In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K}$.

8.1. Consider the affine transformation

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Explain why it is a rotation. Calculate the cosine of the rotation angle and determine ϕ^{-1}.

8.2. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

8.3. Consider the affine transformation

$$\phi(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}.$$

Explain why it is an isometry. Discuss the type of the isometry in terms of the parameter λ.

8.4. Using rotations around the coordinate axes, give a parametrization of a cone containing the line $\ell = \{(0,t,t) : t \in \mathbb{R}\}$ and with axis the z-axis.

8.5. Using Euler-Rodrigues formula, write down the matrix form of a rotation around the axis $\mathbb{R}\mathbf{v}$ where $\mathbf{v} = (1,1,0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.

8.6. A rotation with angle θ around the origin after a reflection in the x-axis is a reflection in the line $y = \tan(\theta/2)x$. (Hint: this is Lemma 7.15, you need to fill in the details in that proof)

8.7. Let $\mathrm{Rot}_{\theta_1,C_1}$ and $\mathrm{Rot}_{\theta_2,C_2}$ be rotations with angles θ_1,θ_2 and centers C_1,C_2 respectively. Show that the composition of these two rotations is a (possibly trivial) translation if and only if $\theta_1+\theta_2 = 2k\pi$ for some integer k.

8.8. Discuss, in dimension 2, the possible isometries obtained by composing a reflection in a line ℓ_1 with a reflection in a line ℓ_2 in terms of the relative positions of the two lines.