Math189R SU17
Homework 5
Monday, June 5, 2017

Feel free to work with other students, but make sure you write up the homework and code on your own (no copying homework *or* code; no pair programming). Feel free to ask students or instructors for help debugging code or whatever else, though.

1 (Murphy 12.5 - Deriving the Residual Error for PCA) It may be helpful to reference section 12.2.2 of Murphy.

(a) Prove that

$$\left\|\mathbf{x}_i - \sum_{j=1}^{k} z_{ij}\mathbf{v}_j\right\|^2 = \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j.$$

Hint: first consider the case when $k = 2$. Use the fact that $\mathbf{v}_i^\top \mathbf{v}_j$ is 1 if $i = j$ and 0 otherwise. Recall that $z_{ij} = \mathbf{x}_i^\top \mathbf{v}_j$.

(b) Now show that

$$J_k = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \mathbf{v}_j^\top \mathbf{x}_i \mathbf{x}_i^\top \mathbf{v}_j\right) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i^\top \mathbf{x}_i - \sum_{j=1}^{k} \lambda_j.$$

Hint: recall that $\mathbf{v}_j^\top \boldsymbol{\Sigma} \mathbf{v}_j = \lambda_j \mathbf{v}_j^\top \mathbf{v}_j = \lambda_j$.

(c) If $k = d$ there is no truncation, so $J_d = 0$. Use this to show that the error from only using $k < d$ terms is given by

$$J_k = \sum_{j=k+1}^{d} \lambda_j.$$

Hint: partition the sum $\sum_{j=1}^{d} \lambda_j$ into $\sum_{j=1}^{k} \lambda_j$ and $\sum_{j=k+1}^{d} \lambda_j$.

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(a)

$$\left\| x_i - \sum_{j=1}^{k} z_{ij} \vec{v_j} \right\|_2^2 = \left(\vec{x_i} - \sum_{j=1}^{k} z_{ij} \vec{v_j}\right)^T \left(\vec{x_i} - \sum_{j=1}^{k} z_{ij} \vec{v_j}\right)$$

$$= \vec{x_i}^T \vec{x_i} - 2 \sum_{j=1}^{k} z_{ij} \vec{v_j}^T \vec{x_i} + \left(\sum_{j=1}^{k} z_{ij} \vec{v_j}\right)^T \left(\sum_{j=1}^{k} z_{ij} \vec{v_j}\right)$$

$$= \vec{x_i}^T \vec{x_i} - 2 \sum_{j=1}^{k} z_{ij} v_j^T \vec{x_i} + \sum_{j=1}^{k} \vec{v_j}^T z_{ij}^T z_{ij} \vec{v_j}$$

$$= \vec{x_i}^T \vec{x_i} - 2 \sum_{j=1}^{k} z_{ij} \vec{v_j}^T \vec{x_i} + \sum_{j=1}^{k} \vec{v_j}^T \vec{x_i} \vec{x_i}^T \vec{v_j}$$

$$= \vec{x_i}^T \vec{x_i} - \sum_{j=1}^{k} v_j^T x_i x_i^T v_j$$, as desired.

(b)

$$J_k = \frac{1}{n} \sum_{i=1}^{n} \left(\vec{x_i}^T \vec{x_i} - \sum_{j=1}^{k} \vec{v_j}^T \vec{x_i} \vec{x_i}^T \vec{v_j} \right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \vec{x_i}^T \vec{x_i} - \sum_{j=1}^{k} \vec{v_j}^T \frac{1}{n} \left(\sum_{i=1}^{n} \vec{x_i} \vec{x_i}^T \right) \vec{v_j}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \vec{x_i}^T \vec{x_i} - \sum_{j=1}^{k} \lambda_j$$ as desired.

(c) Because $J_d = 0$, $\sum_{j=1}^{d} \lambda_j = \frac{1}{n} \sum_{i=1}^{n} \vec{x_i}^T \vec{x_i}$. So

$$J_k = \frac{1}{n} \sum_{i=1}^{n} \vec{x_i}^T \vec{x_i} - \sum_{j=1}^{d} \lambda_j + \sum_{j=k+1}^{d} \lambda_j = \sum_{j=k+1}^{d} \lambda_j .$$

2 (ℓ_1-Regularization) Consider the ℓ_1 norm of a vector $\mathbf{x} \in \mathbb{R}^n$:

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i|.$$

Draw the norm-ball $B_k = \{\mathbf{x} : \|\mathbf{x}\|_1 \le k\}$ for $k = 1$. On the same graph, draw the Euclidean norm-ball $A_k = \{\mathbf{x} : \|\mathbf{x}\|_2 \le k\}$ for $k = 1$ behind the first plot. (Do not need to write any code, draw the graph by hand).

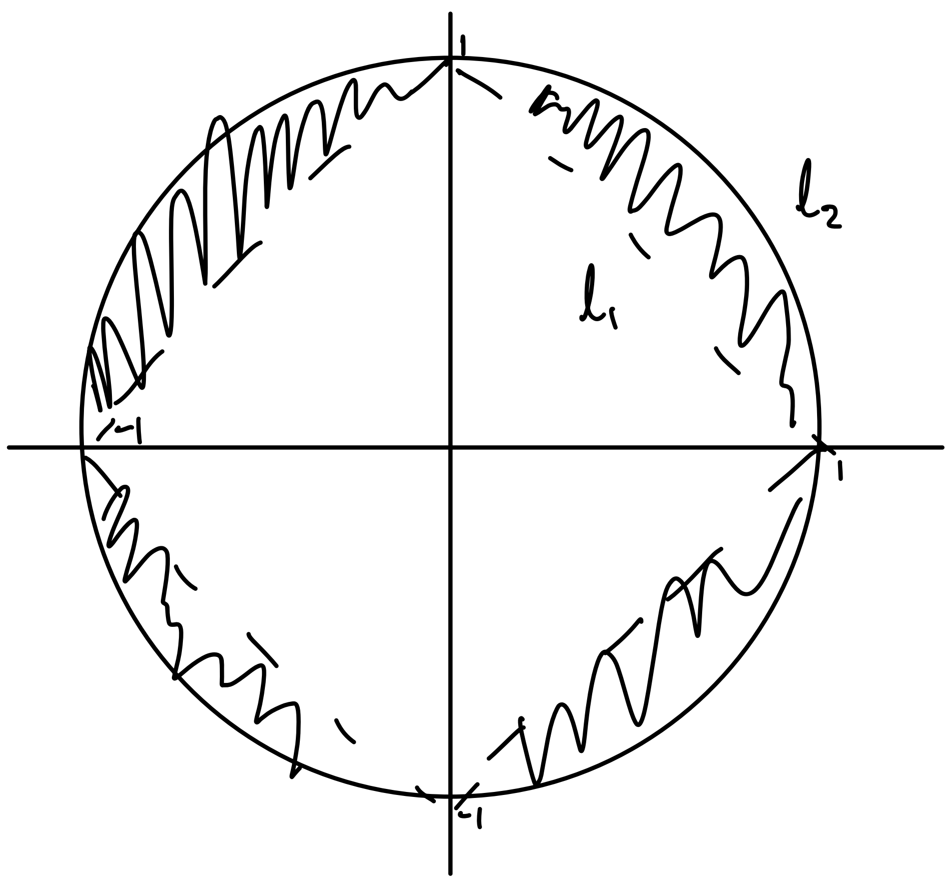
Show that the optimization problem

$$\begin{aligned} \text{minimize: } & f(\mathbf{x}) \\ \text{subj. to: } & \|\mathbf{x}\|_p \le k \end{aligned}$$

is equivalent to

$$\text{minimize: } f(\mathbf{x}) + \lambda\|\mathbf{x}\|_p$$

(hint: create the Lagrangian). With this knowledge, and the plots given above, argue why using ℓ_1 regularization (adding a $\lambda\|\mathbf{x}\|_1$ term to the objective) will give sparser solutions than using ℓ_2 regularization for suitably large λ.



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Minimize: $f(\vec{x})$

Subject to: $\|\vec{x}_p\| \leq k$.

$$\inf_{x} \sup_{\lambda \geq 0} L(\vec{x}, \lambda) = \inf_{x} \sup_{\lambda \geq 0} f(x) + \lambda \left(\|\vec{x}\|_p - k\right).$$

$$\sup_{\lambda \geq 0} \inf_{\vec{x}} f(\vec{x}) + \lambda \left(\|\vec{x}\|_p - k\right) = \sup_{\lambda \geq 0} g(\lambda).$$

reduces to the problem of

minimizing $f(\vec{x}) + \lambda \|\vec{x}\|_p$.

It is clear this quantity is minimized when more weights are zero.

Extra Credit (Lasso) Show that placing an equal zero-mean Laplace prior on each element of the weights $\boldsymbol{\theta}$ of a model is equivelent to ℓ_1 regularization in the Maximum-a-Posteriori estimate

$$\text{maximize: } \mathbb{P}(\boldsymbol{\theta}|\mathcal{D}) = \frac{\mathbb{P}(\mathcal{D}|\boldsymbol{\theta})\mathbb{P}(\boldsymbol{\theta})}{\mathbb{P}(\mathcal{D})}.$$

Note the form of the Laplace distribution is

$$\text{Lap}(x|\mu, b) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$$

where μ is the location parameter and $b > 0$ controls the variance. Draw (by hand) and compare the density $\text{Lap}(x|0,1)$ and the standard normal $\mathcal{N}(x|0,1)$ and suggest why this would lead to sparser solutions than a Gaussian prior on each elements of the weights (which correspond to ℓ_2 regularization).

We seek to maximize $P(\theta | D) = \dfrac{P(D|\theta)P(\theta)}{P(D)}$

$$= \log P(D|\theta) + \log P(\theta) - \log P(D),$$

or minimize $-\log P(D|\theta) - \log P(\theta)$.

Given a prior $\theta_i \sim \text{Lap}(0, b)$.

$$-\log P(\theta) = -\log \prod_i \exp\left(-\frac{|\theta_i|}{b}\right) + C$$

$$= \frac{1}{b}\sum_i |\theta_i| + Z$$

$$= \lambda \|\theta\|_1 + Z.$$

Our original problem is equivalent to

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Minimizing $-\log P(D|\theta) + \lambda\|\theta\|_1$, as desired.