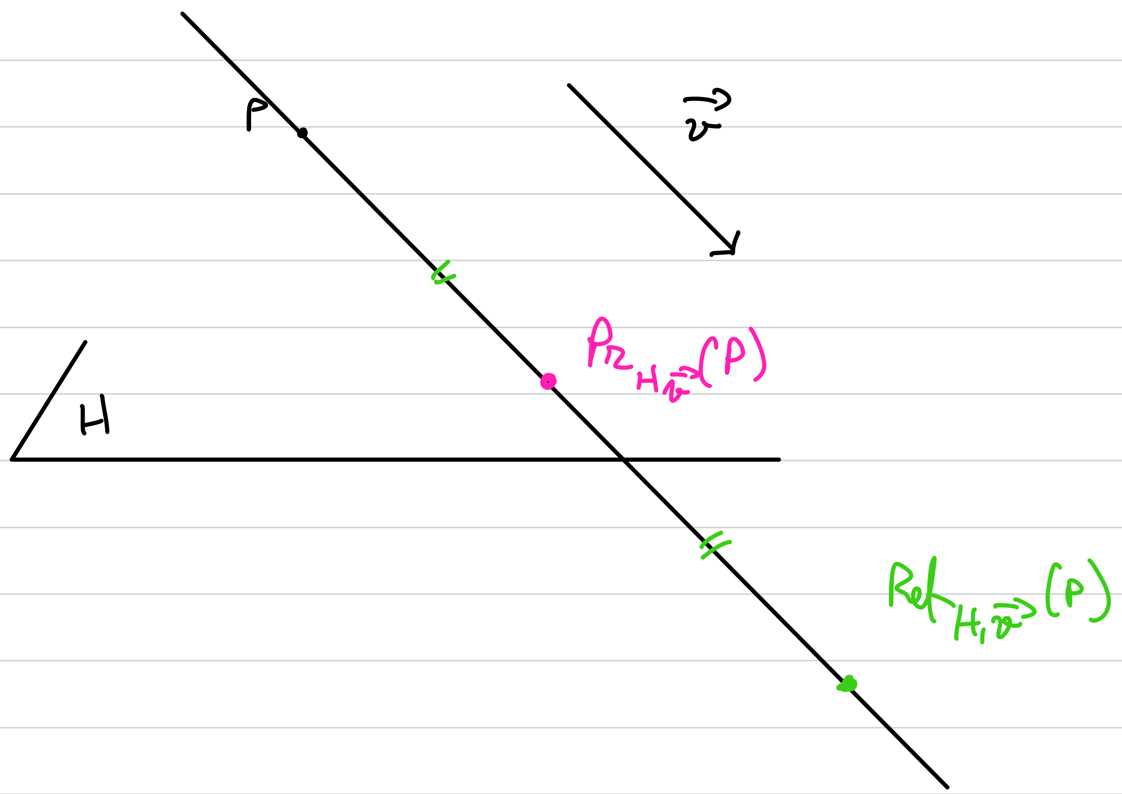
Seminar 6 - Projections & reflections

H - hyperplane

$H: a_1 x_1 + \ldots + a_n x_n + a_{n+1} = 0$

$\vec{v} \in V^n$



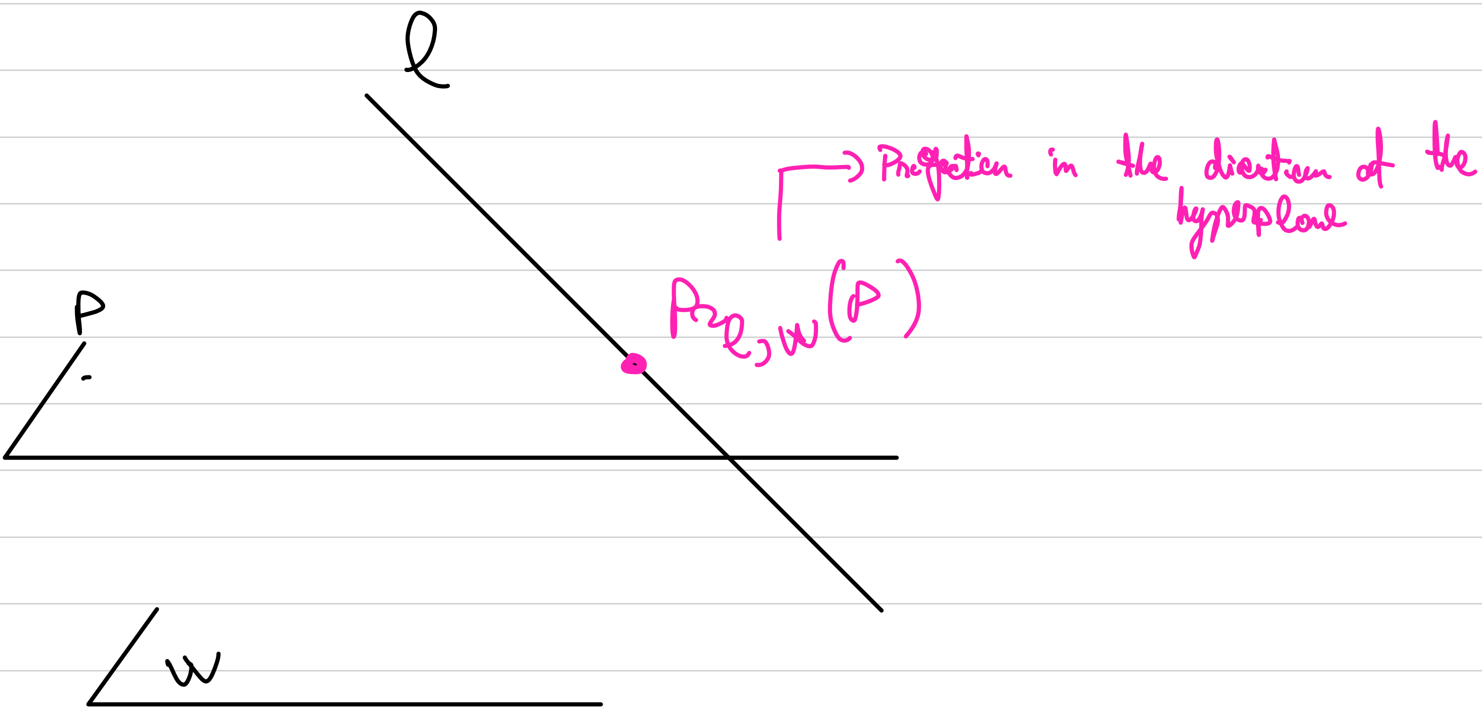
$$\boxed{Pr_{H,\vec{v}}(P) = \overbrace{\left(I_n - \frac{v \cdot a^t}{v^t \cdot a} \right)}^{\to \text{ matrice}} \cdot P - \frac{a_{n+1}}{v^T \cdot a} \cdot v}$$

$\to a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$$Ref_{H,\vec{v}}(P) = 2\, Pr_{H,v}(P) - P$$

$$= \left(2 I_n - 2\cdot\frac{v a^T}{v^T a}\right) P - \frac{2\, a_{n+1}}{v^T a}\cdot v - \underbrace{P}_{I_n \cdot P}$$

$$= \left(I_n - \frac{2\, v a^T}{v^T a}\right) P - \frac{2\, a_{n+1}}{v^T a}\cdot v$$



$W: a_1 x_1 + \dots + a_n x_n = 0$

$$Pr_{\ell,w}(P) = \frac{v \cdot a^T}{v^T \cdot a} \cdot P + \left(I_n - \frac{v \cdot a^T}{v^T \cdot a} \right) \cdot Q$$

$v \in D(\ell)$, $Q \in \ell$

7.2. Determine the orthogonal projection of the line

$$\ell : \begin{cases} 2x - y - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

with the plane

$$\Pi : x + 2y - z = 0$$

(do this by deriving the matrix form of the projection)

n - normal vector : $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$$\ell : \begin{cases} 2x - y - 1 = 0 \\ x + y - z + 1 = 0 \end{cases}$$

$$\begin{pmatrix} 2 & -1 & 0 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 2 & -1 & 0 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & -3 & 2 & -3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 1 \end{pmatrix}$$

$$\Leftrightarrow l: \begin{cases} x - \frac{1}{3}z = 0 \\ y - \frac{2}{3}z + 1 = 0 \end{cases} \Leftrightarrow \begin{cases} x = t \\ y = -1 + 2t \end{cases}$$

$$t = \frac{1}{3}z$$

$$\Leftrightarrow l: \begin{cases} x = t \\ y = -1 + 2t \\ z = 3t \end{cases}$$

$$\vec{n}(1, 2, -1)$$

$$Pr_{\pi, \vec{n}}(P) = \left(I_3 - \frac{n \cdot n^T}{n^T \cdot n} \right) \cdot P$$

$$n \cdot n^T = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix}$$

$$n^T \cdot n = \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 6$$

$$Pr_{\pi}^{\perp}(p) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{6} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \right) \cdot p$$

$$= \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \cdot p$$

$$Pr_{\pi}^{\perp}(l) = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \cdot \begin{bmatrix} t \\ -1+2t \\ 3t \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 5 & -2 & 1 \\ -2 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} t \\ -1+2t \\ 3t \end{bmatrix}$$

... cava ceva, calculezi tu

7.4. H - hyperplane

v - vector, $v \nparallel H$

Use the deduced matrix form to show that

a) $Pr_{H,v} \circ Pr_{H,v} = Pr_{H,v}$

b) $Refl_{H,v} \circ Refl_{H,v} = Id$

$$Pr_{H,v}(P) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) P - \frac{a_{n+1}}{v^T a} \cdot v$$

We want to show that $Pr_{H,v}(Pr_{H,v}(P)) = Pr_{H,v}(P)$

$$Pr_{H,v}(Pr_{H,v}(P)) = \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) \cdot \left(\left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) P - \frac{a_{n+1}}{v^T \cdot a} \cdot v\right) - \frac{a_{n+1}}{v^T \cdot a} \cdot v$$

$$= \left(I_n \left(I_n - \frac{v \cdot a^T}{v^T \cdot a}\right) P - I_n \cdot \frac{a_{n+1}}{v^T \cdot a} \cdot v\right.$$

$$- \frac{v \cdot a^T}{x^T \cdot a} \cdot \left(I_n - \frac{v \cdot a^T}{x^T \cdot a} \right) P + \frac{v \cdot a^T}{x^T \cdot a} \cdot \frac{a_{n+1}}{x^T \cdot a} \cdot v$$

$$- \frac{a_{n+1}}{x^T \cdot a} \cdot v$$

$$= \left(I_n - I_n \cdot \frac{v \cdot a^T}{x^T \cdot a} \right) P - \frac{a_{n+1}}{x^T \cdot a} \cdot v$$

$$- \left(\frac{v \cdot a^T}{x^T \cdot a} - \left(\frac{v \cdot a^T}{x^T \cdot a} \right)^2 \right) P + v \left(\cancel{\frac{v \cdot a^T \cdot a_{n+1}}{(x^T \cdot a)^2}} - \cancel{\frac{a_{n+1}}{x^T \cdot a}} \right)$$

$$= \left(I_n - \frac{v \cdot a^T}{x^T \cdot a} \right) P - \frac{a_{n+1}}{x^T \cdot a} \cdot v$$

$$- \left(\frac{v \cdot a^T \cdot x^T \cdot a - (v \cdot a^T)^2}{(x^T \cdot a)^2} \right) P$$

$$v \cdot \underbrace{a^T \cdot v}_{\text{scalar}} \cdot a^T = (a^T \cdot v) \cdot (v \cdot a^T)$$

$$a^T x = (a^T \cdot x)^T = x^T \cdot a$$

$$\underbrace{v \cdot a^T}_{\text{scalar}} \cdot v = a^T \cdot v \cdot v = (v^T \cdot a) \cdot v$$

...

$$Pr^2_{H_v}(P) = \left(I_n - \frac{2 \cdot v \cdot a^T}{v^T \cdot a} + \frac{1}{v^T \cdot a} \cdot v \cdot a^T \right) \cdot P$$

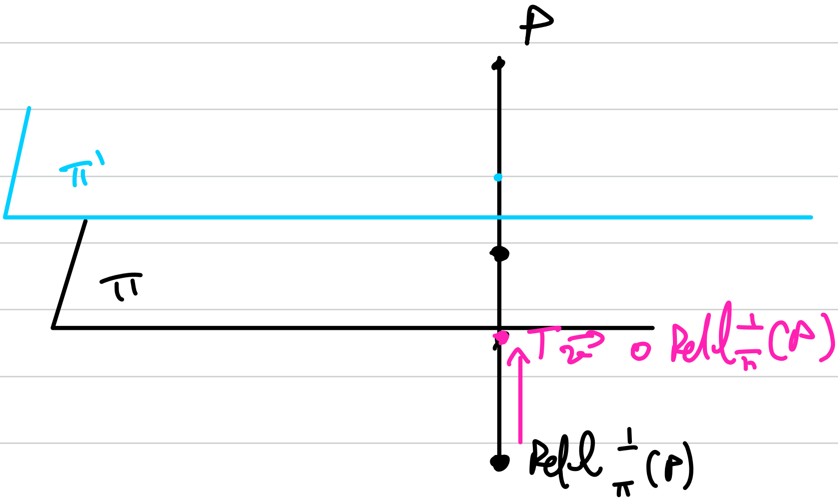
... incomplet ...

7.6. Π - plane with normal vector $\vec{n}$ and equation:

$$\Pi : \langle \vec{n}, \vec{x} \rangle = c$$

$$\Leftrightarrow n_1 x_1 + n_2 x_2 + n_3 x_3 - c = 0$$

Show that the composition of the orth. reflection in Π followed by a translation with $\vec{v} \parallel \vec{n}$ is a reflection in the plane Π' and deduce the plane's eqn.



$$Refl^{\perp}_{\pi}(P) = 2\left(I_n - \frac{\vec n \cdot \vec n^T}{\vec n^T \cdot \vec n}\right)P + \frac{2c}{\vec n^T \cdot \vec n}\vec n$$

$\vec v \parallel \vec n \Rightarrow \exists k$ scalar: $\vec v = k\vec n$

$$T_{\vec v} \circ Refl^{\perp}_{\pi}(P) = \left(I_n - 2\frac{\vec n \cdot \vec n^T}{\vec n^T \cdot \vec n}\right)\cdot P + \frac{2c}{\vec n^T \cdot \vec n}\cdot \vec n + k\cdot \vec n$$

$$= \left(I_n - 2\frac{\vec n\, \vec n^T}{\vec n^T \vec n}\right)\cdot P + \frac{\underbrace{2c + k\,\vec n^T \cdot \vec n}_{=2c'}}{\vec n^T \cdot \vec n}\cdot \vec n$$

$= Ref^{\perp}_{\pi'}(P)$, where:

$\pi' : \langle \vec n, \vec x \rangle = c'$