"We learn more by looking for the answer to a question and not finding it

than we do from learning the answer itself."

– Lloyd Alexander, author

Problems in Linear Differential Equations

1 Introduction

1.1 Problem: exponential increasing

Check that the function $\varphi : \mathbb{R} \to \mathbb{R}$, given by the expression $\varphi(t) = 2e^{3t}$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x' = 3x, \quad x(0) = 2.$$

Represent the corresponding solution curve and describe its long-term behavior.

1.2 Problem: oscillations with constant amplitude

Check that the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \frac{1}{6}\sin t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x'' + x = 0, \quad x(0) = 0, \quad x'(0) = \frac{1}{6}.$$

Represent the corresponding solution curve and describe its long-term behavior.

Describe the motion of the simple planar gravity pendulum.

1.3 Problem: oscillations with exponentially decreasing amplitude

Check that the function $\varphi(t) = \frac{1}{6}e^{-2t}\cos t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x'' + 4x' + 5x = 0, \quad x(0) = \frac{1}{6}, \quad x'(0) = -\frac{1}{3}.$$

Represent this solution curve and describe its long-term behavior.

Describe the motion of the planar gravity damped pendulum.

1.4 Problem*: periodic response to a periodic force

Check that the function $\varphi(t) = \sin 2.2t - \sin 2t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x'' + 4.84x = -0.84\sin 2t, \quad x(0) = 0, \quad x'(0) = 0.2.$$

What are the minimal periods of the force (the function $f(t) = -0.84\sin 2t$) and, respectively, of the response (the function $\varphi(t) = \sin 2.2t - \sin 2t$)?

1.5 Problem*: quasiperiodic response to a periodic force

Check that the function $\varphi(t) = \sin\sqrt{6}t - \sin 2t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x'' + 6x = -2\sin 2t, \quad x(0) = 0, \quad x'(0) = \sqrt{6} - 2.$$

1.6 Problem: unbounded response to a bounded force

Check that the function $\varphi(t) = t\sin t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem:

$$x'' + x = 2\cos t, \quad x(0) = 0, \quad x'(0) = 0.$$

Represent this solution curve and describe its long-term behavior.

Describe the motion of the simple forced planar gravity pendulum.

1.7 Problem

Decide whether $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \cos t$ for all $t \in \mathbb{R}$, is a solution of one of the differential equations:

$$x' + x = 0, \quad x'' - x = 0, \quad x''' + x' = 0, \quad x^{(4)} + x'' = 0.$$

1.8 Problem: finding constant solutions for nonlinear equations

Find all constant solutions of the differential equations:

$$\begin{array}{llll} a) & x' = x - x^3, & b) & x' = \sin x, \\ c) & x' = \dfrac{x+1}{2x^2+5}, & d) & x' = x^2 + x + 1, \\ e) & x' = x + 4x^3, & f) & x' = -1 + x + 4x^3. \end{array}$$

1.9 Problem: particular solutions for linear nonhomogeneous equations

(i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be defined as:

$$x_1(1) = 1, \quad x_2(t) = t, \quad x_3(t) = t^2, \quad \forall t \in \mathbb{R}.$$

Prove that they are linearly independent in the real linear space $C(\mathbb{R})$.

(ii) Find a solution of the form $x(t) = at^2 + bt + c$, with $a, b, c \in \mathbb{R}$ of

$$x' - 5x = 2t^2 + 3.$$

1.10 Problem: particular solutions for linear nonhomogeneous equations

(i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be defined as:

$$x_1(1) = \cos t, \quad x_2(t) = \sin t, \quad x_3(t) = e^t, \quad \forall t \in \mathbb{R}.$$

Prove that they are linearly independent in the real linear space $C(\mathbb{R})$.

(ii) Find a solution of the form $x(t) = a\cos t + b\sin t + ce^t$, with $a, b, c \in \mathbb{R}$ of

$$x''' + x = -3\sin t + 2e^t.$$

1.11 Problem: linear homogeneous equations with constant coefficients

Find solutions of the form $x(t) = e^{rt}$ with $r \in \mathbb{R}$ of:

$$\begin{array}{rr} a) \quad x'' - 5x' + 6x = 0, & b) \quad x''' - 5x'' + 6x' = 0, \\ c) \quad x^{(4)} - 5x''' + 6x'' = 0, & d) \quad x'' + 9x = 0, \\ e) \quad x'' + x' + x = 0. & \end{array}$$

1.12 Problem: linear homogeneous equations with nonconstant coefficients

Find solutions of the form $x(t) = t^r$ with $r \in \mathbb{R}$ of:

$$\begin{array}{rr} a) \quad t^2x'' - 4tx' + 6x = 0, & b) \quad t^2x'' + tx' - x = 0, \\ c) \quad t^2x'' - x = 0, & d) \quad t^2x'' + x = 0, \\ e) \quad t^2x'' - tx' + x = 0. & \end{array}$$

1.13 Problem: integrating a differential equation

Integrate the following differential equations.

$$\begin{array}{l} a) \quad x' = 0, \quad b) \quad x' = 2t, \quad c) \quad x' = \sin t, \\ d) \quad x' = 2t + \sin t, \quad e) \quad x' = e^{2t}\cos t, \\ f) \quad x' = (t^2 - 5t + 7)\sin t, \quad g) \quad x' = e^{t^2}, \\ h) \quad x'' = -3, \quad i) \quad x''' = 0, \\ j) \quad tx' + x = 0, \quad k) \quad tx' + x = 1, \\ l) \quad t^3x' + 3t^2x = 1, \quad m) \quad 2xx' = -2t, \\ n) \quad x'e^t + xe^t = 0, \quad o) \quad x'e^{2t} + 2xe^{2t} = 0. \end{array}$$

1.14 Problem: the general solution of $x' = ax$

Fix $a \in \mathbb{R}^*$. Justify that the formula

$$x = ce^{at}, \quad c \in \mathbb{R}$$

describes the set of all solutions of the differential equation

$$x' = ax.$$

Hint: Consider a new unknown $y = xe^{-at}$ and check that $y' = 0$.

1.15 Problem*: the general solution of $x'' = a^2x$

Fix $a \in \mathbb{R}^*$. Justify that the formula

$$x = c_1e^{at} + c_2e^{-at}, \quad c_1, c_2 \in \mathbb{R}$$

describes the set of all solutions of the differential equation

$$x'' = a^2x.$$

Hint: Consider a new unknown $y = x' - ax$ and check that $y' = -ay$.

1.16 Problem*: the general solution of $x'' = -x$

Justify that the formula

$$x = c_1 \cos t + c_2 \sin t, \quad c_1, c_2 \in \mathbb{R}$$

describes the set of all solutions of the differential equation

$$x'' = -x.$$

Hint: Consider two new unknowns $y = x\cos t - x' \sin t$, $z = x' \cos t + x \sin t$ and check that $y' = z' = 0$.

2 Understanding the Fundamental Theorems

2.1 Problem: what is an IVP?

How many solutions do each of the following problems have?

$$\begin{aligned}
&a) \quad x'' + t^2 x = 0, \quad x(0) = 0,\\
&b) \quad x'' + t^2 x = 0, \quad x(0) = 0, \quad x'(0) = 0,\\
&c) \quad x'' + t^2 x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1.
\end{aligned}$$

2.2 Problem.

We consider $x' + \frac{1}{t^2}x = 0$, $t \in (0, \infty)$. First check that $x = e^{\frac{1}{t}}$ is a solution. Then find the general solution.

2.3 Problem.

We consider $x' + 2x = 0$. First find a solution of the form e^{rt}, with $r \in \mathbb{R}$. Then find the general solution.

2.4 Problem.

We consider $x'' - 2x = 0$. First find two solutions of the form e^{rt}, with $r \in \mathbb{R}$. Then find the general solution.

2.5 Problem.

Find the general solution of each of the following equations, looking first for some solutions of the form $x = t^r$, with $r \in \mathbb{R}$.

$$\begin{aligned}
&a) \quad t^2 x'' - 8tx' + 20x = 0, \quad t \in (0, \infty),\\
&b) \quad t^2 x'' - 6x = 0, \quad t \in (0, \infty),\\
&c) \quad t^2 x'' + tx' + x = 0, \quad t \in (0, \infty).
\end{aligned}$$

2.6 Problem.

Determine whether the given functions form the general solution of the differential equation:

a) Is $x = c_1 e^t + c_2 e^{-t}$, $\quad c_1, c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$?

b) Is $x = c_1 \cosh t + c_2 \sinh t$, $\quad c_1, c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$?

Recall that $\cosh t = \frac{e^t + e^{-t}}{2}$ and $\sinh t = \frac{e^t - e^{-t}}{2}$.

2.7 Problem.

a) Verify that $y_1 = x$ and $y_2 = e^{-2x}$ are solutions of $(2x+1)y''+4xy'-4y = 0$.

b) Find the solution of the Initial Value Problem:

$$(2x + 1)y'' + 4xy' - 4y = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

2.8 Problem.

Check that e^{t^2} is a solution of $x' - 2tx = 0$.
Find a constant solution of $x' - 2tx = -2t$.
Find the general solution of $x' - 2tx = -2t$ and, respectively, of $x' - 2tx = t$.

2.9 Problem.

a) Find a particular solution of the form $x_p = ae^t$ (with $a \in \mathbb{R}$) for the equation $x' - 2x = e^t$.

b) Find a particular solution of the form $x_p = be^{-t}$ (with $b \in \mathbb{R}$) for the equation $x' - 2x = e^{-t}$.

c) Using the Superposition Principle, and parts (a) and (b), find a particular solution for the equation $x' - 2x = 5e^t - 3e^{-t}$.

d) Find the general solution of $x' - 2x = 5e^t - 3e^{-t}$.

3 First Order Linear Differential Equations

3.1 Problem.

Find the general solution of the following first-order linear homogeneous equations using the separation of variables method:

$$a)\quad x' - 2tx = 0, \quad t \in \mathbb{R}, \qquad b)\quad x' + \frac{1}{t}x = 0, \quad t \in (0, \infty),$$
$$c)\quad x' - \frac{1}{t}x = 0, \quad t \in (0, \infty), \qquad d)\quad x' - \frac{3}{t}x = 0, \quad t \in (0, \infty).$$

3.2 Problem.

Find the general solution of the following first-order linear nonhomogeneous equations using the separation of variables method and the Lagrange method:

$$a)\quad x' + \frac{1}{t^2}x = 1 + \frac{1}{t}, \quad t \in (0, \infty),$$
$$b)\quad x' + 2tx = e^{-t^2 - t}, \quad t \in \mathbb{R},$$
$$c)\quad x' + \frac{2t}{1+t^2}x = 3, \quad t \in \mathbb{R},$$
$$d)\quad x' - \frac{2}{t}x = t^2 \sin(2t) - \frac{4}{t^3}, \quad t \in (0, \infty).$$

3.3 Problem.

Find in two ways the general solution of

$$a)\quad x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1}, \quad t \in (0, \infty), \qquad b)\quad x' - x = e^{t-1}, \quad t \in \mathbb{R}.$$

3.4 Problem.

Find the general solution of the following equations by reducing their order:

$$a)\quad x''' - x'' = 0, \quad t \in \mathbb{R}, \qquad b)\quad x'' = \frac{2}{t}x', \quad t \in (0, \infty).$$

3.5 Problem.

Let $f : [0, \infty) \to \mathbb{R}$ be a C^1 function that verifies

$$f'(t) \leq 5f(t), \quad t \in [0, \infty), \quad \text{and} \quad f(0) = 2.$$

Prove that

$$f(t) \leq 2e^{5t}, \quad t \in [0, \infty).$$

4 Linear Homogeneous Differential Equations with Constant Coefficients

4.1 Problem.

Find the general solution of each of the following differential equations:

a) $x' - 3x = 0,$	*b*) $x' + 6x = 0,$
c) $x'' + x' - 6x = 0,$	*d*) $x'' - 6x = 0,$
e) $x'' + 4x' + 4x = 0,$	*f*) $x'' = 0,$
g) $x'' + x' + x = 0,$	*h*) $x'' + 9x = 0,$
i) $x''' = 0,$	*j*) $x^{(n)} = 0,$
k) $x''' - 6x'' + 11x' - 6x = 0,$	*l*) $x^{(4)} - x = 0.$

4.2 Problem.

Find the linear homogeneous differential equation with constant coefficients and minimal order that has the following functions as solutions. Write the general solution of the differential equation found.

a) e^{-3t} and e^{5t},	*b*) $5e^{-3t}$ and $-3e^{5t}$,	*c*) $5e^{-3t} - 3e^{5t}$,
d) $5te^{-3t}$ and $-3e^{5t}$,	*e*) $5e^{-3t}$ and $-3te^{5t}$,	*f*) $(5 - 3t)e^{-3t}$,
g) $(5 - 3t + 2t^2)e^{-3t}$,	*h*) $\sin 3t$,	*i*) $t - \sin 3t$,
j) $-t \sin 3t$,	*k*) $e^{5t} \sin 3t$,	*l*) $e^{-3t} \sin 3t$,
m) $t^7 + 1$,	*n*) $5t - 3e^{5t}$,	*o*) $(t - 1)^2$,
p) $2 \cos^2 t$,	*r*) $\sin^2 t$,	*s*) $(e^t)^2$.

4.3 Problem.

Decide whether the following statements are true or false:

a) There exists a linear homogeneous differential equation with constant coefficients of order 7 that has as solutions $(t^3 + 2t^4) \cos 2t$ and te^{-t}.

b) There exists a linear homogeneous differential equation with constant coefficients that has as solution $1/t$.

c) There exists a linear homogeneous differential equation with constant coefficients that has as solution e^{t^2}.

d) There exists a linear homogeneous differential equation with constant coefficients that has as solution $t/(1 + t^2)$.

4.4 Problem.

Find the solution for each of the following IVPs (Initial Value Problems), where $\eta, \lambda \in \mathbb{R}$ are fixed parameters.

$$\begin{aligned}
&a) \quad x' = 3x, \quad x(0) = 2,\\
&b) \quad x'' + x = 0, \quad x(0) = 0, \quad x'(0) = \frac{1}{6},\\
&c) \quad x'' + \pi^2 x = 0, \quad x(0) = 0, \quad x'(0) = \eta,\\
&d) \quad x'' + \lambda x' = 0, \quad x(0) = 0, \quad x'(0) = \eta.
\end{aligned}$$

4.5 Problem.

Find all solutions for each of the following BVPs (Boundary Value Problems):

$$\begin{aligned}
&a) \quad x'' + x = 0, \quad x(0) = x(\pi) = 0,\\
&b) \quad x'' + x = 0, \quad x(0) = x(1) = 0,\\
&c) \quad x'' + \pi^2 x = 0, \quad x(0) = x(1) = 0,\\
&d) \quad x'' + \pi^2 x = 0, \quad x(0) = x(2) = 0.
\end{aligned}$$

4.6 Problem.

Let $\mu, \omega > 0$. Prove that each solution of $x'' + \mu x' + \omega^2 x = 0$ tends to 0 as $t \to \infty$.

4.7 Problem.

Let $\mu, \omega > 0$. Prove that all solutions of $x'' + \omega^2 x = 0$ are periodic and write their minimal period. Justify that any non-null solution of $x'' + \mu x' + \omega^2 x = 0$ is neither periodic, nor bounded on $\mathbb{R}$.

4.8 Problem.

Find $\lambda > 0$ such that there exist non-null solutions of

$$x'' + \lambda x = 0, \quad x(-\pi) = x(\pi), \quad x'(-\pi) = x'(\pi).$$

4.9 Problem.

Find $\lambda > 0$ such that there exist non-null solutions of

$$x'' + \lambda x = 0, \quad x(0) = x(\pi) = 0.$$

5 Linear Nonhomogeneous Differential Equations with Constant Coefficients

5.1 Problem.

Let $A_1, A_2, \lambda, a, b, \alpha, \beta \in \mathbb{R}$ and $f \in C(\mathbb{R})$. We consider the differential equation and its characteristic equation

$$x'' + A_1 x' + A_2 x = f(t), \quad r^2 + A_1 r + A_2 = 0.$$

Prove the following propositions.

a) Let $f(t) = a$ be constant. There is a constant particular solution $x_p = b$ if and only if $A_2 \neq 0$.

b) Let $f(t) = ae^{\lambda t}$. There is a particular solution of the form $x_p = be^{\lambda t}$ if and only if λ is not a root of the characteristic equation.

c) Let $f(t) = ae^{\lambda t}$. If λ is a simple root of the characteristic equation then there is a particular solution of the form $x_p = bte^{\lambda t}$.

c) Let $f(t) = ae^{\lambda t}$. If λ is a double root of the characteristic equation then there is a particular solution of the form $x_p = bt^2 e^{\lambda t}$.

d) Let $f(t) = a_1 e^{\alpha t}\cos(\beta t) + a_2 e^{\alpha t}\sin(\beta t)$. If $\alpha + i\beta$ is not a root of the characteristic equation then there is a particular solution of the form $x_p = b_1 e^{\alpha t}\cos(\beta t) + b_2 e^{\alpha t}\sin(\beta t)$.

e) Let $f(t) = a_1 e^{\alpha t}\cos(\beta t) + a_2 e^{\alpha t}\sin(\beta t)$. If $\alpha + i\beta$ is a root of the characteristic equation then there is a particular solution of the form $x_p = t(b_1 e^{\alpha t}\cos(\beta t) + b_2 e^{\alpha t}\sin(\beta t))$.

5.2 Problem.

Decide whether the following statements are true or false:

a) All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim\limits_{t\to\infty} x(t) = 1$.

b) The solution of the IVP $x''+4x = 1$, $x(0) = \frac{5}{4}$, $x'(0) = 0$ satisfies $x(\pi) = \frac{5}{4}$.

c) The equation $x' = 3x + t^3$ admits a polynomial solution. (Hint: Look for a polynomial solution of degree 3.)

5.3 Problem.

Let $\lambda \in \mathbb{R}$ be a parameter. Find the general solution of $x'' - x = e^{\lambda t}$ knowing that, depending on λ, it has a particular solution either of the form $ae^{\lambda t}$ or of the form $ate^{\lambda t}$.

5.4 Problem.

Let $\omega > 0$ be a parameter and denote $\varphi(\cdot, \omega)$ the solution of the IVP:

$$x'' + x = \cos(\omega t), \quad x(0) = 0, \quad x'(0) = 0.$$

a) When $\omega \neq 1$, find a solution of the form $x_p(t) = a\cos(\omega t) + b\sin(\omega t)$ for $x'' + x = \cos(\omega t)$. Determine the real coefficients a and b.

b) Find a solution of the form $x_p(t) = t(a\cos t + b\sin t)$ for $x'' + x = \cos t$.

c) Find $\varphi(\cdot, \omega)$ for any $\omega > 0$.

d) Prove that $\lim\limits_{\omega \to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

5.5 Problem.

Let $\alpha > 0$ and $\varphi(\cdot, \alpha)$ be the solution of the IVP:

$$x'' - 4x = e^{\alpha t}, \quad x(0) = 0, \quad x'(0) = 0.$$

a) When $\alpha \neq 2$, find a solution of the form $x_p(t) = ae^{\alpha t}$ for $x'' - 4x = e^{\alpha t}$. Determine the real coefficient a.

b) Find a solution of the form $x_p(t) = te^{2t}$ for $x'' - 4x = e^{2t}$.

c) Find $\varphi(\cdot, \alpha)$ for any $\alpha > 0$.

d) Prove that $\lim\limits_{\alpha \to 2} \varphi(t, \alpha) = \varphi(t, 2)$ for each $t \in \mathbb{R}$.

6 Linear Homogeneous Planar Systems with Constant Coefficients

6.1 Problem.

Let $A \in M_2(\mathbb{R})$. Using both the characteristic equation method and reduction to a second-order equation, find the general solution of the system $X' = AX$ in each of the following cases. Also, find a fundamental matrix solution and, finally, find e^{tA}, the principal matrix solution.

$a)\quad A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$

$b)\quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$c)\quad A = \begin{bmatrix} 4 & -5 \\ 1 & -2 \end{bmatrix}$

$d)\quad A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$

$e)\quad A = \begin{bmatrix} 5 & -9 \\ 2 & -1 \end{bmatrix}$

$f)\quad A = \begin{bmatrix} 5 & -3 \\ 8 & -6 \end{bmatrix}$

$g)\quad A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$

$h)\quad A = \begin{bmatrix} 1 & -4 \\ 2 & 5 \end{bmatrix}$

$i)\quad A = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$

$j)\quad A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

$k)\quad A = \begin{bmatrix} 0 & 4 \\ 5 & 1 \end{bmatrix}$

$l)\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$m)\quad A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$

$n)\quad A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$

7 Exam Problems

7.1 Problem.

Let $f \in C(\mathbb{R})$ and $\eta \in \mathbb{R}$. Write the solution of the IVP:

$$x' + x = f(t), \quad x(0) = \eta.$$

7.2 Problem.

Write the general solution of the following equations:

$$\begin{aligned} &a) \quad x' - 3t^2 x = t^3, \\ &b) \quad x' - 3t^2 x = f(t), \quad \text{where } f \in C(\mathbb{R}). \end{aligned}$$

7.3 Problem.

Find the solution of the IVP:

$$\begin{aligned} &a) \quad x' + 2tx = t, \quad x(0) = 0, \\ &b) \quad x' + tx = 1, \quad x(0) = 0, \\ &c) \quad x'' + 4x = 1, \quad x(0) = 1, \quad x'(0) = 0. \end{aligned}$$

7.4 Problem.

Let $a \in \mathbb{R}^*$. Find the general solution of:

$$x' - ax = at - 1.$$

7.5 Problem.

Using Euler's formula, compute:

$$e^{it}, \quad e^{i\pi}, \quad e^{i\pi/2}, \quad e^{(-1+i)t}.$$

7.6 Problem.

Find the linear homogeneous differential equation with constant coefficients that has the general solution:

$$x(t) = c_1 e^{-t} \cos t + c_2 e^{-t} \sin t, \quad c_1, c_2 \in \mathbb{R}.$$

7.7 Problem.

Find a linear homogeneous differential equation that has as solution the function:

$$(2te^t)^2.$$

7.8 Problem.

Find a linear homogeneous differential equation that has as solution the function:

$$1 + t(1 + e^{-t}).$$

7.9 Problem.

Let $k, \eta \in \mathbb{R}$ be fixed parameters. Find the solution of the IVP:

$$x' = k(21 - x), \quad x(0) = \eta.$$

7.10 Problem.

We consider the differential equation:

$$x'' - x = te^{-2t}.$$

a) Find a particular solution of the form $x_p(t) = (at + b)e^{-2t}$.

b) Find the general solution.

c) Find the solution satisfying the initial conditions $x(0) = 0, \quad x'(0) = 0$.

7.11 Problem.

Let $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined for each $x \in C^2(\mathbb{R})$ by:

$$L(x)(t) = x''(t) - 2x'(t) + x(t), \quad t \in \mathbb{R}.$$

a) Prove that L is a linear map. What is the dimension of its kernel?

b) Find the general solution of $x'' - 2x' + x = \cos 2t$, knowing that it has a particular solution of the form $x_p = a\cos 2t + b\sin 2t$.

c) Let $f_1(t) = e^{2t}$ and $f_2(t) = e^{-2t}$.
Find a particular solution $x_p = af_1 + bf_2$ of $L(x) = 3f_1 + 5f_2$.

7.12 Problem.

Let $a_1, a_2 \in C(\mathbb{R})$ and define $L : C^2(\mathbb{R}) \to C(\mathbb{R})$ for each $x \in C^2(\mathbb{R})$ by:

$$L(x)(t) = x''(t) + a_1(t)x'(t) + a_2(t)x(t), \quad t \in \mathbb{R}.$$

Prove that L is a linear map.

Let $\Phi : \ker L \to \mathbb{R}^2$ be defined for each $x \in \ker L$ by:

$$\Phi(x) = (x(0), x'(0)).$$

Prove that Φ is an isomorphism of linear spaces. What is the dimension of $\ker L$?

7.13 Problem.

a) State the Fundamental Theorem for linear homogeneous second-order differential equations.

b) Decide whether the following statement is true or false, and justify your answer:

> "The general solution of the differential equation $x'' - 9x = 0$ is $x(t) = c_1 \cosh 3t + c_2 \sinh 3t$, where c_1, c_2 are arbitrary real constants."

7.14 Problem.

Let $a \in \mathbb{R}$ be a fixed parameter. Find the general solution of the differential equation:

$$x'' - x = e^{at},$$

knowing that, depending on a, it has a particular solution either of the form $x_p = be^{at}$, or of the form $x_p = bte^{at}$.

7.15 Problem.

Let $a > 0$ and $b \in \mathbb{R}$ be fixed parameters. Write the general solution of:

$$x'' - a^2 x = e^{bt}.$$

7.16 Problem.

Let $a \in \mathbb{R} \setminus \{0, 1\}$ be a fixed parameter. Find the general solution of:

$$x' + ax = -at + 1,$$

$$x'' - ax' + (a-1)x = 0.$$

7.17 Problem.

Find the solution of the IVP:

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Describe the motion of a pendulum governed by this IVP. Determine after how much time the pendulum returns to its initial position, i.e. find $T > 0$ such that $\theta(T) = \frac{\pi}{2}$ and $\theta'(T) = 0$.

7.18 Problem.

We consider the differential equation:

$$x'' + 4x = \cos 2t.$$

a) Find a particular solution of the form $x_p = t(a\cos 2t + b\sin 2t)$.

b) Find the general solution.

c) Describe the motion of a spring-mass system governed by this equation.

7.19 Problem.

Let $k, m > 0$ be fixed parameters. Describe the motion of a spring-mass system whose equation is:

$$x'' + \frac{k}{m}x = 0.$$

7.20 Problem.

Find the solution of the IVP:

$$x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = -2.$$

Represent the corresponding integral curve and describe its long-term behavior.

7.21 Problem.

Let $a > 0$ be a fixed parameter. Consider the differential equation:

$$x'' + 2ax' + 4x = 0.$$

Write the general solution and describe the long-term behavior (in the future) of a nonnull solutions Discuss with respect to the parameter a.

7.22 Problem.

a) Find the general solution of the differential equation:

$$\varphi'' + \frac{9}{4}\varphi = 0.$$

b) Decide whether the following statement is true or false:

"All solutions of $\varphi'' + \frac{9}{4}\varphi = 0$ are periodic with a period $T = 4\pi$."

7.23 Problem.

We say that a differential equation exhibits resonance when all its solutions are unbounded. For what values of the mass m will the following equation exhibit resonance?

$$mx'' + 25x = 12\cos(36\pi t).$$

7.24 Problem.

Find the general solution of:

$$\ddot{\theta} + \dot{\theta} + \theta = 0.$$

Prove that any solution verifies:

$$\lim_{t\to\infty} \theta(t) = 0.$$

7.25 Problem.

Let $\alpha \in \mathbb{R}$ be a fixed parameter. Describe the long-term behavior (in the future) of the function:

$$x(t) = e^{\alpha t}\cos 2t, \quad t \in \mathbb{R}.$$

7.26 Problem.

Let $\gamma > 0$. Decide if the following statement is true:

"Any solution of $x'' + \gamma x' + 9x = 0$ satisfy $\lim_{t\to\infty} x(t) = 0$."

7.27 Problem.

Find $\gamma \in \mathbb{R}$ such that any solution of the following differential equation is periodic.

$$x'' + \gamma x' + 9x = 0.$$

7.28 Problem.

Consider the differential equation:

$$t^2x'' + 2tx' - 2x = 0, \quad t \in (0, \infty).$$

a) Find solutions of the form $x(t) = t^r$ where $r \in \mathbb{R}$ is to be determined.

b) Specify the type of the equation and find its general solution.

c) Find the solution of the IVP:

$$t^2x'' + 2tx' - 2x = 0, \quad x(1) = 0, \quad x'(1) = 1.$$

7.29 Problem.

Find the general solution of the differential equation:

$$x^2u'' - 6xu' + 10u = 0.$$

where the unknown function is denoted by u of independent variable x. Hint: Look for solutions of the form $u = x^r$ with $r \in \mathbb{R}$.

7.30 Problem.

Find the solution of the following Initial Value Problem:

$$y'' - \frac{y'}{x} = x^2, \quad y(2) = 0, \quad y'(2) = 4.$$

7.31 Problem.

a) Find a particular solution of the form $x_p(t) = at^2e^t$ for:

$$x'' - 2x' + x = e^t.$$

b) Find a constant solution for:

$$x'' - 2x' + x = 5.$$

c) Find the general solution of the differential equation:

$$x'' - 2x' + x = 10 + 5e^t.$$

d) Find the solution of the IVP:

$$x'' - 2x' + x = 5, \quad x(0) = 5, \quad x'(0) = 0.$$

7.32 Problem.

Decide if the following statement is true:

"The following Boundary Value Problem has at least one solution:

$$x'' + 9x = 0, \quad x(0) = 0, \quad x(\pi) = 9."$$

7.33 Problem.

We use the notation $L(x) = x'' + 25x$.

a) Find the solution of the IVP:

$$L(x) = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

Represent this solution curve and describe its long-term behavior (in the future).

b) Let $\varphi_1(t) = t\cos(5t)$ and $\varphi_2(t) = t\sin(5t)$. Compute:

$$L(5), \quad L(\varphi_1), \quad L(\varphi_2).$$

c) Find a constant solution for:

$$L(x) = 5.$$

d) Find the general solution of:

$$L(x) = 25 - 25\sin(5t).$$

7.34 Problem.

Consider the differential equation:

$$x' + \frac{1}{t^2}x = 0, \quad t \in (-\infty, 0).$$

a) Check that $x = e^{1/t}$ is a solution of this equation.

b) Find the solution of the IVP:

$$x' + \frac{1}{t^2}x = 0, \quad x(-1) = 1.$$

c) Find the general solution of:

$$x' + \frac{1}{t^2}x = 1 + \frac{1}{t}, \quad t \in (-\infty, 0).$$

7.35 Problem.

Find in two ways the general solution of:

$$x' + \frac{1}{t}x = -2, \quad t \in (0, \infty).$$

7.36 Problem.

Consider the differential equation:

$$x' + \frac{2t}{1+t^2}x = 3.$$

a) Find its general solution.

b) Find its solution satisfying $x(0) = 1$. Is this solution bounded? What about other solutions?

7.37 Problem.

We consider the system $X' = A(t)X + f(t)$, where:

$$A(t) = \begin{bmatrix} 5 & 4e^t \\ -7e^{-t} & -7 \end{bmatrix}, \quad f(t) = \begin{bmatrix} -t \\ 1 \end{bmatrix}.$$

a) Prove that $U(t)$ is a fundamental matrix solution for $X' = A(t)X$:

$$U(t) = \begin{bmatrix} -4e^{-2t} & e^t \\ 7e^{-3t} & -1 \end{bmatrix}.$$

b) Find the principal matrix solution of $X' = A(t)X$.

c) Find the solutions of $X' = A(t)X$ satisfying $X(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

d) Find the solution of $X' = A(t)X + f(t)$ satisfying $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

7.38 Problem.

Let $n \geq 1$ and $A \in M_n(\mathbb{R})$. Assume that A has an eigenvalue $\lambda \in \mathbb{C}$ and two linearly independent vectors $v_1, v_2 \in \mathbb{C}^n$ such that:

$$(A - \lambda I_n)v_1 = 0, \quad (A - \lambda I_n)v_2 = v_1.$$

Prove that:

$$\varphi_1(t) = e^{\lambda t}v_1, \quad \varphi_2(t) = e^{\lambda t}(tv_1 + v_2)$$

are two linearly independent solutions of the linear system $X' = AX$.

7.39 Problem.

Show that any solution of the system $X' = AX$ satisfies $\lim_{t\to\infty} X(t) = 0$, in each of the following cases:

$$a) \quad A = \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 2 & -5 \end{bmatrix},$$

$$b) \quad A = \begin{bmatrix} -5 & 3 \\ -3 & 1 \end{bmatrix},$$

$$c) \quad A = \begin{bmatrix} -5 & 9 \\ -2 & 1 \end{bmatrix}.$$

7.40 Problem.

Find the general solution and then show that any solution of the system $X' = AX$ is bounded for $t \in [0, \infty)$, in each of the following cases:

$$a) \quad A = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{bmatrix},$$

$$b) \quad A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix}.$$

7.41 Problem.

Let $A = \begin{bmatrix} -5 & 1 \\ -1 & 1 \end{bmatrix}$. Prove that any solution of the system $X' = AX$ satisfies:

$$\lim_{t \to \infty} X(t) = 0. \tag{1}$$

7.42 Problem.

Let $A \in M_2(\mathbb{R})$.

a) Let $\eta \in \mathbb{R}^2$. Write a representation formula for the solution of the IVP:

$$X' = AX, \quad X(0) = \eta. \tag{2}$$

b) Using the definition of the matrix exponential, compute:

$$e^{t\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}, \quad e^{t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}, \quad e^{t\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}.$$

c) Let $\lambda \in \mathbb{R}$ be an eigenvalue of A and $v_1 \in \mathbb{R}^2$ be an eigenvector of A corresponding to λ. Assume there exists $v_2 \in \mathbb{R}^2$ such that:

$$(A - \lambda I_2)v_2 = v_1. \tag{3}$$

Prove that $\{v_1, v_2\}$ are linearly independent and that:

$$\varphi_1(t) = e^{\lambda t} v_1,$$
$$\varphi_2(t) = e^{\lambda t}(t v_1 + v_2)$$

are solutions of the system $X' = AX$.

d) Take:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}. \tag{4}$$

Apply (c) to find two solutions of the system $X' = AX$. Write the matrix solution whose columns are these solutions. Is this a fundamental matrix solution? Find the principal matrix solution. Compute e^{tA}.