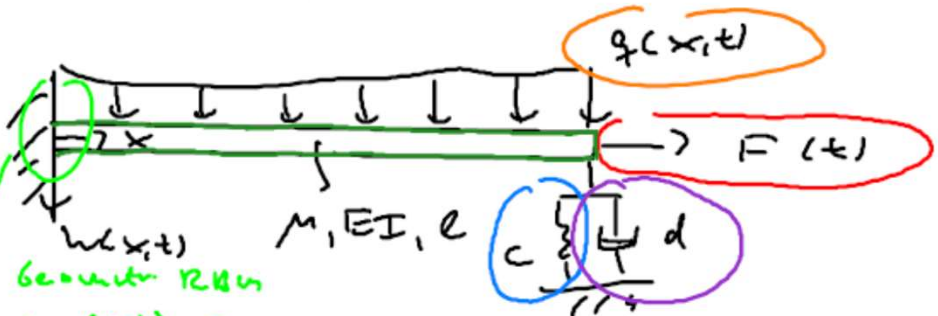
Kontinuumsmechanik VL 9

Bsp. zu Balkenschwingungen



Geometr. RBen

$w(0,t) = 0$

$w'(0,t) = 0$

$$T = \int_0^l \frac{1}{2} \mu \dot{w}^2 \, dx$$

$$U = \int_0^l \frac{1}{2} EI w''^2 \, dx + \int_0^l \frac{1}{2} F w'^2 \, dx + \frac{1}{2} c w^2(l)$$

$$\delta W = \int_0^l q \, \delta w \, dx - d \, \dot{w}(l) \, \delta w(l)$$

$$\delta \int_{t_0}^{t_1} L \, \mathrm{d}t + \int_{t_0}^{t_1} \delta W \, \mathrm{d}t = 0$$ Prinzip von Hamilton

1. Durchführen der Variation $\delta(\ldots)$:

Nach Zusammenfassen $\iint(\ldots)\,dx\,dt + \int(\ldots)\,dt$ =0

$$\int_{t_0}^{t_1}\int_0^{l}\left(m\dot{w}\,\delta\dot{w} - EI w''\,\delta w'' - F w'\,\delta w' + q\,\delta w\right)dx\,dt$$

$$+\int_{t_0}^{t_1}\left(-c\,w(l)\,\delta w(l) - d\,\dot{w}(l)\,\delta w(l)\right)dt = 0$$

2. Partielle Integration + geometrische Randbedingungen:

$$\delta\dot{w},\ \delta w',\ \delta w'' \rightarrow \delta w$$

Partielle Integration: $\int_a^b u\dot{v}\,dt = uv\Big|_a^b - \int_a^b \dot{u}\, v\,dt \qquad \int_a^b uv'\,dx = uv\Big|_a^b - \int_a^b u'v\,dx$ (1)

$$\int_{t_0}^{t_1}\int_0^{l}\left(\mu\dot w\,\delta\dot w - EI\,w''\,\delta w'' - F\,w'\,\delta w' + q\,\underset{4}{\delta w}\right)dx\,dt$$

$$+\int_{t_0}^{t_1}\left(-\underset{5}{c\,w(l)}\,\delta w(l) - \underset{6}{d\,\dot w(l)}\,\delta w(l)\right)dt = 0$$

$$\int_0^{l}\int_{t_0}^{t_1}\mu\dot w\,\delta\dot w\,dt\,dx = \underbrace{\int_0^{l}\cancel{\mu\dot w\,\delta w\Big|_{t_0}^{t_1}}\,dx}_{=0} - \int_0^{l}\int_{t_0}^{t_1}\mu\ddot w\,\underset{1}{\delta w}\,dt\,dx$$

Keine Variation
an Zeitgrenzen

$$\int_{t_0}^{t_1}\int_0^{l}\left(m\ddot{w}\,\delta w - EI w''\,\delta w'' - F w'\,\delta w' + q\,\underset{(4)}{\delta w}\right)dx\,dt$$

$$+\int_{t_0}^{t_1}\left(-\underset{(5)}{c\,w(l)}\,\delta w(l) - \underset{(6)}{d\,\dot{w}(l)}\,\delta w(l)\right)dt = 0$$

$$-\int_{t_0}^{t_1}\int_0^{l} EI w''\,\delta w''\,dx\,dt = -\int_{t_0}^{t_1} EI w''\,\underbrace{\delta w'\Big|_0^{l}}_{=0}dt + \int_{t_0}^{t_1}\int_0^{l} EI w'''\,\delta w'\,dx\,dt$$

$$= -\int_{t_0}^{t_1} EI w''\,\overset{(3)}{\underbrace{\delta w'\Big|_0^{l}}_{=0}}dt + \int_{t_0}^{t_1} EI w'''\,\overset{(7)}{\underbrace{\delta w\Big|_0^{l}}_{=0}}dt - \int_{t_0}^{t_1}\int_0^{l} \underset{(2)}{EI w^{IV}}\,\delta w\,dx\,dt$$

$$-\int_{t_0}^{t_1}\int_0^{l} F w'\,\delta w'\,dx\,dt = -\int_{t_0}^{t_1} F w'\,\overset{(8)}{\underbrace{\delta w\Big|_0^{l}}_{=0}}dt + \int_{t_0}^{t_1}\int_0^{l} \underset{(1)}{F w''}\,\delta w\,dx\,dt$$

Geometrische Randbedingungen $\quad w(0,t)=0 \quad w'(0,t)=0$

geometr. RBen $\qquad w(0,t)=0 \qquad w'(0,t)=0$

Virtuelle Verschiebungen sind verträglich mit den Bindungen

$$\delta w(0) = 0 \qquad \delta w'(0) = 0$$

3. Sortieren/Auswerten:

$$\int_{t_0}^{t_1}\int_0^l (\dots)\,\delta w\,dx\,dt + \int_{t_0}^{t_1}\left\{(\dots)\,\delta w(l) + (\dots)\,\delta w'(l)\right\}dt = 0$$

$$\int_{t_0}^{t_1}\int_0^l \underbrace{\Big(\underset{(1)}{-\mu\ddot{w}} \underset{(2)}{- EI w^{IV}} \underset{(3)}{+ F w''} \underset{(4)}{+ q}\Big)}_{\overset{!}{=}0}\delta w\,dx\,dt + \int_{t_0}^{t_1}\Big\{\overbrace{\Big(\underset{(5)}{-c w(l)} \underset{(6)}{- d\dot{w}(l)} \underset{(7)}{+ EI w'''(l)} \underset{(8)}{- F w'(l)}\Big)}^{\overset{!}{=}0}\delta w(l)$$

$$+ \Big(\underbrace{-EI w''(l)}_{\underset{\overset{!}{=}0}{(9)}}\Big)\delta w'(l)\Big\}dt = 0$$

$$\int_{t_0}^{t_1}\int_0^l \underbrace{\Big(\underset{①}{-\mu\ddot{w}}\;\underset{②}{-EI\,w^{IV}}\;\underset{③}{+F\,w''}\;\underset{④}{+q}\Big)}_{\overset{!}{=}0}\,\delta w\,dx\,dt + \int_{t_0}^{t_1}\Big\{\overbrace{\Big(\underset{⑤}{-c\,w(l)}\;\underset{⑥}{-d\,\dot{w}(l)}\;\underset{⑦}{+EI\,w'''(l)}\;\underset{⑧}{-F\,w'(l)}\Big)}^{\overset{!}{=}0}\,\delta w(l)$$

$$+\Big(\underbrace{-EI\,w''(l)}_{⑨\;\overset{!}{=}0}\Big)\,\delta w'(l)\Big\}\,dt = 0$$

δw, $\delta w(l)$, $\delta w'(l)$ sind unabhängig voneinander

$$\int\int\underbrace{(\quad)}_{\overset{!}{=}0}\delta w\,dx\,dt + \int\underbrace{(\quad)}_{\overset{!}{=}0}\delta w(l)\,dt + \int\underbrace{(\quad)}_{\overset{!}{=}0}\delta w'\,dt = 0$$

$\int\int(\quad)\,\delta w\,dx\,dt \to$ Feldgleichung: $\mu\ddot{w} - F\,w'' + EI\,w^{IV} = q$

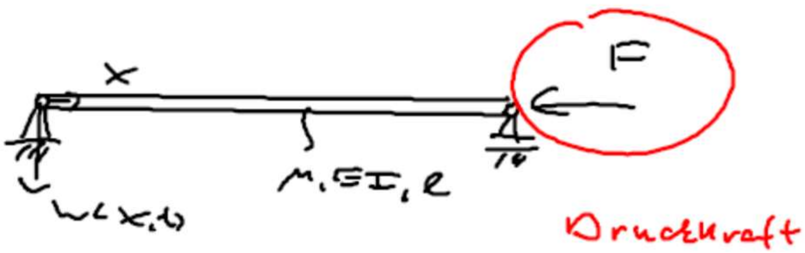
$\int(\quad)\,\delta w(l)\,dt \to$ dyn. RB: $c\,w(l) - d\,\dot{w}(l) + EI\,w'''(l) - F\,w'(l) = 0$

(Querkraft bei *x=l*)

$\int(\quad)\,\delta w'(l)\,dt \to$ dyn. RB: $EI\,w''(l) = 0$ (Moment bei *x=l*)

+ geom. RBen (von oben): $w(0) = 0$ $w'(0) = 0$

4.2.5 Knickproblem aus Sicht der Dynamik



$$T = \frac{1}{2}\int_0^\ell \mu \dot{w}^2 \, dx$$

$$U = \frac{1}{2}\int_0^\ell EI\, w''^2 \, dx + \frac{1}{2}\int_0^\ell (-F)\, w'^2 \, dx$$

$$\delta W = 0$$

1. Durchführen der Variation $\delta(\ldots)$:

$$\int_{t_0}^{t_1}\int_0^\ell \left(\mu \dot{w}\, \delta \dot{w} - EI\, w''\, \delta w'' + F\, w'\, \delta w'\right) dx\, dt = 0$$

2. Partielle Integration + geometrische Randbedingungen:

$$\int_{t_0}^{t_1}\int_0^\ell \left(-\mu \ddot{w} - EI\, w^{IV} - F\, w''\right) \underset{①}{\delta w}\, dx\, dt + \int_{t_0}^{t_1} \left\{ \underset{③}{\overset{②}{\left(-EI\, w''\, \delta w'\right)\Big|_0^\ell}} + \left(EI\, w'''\, \underset{=0}{\overset{=0}{\delta w}}\right)\Big|_0^\ell + \left(F\, w'\, \underset{=0}{\overset{=0}{\delta w}}\right)\Big|_0^\ell \right\} dt = 0$$

Geometr. RB

$\boxed{w(0,t) = w(\ell,t) = 0}$

$\leadsto$ $\delta w(0) = 0$ $\quad \delta w(\ell) = 0$

geometr. RB

$$\int_{t_0}^{t_1}\int_0^l \left(-\mu\ddot w - EI w^{IV} - F w''\right)\underset{①}{\delta w}\,dx\,dt + \int_{t_0}^{t_1}\Big\{\left(-EI w''\,\underset{③}{\overset{②}{\delta w'}}\right)\Big|_0^l + \underset{=0}{\overset{=0}{\left(EI w'''\,\delta w\right)\Big|_0^l}} + \underset{=0}{\overset{=0}{\left(F w'\,\delta w\right)\Big|_0^l}}\Big\}\,dt=0$$

Geometr. RB

$$\boxed{w(0,t) = w(l,t) = 0}$$ geometr. RBn

$\hookrightarrow$ $\delta w(0) = 0 \qquad \delta w(l) = 0$

3. Sortieren/Auswerten:

$\underset{①}{\delta w}, \quad \underset{②}{\delta w'(l)}, \quad \underset{③}{\delta w'(0)}$ alle unabhängig

① Feldgleichung: $\boxed{\mu\ddot w + F w'' + EI w^{IV} = 0}$ (Saite mit vorg. Vorsp. + Balken)

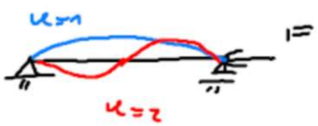
② dyn. RBen $\boxed{w''(0) = w''(l) = 0}$

③

RWP vgl. 4. Ordn. + 4 RBen

Lösung des Randwertproblems:

$$\text{Ansatz:} \quad w(x,t) = \overbrace{\underbrace{\sin k\pi \frac{x}{\ell}}_{\text{Eigenform}}}^{W(x)} \cdot \overbrace{\sin \omega_k t}^{p(t)} \qquad k = 1, 2, \dots$$



↳ Eigenkreisfrequenz ω_k, unbekannt

$W(x) = \sin k\pi \frac{x}{\ell}$ erfüllt alle RBen $\big(W(0) = W(\ell) = 0$, $W''(0) = W''(\ell) = 0 \big)$

In Feldgleichung einsetzen:

$$\ddot{w}(x,t) = \sin k\pi \frac{x}{\ell} \, (-\omega_k)^2 \sin \omega_k t$$

$$w^{IV}(x,t) = \left(\frac{k\pi}{\ell}\right)^4 \sin k\pi \frac{x}{\ell} \, \sin \omega_k t$$

$$\left(-\mu \omega_k^2 - F \left(\frac{k\pi}{\ell}\right)^2 + EI \left(\frac{k\pi}{\ell}\right)^4 \right) \cancel{\sin k\pi \frac{x}{\ell}} \; \cancel{\sin \omega_k t} = 0$$

Auflösen nach ω_k:

$$\omega_k^2 = \frac{EI \left(\frac{k\pi}{\ell}\right)^4 - F \left(\frac{k\pi}{\ell}\right)^2}{\mu}$$

F Druckkraft

$$\omega_k^2 = \frac{EI\left(\frac{k\pi}{l}\right)^4 - F\left(\frac{k\pi}{l}\right)^2}{\mu}$$

- Negatives F (hier Zugkraft) erhöht ω_k
- Positives F (hier Druckkraft) verkleinert ω_k

Im Grenzfall wird $\omega_1^2 = 0$; daraus

$$F_{krit} = EI\left(\frac{\pi}{l}\right)^2$$

Dies entspricht der in der Festigkeitslehre für diesen Fall berechneten Knicklast.