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The Gamma Distribution

Study Notes | Written by Larry Cui

1 Prologue: waiting time variable

Under a Poisson distribution $p_X(k) = \dfrac{e^{-\lambda y}(\lambda y)^k}{k!}$, if we want to know the waiting time distribution of the next occurrence, we would differentiate the cdf of all occurrences probability during the certain period of y:

$$F_Y(y) = [1 - P(Y = 0)] = 1 - e^{-\lambda y} \qquad \Rightarrow \qquad f_Y(y) = \frac{d}{dy}F_Y(y) = \lambda e^{-\lambda y}, \quad y > 0$$

A little modification of this idea can bring it to a broader application: the waiting time distribution for the rth event occurrence.

lemma Suppose that Poisson events are occurring at the constant rate of λ per unit time. Waiting time Y for the rth event has a pdf

$$f_Y(y) = \frac{\lambda^r}{(r-1)!}y^{r-1}e^{-\lambda y}, \quad y > 0$$

Proof: To construct this formula, first of all, we know that for rth event happening exactly on time point y, $(r - 1)$ events must happen first within $0 \backsim y$ period.

$$\sum_{k=0}^{r-1} e^{-\lambda y}\frac{(\lambda y)^k}{k!} \tag{1}$$

$$F_Y(y) = 1 - \sum_{k=0}^{r-1} e^{-\lambda y}\frac{(\lambda y)^k}{k!} \tag{2}$$

Eq. (1) is the probability of $(r - 1)$ events happening within y time period, and Eq. (2) is apparently the probability of rth and more events happening during the same period. Differentiating $F_Y(y)$, the meaning of which is to find the changing rate of F, gives us the pdf of the rth event happening on time point y. Someone may argue "why rth event?" After all, $F_Y(Y)$ is the cdf for rth and more events. Well, rth event is the closest to happen, and for $(r + 1)$th and above, we will have updated pdf for them.

By differentiating:

$$\begin{aligned} f_Y(y) = F_Y'(y) &= \frac{d}{dy}\left[1 - \sum_{k=0}^{r-1} e^{-\lambda y}\frac{(\lambda y)^k}{k!}\right] \\ &= \sum_{k=0}^{r-1} \lambda e^{-\lambda y}\frac{(\lambda y)^k}{k!} - \sum_{k=1}^{r-1} \lambda e^{-\lambda y}\frac{(\lambda y)^{k-1}}{(k-1)!} \qquad \text{Remark: } \left[\frac{(\lambda y)^0}{0!}\right]' = 0 \\ &= \sum_{k=0}^{r-1} \lambda e^{-\lambda y}\frac{(\lambda y)^k}{k!} - \sum_{k=0}^{r-2} \lambda e^{-\lambda y}\frac{(\lambda y)^k}{k!} \\ &= \frac{\lambda^r}{(r-1)!} y^{r-1} e^{-\lambda y} \end{aligned}$$

2 Gamma Function

Definition

$$\Gamma(r) = \int_0^\infty y^{r-1} e^{-y}\, dy$$

Although we derive this function from the analysis of waiting time for rth event, for the function itself, r is not necessarily to be any integers, but real number. We also have some interesting features for this gamma function:

- $\Gamma(1) = 1$
  proof: $\Gamma(1) = \displaystyle\int_0^\infty y^{1-1} e^{-y}\, dy = \int_0^\infty e^{-y}\, dy = 1$
- $\Gamma(r) = (r-1)\Gamma(r-1)$
  proof: $\displaystyle\int_0^\infty y^{r-1} e^{-y}\, dy = -y^{r-1}e^{-y}\Big|_0^\infty + \int_0^\infty (r-1) y^{r-2} e^{-y}\, dy = (r-1)\int_0^\infty y^{r-2} e^{-y}\, dy$
- if r is an integer, then $\Gamma(r) = (r-1)!$
  proof: this is the result of direct application of the above two features.

3 Gamma pdf

Definition a random variable Y is said to have the gamma pdf with paramaters r and λ if

$$f_Y(y) = \underbrace{\frac{\lambda^r}{\Gamma(r)}}_{\text{constant}} y^{r-1} e^{-\lambda y}, \quad y \geqslant 0$$

Proof: $f_Y(y)$ is always positive, so the rest is to prove the sum adds up to 1. If we let $y' = \lambda y, dy' = \lambda\, dy$, the right part of the above equation

$$\frac{1}{\Gamma(r)}\int_0^\infty \lambda^r y^{r-1} e^{-\lambda y}\, dy = \frac{1}{\Gamma(r)}\int_0^\infty y'^{(r-1)} e^{-y'}\, dy' = \frac{1}{\Gamma(r)} \cdot \Gamma(r) = 1$$

The expected value and variance of Gamma pdf:

$$\begin{aligned}
E(X) &= \frac{1}{\Gamma(r)} \int_0^\infty y \cdot \lambda^r y^{r-1} e^{-\lambda y}\, dy \\
&= \frac{1}{\Gamma(r)} \frac{1}{\lambda} \int_0^\infty u^r e^{-u}\, du && \text{Remark: } u = \lambda y, du = \lambda\, dy \\
&= \frac{1}{\Gamma(r)} \frac{1}{\lambda} \cdot \Gamma(r+1) \\
&= \frac{r}{\lambda} && \text{Remark: } \Gamma(r+1) = r\Gamma(r)
\end{aligned}$$

$$\begin{aligned}
Var(X) = E(X^2) - E(X)^2 &= \frac{1}{\Gamma(r)} \int_0^\infty y^2 \cdot \lambda^r y^{r-1} e^{-\lambda y}\, dy - \frac{r^2}{\lambda^2} \\
&= \frac{1}{\Gamma(r)} \frac{1}{\lambda^2} \int_0^\infty u^{r+1} e^{-u}\, du - \frac{r^2}{\lambda^2} && \text{Remark: } u = \lambda y, du = \lambda\, dy \\
&= \frac{1}{\Gamma(r)} \frac{1}{\lambda^2} \cdot \Gamma(r+2) - \frac{r^2}{\lambda^2} \\
&= \frac{r}{\lambda^2} && \text{Remark: } \Gamma(r+2) = (r+1)r\Gamma(r)
\end{aligned}$$

4 Additive Property and *mgf*

The additive property of a variable means the sum of two or more independent variables "reproduce" the similar pdf. Usually we see binomial, Poisson or normal variables behave this way, because in essence they all have "normal like" pdf. Most other variables, however, don't have such property, due to the central limit theorem.

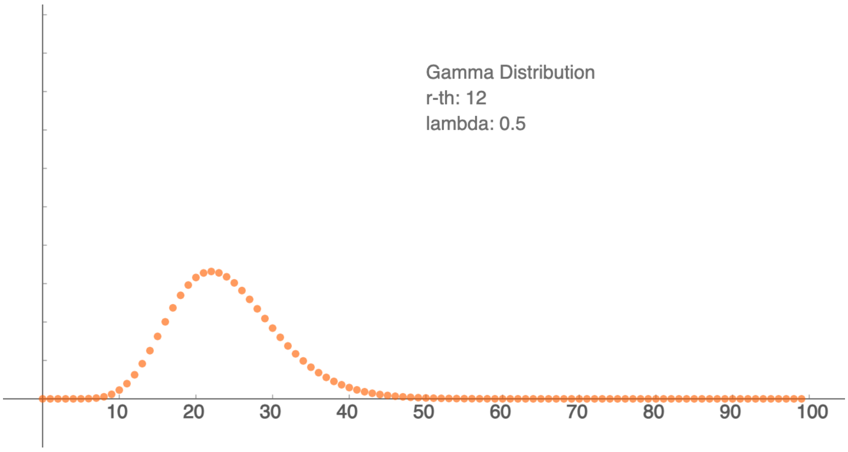

Figure 1: Gamma Distribution *pdf*

Additive Property of Gamma pdf

Suppose two independent variables U has the gamma pdf with parameters r and λ, V with s and the same λ. Then $U + V$ has a gamma pdf with parameters $r + s$ and λ.

Proof(direct, weak version): Let $t = u + v$, then pdf of t is

$$
\begin{aligned}
f_{U+V}(t) &= \int_0^t f_U(u) f_V(t-u)\, du && \text{Remark: both } u, v \geqslant 0 \text{, so the scope of } u \text{ is } (0,t) \\
&= \frac{\lambda^{r+s}}{\Gamma(r)\Gamma(s)} \int_0^t u^{r-1} e^{-\lambda u} (t-u)^{s-1} e^{-\lambda(t-u)}\, du \\
&= \frac{\lambda^{r+s} e^{-\lambda t}}{\Gamma(r)\Gamma(s)} \int_0^t u^{r-1} (t-u)^{s-1}\, du \\
&= \frac{\lambda^{r+s} e^{-\lambda t} t^{r+s-1}}{\Gamma(r)\Gamma(s)} \int_0^1 x^{r-1} (1-x)^{s-1}\, dx && \text{Let } x = u/t, x \in (0,1)
\end{aligned}
$$

we can tell from the above formula that $e^{-\lambda t} t^{r+s-1}$ is the only changing part of the function and matches a gamma pdf with $r = r + s$ and $y = t$, the rest is constant.

Strong version: another way to prove the additive property is by mgf. First of all, we find the $M_Y(t)$ of a gamma pdf

$$
\begin{aligned}
M_Y(t) &= \int_0^\infty e^{ty} f_Y(y)\, dy \\
&= \frac{1}{\Gamma(r)} \int_0^\infty e^{ty} \cdot \lambda^r y^{r-1} e^{-\lambda y}\, dy \\
&= \frac{\lambda^r}{(\lambda - t)^r} \cdot \left[\frac{1}{\Gamma(r)} \int_0^\infty (\lambda - t)^r y^{r-1} e^{-(\lambda - t) y}\, dy \right] \\
&= \left(1 - \frac{t}{\lambda}\right)^{-r} && \text{in brackets is the integral of a gamma pdf with } \lambda = \lambda - t
\end{aligned}
$$

Then we have

$$
M_U(t) = \left(1 - \frac{t}{\lambda}\right)^{-r} \quad \text{and} \quad M_V(t) = \left(1 - \frac{t}{\lambda}\right)^{-s}
$$

and

$$
M_{U+V}(t) = M_U(t) \cdot M_V(t) = \left(1 - \frac{t}{\lambda}\right)^{-(r+s)}
$$

From this mgf, we conclude the pdf of new variable $U + V$ has gamma pdf with $r = r + s$ and λ.