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Reading Notes: Moment Generating Function

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By definition, *moment generating function* ($M_x(t)$) takes the form as follows:

Definition

$$M_x(t) = E(\mathsf{e}^{tx}) \text{ , and}$$

$$\text{for discrete variable } x\text{: } E(\mathsf{e}^{tx}) = \sum_k \mathsf{e}^{tx} p_x(k)$$

$$\text{for continuous variable } x\text{: } E(\mathsf{e}^{tx}) = \int_{-\infty}^{\infty} \mathsf{e}^{tx} f_x(x) dx$$

The first application of the *mgf* is to find "moments":

Theorem 1

$$M_x^{(r)}(t) = E(X^r) \text{ , when } t = 0$$

Proof:

For $r = 1$,
$$\begin{aligned} M_x^{(1)}(t) &= \frac{d}{dt} \int_{-\infty}^{\infty} \mathsf{e}^{tx} f_x(x) dx \\ &= \int_{-\infty}^{\infty} \frac{d}{dt} \mathsf{e}^{tx} f_x(x) dx \\ &= \int_{-\infty}^{\infty} x\mathsf{e}^{tx} f_x(x) dx \end{aligned}$$

For $r = 2$,
$$\begin{aligned} M_x^{(2)}(t) &= \frac{d^2}{dt^2} \int_{-\infty}^{\infty} \mathsf{e}^{tx} f_x(x) dx \\ &= \int_{-\infty}^{\infty} \frac{d^2}{dt^2} \mathsf{e}^{tx} f_x(x) dx \\ &= \int_{-\infty}^{\infty} x^2\mathsf{e}^{tx} f_x(x) dx \end{aligned}$$

Let $t = 0$, the above equation equals to $M_x^{(1)}(0) = \int_{-\infty}^{\infty} x\mathsf{e}^{0x} f_x(x) dx = E(X)$, and $M_x^{(2)}(0) = \int_{-\infty}^{\infty} x^2\mathsf{e}^{0x} f_x(x) dx = E(X^2)$, respectively.

Theorem 1 can also be interpreted directly from *mgf*'s definition. If we use Taylor Series to expand the e^{tx}, and evaluate superscript x at 0, we get:

$$\begin{aligned} M_x(t) = E(\mathrm{e}^{tx}) &= E(\mathrm{e}^{tx}x^0 + \frac{t\mathrm{e}^{tx}}{1!}x^1 + \frac{t^2\mathrm{e}^{tx}}{2!}x^2 + \frac{t^3\mathrm{e}^{tx}}{3!}x^3 + \cdots) \\ &= E(\mathrm{e}^0x^0 + \frac{t\mathrm{e}^0}{1!}x^1 + \frac{t^2\mathrm{e}^0}{2!}x^2 + \frac{t^3\mathrm{e}^0}{3!}x^3 + \cdots) \\ &= E(1) + \frac{t}{1!}E(x^1) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \cdots \end{aligned}$$

Obviously, $M_x^{(r)}(t) = E(x^r) + \frac{t}{1!}E(x^{r+1}) + \frac{t^2}{2!}E(x^{r+2}) + \frac{t^3}{3!}E(x^{r+3}) + \cdots$. If we let $t = 0$, we reach the equation easily at $M_x^{(r)}(0) = E(x^r)$, for the rest parts reduce to 0.

Theorem 2

Suppose that W_1 and W_2 are random variables for which $M_{w_1}(t) = M_{w_2}(t)$ for some interval of t's containing 0. Then $f_{w_1}(w) = f_{w_2}(t)$.

The proof of Theorem 2 requires further knowledge on characteristic functions, so I will come back to this issue later.

Theorem 3a

Let W be a random variable with moment generating function $M_w(t)$. Let $V = aW + b$. Then,

$$M_v(t) = \mathrm{e}^{bt}M_w(at)$$

Proof:

We presume here the variable W is continuous and the proof is as follows (for discrete variables, the underlying logic is the same):

$$\begin{aligned} M_v(t) &= \int_{-\infty}^{\infty} \mathrm{e}^{tV} f(w)dw \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{t(aW+b)} f(w)dw \\ &= \mathrm{e}^{bt}\int_{-\infty}^{\infty} \mathrm{e}^{atW} f(w)dw \\ &= \mathrm{e}^{bt}M_w(at) \end{aligned}$$

Theorem 3b

Let $W_1, W_2, \ldots, W_n$ be independent random variables, and $W = W_1 + W_2 + \cdots + W_n$, then:

$$M_w(t) = M_{w_1}(t) \cdot M_{w_2}(t) \cdots M_{w_n}(t)$$

Proof:

We only need to prove the situation of $W = X + Y$, based on which three or more terms can easily be proved by induction. We know from the definition that $M_w(t) = E(\mathsf{e}^{tw})$, since w is the sum of x and y, we have $M_w(t) = E(\mathsf{e}^{t(x+y)})$.

$f(w)$ takes the value when $X = x$, and $Y = y$, i.e., $f(w) = f(X = x, Y = y)$, but the condition *independent* tells us that $f(X = x, Y = y) = f(x)f(y)$. As a result,

$$\begin{aligned} E(\mathsf{e}^{t(x+y)}) &= \int \mathsf{e}^{t(x+y)} f(w)dw \\ &= \int\int \mathsf{e}^{t(x+y)} f(x)f(y)dxdy \\ &= \int \mathsf{e}^{tx} f(x)dx \cdot \int \mathsf{e}^{ty} f(y)dy \\ &= M_x(t) \cdot M_y(t) \qquad\qquad \text{(Proved!)} \end{aligned}$$