

## Seminar 2 5

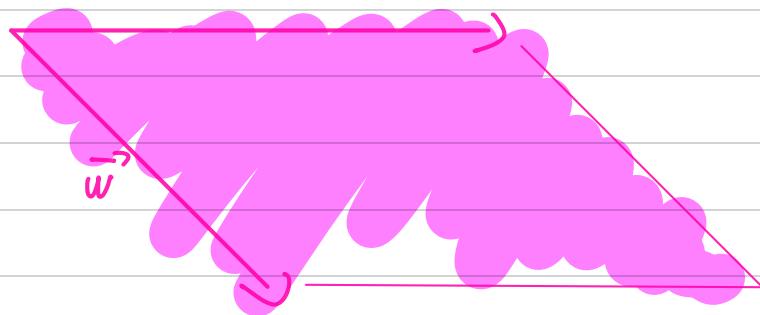
### Cross product

$$\vec{v}, \vec{w} \in \mathbb{V}^3 \Rightarrow \vec{v} \times \vec{w} \in \mathbb{V}^3$$

$$\bullet |\vec{v} \times \vec{w}| = |\vec{v}| \cdot |\vec{w}| \cdot \sin(\hat{\vec{v}}, \vec{w})$$

$$\vec{v}$$

= area of parallelogram



$$\bullet \vec{v} \times \vec{w} \perp \vec{v}, \vec{v} \times \vec{w} \perp \vec{w}$$

• orientation: right-hand rule  
(spew rule)

↪ middle finger up!

### Properties of $\cdot \times$ :

$$\bullet (\vec{v}_1 + \vec{v}_2) \times \vec{w} = \vec{v}_1 \times \vec{w} + \vec{v}_2 \times \vec{w}$$

$$\bullet (\lambda \vec{v}) \times \vec{w} = \lambda (\vec{v} \times \vec{w})$$

$$\bullet \vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$$

- $\vec{v} \times \vec{v} = 0$   
 $\hookrightarrow \sin(\pi, \theta) = 0$

If we assume the reference frame to be right-orthonormal  
and we take  $v(x_1, y_1, z_1)$

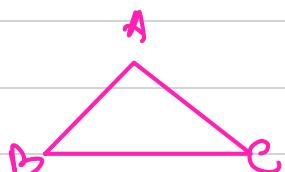
$$w(x_2, y_2, z_2)$$

$$v \times w = \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} i + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} j + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k$$

Applications :

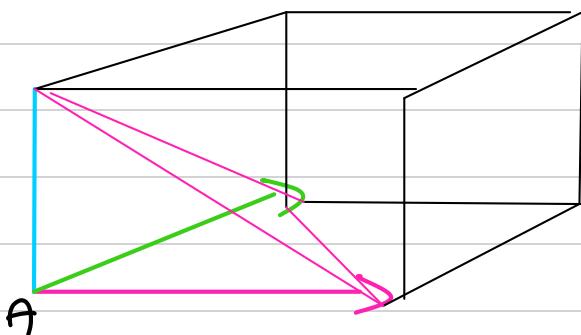
• area:



$$A_{\text{Area}} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$$

$\hookrightarrow$  half the A of

• Volume:



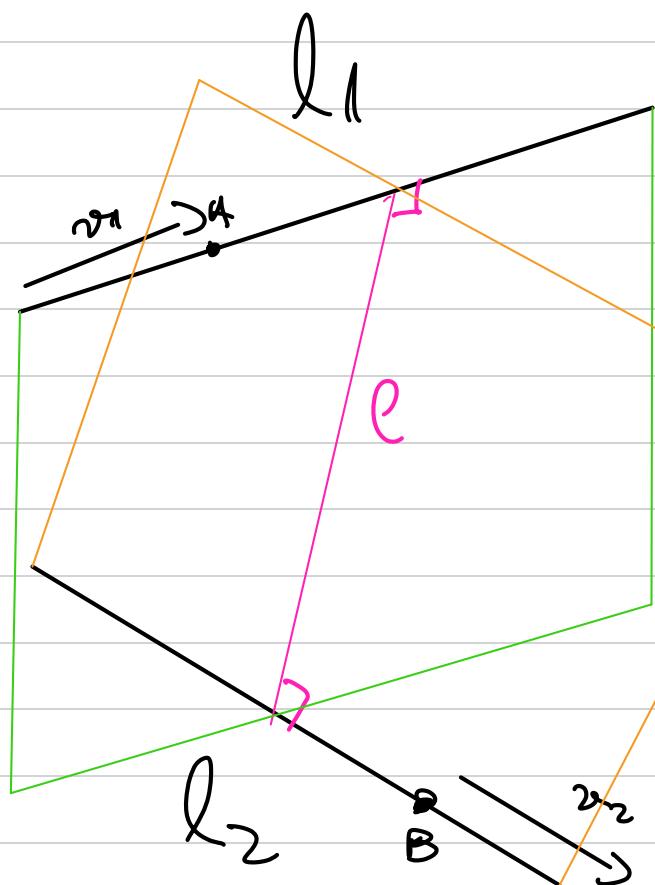
$$\text{Val}(ABCD A'B'C'D') = \{\vec{AB}, \vec{AC}, \vec{AD}\}$$

$$= (\vec{AB} \times \vec{AC}) \vec{AD}$$

$$\text{Val of a tetrahedron} = \frac{1}{6} \{ \vec{AB}, \vec{AC}, \vec{AA'} \}$$

$$= \text{dist}(A', \text{ABD}) \cdot A_{\text{ABD}} \cdot \frac{1}{3}$$

The common perpendicular of 2 skew lines



$$A \in l_1$$

$$B \in l_2$$

$$v_1 \in D(l_1)$$

$$v_2 \in D(l_2)$$

$\pi_1$  = plane given by  
 $A, \vec{v}_1, \vec{v}_1 \times \vec{v}_2$

$\pi_2$  = plane given by  
 $B, \vec{v}_2, \vec{v}_1 \times \vec{v}_2$

The common perp.

$$l : \begin{cases} \pi_1 : \dots \\ \pi_2 : \dots \end{cases}$$

5.1. Consider :

$$\begin{aligned}\vec{a} &= (3, -1, -2) \\ \vec{b} &= (1, 2, 1)\end{aligned}$$

Calculate :

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ 1 & 2 & 1 \end{vmatrix} = -i - 2j + 6k - (-k - 4i + 3j) \\ = 5i - 5j + 4k$$

$$\begin{aligned}(\vec{2a} + \vec{b}) \times \vec{b} &= 2\vec{a} \times \vec{b} + \underbrace{\vec{b} \times \vec{b}}_0 \\ &= 2(\vec{a} \times \vec{b}) \\ &= 6i - 10j + 14k\end{aligned}$$

$$(\vec{2a} + \vec{b}) \times (\vec{2a} - \vec{b}) -$$

$$\begin{aligned}&= 2\vec{a} \times (\vec{2a} - \vec{b}) + \vec{b} \times (\vec{2a} - \vec{b}) \\&= -(\vec{2a} - \vec{b}) \times \vec{2a} - (\vec{2a} - \vec{b}) \times \vec{b} \\&= -\underbrace{2\vec{a} \times \vec{2a}}_0 + \vec{b} \times \vec{2a} - \underbrace{2\vec{a} \times \vec{b}}_0 + \underbrace{\vec{b} \times \vec{b}}_0 \\&\quad + \vec{b} \times \vec{2a} \\&= 2(\vec{b} \times \vec{2a}) = 2(\vec{a} \times \vec{b}) = \\&= -12i + 20j - 28k\end{aligned}$$

Multiple choice ???

5.2. A(1, 2, 0), B(3, 0, -3), C(5, 2, 6), D(1, 0, 1)

a) Area of  $\triangle ABC$ ;  $\text{Dist}(C, AB)$

b) Vol (ABCD),  $\text{dist}(D, ANC)$   
L-shaped volume

c) the common perp. of AB and CD

a)  $A_{ABC} = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$$\vec{AB} (2, -2, -3)$$

$$\vec{AC} (4, 0, 6)$$

$$|\vec{AB} \times \vec{AC}| = \left| \begin{vmatrix} i & j & k \\ 2 & -2 & -3 \\ 4 & 0 & 6 \end{vmatrix} \right|$$

$$= \left| -12i - 12j - (-3k + 12j) \right|$$

$$= \left| -12i - 24j + 3k \right|$$

$$= |4(-3i - 6j + k)|$$

$$= 4\sqrt{(5+36+4)}$$

$$= 4 \cdot 7 = 28$$

$$A_{ABC} = \frac{1}{2}$$

$$A_{ABD} = \frac{\text{dist}(C, AB) \cdot |AB|}{2}$$

$$|AB| = \sqrt{14}$$

$$2D = \text{dist}(C, AB) \cdot \sqrt{14}$$

$$\Rightarrow \text{dist}(C, AB) = \frac{2D \sqrt{14}}{\sqrt{14}}$$

$$\hookrightarrow V_{ABCD} = \frac{1}{6} \left\{ \vec{AB}, \vec{AC}, \vec{AD} \right\}$$

$$\vec{AB} (2, -2, -2)$$

$$\vec{AC} (4, 0, 6)$$

$$\vec{AD} (0, -2, 1)$$

$$\left| (\vec{AB} \times \vec{AC}) \cdot \vec{AD} \right| =$$

$$= \left| (-12, -24, 8) \cdot (0, -2, 1) \right|$$

$$= 96$$

$$\Rightarrow V_{ABCD} = \frac{56}{6} = \frac{28}{3}$$

$$\frac{2\delta}{3} = \frac{1}{3} \cdot \text{dist}(D, ABC) \cdot A_{ABC}$$

$$A_{ABC} = \frac{1}{2} |AB \times AC| = 4$$

$$\Rightarrow \text{dist}(D, ABC) = 2$$

General formula:

$$\Pi: Ax + By + Cz + D = 0$$

$$\text{dist}(P, \Pi) = \frac{|Ax_p + By_p + Cz_p + D|}{\sqrt{A^2 + B^2 + C^2}}$$

c)  $\ell: \begin{cases} \bar{\Pi}_1 \\ \bar{\Pi}_2 \end{cases} \dots$

5.5. (Law of sines)

for ABC .

$$\vec{u} = \vec{AB}, \vec{v} = \vec{BC}, \vec{w} = \vec{CA}$$

Show that  $\vec{u} \times \vec{v} = \vec{v} \times \vec{w} = \vec{w} \times \vec{u}$

and deduce the law of sines :

$$\frac{c}{\sin A} = \frac{b}{\sin B} = \frac{a}{\sin C}$$

$$\vec{u} = (-\vec{v} - \vec{w})$$

$$\begin{aligned}\vec{u} \times \vec{v} &= (-\vec{v} - \vec{w}) \times \vec{v} = \underbrace{-\vec{v} \times \vec{v}}_0 - \vec{w} \times \vec{v} \\ &= \vec{v} \times \vec{w}\end{aligned}$$

$$\begin{aligned}\vec{w} \times \vec{u} &= -\vec{u} \times \vec{w} = (\vec{v} + \vec{w}) \times \vec{w} \\ &= \vec{v} \times \vec{w} + \vec{w} \times \vec{w} = \vec{v} \times \vec{w}\end{aligned}$$

$$|\vec{u} \times \vec{v}| = |\vec{v} \times \vec{w}| \Rightarrow$$

$$\Rightarrow |\vec{AB} \times \vec{BC}| = |\vec{BC} \times \vec{CA}|$$

$$|\vec{AB}| |\vec{BC}| \sin B = |\vec{BC}| |\vec{CA}| \sin C$$

$$\Rightarrow \frac{|\vec{AB}| \cdot |\vec{BC}|}{\sin C} = \frac{|\vec{BC}| \cdot |\vec{CA}|}{\sin B}$$

$$\Rightarrow \frac{(AB)}{\sin C} = \frac{|c|}{\sin B}$$

Proof of the 'Jacobi' identity

$$(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{c} \times \vec{a}) \times \vec{b} = 0$$

$$\text{Lemma: } (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{b} & \vec{a} \\ b \cdot c & a \cdot c \end{vmatrix} \\ = \langle a \cdot c \rangle \vec{b} - \langle b \cdot c \rangle \vec{a}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \vec{b} & \vec{c} \\ b \cdot a & c \cdot a \end{vmatrix}$$

$$\begin{aligned} \text{Proof: } & a(x_1, y_1, z_1) \\ & b(x_2, y_2, z_2) \\ & c(x_3, y_3, z_3) \end{aligned}$$

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} i - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} j \\ &+ \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} k \end{aligned}$$

Just do the calculations

$$\left( \text{even for } (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} i & j & k \\ \dots & \dots & \dots \\ x_c & y_c & z_c \end{vmatrix} \right)$$

Bock für Jocelin:

$$(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{c} \times \vec{a}) \times \vec{b}$$

$$= \dots = 0$$