Low rank representation, RPCA and TILT

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Robust Principal Component Analysis on image processing

Suppose the image, which we're interested, has high repeatability along horizontal or vertical direction. We use a convex program to separate the low-rank part and block-sparse part of the observed matrix. The decomposition involves the following model:

$$\boldsymbol{I} = \boldsymbol{A} + \boldsymbol{E}$$

where $\boldsymbol{I}$ is the observation matrix with a low-rank component $\boldsymbol{A}$ and block-sparse component $\boldsymbol{E}$.

Then the problem here is to recover the background, a low rank matrix $\boldsymbol{A}$ from corrupted observations. That is, for appropriate γ, we aim to solve the optimization problem:

$$\min_{\boldsymbol{A},\boldsymbol{E}} \operatorname{rank}(\boldsymbol{A}) + \gamma \|\boldsymbol{E}\|_0 \quad \text{subject to} \;\; \boldsymbol{I} = \boldsymbol{A} + \boldsymbol{E},$$

where $\|\boldsymbol{E}\|_0$ denotes the number of non-zero entries in $\boldsymbol{E}$.

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However, the optimization problem is not convex. Thus, we reformulate the problem as:

$$\min_{\boldsymbol{A},\boldsymbol{E}} \|\boldsymbol{A}\|_* + \gamma\|\boldsymbol{E}\|_1 \quad \text{subject to } \boldsymbol{I} = \boldsymbol{A} + \boldsymbol{E}.$$

Note that $\|\boldsymbol{A}\|_*$ is the nuclear norm of $\boldsymbol{A}$, and is defined by

$$\|\boldsymbol{A}\|_* := \sum_{i=1}^{r} \sigma_i,$$

where r is the rank of $\boldsymbol{A}$, and σ_i are the singular values of $\boldsymbol{A}$. Now, the problem is a convex minimization problem. Apply the penalty parameter μ, the problem becomes

$$\min_{\boldsymbol{A},\boldsymbol{E}} \|\boldsymbol{A}\|_* + \gamma\|\boldsymbol{E}\|_1 + \frac{\mu}{2}\|\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{E}\|_F^2,$$

for appropriate γ and μ.

Solving RPCA by Alternating Direction Method

From the above discussion. We set $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{E}^0 = \mathbf{0}$, the basic ADM iteration scheme for our problem gives that

$$\begin{aligned} \boldsymbol{A}^{(k+1)} &= \arg\min_{\boldsymbol{A}} \ \|\boldsymbol{A}\|_* + \frac{\mu}{2}\|\boldsymbol{I} - \boldsymbol{A} - \boldsymbol{E}^{(k)}\|_F^2, \\ \boldsymbol{E}^{(k+1)} &= \arg\min_{\boldsymbol{E}} \ \gamma\|\boldsymbol{E}\|_1 + \frac{\mu}{2}\|\boldsymbol{I} - \boldsymbol{A}^{(k+1)} - \boldsymbol{E}\|_F^2. \end{aligned}$$

Thus after some mathematics calculation, we can obtain the iterative method:

$$\begin{aligned} \boldsymbol{A}^{(k+1)} &= \mathrm{SVT}_{\frac{1}{\mu}}(\boldsymbol{I} - \boldsymbol{E}^{(k)}), \\ \boldsymbol{E}^{(k+1)} &= \mathrm{sign}(\boldsymbol{I} - \boldsymbol{A}^{(k+1)}) \odot \max\{|\boldsymbol{I} - \boldsymbol{A}^{(k+1)}| - \frac{\gamma}{\mu}, 0\}, \end{aligned}$$

by setting $\boldsymbol{A}^0 = \boldsymbol{I}$, $\boldsymbol{E}^0 = \mathbf{0}$.

Singular Value Decomposition

SVD Theorem :
Let $\boldsymbol{M} \in \mathbb{R}^{m \times n}$, the SVD of $\boldsymbol{M}$ is the factorization in the form

$$\boldsymbol{M} = \mathsf{U}\Sigma\mathsf{V}^\top,$$

where the column vectors of $\mathsf{U} \in \mathbb{R}^{m \times m}$ are orthonormal, $\Sigma \in \mathbb{R}^{m \times n}$ is diagonal with all non-negative entries called the singular values of $\boldsymbol{M}$ and the column vectors of $\mathsf{V} \in \mathbb{R}^{n \times n}$ are also orthonormal.

Singular value thresholding(SVT) :
Let $\boldsymbol{M} \in \mathbb{R}^{m \times n}$, suppose $\mathsf{U}\Sigma\mathsf{V}^\top$, the SVD of $\boldsymbol{M}$. The singular value thresholding (SVT) of $\boldsymbol{M}$ with threshold $\sigma > 0$ is defined by

$$\mathsf{SVT}_\sigma(M) = \mathsf{U}\mathsf{D}_\sigma(\Sigma)\mathsf{V}^\top,$$

where

$$\mathsf{D}_\sigma(\Sigma)_{ii} = \max\{\Sigma_{ii} - \sigma, 0\}.$$

Denoising using RPCA

Although we supposed that the background of the given image to be high repeatability, which is not achieved in many cases, we can still use the model to denoise after choosing appropriate parameters.

Background recovering using RPCA

The photo on the left is the image we observed, and the right is the result after we did RPCA. It can be seen that not only the middle part is filled in the corners too. The processing is that we first divide the RGB three-layer image into RPCA, and then combine the results.

Deformed and Corrupted Low-Rank Textures

Deformed Low-Rank Textures The image $\boldsymbol{I}(x, y)$ that we observe from a certain viewpoint is a transformed version of the original low-rank texture function $\boldsymbol{I}^0(x, y)$:

$$\boldsymbol{I}(x, y) = \boldsymbol{I}^0 \circ \tau^{-1}(x, y),$$

where $\tau \in \mathbb{R}^2 \to \mathbb{R}^2$ belongs to a certain Lie group $\mathbb{G}$. Here, we assume that $\mathbb{G}$ is either the 2D affine group, or the homography group acting linearly on the image domain.

Corrupted Low-Rank Textures The observed image of the texture might be corrupted by noise and occlusions or contain some pixels from the surrounding background. We can model such deviations by an error matrix E as follows:

$$\boldsymbol{I} = \boldsymbol{I}^0 + \boldsymbol{E}.$$

Transform Invariant Low-rank Texture (TILT)

Given a deformed and corrupted image of a low-rank texture: $\boldsymbol{I} = (\boldsymbol{I}^0 + \boldsymbol{E}) \circ \tau^{-1}$, recover the low-rank texture $\boldsymbol{I}^0$ and the domain transformation $\tau \in \mathbb{G}$.
The above formulation naturally leads to the following optimization problem:

$$\min_{\boldsymbol{I}^0, \boldsymbol{E}, \tau} \ \operatorname{rank}(\boldsymbol{I}^0) + \gamma \|\boldsymbol{E}\|_0 \quad \text{subject to } \ \boldsymbol{I} \circ \tau = \boldsymbol{I}^0 + \boldsymbol{E}.$$

However, the optimization problem is not convex. Thus, we reformulate the problem as:

$$\min_{\boldsymbol{I}^0, \boldsymbol{E}, \Delta\tau} \ \|\boldsymbol{I}^0\|_* + \gamma \|\boldsymbol{E}\|_1 \quad \text{subject to } \boldsymbol{I} \circ \tau + \nabla \boldsymbol{I} \Delta\tau = \boldsymbol{I}^0 + \boldsymbol{E},$$

where $\nabla \boldsymbol{I}$ is the Jacobian (derivatives of the image with respect to the transformation parameters).

ADM for TILT model

Due to the special structure of our problem, the optimization problem can be solved in a single step. More precisely, the solutions can be expressed explicitly as follows:

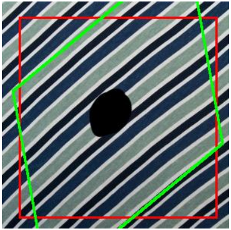
$$
\begin{aligned}
\boldsymbol{I}^{0(k+1)} &= \mathrm{SVT}_{\frac{1}{\mu}}(\boldsymbol{M}^{(k)} - \boldsymbol{E}^{(k)}), \\
\boldsymbol{E}^{(k+1)} &= \mathrm{sign}(\boldsymbol{M}^{(k)} - \boldsymbol{I}^{0(k+1)}) \odot \max\{|\boldsymbol{M}^{(k)} - \boldsymbol{I}^{0(k+1)} - \frac{\gamma}{\mu}|, 0\}, \\
\Delta\tau^{(k+1)} &= (\nabla \boldsymbol{I} \circ \tau^{(k)})^{\dagger}(-\boldsymbol{I} \circ \tau^{(k)} + \boldsymbol{I}^{0(k+1)} + \boldsymbol{E}^{(k+1)}),
\end{aligned}
$$

where $\boldsymbol{M}^{(k)} = \boldsymbol{I} \circ \tau^{(k)} + (\nabla \boldsymbol{I}^{(k)} \Delta\tau^{(k)}) \circ \tau^{(k)}$.
While ADM iteration above converges, then do ADM again by

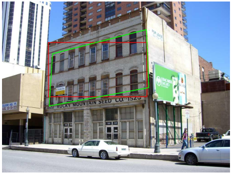
$$\tau^{(k+1)} = \tau^{(k)} + \Delta\tau.$$

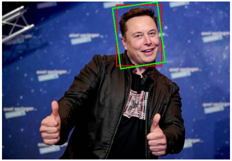
Note that $\boldsymbol{X}^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of $\boldsymbol{X}$.

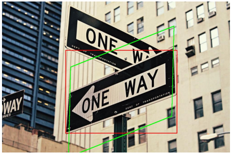
ADM for TILT model

Experimental Results of TILT







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