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The Geometric Distribution

Study Notes | Written by Larry Cui

Consider a series of independent trials and each has one of two outcomes, success or failure. If p is the probability of success, the geometric distribution means the probability at which the first success occurs. Obviously, a formula for the pdf of this distribution is:

Geometric Distribution pdf

$$p_X(k) = (1-p)^{k-1}p, \qquad k = 1, 2, \ldots$$

1 cdf of the geometric distribution:

$$\begin{aligned} F_X(t) = P(X \leqslant t) &= \sum_{k=1}^{t} (1-p)^{k-1}p \\ &= p\sum_{k=1}^{t}(1-p)^{k-1} \\ &= p \cdot \frac{1-(1-p)^t}{1-(1-p)} \\ &= 1-(1-p)^t \end{aligned}$$

2 $E(X)$ of the geometric distribution:

$$\begin{aligned} E(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}p \\ &= p\sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \end{aligned}$$

Now take a look at the sum part, we can tell it's the derivative of another series:

$$\left[\sum_{k=1}^{\infty} -(1-p)^k\right]' = \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1}$$

But the closed form of the formula within the brackets is $\dfrac{p-1}{p}$, so

$$\sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} = \left(\frac{p-1}{p}\right)' = \frac{1}{p^2}$$

and

$$E(X) = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

3 Var(X) of the geometric distribution:

The key step to find variance is to find $E(X^2)$. By the same technique, we have

$$\begin{aligned}\left[\sum_{k=1}^{\infty}(1-p)^{k+1}\right]'' &= \sum_{k=1}^{\infty}(k+1)k \cdot (1-p)^{k-1}p \\ &= \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} + k \cdot (1-p)^{k-1}\end{aligned}$$

The second term on the right side of the equation is $\frac{1}{p^2}$, and we can also use a closed form to represent the left side, so the equation becomes

$$\begin{aligned}\left[\frac{(1-p)^2}{p}\right]'' &= \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} + \frac{1}{p^2} \\ \frac{2}{p^3} &= \sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} + \frac{1}{p^2}\end{aligned}$$

so

$$E(X^2) = p\sum_{k=1}^{\infty} k^2 \cdot (1-p)^{k-1} = p \cdot \left(\frac{2}{p^3} - \frac{1}{p^2}\right) = \frac{2-p}{p^2}$$

and

$$VarX = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

4 *mgf* of the geometric distribution:

By definition, the moment generating function of a distribution function is the expected value of e^{tx}:

$$\begin{aligned}M_k(t) = E(e^{tk}) &= \sum_{k=1}^{\infty} e^{tk} \cdot (1-p)^{k-1}p \\ &= \frac{p}{1-p}\sum_{k=1}^{\infty} e^{tk} \cdot (1-p)^k \\ &= \frac{p}{1-p} \cdot \frac{e^t(1-p)}{1-e^t(1-p)} \\ M_k(t) &= \frac{pe^t}{1-(1-p)e^t}\end{aligned}$$