General Formulation for Magnetic Forces in Linear Materials and Permanent Magnets

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A previous work described how a magneto-mechanical tensor can be obtained from the application of the virtual work principle to the energy density or co-energy density in any given media. This paper describes how the volume and surface force densities can be obtained from this tensor. Taking these force densities into account, we can calculate the total force over any given magnetic medium that is surrounded by any other magnetic medium. This represents a generalization of the classic Maxwell theorem. With this methodology, it is possible to calculate the surface forces in bodies in contact. We demonstrate this by showing the equivalence of the results obtained by this procedure and the equivalent source method and the Kelvin expression method. These methods could previously be applied only to bodies surrounded by air. We also show that this procedure is equivalent to other methods used to calculate the forces between linear magnetic materials (Korteweg–Helmholtz's tensor, ponderomotive force density). We have compared our method with results from continuum design sensitivity analysis, a method that allows for the calculation of global and local forces in magnetic media in contact.

Index Terms—**Magnetic forces, magnetic materials, magnetostatics, Maxwell's tensor.**

I. Introduction

THE analysis of the distribution of magnetic forces and the calculation of the total force in a continuous medium are still relevant problems in many fields of investigation. This work focuses on the field of magnetostatics, which is of special interest in the design of electromechanical systems. In this area, the calculation of forces is carried out by following four basic procedures.

The application of the virtual work principle (VWP) is well documented and has been developed, both in a general sense [1]–[4] and with reference to specific applications [5]–[8], and the procedure is widely used.

The method known as Maxwell's stress tensor (MST) consists in obtaining the total force over an isolated body in air by the integration of the classic Maxwell tensor over any closed surface within which the body is located. When this procedure is extended to bodies that are in contact with each other it is necessary to introduce an artificial thin shell of empty space that surrounds the material. However, as this breaks with the hypothesis of the short range of the distribution of magnetic forces in materials, the results obtained are only very approximate. A classic result [9], [10] is that, in this case, the total force over the isolated element may be obtained by adding an integral of volume force density to an integral of surface force density. The volume force density can be calculated as the divergence of a magneto-mechanical tensor in the medium, whereas the surface force density is the difference between this tensor in the medium and the Maxwell's tensor in air. The magneto-mechanical tensor in the medium is not univocally defined, and therefore, neither are the corresponding surface forces and densities of forces. The limits of the VWP and MST procedures are well documented and their results have been compared [11], [12]. In [13], a magneto-mechanical tensor in bodies by application of VWP to the densities of energy and co-energy has been deduced. This general tensor depends on the co-energy density in the medium. The obtained tensor reproduces also the classic Maxwell stress tensor in air.

A third process is known as the equivalent sources method (ESM) and consists in the integration of a volume magnetic force density and of its associated surface density. This method uses two volume magnetic force densities. The force density may be described as the interaction of the magnetic field $\mathbf{H}$ with a virtual magnetic charge that depends on the divergence of the magnetization $\mathbf{m}$ of the material, or, it may be described as the interaction of the magnetic induction $\mathbf{B}$ with the virtual magnetizing current $\mathbf{j}_m$, that depends on the rotational of the magnetization of the material. The corresponding surface terms include a density of magnetic surface charge and a density of magnetizing surface current, respectively. This method is limited to cases in which the body is surrounded by air. These procedures and their application to the calculation by finite elements have been widely described [14], [15].

Lastly, the value of the classic Kelvin density of force (KV) depends on the derivation in the direction of the magnetization of the magnetic field and the value of the corresponding surface density depends on the normal component of the magnetization of the material [16]. The equivalence of the procedures ESM and KV in the calculation of the total force has been shown in [17]. However, experimental differences between the deformations calculated from the force densities obtained by these methods have been observed [18].

Recently, the calculation of magnetic forces based on continuum design sensitivity analysis (CDSA) has been documented [19], [20]. This method allows for the determination of volume and surface force densities in bodies in contact and the integration of these densities over their surface and volume, respectively. The method therefore avoids the limitation described in the MST procedure. Although the CDSA procedure

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was developed following a different mathematical apparatus and methodology, it can also be used to calculate the surface forces in bodies in contact.

The main objective of this work is to establish a method for the calculation of volume and surface force densities in magnetic media by defining only a magneto-mechanical tensor in any given magnetic media. This method permits the calculation of the total force over a body without it being necessary that the body is surrounded by air and, therefore, the calculation of the volume force density and the surface force densities in bodies that are in contact. This method obviously requires the definition of a magneto-mechanical tensor in a general magnetic media.

There are currently various ways of representing magneto-mechanical tensors in materials. Reference [21] shows the use of the classic tensor [22], [23] for a macroscopic polarizable media in equilibrium given as the equation

$$T_{ij} = H_i B_j - \frac{1}{2}\mu_0 \mathbf{H}^2 \delta_{ij}.$$

This tensor allows for the definition by symmetrization of an equivalent stress tensor [24] in a unified magneto-elastic coupling formulation. It is in this context that [25] uses the tensor

$$T_{ij} = B_i H_j - \left(\frac{1}{2}\frac{\mathbf{B}^2}{\mu_0} - \mathbf{B}\cdot\mathbf{m}\right)\delta_{ij}$$

in which vector $\mathbf{m}$ is the magnetization of the material, and this tensor is similar to that which can be deduced from the more general expressions obtained in [26].

The tensor obtained in [13] has the general form

$$T_{ij} = B_i H_j + G\delta_{ij}$$

in which G is the volume density of co-energy (magnetic free enthalpy).

For linear magnetic media, in which the density of the co-energy G can be obtained analytically, the tensor introduced in [13] adopts a form that differs from the tensors previously mentioned, while still maintaining formal similarities to one another. However, as mentioned before, the definition of a magneto-mechanical tensor in a material is not univocal and the calculation of the total force over a body depends on the corresponding surface forces that are derived from the definition of this tensor.

The aim of this work is to obtain the total magnetic forces in rigid media in equilibrium from the tensor defined in [13]. This supposes the calculation and the analysis of the volume densities and surface forces for a magnetic body that is in contact with another magnetic medium. The general expressions obtained are applied to the particular case of a body that is surrounded by air, and this reproduces the densities of forces and the surface force values usually employed in ESM and KV. Also, a comparison is made between the expressions obtained for the surface forces and those already documented in previous works concerning forces over magnetic bodies in contact.

II. Density of Magnetic Force in a Material Medium

Reference [13] shows the calculation of magneto-mechanical tensors in magnetic media in equilibrium for magnetostatic processes by the application of VWP, for either the variation of the volume density of free magnetic energy $F(\mathbf{B})$ or the volume density or free magnetic enthalpy, which is also called magnetic co-energy, $G(\mathbf{H})$. The densities $F(\mathbf{B})$ and $G(\mathbf{H})$ are defined by the following relations:

$$dF = \mathbf{H}\cdot d\mathbf{B} \tag{1}$$

$$G = F - \mathbf{H}\cdot\mathbf{B} \tag{2}$$

$$dG = -\mathbf{B}\cdot d\mathbf{H}. \tag{3}$$

The magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ verify the equations

$$\vec{\nabla}\times\mathbf{H} = \mathbf{j} \tag{4}$$

$$B = \mu_0(\mathbf{H} + \mathbf{m}) \tag{5}$$

$$\vec{\nabla}\cdot\mathbf{B} = 0 \tag{6}$$

in which the vectors $\mathbf{j}$ and $\mathbf{m}$ are, respectively, the free current density and the magnetization density. In a material without permanent magnetization, vector $\mathbf{m}$ is proportional to field $\mathbf{H}$

$$\mathbf{m} = \chi(\mathbf{H})\mathbf{H} \Rightarrow \mathbf{B} = \mu(\mathbf{H})\mathbf{H} = \mu_0(1+\chi(\mathbf{H}))\mathbf{H} \tag{7}$$

where $\chi(\mathbf{H})$ is the magnetic susceptibility, μ_0 is the magnetic permeability of free space, and $\mu(\mathbf{H})$ is the magnetic permeability of the material.

By the virtual variation of the co-energy and by maintaining the magnetic field $\mathbf{H}$ constant, in accordance with VWP, the following magneto-mechanical tensor is obtained:

$$T_{ij} = B_i H_j + G\delta_{ij}. \tag{8}$$

In air, the expression (8) reproduces the classic Maxwell stress tensor

$$T_{ij} = \mu_0 H_i H_j - \frac{1}{2}\mu_0\mathbf{H}^2\delta_{ij}. \tag{9}$$

In linear magnetic materials, the magnetic susceptibility does not depend on magnetic field $\mathbf{H}$; in these materials, the more general expression of the magnetic induction is

$$\mathbf{B} = \mu\mathbf{H} + \mu_0\mathbf{m}_p \tag{10}$$

where $\mathbf{m}_p$ is the magnetization of the permanent magnet and in these media the density of co-energy can be obtained by the integration of (3). Taking the density of the co-energy to be zero when the induction $\mathbf{B}$ is zero (and therefore the density of the energy is also zero), the following equation is obtained

$$G = -\frac{\mathbf{B}^2}{2\mu}. \tag{11}$$

The volume density of magnetic force is calculated as the divergence of the magneto-mechanical tensor (8)

$$f_i = \partial_j T_{ij}. \tag{12}$$

The general expression for this force density in any magnetic material, following (12) is

$$f = (\mathbf{H}\cdot\vec{\nabla})\mathbf{B} + (\vec{\nabla}\cdot\mathbf{H})\mathbf{B} + \vec{\nabla}G. \tag{13}$$

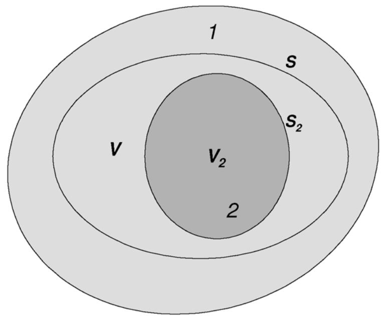

Fig. 1. General arrangement for the calculation of forces in magnetic media in equilibrium.

III. Calculation of Magnetic Forces Over a Rigid Medium in Equilibrium

The integral of the expression (8) over a closed surface S in a magnetic medium is equal to the magnetic force $\mathbf{F}$ that the medium exerts over the volume V included in the surface S. In a steady magnetostatic state, this force is equal and opposite to the force that the medium contained in the closed surface S exerts over the rest of the material. In this way, as a general arrangement, the reader is asked to considerer two media in magnetostatic equilibrium, media 1 and 2, medium 2 being totally contained in medium 1. In Fig. 1, S represents any surface in medium 1 that completely contains medium 2. V represents the volume bordering S and S_2 is the surface that borders medium 2, being V_2 its volume.

Therefore, taking into account the discontinuity of the tensor on surface S_2, the following equation is valid:

$$\begin{aligned} F_i = \oint_S T_{ij}^{(1)} n_j dS &= \int_{V-V_2} f_i^{(1)} dV \\ &+ \oint_{S_2} \left(T_{ij}^{(1)} - T_{ij}^{(2)} \right) n_j dS_2 \\ &+ \int_{V_2} f_i^{(2)} dV = \int_{V-V_2} f_i^{(1)} dV + F_i^{(2)} \end{aligned} \tag{14}$$

in which the superindices (p) refer to medium $p, p = 1, 2$, where the magneto-mechanical tensors and the corresponding force densities are calculated. Vector $\mathbf{F}^{(2)}$ represents the total force acting on body 2 when it is totally contained in medium 1. Vector $\mathbf{F}$ represents the total force that medium 1 exerts over volume V that is within surface S. Normal unitary vector $\mathbf{n}$ points away from the corresponding surface element. The first term of (14) represents the total force over the volume surrounded by S which is outside medium 2. The second term represents the total force over the surface of medium 2 and the third term represents the total force over the volume of medium 2. The total force over the body 2 is obtained by adding the second and third terms.

If medium 1 is a homogeneous medium of uniform linear nonpermanent magnetization and there are no distributions of current included in the volume enclosed by S and outside of medium 2, the first term in (14) vanishes. Given these conditions, the following equation is verified:

$$\begin{aligned} F_i = F_i^{(2)} &= \oint_S T_{ij}^{(1)} n_j dS \\ &= \oint_{S_2} \left(T_{ij}^{(1)} - T_{ij}^{(2)} \right) n_j dS_2 + \int_{V_2} f_i^{(2)} dV. \end{aligned} \tag{15}$$

Expression (15) leads to the classic Maxwell's theorem when medium 1 is air and the corresponding tensor is given by (9).

By following this process, the values of the forces that act over a medium in magnetostatic equilibrium can be obtained. The general expression (14) can be used in any arrangement, but its utility is conditioned by the ease with which the integral $\int_{V-V_2} f_i^{(1)} dV$ can be calculated.

In any case, the forces on the surface between the two media can be easily calculated from the magneto-mechanical tensor (8). As a general rule, the continuity condition on the surface applies

$$\mathbf{B}^{(1)}.\mathbf{n} = \mathbf{B}^{(2)}.\mathbf{n} = B_n \tag{16}$$

The general surface density values of force on points on the surface that separates the two magnetic media are given by the equation

$$\mathbf{f}_s^{(2)} = (T^{(1)} - T^{(2)})\mathbf{n} = [(\mathbf{H} \cdot \mathbf{n})\mathbf{B}]_2^1 + [G]_2^1 \mathbf{n} \tag{17}$$

in which the symbol $[A]_2^1 = A^{(1)} - A^{(2)}$ represents the jump or discontinuity of the magnitude A on the surface between media 1 and 2.

The general expression for the total force obtained by the variation of the co-energy that act over a magnetic medium 2 that is totally enclosed by magnetic medium 1 is

$$\begin{aligned} F_i^{(2)} &= \oint_{S_2} \left([(\mathbf{H} \cdot \mathbf{n}) B_i]_2^1 + [G]_2^1 n_i \right) dS_2 \\ &\quad + \int_{V_2} ((\mathbf{H} \cdot \vec{\nabla}) B_i + (\vec{\nabla} \cdot \mathbf{H}) B_i + \partial_i G) dV. \end{aligned} \tag{18}$$

IV. Surface Force Densities Between Two Media of Linear Magnetization in Contact

Expression (17) can be applied in media of linear magnetization, that is, in media in which the induction $\mathbf{B}$ depends on the magnetic field $\mathbf{H}$ in the form given by (10). For these media, taking (16) into account, the following expression is verified:

$$\begin{aligned} \mathbf{f}_s^{(2)} &= (T^{(1)} - T^{(2)})\mathbf{n} \\ &= [(\mathbf{H} \cdot \mathbf{n})\mathbf{B}]_2^1 - \frac{1}{2} \left[\frac{\mathbf{B}^2}{\mu} \right]_2^1 \mathbf{n}. \end{aligned} \tag{19}$$

In those cases in which one can apply the following continuity condition for the tangential components of the magnetic field $\mathbf{H}$ in points on the surface:

$$[\mathbf{H} - (\mathbf{H} \cdot \mathbf{n})\mathbf{n}]_2^1 = \mathbf{0} \tag{20}$$

the surface force density may be expressed as follows:

$$\mathbf{f}_s^{(2)} = B_n^2\left[\frac{1}{\mu}\right]_2^1\mathbf{n} + \mu_0 B_n\left[\frac{\mathbf{m}\cdot\mathbf{n}}{\mu}\right]_2^1\mathbf{n} - [(\mathbf{H}\cdot\mathbf{n})(\mathbf{B}-B_n\mathbf{n})]_2^1 - \frac{1}{2}\left[\frac{\mathbf{B}^2}{\mu}\right]_2^1\mathbf{n}. \quad (21)$$

It is important to note that the surface density (21) has to be included when working with the volume force density given by the (13) and when the continuity conditions (16) and (20) are accepted.

It is useful to develop the (21) for those particular cases in which the materials are not susceptible to induced magnetization, $\chi = 0$ (permanent magnets) and for linear materials without permanent magnetization, $\mathbf{m}_p = \mathbf{0}$.

A. Surface Force Densities Between Two Linear Media Without Permanent Magnetization

In these media, (21) takes the following form:

$$\mathbf{f}_s^{(2)} = \left[\frac{1}{\mu}\right]_2^1 B_n^2\mathbf{n} - \frac{1}{2}\left[\frac{\mathbf{B}^2}{\mu}\right]_2^1\mathbf{n} \quad (22)$$

so that the force surface densities, in homogeneous media without permanent magnetization are always normal on the surfaces of the media in contact.

B. Surface Force Density on a Permanent Magnet in a Linear Media Without Permanent Magnetization

In the case in which medium 2 is a permanent magnet, $\mu_2 = \mu_0$, and medium 1 is a linear medium without permanent magnetization, the following equation is verified:

$$\mathbf{f}_s^{(2)} = B_n\left[B_n\left(\frac{1}{\mu_1}-\frac{1}{\mu_0}\right) - (\mathbf{m}_2\cdot\mathbf{n})\right]\mathbf{n} + \left((\mathbf{m}_2\cdot\mathbf{n})\mathbf{B}_2^t - B_n\mathbf{m}^t\right) - \frac{1}{2}\left[\frac{\mathbf{B}^2}{\mu}\right]_2^1\mathbf{n}. \quad (23)$$

In (23), the super index t indicates the components tangential to the surface of the fields $\mathbf{B}$ and $\mathbf{m}$. When medium 1 is air, expression (23) becomes

$$\mathbf{f}_s^{(2)} = -B_n\mathbf{m}_2 + (\mathbf{m}_2\cdot\mathbf{n})\mathbf{B}_2^t - \frac{1}{2}\left[\frac{\mathbf{B}^2}{\mu}\right]_2^1\mathbf{n}. \quad (24)$$

C. Surface Force Density Between Two Permanent Magnets

By applying (21) to this case, the following equation are obtained:

$$\mathbf{f}_s^{(2)} = B_n[\mathbf{m}\cdot\mathbf{n}]_2^1\mathbf{n} + \mu_0[\mathbf{m}\cdot\mathbf{n}]_2^1\mathbf{H}^t - \mu_0[(\mathbf{H}\cdot\mathbf{n})\mathbf{m}^t]_2^1 - \frac{1}{2\mu_0}[\mathbf{B}^2]_2^1\mathbf{n}. \quad (25)$$

The general expressions (19) or (21) and their particular application to different media (22)–(25) must be considered as surface contributions to the total force when the volume force density is given by (13). It has already been pointed out that the expression of total force as the sum of surface and volume forces is not univocally defined. In fact, the ESM and KV methods deal with different volume force densities and, therefore, different surface force densities. Points V and VI develop the expressions of total force given by (18), by testing whether they match the value of the forces obtained by the ESM and KV methods for media in air, and by deducing generalizations from these procedures in any given media. Point VII shows a comparison between this methodology and the CDSA procedure.

However, in the case that medium 2 is a linear medium without permanent magnetization in which the density of current $\mathbf{j}$ is zero, and with uniform magnetic susceptibility, it is easy to show that force density (13) vanishes:

$$\mathbf{f} = (\mathbf{H}.\vec{\nabla})\mathbf{B} + \vec{\nabla}G = \mu(\mathbf{H}.\vec{\nabla})\mathbf{H} - \frac{\mu}{2}\vec{\nabla}\mathbf{H}^2 = -\mu\mathbf{H}\wedge(\bar{\nabla}\wedge\mathbf{H}) = \mathbf{0} \quad (26)$$

so that the total force over the body surrounded by any given medium 1 is calculated as the integral over the surface of the body of the general expressions (17) and (19), or, allowing for the continuity conditions indicated, by using expression (21).

Moreover, if medium 1 is also a medium without permanent magnetization, the expression (17) in this case also coincides with the expression of Korteweg–Helmholtz's tensor for linear media without permanent magnetization [27]. This same result when applied to linear materials without permanent magnetization is referred to in other works as the ponderomotive force density acting upon the interface of two magnetic media [28].

V. Expression of the Total Force as a Function of Kelvin's Density

The expression of the force density (13) can be developed in a homogeneous medium by the use of the vectorial identity

$$(\mathbf{H}\cdot\vec{\nabla})\mathbf{B} = \frac{1}{2}\vec{\nabla}\wedge(\mathbf{H}\wedge\mathbf{B}) + \frac{1}{2}\vec{\nabla}(\mathbf{H}\cdot\mathbf{B}) - \frac{1}{2}\mathbf{B}(\vec{\nabla}\cdot\mathbf{H}) - \frac{1}{2}\mathbf{B}\wedge(\vec{\nabla}\wedge\mathbf{H}) - \frac{1}{2}\mathbf{H}\wedge(\vec{\nabla}\wedge\mathbf{B}) \quad (27)$$

that, when considering relation (5) and allowing that the density of current $\mathbf{j}$ is zero in the medium, leads to the general expression

$$\mathbf{f} = \mu_0(\mathbf{m}\cdot\vec{\nabla})\mathbf{H} + \vec{\nabla}\left(G + \mu_0\frac{\mathbf{H}^2}{2}\right) + \vec{\nabla}\wedge(\mathbf{B}\wedge\mathbf{H}). \quad (28)$$

Substituting in (18) the volume force density for the value given in (28), the following equation is obtained:

$$\mathbf{F}_2 = \int_{V_2}\mu_0(\mathbf{m}_2\cdot\vec{\nabla})\mathbf{H}_2 dV + \oint_{S_2}\left[\left(G_1 + \mu_0\frac{\mathbf{H}_2^2}{2}\right)\mathbf{n} + \mathbf{B}_1(\mathbf{H}_1\cdot\mathbf{n}) - \mathbf{H}_2(\mathbf{B}_2\cdot\mathbf{n})\right]dS_2 \quad (29)$$

in which, so as to achieve greater simplicity, the subindices 1 and 2 represent now the values in media 1 and 2. Expression (29) represents the total force over any given homogeneous magnetic medium 2 surrounded by any given magnetic medium 1.

This formula (29) is the generalization of Kelvin's classic expression, given that in the case that medium 1 is air, and supposing the continuity of the tangential components of field $\mathbf{H}$ in points on the surface (20), it is easy to obtain the classic Kelvin's expression:

$$\mathbf{F}_2 = \int_{V_2} \mu_0(\mathbf{m}_2 \cdot \vec{\nabla})\mathbf{H}_2 dV + \oint_{S_2} \mu_0 \frac{(\mathbf{m}_2 \cdot \mathbf{n})^2}{2} d\mathbf{S}_2. \quad (30)$$

VI. Expression of Total Force as a Function of the Densities of Equivalent Sources

Using the vectorial identity (27), in a similar way to its use in point V, it is possible to obtain the following integral relation:

$$\begin{aligned}\mathbf{F}_2 = \int_{V_2} \Big(&-\frac{\mu_0}{2}(\vec{\nabla} \cdot \mathbf{m}_2)\mathbf{H}_2 \\ &+\frac{1}{2}(\vec{\nabla} \wedge \mathbf{m}_2) \wedge \mathbf{B}_2 \Big) dV \\ &+ \oint_{S_2} \left(G_1 + \frac{1}{2}(\mathbf{B}_2 \cdot \mathbf{H}_2) + \mu_0 \frac{\mathbf{m}_2^2}{4} \right) d\mathbf{S}_2 \\ &+ \oint_{S_2} \Big(\mathbf{B}_1(\mathbf{H}_1 \cdot \mathbf{n}) - \frac{\mu_0}{2}\mathbf{m}_2(\mathbf{m}_2 \cdot \mathbf{n}) \\ &-\frac{1}{2}\mathbf{B}_2(\mathbf{H}_2 \cdot \mathbf{n}) - \frac{1}{2}\mathbf{H}_2(\mathbf{B}_2 \cdot \mathbf{n}) \Big) dS_2. \end{aligned} \quad (31)$$

The general expression (31) gives the total force over a magnetic medium 2 surrounded by any given medium 1, without currents, calculating the volume force density as a function of the divergence and the rotational of the magnetization of material. This expression can be applied to the particular case in which medium 1 is air, and by applying the usual continuity conditions (16) and (20) the following expression can be obtained:

$$\begin{aligned}\mathbf{F}_2 = \int_{V_2} \Big(&-\frac{\mu_0}{2}(\vec{\nabla} \cdot \mathbf{m}_2)\mathbf{H}_2 \\ &+\frac{1}{2}(\vec{\nabla} \wedge \mathbf{m}_2) \wedge \mathbf{B}_2 \Big) dV \\ &+ \oint_{S_2} \Big(-\frac{1}{2}(\mathbf{n} \wedge \mathbf{m}_2) \wedge \left(\frac{\mathbf{B}_2 + \mathbf{B}_1}{2} \right) \\ &+\frac{\mu_0}{2}(\mathbf{m}_2 \cdot \mathbf{n}) \left(\frac{\mathbf{H}_2 + \mathbf{H}_1}{2} \right) \Big) dS_2. \end{aligned} \quad (32)$$

On the other hand, the ESM procedures lead to the well-known expressions:

$$\begin{aligned}\mathbf{F}_2^{(\rho)} &= \int_{V_2} -\mu_0(\vec{\nabla} \cdot \mathbf{m}_2)\mathbf{H}_2 dV \\ &\quad + \oint_{S_2} \mu_0(\mathbf{m}_2 \cdot \mathbf{n}) \left(\frac{\mathbf{H}_2 + \mathbf{H}_1}{2} \right) dS_2 \\ &= \int_{V_2} \rho_m \mathbf{H}_2 dV + \oint_{S_2} \rho_m^S \left(\frac{\mathbf{H}_2 + \mathbf{H}_1}{2} \right) dS_2 \end{aligned} \quad (33)$$

in which $\rho_m = -\mu_0(\vec{\nabla}.\mathbf{m}_2)$ and $\rho_m^S = \mu_0(\mathbf{m}_2.\mathbf{n})$ are volume and surface densities of equivalent magnetic charge, and

$$\begin{aligned}\mathbf{F}_2^{(j)} &= \int_{V_2} ((\vec{\nabla} \wedge \mathbf{m}_2) \wedge \mathbf{B}_2) dV \\ &\quad + \oint_{S_2} \left(-(\mathbf{n} \wedge \mathbf{m}_2) \left(\frac{\mathbf{B}_2 + \mathbf{B}_1}{2} \right) \right) dS_2 \\ &= \int_{V_2} (\mathbf{j}_m \wedge \mathbf{B}_2) dV + \oint_{S_2} \left(\mathbf{j}_m^S \wedge \left(\frac{\mathbf{B}_2 + \mathbf{B}_1}{2} \right) \right) dS_2 \end{aligned} \quad (34)$$

in which $\mathbf{j}_m = (\vec{\nabla} \wedge \mathbf{m}_2)$ and $\mathbf{j}_m^S = -(\mathbf{n} \wedge \mathbf{m}_2)$ are volume and surface densities of equivalent current, in such a way that (32) can be reduced to

$$\mathbf{F}_2 = \frac{1}{2}\mathbf{F}_2^{(\rho)} + \frac{1}{2}\mathbf{F}_2^{(j)}. \quad (35)$$

This confirms that the method employed in this work is in accordance with the methods of equivalent magnetic charge and equivalent magnetizing current. This also shows that (31) is an extension of the ESM that allows for its application in the case that the magnetic material is not surrounded by air.

VII. Comparison With the CDSA Procedure

References [19] and [20] show a general procedure, called continuum design sensitivity analysis (CDSA), for the calculation of global and local forces in continuous media that can be applied to linear and no linear magnetostatic problems. The energy sensitivity function, that is a virtual variation of the energy and the co-energy of the system with respect to the design variable, includes terms of volume of the two magnetic media and their contact surfaces. Each medium is described by the distribution of its magnetic permeability, permanent magnetization and current. This allows for the calculation of forces corresponding to each interaction, by a suitable variation of the energy or the co-energy of the system. So, the force between linear or nonlinear media, without permanent magnetization, is obtained as a variation of the co-energy of the system, and its result is the surface distribution expressed by the following equation:

$$\mathbf{f}_{s,2}^{(\mathrm{CDSA})} = \frac{1}{2}\left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right)(\mathbf{B}_2 \cdot \mathbf{B}_1)\mathbf{n} \quad (36)$$

and a volume distribution

$$\mathbf{f}_2^{(\mathrm{CDSA})} = \frac{1}{2}\vec{\nabla}\left(\frac{1}{\mu_2} \right)\mathbf{B}_2^2. \quad (37)$$

Expression (17) gives, following the procedure described in this work, the surface force in all cases. It can easily be shown that expressions (36) and (17) are equivalent, when the co-energy densities in media 1 and 2 in (17) are calculated by (11), that is to say, when the magnetic behavior is taken to be linear. In the same way, (37) can be easily obtained from the general expression of volume force density (13).

In [20], the distribution of forces for a system formed by two permanent magnets is calculated, as a variation of the energy of the system (i.e., by maintaining the magnetic flux constant in the virtual displacement, in agreement with VWP). As a consequence, the global force over magnet 2 is

$$\mathbf{F}_2^{(\mathrm{CDSA})} = \oint_{S_2} [(\mathbf{m}_2 - \mathbf{m}_1) \cdot \mathbf{B}_2]\mathbf{n} dS_2. \tag{38}$$

Expression (38) shows that the global force over the magnet, applying CDSA, is obtained only from the surface terms and there are no volume forces. By application of actual procedures based in the integration of surface and volume densities (KV, ESM), this result is only obtained when using the ESM procedure and if the magnetization of the magnet is taken to be uniform.

Using the methodology followed in this work, volume force densities that have a value zero for any distribution of magnetization are not obtained. For this reason, the forces on magnets, in a general configuration, shows components of both surface and volume.

One of the relevant aspects of work [20] is the definition of a sole function from which the forces in the system can be obtained, by variation of the energy or the co-energy. In the same way, [13] showed magneto-mechanical tensors obtained either from the energy or the co-energy. The surface and volume force densities obtained from the corresponding tensors generally gave the same results. It is easy to demonstrate a formal relation between the global force values calculated from the tensors obtained by a variation of the energy (F) or the co-energy (G)

$$\mathbf{F}_2^F = \mathbf{F}_2^G + \oint_{S_2} (\mathbf{n} \wedge (\mathbf{B}_1 \wedge \mathbf{H}_1)) dS_2. \tag{39}$$

Equation (39) indicates that the forces calculated by varying the co-energy or the energy are always equal except when the medium 1, which surrounds the body studied, has a permanent magnetization. In this case, it is necessary to add to the general expressions the surface contribution indicated in expression (39).

In practice, this case is only relevant in the interaction between permanent magnets and in the absence of any other source of magnetic energy.

The application of (39) to expression (31) determines the value of the force between two magnets, obtained by a variation of their magnetic energy. This force is expressed in terms of the charge and current equivalents, which in turn depend on the spatial distribution of magnetization.

As has been previously indicated, it has not, as yet, been possible to establish an equivalence between the CDSA procedure and the procedure detailed in this work for the interaction between two permanent magnets of nonuniform magnetization. The application of expression (31) and (39) to a permanent uniform magnet in interaction with another magnet leads to a expression of force that includes term (38) together with other terms.

Reference [20] does not show results relative to the forces of interaction between linear or nonlinear media with permanents magnets that can be contrasted with the methodology presented in this paper.

VIII. Conclusion

This work presents a procedure by means of which it is possible to calculate the total forces over a magnetic media surrounded by another magnetic media from the magneto-mechanical tensor obtained by the application of the VWP to the density of the co-energy. The way in which the volume force density and its corresponding surface force density are calculated allows for the ESM and KV to be extended to include the case in which a body is not surrounded by air, and therefore makes it possible to determine the surface forces in materials in contact.

It has been shown that the application of this procedure to the case in which a body is surrounded by air reproduces the results given by the standard ESM and KV procedures. It has also been demonstrated that the expressions for the surface forces obtained from this tensor reproduce the results given by the classic expressions for the forces between linear nonpermanent magnetic media.

An analysis of the results obtained by the CDSA procedure, using the methodology described in this work, has been carried out. The forces given by the interaction of two linear media are equivalent in both procedures. In the case of nonlinear media the results are also equivalent if the expression of co-energy density G is treated as if it were linear. The application of the methodology presented in this work for the interaction of two general permanent magnets leads to volume force densities.

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