Design and Analysis of Algorithms

Part I: Divide and Conquer

Lecture 6: Counting Inversion Problem and Polynomial Multiplication Problem

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Outline

- Review to Divide-and-Conquer Paradigm
- Counting Inversions Problem
  - Problem definition
  - A brute force algorithm
  - A divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm
- Polynomial Multiplication Problem
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  - The first divide-and-conquer algorithm
  - An improved divide-and-conquer algorithm
  - Analysis of the divide-and-conquer algorithm

Review to Divide-and-Conquer Paradigm

- Divide-and-conquer (D&C) is an important algorithm design paradigm.
  - **Divide**
    Dividing a given problem into two or more subproblems (ideally of approximately equal size)
  - **Conquer**
    Solving each subproblem (directly if small enough or recursively)
  - **Combine**
    Combining the solutions of the subproblems into a global solution

Review to Divide-and-Conquer Paradigm

- In Part I, we will illustrate Divide-and-Conquer using several examples:
  - Maximum Contiguous Subarray (最大子数组)
  - Counting Inversions (逆序计数)
  - Polynomial Multiplication (多项式乘法)
  - QuickSort and Partition (快速排序与划分)
  - Randomized Selection (随机化选择)
  - Lower Bound for Sorting (基于比较的排序下界)

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Counting inversions

Music site tries to match your song preferences with others.

- You rank *n* songs.
- Music site consults database to find people with similar tastes.

Similarity metric: number of inversions between two rankings.

- My rank: 1, 2, …, n.
- Your rank: $a_1, a_2, \ldots, a_n$.
- Songs i and j are inverted if $i < j$, but $a_i > a_j$.

	A	B	C	D	E
Me	1	2	3	4	5
You	1	3	4	2	5

Formal Definition

- Input: An array of reals $A[1\ldots n]$
- Output: The total number of inversions, namely

$$\sum_{1 \le i < j \le n} X_{i,j}$$

$$X_{i,j} = \begin{cases} 1, & A[i] > A[j] \\ 0, & A[i] \le A[j] \end{cases}$$

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A Brute Force Algorithm

List each pair i < j and count the inversions.

Input: L
Output: r
$r \leftarrow 0$;
for $i \leftarrow 1$ *to* $L.length$ **do**
    for $j \leftarrow i+1$ *to* $L.length$ **do**
        if $L[i] > L[j]$ **then**
            $r \leftarrow r+1$;
        end
    end
end
return r;

$O(n^2)$ comparisons and additions.

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Review to Merge Sort

Mergesort(A, left, right)

if *left* $<$ *right* **then**
    center $\leftarrow \lfloor$(left + right)/2$\rfloor$;
    Mergesort(A, left, center);
    Mergesort(A, center+1, right);
    "Merge" the two sorted arrays;
end

- To sort the entire array $A[1 \ldots n]$, we make the initial call Mergesort(A, 1, n).
- Key subroutine: "Merge"

Counting inversions: divide-and-conquer

- Divide: separate list into two halves A and B.
- Conquer: recursively count inversions in each list.
- Combine: count inversions (a, b) with a∈A and b∈B.
- Return sum of three counts.

Input

14	7	18	3	10	19	11	23	2	25	16	17

14	7	18	3	10	19

11	23	2	25	16	17

Count inversions in left half A

14-7,14-3,14-10,7-3,18-3,18-10

Count inversions in right half B

11-2,23-2,23-16,23-17,25-16,25-17

Output

6+6+13 =25

Count inversions (a,b) with a∈A and b∈B

14-11,14-2,7-2,18-11,18-2,18-16,18-17,3-2,10-2,19-11,19-2,19-16,19-17

How to combine two subproblems?

Q. How to count inversions (a, b) with a ∈ A and b ∈ B?

A. Easy if A and B are sorted!

Warmup algorithm.

- Sort A and B.
- For each element b ∈ B,
  - binary search in A to find how many elements in A are greater than b.

Sort A

3	7	10	14	18	19

Sort B

2	11	16	17	23	25

6	3	2	2	0	0

Count for b∈B

Inversions between A and B: 6+3+2+2=13

Combine two subproblems: Improvement

Count inversions (a, b) with a ∈ A and b ∈ B, assuming A and B are sorted.

- Scan A and B from left to right.
- Compare a_i and b_j.
  - If $a_i < b_j$, then a_i is not inverted with any element left in B.
  - If $a_i > b_j$, then b_j is inverted with every element left in A.
- Append smaller element to sorted list C.

i = 0

Two sorted halves

3	**7**	**10**	**14**	**18**	**19**

2	**11**	**16**	**17**	**23**	**25**
6	3	2	2	0	

Auxiliary array

2	**3**	**7**	**10**	**11**	**14**	**16**	**17**	**18**	**19**	**23**	**25**

Total: 6+3+2+2+0+0 = 13

Combine two subproblems: Improvement

Merge-and-Count(A, B)

Input: A, B
Output: r, L
$r \leftarrow 0, L \leftarrow \emptyset$;
while *both* A *and* B *are not empty* **do**
    // Let a and b represent the first element of A and B, repectively
    if $a < b$ **then**
        Move a to the back of L;//$A.length$ is decreased by 1;
    end
    else
        Increase r by $A.length$;
        Move b to the back of L;
    end
end
if A *is not empty* **then**
    Move A to the back of L;
end
else
    Move B to the back of L;
end
return L, r;

Combine two subproblems: Improvement

- For every element in A and B,
  - Only O(1) times operations are executed.

- Function *Sort-and-Count(A,B)* can be executed in O(n) time where n is the number of elements in A and B.

The Complete Divide-and-Conquer Algorithm

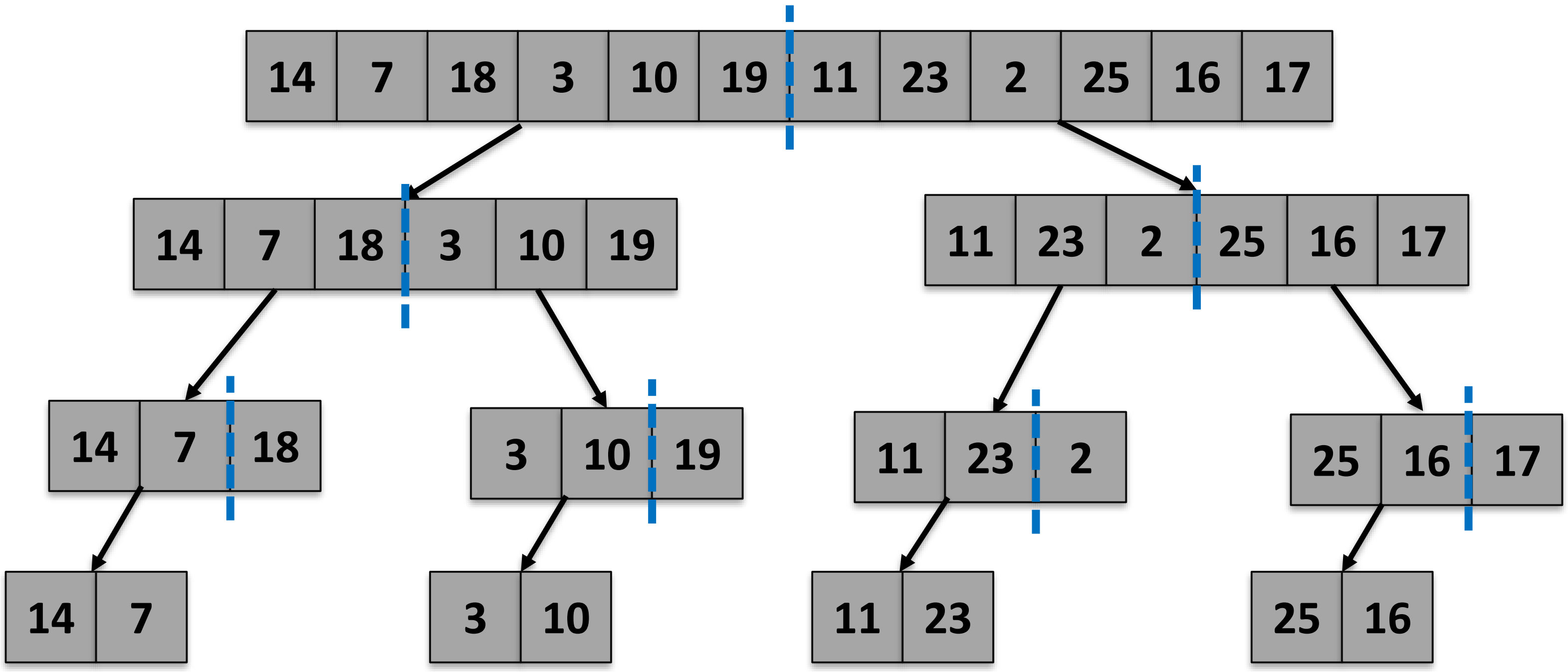
Sort-and-Count(L)

Input: L
Output: r_L, L
if $L.length = 1$ **then**
  return $0, L$;
end
Divide L into two halves A and B;
$(r_A, A) \leftarrow$Sort-and-Count(A);$//T(\lceil \frac{n}{2} \rceil)$
$(r_B, B) \leftarrow$Sort-and-Count(B);$//T(\lfloor \frac{n}{2} \rfloor)$
$(r_L, L) \leftarrow$Merge-and-Count(A, B);$//O(n)$
return $r_A + r_B + r_L, L$;

$$T(n) = \begin{cases} O(1), & if\ n = 1 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(n) & otherwise \end{cases}$$

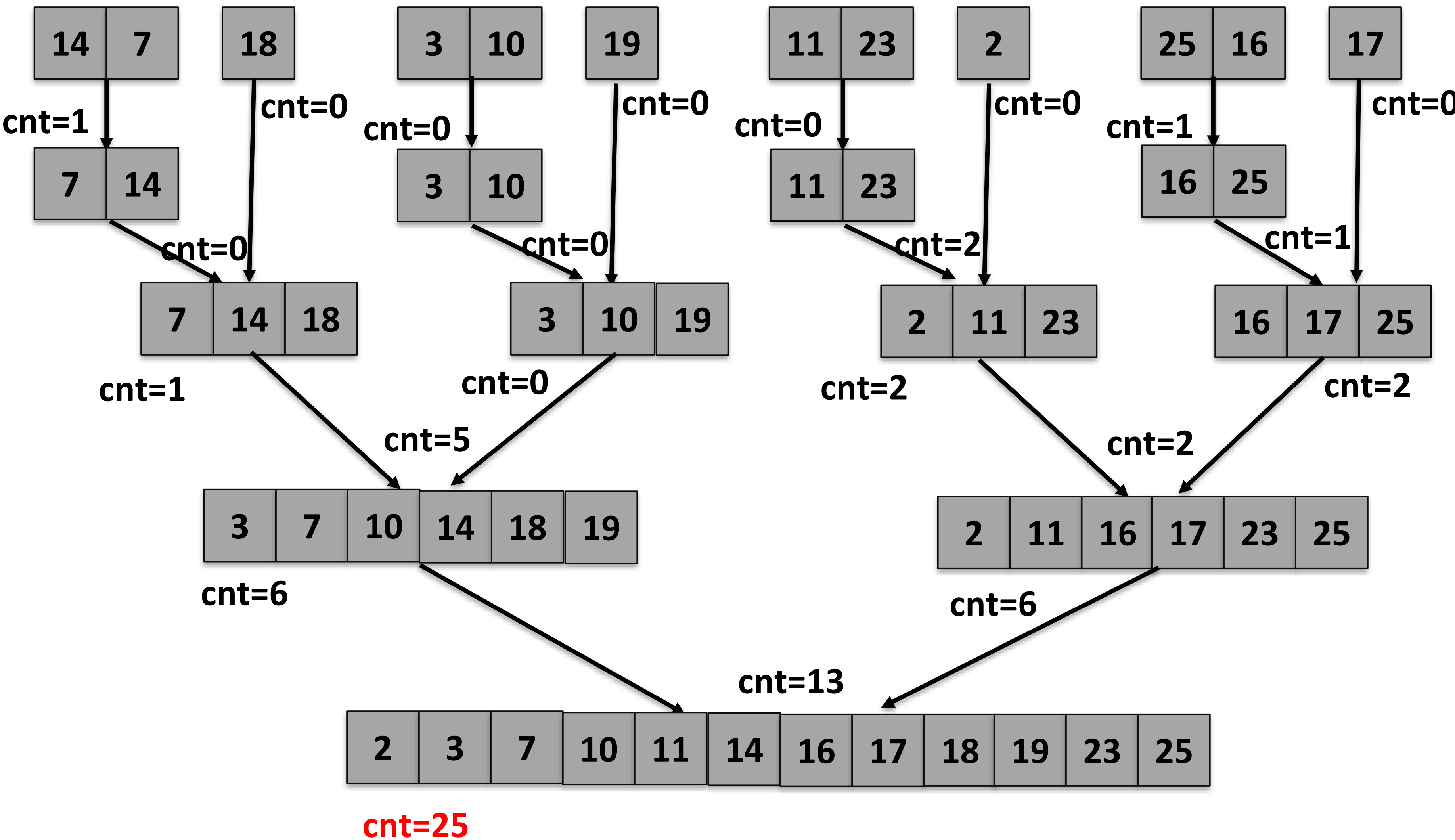
Example

Divide



Example

Conquer



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Analysis of the D&C Algorithm

Proposition. The sort-and-count algorithm counts the number of inversions in a permutation of size n in $O(n \log n)$ time.

Proof. The worst-case running time T(n) satisfies the recurrence:

$$T(n) = \begin{cases} O(1), & if\ n = 1 \\ T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + O(n) & otherwise \end{cases}$$

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The Polynomial Multiplication Problem

Definition (Polynomial Multiplication Problem)

Given two polynomials

$$\begin{aligned} A(x) &= a_0 + a_1 x + \cdots + a_n x^n \\ B(x) &= b_0 + b_1 x + \cdots + b_m x^m \end{aligned}$$

Compute the product $A(x)B(x)$

Example

$$\begin{aligned} A(x) &= 1 + 2x + 3x^2 \\ B(x) &= 3 + 2x + 2x^2 \\ A(x)B(x) &= 3 + 8x + 15x^2 + 10x^3 + 6x^4 \end{aligned}$$

- Assume that the coefficients a_i and b_i are stored in arrays A[0…n] and B[0…m]
- Cost: number of scalar multiplications and additions

What do we need to compute exactly?

Define

- $A(x) = \sum_{i=0}^{n} a_i x^i$
- $B(x) = \sum_{i=0}^{m} b_i x^i$
- $C(x) = A(x)B(x) = \sum_{k=0}^{n+m} c_k x^k$

Then

- $c_k = \sum_{0 \le i \le n, 0 \le j \le m, i+j=k} a_i b_j$, for all $0 \le k \le m+n$

Definition

The vector $(c_0, c_1, \ldots, c_{m+n})$ is the convolution of the vectors $(a_0, a_1, \ldots, a_n)$ and $(b_0, b_1, \ldots, b_m)$

- We need to calculate convolutions. This is a major problem in digital signal processing

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A Direct (Brute Force) Approach

To ease analysis, assume n = m.

- $A(x) = \sum_{i=0}^{n} a_i x^i$
- $B(x) = \sum_{i=0}^{m} b_i x^i$
- $C(x) = A(x)B(x) = \sum_{k=0}^{2n} c_k x^k$ with

$$c_k = \sum_{0 \le i,j \le n, i+j=k} a_i b_j \,, for\ all\ 0 \le k \le 2n$$

Direct approach: Compute all c_k's using the formula above.

- Total number of multiplications: $O(n^2)$
- Total number of additions: $O(n^2)$
- Complexity: $O(n^2)$

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The First Divide-and-Conquer: Divide

Assume n is a power of 2

Define

$$A_0(x) = a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$$

$$A_1(x) = \mathrm{a}_{\frac{\mathrm{n}}{2}} + \mathrm{a}_{\frac{\mathrm{n}}{2}+1}\mathrm{x} + \cdots + \mathrm{a}_{\mathrm{n}}\mathrm{x}^{\frac{\mathrm{n}}{2}}$$

$$A(x) = A_0(x) + A_1(x) x^{\frac{n}{2}}$$

Similarly, define $B_0(x)$ and $B_1(x)$ such that

$$B(x) = B_0(x) + B_1(x) x^{\frac{n}{2}}$$

$$A(x)B(x) = A_0(x)B_0(x) + A_0(x)B_1(x)x^{\frac{n}{2}} + A_1(x)B_0(x)x^{\frac{n}{2}} + A_1(x)B_1(x)x^n$$

The original problem (of size n) is divided into 4 problems of input size n/2

Example

$$A(x) = 2 + 5x + 3x^2 + x^3 - x^4$$
$$B(x) = 1 + 2x + 2x^2 + 3x^3 + 6x^4$$
$$A(x)B(x) = 2 + 9x + 17x^2 + 23x^3 + 34x^4$$
$$+39x^5 + 19x^6 + 3\mathrm{x}^7 - 6\mathrm{x}^8$$
$$A_0(x) = 2 + 5x, A_1(x) = 3 + x - x^2$$
$$A(x) = A_0(x) + A_1(x)x^2$$
$$B_0(x) = 1 + 2x, B_1(x) = 2 + 3x + 6x^2$$
$$B(x) = B_0(x) + B_1(x)x^2$$
$$A_0(x)B_0(x) = 2 + 9x + 10x^2$$
$$A_1(x)B_1(x) = 6 + 11x + 19x^2 + 3x^3 - 6x^4$$
$$A_0(x)B_1(x) = 4 + 16x + 27x^2 + 30x^3$$
$$A_1(x)B_0(x) = 3 + 7x + x^2 - 2x^3$$
$$A_0(x)B_1(x) + A_1(x)B_0(x) = 7 + 23x + 28x^2 + 28x^3$$

$$A_0(x)B_0(x) + \big(A_0(x)B_1(x) + A_1(x)B_0(x)\big)x^2 + A_1(x)B_1(x)x^4$$
$$= 2 + 9x + 17x^2 + 23x^3 + 34x^4 + 39x^5 + 19x^6 + 3x^7 - 6x^8$$

The First Divide-and-Conquer: Conquer

Conquer: Solve the four subproblems

- Compute $A_0(x)B_0(x), A_0(x)B_1(x), A_1(x)B_0(x), \mathrm{A}_1(\mathrm{x})\mathrm{B}_1(\mathrm{x})$ by recursively calling the algorithm 4 times

Combine

- Add the following four polynomials

$$
\begin{aligned}
A_0(x)B_0(x) &+ A_0(x)B_1(x)x^{\frac{n}{2}} \\
&+ A_1(x)B_0(x)x^{\frac{n}{2}} \\
&+ A_1(x)B_1(x)x^{n}
\end{aligned}
$$

- Takes O(n) operations

The First Divide-and-Conquer Algorithm

PolyMulti1$(A(x), B(x))$

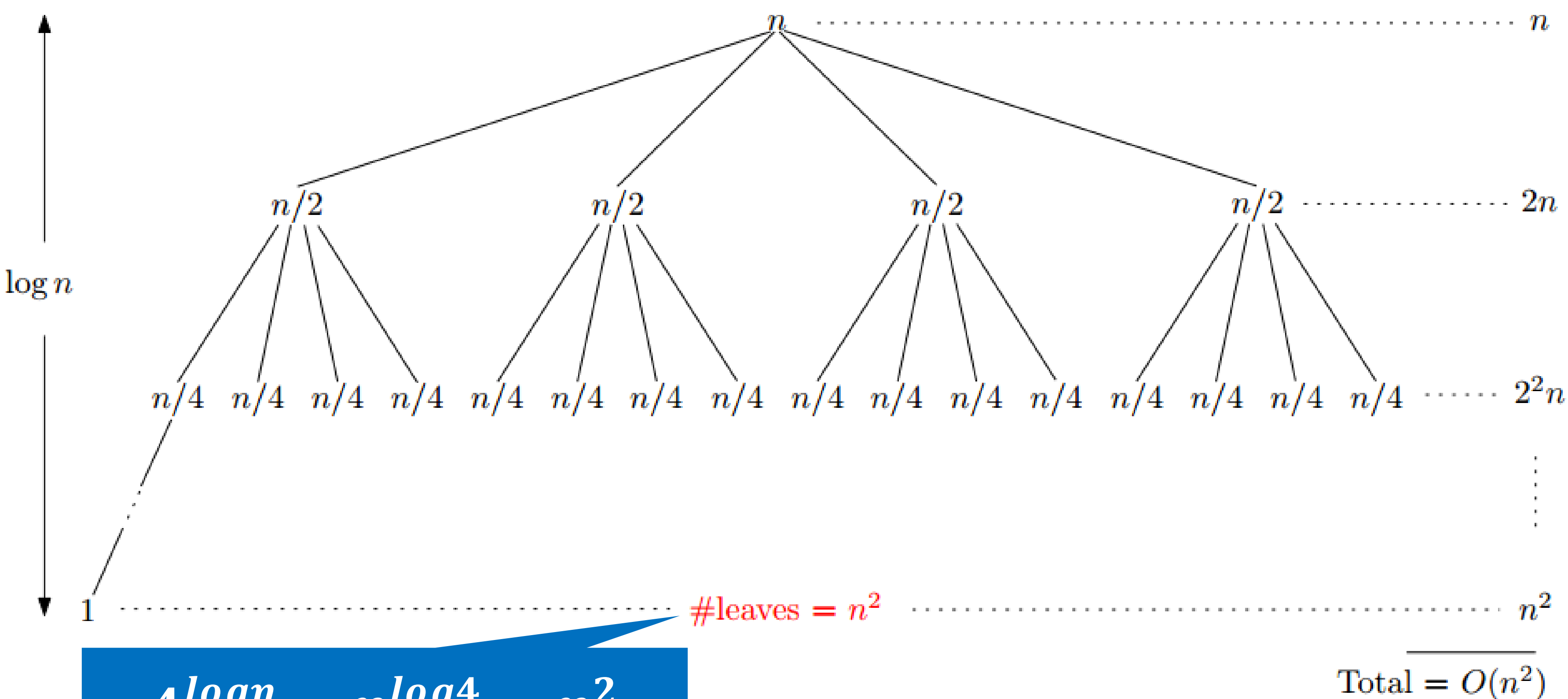
Input: $A(x), B(x)$
Output: $A(x) \times B(x)$
$A_0(x) \leftarrow a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$;
$A_1(x) \leftarrow a_{\frac{n}{2}} + a_{\frac{n}{2}+1} x + \cdots + a_n x^{\frac{n}{2}}$;
$B_0(x) \leftarrow b_0 + b_1 x + \cdots + b_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$;
$B_1(x) \leftarrow b_{\frac{n}{2}} + b_{\frac{n}{2}+1} x + \cdots + b_n x^{\frac{n}{2}}$;
$U(x) \leftarrow$PolyMulti1$(A_0(x), B_0(x))$;$//T(n/2)$
$V(x) \leftarrow$PolyMulti1$(A_0(x), B_1(x))$;$//T(n/2)$
$W(x) \leftarrow$PolyMulti1$(A_1(x), B_0(x))$;$//T(n/2)$
$Z(x) \leftarrow$PolyMulti1$(A_1(x), B_1(x))$;$//T(n/2)$
return $(U(x) + [V(x) + W(x)]x^{\frac{n}{2}} + Z(x)x^n)$;$//O(n)$

$$T(n) = \begin{cases} 4T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

Analysis of Running Time

Assume that n is a power of 2

$$T(n) = \begin{cases} 4T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$



$4^{logn} = n^{log4} = n^2$

Analysis of Running Time

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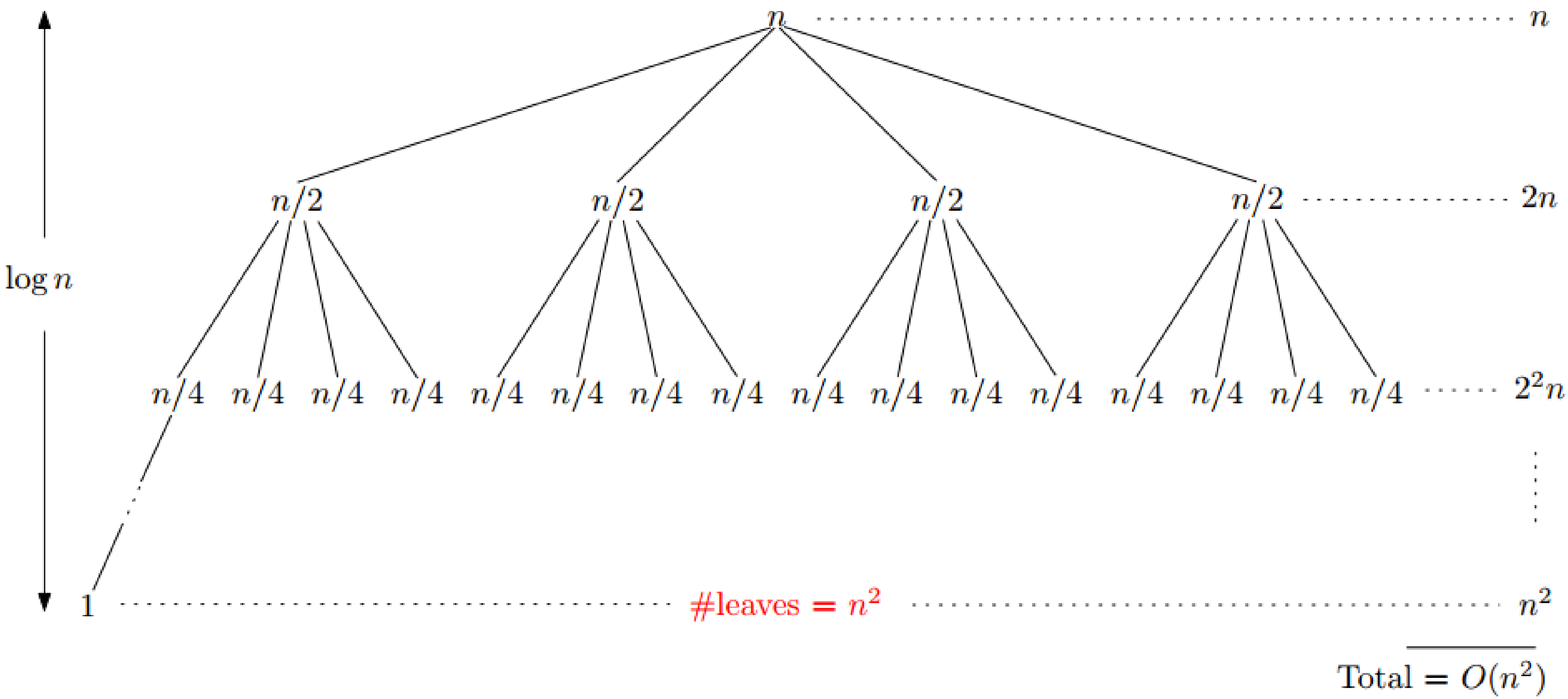

Same order as the brute force approach! No improvement!

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Two Observations

Observation 1:

What we really need are the following 3 terms:

$$A_0 B_0, \; A_0 B_1 + A_1 B_0, \; A_1 B_1?$$

Instead of the following 4 terms:

$$A_0 B_0, \; A_0 B_1, \; A_1 B_0, \; A_1 B_1?$$

Observation 2:

The three terms can be obtained using only 3 multiplications:

$$\begin{aligned} Y &= (A_0 + A_1)(B_0 + B_1) \\ U &= A_0 B_0 \\ Z &= A_1 B_1 \end{aligned}$$

- *U and Z are what we originally wanted*
- $A_0 B_1 + A_1 B_0 = Y - U - Z$

The improved Divide-and-Conquer Algorithm

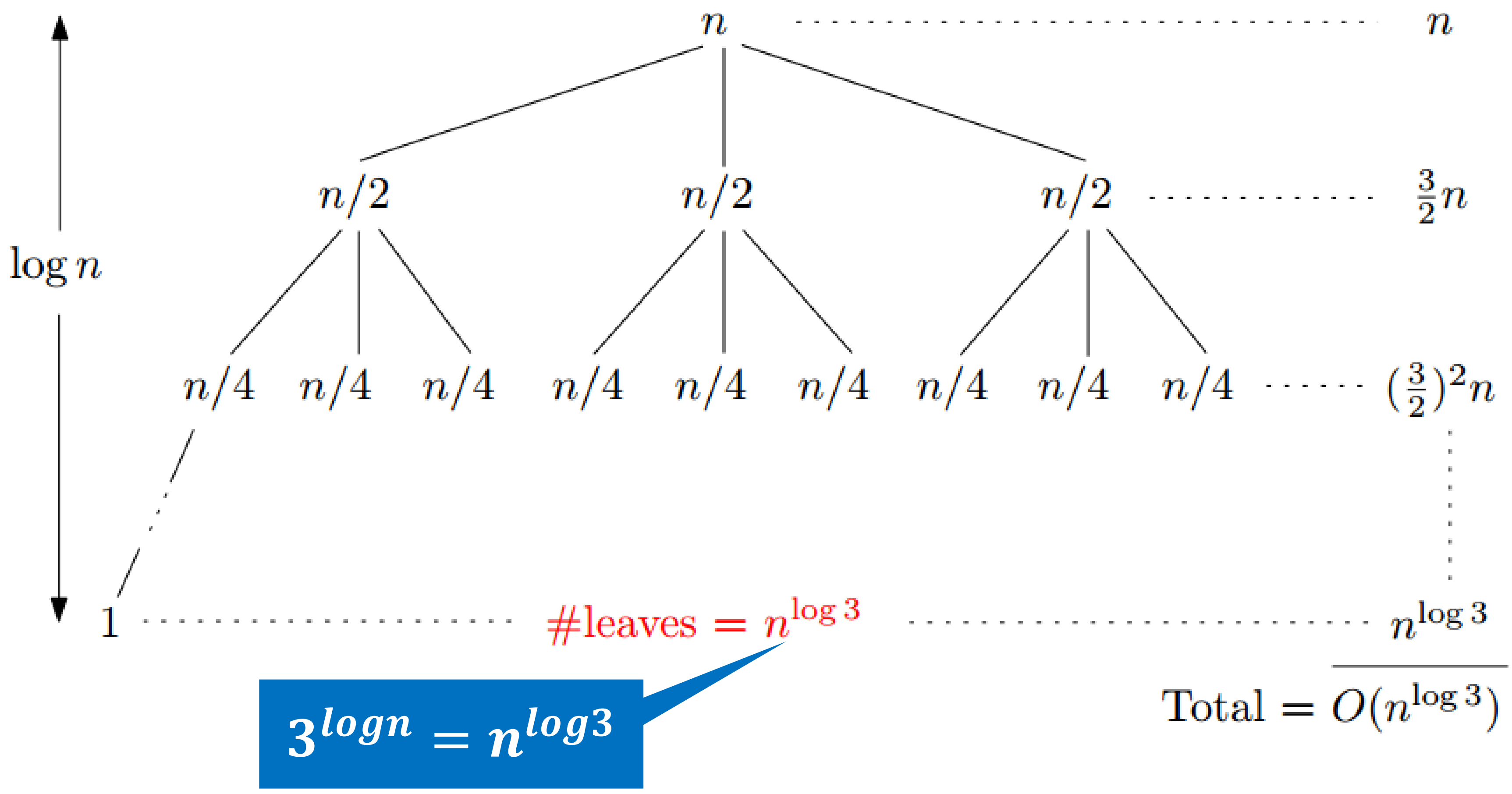
$\text{PolyMulti2}(A(x), B(x))$

Input: $A(x), B(x)$
Output: $A(x) \times B(x)$
$A_0(x) \leftarrow a_0 + a_1 x + \cdots + a_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$;
$A_1(x) \leftarrow a_{\frac{n}{2}} + a_{\frac{n}{2}+1} x + \cdots + a_n x^{n-\frac{n}{2}}$;
$B_0(x) \leftarrow b_0 + b_1 x + \cdots + b_{\frac{n}{2}-1} x^{\frac{n}{2}-1}$;
$B_1(x) \leftarrow b_{\frac{n}{2}} + b_{\frac{n}{2}+1} x + \cdots + b_n x^{n-\frac{n}{2}}$;
$Y(x) \leftarrow \text{PolyMulti2}(A_0(x) + A_1(x), B_0(x) + B_1(x));//T(n/2)$
$U(x) \leftarrow \text{PolyMulti2}(A_0(x), B_0(x));//T(n/2)$
$Z(x) \leftarrow \text{PolyMulti2}(A_1(x), B_1(x));//T(n/2)$
return $(U(x) + [Y(x) - U(x) - Z(x)]x^{\frac{n}{2}} + Z(x)x^{2\frac{n}{2}});//O(n)$

$$T(n) = \begin{cases} 3T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$

Running Time of the Improved Algorithm

$$T(n) = \begin{cases} 3T(n/2) + n, & \text{if } n > 1, \\ 1, & \text{if } n = 1. \end{cases}$$



Running Time of the Improved Algorithm

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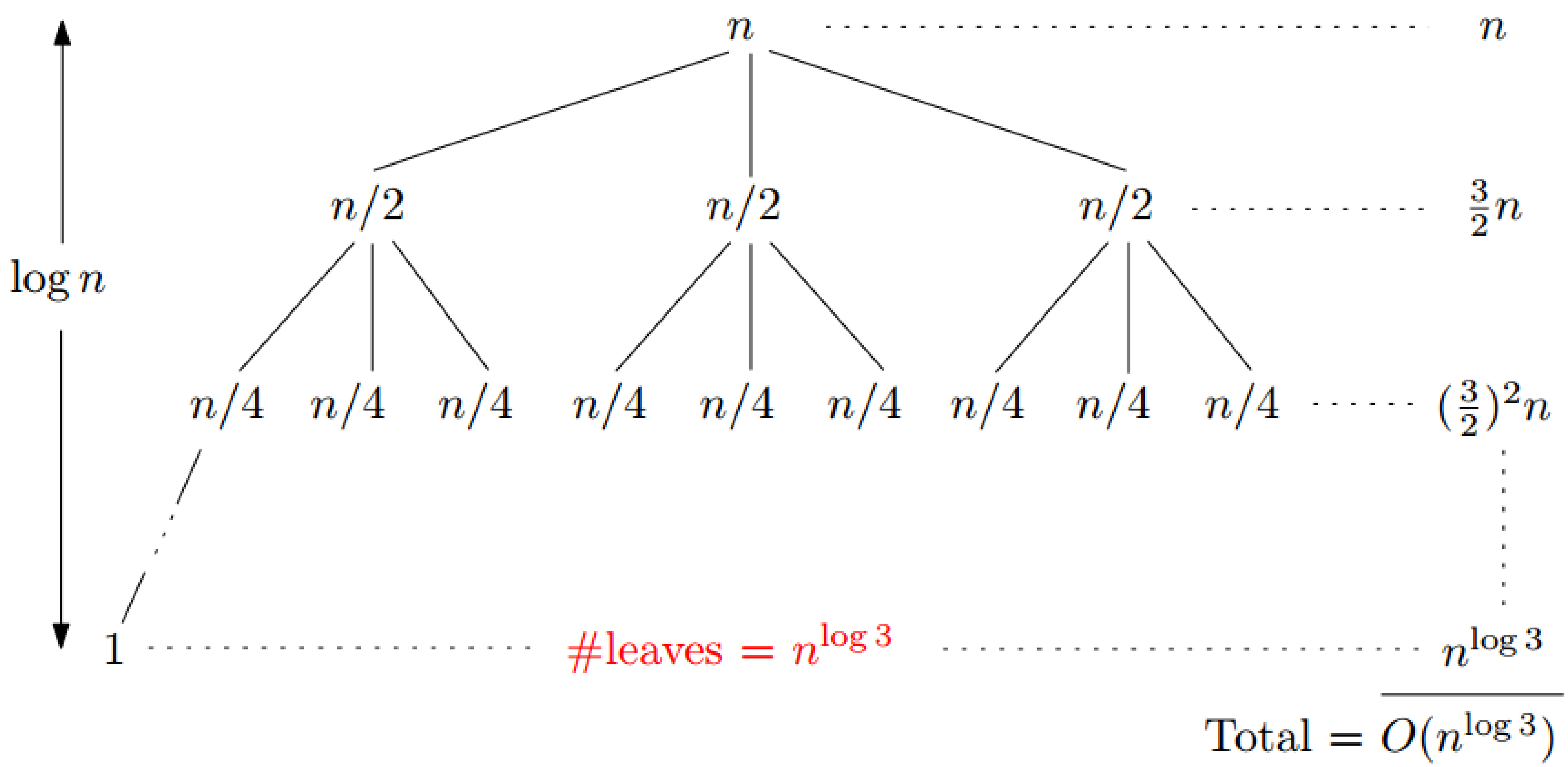

The second method is much better!

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Analysis of the D&C algorithm

- The divide-and-conquer approach does not always give you the best solution
  - Our original algorithm was just as bad as brute force
- There is actually an $O(n \log n)$ solution to the polynomial multiplication problem
  - It involves using the Fast Fourier Transform algorithm as a subroutine
  - The FFT is another classic divide-and-conquer algorithm(check Chapt 30 in CLRS if interested)
- The idea of using 3 multiplications instead of 4 is used in large-integer multiplications
  - A similar idea is the basis of the classic Strassen matrix multiplication algorithm (CLRS 4.2)

谢谢

BDA

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