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Stirling's Formula to De Moivre & Laplace Theorem

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Abstract

Stirling formula and De Moivre Laplace theorem are important intermediate steps toward the central limit theorem. In many undergraduate statistics textbooks, however, step by step proof is often omitted, which sometimes poses quite a huge challenge to students who want to fully understand the logic behind the formulas. This note is trying to fill the gap between unusually looking equations and the proofs behind them. During the process of writing this note, I referred frequently to [**balazs2014stirling**] and Jacek Cichon's "Stirling Approximation Formula", to whom a huge debt is owed.

1 Stirling's Formula

Stirling's formula is necessary for the proof of De Moivre Laplace Theorem. It's an approximation of the n factorial:

Stirling's Formula

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

2 A Rough Estimate

If we take logarithm of $n!$, the product of "$1 \cdot 2 \cdot 3 \cdots n$" becomes the sum of logarithms (let's name it as " $S(n)$ "), i.e., $S(n) = \ln 1 + \ln 2 + \cdots + \ln n$. A straight forward comparisons below:

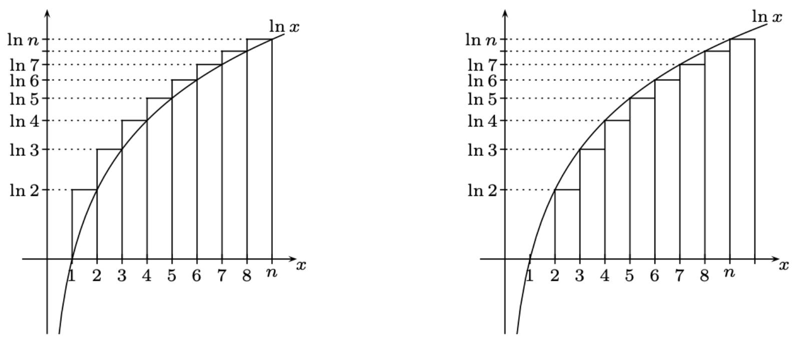

Figure 1: $S(n)$ vs. $\ln x$

We can directly derive the following bounds:

$$\int_1^n \ln x dx \leqslant S(n) \leqslant \int_1^{n+1} \ln x dx \tag{1}$$

Integrate the left side of the above equation, we have

$$\int_1^n \ln x dx = (x \ln x - x)\Big|_1^n = n \ln n - (n-1)$$

Integrate the rigth side gives us

$$\int_1^{n+1} \ln x dx = (n+1)\ln(n+1) - n$$

We have new equation from eq.(1) as

$$\ln \frac{n^n}{e^{n-1}} \leqslant \ln n! \leqslant \ln \frac{(n+1)^{n+1}}{e^n}$$

$$\frac{n^n}{e^{n-1}} \leqslant n! \leqslant \frac{(n+1)^{n+1}}{e^n}$$

Take a second look at the right side of the above equation, we can simplify it

$$\begin{aligned}\frac{(n+1)^{n+1}}{e^n} &= \left(\frac{n+1}{e}\right)^n (n+1) \\ &= \left(\frac{n}{e} \cdot \frac{n+1}{n}\right)^n (n+1) \\ &= \left(\frac{n}{e}\right)^n \cdot \left(1 + \frac{1}{n}\right)^n (n+1)\end{aligned}$$

as $n \to \infty$, $\left(1+\frac{1}{n}\right)^n = e$, the right side becomes $e(n+1)\left(\frac{n}{e}\right)^n$, we have a shorter form of eq. (1) as

$$e\left(\frac{n}{e}\right)^n \leqslant n! \leqslant e(n+1)\left(\frac{n}{e}\right)^n$$

which is saying that

$$n! = f(n)\left(\frac{n}{e}\right)^n \tag{2}$$

for a function $f(n)$ where $e \leqslant f(n) \leqslant e(n+1)$.

3 Finding Constant C

We re-arrange eq. (2) by dividing both sides with $\sqrt{2n}\left(\frac{n}{e}\right)^n$:

$$\frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n} = \frac{f(n)}{\sqrt{2n}}$$

now we need to find a limit C (if there's one) of the left side when n approaches to 0, then we get our result for $n! \approx C\sqrt{2n}\left(\frac{n}{e}\right)^n$.

3.1 The Integration of $\int \sin^n x\, dx$

We start from the integration of $\sin^n x\, dx$:

$$\begin{aligned}
\int \sin^n x\, dx &= \int \sin^{n-1} x \sin x\, dx \\
&= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x \cos^2 x\, dx \\
&= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x(1-\sin^2 x)\, dx \\
&= -\sin^{n-1} x \cos x + \int (n-1)\sin^{n-2} x\, dx - \int (n-1)\sin^n x\, dx \\
n\int \sin^n x\, dx &= -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx
\end{aligned}$$

so we have,

$$\int \sin^n x\, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx$$

If we evaluate from $0 \sim \frac{\pi}{2}$, the above equation reduces to

$$\int_0^{\pi/2} \sin^n x dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x dx$$

Let S be the integration sum, we need two different equations to address two situations, one for even and one for odd:

(a) for $n = k$,

$$\begin{aligned}
S_{even} &= \int_0^{\pi/2} \sin^{2k} x\, dx \\
&= \frac{2k-1}{2k}\cdot\frac{2k-3}{2k-2}\cdots\frac{1}{2}\cdot\int_0^{\pi/2}\sin^0 x\, dx \\
&= \frac{\pi}{2}\prod_{k=1}^{n/2}\frac{2k-1}{2k}
\end{aligned}$$

(b) for $n = 2k+1$,

$$\begin{aligned}
S_{odd} &= \int_0^{\pi/2} \sin^{2k+1} x\, dx \\
&= \frac{2k}{2k+1}\cdot\frac{2k-2}{2k-1}\cdots\frac{2}{3}\cdot\int_0^{\pi/2}\sin^1 x\, dx \\
&= \prod_{k=1}^{(n-1)/2}\frac{2k}{2k+1}\cdot(-\cos x)\Big|_0^{\pi/2} \\
&= \prod_{k=1}^{(n-1)/2}\frac{2k}{2k+1}
\end{aligned}$$

3.2 Wallis Product Formula

Wallis Formula

$$\prod_{n=1}^{\infty} \frac{2n}{2n-1} \frac{2n}{2n+1} = \frac{\pi}{2}$$

The left side of Wallis formula is actually $\frac{\pi}{2} \cdot \frac{S_{odd}}{S_{even}}$ for $x \in (0, \pi/2)$. If we can prove that $\frac{S_{odd}}{S_{even}} = 1$ as $n \to \infty$, we are done with the proof. First of all, we know that

$$0 < \sin^{2k+2} x < \sin^{2k+1} x < \sin^{2k} x$$

$$0 < S_{2k+2} < S_{2k+1} < S_{2k}$$

divide by S_{2k} on each term, we have

$$0 < \frac{S_{2k+2}}{S_{2k}} < \frac{S_{2k+1}}{S_{2k}} < 1$$

Because $\lim_{k\to\infty} \frac{S_{2k+2}}{S_{2k}} = \lim_{k\to\infty} \frac{2k+1}{2k+2} = 1$, we can apply the squeeze theorem here that

$$1 < \frac{S_{2k+1}}{S_{2k}} < 1 \quad \text{when } k \to \infty$$

$$\text{so} \quad \frac{S_{odd}}{S_{even}} = 1 \quad \text{when} k \to \infty$$

3.3 Next Step

Now we have all tools we need to crack down the constant C. First, let's re-write the Wallis formula in a compact way:

$$\begin{aligned}
\prod_{n=1}^{n} \frac{2n}{2n-1} \frac{2n}{2n+1} &= 2^{2n}(n!)^2 \prod_{n=1}^{n} \frac{1}{2n-1} \frac{1}{2n+1} \\
&= 2^{2n}(n!)^2 \cdot \frac{2 \cdot 4 \cdots 2n}{(2n)!} \cdot \frac{2 \cdot 4 \cdots 2n}{(2n+1)!} \\
&= \frac{2^{2n}(n!)^2 2^{2n}(n!)^2}{((2n)!)^2(2n+1)}
\end{aligned}$$

so

$$\prod_{n=1}^{n} \frac{2n}{2n-1} \frac{2n}{2n+1} = \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} \tag{3}$$

Since we pick C so $n! \approx C\sqrt{2n}\left(\frac{n}{e}\right)^n$, the equation still holds if we substitute n by $2n$: $(2n)! \approx C\sqrt{4n}\left(\frac{2n}{e}\right)^{2n}$. Now we use C terms instead of $n!$ and $2n!$ and re-write eq.(3)

as follows:

$$\begin{aligned}\frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} &= \frac{2^{4n}(C\sqrt{2n}\left(\frac{n}{e}\right)^n)^4}{(C\sqrt{4n}(\frac{2n}{e})^{2n})^2(2n+1)} \\ &= \frac{2^{4n}C^4 4n^2\left(\frac{n}{e}\right)^{4n}}{C^2 4n(\frac{2n}{e})^{4n}(2n+1)} \\ &= \frac{2^{4n}C^2 n\left(\frac{1}{2}\right)^{4n}}{2n+1} \\ &= C^2\frac{n}{2n+1}\end{aligned}$$

from Wallis formula, we know

$$\begin{aligned}\lim_{n\to\infty} C^2\frac{n}{2n+1} &= \frac{\pi}{2} \\ C^2\cdot\frac{1}{2} &= \frac{\pi}{2} \\ C &= \sqrt{\pi}\end{aligned}$$

4 Rethink of C

We've known that Stirling's formula is the limit of $n!$ when $n \to \infty$, so is the C. However, if n is not a very large number, what's the boundaries of C, i.e.,

$$? \leqslant \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n} \leqslant ?$$

Let

$$a_n = \frac{n!}{\sqrt{2n}\left(\frac{n}{e}\right)^n} \qquad \text{and} \qquad b_n = \ln a_n$$

we get the equation of $b_n - b_{n+1}$ as follows:

$$\begin{aligned}b_n - b_{n+1} &= \ln n! - \frac{1}{2}\ln 2n - n(\ln n - 1) - \ln(n+1)! + \frac{1}{2}\ln(2n+2) + (n+1)(\ln(n+1) - 1) \\ &= -\ln(n+1) + \frac{1}{2}\ln\left(\frac{n+1}{n}\right) + (n+1)(\ln(n+1)) - n\ln n - 1 \\ &= n\ln(n+1) - n\ln n + \frac{1}{2}\ln\left(\frac{n+1}{n}\right) - 1 \\ &= \left(n+\frac{1}{2}\right)\ln\left(\frac{n+1}{n}\right) - 1\end{aligned}$$

Now we need some techniques from Taylor Series expansion. First of all, we use $t = \dfrac{1}{2n+1}$ to re-write the above equation as

$$b_n - b_{n+1} = \frac{1}{2t}\ln\frac{1+t}{1-t} - 1$$

then we use polynomial expansion of $\ln(1+t)$:

$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots = -\sum_{k=1}^{\infty}(-1)^k \frac{t^k}{k}$$

and of $\ln(1-t)$:

$$\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \cdots = -\sum_{k=1}^{\infty} \frac{t^k}{k}$$

so

$$\begin{aligned} b_n - b_{n+1} &= \frac{1}{2t}\left(-\sum_{k=1}^{\infty}(-1)^k \frac{t^k}{k} + \sum_{k=1}^{\infty} \frac{t^k}{k}\right) - 1 \\ &= \frac{1}{2t}\sum_{k=0}^{\infty} \frac{2t^{2k+1}}{2k+1} - 1 \\ &= \frac{1}{t}\sum_{k=1}^{\infty} \frac{t^{2k+1}}{2k+1} \end{aligned}$$

Because we are constructing t in such a way that no matter what value n takes from positive integers, t will always be positive either, we know that d_n is decreasing.

Further, we look at another feature of $d_n - d_{n+1}$ when we change t back to n:

$$\begin{aligned} b_n - b_{n+1} &= \frac{1}{t}\sum_{k=1}^{\infty} \frac{t^{2k+1}}{2k+1} \\ &= (2n+1)\sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k+1}} \\ &= \sum_{k=1}^{\infty} \frac{1}{2k+1}\frac{1}{(2n+1)^{2k}} \end{aligned} \tag{4}$$

We can derive two inequality forms from eq. (4):

$$\begin{aligned} b_n - b_{n+1} &< \sum_{k=1}^{\infty} \frac{1}{3}\frac{1}{(2n+1)^{2k}} \\ &= \frac{1}{3}\frac{1}{(2n+1)^2} \cdot \frac{1}{1 - \frac{1}{(2n+1)^2}} \\ &= \frac{1}{12}\frac{1}{n(n+1)} \end{aligned}$$

and

$$b_n - b_{n+1} > \sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2k+1}} \quad [1]$$

[1] I haven't figured out how to simplify the right side of the above equation.

Observe that (a telescoping sum):

$$
\begin{aligned}
b_1 - b_n &= (b_1 - b_2) + (b_2 - b_3) + (b_3 - b_4) + \cdots + (b_{n-1} - b_n) \\
&< \frac{1}{12} \sum_{m=1}^{n-1} \frac{1}{m(m+1)} = \frac{1}{12}\left(1 - \frac{1}{n}\right) \quad \text{and,} \\
&> \sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2k+1}}
\end{aligned}
$$

hence

$$
b_n > b_1 - \frac{1}{12}\left(1 - \frac{1}{n}\right) = \frac{11}{12} - \frac{1}{2}\ln 2 + \frac{1}{12n}
$$

and

$$
b_n < b_1 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2k+1}} = 1 - \frac{1}{2}\ln 2 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^{2k+1}}
$$