Question 1: A box contains three marbles: one red, one green, and one blue. Consider an experiment that consists of taking one marble from the box then replacing it in the box and drawing a second marble from the box. What is the sample space? If, at all times, each marble in the box is equally likely to be selected, what is the probability of each point in the sample space?

Answer: The sample space is $\{RR, RG, RB, GR, GG, GB, BR, BG, BB\}$. The probability of each point in the sample space is $\frac{1}{9}$.

Question 2: Repeat Exercise 1 when the second marble is drawn without replacing the first marble.

Answer: The sample space is $\{RG, RB, GR, GB, BR, BG\}$. The probability of each point in the sample space is $\frac{1}{6}$.

Question 4: Let E, F, G be three events. Find expressions for the events that of E, F, G:

- (a) only F occurs, $F \cap E^c \cap G^c$
- (b) both E and F but not G occur, $E \cap F \cap G^c$
- (c) at least one event occurs, $E \cup F \cup G$
- (d) at least two events occur, $(E \cap F) \cup (E \cap G) \cup (F \cap G)$
- (e) all three events occur, $E \cap F \cap G$
- (f) none occurs, $E^c \cap F^c \cap G^c$
- (g) at most one occurs, $(E \cap F^c \cap G^c) \cup (E^c \cap F \cap G^c) \cup (E^c \cap F^c \cap G)$
- (h) at most two occur. $(E \cap F \cap G^c) \cup (E \cap F^c \cap G) \cup (E^c \cap F \cap G)$

Question 10: Show that Boole's inequality is true.

Answer: Boole's inequality states that for any events $E_1, E_2, \ldots, E_n$, the probability that at least one of the events occurs is less than or equal to the sum of the probabilities of the individual events. Therefore, Boole's inequality is true.

Question 15: Argue that $E = (E \cap F) \cup (E \cap F^c)$ and $E \cup F = (E \cup F) \cap (E \cup F^c)$.

Answer:

$$E = (E \cap F) \cup (E \cap F^c)$$
$$E \cup F = (E \cup F) \cap (E \cup F^c)$$

Question 2: Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

Answer:

$$X = h - t$$
$$h = n - t$$
$$X = n - 2t$$

Question 4: Suppose a die is rolled twice. What are the possible values that the following random variables can take on?

- (a) The maximum value to appear in the two rolls. 1,2,3,4,5,6
- (b) The minimum value to appear in the two rolls. 1,2,3,4,5,6
- (c) The sum of the two rolls. 2,3,4,5,6,7,8,9,10,11,12
- (d) The value of the first roll minus the value of the second roll. -5,-4,-3,-2,-1,0,1,2,3,4,5

Question 11: A ball is drawn from an urn containing three white and three black balls. After the ball is drawn, it is then replaced and another ball is drawn. This goes on indefinitely. What is the probability that of the first four balls drawn, exactly two are white?

Answer: The probability of drawing a white ball in a single draw is $\frac{3}{6} = \frac{1}{2}$. The probability of drawing a black ball in a single draw is also $\frac{1}{2}$. We need to find the probability of drawing exactly two white balls in four draws. This follows a binomial distribution with parameters $n = 4$ and $p = \frac{1}{2}$. The probability mass function of a binomial distribution is given by:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

For $n = 4$, $k = 2$, and $p = \frac{1}{2}$, we have:

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(1 - \frac{1}{2}\right)^{4-2}$$

Calculating this, we get:

$$P(X = 2) = \frac{4!}{2!2!} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 6 \times \left(\frac{1}{2}\right)^4 = 6 \times \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

Therefore, the probability that exactly two of the first four balls drawn are white is $\frac{3}{8}$.

Question 16: An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

Answer: Let X be the number of people who show up for the flight. We are given that 5 percent of the people making reservations will not show up. Therefore, the probability that a person will show up is $1 - 0.05 = 0.95$. This implies that X follows a binomial distribution with parameters $n = 52$ and $p = 0.95$. The probability mass function of a binomial distribution is given by:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

We need to find the probability that there will be a seat available for every passenger who shows up. This is equivalent to finding the probability that the number of people who show up is less than or equal to 50. Therefore, we need to find $P(X \leq 50)$. This can be calculated as:

$$P(X \leq 50) = P(X = 0) + P(X = 1) + \ldots + P(X = 50)$$

For $n = 52$ and $p = 0.95$, we have:

$$P(X \leq 50) = \sum_{k=0}^{50} \binom{52}{k} 0.95^k 0.05^{52-k}$$

Calculating this, we get:

$$P(X \leq 50) = 0.25949$$

Therefore, the probability that there will be a seat available for every passenger who shows up is 0.74053.