## 11.1 Warm-Up Exercises

**11.1.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}.$$

Write down the equations of the tangent planes in the intersection points.

**11.2.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

**11.3.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

**11.4.** For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S} : \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[ \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

a) Give an equation of $\mathcal{S}$.

b) Find the parameters of the point $P(3, \sqrt{3}, 1)$.

c) Calculate a parametrization of the tangent plane $\mathcal{T}_P\mathcal{S}$ using partial derivatives.

d) Give an equation of $\mathcal{T}_P\mathcal{S}$.

**11.1.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{4,2\sqrt{3},2} : \frac{x^2}{16} + \frac{y^2}{12} + \frac{z^2}{4} - 1 = 0 \quad \text{with the line} \quad \ell : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}.$$

Write down the equations of the tangent planes in the intersection points.

$\mathcal{E} \cap \ell =$ points on $\ell$ satisfying the eq. of the ellipsoid

$$\frac{(4+2t)^2}{16} + \frac{(-6-3t)^2}{12} + \frac{(-2-2t)^2}{4} - 1 = 0$$

$$\Leftrightarrow \quad (2+t)^2 + (1+t)^2 - 1 = 0$$

$$\Leftrightarrow \quad (t+1)(t+2) = 0$$

the parameters $t=-1$ and $t=-2$ correspond to points on $\ell$ which satisfy the eq. of $\mathcal{E}$

$t=-1$ corresponds to $P_1(2,-3,0)$

$t=-2$ — " — $P_2(0,0,2)$

The tangent planes in these points are

$$T_{P_1}\mathcal{E} : \frac{2x}{16} - \frac{3y}{12} + \frac{0 \cdot z}{4} - 1 = 0 \quad \Leftrightarrow \quad x - 2y - 8 = 0$$

$$T_{P_2}\mathcal{E} : \frac{0x}{16} + \frac{0 \cdot y}{12} + \frac{2 \cdot z}{4} - 1 = 0 \quad \Leftrightarrow \quad z = 2$$

**11.2.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,3,4} : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$$

with planes parallel to the coordinate planes. Treat the various cases separately.

Intersection with planes $\parallel$ $Oxy$: $\quad \mathcal{E} \cap z = h$ for some $h \in \mathbb{R}$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 - \frac{h^2}{16}$$

$$\Leftrightarrow \quad \frac{x^2}{4\left(1-\frac{h^2}{16}\right)} + \frac{y^2}{9\left(1-\frac{h^2}{16}\right)} = 1 \quad \text{(in the plane } z=h\text{)}$$

which gives

- the empty set if $1 < \frac{h^2}{16} \Leftrightarrow h \in \left]-\infty, -4\right[ \cup \left]4, \infty\right[$
- the point $(0,0,4)$ if $h=4$
- the point $(0,0,-4)$ if $h=-4$
- an ellipse if $h \in (-4,4)$
  with semi-major axis $\sqrt{9\left(1-\frac{h^2}{16}\right)}$
  and semi-minor axis $\sqrt{4\left(1-\frac{h^2}{16}\right)}$

The other cases are treated similarly.

**11.3.** Determine the intersection of the ellipsoid

$$\mathcal{E}_{2,\sqrt{3},3} : \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \quad \text{with the line} \quad \ell : x = y = z.$$

Write down the equations of the tangent planes in the intersection points.

$$\mathcal{E} \cap \ell : \begin{cases} \frac{x^2}{4} + \frac{y^2}{3} + \frac{z^2}{9} = 1 \\ y = x \\ z = x \end{cases}$$

$$\Rightarrow \frac{x^2}{4} + \frac{x^2}{3} + \frac{x^2}{9} = 1$$

$$\Rightarrow 25x^2 = 4 \cdot 9$$

$$\Rightarrow x = \pm \frac{6}{5}$$

$\Rightarrow$ we obtain two points: $P_1\left(\frac{6}{5}, \frac{6}{5}, \frac{6}{5}\right)$ and $P_2\left(-\frac{6}{5}, -\frac{6}{5}, -\frac{6}{5}\right)$

- The tangent planes in these points are

$$T_{P_1}\mathcal{E} : \frac{\frac{6}{5}x}{4} + \frac{\frac{6}{5}y}{3} + \frac{\frac{6}{5}z}{9} = 1$$

$$\Leftrightarrow \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = \frac{5}{6}$$

and $T_{P_2}\mathcal{E} : \frac{x}{4} + \frac{y}{3} + \frac{z}{9} = -\frac{5}{6}$

**11.4.** For the surface $\mathcal{S}$ with parametrization

$$\mathcal{S}: \begin{cases} x = 4\cos(s)\cos(t) \\ y = 4\sin(s)\cos(t) \\ z = 2\sin(t) \end{cases} \qquad s \in [0, 2\pi[ \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}[$$

a) Give an equation of $\mathcal{S}$.

b) Find the parameters of the point $P(3, \sqrt{3}, 1)$.

c) Calculate a parametrization of the tangent plane $\mathcal{T}_P\mathcal{S}$ using partial derivatives.

d) Give an equation of $\mathcal{T}_P\mathcal{S}$.

- $S : \frac{x^2}{16} + \frac{y^2}{16} + \frac{z^2}{4} = 1$

- $P(3, \sqrt{3}, 1)$ $\quad \sin(t) = \frac{1}{2} \Rightarrow t = \frac{\pi}{6}$

$$\Rightarrow \begin{cases} 3 = 4\frac{\sqrt{3}}{2}\cos(s) \\ \sqrt{3} = 4\frac{\sqrt{3}}{2}\sin(s) \end{cases} \Rightarrow \begin{cases} \cos(s) = \frac{\sqrt{3}}{2} \\ \sin(s) = \frac{1}{2} \end{cases} \Rightarrow s = \frac{\pi}{6}$$

So P is obtained with the parameters $(t, s) = (\frac{\pi}{6}, \frac{\pi}{6})$

- $T_pS = p + \langle \frac{\partial\sigma}{\partial t}(p), \frac{\partial\sigma}{\partial s}(p) \rangle$ where $\sigma(t,s) = \begin{bmatrix} 4\cos(s)\cos(t) \\ 4\sin(s)\cos(t) \\ 2\sin(t) \end{bmatrix}$

$$\frac{\partial\sigma}{\partial s}(p) = \begin{bmatrix} -4\sin(s)\cos(t) \\ 4\cos(s)\cos(t) \\ 0 \end{bmatrix}(p) = \begin{bmatrix} -\sqrt{3} \\ 3 \\ 0 \end{bmatrix}$$

$$\frac{\partial\sigma}{\partial t}(p) = \begin{bmatrix} -4\cos(s)\sin(t) \\ -4\sin(s)\sin(t) \\ 2\cos(t) \end{bmatrix}(p) = \begin{bmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{bmatrix}$$

$$\Rightarrow T_pS : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ \sqrt{3} \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} -\sqrt{3} \\ 3 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -\sqrt{3} \\ -1 \\ \sqrt{3} \end{bmatrix}$$

- $T_pS : \frac{3x}{16} + \frac{\sqrt{3}y}{16} + \frac{z}{4} = 1$