

Gmacs

A generalized size-structured stock assessment model

The Gmacs Development Team

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Indices

Symbol	Description
g	group
h	sex
i	year
j	time step (years)
k	gear or fleet
ℓ	index for size class
m	index for maturity state
o	index for shell condition

Estimable parameters

Symbol	Description
M_0	Initial instantaneous natural mortality rate
\bar{R}	Average recruitment
\ddot{R}	Initial recruitment
α_r	Mode of size-at-recruitment
β_r	Shape parameter for size-at-recruitment
R_0	Unfished average recruitment
κ	Recruitment compensation ratio
α_h	Mode of size-at-recruitment
β_h	Shape parameter for size-at-recruitment
φ_h	Instantaneous natural mortality rate
ν	Instantaneous natural mortality rate
ξ	Discard mortality rate

$$\Theta = \{M_0, \bar{R}, \ddot{R}, \alpha_r, \beta_r, R_0, \kappa, \alpha_h, \beta_h, \varphi_h, \nu, \xi\}.$$

Growth parameters

$$M_{0,h} > 0, \bar{R} > 0, \ddot{R} > 0, \alpha_r > 0, \beta_r > 0, R_0 > 0, \kappa > 1.0$$

Symbol	Description
α_h	Mode of size-at-recruitment
β_h	Shape parameter for size-at-recruitment
φ_h	Instantaneous natural mortality rate

$$\Phi = \{\alpha_h, \beta_h, \varphi_h\}.$$

Other variables

Symbol	Description
$w_{h,\ell}$	Mean weight at length
$m_{h,\ell}$	Average proportion mature at length

Cubic splines

A spline is a numeric function that is piecewise-defined by polynomial functions, and which possesses a sufficiently high degree of smoothness at the places where the polynomial pieces connect (which are known as knots). A cubic spline is constructed of piecewise third-order polynomials. The second derivative of each polynomial is commonly set to zero at the endpoints, since this provides a boundary condition that completes the system of $m - 2$ equations. This produces a so-called “natural” cubic spline and leads to a simple tridiagonal system which can be solved easily to give the coefficients of the polynomials. However, this choice is not the only one possible, and other boundary conditions can be used instead.

Selectivity, retention and fishing mortality

Symbol	Description
$a_{h,i,k}$	Length at 50% selectivity
$\sigma_{h,i,k}^s$	Standard deviation in length at selectivity
$s_{h,i,k,\ell}$	Length at 50% selectivity in length interval ℓ
$r_{h,i,k}$	Length at 50%
$\sigma_{h,i,k}^y$	Standard deviation in length at retention
$y_{h,i,k,\ell}$	Length at 50% retention in length interval ℓ
$\xi_{h,k}$	Discard mortality rate for sex h and gear k
$\nu_{h,i,k,\ell}$	Vulnerability due to fishing mortality for sex h
f_k	
$\Psi_{k,i}$	
$F_{k,i}$	
$f_{h,i}$	

Selectivity and retention

The probability of catching an animal of sex h , in year i , in fishery k , of length ℓ (i.e. selectivity) is

$$s_{h,i,k,\ell} = \left(1 + \exp\left(-(\ell - a_{h,i,k})/\sigma_{h,i,k}^s\right)\right)^{-1}.$$

The probability of an animal of sex h , in year i , in fishery k , of length ℓ being retained is

$$y_{h,i,k,\ell} = \left(1 + \exp\left(-(r_{h,i,k} - \ell)/\sigma_{h,i,k}^y\right)\right)^{-1}.$$

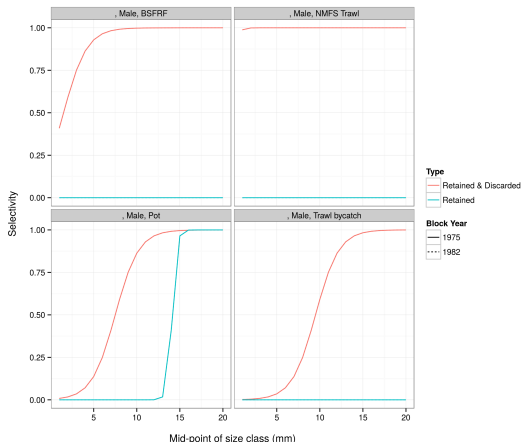
The joint probability of vulnerability due to fishing and discard mortality is

$$\nu_{h,i,k,\ell} = s_{h,i,k,\ell} [y_{h,i,k,\ell} + (1 - y_{h,i,k,\ell})\xi_{h,k}],$$

where $\xi_{h,k}$ is the discard mortality rate for sex h in fishery k .

Selectivity and retention

Assuming that selectivity for the NMFS trawl fishery is split into two blocks (1975-1981 and 1982-2014) and that retention is constant with time $y_{h,i,k,\ell} = y_{h,k,\ell}$



Fishing mortality

$$\mathbf{F}_{k,i} = \exp \left(\bar{\mathbf{f}}_k + \mathbf{\Psi}_{k,i} \right),$$

$$\mathbf{f}_{h,i} = \sum_k \mathbf{F}_{k,i} \nu_{h,i,k,\ell},$$

$$C_i = \sum_{\ell} \left[\mathbf{n} w_{h,\ell} \frac{F_i}{Z_{i,\ell}} \left(1 - e^{-Z_{i,\ell}} \right) \right] \quad \text{where} \quad Z_{h,i} = M_{h,i} + F_{h,i}.$$

Growth

The average molt increment from size class ℓ to ℓ' is assumed to be sex-specific and is defined by the linear function

$$a_{h,\ell} = \frac{\alpha_h + \beta_h \ell}{\varphi_h}.$$

The probability of transitioning from size class ℓ to ℓ' assumes that variation in molt increments follows a gamma distribution

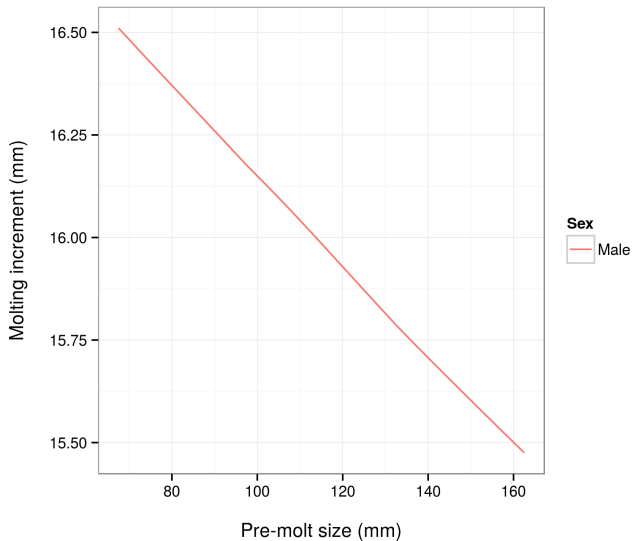
$$p(\ell'|\ell)_h = \mathbf{G}_h = \int_{\ell}^{\ell+\Delta\ell} \frac{\ell^{a_{h,\ell}-1} \exp\left(-\frac{\ell}{\varphi_h}\right)}{\Gamma(a_{h,\ell}) \ell^{a_{h,\ell}}} \quad \text{where} \quad \Delta\ell = \ell' - \ell.$$

Specifically

$$\mathbf{G} = G_{\ell,\ell'} = \begin{pmatrix} G_{1,1} & G_{1,2} & \dots & G_{1,n} \\ G_{2,1} & G_{2,2} & \dots & G_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n,1} & G_{n,2} & \dots & G_{n,n} \end{pmatrix} \quad \text{where} \quad \sum_{\ell'} G_{\ell,\ell'} = 1 \quad \forall \ell.$$

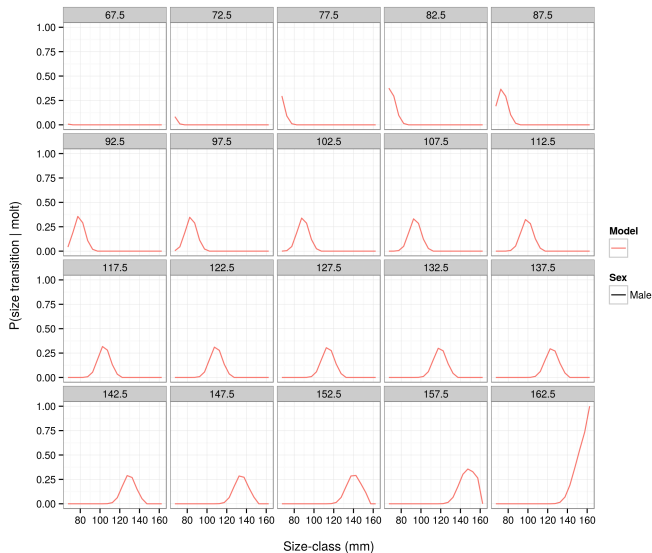
Growth increment

Growth increment



Growth

Growth transition



Natural mortality

Symbol	Description
$M_{0,h}$	
σ_M	Standard deviation of natural mortality
δ_i	Natural mortality deviate
$M_{h,i}$	Natural mortality by sex h and year i

Natural mortality options

Natural mortality (M) is assumed to be sex-specific (h), size-independent (ℓ), and may or may not be constant over time (i). The options currently available in Gmacs include:

- 1 Constant natural mortality ($M_{h,i} = M_{0,h}$)
- 2 Random walk (deviates constrained by variance σ_M^2)
- 3 Cubic Spline (deviates constrained by nodes and node placement)
- 4 Blocked changes (deviates constrained by variance at specific knots)

Natural mortality: option 2

If time-varying natural mortality is specified using the **random walk** option, the model constrains $M_{h,i}$ to be a random-walk process with variance σ_M^2

$$M_{h,i+1} = \begin{cases} M_{0,h} \\ M_{h,i}e^{\delta_i} \end{cases},$$

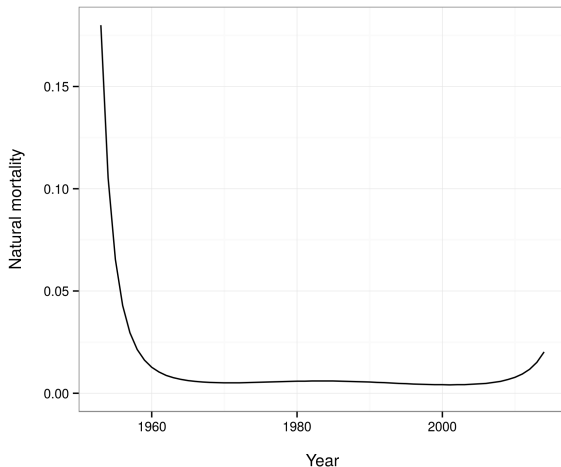
where

$$\delta_i \sim \mathcal{N}(0, \sigma_M^2).$$

A time-varying natural mortality can be estimated for all years (i), or for specified blocks of years ($\iota \in i$).

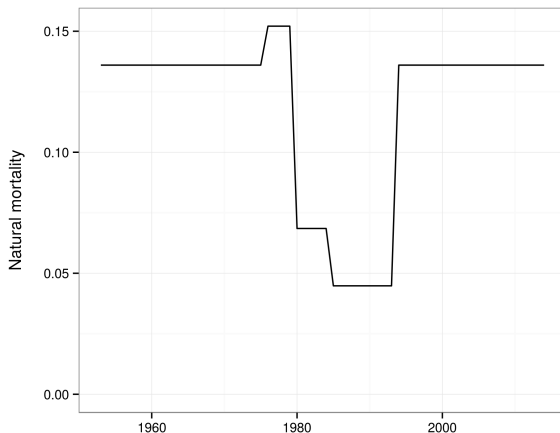
Natural mortality: option 3

If time-varying natural mortality is specified using the **cubic spline** option, the model constrains $M_{h,i}$ to be a cubic spline process at specified knots. For example, setting



Natural mortality: option 4

If time-varying natural mortality is specified using the **blocked changes** option, the model constrains $M_{h,i}$ by the variance (σ_M^2). For example, setting $\sigma_M^2 = 0.04$ and four specific years (1976, 1980, 1985, 1994) we get



Growth and survival

Growth and survival

$$\mathbf{A}_h = \mathbf{G}_h [\exp(-\mathbf{M}_{0,h})\mathbf{I}] .$$

Assuming a non-zero steady-state fishing mortality rate vector $\mathbf{f}_{h,i}$, the equilibrium growth and survival process is represented by

$$\mathbf{B}_{h,i} = \mathbf{G}_h [\exp(-M_{h,i} - \mathbf{f}_{h,i})\mathbf{I}] .$$

The vector $\mathbf{f}_{h,i}$ represents all mortality associated with fishing, including discards in directed and non-directed fisheries.

Survivorship to length

Assuming unit recruitment, then the growth and survivorship in unfished and fished conditions is given by the solutions to the matrix equations

$$\mathbf{u}_h = -(\mathbf{A}_h - \mathbf{I})^{-1}(p(\mathbf{r})),$$

$$\mathbf{v}_h = -(\mathbf{B}_h - \mathbf{I})^{-1}(p(\mathbf{r})).$$

The vectors \mathbf{u}_h and \mathbf{v}_h represent the unique equilibrium solution for the numbers per recruit in each size category. The total unfished numbers in each size category is defined as $R_0\mathbf{u}_h$.

Steady-state conditions

$$B_0 = R_0 \sum_h \lambda_h \sum_{\ell} \mathbf{u}_h w_{h,\ell} m_{h,\ell},$$

$$\tilde{B} = \tilde{R} \sum_h \lambda_h \sum_{\ell} \mathbf{v}_h w_{h,\ell} m_{h,\ell}.$$

Recruitment

Recruitment size-distribution

$$\alpha = \frac{\alpha_r}{\beta_r},$$

$$p(\mathbf{r}_i) = \int_{x_\ell - 0.5\Delta x}^{x_\ell + 0.5\Delta x} \frac{x^{\alpha-1} \exp\left(\frac{x}{\beta_r}\right)}{\Gamma(\alpha)x^\alpha} dx,$$

$$\mathbf{r}_{h,i} = 0.5p(\mathbf{r}_i)\ddot{R}.$$

Population dynamics

Initial numbers at length

$$\mathbf{n}_{h,i} = \left[-(\mathbf{A}_h - \mathbf{I})^{-1} \mathbf{r}_{h,i} \right] e^{\nu} \quad \text{where } i = 1.$$

The numbers in each size-class in the following time-step ($\mathbf{n}_{h,i+1}$) is the product of the numbers in each size-class in the previous time-step ($\mathbf{n}_{h,i}$), size-specific growth and survival ($\mathbf{A}_{h,i}$), plus new recruits ($\mathbf{r}_{h,i}$)

$$\mathbf{n}_{h,i+1} = \mathbf{n}_{h,i} \mathbf{A}_{h,i} + \mathbf{r}_{h,i} \quad \text{where } i > 1.$$

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Data weighting

Log-likelihood, likelihood, distribution

$$\ell(\mu, \sigma^2, \lambda; x) = \lambda \left(-\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} (x - \mu)^2 \right)$$

$$p(x|\mu, \sigma^2, \lambda) = \exp(\lambda) (2\pi\sigma^2)^{-\frac{1}{2}} \exp \left[-\frac{1}{2\sigma^2} (x - \mu)^2 \right]$$

$$x|\mu, \sigma^2, \lambda \sim \mathcal{N}(\mu, \lambda\sigma^2)$$