

Seminar 4

$\vec{v}, \vec{w} \in V$

$$\vec{v} \cdot \vec{w} = \begin{cases} 0, & \vec{v} = 0 \text{ or } \vec{w} = 0 \\ |\vec{v}| \cdot |\vec{w}| \cdot \cos(\vec{v}, \vec{w}) \end{cases}$$

$$\vec{v} \perp \vec{w} \Rightarrow \vec{v} \cdot \vec{w} = 0$$

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2$$

Say we work in E^n

From now on, we will work over orthonormal reference frames

orthonormal = orthogonal + normalized

$$R = (0, \vec{v}_1, \dots, \vec{v}_n)$$

- orthogonal : $\forall i \neq j : \vec{v}_i \perp \vec{v}_j$

- normalized : $|\vec{v}_1| = \dots = |\vec{v}_n| = 1$

If $\vec{v}, \vec{w} \in V$

$$\{\vec{v}\}_R = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\{\vec{w}\}_R = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

$$\vec{v} \cdot \vec{w} = \{\vec{v}\}_{\mathbb{R}}^+ \cdot \{\vec{w}\}_{\mathbb{R}}^+ = \alpha_1 b_1 + \alpha_2 b_2 + \dots + \alpha_m b_m$$

u.t. \vec{m}, \vec{n} unit vectors
 $\angle(\vec{m}, \vec{n}) = 120^\circ$

Determine: $\vec{a} \cdot \vec{b}$ if

$$\begin{aligned}\vec{a} &= 2\vec{m} + 4\vec{n} \\ \vec{b} &= \vec{m} - \vec{n}\end{aligned}$$

$$\vec{a} \cdot \vec{m} = |\vec{a}| \cdot |\vec{m}| \cdot \cos(120^\circ) = -\frac{1}{2}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\vec{a}, \vec{b})$$

$$\begin{aligned}|\vec{a}| &= \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{(2\vec{m} + 4\vec{n})^2} \\ &= \sqrt{4\vec{m} \cdot \vec{m} + 16\vec{n} \cdot \vec{n} + 16\vec{m} \cdot \vec{n}} \\ &= \sqrt{20 - 8} = \sqrt{12} = 2\sqrt{3}\end{aligned}$$

$$|\vec{b}| = \sqrt{\vec{b} \cdot \vec{b}} = \sqrt{(\vec{m} - \vec{n})^2} = \sqrt{2 + 1} = \sqrt{3}$$

$$\begin{aligned}\vec{a} \cdot \vec{b} &= (2\vec{m} + 4\vec{n})(\vec{m} - \vec{n}) \\ &= 2\vec{m}^2 - 2\vec{m} \cdot \vec{n} + 4\vec{n} \cdot \vec{m} - 4\vec{n}^2 \\ &= 2 - 1 - 4 = 2 - 5 \\ &\leq -3\end{aligned}$$

$$\begin{aligned}-3 &= -2\sqrt{3} \cdot \cos(\vec{a}, \vec{b}) \Rightarrow \cos(\vec{a}, \vec{b}) = -\frac{-3}{2\sqrt{3} \cdot \sqrt{3}} \\ &= -\frac{1}{2}\end{aligned}$$

$$\alpha_1, \alpha_2 = 120^\circ$$

\mathbb{E}^m

Take a hyperplane H in \mathbb{E}^m

$$H: \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_m x_m + b = 0$$

$$D(H): \underbrace{\alpha_1 x_1 + \dots + \alpha_m x_m}_{} = 0$$

$$(\alpha_1 \dots \alpha_m) \cdot (x_1 \dots x_m) = 0$$

$$D(H)^{\perp} = \left\{ \vec{w} \in V \mid \vec{w} \perp \vec{v}, \vec{v} \in D(H) \right\}$$

$$= \langle \vec{n}_{\pi} \rangle \quad \text{if } \vec{v} \in D(H)$$

$$\text{Ex: } \pi: 2x + 6y - 3z + 4 = 0$$

$$\vec{n}_{\pi} (2, 6, -3)$$

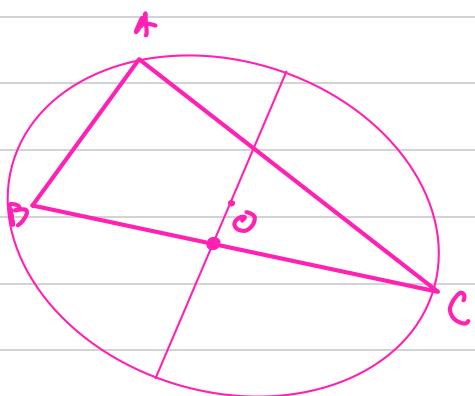


The normal vector is given by the equation. + it gives the orientation

u. 2. $A(1, 2)$
 $B(3, -2)$
 $C(5, 6)$

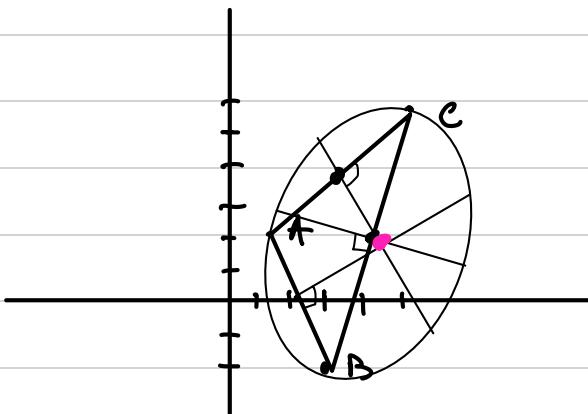
- a) Calculate the circumcenter of $\triangle ABC$
- b) Calculate the orthocenter of $\triangle ABC$
- c) Find the eq. of the angle bisector.

circumcenter = \cap of \perp bisectors
 \hookrightarrow medians



orthocenter = \cap of the heights

$A(1, 2)$
 $B(3, -2)$
 $C(5, 6)$



M - midpoint of BC $\rightarrow M = (4, 2)$

N - mid point at $AC \Rightarrow N(3, 4)$

$$m_{BC} = \frac{\vec{t}_C - \vec{t}_B}{x_C - x_B} = \frac{2}{2} = u \Rightarrow m_{ON} = -\frac{1}{u}$$

$$m_{AC} = \frac{\vec{t}_C - \vec{t}_A}{x_C - x_A} = \frac{u}{u} = 1 = m_{ON} = -1$$

OM: $\vec{t} - \vec{t}_M = m(x - x_M)$
 $\vec{t} - 2 = -\frac{1}{u}(x - u) \Rightarrow \vec{t} = -\frac{x+12}{u}$

ON: $\vec{t} - u = -1(x \rightarrow)$
 $\vec{t} = -x + 4$

$$OM \wedge ON = \left\{ \begin{array}{l} \vec{t} = \frac{-x+12}{u} \quad | \cdot u \\ \vec{t} = -x + 4 \\ \end{array} \right. \begin{array}{l} \Leftrightarrow -x + 4 = \frac{-x+12}{u} \\ -4x + 2u = -x + 12 \\ -3x = -16 \\ x = \frac{16}{3} \end{array}$$

$$\vec{t} = \frac{5}{3}$$

$$\Rightarrow O\left(\frac{16}{3}, \frac{5}{3}\right)$$

b) Let $M \in AC$: $\vec{BM} \perp \vec{AC} \Rightarrow m_{\vec{BM}} = -\frac{1}{m_{\vec{AC}}}$

$$m_{\vec{AC}} = \frac{\vec{x}_C - \vec{x}_A}{x_C - x_A} = \frac{4}{4} = 1 \Rightarrow m_{\vec{BM}} = -1$$

NG BC $\{ \dots \} m_{AN} = -\frac{1}{u}$

$$AN: \vec{y} - \vec{y}_A = m_{AN}(x - x_A)$$

$$\vec{y} - 2 = -\frac{1}{u}(x - 1)$$

$$AN: \vec{y} = \frac{-x+1}{u} + 2$$

$$BM: \vec{y} - \vec{y}_B = m_{BM}(x - x_B)$$

$$\vec{y} + 2 = -x + 3$$

$$BAU: \vec{y} = 1 - x$$

$$AN \cap BM = \begin{cases} \vec{y} = \frac{-x+1}{u} + 2 \\ \vec{y} = 1 - x \end{cases}$$

$$\Leftrightarrow 1 - x = \frac{-x+1}{u} + 2$$

$$-1 + x = -\frac{x+1}{u}$$

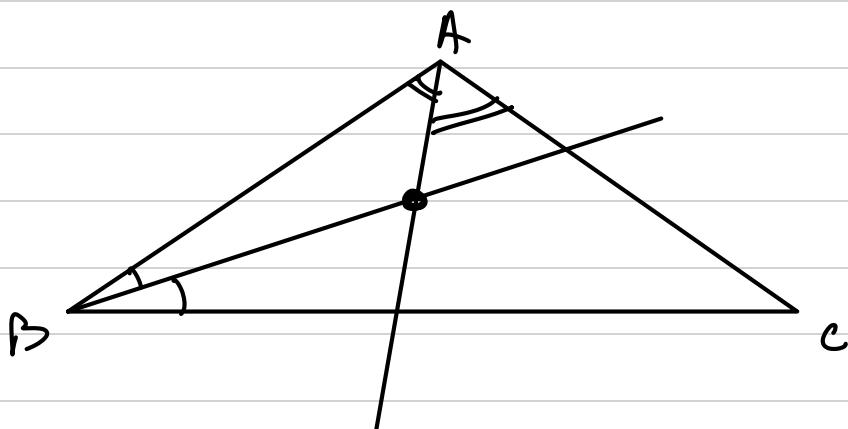
$$-4 - 4x = -x + 1$$

$$-3x = 5$$

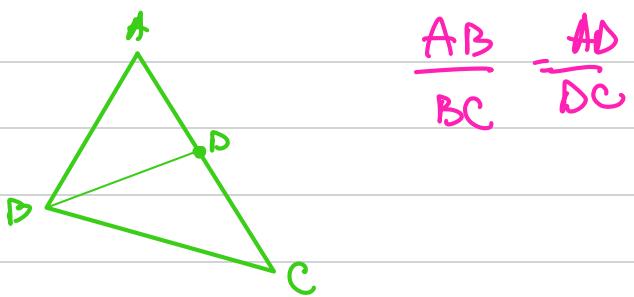
$$x = -\frac{5}{3}$$

$$\gamma = \frac{D}{3} \Rightarrow$$

$$\Rightarrow \text{AN} \cap \text{BM} = O\left(-\frac{5}{3}, \frac{8}{3}\right)$$



First approach: Bisection th.



$$AB = \sqrt{2^2 + 6^2} = \sqrt{20} = 2\sqrt{5}$$

$$BC = \sqrt{2^2 + 3^2} = \sqrt{13}$$

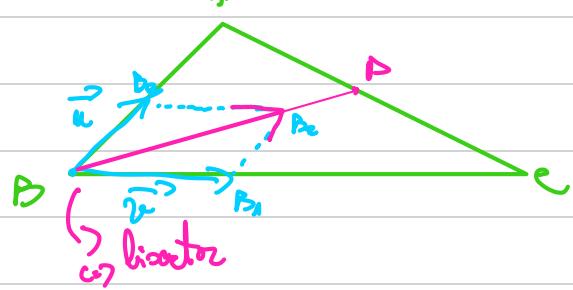
→ th. from 1st week

$$\frac{AD}{DC} = \frac{\sqrt{5}}{\sqrt{13}} \Rightarrow \vec{BD} = \frac{\sqrt{5} \vec{BC} + \sqrt{13} \cdot \vec{BA}}{\sqrt{5} + \sqrt{13}}$$

$$= \frac{1}{\sqrt{5} + \sqrt{13}} \cdot (\sqrt{5}(2, 3) + \sqrt{13}(-2, 6))$$

$$BD : \begin{cases} x = 3 + 2 \cdot \frac{2\sqrt{5} - 2\sqrt{13}}{\sqrt{5} + \sqrt{13}} \\ y = -2 \cdot 2 \cdot \frac{3\sqrt{5} - 4\sqrt{13}}{\sqrt{5} + \sqrt{13}} \end{cases}$$

2nd approach: Parallelogram rule



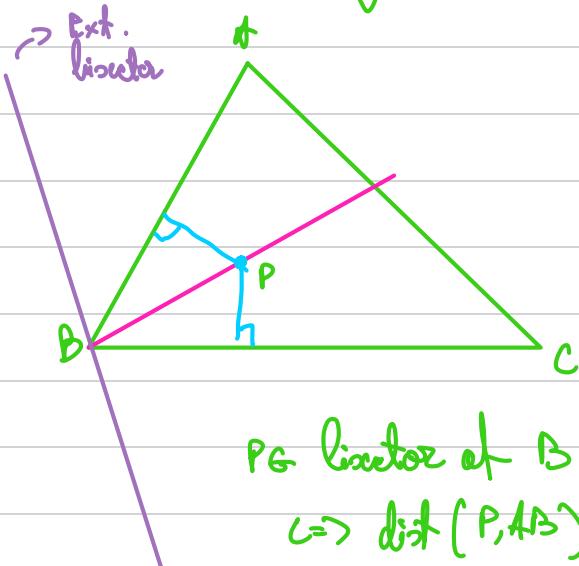
The direction of the bisector is given by :

$$\frac{\vec{BA}}{|\vec{BA}|} + \frac{\vec{BC}}{|\vec{BC}|} \Rightarrow \text{True bc. } B, B_1, B_2, B_3 = \text{ rhombus}$$

$$\frac{1}{2\sqrt{5}} (-2, 4) + \frac{1}{2\sqrt{14}} (2, 3)$$

$$\text{So. } \vec{BD} = \begin{cases} x = 3 + 2 \left(-\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{14}} \right) \\ y = -2 + 2 \left(\frac{2}{\sqrt{5}} + \frac{4}{\sqrt{14}} \right) \end{cases}$$

3rd approach : using distances



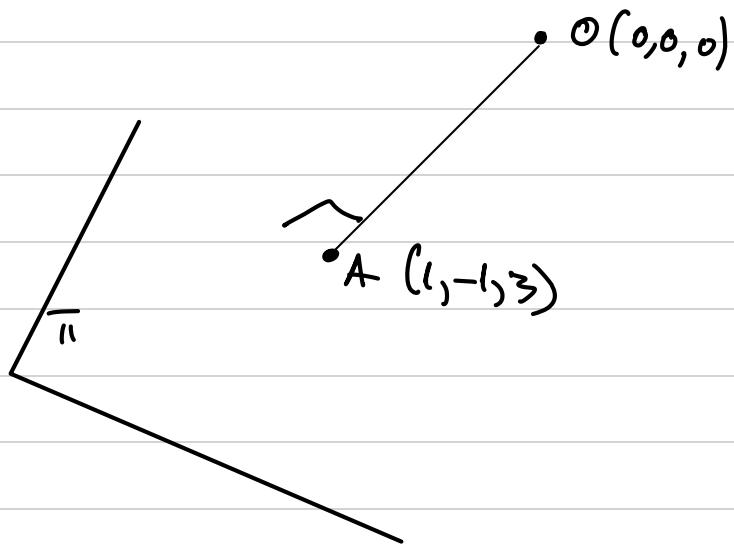
P is bisector of BC \Leftrightarrow
 $\Leftrightarrow \text{dist}(P, AB) = \text{dist}(P, BC)$

This gives two lines l_1 and l_2 .

To decide which one is the initial bisector we use the half-plane rule

→ Plugging the coordinates of A and C and pick the line that separates them.

4.6. Determine a Cartesian equation of the plane π if A $(-1, 3)$ is the orthogonal projection at the origin on π



$$\pi : x - y + 3z + d = 0$$

$$\begin{aligned} A \in \pi &\Rightarrow 1 + 1 + 9 + d = 0 \Rightarrow d = -11 \\ &\Rightarrow d = -11 \end{aligned}$$

$$\Rightarrow x - y + 3z - 11 = 0$$

$$OA \perp \pi \Rightarrow$$

$$\Rightarrow \vec{OA} \in D(\pi)$$

$\hookrightarrow \vec{OA}$ is a normal vector for π

\Rightarrow we can use its coordinates for the eq. of π

u.5. Consider the vector $\vec{v}(x, y, z)$ so that

$$\vec{v} \perp \vec{a}(1, -2, -3)$$

$$\vec{v} \perp \vec{b}(0, 1, 1)$$

$$\langle v, \vec{ox} \rangle \leq \frac{\pi}{2} \Rightarrow \cos(v, \vec{ox}) > 0$$

$$|\vec{v}| = 26$$

$$\vec{v} \cdot \vec{a} = 0 \Leftrightarrow 4x - 2y - 3z = 0$$

$$\vec{v} \cdot \vec{b} = 0 \Leftrightarrow y + 3z = 0$$

$$\vec{v} \cdot \vec{ox}(1, 0, 0) = (\vec{v} \cdot 1 \cdot \cos(v, \vec{ox})) > 0$$

$$\vec{v} \cdot \vec{ox} > 0 \Rightarrow (x, y, z)(1, 0, 0) > 0 \Rightarrow x > 0$$

$$v: \left\{ \begin{array}{l} 4x - 2y - 3z = 0 \\ y + 3z = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} 4x - y = 0 \Rightarrow x = \frac{y}{4} \\ z = -\frac{y}{3} \end{array} \right.$$

$$\cos(v, \vec{ox}) > 0$$

$$\Rightarrow x > 0$$

$$|\vec{v}| = 26 \Rightarrow \sqrt{x^2 + y^2 + z^2} = \sqrt{x^2 + 4^2 + 8^2} = 26$$

$$\left. \begin{aligned} &\Leftrightarrow \begin{cases} 4x - y - 6z = 0 \\ \sqrt{x^2 + y^2 + z^2} = 26 \end{cases} \\ &x > 0 \end{aligned} \right\}$$

$$\rightarrow \sqrt{\left(\frac{y}{4}\right)^2 + y^2 + \left(-\frac{y}{3}\right)^2} = 26$$

$$\Leftrightarrow \sqrt{\frac{y^2}{16} + y^2 + \frac{y^2}{9}} = 26$$

$$\Leftrightarrow \sqrt{9y^2 + 144y^2 + 16y^2} = 26$$

144

$$\Leftrightarrow \frac{13|y|}{12} = 26$$

$$|y| = 24$$

$$\Rightarrow |y| = \pm 24$$

$$x = \frac{y}{4} \Rightarrow x = \pm 6$$

$$x > 0 \Rightarrow x = 6, y = 24$$

$$z = -\frac{4}{3} \Rightarrow z = -8$$

$\vartheta(6, 24, -8)$