10.1 Warm-Up Exercises

10.1. Check the calculations in Examples 10.2, 10.3 and 10.4.

10.2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in \mathrm{SO}(2)$ which diagonalizes A.

a) $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 5 & -13 \\ -15 & 5 \end{bmatrix}$

c) $\begin{bmatrix} 7 & -2 \\ -2 & 5/3 \end{bmatrix}$

10.3. For the curve with equation $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$

a) find the isometric canonical equation;

b) find the affine canonical equation using Lagrange's method and indicate the corresponding affine change of coordinates.

10.2. For each of the following matrices A, write down a quadratic equation with associated matrix A and find the matrix $M \in SO(2)$ which diagonalizes A.

a) $\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$

- a quadratic equation with associated matrix is $6x^2 + 9y^2 + 4xy + 42 = 0$
- the matrix M is the base change matrix $M_{e',e}$ where e is the basis with respect to which the matrix A is given and e' is an orthonormal basis of eigenvectors

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda-6 & -2 \\ -2 & \lambda-9 \end{vmatrix} = (\lambda-6)(\lambda-9) - 4 = \lambda^2 - 15\lambda + 54 - 4 = 0$$

$$\lambda^2 - 15\lambda + 50 = 0 \iff (\lambda-10)(\lambda-5) = 0$$

so, the eigenvalues are 5 and 10

$\lambda = 5$ $\quad A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 5 \begin{bmatrix} x \\ y \end{bmatrix} \iff (5 \cdot I - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\iff \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \quad \text{so } -x - 2y = 0$$

$$\Rightarrow x = -2y$$

the eigenspace for the eigenvalue $\lambda = 5$ is

$$\left\{ \begin{bmatrix} -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\rangle$$

$\lambda = 10$ $\quad (10 I_2 - A)\begin{bmatrix} x \\ y \end{bmatrix} = 0 \iff \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{so } -2x + y = 0 \Rightarrow y = 2x$$

the eigenspace for the eigenvalue $\lambda = 10$ is

$$\left\{ \begin{bmatrix} t \\ 2t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\rangle$$

$\Rightarrow$ an orthonormal basis of eigenvectors is $\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

so M could be $M_{e,e'} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad M \cdot M^t \left(= M^2\right) = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I_2$

$\det M = \frac{1}{5} \cdot (-5) = -1$

$\Downarrow$

we can replace $\frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ by $\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

then $M_{e,e'} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ has $M \cdot M^t = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I_2$

and $\det M = \frac{1}{5} 5 = 1$

$\Rightarrow M \in SO(2)$

Moreover $\quad M^{-1} A M = M^t A M = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 10 & 10 \\ -5 & 20 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 & 0 \\ 0 & 50 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

10.3. For the curve with equation $x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$

a) find the isometric canonical equation;

associated matrix is $Q = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

eigenvalues and eigenvectors are

$$\left.\begin{array}{ll} \lambda_1 = 3 & v_1 = (-1,1) \\ \lambda_2 = -1 & v_2 = (1,1) \end{array}\right\} \Rightarrow$$

orthonormal basis of eigenvectors is

$$B' = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

base change matrix

$$M = M_{BB'} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$ with determinant -1

we change the sign of first vector

$$B' = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

then $M = M_{BB'} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ is a rotation by $\theta = -45^\circ$

the equation of our quadratic curve changes as follows

$$[x' \ y'] M^T Q M \begin{bmatrix} x' \\ y' \end{bmatrix} + [-6 \ 2] M \begin{bmatrix} x' \\ y' \end{bmatrix} + 1 = 0$$

$$\Leftrightarrow \quad [x' \ y'] \underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} x' \\ y' \end{bmatrix} + [-6 \ 2]\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + 1 = 0$$

$$[x' \ y'] \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix} + \frac{1}{\sqrt{2}}[-8 \ -4]\begin{bmatrix} x' \\ y' \end{bmatrix} + 1 = 0$$

$$\Leftrightarrow \quad 3x'^2 - y'^2 - \frac{8}{\sqrt{2}}x' - \frac{4}{\sqrt{2}}y' + 1 = 0$$

$$\Leftrightarrow \quad 3\left(x'^2 - 2\cdot\frac{4}{3\sqrt{2}}x' + \left(\frac{4}{3\sqrt{2}}\right)^2\right) - \underbrace{\left(\frac{4}{3\sqrt{2}}\right)^2}_{=\frac{8}{9}} - \left(y'^2 - 2\cdot\frac{2}{\sqrt{2}}y' + \left(\frac{2}{\sqrt{2}}\right)^2\right) + \underbrace{\left(\frac{2}{\sqrt{2}}\right)^2}_{=2} + 1 = 0$$

$$\Leftrightarrow \quad 3\Big(\underbrace{x' - \frac{4}{3\sqrt{2}}}_{x''}\Big)^2 - \Big(\underbrace{y' - \frac{2}{\sqrt{2}}}_{y''}\Big)^2 \underbrace{- \frac{8}{9} - 2 + 1}_{-\frac{17}{9}} = 0$$

$$\Leftrightarrow \quad 3x''^2 - y''^2 = \frac{17}{9}$$

$$\mathcal{H}_{\frac{17}{27}, \frac{17}{9}} : \quad \frac{x''^2}{\frac{17}{27}} - \frac{y''^2}{\frac{17}{9}} = 1$$

b) find the affine canonical equation using Lagrange's method and indicate the corresponding affine change of coordinates.

$$x^2 - 4xy + y^2 - 6x + 2y + 1 = 0$$

$$\Leftrightarrow \quad (x^2 - 4xy + 4y^2) - 4y^2 + y^2 - 6x + 2y + 1 = 0$$

$$\Leftrightarrow \quad (\underbrace{x - 2y}_{x_1})^2 - 3y^2 - 6x + 2y + 1 = 0 \qquad y_1 = y \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Leftrightarrow \quad x_1^2 - 3y_1^2 - 6x_1 + 2y_1 + 1 = 0$$

$$\Leftrightarrow \quad (x_1^2 - 6x_1 + 9) - 9 - \left(3y_1^2 + 2y_1\cdot\sqrt{3}\cdot\frac{1}{\sqrt{3}} + \frac{1}{3}\right) + \frac{1}{3} + 1 = 0$$

$$\Leftrightarrow \quad (\underbrace{x_1 - 3}_{x_2})^2 - (\underbrace{\sqrt{3}y_1 + \frac{1}{\sqrt{3}}}_{y_2})^2 - \frac{23}{3} = 0 \qquad \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} -3 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\iff \quad x_2^2 - y_2^2 = \frac{23}{3}$$

$$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{bmatrix} \sqrt{\frac{23}{3}} & 0 \\ 0 & \sqrt{\frac{23}{3}} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\iff \quad \underbrace{\left(\frac{x_2}{\sqrt{\frac{23}{3}}}\right)}_{x_3}^2 - \underbrace{\left(\frac{y_2}{\sqrt{\frac{23}{3}}}\right)}_{y_3}^2 = 1$$

$$\iff \quad x_3^2 - y_3^2 = 1$$ hyperbola

So $$\begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{bmatrix} \sqrt{\frac{23}{3}} & 0 \\ 0 & \sqrt{\frac{23}{3}} \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -3 \\ \frac{1}{\sqrt{3}} \end{bmatrix} \right)$$

$$= \sqrt{\frac{23}{3}} \begin{bmatrix} 1 & -2 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \sqrt{\frac{23}{3}} \begin{bmatrix} -3 \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$ (the affine change of coordinates)