Sunday, June 15, 2025 3:59 PM

$$\begin{cases} \dot{x} = -y(x^2+y^2) \\ \dot{y} = x(x^2+y^2) \end{cases} \qquad \varphi(t,1,0) = (\cos t, \sin t), \quad \forall t \in \mathbb{R}$$

$\begin{cases} \dot{X} = f(X) \\ \eta \in \mathbb{R}^n \end{cases}$ $f : \mathbb{R}^n \to \mathbb{R}^n$ C^1

$\boxed{\text{The IVP } \begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases} \text{ has a unique sol.}}$ denoted by $\varphi(t,\eta)$

$\boxed{n=2}$

$\varphi(t,1,0)$ denotes the unique sol. of the IVP (1) $\begin{cases} \dot{x} = -y(x^2+y^2) \\ \dot{y} = x(x^2+y^2) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$.

We have to check that $x = \cos t$, $y = \sin t$ verifies the IVP (1) $\Longleftrightarrow$

$$\begin{cases} (\cos t)' = -\sin t(\cos^2 t + \sin^2 t), & \forall t \in \mathbb{R} \\ (\sin t)' = \cos t(\cos^2 t + \sin^2 t), & \forall t \in \mathbb{R} \\ \cos 0 = 1 \\ \sin 0 = 0 \end{cases} \Longleftrightarrow \begin{cases} -\sin t = -\sin t \cdot 1, & \forall t \in \mathbb{R} \\ \cos t = \cos t \cdot 1, & \forall t \in \mathbb{R} \\ 1 = 1 \\ 0 = 0 \end{cases}$$

TRUE.

$\varphi(t,2,0) = (2\cos(4t), 2\sin(4t))$ $\forall t \in \mathbb{R}$.

$x = 2\cos(4t)$, $y = 2\sin(4t)$

$$[2\cos(4t)]' = -2\sin(4t)\underbrace{\left(4\cos^2(4t) + 4\sin^2(4t)\right)}_{4}$$

TRUE

$\varphi(t,3,0) =$

$x(0) = 3$ $\quad x = 3\cos[\,]$, $y = 3\sin[\,]$

$$-3\sin[\,] \cdot \underbrace{[\,]'}_{9} = -3\sin[\,] \cdot \underbrace{(9\cos^2 + 9\sin^2)}_{9}$$

$x = 3\cos(9t)$ $\quad y = 3\sin(9t)$

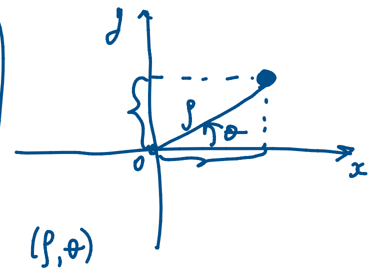
$\varphi(t,3,0) = (3\cos(9t), 3\sin(9t))$. $\forall t \in \mathbb{R}$

$$\varphi(t,3,0) = \big(3\cos(9t),\ 3\sin(9t)\big). \qquad t \in \mathbb{R}$$

$$\begin{cases} \dot x = -y(x^2+y^2) \\ \dot y = x(x^2+y^2) \end{cases}$$

$$\big(\rho(t),\ \theta(t)\big)$$

$$\boxed{\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \iff \begin{cases} \rho^2 = x^2+y^2 \\ \tan\theta = \dfrac{y}{x} \end{cases}}$$



(ρ,θ)

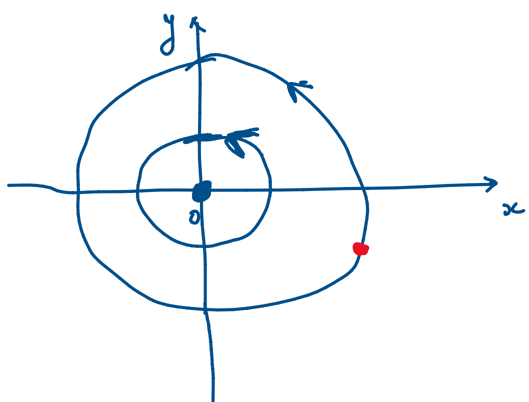
$$\begin{cases} 2\rho\dot\rho = 2x\dot x + 2y\dot y \quad \Big|:2 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot y x - y\dot x}{x^2} \end{cases}$$

$$\begin{cases} \rho\dot\rho = -xy(x^2+y^2) + xy(x^2+y^2) \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{x^2(x^2+y^2) + y^2(x^2+y^2)}{x^2} \end{cases}$$

$$\begin{cases} \dot\rho = 0 \\ \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\rho^4}{\rho^2\cos^2\theta} \end{cases}$$

$$\begin{cases} \dot\rho = 0 \\ \dot\theta = \rho^2 \end{cases} \qquad \begin{cases} \rho = c_1,\ c \in \mathbb{R} \\ \theta = c_1^2 t + c_2 \end{cases}$$

$\dot\theta = c^2 > 0$



$\rho(t) = 1$, $\theta(t) = t + c_2$,

$\rho = 2$

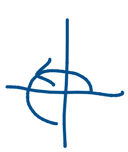
$\vdots$

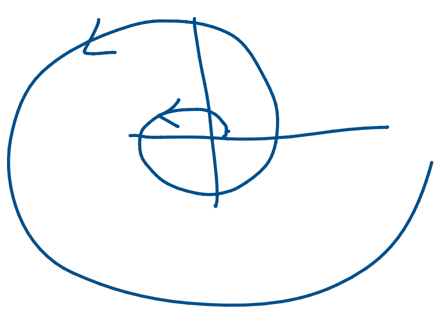
pct. de ech.

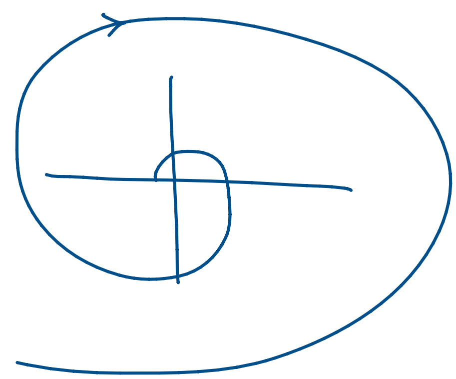
$$\begin{cases} -y(x^2+y^2) = 0 \\ x(x^2+y^2) = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$$

the only ech. point is $(0,0)$.

$\dot\rho > 0$







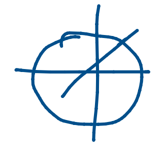
$$\begin{cases} x = c_1\cos(c_1^2 t + c_2) \\ y = c_1\sin(c_1^2 t + c_2) \end{cases}$$

$c_1, c_2 \in \mathbb{R}$ the gen. sol. of the system.

any sol. is periodic.

$\varphi(t,\eta,0)$ $\quad x(0)=\eta\ ,\ y(0)=0$

$$\begin{cases} x(0) = c_1\cos(c_2) \\ y(0) = c_1 \sin(c_2) \end{cases} \Rightarrow \begin{cases} c_1\cos(c_2)=\eta \\ c_1\sin(c_2)=0 \end{cases} \quad \begin{matrix} c_2=0 \\ c_1=\eta \end{matrix}$$



$$\varphi(t,\eta,0) = \left(\eta\cos(\eta^2 t),\ \eta\sin(\eta^2 t)\right).$$

$$\begin{cases} c_2=\pi \\ c_1=-\eta \\ x=-\eta\cos(\eta^2 t+\pi) = \eta\cos(\eta^2 t) \\ y=-\eta\sin(\eta^2 t+\pi) = \eta\sin(\eta^2 t) \end{cases}$$

$$\ddot{x} = f(\dot{x},x) \qquad \begin{cases} x \\ y=\dot{x} \end{cases} \Rightarrow \quad \dot{y}=\ddot{x}=f(\dot{x},x)=f(y,x)$$

$$\begin{cases} \dot{x}=y \\ \dot{y}=f(y,x) \end{cases} \qquad X=\begin{pmatrix} x \\ y \end{pmatrix} \qquad F(X)=F(x,y)=\begin{pmatrix} y \\ f(y,x) \end{pmatrix}$$

$$\Leftrightarrow \quad \dot{X}=F(X). \qquad F:\mathbb{R}^2\to\mathbb{R}^2$$

$\theta''+w^2\sin\theta=0$ second order nonlinear d.e.

$$\begin{cases} x=\theta \\ y=\theta' \end{cases} \qquad \begin{cases} \dot{x}=y \\ \dot{y}=-w^2\sin x \end{cases} \qquad \frac{dy}{dx} = \frac{-w^2\sin x}{y}$$

$$\begin{cases} y=0 \\ -w^2\sin x=0. \end{cases} \Leftrightarrow \begin{cases} y=0 \\ x=k\pi,\ k\in\mathbb{Z} \end{cases} \Leftrightarrow \quad (k\pi,0),\ k\in\mathbb{Z}.$$

$$\left.\begin{matrix} y=\theta' \Rightarrow y'=\theta'' \\ \theta''=-w^2\sin\theta \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y'=-w^2\sin\theta \\ \theta=x \end{matrix}\right\} \Rightarrow \quad y'=-w^2\sin x.$$

$f:\mathbb{R}\to\mathbb{R}$, $\quad f(x)=2x(1-x)=2x-2x^2$

fixed points: $\quad f(x)=x \Leftrightarrow 2x-2x^2=x \Leftrightarrow 2x^2-x=0 \Leftrightarrow x(2x-1)=0$

... points $\quad x_1^*=0$ and $x_2^*=\frac{1}{2}$.

fixed points: $f(x) = x \iff 2x$ ...

two fixed points $\eta_1^* = 0$ and $\eta_2^* = \frac{1}{2}$.

the linearization method: $f'(x) = 2 - 4x$ $\quad f'(0) = 2, \; f'(\frac{1}{2}) = 0$

$|f'(\frac{1}{2})| < 1 \Rightarrow \frac{1}{2}$ is an attractor (is a super-attractor).

$f(x) = 2x(1-x) = 2x - 2x^2$

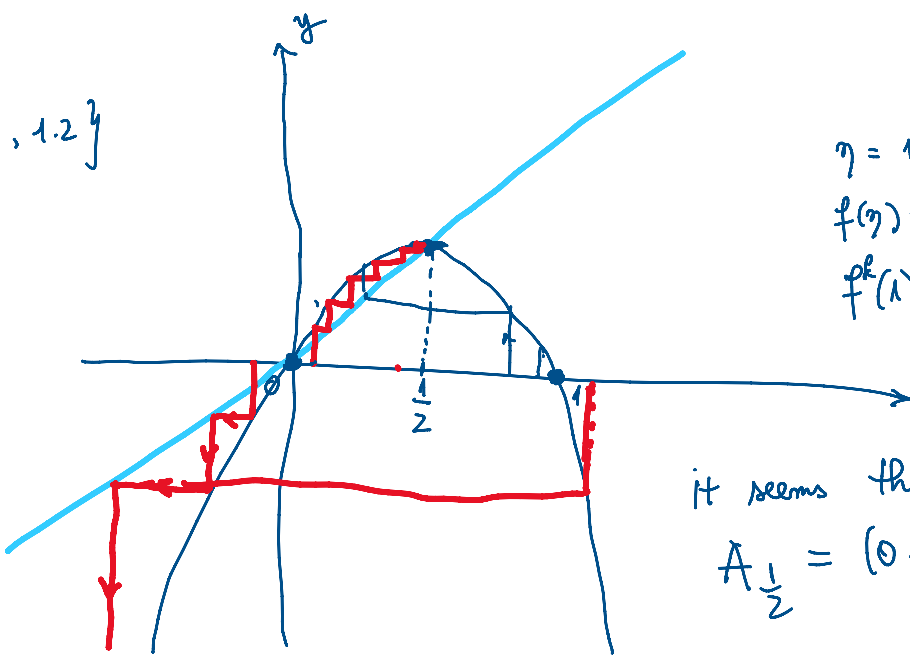
$f(\frac{1}{2}) = \frac{1}{2}$

$\eta \in \{-0.2, \; 0.2, \; 0.4, \; 1.2\}$

$\frac{1}{2}$ is an attractor fixed point

$A_{\frac{1}{2}}$ the basin of attraction of $\frac{1}{2}$

$(0,1) \subseteq A_{\frac{1}{2}}$



$\eta = 1$

$f(\eta) = f(1) = 0$

$f^k(1) = 0, \; \forall k \geq 1$

$\Downarrow$

$1 \notin A_{\frac{1}{2}}$

it seems that

$A_{\frac{1}{2}} = (0,1)$.

$\mathcal{L}(x)(t) = x'''(t) + t x'(t) + 5t^2 x(t)$ $\qquad \mathcal{L} : C^3(\mathbb{R}) \to C(\mathbb{R})$

→ $\mathcal{L}$ is a linear map

→ the dimension of the kernel of $\mathcal{L}$ is 3.

e^{At} $\qquad \begin{cases} \dot{X} = AX \\ X(0) = \eta \end{cases}$ $\qquad X(t) = e^{At}\eta$ ..

$A = PBP^{-1} \Rightarrow A^k = PB^kP^{-1}$, $\quad e^{At} = Pe^{Bt}P^{-1}$

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k$$

$f(x) = 1 + mx^2 \qquad m = ? \qquad \left\{ \frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4} \right\}$ is a 2-cycle.

$\gamma_\eta = \{ \eta, f(\eta), f(f(\eta)), f(f(f(\eta))), \dots \}$ orbit

$\{ \eta, f(\eta), \eta, f(\eta) \dots \}$ 2-cycle

$\{ \eta_1, \eta_2 \}$ is a 2cycle $\Leftrightarrow$ $\eta_2 = f(\eta_1)$ and $\eta_1 = f(\eta_2)$.

γ_η is a 2-cycle, by def, if η is a fixed point of f^2 and it is not a fixed point of f.

$\eta = \frac{1+\sqrt{5}}{4} \qquad \frac{1-\sqrt{5}}{4} = f(\eta) \qquad f(f(\eta)) = \eta$

$$f\left(\frac{1+\sqrt{5}}{4}\right) = \frac{1-\sqrt{5}}{4} \qquad f\left(\frac{1-\sqrt{5}}{4}\right) = \frac{1+\sqrt{5}}{4}$$

$f(x) = 1 + mx^2 \qquad f\left(\frac{1+\sqrt{5}}{4}\right) = 1 + m \cdot \left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 + m \cdot \frac{1+2\sqrt{5}+5}{16} =$

$$= \frac{16 + 6m + 2m\sqrt{5}}{16} = \frac{1-\sqrt{5}}{4} \Rightarrow 16 + 6m + 2m\sqrt{5} = 4 - 4\sqrt{5}$$

$$\Rightarrow 2m(3+\sqrt{5}) = -4(3+\sqrt{5}) \Rightarrow \boxed{m = -2}$$

$\Rightarrow f(x) = 1 - 2x^2 \qquad f\left(\frac{1-\sqrt{5}}{4}\right) = 1 - 2 \cdot \left(\frac{1-\sqrt{5}}{4}\right)^2 = 1 - 2 \cdot \frac{1-2\sqrt{5}+5}{16} =$

$$= 1 - \cancel{4}^{1} \cdot \frac{3-\sqrt{5}}{\cancel{16}_{4}} = \frac{1+\sqrt{5}}{4} \quad \checkmark$$

$f(x) = 1 - 2x^2 \qquad \left\{ \frac{1+\sqrt{5}}{4}, \frac{1-\sqrt{5}}{4} \right\}$ 2-cycle

$$\left| f'\left(\frac{1+\sqrt{5}}{4}\right) \cdot f'\left(\frac{1-\sqrt{5}}{4}\right) \right| = \left| +(1+\sqrt{5}) \cdot (1-\sqrt{5}) \right| = |-4| = 4 > 1 \Rightarrow$$

$f'(x) = -4x$

... ttractor.

$f'(x) = -4x$

$\Rightarrow$ the cycle is not an attractor.

$$\begin{cases} y' = 1 + x y^2 \\ y(0) = 0 \end{cases}$$

Euler's numerical formula on $[0,1]$, $h = 0.02$

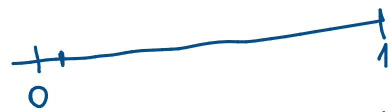
$$\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

$y_{k+1} = y_k + h f(x_k, y_k)$

x_0, y_0

$x_{k+1} = x_k + h$

$f(x,y) = 1 + x y^2$



$$\begin{cases} \boxed{x_0 = 0, \quad y_0 = 0} \\ x_{k+1} = x_k + 0.02 \\ y_{k+1} = y_k + 0.02\left(1 + x_k y_k^2\right), \quad k = \overline{0,49} \end{cases}$$

$x_0 = 0 \quad x_1 = 0.02 \quad x_2 = 0.04 \quad x_3 = 0.06 \quad \dots \quad x_n = 1$

$\varphi(0.5)$ $\qquad x_k = 0.02 \cdot k$ $\qquad 0.02 \cdot k = 0.5 \Rightarrow k = \frac{0.5}{0.02} = \frac{50}{2} = 25.$

$x_{25} = 0.5$

$\varphi(0.04)$ $\qquad x_2 = 0.04 \qquad y_2 \approx \varphi(x_2) \qquad y_2 = ?$

$x_1 = 0.02$

$y_0 = 0 \qquad y_1 = y_0 + 0.02\left(1 + x_0 y_0^2\right) = 0.02$

$y_2 = y_1 + 0.02\left(1 + x_1 y_1^2\right) = 0.02 + 0.02\left(1 + 0.02^3\right) =$

$= 0.02 + 0.02 + 0.02^4 = 0.04 + 0.00000016 =$

$= 0.04000016$

$\varphi(t, \eta)$ flow $\qquad \begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$

$H(\varphi(t, \eta)) = H(\eta), \quad \forall t \in, \forall \eta \in \mathbb{R}^n$