Week 3+ : Warm-up exercises

! Week 3 – 6/7 : Sunt CELE MAI DE BAZĂ !

↳ ec. dreptei

↳ ec. planului

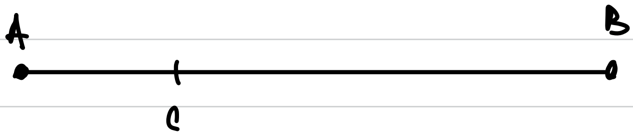
+ mai pe la final chestii cu curbe, chestii care se leagă de analiză

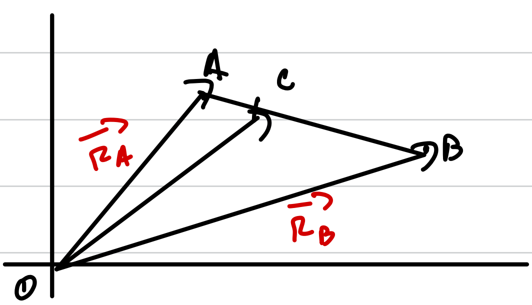
• nu mai e limită superioară de puncte neapărat (1 pt max la nota finală)

• recuperare seminar 0 - online, prob. săpt. asta

1.1. Let $\lambda = \frac{CA}{CB}$ - ratio in which $C \in [AB]$ divides it.

Prove: $\vec{OC} = \frac{1}{1+\lambda}\vec{OA} + \frac{\lambda}{1+\lambda}\vec{OB}$





$\vec{OC} = \vec{OA} + \vec{AC}$

$\lambda = \frac{AC}{CB} \Rightarrow AC = \lambda CB \Rightarrow \vec{AC} = \lambda \cdot \vec{CB} \quad \Big| + \lambda\vec{AC}$

$\Rightarrow (\lambda+1)\vec{AC} = (\lambda+1)(\vec{CB} + \vec{AC})$

$= (\lambda+1) \cdot \vec{AB}$

$$\Rightarrow \vec{OC} = \vec{OA} + \lambda\vec{AB} =$$

$$= \frac{\lambda}{\lambda+1}\left(\vec{AO} + \vec{OB}\right) + \vec{OA}$$

$$= \frac{\lambda}{\lambda+1}\vec{OB} + \left(1 - \frac{\lambda}{\lambda+1}\right)\vec{OA}$$

$$= \frac{\lambda}{\lambda+1}\vec{OB} + \frac{1}{\lambda+1}\vec{OA}$$

Alternatively: (with pos. vectors)

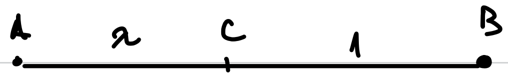
$$\vec{AC} = \lambda\vec{CB}$$

$$\vec{r_C} - \vec{r_A} = \lambda\left(\vec{r_B} - \vec{r_C}\right)$$

$$(\lambda+1)\vec{r_C} = r_A + \lambda\vec{r_B}$$

$$\vec{r_C} = \frac{\vec{r_A} + \lambda\vec{r_B}}{1+\lambda}$$

Mia's way – Trick 4 memory



not on the Exam paper

!

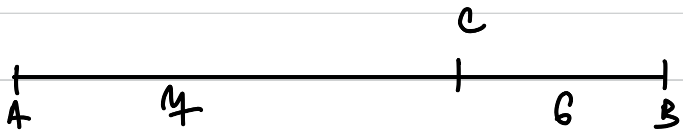
O

$$\lambda\vec{r_B} + \vec{r_A} = (\lambda+1)\vec{r_C}$$

$$\vec{r_C} = \frac{\vec{r_A} + \lambda\vec{r_B}}{\lambda+1}$$

Ex: $C \in [AB]$

$$\frac{AB}{BC} = \frac{13}{6}$$



$$\vec{r}_C = \frac{7\vec{r}_B + 6\vec{r}_A}{13}$$

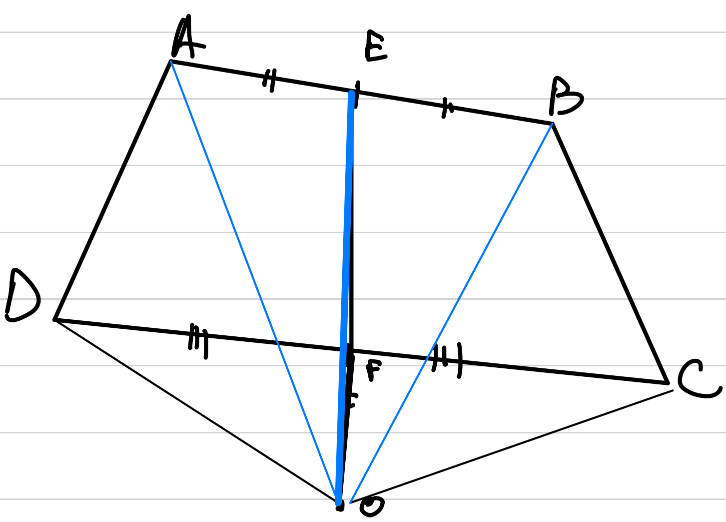
Reversing the scales

1.3. ABCD quadrilateral

E midpoint of $[AB]$

F midpoint of $[CD]$

$$\vec{EF} = \frac{1}{2}\left(\vec{AD} + \vec{BC}\right)$$



$$\vec{EF} = \vec{r}_F - \vec{r}_E$$

$$\vec{r}_F = \frac{\vec{r}_D + \vec{r}_C}{2}$$

$$\vec{r}_E = \frac{\vec{r}_B + \vec{r}_A}{2}$$

$$\vec{EF} = \frac{r_D + r_C}{2} - \frac{r_B + r_A}{2}$$

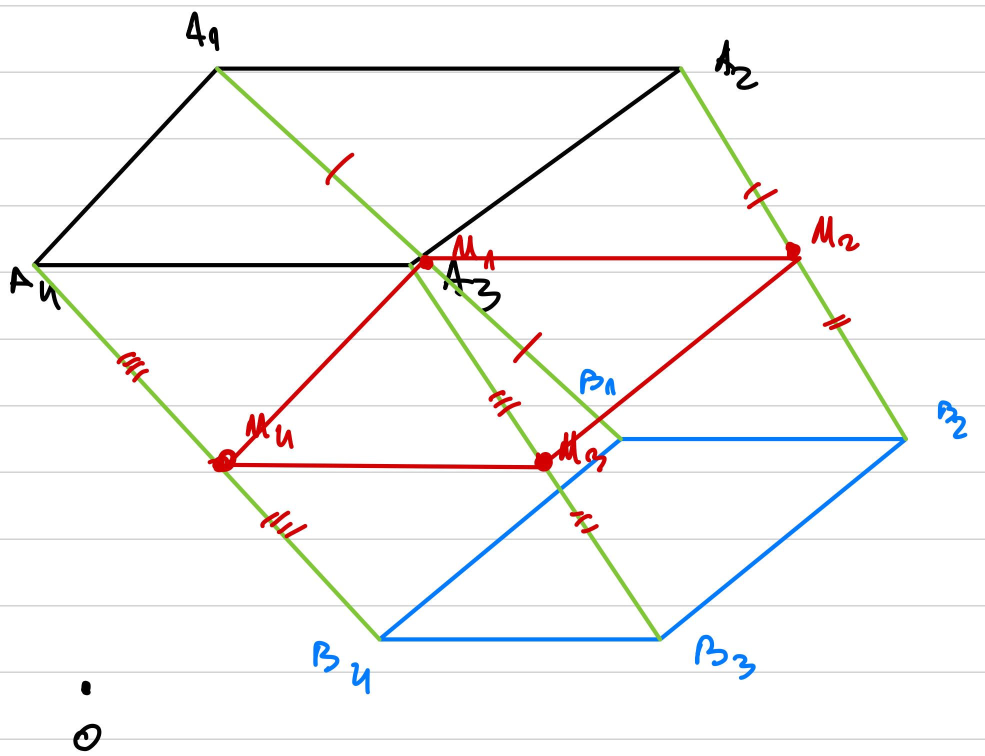
$$= \frac{\vec{r_D} - \vec{r_A}}{2} + \frac{\vec{r_C} - \vec{r_B}}{2}$$

$$= \frac{1}{2}\left(\vec{AD} + \vec{BC}\right)$$

1.5. $\left.\begin{matrix} A_1A_2A_3A_4 \\ B_1B_2B_3B_4 \end{matrix}\right\}$ parallelog.

Show that the midpoints M_1, M_2, M_3, M_4 of $\{A_1B_1\}$, $\{A_2B_2\}$, $\{A_3 B_3\}$, $\{A_4 B_4\}$

are vertices of a parallelogr.



$$\vec{R_{M_1}} = \frac{\vec{R_{A_1}} + \vec{R_{B_1}}}{2}$$

$$\vec{R_{M_2}} = \frac{\vec{r_{A_2}} + \vec{r_{B_2}}}{2}$$

$$\vec{R_{M_3}} = \frac{\vec{r_{A_3}} + \vec{r_{B_3}}}{2}$$

$$\vec{r_{M_4}} = \frac{\vec{r_{A_4}} + \vec{r_{B_4}}}{2}$$

$$\overrightarrow{M_4M_3} = \vec{r}_{M_3} - \vec{r}_{M_4} = \frac{1}{2}\left(\overrightarrow{A_4A_3} + \overrightarrow{B_4B_3}\right)$$

$A_1A_2A_3A_4$ - parallelog. $\Rightarrow \overrightarrow{A_4A_3} = \overrightarrow{A_1A_2}$

$B_1B_2B_3B_4$ - parallelog $\Rightarrow \overrightarrow{B_4B_3} = \overrightarrow{B_1B_2}$

$$\overrightarrow{M_4M_3} = \frac{1}{2}\left(\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2}\right) = \overrightarrow{M_1M_2}$$

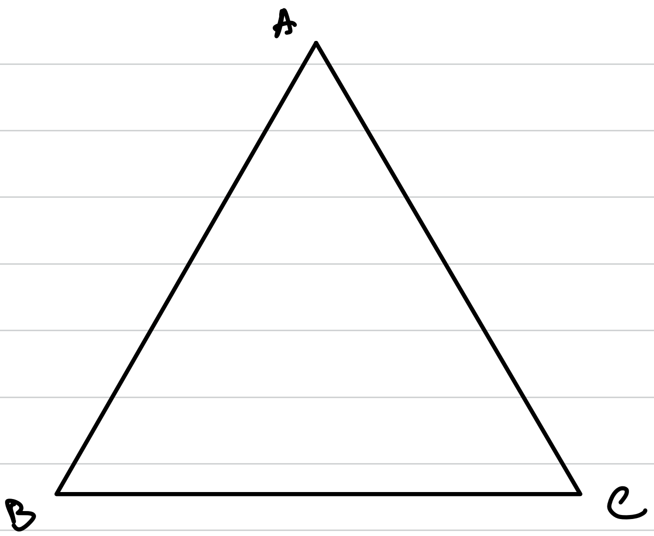
↳ same length
↳ same direction $\Rightarrow$

$\Rightarrow \overrightarrow{M_4M_3} \underline{\underline{\parallel}} \overrightarrow{M_1M_2} \Rightarrow M_1M_2M_3M_4$ - parallelog

↳ $\parallel$ AND $=$

INTERLUDE

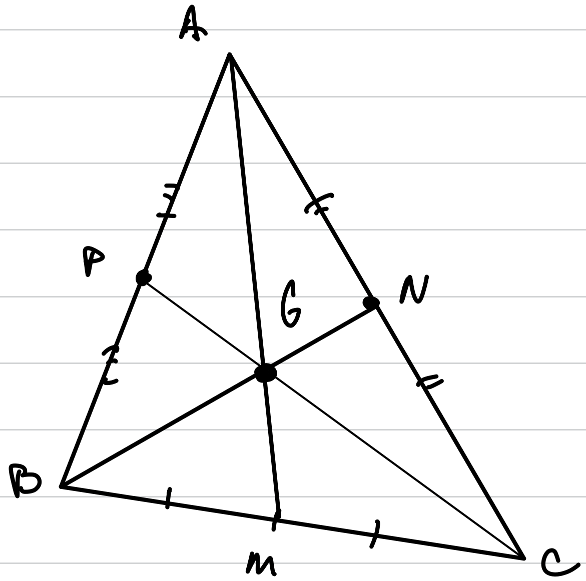
1.6. Starting from Hilbert's axioms prove that the base angles at an isosceles triangle are congruent.



$AB = AC$
$AC = AB$
$\widehat{BAC} = \widehat{CAB}$

Ax III.6 $\Rightarrow \widehat{ABC} \equiv \widehat{ACB}$

1.7. Using vectors show that the medians in a Δ are <u>concurrent</u> and deduce the ratio in which the common intersection point divides the medians.



The vector eq. of a line through points A and B

$\forall T \in \ell\ \exists!\ \lambda \in \mathbb{R}$:

$$\boxed{\vec{r}_T = \lambda \vec{r}_A + (1-\lambda)\vec{r}_B}$$

$\cdot\ O$

Let M, N, P - midpoints of the sides

Let $\{G\} = AM \cap BN$

We will find $\vec{r}_G$ and prove $G \subset CP$

$$\vec{BA} + \vec{BC} = 2\vec{BN}$$

$$\vec{AC} + \vec{AB} = 2\vec{AM}$$

$$\vec{BA} + \vec{BC} + \vec{AC} + \vec{AB} = 2(\vec{BN} + \vec{AM})$$

$$\vec{BC} + \vec{AC} = 2(\vec{BN} + \vec{AM}) \quad \Big|\cdot(-1)$$

$$\vec{CB} + \vec{CA} = 2(\vec{NB} + \vec{MA})$$

$$2\vec{CP} = 2(\vec{NB} + \vec{MA})$$

$$\vec{CP} = \vec{NB} + \vec{MA}$$

$$\vec{CP} = \vec{NG} + \vec{GB} + \vec{MG} + \vec{GA}$$

$$\boxed{\text{Blocaj la tablă + mental}}$$

$$G \in AM \Rightarrow \vec{r_G} = \lambda \vec{r_A} + (1-\lambda)\vec{r_M}$$
$$= \lambda \vec{r_A} + \frac{1-\lambda}{2}\vec{r_B} + \frac{1-\lambda}{2}\vec{r_C}$$

$$G \in BN \Rightarrow \vec{r_G} = \mu \vec{r_B} + (1-\mu)\vec{r_N}$$
$$= \mu \vec{r_B} + \frac{1-\mu}{2}\vec{r_A} + \frac{1-\mu}{2}\vec{r_C}$$

$$\left(\lambda - \frac{1-\mu}{2}\right)\vec{r_A} + \left(\frac{1-\lambda}{2} - \mu\right)\vec{r_B} +$$
$$+ \left(\frac{1-\lambda}{2} - \frac{1-\mu}{2}\right)\vec{r_C} = 0$$

Fix the origin in A:

$$\vec{r_A} = \vec{0}, \quad \vec{r_B} = \vec{AB}, \quad \vec{r_C} = \vec{AC}$$

lin. independent

So:

$$\left(\frac{1-\lambda}{2}-\mu\right)\vec{AB}+\left(\frac{1-\lambda}{2}-\frac{1-\mu}{2}\right)\vec{AC}=0$$

ΔABC is non-degenerate $\Rightarrow$ $\vec{AB}$, $\vec{AC}$ lin. indep. $\Rightarrow$

Recap Base Changes from Lin. Alg.

$$\Rightarrow \begin{cases}\frac{1-\lambda}{2}-\mu=0\\ \frac{1-\lambda}{2}-\frac{1-\mu}{2}=0 \Rightarrow \mu=\lambda\end{cases}$$

$$\Rightarrow 1-\lambda=2\lambda \Rightarrow \lambda=\frac{1}{3}$$

So:

$$\vec{r_G}=\frac{1}{3}\vec{r_A}+\frac{1}{3}\vec{r_B}+\frac{1}{3}\vec{r_C}=$$

$$=\frac{\vec{r_A}+\vec{r_B}+\vec{r_C}}{3}$$

$$\vec{CG}=\vec{r_G}-\vec{r_C}=\frac{\vec{r_A}+\vec{r_B}-2\vec{r_C}}{}$$

$$\vec{CP}=\vec{r_P}-\vec{r_C}=\frac{\vec{r_A}+\vec{r_B}}{2}-\vec{r_C}$$

$\overrightarrow{CP} = \frac{3}{2} \Rightarrow \overline{CG} \Rightarrow C, P, G$ - colinear $\Rightarrow$

$\Rightarrow G = AM \cap BN \cap CP$ and $CP = \frac{3}{2} CG$

1.8. G, H, O collinear