Seminar 10 - Classification of conics & quadrics

$$Q: a_{11} x^2 + 2 a_{12} xy + a_{22} y^2 + 2 a_{10} x + 2a_{20} y + a_{00}$$

↳ $a_{\text{①②}}$ ↳ second term

↳ first term

1 - x
2 - y
0 - None

$$Q: \begin{pmatrix} x & y \end{pmatrix} \underbrace{\begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x \\ y \end{array} & \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \end{array}}_{M(Q)} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2a_{10} & 2a_{20} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \hat{c}$$

↳ mat. of conic

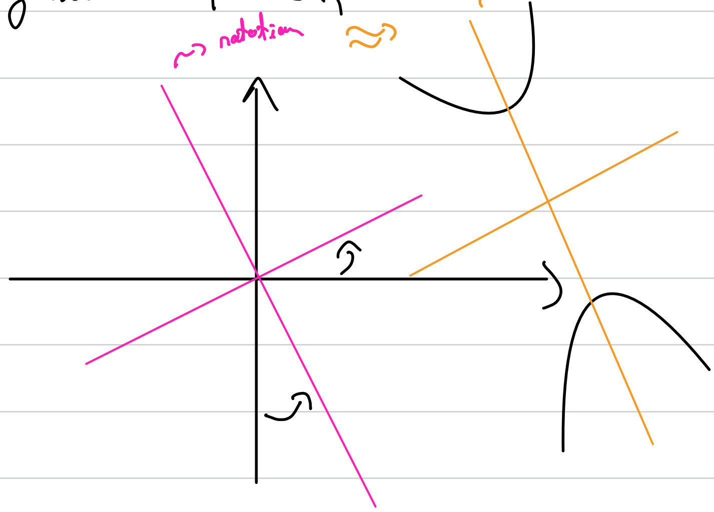
$B = (\vec{i}, \vec{j})$ - the initial basis

We will construct a basis B' of eigenvectors of $M(Q)$ ⤳ right oriented and orthonormal

Then we will have:

$$\underbrace{M_{BB'}^{-1}}_{\uparrow} \cdot M(Q) \cdot M_{BB'} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

↳ $M_{BB'}^{t}$ because $M_{BB'} \in SO(2)$



Now, after this rotation, the equation of Q is:

$$Q: \lambda_1 x'^2 + \lambda_2 y'^2 + b_{1,0} x' + b_{2,0} y' + b_{00} = 0$$

$$\hookrightarrow \begin{pmatrix} x \\ y \end{pmatrix} = M_{BB'} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Now we force squares in x' and y' $\Rightarrow$ the isometric canonical form

10.2. For each of the following equations write down the associated matrix and bring the equation to an isometric canonical form
$\rightarrow$ forced to use this

a) $-x^2+xy-y^2+x=0$

b) $6xy+x-y=0$

a) $$Q: \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$M(Q) = \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}$$

$$P_{M(Q)}(x) = \det\left(M(Q) - x I_2\right)$$

$$= \begin{vmatrix} -1-x & \frac{1}{2} \\ \frac{1}{2} & -1-x \end{vmatrix}$$

$$= (x+1)^2 - \frac{1}{4} =$$

$$= x^2 + 2x + \frac{3}{4}$$

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{4-3}}{2} = \frac{-2 \pm 1}{2} \begin{cases} -\frac{3}{2} \\ -\frac{1}{2} \end{cases}$$

$$\left(M \cdot x = \lambda \cdot x \quad \text{or} \quad (M - \lambda I_2) \cdot x = 0 \right)$$

$$\lambda_1 = -\frac{3}{2} \Rightarrow \begin{pmatrix} -1-\left(-\frac{3}{2}\right) & \frac{1}{2} \\ \frac{1}{2} & -1-\left(-\frac{3}{2}\right) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow y = -x$$

$$\Rightarrow S\left(-\frac{3}{2}\right) = \langle(-1,1)\rangle$$

We choose $\vec{v_1} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\lambda = -\frac{1}{2} \Rightarrow \begin{pmatrix} -1-\left(-\frac{1}{2}\right) & \frac{1}{2} \\ \frac{1}{2} & -1-\left(-\frac{1}{2}\right) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-x + y = 0$, so $y = x$

$S(\lambda_2) = \langle(1,1)\rangle$

We choose $\vec{v_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If we choose $\mathcal{B}' = \left(\frac{1}{\sqrt{2}}(-1,1), \frac{1}{\sqrt{2}}(1,1)\right)$

We get $M_{\mathcal{B}\mathcal{B}'} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

But $\det M_{\mathcal{B}\mathcal{B}'} = -1$

To get $M_{\mathcal{B}\mathcal{B}'} \in SO(2)$, we flip $\vec{v_1}$ and $\vec{v_2}$, so:

$$\mathcal{B}' = \left(\frac{1}{\sqrt{2}} (1,1) \, , \, \frac{1}{\sqrt{2}} (-1,1) \right)$$

$$M_{\mathcal{B}\mathcal{B}'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

So we have :

$$\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{M_{\mathcal{B}\mathcal{B}'}}_{= [id]_{\mathcal{B}'\mathcal{B}}} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$Q : \begin{pmatrix} x \\ y \end{pmatrix}^t \cdot M(Q) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$Q : \begin{pmatrix} x' & y' \end{pmatrix} \cdot M_{\mathcal{B}\mathcal{B}'}^t \cdot M(Q) \cdot M_{\mathcal{B}\mathcal{B}'} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot M_{\mathcal{B}\mathcal{B}'} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$Q : \begin{pmatrix} x' & y' \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

$$\begin{pmatrix} 1, 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$Q : \begin{pmatrix} x' & y' \end{pmatrix} \cdot \frac{1}{2} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} +$$

$$\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$Q : \begin{pmatrix} x' & y' \end{pmatrix} \cdot \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{3}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} x' & -y' \end{pmatrix} = 0$$

$Q: -\frac{1}{2}x'^2 - \frac{3}{2}y'^2 + \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{2}}y' = 0$

$Q: \left(-\frac{1}{2}x'^2 + \frac{1}{\sqrt{2}}x'\right) + \left(-\frac{3}{2}y'^2 - \frac{1}{\sqrt{2}}y'\right) = 0$

$= -\frac{1}{2}\left(x'^2 - \sqrt{2}\,x'\right) - \frac{3}{2}\left(y'^2 - \frac{2}{3\sqrt{2}}y\right) = 0$

$= -\frac{1}{2}\left(x'^2 - 2\cdot x'\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\right) - \frac{3}{2}\left(y'^2 - 2y\frac{1}{3\sqrt{2}} + \frac{1}{18}\right)$

$+\frac{1}{4} + \frac{1}{12} = 0$

$$\frac{1}{2}\underbrace{\left(x' - \frac{1}{\sqrt{2}}\right)}_{=x''}{}^2 + \frac{3}{2}\underbrace{\left(y^2 - \frac{1}{3\sqrt{2}}\right)}_{=y''}{}^2 = \frac{1}{3}$$

$$\begin{cases} x'' = x' - \frac{1}{\sqrt{2}} \\ y'' = y' - \frac{1}{3\sqrt{2}} \end{cases}$$

$$\frac{1}{2}x''^2 + \frac{3}{2}y''^2 = \frac{1}{3} \;\Big|\cdot 3$$

$$\frac{x''^2}{\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2} + \frac{y''^2}{\left(\frac{\sqrt{2}}{3}\right)^2} = 1$$

[illegible] canonical form (every step was an ISOMETRY)

If we don't want an isometric canonical form (an affine one suffices) then we can use the Lagrange method

Step I: Form a square using one of the pure quadratic terms and all of the mixed terms containing it

The new variable is inside the newly formed square

Step II: Form the remaining squares

Q: $-x^2+xy-y^2+x=0$

$$(-x^2+xy)-y^2+x=0$$

$$-\left(x^2-2x\frac{y}{2}+\frac{y^2}{4}\right)+\left(\frac{y^2}{4}-y^2+x=0\right.$$

$$-\underbrace{\left(x-\frac{y}{2}\right)^2}_{x'}-\frac{3y^2}{4}+x=0$$

$$x'=x-\frac{y}{2}$$
$$y'=y$$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}1 & -\frac{1}{2}\\0 & 1\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{cases}x=x'+\frac{y'}{2}\\ y=y'\end{cases}$$

$$-x'^2-\frac{3y'^2}{4}+x'+\frac{y'}{2}=0$$

$$\left(-x'^2+x'\right)+\left(-\frac{3y'^2}{4}+\frac{y'}{2}\right)=0$$

$$-\left(x'^2-2x'\cdot\frac{1}{2}+\frac{1}{4}\right)+\left(-\frac{3}{4}\right)\left(y'^2-2y'\frac{1}{3}+\frac{1}{9}\right)+\frac{1}{4}+\frac{1}{12}=0$$

$$-\underbrace{\left(x'-\frac{1}{2}\right)}_{x''}^2-\frac{3}{4}\underbrace{\left(y'-\frac{1}{3}\right)}_{y''}^2+\frac{1}{3}=0$$

$$\begin{cases} x''=x'-\frac{1}{2} \\ y''=y'-\frac{1}{3} \end{cases}$$

$$\Leftrightarrow \quad x''^2+\frac{3}{4}y''^2=\frac{1}{3}$$

$$3x''^2+\frac{9}{4}y''^2=1$$

$$\frac{x''^2}{\left(\frac{1}{\sqrt{3}}\right)^2}+\frac{y''^2}{\left(\frac{2}{3}\right)^2}=1$$

$$\begin{cases} x'''=\dfrac{x''}{\frac{1}{\sqrt{3}}} \\ y'''=\dfrac{y''}{\frac{2}{3}} \end{cases}$$

$$Q: x'''^2+y'''^2=1$$

b) $6xy+x-y=0$

(use the Lagrange method)

To make the squares appear we use:

$x=x'+y'$

$y=y'$

$$\Rightarrow \begin{cases} x' = x - y \\ y' = y \end{cases}$$

$$\Rightarrow 6(x' + y') \cdot y' + x' + y' - y' = 0$$

$$\Leftrightarrow 6x'y' + 6y'^2 + x' = 0$$

$$= 6\left(y'^2 + 6x'y'\right) + x' = 0$$

$$= 6\left(y'^2 + 2x'y' \cdot \frac{1}{2} + \left(\frac{x'}{2}\right)^2\right) + x' - 6\left(\frac{x'}{2}\right)^2$$

$$= 6\underbrace{\left(\frac{x'}{2} + y'\right)}_{y''}^2 + x' - \frac{3x'^2}{2}$$

$$\begin{cases} y'' = \frac{x'}{2} + y' \\ x'' = x' \end{cases} \Rightarrow \begin{cases} y' = y'' - \frac{x''}{2} \\ x' = x'' \end{cases}$$

$$= 6y''^2 + x'' - \frac{3x''^2}{2} = 0$$

$$= 12y''^2 + 2x'' - 3x''^2 = 0$$

$$= 12y''^2 - 3\left(x''^2 - \frac{2}{3}x'' + \frac{1}{3}\right) + \frac{1}{3} = 0$$

$$= 12y''^2 - 3\underbrace{\left(x'' - \frac{1}{3}\right)}_{x'''}^2 + \frac{1}{3}$$

$$\begin{cases} x''' = x'' - \frac{1}{3} \\ y''' = y'' \end{cases}$$

$$= 12 y'''^2 - 3x'''^2 + \frac{1}{3} = 0$$

$$12 y'''^2 - 3x'''^2 = -\frac{1}{3} \quad \Big| \cdot -3$$

$$-36 y'''^2 + 9x'''^2 = 1$$

$$\left(\frac{x'''}{\frac{1}{3}}\right)^2 - \left(\frac{y'''}{\frac{1}{6}}\right)^2 = 1$$

↳ Hyperbola

$$\begin{cases} x^{(4)} = \dfrac{x'''}{\frac{1}{3}} \\ y^{(4)} = \dfrac{y'''}{\frac{1}{6}} \end{cases}$$

$$Q: \; x^{(4)2} - y^{(4)2} = 1$$

Finding the shape of the conic

$\hat{D}$	D	T	Conic
$\hat{D}=0$	$D>0$		point
	$D=0$		2 lines or $\emptyset$
	$D<0$		2 lines
$\hat{D}\neq 0$	$D>0$	$\hat{D}T<0$	ellipse
	$D>0$	$\hat{D}T>0$	$\emptyset$
	$D=0$		parabola
	$D<0$		hyperbola

$Q: \; a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{10}x + 2a_{20}y + a_{00} = 0$

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}$$

$$\hat{D} = \begin{vmatrix} a_{00} & a_{10} & a_{20} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{12} & a_{22} \end{vmatrix}$$

$T = a_{11} + a_{22}$

$-x^2 + xy - y^2 + x = 0$

$$D = \begin{vmatrix} -1 & \frac{1}{2} \\ \frac{1}{2} & -1 \end{vmatrix} = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

$$\hat{D} = \begin{vmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & \frac{1}{2} & -1 \end{vmatrix} = +\frac{1}{4} \neq 0$$

$T = -2 \Rightarrow \hat{D}T < 0 \Rightarrow$ ellipse

$6xy + x - y = 0$

$$D = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = -9$$

$$\hat{D} = \begin{vmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & 3 \\ -\frac{1}{2} & 3 & 0 \end{vmatrix} = -\frac{3}{2} \neq 0 \Rightarrow \text{hyperbola}$$