Technische Universität Berlin

R Portfolio Optimization Project

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Contents

Introduction text to our project.

1 Minimum-Variance-Portfolio

An investment strategy based on a Minimum Variance Portfolio seeks to minimize the risk of an investment. There is thus no desired target-return or stock-forecast to be considered. The portfolio optimization process can be described as pure risk-minimization, where the goal is a determination of the weight-distribution yielding the lowest possible risk at any time. Graphically, one can think of a Minimum Variance Portfolio as the most left point of the mean-variance-frontier.

1.1 Theory

As written in the paper by DeMiguel et al.[DGR09], the weights were chosen according to the portfolio that minimizes the variance of return, e.g.

$$\min_{w_t} w_t^{\perp} \Sigma_t w_t \tag{1}$$

under the restriction that

$$1_N^{\perp} w_t \overset{!}{=} 1 \tag{2}$$

and

$$\Sigma_t w_t \overset{!}{=} 1, \tag{3}$$

where $w_t \in \mathbb{R}^N$ is the weight-vector at time $t \in \{1, 2, \ldots, T\}$ with $T \in \mathbb{N}$ period under observation and $N \in \mathbb{N}$ number of assets considered. $\Sigma_t \in \mathbb{N} \times \mathbb{N}$ names the covariance matrix of excess-returns $R_\tau \in \mathbb{R}^{N \times \tau}$ at time t for a subset τ of the period of Observation $\tau \subseteq \{1, 2, \ldots, T\}$. In the equation above, 1 is thought of as the N-dimensional vector containing 1s.

We now want to investigate the restriction $\min_{w_t}$for one fixed time-period T, such that $\tau = T$. As a result, weights are not time-dependent anymore. Note that when constructing a Minimum Variance Portfolio, weights are in fact time-dependent. However, for the sake of legibility, we will for now treat the case where $w_t = w$.

Considering three risky assets and a weight vector $w \in \mathbb{R}^3$, such that

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{4}$$

the minimal variance restraint can then be interpreted as follows[Ziv13, p. 7]:

$$\begin{aligned} \min_{w_1, w_2, w_3} \sigma^2_{w_1, w_2, w_3} =& w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + w_3^2 \sigma_3^2 + \\ & 2 w_1 w_2 \sigma_{12} + 2 w_1 w_3 \sigma_{13} + 2 w_2 w_3 \sigma_{23} \end{aligned} \tag{5}$$

Here, the components' variance σ_i^2 and standard deviation $\sigma_{i,j}$ for $i, j \in \{1, 2, 3\}$ are used to describe the relationship. The Lagrange-function of this problem can be written as

$$\begin{aligned} L(w_1, w_2, w_3, \lambda) =& w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + \\ & 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23} \\ & +\lambda(w_1 + w_2 + w_3 - 1) \end{aligned} \tag{6}$$

Now, the component-wise deviates can be found and be set to equal zero, yielding the first order conditions.

$$\begin{aligned} 0 &= \frac{\partial L}{\partial w_1} = 2w_1\sigma_1^2 + 2w_2\sigma_{12} + 2w_3\sigma_{12} + \lambda \\ 0 &= \frac{\partial L}{\partial w_2} = 2w_2\sigma_2^2 + 2w_1\sigma_{12} + 2w_3\sigma_{23} + \lambda \\ 0 &= \frac{\partial L}{\partial w_3} = 2w_3\sigma_3^2 + 2m_1\sigma_{13} + 2w_2\sigma_{23} + \lambda \\ 0 &= \frac{\partial L}{\partial \lambda} = w_1 + w_2 + w_3 - 1 \end{aligned} \tag{7}$$

In this, λ is the Lagrange multiplier. Equation 7 can be expressed as a system of linear equations:

$$\begin{pmatrix} 2\sigma_1^2 & 2\sigma_{12} & 2\sigma_{13} & 1 \\ 2\sigma_{12} & 2\sigma_2^2 & 2\sigma_{23} & 1 \\ 2\sigma_{13} & 2\sigma_{23} & 2\sigma_3^2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{8}$$

Now we can clearly see that the equation has the form of

$$\begin{pmatrix} 2\Sigma & 1 \\ 1^\top & 0 \end{pmatrix} \begin{pmatrix} w \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9}$$

where Σ is just the covariance-matrix of the matrix of excess-returns R_τ.

1.2 Solving the system of linear equations

Above equation is of form

$$A \cdot x = b, \tag{10}$$

thus a system of linear differential equations. Since $A = \Sigma$ and $b = 1$ are known, we can solve this system to obtain $x = w_t$ the weight-vector at time t. Note that the weight-vector $\mathbf{x}$ may contain negative which are smaller than zero. These negative weights are interpreted as short sales[1]. While short sales are in reality strongly regulated and restricted, the Minimum Variance Portfolio created in this work will not consider any restrictions.

[1] `https://en.wikipedia.org/wiki/Short_(finance)`

1.3 in-sample sharpe ratio

The in-sample sharpe ratio for one strategy k is a basic measure of performance. Only one weights-vector is computed for all results. According to DeMiguel et al. [DGR09, p. 1928] it can be computed by

$$\widehat{SR}_k^{IS} = \frac{\hat{\mu}_k^{IS\perp}\ \hat{w}_k}{\left(\hat{\mu}_k^{\perp}\ \Sigma\ \hat{\mu}_k\right)^{1/2}}. \tag{11}$$

Here, $\hat{\mu}_k^{IS\perp}$ is the mean-returns vector for strategy k, whereas Σ is the covariance-matrix of the excess-returns matrix R. $\hat{w}_k$ is the mean weights-vector of strategy k.

1.4 Implementation

In this section we want to display and discuss the implementation of the approach described in the previous chapter in the R programming language. We will compute several performance measures of the Minimal Variance Portfolio we're about to derive from the data.

1.5 Datasets and data preparation

The first dataset upon which this analysis is based is called "Ten sector portfolios of the S&P 500 and the US equity market portfolio". It has been created by Roberto Wessels and was obtained from Mendeley Data[2]. Note that for this dataset, it is necessary to substract the Treasury Bill-rates from each entry to obtain adjusted returns stripped of the risk-free rate. We call this dataset **TSP**.

The other dataset concerned is provided by Kenneth French and downloaded from his website[3]. The dataset is called "SMB and HML portfolios and the US equity market portfolio" and contains the factors "Small Minus Big" (SMB) and "High Minus Low" (HMB) from the Fama-French three-factor model [4] as well as the whole S&P500 portfolio. We call this dataset short **SHS** This dataset is already cleared from the risk-free rate, thus a Treasure Bill-rate adjustment is not necessary.
In listing 1, it is showcased how data is imported and loaded into a dataframe. In line 1, the .csv-file containing the data is loaded via the read.csv function, where a separator as well as the header-parameter are specified.

```
1 return_matrix = read.csv('../ten-roberto-wessels.csv', header=TRUE, sep=',')
2 sub_return_matrix = subset(return_matrix, select = -c(13))
3 sub_return_matrix = apply(sub_return_matrix[,-1], 2, '-', return_matrix
      [,13])
4 return_matrix = cbind(return_matrix$Date, sub_return_matrix)
5 colnames(return_matrix)[1] = 'Date'
```

Listing 1: Loading the data and subtracting the risk-free rate from the columns of returns yields a dataframe containing the matrix of excess-returns.

[2]Data can be downloaded here. It's also attached to this report.
[3]The dataset from Kenneth Frenchs' website can be downloaded here and is also attached to this report.
[4]For an introduction to the Fama-French three-factor model, see Wikipedia.

In case of dataset TSP, it is necessary to subtract the risk-free rate from the returns. To do this, a subset of the return_matrix is created in line 2, excluding the monthly risk-free rate which is stored in column 13 of the return_matrix. In line 3, the risk-free rate is sutracted from every column the return_matrix' subset, except for the column containing the date. Line 4 shows the reconstruction of the return_matrix, now containing risk-free returns. After the dates' column-name has been reset in line 5, the return_matrix now contains only risk-free returns.

1.5.1 Computing the in-sample sharpe ratio

A very basic way of measuring portfolio performance is provided by the in-sample sharpe ratio. By setting $\tau = \{1, 2, \ldots, T\}$, the entire time-series is considered when computing the covariance matrix of the matrix of excess-returns Σ_t, effectively removing the weight-vectors' time-dependency.

Following the procedure leading to the system of linear equations in equation10, we first compute a covariance matrix of the entire risk-free return-matrix. Line 1 of listing 2 does just this, excluding the column containing the time-series dates. A vector of dimension equal to the amount of assets listed in the return_matrix is constructed, where every component of the vector equals one.

```
1 cov_return_matrix = cov(return_matrix[,-1])
2 b = vector(length=nA) + 1
3 weights = solve(cov_benchmark_matrix, b)
4 weights = weights/abs(sum(weights))
5 mean_returns = data.matrix(apply(benchmark_matrix[,-1], 2, mean))
6 weights = data.matrix(weights)
7 sharpe_ratio_IS = t(mean_returns) %*% weights / sqrt(t(weights)%*%cov_
    benchmark_matrix%*%weights)
8 print(sharpe_ratio_IS)
```

Listing 2: This example shows how the covariance-matrix of a returns-matrix is computed in R.

Line 3 of listing 2 uses the `solve()` function to solve the linear equation by inverting the covariance matrix. The resulting `weights`-vector is normalized in line 4. A vector containing the mean-values of `return_matrix` is created in line 5 and the `weights`-vector is converted to a data.matrix in line 6. This is done to facilitate the vector- and matrix-multiplication performed in line 7 to calculate the in-sample sharpe ratio according to 11, which is printed in line 7.

Listing 2 shows the computation of the covariance matrix Σ derived from the matrix of excess-returns, named return_matrix in this example.

```
b = vector(length=nA) + 1
x = solve(cov_return_matrix, b)
x = x/sum(x)
```

Listing 3: Code example shows how to solve a system of linear equations

1.6 Discussion

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2 Bayes-Stein-Portfolio

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2.1 Theory

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2.2 Execution

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3 Sample-based mean-variance-Portfolio

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References

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