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The Negative Binomial Dsitribution

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Follow the logic of geometric distribution, if we want to study the probability of r^{th} success in k^{th} of a series of trials, it must be the case that $(r-1)$ success occur during the first $(k-1)$ trials and the r^{th} happens on exactly the k^{th} trial.

Negative Binomial Distribution pdf

$$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \qquad k = r, r+1, \dots$$

If we let X be the sum of independent variables $X_1, X_2, \dots, X_r$, and let $X \to \infty$, the negative binomial can be interpreted as r successes happen one after another, and each of which follows the geometric distribution model. This perspective won't give us a better form to the pdf, but will greatly simplify the calculation of expected values.

E(X), Var(X) and mgf Let $p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots$ Then

a. $M_X(t) = M_{X_1}(t) M_{X_2}(t) \dots M_{X_r}(t) = \left[\dfrac{pe^t}{1-(1-p)e^t} \right]^r$

b. $E(X) = E(X_1) + E(X_2) + \dots + E(X_r) = \dfrac{r}{p}$

c. $Var(X) = Var(X_1) + Var(X_2) + \dots + Var(X_r) = \dfrac{r(1-p)}{p^2}$

Algebraic Approach: the sum of pdf

There's an algebraic approach to the $E(X)$ and $\mathrm{Var}(X)$, which centres around the verification of the pdf of the function. Let's take a look at the following lemma:

lemma (a)

$$\binom{k+m}{m} = \binom{k+m-1}{m} + \binom{k+m-1}{m-1}$$

Proof: a simple expansion of the combinations gives us result. Furthermore, we can also see this equation by some intuition: if you want pick m items from $k+m$ pool, you can first split the pool into one item and the other. The final pick consists of two scenarios: 1) excluding this one item: $\binom{k+m-1}{m}$; and 2) including this one item: $\binom{k+m-1}{m-1}$.

$$\begin{aligned}\binom{k+m-1}{m}+\binom{k+m-1}{m-1} &= \frac{(k+m-1)!}{m!(k-1)!}+\frac{(k+m-1)!}{(m-1)!k!} \\ &= \frac{(k+m-1)!k+(k+m-1)!m}{m!k!} \\ &= \frac{(k+m)!}{m!k!}\end{aligned}$$

Intermediate function: consider the function

$$f_m(z)=\sum_{k=0}^{\infty}\binom{k+m}{m}z^k$$

Use lemma (a), we can expand it as

$$f_m(z)=\sum_{k=0}^{\infty}\binom{k+m-1}{m-1}z^k+\sum_{k=0}^{\infty}\binom{k+m-1}{m}z^k$$

Because $\binom{m-1}{m}=\frac{(m-1)(m-2)\cdots(m-m)}{m!(-1)!}=0$ by definition, we can discard the $k=0$ item from the second term. A little re-arrangement give the following form for the second term:

$$\sum_{k=1}^{\infty}\binom{k+m-1}{m}z^k=z\sum_{k=1}^{\infty}\binom{k+m-1}{m}z^{k-1}=zf_m(z)$$

so

$$f_m(z)=f_{m-1}(z)+zf_m(z)\quad\Rightarrow\quad f_m(z)=\frac{f_{m-1}(z)}{1-z}$$

But since $f_0(z)=\sum_{k=0}^{\infty}\binom{k}{0}z^k=\frac{1}{1-z}$, we know $f_m(z)=\frac{1}{(1-z)^{m+1}}$.

To verify the sum of the pdf equals to 1, we use the above function to get (here we let $m=r-1, k'=k-r$):

$$\begin{aligned}\sum_{k=r}^{\infty}\binom{k-1}{r-1}p^r(1-p)^{k-r} &= p^r\cdot\sum_{k'=0}^{\infty}\binom{k'+m}{m}(1-p)^{k'} \qquad (1)\\ &= p^r\cdot f_m(1-p) \\ &= p^r\cdot\frac{1}{(1-1+p)^r}=1 \qquad \text{Remark: } r=m+1\end{aligned}$$

Now we can go on with $E(X)$:

$$\begin{aligned}
E(X) &= \sum_{k=r}^{\infty} \binom{k-1}{r-1} k p^r (1-p)^{k-r} \qquad (2)\\
&= p^r \cdot \sum_{k'=0}^{\infty} (k'+r) \binom{k'+m}{m} (1-p)^{k'}\\
&= p^r \cdot \sum_{k'=0}^{\infty} r \binom{k'+r}{r} (1-p)^{k'} && \text{Remark: } m = r-1\\
&= rp^r \cdot \frac{1}{(1-1+p)^{r+1}} = \frac{r}{p}
\end{aligned}$$

In order to find $\mathrm{Var}(X)$, we need a little trick here: find $E[X(X+1)]$ first. As above, we start by lay out the formula:

$$\begin{aligned}
E[X(X+1)] &= \sum_{k=r}^{\infty} k(k+1) \binom{k-1}{r-1} p^r (1-p)^{k-r}\\
&= p^r \cdot \sum_{k'=0}^{\infty} (k'+r+1)(k'+r) \binom{k'+m}{m} (1-p)^{k'} && \text{Remark: } k = k'+r\\
&= p^r \cdot \sum_{k'=0}^{\infty} (k'+r+1)(k'+r) \binom{k'+r-1}{r-1} (1-p)^{k'} && \text{Remark: } m = r-1\\
&= r(r+1)p^r \cdot \sum_{k'=0}^{\infty} \binom{k'+r+1}{r+1} (1-p)^{k'}\\
&= r(r+1)p^r \cdot \frac{1}{(1-1+p)^{r+2}}\\
&= \frac{r(r+1)}{p^2}
\end{aligned}$$

so we have

$$\mathrm{Var}(X) = E[X(X+1)] - E(X)^2 - E(X) = \frac{r(r+1)}{p^2} - \frac{r^2}{p^2} - \frac{r}{p} = \frac{r(1-p)}{p^2}$$

A second way:

We can also find the solution to Equation (1) without having recourse to intermediate function $f_m(z)$. First of all, we re-arrange the expected value formula as:

$$\begin{aligned}
E(X) &= \sum_{k=r}^{\infty} \binom{k-1}{r-1} k p^r (1-p)^{k-r}\\
&= p^r \cdot \sum_{k'=0}^{\infty} (k'+r) \binom{k'+r-1}{r-1} (1-p)^{k'} && \text{Remark: } k' = k-r\\
&= rp^r \cdot \sum_{k'=0}^{\infty} \binom{k'+(r+1)-1}{k'} (1-p)^{k'}\\
&= rp^r \cdot \sum_{k'=0}^{\infty} \binom{k'+(r+1)-1}{k'} (-1)^{k'} (p-1)^{k'}
\end{aligned}$$

We need Taylor/Maclaurin series for help here. Recall

The Maclaurin series for $f(x)$

wherever it converges, can be expressed as:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

and for $f(x) = 1/(1+x)^n$:

$$\frac{1}{(1+x)^n} = 1 - nx + \frac{(-n)(-n-1)}{2!}x^2 + \cdots = \sum_{k=0}^{\infty} \binom{n+k-1}{k} (-1)^k x^k$$

We can see that the last part of $E(X)$ is exactly the sum of Maclaurin series with $n = r+1$ and $x = p-1$, so

$$E(X) = rp^r \cdot \frac{1}{(1+p-1)^{r+1}} = \frac{r}{p}$$