# COMP3711

Design and Analysis of Algorithms

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A summary notes for revision

Fall 2024-2025

# Contents

# 1 Prerequisites

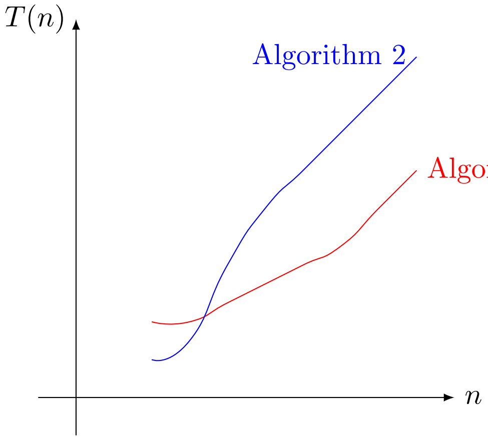
## 1.1 Input size of Problems

| | |
|---|---|
| **Input size** | how large the input is. |
| **Assumption** | 1. any number can be stored in a computer word |
| | 2. each arithmetic operation takes constant time |
| **Examples** | Sorting: Size of the list or array |
| | Graph problems: Numbers of vertices and edges |
| | Searching: Number of input keys |

## 1.2 Asymptotic Notation

1. Running time/ Cost of algorithms
   i. a function of input size: $T(n)$
   ii. number of operations (e.g., comparisons between two numbers)
   iii. using **asymptotic notation**, which ignores constants and non-dominant growth terms

2. Intuitions

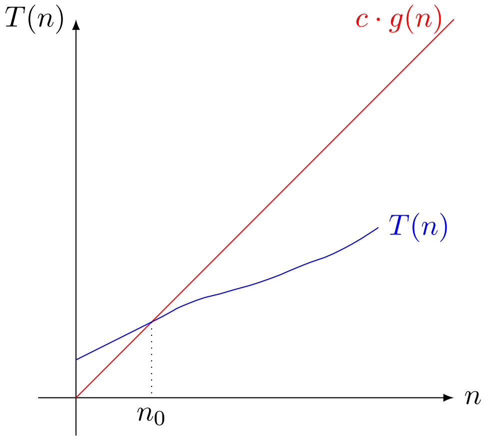



From the figure above, Algorithm 2 is better for large $n$.

3. Rigorous definition of asymptotic notation

| | |
|---|---|
| Upper bound $T(n) = O(f(n))$ | if $\exists c > 0$ and $n_0 \geq 0$ such that $\forall n \neq n_0, T(n) \leq cf(n)$ |
| Lower bound $T(n) = \Omega(f(n))$ | if $\exists c > 0$ and $n_0 \neq 0$ such that $\forall n \neq n_0, T(n) \geq cf(n)$ |
| Tight bound $T(n) = \Theta(f(n))$ | if $T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$ |

4. Big-O Notation

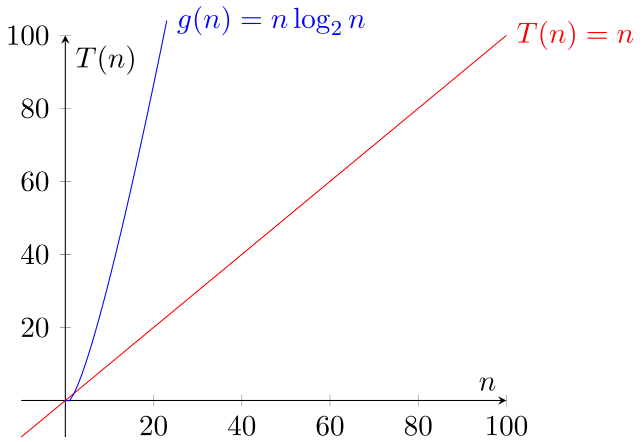



$$T(n) = O(g(n)) \iff \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq T(n) \leq c \cdot g(n)$$

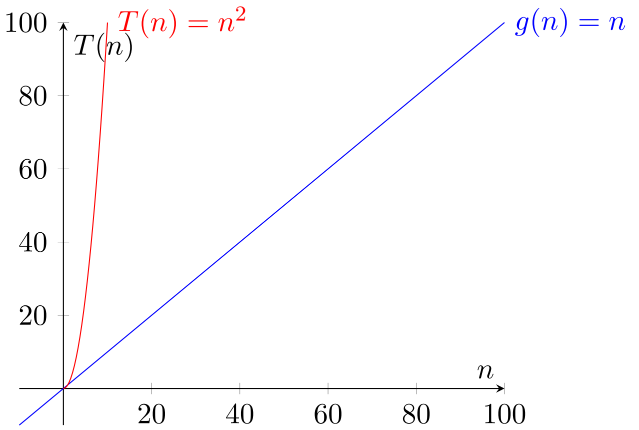
Below are some examples of Big-O notation proofs

(a) $T(n) = n, g(n) = n \log_2 n$
We wish to proof $T(n) = n \in O(n \log_2 n)$.
Choose $c = 1, n_0 = 2$, for all $n \geq 2 = n_0$,

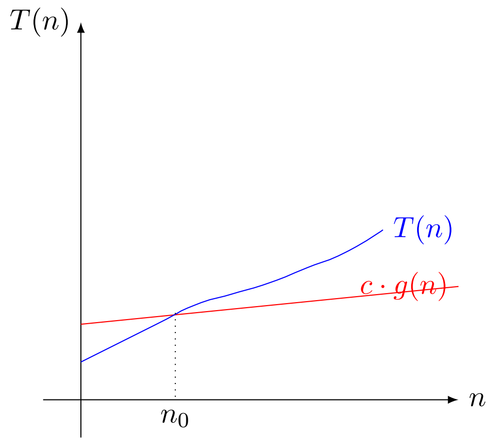
$$1 \leq \log_2 n \iff n \leq n \log_2 n \iff n \leq c \cdot n \log_2 n$$




(b) $T(n) = n^2, g(n) = n$
We wish to proof $T(n) = n^2 \notin O(g(n))$ by contradiction.
Suppose there exists some $c$ and $n_0$ such that for all $n \geq n_0$, $n^2 \leq c \cdot n$. Then, $n \leq c$, $\forall n \geq n_0$, which is not possible as $c$ is a constant and $n$ can be arbitrarily large.

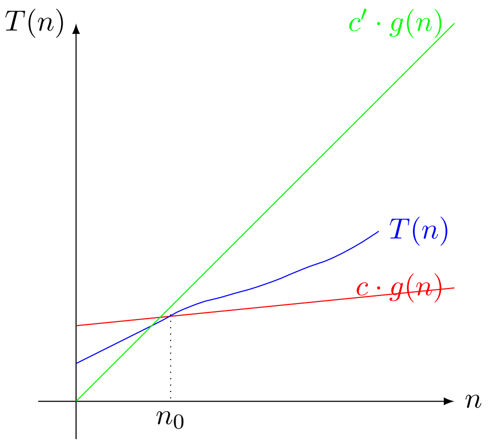



5. Big-$\Omega$ Notation




$$T(n) = \Omega(g(n)) \iff \exists n_0 > 0 \text{ s.t. } \forall n \geq n_0, 0 \leq c \cdot g(n) \leq T(n)$$

6. Big-$\Theta$ Notation




$$T(n) = \Theta(f(n)) \iff T(n) = O(f(n)) \text{ and } T(n) = \Omega(f(n))$$

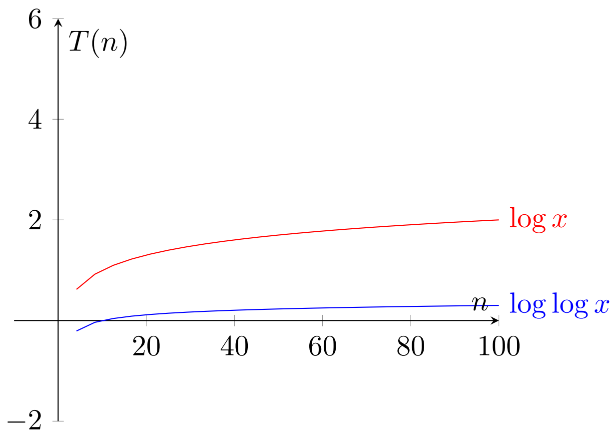
7. Implementation and experimentation are needed sometimes
   If algorithm A is $T_1(n) = 10n \in \Theta(n)$, algorithm B is $T_2(n) = 1000n \in \Theta(n)$, but algorithm A is superior in practice. In this case, Implementation and experimentation are needed.

8. Basic facts on exponents and logarithms
   (a) $2^{2n} \neq \Theta(2^n)$, proof: set $x = 2^n$, then $x^2 \neq \Theta(x)$
   (b) $2^{n+2} = 4 \cdot 2^n = \Theta(2^n)$
   (c) $\log_a(n^b) = \dfrac{b \log n}{\log a} = \Theta(\log n)$
   (d) $\log_b a = \dfrac{1}{\log_a b}$
   (e) $a^{\log_b n} = n^{\log_b a}$

9. Important note on growth of functions

$$k < \log n < n^a < n \log n < n^b < c^n$$

   ,where $k, c \in \mathbb{R}, 0 < a < 2, b \geq 2$ are constants
   (a) $999^{999^{999}} = \Theta(1)$
   (b) $\log \log n = O(\log n)$, proof: for $n \geq 2$, $\log \log n \leq \log n$




   (c) $n \log n = O(\dfrac{n^2}{\log n})$

   proof: To show $n \log n = O(\dfrac{n^2}{\log n})$, it suffices to show that there exists a $C > 0$, such that

$n \log n < C \cdot \dfrac{n^2}{\log n}$ for sufficiently large $n$.

$$\begin{aligned} n \log n &< C \cdot \frac{n^2}{\log n} \\ \iff (\log n)^2 &< C \cdot n \end{aligned}$$

It's obvious that for large $n$, $\log(n) < n^\epsilon$ for $\epsilon > 0$, then we can pick $\epsilon = \dfrac{1}{2}$

$$\begin{aligned} \log n &< n^{\frac{1}{2}} \\ (\log n)^2 &< n \end{aligned}$$

Since $C > 0$, we can see $(\log n)^2 < n < C \cdot n$. We are done.

10. Extra Examples
    (a) $1000n + n \log n = O(n \log n)$
    (b) $n^2 + n \log(n^3) = n^2 + 3n \log n = O(n^2)$
    (c) $n^3 = \Omega(n)$
    (d) $n^3 = O(n^{10})$
    (e) Let $f(n)$ and $g(n)$ be non-negative functions. Using basic definition of $\Theta$-notation, proof that $\max\{f(n), g(n)\} = \Theta(f(n) + g(n))$
        i. Step 1: proof $\max\{f(n), g(n)\} = O(f(n) + g(n))$
        For all $n$, $\max\{f(n), g(n)\}$ is either equal to $f(n)$ or equal to $g(n)$. So we can deduce that $\max\{f(n), g(n)\} \leq f(n) + g(n)$. Therefore, $\max\{f(n), g(n)\} = O(f(n) + g(n))$.
        ii. Step 2: proof $\max\{f(n), g(n)\} = \Omega(f(n) + g(n))$
        Note that $\max\{f(n), g(n)\} \geq f(n)$ and $\max\{f(n), g(n)\} \geq g(n)$. So

$$\begin{aligned} \max\{f(n), g(n)\} + \max\{f(n), g(n)\} &\geq f(n) + g(n) \\ 2 \cdot \max\{f(n), g(n)\} &\geq f(n) + g(n) \\ \max\{f(n), g(n)\} &\geq \frac{1}{2}(f(n) + g(n)) \end{aligned}$$

        Then, we have $\max\{f(n), g(n)\} = \Omega(f(n) + g(n))$
    (f) if $A = \log \sqrt{n}$, $B = \sqrt{\log n}$, then $A = \Omega(B)$
    proof: $A = \log \sqrt{n} = \dfrac{1}{2} \log n = \Theta(\log n)$, $B = \sqrt{\log n} = \Theta(\log n)$. We can simply deduce that $\log \sqrt{n} = \Omega(\sqrt{\log n})$
    (g) Bounds of series - Arithmetic Series
    Proof that $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n = \Theta(n^2)$
        i. Approach 1: use formula $\sum_{i=1}^{n} i = \dfrac{n(1+n)}{2} = \Theta(n^2)$
        ii. Approach 2
            A. Step 1: proof $\sum_{i=1}^{n} i = O(n^2)$

$$\begin{aligned} \sum_{i=1}^{n} i &= 1 + 2 + 3 + \cdots + (n-1) + n \\ &\leq n + n + ... + n \\ &= \sum_{i=1}^{n} n \\ &= n \cdot n \\ &= n^2 = O(n^n) \end{aligned}$$

B. Step 2: proof $\sum_{i=1}^{n} i = \Omega(n^2)$

$$\begin{aligned}\sum_{i=1}^{n} i &= 1+2+3+\cdots+(n-1)+n \\ &\geq 0+0+\cdots+0+...+\frac{n}{2}+(\frac{n}{2}+1)+\cdots+n \\ &\geq \frac{n}{2}\cdot\frac{n}{2} \\ &= \frac{n^2}{4} = \Omega(n^2)\end{aligned}$$

Then, we can say that $\sum_{i=1}^{n} i = \Theta(n^2)$

(h) Bounds of series - Polynomial Series
Proof that $\sum_{i=1}^{n} i^c = 1^c + 2^c + 3^c + \cdots + (n-1)^c + n^c = \Theta(n^{c+1})$
(The proof is more or less the same as the approach 2 of arithmetic series.)

(i) Bounds of series - Harmonic Series $H_n$
Proof that $H_n = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$.
Let $k = \log_2 n$, then $n = 2^k$.

| index | lower bound | parts of $H_n$ | upper bound |
|---|---|---|---|
| 0 | $\frac{1}{2}$ | 1 | 1 |
| 1 | $2 \times \frac{1}{4} = \frac{1}{2}$ | $\frac{1}{2} + \frac{1}{3}$ | $2 \times \frac{1}{2} = 1$ |
| 2 | $4 \times \frac{1}{8} = \frac{1}{2}$ | $\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}$ | $4 \times \frac{1}{4} = 1$ |
| | ... | ... | ... |
| k-1 | $2^{k-1} \times \frac{1}{2^k} = \frac{1}{2}$ | $\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} \cdots + \frac{1}{2^k - 1}$ | $2^{k-1} \times \frac{1}{2^{k-1}} = 1$ |
| k | 0 | $\frac{1}{2^k} = \frac{1}{n}$ | 1 |

Therefore, $H_n < \sum_{i=0}^{k} 1 = k+1 = \log_2 n + 1 = O(\log n)$ and $H_n > \sum_{i=0}^{k-1} \frac{1}{2} + 0 = \frac{k}{2} = \frac{\log_2 n}{2} = \Omega(\log n)$. So, $H_n = \sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)$.

## 1.3 Introduction to Algorithms

1. What is an algorithm?
   An algorithm is an explicit, precise, unambiguous, mechanically-executable sequence of elementary instructions.

2. Examples of algorithms

   (a) adding two numbers
   Input: 2 numbers $x = \overline{x_n x_{n-1} \cdots x_1}, y = \overline{y_n y_{n-1} \cdots y_1}$.
   Output: A number $z = \overline{z_{n+1} z_n \cdots z_1}$, such that $z = x + y$.

```
/*We assume x, y are arrays of length n, z is of length n+1 */
int c = 0; // offset
for (int i = 0; i < n; ++i){
   z[i] = x[i] + y[i] + c;
   if (z[i] >= 10) {
      c = 1;
      z[i] = z[i] - 10;
   }else c = 0;
}
z[n] = c;
```