Independence of the Axiom of Choice, Permutation models

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Outline

Historical Remarks

- Permutation Models were introduced by Fraenkel *1922-37*
- Precise version by Motowski *1938-39*
- A Version with Filters by Specker *1957*
- Additional Contributions Doss (*1945*), Mendelson (*1948,1956*), Jesenin-Vol'pin (*1954*), Shoenfield (*1955*), Fraisse (*1958*)

Goal of this Talk

- Want to show that the Axiom of Choice is *independent* from the other Axioms
- See Chapter 5 of [1] for the result in ordinary set theory
- This talk will introduce Set theory with atoms
- Establish the independence of the Axiom of Choice in this context

ZFA

Recall Axiom of Choice

For every family $\mathscr{F}$ of nonempty sets, there is a function f such that $f(s) \in S$ for each set S in the family $\mathscr{F}$. We call f a *choice function* on $\mathscr{F}$.

Well-ordering Principle

Also known as *Zermelo's Theorem*.
Every set can be well-ordered, i.e. for an ordering $<$ of a set S every nonempty set $X \subseteq S$ has a least element regarding $<$.

Definition ZFA

The set theory with atoms is a modified version of set theory, and is characterized by the fact that it admits objects other than sets, *atoms*.

ZFA

- Atoms are objects without any elements
- Language of ZFA consists of $=$ and $\in$
- Includes constant symbols $\emptyset$ (empty set) and A (set of Atoms)
- *Empty set* $\emptyset$
$$\nexists x \; (x \in \emptyset)$$
- *Atoms* A
$$\forall z[z \in A \leftrightarrow z \neq \emptyset \wedge \nexists x \; (x \in z)]$$

Elements of A are called *atoms*, sets are all objects which are not atoms

Axioms of ZF

- A1. Extensionality
- A2. Pairing
- A3. Comprehension
- A4. Union
- A5. Power-Set
- A6. Replacement
- A7. Infinity
- A8. Regularity

Changes to the Axioms

- A1. Extensionality

$$(\forall \textit{ set } X)(\forall \textit{ set } Y)[\forall\, u\, (u \in X \;\leftrightarrow\; u \in Y) \;\leftrightarrow\; X = Y]$$

- instead of:

$$\forall u\, (u \in X \leftrightarrow u \in Y) \rightarrow X = Y$$

- A8. Regularity

$$(\forall \textit{ nonempty } S)(\exists x \in S)[x \cap S = \emptyset]$$

- instead of

$$(\forall S \neq \emptyset)(\exists x \in S)[x \cap S = \emptyset]$$

Note

'X is nonempty' is not the same as '$X \neq \emptyset$', only if X is a set.
Some options only make sense for sets: $\cup X$ $\mathcal{P}(X)$
Some also for atoms: $\{x, y\}$

Problem 1

- If $A = \emptyset$ we get ZF
- But we are more interested in the case A is not empty

Problem 1
(ZFA+Axiom of Choice+A is infinite) is consistent provided that ZF is consistent

Developement of ZFA

Recall

A set S is *transitive* if $\forall x (x \in S \rightarrow x \subseteq S)$
The ordinal numbers are representatives of well-ordered sets, ordinal α is the set of all smaller ordinals $\alpha = \{\beta : \beta < \alpha\}$

- Ordinals in ZFA do not contain Atoms
- A transitive set does not necessarily contain $\emptyset$

Kernel

Define rank of sets

For any set S, let $\mathcal{P}^{\alpha}(S)$ defined as

$$\begin{aligned}
\mathcal{P}^{0}(S) &= \emptyset \\
\mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^{\alpha}(S) \cup \mathcal{P}(\mathcal{P}^{\alpha}(S)) \\
\mathcal{P}^{\alpha}(S) &= \bigcup_{\beta<\alpha} \mathcal{P}^{\beta}(S) \\
\mathcal{P}^{\infty}(S) &= \bigcup_{\alpha\in On} \mathcal{P}^{\alpha}(S)
\end{aligned}$$

- We then have $V = \mathcal{P}^{\infty}(A)$
- The class $\mathcal{P}^{\infty}(\emptyset)$ is called the *kernel*
- all ordinals are in the kernel

Part 1

- Set theory with atoms, ZFA
- Changes to regular ZF
- Consistency of ZFA (Problem 1)
- Developement of ZFA
- Kernel

Motivation Permutation models

Idea behind Permutation models

Axioms of ZFA do not distinguish between the atoms, use them to construct models in which the set A has no well-ordering.

Reminder

The Well-ordering Principle and the Axiom of Choice are equivalent, [1, p. 10] Theorem 2.1

Definition Permutation model

Let π be a permutation of the set A. Using the hierarchy of $\mathcal{P}^{\alpha}(A)$'s, we can define πx for every x as follows :

$$\pi(\emptyset) = \emptyset, \qquad \pi(x) = \pi'' x = \{\pi(y) : y \in X\}$$

Some facts about Permutation models

(a) $x \in y \leftrightarrow \pi x \in \pi y$.

(b) $\Phi(x_1, ..., x_n) \leftrightarrow \Phi(\pi x_1, ..., \pi x_n)$.

(c) $rank(x) = rank(\pi x)$.

(d) $\pi\{x, y\} = \{\pi x, \pi y\}$, $\pi(x, y) = (\pi x, \pi y)$.

(e) If R is a relation, then πR is a relation and $(x, y) \in R \leftrightarrow (\pi x, \pi y) \in \pi R$.

(f) If f is a function on X, then πf is a function on πX and $(\pi f)(\pi x) = \pi(f(x))$.

(g) $\pi x = x$ for every x in the kernel.

(h) $(\pi * \rho)x = \pi(\rho(x))$.

Definiton Normal Filter

Normal Filter

Let $\mathscr{G}$ be a group of permutations of A. A set $\mathscr{F}$ of subgroups of $\mathscr{G}$ is a normal filter on $\mathscr{G}$ if for all subgroups H,K of $\mathscr{G}$.

(i) $\mathscr{G} \in \mathscr{F}$

(ii) if $H \in \mathscr{F}$ and $H \subseteq K$, then $K \in \mathscr{F}$

(iii) if $H \in \mathscr{F}$ and $K \in \mathscr{F}$, then $H \cap K \in \mathscr{F}$

(iv) if $\pi \in \mathscr{G}$ and $H \in \mathscr{F}$, then $\pi H \pi^{-1} \in \mathscr{F}$

(v) for each $a \in A$, $\{\pi \in \mathscr{G} : \pi a = a\} \in \mathscr{F}$

Definition Permutation Model

Transitivity

A set S is transitive if

$$\forall x(x \in S \to x \subseteq S)$$

.

Similarly, a transitive class is a class which satisfies this condition.

Symmetric of x

For each x, let

$$sym_{\mathscr{G}}(x) = \{\pi \in \mathscr{G} : \pi x = x\}$$

$sym_{\mathscr{G}}(x)$ is a subgroup of $\mathscr{G}$.
Let $\mathscr{G}$ and $\mathscr{F}$ be fixed. We say that x is *symmetric* if $sym(x) \in \mathscr{F}$

THM 4.1

Permutation Model

The class

$$\mathscr{V} = \{x : x \text{ is symmetric and } x \subseteq \mathscr{V}\}$$

consists of all *hereditarily symmetric* objects.
We all $\mathscr{V}$ a*permutation model*

Theorem 4.1

$\mathscr{V}$ is a transitive model of ZFA; $\mathscr{V}$ contains all the elements of the kernel and also $A \in \mathscr{V}$.

Definition Normal Ideal

Normal ideal

Let $\mathscr{G}$ be a group of permutations of A. A family I of subsets of A is a *normal ideal* if for all subsets E, F of A:

(i) $\emptyset \in I$

(ii) if $E \in I$ and $F \subseteq E$, then $F \in I$

(iii) if $E \in I$ and $F \in I$, then $E \cup F \in I$

(iv) if $\pi \in \mathscr{G}$ and $E \in I$, then $\pi'' E \in I$

(v) for each $a \in A$, $\{a\} \in I$

Fix

Fix

For each x, let

$$fix_{\mathscr{G}}(x) = \{\pi \in \mathscr{G} : \pi y = y \text{ for all } y \in x\},$$

$fix_{\mathscr{G}}(x)$ is a subgroup of $\mathscr{G}$.

Fix Permutation Model

Let $\mathscr{F}$ be the filter on $\mathscr{G}$ generated by the subgroups $fix_{\mathscr{G}}(E)$, $E \in I$.
$\mathscr{F}$ is a normal filter, and so it defines a permutation model $\mathscr{V}$.
Note that x is symmetric if and only if there exists $E \in I$ such that

$$fix_{\mathscr{G}}(E) \subseteq sym(x)$$

. We say that E is a *support* of x.

Well ordered ?

- Let $\mathscr{V}$ be a permutation model
- $\mathscr{V}$ contains all elements of the kernel
- this implies that the Axiom of Choice holds in the kernel

$$\forall x \in \mathscr{P}^{\infty}(\emptyset) \text{ can be well-ordered}$$

Therefore any $x \in \mathscr{V}$ can be well-ordered $\Leftrightarrow$ there exists a one-to-one mapping $f : x \to \mathscr{P}^{\infty}(\emptyset)$
But $\pi f = f \Leftrightarrow \pi \in fix_{\mathscr{G}}(x)$
Concluding together with Theorem 4.1

$$\mathscr{V} \models (\text{x can be well-ordered}) \leftrightarrow fix_{\mathscr{G}}(x) \in \mathscr{F}$$

Problem 2

Use equation above

If x can be well ordered in $\mathscr{V}$ then $\mathscr{P}(x)$ can be well ordered in $\mathscr{V}$.

Part 2

- Permutation models
- Properties/Facts of Permutation models
- Normal Filter
- Symmetric of x $sym_{\mathscr{G}}(x)$
- Theorem 4.1
- Normal Ideal
- Fix $fix_{\mathscr{G}}(x)$
- Well ordered ?
- Problem 2

Basic Fraenkel Model

Motivation

Simple example of a permutation model that does not satisfy the Axiom of Choice

- Assume that the set of Atoms A is countable (infinite)
- $\mathscr{G}$ is the group of all permuations of A
- I be the set of all finite subsets of A
- Let $\mathscr{V}$ be the corresponding permutation model
- x is symmetric if and only if there is a finite $E \subseteq A$ such that

$$\pi x = x \text{ whenever } \pi a = a \ \forall a \in E$$

- The subgroup $\mathit{fix}_{\mathscr{G}}(A)$ is not in the filter generated by $\{\mathit{fix}_{\mathscr{G}}(E) : E \subseteq A \text{finite}\}$
- For every finite $E \subset A$, one can easily find $\pi \in \mathscr{G}$ such that

$$\pi \in \mathit{fix}_{\mathscr{G}}(E) \text{ and } \pi \notin \mathit{fix}_{\mathscr{G}}(A)$$

- Together with the earlier result, it follows that the set has no well-ordering in the model $\mathscr{V}$

Theorem 4.2

The Axiom of Choice is unprovable in set theory with atoms.

Problem 3

In this model, the family $S = \{\{a, b\} : a, b \in A\}$ has no choice function. Consequently, A cannot be linearly ordered.

Second Fraenkel Model

Motivation

Constructing a model in which the Axiom of Choice fails even for countable families of pairs.

- Assume that A is countable and divide it into countably many disjoint pairs :

$$A = \bigcup_{i=1}^{\infty} P_n,\ \ P_n = \{a_n, b_n\},\ \ n = 0, 1, ...$$

- Let $\mathscr{G}$ be the group of all permutations of A which preserve the pairs P_n

$$\pi(\{a, b\}) = \{a, b\},\ \ n = 0, 1, ...$$

- Let I be the ideal of finite subsets of A $\rightarrow$ normal ideal

- A set x is symmetric if and only if there is k such that $\pi x = x$ whenever $\pi \in \mathscr{G}$ and

$$\pi a_0 = a_0, \pi b_0 = b_0, ..., \pi a_k = a_k, \pi b_k = b_k$$

- Let $\mathscr{V}$ be the permutation model determined by $\mathscr{G}$ and I. Then $\mathscr{V}$ has the following properties:
  (a) Each P_n is in $\mathscr{V}$
  (b) The sequence $\langle P_n : n \in \omega \rangle$ is in $\mathscr{V}$, thus the set $\{P_n : n \in \omega\}$ is countable
  (c) There is no function $f \in \mathscr{V}$ such that $dom(f) = \omega$ and $f(n) \in P_n$ for each n

- Thus in the model $\mathscr{V}$ there is no choice function on the countable family $\{P_n : n \in \omega\}$ and we get the following Theorem :

Theorem 4.3

The Axiom of Choice for countable families of pairs is unprovable in set theory with atoms.

Part 3

- **Basic Fraenkel Model**
- Theorem 4.2
- Problem 3
- **Second Fraenkel Model**
- Theorem 4.3

Summary

- Set Theory with Atoms, ZFA
- Normal Filters and Normal Ideal
- Permutation models in ZFA
- First and second Fraenkel model

Outlook

Chapter 4.5 in [1] The ordered Mostowski model violates the Axiom of Choice but preserves the weaker Ordering Principle.
Shows that the Axiom of Choice is independent from the Ordering Principle in ZFA.
Chapter 5 in [1] establishes these properties for 'normal' set theory, ZF.
See also [2, 4, 3] for further reading.

Sources

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