatrix} i & j & k \\ 6 & 0 & 1 \\ 1.5 & 2 & 1 \end{vmatrix} \right| = |-2i - 4.5j + 12k| = \sqrt{4 + \frac{81}{4} + 144}$$

$$= \frac{\sqrt{673}}{2}$$

$$|\overrightarrow{AB}| = \sqrt{37}$$

$$|\overrightarrow{AC}| = \sqrt{\frac{9}{4} + 4 + 1} = \frac{\sqrt{29}}{2} \quad \Rightarrow h_2 = \frac{\sqrt{673}}{\sqrt{29}}$$

5.3. Consider the vectors $\mathbf{a}(2,-3,1)$, $\mathbf{b}(-3,1,2)$ and $\mathbf{c}(1,2,3)$. Calculate $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$.

$$(a\times b)\times c = \langle a\cdot c\rangle b - \langle b\cdot c\rangle a = (2-6+3)b - (-3+2+6)a$$

$$= -\begin{bmatrix}-3\\1\\2\end{bmatrix} - 5\begin{bmatrix}2\\-3\\1\end{bmatrix} = \begin{bmatrix}-7\\14\\-3\end{bmatrix}$$

$$a\times(b\times c) = \langle a\cdot c\rangle b - \langle a\cdot b\rangle c = -\begin{bmatrix}-3\\1\\2\end{bmatrix} + 7\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}10\\13\\19\end{bmatrix}$$

5.4. The points $A(1,2,-1)$, $B(0,1,5)$, $C(-1,2,1)$ and $D(2,1,3)$ are given with respect to an orthonormal coordinate system. Are the four points coplanar?

A, B, C, D coplanar $\Leftrightarrow$ $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ linearly dependent

$$\overrightarrow{AB} = \begin{bmatrix}-1\\-1\\6\end{bmatrix}, \quad \overrightarrow{AC} = \begin{bmatrix}-2\\0\\2\end{bmatrix}, \quad \overrightarrow{AD} = \begin{bmatrix}1\\-1\\4\end{bmatrix}$$

$\Updownarrow$

Volume of tetrahedron $ABCD = 0$ $\Leftrightarrow$ $\underbrace{\begin{vmatrix}-1 & -1 & 6\\-2 & 0 & 2\\1 & -1 & 4\end{vmatrix}}_{12-2-8-2=0} = 0$ true

5.5. Consider the tetrahedron with vertices $A(2,-1,1)$, $B(5,5,4)$, $C(3,2,-1)$ and $D(4,1,3)$. Determine

a) the volume of the tetrahedron.

b) the common perpendicular of the sides AB and CD.

a) $\text{Vol} = \left|\frac{1}{6}[\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}]\right| = \left|\frac{1}{6}\begin{vmatrix}3&6&3\\1&3&-2\\2&2&2\end{vmatrix}\right| = |-3| = 3$

b) $AB: \begin{bmatrix}x\\y\\z\end{bmatrix} = \underbrace{\begin{bmatrix}2\\-1\\1\end{bmatrix}}_{A} + t\underbrace{\begin{bmatrix}1\\2\\1\end{bmatrix}}_{\overrightarrow{AB}/3}$ $\qquad CD: \begin{bmatrix}x\\y\\z\end{bmatrix} = \underbrace{\begin{bmatrix}3\\2\\-1\end{bmatrix}}_{C} + \lambda\underbrace{\begin{bmatrix}1\\-1\\4\end{bmatrix}}_{\overrightarrow{CD}}$

$$\frac{\overrightarrow{AB}}{3} \times \overrightarrow{CD} = \begin{vmatrix} i & j & k \\ 1 & 2 & 1 \\ 1 & -1 & 4 \end{vmatrix} = \underbrace{i\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix}}_{9} - \underbrace{j\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}}_{3} + \underbrace{k\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}_{-3} = \underbrace{3\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}}_{= v}$$

common perp :

$$\begin{cases} \begin{vmatrix} x-2 & y+1 & z-1 \\ 1 & 2 & 1 \\ 3 & -1 & -1 \end{vmatrix} = 0 \iff -1(x-2)+4(y+1)-7(z-1)=0 \\ \begin{vmatrix} x-3 & y-2 & z+1 \\ 1 & -1 & 4 \\ 3 & -1 & -1 \end{vmatrix} = 0 \iff 5(x-3)+13(y-2)+2(z+1)=0 \end{cases}$$

$$\iff \begin{cases} -x+4y-7z+13=0 \\ 5x+13y+2z-39=0 \end{cases}$$

5.6. Consider the vectors $\mathbf{a}(2,3,-1)$ and $\mathbf{b}(1,-1,3)$. Determine the vector $\mathbf{p}$ which is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ and for which $\langle \mathbf{p}, 2\mathbf{i}-3\mathbf{j}+4\mathbf{k}\rangle = 51$.

$\boxed{\text{Sol 1}}$ take an arbitrary vector $v(x,y,z)$ and impose the conditions: $v \in \langle a,b\rangle^{\perp} \iff v \perp a$ and $v \perp b$

$\iff v\cdot a=0$ and $v\cdot b=0$

$$\iff \begin{cases} 2x+3y-z=0 \\ x-y+3z=0 \end{cases}$$

this system allows us to express y and z in terms of x

then $v \in \langle a,b\rangle^{\perp} \iff v=v(x,y(x),z(x))$

$\boxed{\text{Sol 2}}$ $\langle a,b\rangle^{\perp} = \langle a\times b\rangle = \left\langle \begin{bmatrix} 8 \\ -7 \\ -5 \end{bmatrix} \right\rangle = \{(8t,-7t,-5t) : t\in\mathbb{R}\}$

$$a\times b = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 8i-7j-5k$$

$$p \in \langle a, b\rangle^{\perp} \iff p = t\begin{pmatrix} 8 \\ -7 \\ -5 \end{pmatrix} \text{ for some } t \in \mathbb{R}$$

$$p \cdot (2i - 3j + 4k) = 51 \iff (t8i - t7j - t5k)\cdot(2i - 3j + 4k) = 51$$

$$\iff 16t + 21t - 20t = 51$$

$$\iff t = \frac{51}{17} = 3$$

5.7. Consider the points $A(2,1,-1)$, $B(3,0,1)$, $C(2,-1,3)$ and $D \in Oy$. Determine the coordinates of D if the volume of a tetrahedron $ABCD$ is 5.

$D \in Oy \Rightarrow D(0,\lambda,0)$

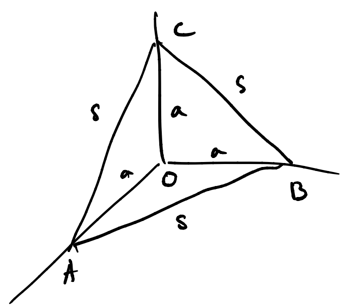
the volume of the tetrahedron is $V = \left|\frac{1}{6}\begin{vmatrix} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & \lambda-1 & 1 \end{vmatrix}\right| = \frac{1}{6}\left|-2+8-8-4(\lambda-1)\right| = \frac{1}{6}\left|2-4\lambda\right|$

so $5 = \frac{|1-2\lambda|}{3} \iff 15 = |1-2\lambda| \iff \begin{cases} \lambda = -7 \\ \lambda = 8 \end{cases}$

5.8. Let π be a plane with normal vector $(1,1,1)$ intersecting the coordinate axes in A, B and C. Give formulas for the volume of the tetrahedron $OABC$ in terms of $h = d(O,\pi)$, $a = |OA|$ and $s = |AB|$ respectively.

we may assume $A(a,0)$, i.e $a > 0$

$\pi : \frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1 \iff x + y + z - a = 0$



$h = d(0,\pi) = \frac{|a|}{\sqrt{3}} = \frac{a}{\sqrt{3}} \Rightarrow a = \sqrt{3}h$

$s = \sqrt{2}a \Rightarrow a = s/\sqrt{2}$

$V = \text{Vol}(OABC) = \frac{1}{6}$ volume cube with side a

$$\Rightarrow V = \frac{a^3}{6} = \frac{\sqrt{3}}{2}h^3 = \frac{s^3}{12\sqrt{2}}$$

$s/\sqrt{6}$

check with Heron: Area $(ABC) = \frac{\sqrt{3}}{4}s^2 \Rightarrow V = \frac{1}{3}\cdot\frac{\sqrt{3}}{4}s^2\cdot h = \frac{s^3}{12\sqrt{2}}$

5.9. Determine the matrices of the linear maps $\phi : \mathbb{V}^3 \to \mathbb{V}^3$ defined by $\phi(\mathbf{v}) = (2\mathbf{i} - \mathbf{j}) \times \mathbf{v}$.

$$\phi(i) = (2i-j)\times i = 2i\times i - j\times i = i\times j = k$$

$$\phi(j) = (2i-j)\times j = 2\, i\times j = 2k$$

$$\phi(k) = (2i-j)\times k = 2i\times k - j\times k = -2j - i$$

$$\Rightarrow [\phi] = \begin{bmatrix} \phi(i) & \phi(j) & \phi(k) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

5.10. Consider the vectors $\mathbf{u} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{k}$. Determine the matrix of the linear map $\phi : \mathbb{V}^3 \to \mathbb{R}$ defined by $\phi(\mathbf{v}) = [\mathbf{v}, \mathbf{u}, \mathbf{w}]$.

$$\phi(i) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 3 \qquad \phi(j) = \begin{vmatrix} 0 & 1 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad \phi(k) = \begin{vmatrix} 0 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow [\phi] = \begin{bmatrix} \phi(i) & \phi(j) & \phi(k) \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}$$