Imperial College
London

COURSEWORK

IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

Optimisation Coursework

Author:
(CID: 01844537)

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Question 1

1(i)

We propose that the function $f(x_1, x_2) = |-x_1 + x_2^2|$ is a function that satisfies:

$$\lim_{|x_1|\to\infty} f(x_1, \alpha x_1) = \lim_{|x_2|\to\infty} f(\alpha x_2, \alpha x_2) = \infty$$

We first prove f's continuity. f is the composition of $g(x_1, x_2) = x_1 + x_2^2$ and the modulus function, both of which are continuous functions on $\mathbb{R}^2 \to \mathbb{R}$. Composition of continuous functions is continuous, hence f is continuous.

Also, we have:

$$\lim_{|x_1|\to\infty} f(x_1, \alpha x_1) = -x_1 + \alpha^2 x_1^2 = \infty$$
$$\lim_{|x_2|\to\infty} f(\alpha x_2, x_2) = -\alpha x_2 + x_2^2 = \infty$$

Hence the proposed conditions are also satisfied.

Now we show that f is not coercive. Let $t \to \infty$. Consider $\boldsymbol{v} = (t^2, t)$, we have that:

$$\|\boldsymbol{v}\| = \sqrt{t^4 + t^2} > t \to \infty$$

But:

$$f\left(t^2, t\right) = -t^2 + t^2 = 0 \neq \infty$$

Therefore we conclude that f is not coercive.

1(ii)

We first calculate $\nabla f(\boldsymbol{x})$:

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} 16x_1^3 - 8x_1x_2 \\ 2x_2 - 4x_1^2 \end{pmatrix}$$

Setting both rows to 0, we get that the stationary points are in the form $(t, 2t^2)$, $t \in \mathbb{R}$. Now, we calculate the Hessian matrix to determine the nature of the stationary points:

$$\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 48x_1^2 - 8x_2 & -8x_1 \\ -8x_1 & 2 \end{pmatrix}$$

Setting $x_2 = 2x_1^2$, the Hessian can be written as:

$$\nabla^2 f(\boldsymbol{x}) = \begin{pmatrix} 32x_1^2 & -8x_1 \\ -8x_1 & 2 \end{pmatrix}$$

The eigenvalues for this matrix are $\lambda_1 = 0, \lambda_2 = 32x_1^2 + 2$. Since both eigenvalues are non-negative, we can conclude that the stationary points are either minimum points or saddle points due to the semi-positive definite hessian. Also, note that we can write f in the following form:

$$f(x_1, x_2) = 4x_1^4 + x_2^2 - 4x_1^2 x_2 + 4 = (x_2 - 2x_1^2)^2 + 4 \geq 4$$

and $f(x_1, 2x_1^2) = 4$, therefore we conclude that stationary points $(x_1, 2x_1^2)$ are global minimal points (since they attain the lower bound of the function). They are non-strict global minima, since the stationary points are non-unique. They are also non-strict local minimum points, since for any ball $B(\boldsymbol{t}^*, \epsilon)$ around an arbitrary stationary point $\boldsymbol{t}^* = (t, 2t^2)$, $\exists \boldsymbol{t'} = (t', 2(t')^2)$ such that $\|\boldsymbol{t}^* - \boldsymbol{t}'\| < \epsilon$ and $f(\boldsymbol{t'}) = f(\boldsymbol{t}^*) = 4$ (such $\boldsymbol{t}^*$ always exists as $2x_1^2$ is a continuous function of x_1).

Question 2

2(i)

We have the following relationship between $\boldsymbol{x}$ and $\boldsymbol{u}$:

$$\begin{aligned} x_0 &= \bar{x} \\ x_i &= \alpha x_{i-1} + d u_i \end{aligned}$$

We can recursively define a relationship between x_i, $\bar{x}$, and d:

$$\begin{aligned} x_0 &= \bar{x} \\ x_1 &= \alpha \bar{x} + d u_1 \\ x_2 &= \alpha x_1 + d u_2 = \alpha^2 \bar{x} + \alpha d u_1 + d u_2 \\ &\vdots \\ x_N &= \alpha^N \bar{x} + \sum_{i=1}^{N} \alpha^{N-i} d u_i \end{aligned}$$

Consider the truncated vector $\boldsymbol{x'}$, where:

$$\boldsymbol{x'} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

This vector can be written in the form as follows:

$$\boldsymbol{x'} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} = \underbrace{\begin{pmatrix} d & 0 & 0 & \cdots & 0 \\ \alpha d & d & 0 & \cdots & 0 \\ \alpha^2 d & \alpha d & d & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \alpha^{N-1} d & \alpha^{N-2} d & \alpha^{N-3} d & \cdots & d \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_N \end{pmatrix}}_{\boldsymbol{u}} - \underbrace{\begin{pmatrix} -\alpha \bar{x} \\ -\alpha^2 \bar{x} \\ -\alpha^3 \bar{x} \\ \vdots \\ -\alpha^N \bar{x} \end{pmatrix}}_{\boldsymbol{b}}$$

Now let us consider the original optimisation problem:

$$\min_{\boldsymbol{u}\in\mathbb{R}^N} ||\boldsymbol{x}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2, \quad \gamma > 0 \tag{1}$$

More specifically, consider the vector $\boldsymbol{x}$: the value of x_0 is fixed to $\bar{x}$, independent of the values of u_i. Therefore, finding $\boldsymbol{x}$ that satisfies (1) is equivalent to finding $\boldsymbol{x}'$ and $\boldsymbol{u}$ that satisfy

$$\min_{\boldsymbol{u}\in\mathbb{R}^N} ||\boldsymbol{x}'||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2, \quad \gamma > 0$$

Since x_0 has no effect on the value of $||\boldsymbol{x}||$. Therefore, once we calculate the value of $||\boldsymbol{x}'||$, inserting $\bar{x}$ at the start of $\boldsymbol{x}'$ would retrieve us the appropriate $\boldsymbol{x}$.

Hence, we now have a regularised linear least squares problem:

$$\min_{\boldsymbol{u}\in\mathbb{R}^N} ||\boldsymbol{A}\boldsymbol{u} - \boldsymbol{b}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2, \quad \gamma > 0$$

From lectures, we know that the solution to this system is:

$$\boldsymbol{u}_{RLS} = (A^T A + \frac{\gamma}{2} I^T I)^{-1} A^T \boldsymbol{b}$$

The solution $\boldsymbol{u}_{RLS}$ exists and is unique, if the matrix $(A^T A + \frac{\gamma}{2} I^T I)$ is invertible, which is equivalent to saying $\text{Null}(A) \cap \text{Null}(I) = 0$. But $\text{Null}(I) = 0$, hence the matrix is always invertible.

To show that $||\boldsymbol{u}^*|| \leq ||\boldsymbol{u}||$, let us first suppose the opposite, $||\boldsymbol{u}^*|| > ||\boldsymbol{u}||$. By definition of $||\boldsymbol{u}||$ being the solution for the unregularised problem, we have that:

$$||\boldsymbol{x}_{\bar{x}}^{\boldsymbol{u}}|| \leq ||\boldsymbol{x}_{\bar{x}}^{\boldsymbol{u}^*}||$$

But then we would have:

$$||\boldsymbol{x}_{\bar{x}}^{\boldsymbol{u}}|| + \frac{\gamma}{2}||\boldsymbol{u}|| < ||\boldsymbol{x}_{\bar{x}}^{\boldsymbol{u}^*}|| + \frac{\gamma}{2}||\boldsymbol{u}^*||$$

which is a contradiction, as $\boldsymbol{u}$ is the solution for the regularised least squares problem.

2(ii)

Using the regularised solution in 2(i) and Python, we generate the following plots with varying γ values:

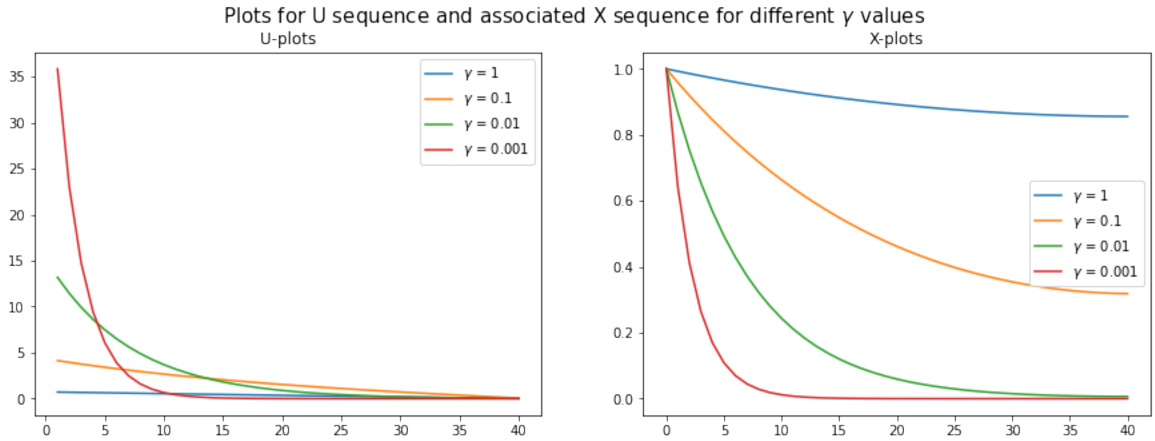

Figure 1: Plots with varying γ values

As we can see in Figure 1, when we increase the γ parameter value, the U plots flatten towards 0 and the X plots flatten towards 1. This is to be expected, as when we increase the value of γ, the weight of the regularising function on the eventual solution increases; the minimiser for the regularising function ($\min\limits_{\boldsymbol{u}} ||\boldsymbol{u}||_2^2$) is just the zero vector. Hence, the solution for:

$$\min_{\boldsymbol{u}\in\mathbb{R}^N} ||\boldsymbol{x}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2$$

would tend towards $\overrightarrow{\boldsymbol{0}}$ vector when γ is large. As for the behaviour of $\boldsymbol{x}$, we consider how x_i is defined:

$$x_i = \alpha x_{i-1} + d u_i$$

When u_i are small, $x_i \approx \alpha x_{i-1}$. In our case, $\alpha = 1$ and $x_0 = \bar{x} = 1$, hence $x_i \approx x_{i-1} \approx \cdots \approx x_0 = 1$ and $\boldsymbol{x} \to \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ vector when γ is large.

Now let us analyse what happens when γ is small; when γ is small, the term $||\boldsymbol{x}||_2^2$ holds a much bigger weight on the equation: $\min_{\boldsymbol{u}\in\mathbb{R}^N} ||\boldsymbol{x}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2$. Consider the minimiser for $||\boldsymbol{x}||_2^2 = ||A\boldsymbol{u} - \boldsymbol{b}||_2^2$. Since A is a lower triangular matrix, it follows that it's invertible and $\min\limits_{\boldsymbol{u}} ||A\boldsymbol{u} - \boldsymbol{b}||_2^2$ is just equal to $A^{-1}\boldsymbol{b}$. For our values of α and d, we can deduce that $A^{-1}\boldsymbol{b}$ is the vector: $\boldsymbol{u_0} = \begin{pmatrix} 100 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ and the associated $\boldsymbol{x}$ vector is equal to $\boldsymbol{x_0} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Hence, when γ is small, one should expect the trajectories for $\boldsymbol{u}$ and $\boldsymbol{x}$ to tend towards the aforementioned vectors. $\boldsymbol{u_0}$ and $\boldsymbol{x_0}$ are plotted in figure 2:

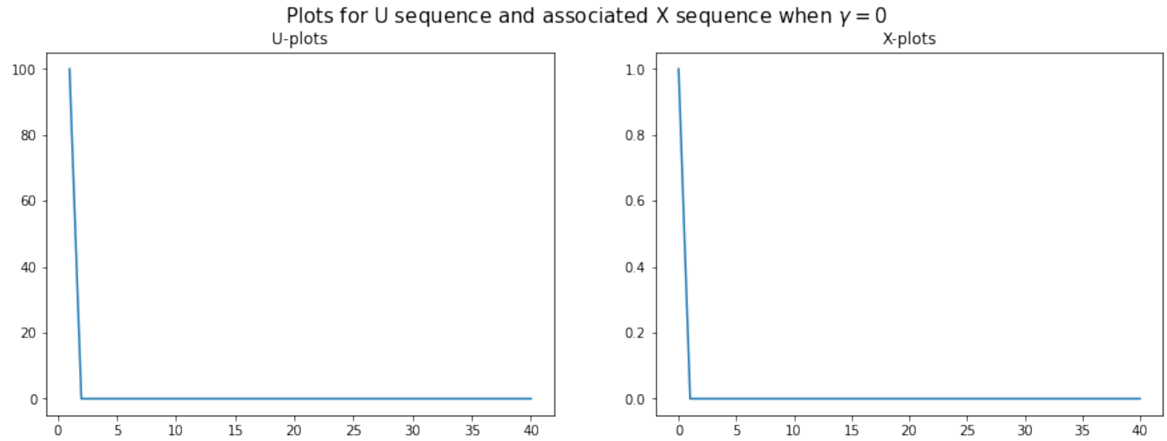

Figure 2: Plots for $\boldsymbol{u}$, $\boldsymbol{x}$, $\gamma = 0$

Observing figure 1 again and comparing it with figure 2, we can see that the trajectories do indeed tend towards $\boldsymbol{u_0}$ and $\boldsymbol{x_0}$ when γ is small.

2(iii)

For the question, we will use the Gradient Descent methods with constant step size 0.1 to derive the minimiser for function f, where:

$$f(\boldsymbol{u}) = ||\boldsymbol{x}_{\bar{\boldsymbol{x}}}^{\boldsymbol{u}}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2 - \delta \sum_{i=1}^{N} log(u_{max} - u_i)$$

Similarly to part (i), finding $\min_{\boldsymbol{u}} f(\boldsymbol{u})$ is equivalent to finding:

$$\min_{\boldsymbol{u}} ||(\boldsymbol{x}')_{\bar{\boldsymbol{x}}}^{\boldsymbol{u}}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2 - \delta \sum_{i=1}^{N} log(u_{max} - u_i)$$

where $\boldsymbol{x}'$ is $\boldsymbol{x}$ with x_0 truncated out:

$$\boldsymbol{x}' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

Transforming $\boldsymbol{x}'$ with the same method as used in (i), the optimisation problem becomes:

$$\min_{\boldsymbol{u}} ||A\boldsymbol{u} - \boldsymbol{b}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2 - \delta \sum_{i=1}^{N} log(u_{max} - u_i)$$

where A and $\boldsymbol{b}$ are the same as described in part (i). Let:

$$g(\boldsymbol{u}) = ||A\boldsymbol{u} - \boldsymbol{b}||_2^2 + \frac{\gamma}{2}||\boldsymbol{u}||_2^2 - \delta \sum_{i=1}^{N} log(u_{max} - u_i)$$

Then:

$$\nabla g(\boldsymbol{u}) = 2A^T A\boldsymbol{u} - 2A^T\boldsymbol{b} + \gamma\boldsymbol{u} + \delta\boldsymbol{u}'$$

where:

$$\boldsymbol{u}' = \begin{pmatrix} \frac{1}{u_{max}-u_1} \\ \frac{1}{u_{max}-u_2} \\ \vdots \\ \frac{1}{u_{max}-u_N} \end{pmatrix}$$

With the $\nabla g(\boldsymbol{u})$ calculated, we are now ready to implement the Gradient Descent method. The plots below are generated with constant step size of $t = 0.1$ and tolerance parameter $\lambda = 0.0001$; the starting vector is chosen to be the $(N \times 1)$-array of 1's:

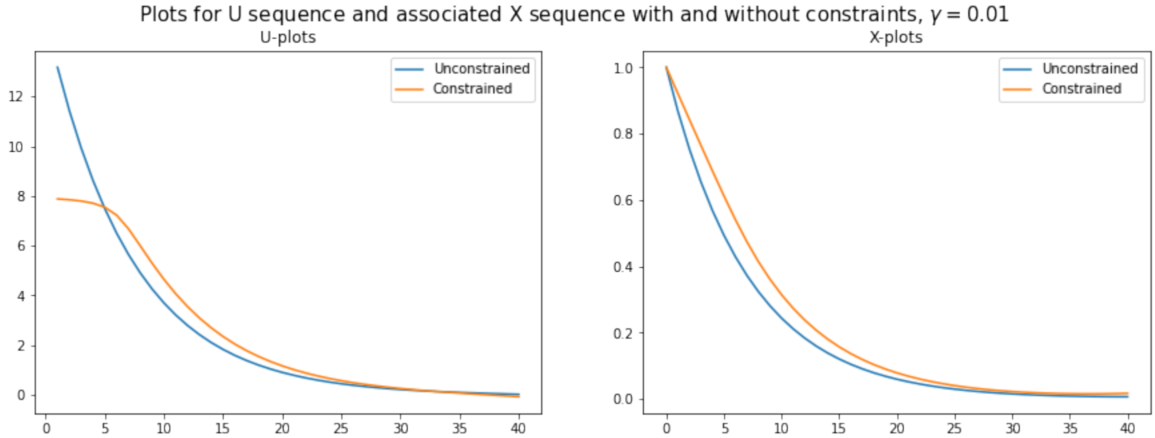

Figure 3: Plots for the constrained and unconstrained problem, $\gamma = 0.01$

Let us first analyse the plots for control sequence $\boldsymbol{u}$. For comparison's sake, let us denote $\boldsymbol{v}$ as the $\boldsymbol{u}$ sequence generated without constraints and $\boldsymbol{w}$ the sequence generated with constraints. At the start of the plot, $\boldsymbol{w}$ expectedly starts below $u_{max} = 8$ (because of constraints on w_i), whereas $\boldsymbol{v}$ starts marginally above u_{max}. The additional log term constrains the values of u_i, as when $u_i \uparrow u_{max}$, it causes the log term to blow up towards infinity. The "loss" of magnitude in the starting w_i's is compensated by a slower descent towards 0. As we can see in the plot, $\boldsymbol{w}$'s trajectory has a flatter gradient than the trajectory of $\boldsymbol{v}$. Trajectories of $\boldsymbol{x}$ mostly overlap, with the constrained trajectory being slightly above the unconstrained one due to the additional log term.

2(iv)

For this question, the function we are trying to minimize is:

$$f(\boldsymbol{u}) = \|\boldsymbol{x}_{\bar{\boldsymbol{x}}}^{\boldsymbol{u}}\|_2^2 + \frac{\gamma_2}{2}\|\boldsymbol{u}\|_2^2 + \gamma_1 \sum_{i=1}^{N} \mathcal{L}_\epsilon(u_i)$$

where $\mathcal{L}_\epsilon$ is the ℓ_1 norm approximation defined as:

$$\mathcal{L}_\epsilon(u_i) = \begin{cases} \frac{1}{2}u_i^2 & \text{if } |u_i| \leq \epsilon \\ \epsilon(|u_i| - \frac{1}{2}\epsilon) & \text{otherwise} \end{cases}$$

We first consider the continuity of $\mathcal{L}_\epsilon$; $\mathcal{L}_\epsilon(x)$ is a piecewise function, and both pieces of the function are continuous in their respective domains, hence the only points where we have to consider are $\pm\epsilon$. We have:

$$\lim_{x \to \epsilon^-} \mathcal{L}_\epsilon(x) = \frac{1}{2}\epsilon^2 = \lim_{x \to \epsilon^+} \mathcal{L}_\epsilon(x) = \epsilon(\epsilon - \frac{1}{2}\epsilon) = \frac{1}{2}\epsilon^2$$
$$\lim_{x \to -\epsilon^+} \mathcal{L}_\epsilon(x) = \frac{1}{2}\epsilon^2 = \lim_{x \to -\epsilon^-} \mathcal{L}_\epsilon(x) = \epsilon(|-\epsilon| - \frac{1}{2}\epsilon) = \frac{1}{2}\epsilon^2$$

as long as $\epsilon > 0$, hence $\mathcal{L}_\epsilon(x)$ is continuous ($\epsilon > 0$).

Now consider the differentiability of $\mathcal{L}_\epsilon(x)$. Again, given that $\epsilon > 0$, the pieces of $\mathcal{L}_\epsilon$ are always differentiable in their respective domains, since x^2 is always differentiable and $\epsilon(|x| - \frac{1}{2}\epsilon)$ is differentiable everywhere apart from 0. The only points where we have to consider are $\pm\epsilon$.

$$\lim_{x \to \epsilon^-} \mathcal{L}_\epsilon^{'}(x) = \epsilon = \lim_{x \to \epsilon^+} \mathcal{L}_\epsilon^{'}(x) = \epsilon(sgn(\epsilon)) = \epsilon$$
$$\lim_{x \to -\epsilon^-} \mathcal{L}_\epsilon^{'}(x) = -\epsilon = \lim_{x \to -\epsilon^+} \mathcal{L}_\epsilon^{'}(x) = \epsilon(sgn(-\epsilon)) = -\epsilon$$

The two derivatives' limits are equal to each other when taking $x \to \pm\epsilon$, hence we conclude that $\mathcal{L}_\epsilon$ is differentiable. Also, when we take $\epsilon \to 0$, $\sum_{i=1}^N \mathcal{L}_\epsilon(u_i) \approx \sum_{i=1}^N \epsilon|u_i|$. Taking ϵ out and combining it with γ_1, we have $\sum_{i=1}^N \epsilon|u_i|$, which is just the ℓ_1 norm. Hence, if ϵ is sufficiently small then it's a good approximation of ℓ_1 norm.

Let us now consider $\mathcal{L}_\epsilon(x)$ does as a penalty function. When $|u_i| \leq \epsilon$, $\mathcal{L}_\epsilon$ penalises them by squaring each term (the constant $\frac{1}{2}$ can be taken out into the γ_1 term), much similar to our previously used penalty function, $\frac{\gamma_2}{2}\|\boldsymbol{u}\|_2^2$. However, when $|u_i| > \epsilon$, $\mathcal{L}_\epsilon$ penalises u_i linearly instead of squaring. Moreover, if $|u_i| \gg \epsilon$, then we have the following inequality:

$$u_i^2 \gg \epsilon(|u_i|) > \epsilon(|u_i| - \epsilon)$$

From which we can see that large u_i's, $\mathcal{L}_\epsilon$ is significantly less penalising than $\frac{\gamma_2}{2}\|\boldsymbol{u}\|_2^2$.

Let us now return to our optimisation problem. Similar to parts (i) and (iii), finding $\boldsymbol{u}$ that minimises $f(\boldsymbol{u})$ is equivalent to finding $\boldsymbol{u}$ that minimises g, where g is defined as:

$$g(\boldsymbol{u}) = \|(\boldsymbol{x}')_{\tilde{\boldsymbol{x}}}^{\boldsymbol{u}}\|_2^2 + \frac{\gamma_2}{2}\|\boldsymbol{u}\|_2^2 + \gamma_1 \sum_{i=1}^{N} \mathcal{L}_\epsilon(u_i)$$

$$\boldsymbol{x}' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

Now we can calculate the derivative of g. $\nabla g(\boldsymbol{u})$ is equal to:

$$\nabla g(\boldsymbol{u}) = 2A^T A\boldsymbol{u} - 2A^T \boldsymbol{b} + \gamma_2 \boldsymbol{u} + \gamma_1 \boldsymbol{\mathcal{L}u}'$$

where:

$$[\boldsymbol{\mathcal{L}u}']_i = \begin{cases} u_i & \text{if } |u_i| \leq \epsilon \\ \epsilon(sgn(u_i)) & \text{otherwise} \end{cases}$$

$[\boldsymbol{\mathcal{L}u}']_i$ denotes the i-th component of $\boldsymbol{\mathcal{L}u}'$. Implementing the backtracking Gradient Descent method with parameters $\alpha = 0.5$, $\beta = 0.5$, $s = 1$ and tolerance number $\lambda = 10^{-4}$, we generate the following plots with $\epsilon = 3, \gamma_1 = 0, \gamma_2 = 10^{-2}$ and $\epsilon = 3, \gamma_2 = 0, \gamma_1 = 10^{-2}$; the starting vector is chosen to be the $(N \times 1)$-array of 1's:

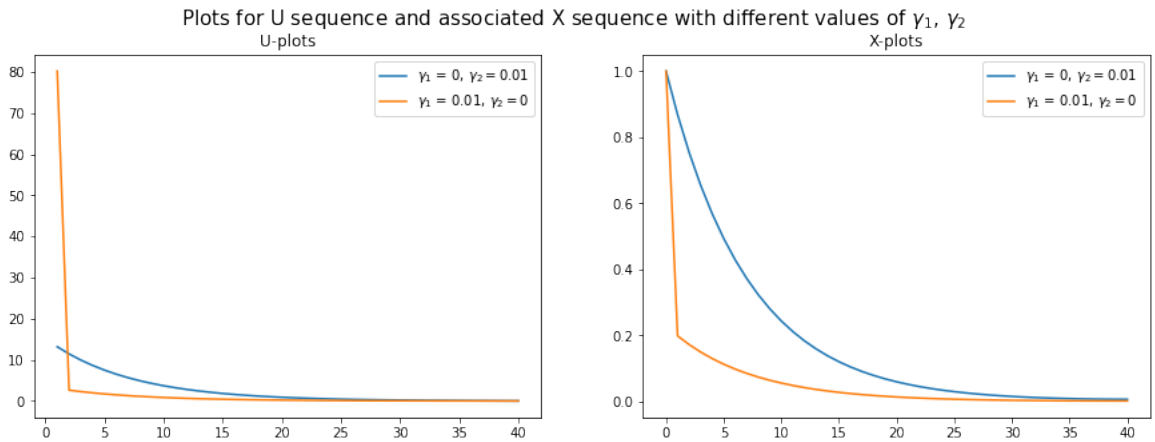

Figure 4: Plots for $\gamma_1 = 0$, $\gamma_2 = 10^{-2}$ and $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$

For the $\boldsymbol{u}$ plots, we can see that when $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$, the trajectory starts very high before rapidly dropping to near 0, whereas when $\gamma_1 = 0$, $\gamma_2 = 10^{-2}$, $\boldsymbol{u}$ starts significantly lower but plateaus towards 0 in a slower and smoother manner. The $\boldsymbol{x}$ plots show a similar behaviour: when $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$, $\boldsymbol{x}$ drops rapidly between x_0 and x_1 before gradually plateauing towards 0; when $\gamma_1 = 0$, $\gamma_2 = 10^{-2}$, $\boldsymbol{x}$ descends towards 0 also in a much more smoother manner. There is a noticeable increase in sparsity in both $\boldsymbol{u}$ and $\boldsymbol{x}$ trajectories when $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$.

Interestingly, the plots generated for $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$ show great resemblance to figure 2, i.e. the minimiser of $||A\boldsymbol{u} - \boldsymbol{b}||_2^2$. Recall our previous minimiser $\boldsymbol{u_0} = \begin{pmatrix} 100 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for

$\|A\boldsymbol{u} - \boldsymbol{b}\|_2^2$, whose leading term $((\boldsymbol{u_0})_1 = 100)$ is very large. When $\gamma_1 = 10^{-2}$, $\gamma_2 = 0$, the optimisation problem becomes:

$$\min_{\boldsymbol{u}} \|\boldsymbol{x}_{\bar{\boldsymbol{x}}}^{\boldsymbol{u}}\|_2^2 + 10^{-2} \sum_{i=1}^{N} \mathcal{L}_\epsilon(u_i)$$

with $10^{-2} \sum_{i=1}^{N} \mathcal{L}_\epsilon(u_i)$ as the sole regularising term. Since $\mathcal{L}_\epsilon$ penalises the u_i with $|u_i| > \epsilon$ weakly, the large leading term $((\boldsymbol{u_0})_1 = 100)$ in $\boldsymbol{u_0}$ is penalised much less in comparison to when $\frac{\gamma_2}{2}\|\boldsymbol{u}\|_2^2$ is used as the regularising function. Hence the solution for

$$\min_{\boldsymbol{u}} \|\boldsymbol{x}_{\bar{\boldsymbol{x}}}^{\boldsymbol{u}}\|_2^2 + 10^{-2} \sum_{i=1}^{N} \mathcal{L}_\epsilon(u_i)$$

would tend more towards for $\boldsymbol{u_0}$, which explains the resemblance between figure 4 and 2.