In the following exercises, all coordinates and components are given with respect to a right-oriented orthonormal frame $\mathcal{K}$.

8.1. Consider the affine transformation

$$\phi(\mathbf{x}) = \frac{1}{\sqrt{13}}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Explain why it is a rotation. Calculate the cosine of the rotation angle and determine ϕ^{-1}.

8.2. Let T be the isometry obtained by applying a rotation of angle $-\frac{\pi}{3}$ around the origin after a transation with vector $(-2,5)$. Determine the inverse transformation, T^{-1}.

8.3. Consider the affine transformation

$$\phi(\mathbf{x}) = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ -2 & -2 & -1 \\ -2 & 1 & 2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} \lambda \\ 0 \\ 1 \end{bmatrix}.$$

Explain why it is an isometry. Discuss the type of the isometry in terms of the parameter λ.

8.4. Using rotations around the coordinate axes, give a parametrization of a cone containing the line $\ell = \{(0,t,t) : t \in \mathbb{R}\}$ and with axis the z-axis.

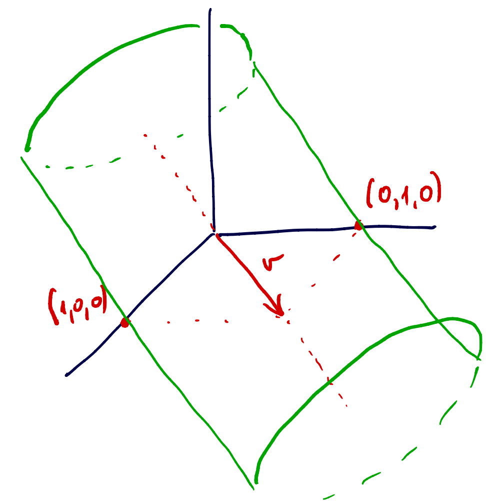
8.5. Using Euler-Rodrigues formula, write down the matrix form of a rotation around the axis $\mathbb{R}\mathbf{v}$ where $\mathbf{v} = (1,1,0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.

8.6. A rotation with angle θ around the origin after a reflection in the x-axis is a reflection in the line $y = \tan(\theta/2)x$. (Hint: this is Lemma 7.15, you need to fill in the details in that proof)

8.7. Let $\mathrm{Rot}_{\theta_1,C_1}$ and $\mathrm{Rot}_{\theta_2,C_2}$ be rotations with angles θ_1,θ_2 and centers C_1,C_2 respectively. Show that the composition of these two rotations is a (possibly trivial) translation if and only if $\theta_1+\theta_2 = 2k\pi$ for some integer k.

8.8. Discuss, in dimension 2, the possible isometries obtained by composing a reflection in a line ℓ_1 with a reflection in a line ℓ_2 in terms of the relative positions of the two lines.

8.5. Using Euler-Rodrigues formula, write down the matrix form of a rotation around the axis $\mathbb{R}\mathbf{v}$ where $\mathbf{v} = (1,1,0)$. Use this matrix form to give a parametrization of a cylinder with axis $\mathbb{R}\mathbf{v}$ and diameter $\sqrt{2}$.



Euler - Rodrigues

$$\mathrm{Rot}_{v,\theta}(x) = \cos\theta\, x + \sin\theta\,(v \times x) + (1-\cos\theta)\underbrace{\langle v, x\rangle v}_{= (v \otimes v)x} \qquad \text{if } |v| = 1$$

$$= \left[\cos\theta\, I_3 + \sin\theta\,(v\times \_) + (1-\cos\theta)\, v\otimes v\right] x$$

Replace v by $\frac{v}{|v|} = \frac{1}{\sqrt{2}}(i+j)$. Then $|v|=1$.

$$[v\times \_] = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ v\times i & v\times j & v\times k \\ \| & \| & \| \\ \frac{1}{\sqrt{2}}(i+j)\times i & \frac{1}{\sqrt{2}}(i+j)\times j & \frac{1}{\sqrt{2}}(i+j)\times k \\ \| & \| & \| \\ -\frac{1}{\sqrt{2}}k & +\frac{1}{\sqrt{2}}k & \frac{1}{\sqrt{2}}(i-j) \end{array}$$

$$v\otimes v = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \mathrm{Rot}_{v,\theta}(x) = \left(\begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{bmatrix} + \frac{\sin\theta}{\sqrt{2}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} + \frac{(1-\cos\theta)}{2} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) x =$$

$$\Rightarrow \quad Rot_{v,\theta}(x) = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & 1-\cos\theta & \sqrt{2}\sin\theta \\ 1-\cos\theta & 1+\cos\theta & -\sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \sqrt{2}\sin\theta & 2\cos\theta \end{bmatrix} x$$

We may obtain this rotation also with

$$Rot_{v,\theta}(x) = Rot_{Oz,-45°} \circ Rot_{Oy,\theta} \circ Rot_{Oz,45°}(x) = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & 1-\cos\theta & \sqrt{2}\sin\theta \\ 1-\cos\theta & 1+\cos\theta & -\sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \sqrt{2}\sin\theta & 2\cos\theta \end{bmatrix} x$$

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}}_{\begin{bmatrix} \cos\theta & 1 & \sin\theta \\ -\cos\theta & 1 & -\sin\theta \\ -\sqrt{2}\sin\theta & 0 & \sqrt{2}\cos\theta \end{bmatrix}} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Since the line $\ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v} = \begin{bmatrix} 1+t \\ t \\ 0 \end{bmatrix}$ is contained in the cylinder (in particular parallel to its axis)

a param. of the cylinder is

$$\phi(t,\theta) = \begin{bmatrix} 1+\cos\theta & 1-\cos\theta & \sqrt{2}\sin\theta \\ 1-\cos\theta & 1+\cos\theta & -\sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \sqrt{2}\sin\theta & 2\cos\theta \end{bmatrix}\begin{bmatrix} 1+t \\ t \\ 0 \end{bmatrix} = \dots$$