

# Independence of the Axiom of Choice, Permutation models

Seminar Mengenlehre PD. C. Gassner

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# PDF Slides

<https://github.com/Lars-B/Seminar-Mengenlehre.git>

# Outline

## Introduction

- Historical Remarks
- Definitions

## Permutation models

- Motivation
- Facts with Proof
- Definition Normal Filter
- Definition Permutation Model
- TMH 4.1
- Definition Normal Ideal

## First and Second Fraenkel Model

- Basic Fraenkel Model
- Second Fraenkel Model

## Sources and further reading

## Historical Remarks

- ▶ Permutation Models were introduced by Fraenkel 1922-37
- ▶ Precise version by Motowski 1938-39
- ▶ A Version with Filters by Specker 1957
- ▶ Additional Contributions Doss (1945), Mendelson (1948,1956), Jesenin-Vol'pin (1954), Shoenfield (1955), Fraisse (1958)

# Goal of this Talk

- ▶ This talk focuses on chapter 4 of [1] 'The axiom of Choice, Jech'
- ▶ Want to show that the Axiom of Choice is *independent* from the other Axioms
- ▶ See Chapter 5 of [1] for the result in ordinary set theory
- ▶ This talk will introduce Set theory with atoms
- ▶ Establish the independence of the Axiom of Choice in this context

# ZFA

## Recall Axiom of Choice

For every family  $\mathcal{F}$  of nonempty sets, there is a function  $f$  such that  $f(S) \in S$  for each set  $S$  in the family  $\mathcal{F}$ . We call  $f$  a *choice function* on  $\mathcal{F}$ .

## Well-ordering Principle

Also known as *Zermelo's Theorem*.

Every set can be well-ordered, i.e. for an ordering  $<$  of a set  $S$  every nonempty set  $X \subseteq S$  has a least element regarding  $<$ .

## Definition ZFA

The set theory with atoms is a modified version of set theory, and is characterized by the fact that it admits objects other than sets, *atoms*.

# ZFA

- ▶ Atoms are objects without any elements
- ▶ Language of ZFA consists of  $=$  and  $\in$
- ▶ Includes constant symbols  $\emptyset$  (empty set) and  $A$  (set of Atoms)
- ▶ *Empty set*  $\emptyset$

$$\nexists x (x \in \emptyset)$$

- ▶ *Atoms*  $A$

$$\forall z [z \in A \leftrightarrow z \neq \emptyset \wedge \nexists x (x \in z)]$$

Elements of  $A$  are called *atoms*, sets are all objects which are not atoms

# Axioms of ZF

- ▶ A1. Extensionality
- ▶ A2. Pairing
- ▶ A3. Comprehension
- ▶ A4. Union
- ▶ A5. Power-Set
- ▶ A6. Replacement
- ▶ A7. Infinity
- ▶ A8. Regularity



# Changes to the Axioms

## ► A1. Extensionality

$$(\forall \text{ set } X)(\forall \text{ set } Y)[\forall u (u \in X \leftrightarrow u \in Y) \leftrightarrow X = Y]$$

## ► instead of:

$$\forall u (u \in X \leftrightarrow u \in Y) \leftrightarrow X = Y$$

## ► A8. Regularity

$$(\forall \text{ nonempty } S)(\exists x \in S)[x \cap S = \emptyset]$$

## ► instead of

$$(\forall S \neq \emptyset)(\exists x \in S)[x \cap S = \emptyset]$$

## Note

' $X$  is nonempty' is not the same as ' $X \neq \emptyset$ ', only if  $X$  is a set.

Some options only make sense for sets:  $\cup X \quad \mathcal{P}(X)$

Some also for atoms:  $\{x, y\}$

## Problem 1

- ▶ If  $A = \emptyset$  we get ZF
- ▶ But we are more interested in the case  $A$  is not empty

### Problem 1

(ZFA+Axiom of Choice+ $A$  is infinite) is consistent provided that  
(ZF + Axiom of Choice) is consistent

# Development of ZFA

## Recall

A set  $S$  is *transitive* if  $\forall x(x \in S \rightarrow x \subseteq S)$

The ordinal numbers are representatives of well-ordered sets,  
ordinal  $\alpha$  is the set of all smaller ordinals  $\alpha = \{\beta : \beta < \alpha\}$

- ▶ Ordinals in ZFA do not contain Atoms
- ▶ A transitive set does not necessarily contain  $\emptyset$

# Kernel

## Define rank of sets

For any set  $S$ , let  $\mathcal{P}^\alpha(S)$  defined as

$$\mathcal{P}^0(S) = \emptyset$$

$$\mathcal{P}^{\alpha+1}(S) = \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S))$$

$$\mathcal{P}^\alpha(S) = \bigcup_{\beta < \alpha} \mathcal{P}^\beta(S)$$

$$\mathcal{P}^\infty(S) = \bigcup_{\alpha \in On} \mathcal{P}^\alpha(S)$$

- ▶ The class  $\mathcal{P}^\infty(\emptyset)$  is called the *kernel*
- ▶ all ordinals are in the kernel

# Part 1

- ▶ Set theory with atoms, ZFA
- ▶ Changes to regular ZF
- ▶ Consistency of ZFA (Problem 1)
- ▶ Developement of ZFA
- ▶ Kernel

# Motivation Permutation models

## Idea behind Permutation models

Axioms of ZFA do not distinguish between the atoms, use them to construct models in which the set  $A$  has no well-ordering.

## Reminder

The Well-ordering Principle and the Axiom of Choice are equivalent, [1, p. 10] Theorem 2.1

## Definition Permutation model

Let  $\pi$  be a permutation of the set  $A$ . Using the hierarchy of  $\mathcal{P}^\alpha(A)$ 's, we can define  $\pi x$  for every  $x$  as follows :

$$\pi(\emptyset) = \emptyset, \quad \pi(x) = \pi''x = \{\pi(y) : y \in X\}$$

## Some facts about Permutation models

- (a)  $x \in y \leftrightarrow \pi x \in \pi y$ .
- (b)  $\Phi(x_1, \dots, x_n) \leftrightarrow \Phi(\pi x_1, \dots, \pi x_n)$ .
- (c)  $\text{rank}(x) = \text{rank}(\pi x)$ .
- (d)  $\pi\{x, y\} = \{\pi x, \pi y\}$ ,  $\pi(x, y) = (\pi x, \pi y)$ .
- (e) If  $R$  is a relation, then  $\pi R$  is a relation and  $(x, y) \in R \leftrightarrow (\pi x, \pi y) \in \pi R$ .
- (f) If  $f$  is a function on  $X$ , then  $\pi f$  is a function on  $\pi X$  and  $(\pi f)(\pi x) = \pi(f(x))$ .
- (g)  $\pi x = x$  for every  $x$  in the kernel.
- (h)  $(\pi * \rho)x = \pi(\rho(x))$ .

# Definiton Normal Filter

## Normal Filter

Let  $\mathcal{G}$  be a group of permutations of  $A$ . A set  $\mathcal{F}$  of subgroups of  $\mathcal{G}$  is a normal filter on  $\mathcal{G}$  if for all subgroups  $H, K$  of  $\mathcal{G}$ .

- (i)  $\mathcal{G} \in \mathcal{F}$
- (ii) if  $H \in \mathcal{F}$  and  $H \subseteq K$ , then  $K \in \mathcal{F}$
- (iii) if  $H \in \mathcal{F}$  and  $K \in \mathcal{F}$ , then  $H \cap K \in \mathcal{F}$
- (iv) if  $\pi \in \mathcal{G}$  and  $H \in \mathcal{F}$ , then  $\pi H \pi^{-1} \in \mathcal{F}$
- (v) for each  $a \in A$ ,  $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathcal{F}$



# Definition Permutation Model

## Transitivity

A set  $S$  is transitive if

$$\forall x (x \in S \rightarrow x \subseteq S)$$

.

Similarly, a transitive class is a class which satisfies this condition.

## Symmetric of $x$

For each  $x$ , let

$$\text{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi x = x\}$$

$\text{sym}_{\mathcal{G}}(x)$  is a subgroup of  $\mathcal{G}$ .

Let  $\mathcal{G}$  and  $\mathcal{F}$  be fixed. We say that  $x$  is *symmetric* if  $\text{sym}(x) \in \mathcal{F}$

# THM 4.1

## Permutation Model

The class

$$\mathcal{V} = \{x : x \text{ is symmetric and } x \subseteq \mathcal{V}\}$$

consists of all *hereditarily symmetric* objects.

We call  $\mathcal{V}$  a *permutation model*

## Theorem 4.1

$\mathcal{V}$  is a transitive model of ZFA;  $\mathcal{V}$  contains all the elements of the kernel and also  $A \in \mathcal{V}$ .

# Definition Normal Ideal

## Normal ideal

Let  $\mathcal{G}$  be a group of permutations of  $A$ . A family  $I$  of subsets of  $A$  is a *normal ideal* if for all subsets  $E, F$  of  $A$ :

- (i)  $\emptyset \in I$
- (ii) if  $E \in I$  and  $F \subseteq E$ , then  $F \in I$
- (iii) if  $E \in I$  and  $F \in I$ , then  $E \cup F \in I$
- (iv) if  $\pi \in \mathcal{G}$  and  $E \in I$ , then  $\pi''E \in I$
- (v) for each  $a \in A$ ,  $\{a\} \in I$

# Fix

## Fix

For each  $x$ , let

$$\text{fix}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} : \pi y = y \text{ for all } y \in x\},$$

$\text{fix}_{\mathcal{G}}(x)$  is a subgroup of  $\mathcal{G}$ .

## Fix Permutation Model

Let  $\mathcal{F}$  be the filter on  $\mathcal{G}$  generated by the subgroups  $\text{fix}_{\mathcal{G}}(E)$ ,  $E \in I$ .

$\mathcal{F}$  is a normal filter, and so it defines a permutation model  $\mathcal{V}$ .

Note that  $x$  is symmetric if and only if there exists  $E \in I$  such that

$$\text{fix}_{\mathcal{G}}(E) \subseteq \text{sym}(x)$$

. We say that  $E$  is a *support* of  $x$ .

## Well ordered ?

- ▶ Let  $\mathcal{V}$  be a permutation model
- ▶  $\mathcal{V}$  contains all elements of the kernel
- ▶ this implies that the Axiom of Choice holds in the kernel

$\forall x \in \mathcal{P}^\infty(\emptyset)$  can be well-ordered

Therefore any  $x \in \mathcal{V}$  can be well-ordered  $\Leftrightarrow$  there exists a one-to-one mapping  $f : x \rightarrow \mathcal{P}^\infty(\emptyset)$

But  $\pi f = f \Leftrightarrow \pi \in \text{fix}_{\mathcal{G}}(x)$

Concluding together with Theorem 4.1

$$\mathcal{V} \models (x \text{ can be well-ordered}) \Leftrightarrow \text{fix}_{\mathcal{G}}(x) \in \mathcal{F}$$

## Problem 2

Use equation above

If  $x$  can be well ordered in  $\mathcal{V}$  then  $\mathcal{P}(x)$  can be well ordered in  $\mathcal{V}$ .

## Part 2

- ▶ Permutation models
- ▶ Properties/Facts of Permutation models
- ▶ Normal Filter
- ▶ Symmetric of  $x$   $\text{sym}_{\mathcal{G}}(x)$
- ▶ Theorem 4.1
- ▶ Normal Ideal
- ▶ Fix  $\text{fix}_{\mathcal{G}}(x)$
- ▶ Well ordered ?
- ▶ Problem 2

# Basic Fraenkel Model

## Motivation

Simple example of a permutation model that does not satisfy the Axiom of Choice

- ▶ Assume that the set of Atoms  $A$  is countable (infinite)
- ▶  $\mathcal{G}$  is the group of all permutations of  $A$
- ▶  $I$  be the set of all finite subsets of  $A$
- ▶ Let  $\mathcal{V}$  be the corresponding permutation model
- ▶  $x$  is symmetric if and only if there is a finite  $E \subseteq A$  such that

$$\pi x = x \text{ whenever } \pi a = a \ \forall a \in E$$



- ▶ The subgroup  $\text{fix}_{\mathcal{G}}(A)$  is not in the filter generated by  $\{\text{fix}_{\mathcal{G}}(E) : E \subseteq A \text{ finite}\}$
- ▶ For every finite  $E \subset A$ , one can easily find  $\pi \in \mathcal{G}$  such that

$$\pi \in \text{fix}_{\mathcal{G}}(E) \text{ and } \pi \notin \text{fix}_{\mathcal{G}}(A)$$

- ▶ Together with the earlier result, it follows that the set has no well-ordering in the model  $\mathcal{V}$

## Theorem 4.2

The Axiom of Choice is unprovable in set theory with atoms.

## Problem 3

In this model, the family  $S = \{\{a, b\} : a, b \in A\}$  has no choice function. Consequently,  $A$  cannot be linearly ordered.

## Second Fraenkel Model

### Motivation

Constructing a model in which the Axiom of Choice fails even for countable families of pairs.

- ▶ Assume that  $A$  is countable and divide it into countably many disjoint pairs :

$$A = \bigcup_{i=1}^{\infty} P_n, \quad P_n = \{a_n, b_n\}, \quad n = 0, 1, \dots$$

- ▶ Let  $\mathcal{G}$  be the group of all permutations of  $A$  which preserve the pairs  $P_n$

$$\pi(\{a, b\}) = \{a, b\}, \quad n = 0, 1, \dots$$

- ▶ Let  $I$  be the ideal of finite subsets of  $A \rightarrow$  normal ideal

- A set  $x$  is symmetric if and only if there is  $k$  such that  $\pi x = x$  whenever  $\pi \in \mathcal{G}$  and

$$\pi a_0 = a_0, \pi b_0 = b_0, \dots, \pi a_k = a_k, \pi b_k = b_k$$

- Let  $\mathcal{V}$  be the permutation model determined by  $\mathcal{G}$  and  $I$ . Then  $\mathcal{V}$  has the following properties:
- (a) Each  $P_n$  is in  $\mathcal{V}$
  - (b) The sequence  $\langle P_n : n \in \omega \rangle$  is in  $\mathcal{V}$ , thus the set  $\{P_n : n \in \omega\}$  is countable
  - (c) There is no function  $f \in \mathcal{V}$  such that  $\text{dom}(f) = \omega$  and  $f(n) \in P_n$  for each  $n$

- ▶ Thus in the model  $\mathcal{V}$  there is no choice function on the countable family  $\{P_n : n \in \omega\}$  and we get the following Theorem :

### Theorem 4.3

The Axiom of Choice for countable families of pairs is unprovable in set theory with atoms.

## Part 3

- ▶ **Basic Fraenkel Model**
- ▶ Theorem 4.2
- ▶ Problem 3
- ▶ **Second Fraenkel Model**
- ▶ Theorem 4.3

# Summary

- ▶ Set Theory with Atoms, ZFA
- ▶ Normal Filters and Normal Ideal
- ▶ Permutation models in ZFA
- ▶ First and second Fraenkel model

## Outlook

Chapter 4.5 in [1] The ordered Mostowski model violates the Axiom of Choice but preserves the weaker Ordering Principle.

Shows that the Axiom of Choice is independent from the Ordering Principle in ZFA.

Chapter 5 in [1] establishes these properties for 'normal' set theory, ZF.

See also [2, 4, 3] for further reading.

# Sources



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