Estimating Group Delay and Phase Delay via Discrete-Time "Analytic" Cross-Correlation

S. Lawrence Marple, Jr.

***Abstract*— Starting with two finite N-point real-valued discrete-time signals, frequency-domain algorithm procedures are provided for estimating the group delay and phase delay between the signals. These delays are extracted from the complex-valued discrete-time "analytic" cross-correlation sequence computed using the two signals. Special adjustments of cross-correlation transform end points are shown to be necessary in order to generate a proper "analytic" cross-correlation sequence. Two cases are considered for delay estimation 1) to sample-interval resolution and 2) to subsample-interval resolution by interpolation.**

I. Introduction

This correspondence was motivated by the desire of the author to understand the source of biases in the estimate of time delays between finite sampled signal records. Using first principles as described in this correspondence, it was discovered that the most common time-delay estimation technique found in most textbooks (a transform approach that typically assumes infinite data records and symmetrically-shaped waveforms [even functions]) fails to apply appropriate adjustment factors to account for finite data records and arbitrarily-shaped waveforms. Application of these adjustment factors to the transform end points will correct the biases that have been observed. Group delay and phase delay are estimated from the envelope and phase of the "analytic" form of the cross-correlation sequence obtained from the two sampled data sequences. The correlation envelope is not assumed here to be a symmetric function as is commonly found in most textbooks. Time-domain approaches for estimating the "analytic" cross-correlation sequence have been presented in the literature [3], [4], but this correspondence will employ a frequency-domain approach, where the correction factors are most easily applied, to exploit the computational efficiency of FFT's.

This correspondence builds on concepts and similar corrections cited in a companion correspondence [5] for computation of complex-valued discrete-time "analytic" signals from finite-duration real-valued signal sequences. As emphasized in the companion correspondence [5], only infinite-duration continuous-time real-valued signals can have complex-valued analytic signals in a formal mathematical sense. A discrete-time complex-valued "analytic" signal for finite-duration discrete-time real-valued signals can be defined if 1) certain portions of the periodic frequency-domain transform are zeroed, and 2) orthogonality is maintained between the real and imaginary components of the complex-valued discrete-time "analytic" signal. Quotes are used with the word "analytic" to emphasize that these conditions are required in order to form a discrete-time analytic-like signal sequence.

II. Continuous-Time Group and Phase Delays from Cross-Correlation Functions

Let $x(t)$ be a real-valued finite energy continuous-time bandpass signal defined over the temporal interval $-\infty < t < \infty$ with continuous-time Fourier transform (CTFT) $X(f)$ that is bandlimited to a frequency interval B Hz wide centered at frequency f_c. This bandpass signal is assumed to have an asymmetrical spectrum about the center frequency for generality, as illustrated in Fig. 1(a), although the spectrum about 0 Hz is complex conjugate symmetric $X(-f) = X^*(f)$. Note that a lowpass signal is represented as a special case by selecting $f_c = B/2$.

The real-valued bandpass signal $x(t)$ may be represented by frequency shifts of a complex-valued baseband signal $b(t) = b^r(t) + jb^i(t)$ with Fourier transform $B(f)$ centered at 0 Hz, as shown in Fig. 1(b). Specifically, the representation is $X(f) = B(f - f_c)/2 + B^*(-f - f_c)/2$. Here, $b^r(t)$ is the real part of the complex baseband signal, and $b^i(t)$ is the imaginary part. Based on this frequency-domain definition, the inverse transform yields the the relationship $x(t) = b^r(t)\cos(2\pi f_c t) - b^i(t)\sin(2\pi f_c t)$. Most textbooks make the assumption that the baseband signal $b(t)$ is narrowband with respect to the center frequency f_c in order to be assured that each sideband shown in Fig. 1(a) does not spillover (alias) across 0 Hz into the other sideband. This is an unnecessarily restrictive assumption. The only condition for the representation of a bandpass signal by a baseband signal is the prevention of aliasing, specifically $B/2 < f_c$. With modern digital filter designs, it is possible to create wideband filters with sharp band edges so that bandpass signals with bandwidths of over 100% of the center frequency are often attainable.

Consider the real-valued bandpass signal $y(t)$ derived from $x(t)$ by imposing an attenuation factor A, a time delay t_d, and a center frequency phase shift θ_c. These three factors are common in propagating signal situations, such as radar, in which free-space propagation loss accounts for A, propagation delay accounts for t_d, and reflection of the signal off an object introduces a phase shift θ_c in the reflected propagating wave. Define the baseband transform $G(f) = A\exp(-j[2\pi f t_d + \theta_c])$, which is nonzero over the interval $-B/2 \le f \le B/2$, with magnitude and phase spectra as shown in Fig. 1(c) and (d). The transform that accounts for the attenuation factor, time delay, and phase shift effect on a bandpass signal centered at f_c will have the form $H(f) = G(f - f_c) + G^*(-f - f_c)$. Therefore, signal $y(t)$ has Fourier transform $Y(f) = H(f)X(f)$ or

$$Y(f) = \frac{1}{2}G(f - f_c)B(f - f_c) + \frac{1}{2}G^*(-f - f_c)B^*(-f - f_c). \quad (1)$$

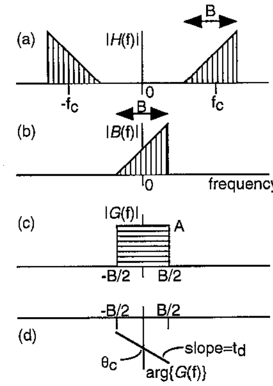

Fig. 1. Real bandpass signal and its complex baseband representation. (a) Magnitude spectrum of real bandpass signal centered at f_c. (b) Magnitude spectrum of complex baseband signal centered at 0 Hz. (c) Magnitude spectrum of baseband linear phase filter. (d) Phase spectrum of baseband linear phase filter.

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The author is with Orincon Corporation, San Diego, CA 92121 USA (e-mail: marple@orincon.com).

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The inverse transform then yields the attenuated, time-delayed, phase-shifted signal as

$$\begin{aligned} y(t) = &Ab^r(t-t_d)\cos(2\pi f_c[t-t_d]+\theta_c) \\ &- Ab^i(t-t_d)\sin(2\pi f_c[t-t_d]+\theta_c). \end{aligned} \tag{2}$$

Note that $y(t)$ could also have been created by passing $x(t)$ through a linear-phase bandpass filter of uniform magnitude response. In this case, A is interpreted as the filter gain, t_d is interpreted as the filter group delay, and θ_c is interpreted as the filter phase delay.

For general filter transform $H(f) = A(f)\exp\{j\theta(f)\}$, the common definitions for group delay t_g and phase delay t_p [1], [7] are

$$t_g = -\frac{\theta'(f_c)}{2\pi} \quad \text{and} \quad t_p = -s\frac{\theta(f_c)}{2\pi f_c} \tag{3}$$

where $\theta'(f_c)$ is the derivative with respect to f evaluated at f_c. For either the propagating signal case or the linear-phase bandpass filter case, $\theta(f) = -2\pi[f-f_c]t_d - \theta_c$. This yields the relationships $t_g = t_d$ and $t_p = \theta_c/(2\pi f_c)$. Thus, the group delay corresponds to either the propagation delay or the filter delay, whereas the phase delay corresponds to the time delay equivalent of the phase shift centered at propagation/filter frequency f_c.

Computing the analytic cross-correlation function is one method to obtain values for A, t_d, and θ_c. To illustrate this computational process, consider first the transform $C_{yx}(f)$ of the cross-correlation function $c_{yx}(t) = \int_{-\infty}^{\infty} y(\tau+t)x(\tau)\,d\tau$, which may be expressed using the correlation theorem of Fourier analysis as

$$\begin{aligned} C_{yx}(f) &= Y(f)X^*(f) \\ &= \frac{A}{2}\exp(-j2\pi[f-f_c]t_d - j\theta_c)|B(f-f_c)|^2 \\ &\quad + \frac{A}{2}\exp(-j2\pi[f+f_c]t_d + j\theta_c)|B(-f-f_c)|^2. \end{aligned} \tag{4}$$

Define $C_{\mathrm{bb}}(f) = |B(f)|^2$ as the Fourier transform of the complex-valued baseband signal autocorrelation $c_{\mathrm{bb}}(t)$, which can be expressed in terms of either its real and imaginary components or its polar components

$$c_{\mathrm{bb}}(t) = c_{\mathrm{bb}}^r(t) + jc_{\mathrm{bb}}^i(t) = e_{\mathrm{bb}}(t)\exp\{j\phi_{\mathrm{bb}}(t)\}. \tag{5}$$

Here, $e_{\mathrm{bb}}(t) = [c_{\mathrm{bb}}^r(t)^2 + c_{\mathrm{bb}}^i(t)^2]^{1/2}$ is the real, positive envelope function, and $\phi_{\mathrm{bb}}(t) = \arctan\{c_{\mathrm{bb}}^i(t)/c_{\mathrm{bb}}^r(t)\}$ is the phase function of the baseband autocorrelation. An autocorrelation function must be complex conjugate symmetric $c_{\mathrm{bb}}(-t) = c_{\mathrm{bb}}^*(t)$ and have its maximum value at $t=0$. This implies that $c_{\mathrm{bb}}^r(t)$ is an even real-valued function, and $c_{\mathrm{bb}}^i(t)$ is an odd real-valued function so that $c_{\mathrm{bb}}^r(0) = e_{\mathrm{bb}}(0) = c_{\mathrm{bb}}(0)$, $c_{\mathrm{bb}}^i(0) = 0$, and $\phi_{\mathrm{bb}}(0) = 0$.

These properties of the baseband autocorrelation make it possible to express the cross-correlation function $c_{yx}(t)$ as either

$$\begin{aligned} c_{yx}(t) = &Ac_{\mathrm{bb}}^r(t-t_d)\cos(2\pi f[t-t_d]+\theta_c) \\ &- Ac_{\mathrm{bb}}^i(t-t_d)\sin(2\pi f[t-t_d]+\theta_c) \end{aligned} \tag{6}$$

or

$$c_{yx}(t) = Ae_{\mathrm{bb}}(t-t_d)\cos(2\pi f[t-t_d] + \phi_{\mathrm{bb}}(t-t_d) + \theta_c). \tag{7}$$

The cross-correlation function at $t = t_d$ reduces to

$$c_{yx}(t_d) = Ae_{\mathrm{bb}}(0)\cos(\theta_c) \tag{8}$$

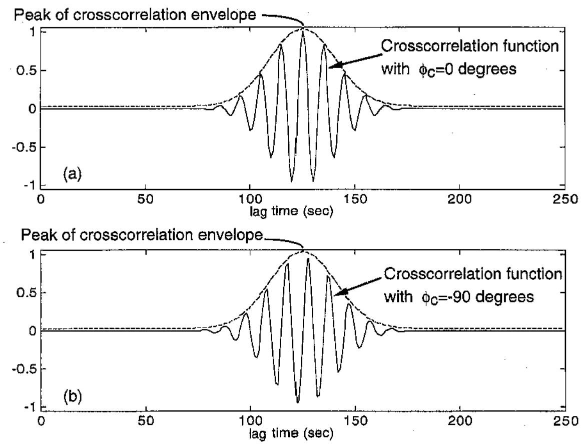

Fig. 2. Illustration of the effect of the center frequency phase shift on the cross-correlation function and its envelope. (a) Envelope peak and cross correlation peak coincide when $\theta = 0$. (b) Cross correlation peak offset from envelope peak when $\theta \neq 0$.

in which the envelope $Ae_{\mathrm{bb}}(0)$ is at its maximum value, but the cross correlation function itself is not at a maximum value unless $\theta_c = 0$ in the phase function $\cos(\theta_c)$. This effect is illustrated in Fig. 2 for the case of a bandpass modulated Gaussian pulse with 125-s delay between x and y. Thus, we cannot rely strictly on the cross-correlation function $c_{yx}(t)$ peak value to determine the actual time delay t_d.

A method to compute the envelope and phase functions of cross-correlation $c_{yx}(t)$ in (7) is to generate the analytic cross-correlation function $c_{yx}^a(t)$ that is defined most simply in the frequency domain as

$$C_{yx}^a(f) = \begin{cases} 2Y(f)X^*(f), & \text{for } f > 0 \\ Y(0)X^*(0), & \text{for } f = 0 \\ 0, & \text{for } f < 0. \end{cases} \tag{9}$$

An inverse transformation will then yield the complex-valued $c_{yx}^a(t)$. For the specific case of a bandpass signal

$$C_{yx}^a(f) = A\exp(-j2\pi[f-f_c]t_d - j\theta_c)|B(f-f_c)|^2 \tag{10}$$

and therefore, the inverse transform is

$$c_{yx}^a(t) = 2Ae_{\mathrm{bb}}(t-t_d)\exp(-j2\pi f[t-t_d] - j\phi_{\mathrm{bb}}(t-t_d) - j\theta_c). \tag{11}$$

The magnitude (envelope function) of the analytic cross-correlation $|c_{yx}^a(t)| = 2Ae_{\mathrm{bb}}(t-t_d)$ will have a peak at $t = t_d$, which yields the propagation (group) delay value. Once the group delay is determined, the center frequency phase shift is determined by evaluating the argument (phase function) of the analytic cross-correlation at the group delay time

$$\arctan\{\mathrm{Im}[c_{yx}^a(t_d)]/\mathrm{Re}[c_{yx}^a(t_d)]\} = -\theta_c. \tag{12}$$

In order to compute the phase delay, as defined in (3), a value for the center frequency is needed. This may be obtained by inspection of the magnitude spectrum of either $|Y(f)|$ or $|X(f)|$, which are computed as part of the formulation of (9) in order to generate the analytic cross-correlation function. The center frequency is simply the average between the observed lower and upper frequencies that define the band edges. For the special case of a symmetric bandpass spectrum, the center frequency may alternatively be obtained as the frequency location of the spectral peak.

III. Estimating Discrete-Time Cross-Correlation Sequence via FFT

Assume that finite N sample data records of the signals $x[n] = x(nT)$ and $y[n] = y(nT)$ are available for cross correlation, where T is the sampling interval. From these data records, a discrete-time cross-correlation sequence $c_{yx}[m] = c_{yx}(mT)$ for time lags $-(N-1) \le m \le (N-1)$ can be generated, for a total of $2N-1$ cross-correlation terms evaluated at a temporal resolution of T-second intervals. Thus, there are twice as many cross-correlation sequence values as there are original signal sample values.

The correlation theorem of discrete-time Fourier analysis [2] may be used to develop a fast computational procedure for the cross-correlation sequence based on the FFT as follows:

- Compute the $2N$-point discrete-time Fourier transforms (DTFT's) using the N-point data sequences $x[n]$ and $y[n]$, which have been zero padded with N zeros each as

$$\begin{aligned} \mathrm{X}[k] &= T \sum_{n=0}^{N-1} x[n] \exp(-j2\pi kn/2N) \\ \mathrm{Y}[k] &= T \sum_{n=0}^{N-1} y[n] \exp(-j2\pi kn/2N) \end{aligned} \tag{13}$$

  over the frequency index interval $0 \le k \le 2N-1$.
- Form the $2N$-point discrete-time cross-correlation transform sequence

$$\mathrm{C}_{yx}[k] = \mathrm{Y}[k]\mathrm{X}^*[k] \quad \text{for } 0 \le k \le 2N-1. \tag{14}$$

- Compute the $2N$-point inverse DTFT

$$\begin{aligned} c_{yx}[m] &= \frac{1}{2NT} \sum_{k=0}^{2N-1} \mathrm{C}_{yx}[k] \exp(+j2\pi mk/2N) \\ &= T \sum_{n=0}^{N-1-|m|} y[n+m]x[n] \end{aligned} \tag{15}$$

  to yield the real-valued $(2N-1)$-point cross-correlation sequence $c_{yx}[m]$ with temporal resolution of T seconds (the $2N$-th point is a zero value).

The following Matlab *common code fragment* will be used in all procedures of this correspondence:

```
X=fft([x;zeros(N,1)]); Y=fft([y;zeros(N,1)]);
C_yx=Y.*conj(X);
```

The following additional Matlab code fragment completes the three steps that compute the standard real-valued discrete-time cross-correlation sequence:

```
c_yx=fftshift(ifft(C_yx))*T;
```

Rarely do the delays to be estimated coincide with one of the evaluated cross-correlation terms, and the cross-correlation sequence needs to be interpolated to finer temporal resolution for better estimation of the group and phase delays. The following trigonometric interpolation procedure [5], [6] will interpolate the cross-correlation sequence from $2N$ terms to $2NM$ terms, where M is the interpolation factor:

- Compute the $2N$-point DTFT's $\mathrm{X}[k]$ and $\mathrm{Y}[k]$ using the N-point data sequences that have been zero padded with N zeros each.
- Form the $2NM$-point split and zero-padded transform sequence

$$\mathrm{C}_{yx}[k] = \begin{cases} \mathrm{Y}[k]\mathrm{X}^*[k], & 0 \le k \le N-1 \\ \frac{1}{2}\mathrm{Y}[N]\mathrm{X}^*[N], & k = N \\ 0, & N+1 \le k \le 2NM-N-1 \\ \frac{1}{2}\mathrm{Y}[N]\mathrm{X}^*[N], & k = 2NM-N \\ \mathrm{Y}[n]\mathrm{X}^*[n], & N+1 \le n \le 2N-1 \\ & k = n+2N(M-1) \end{cases} \tag{16}$$

- Compute the $2NM$-point inverse DTFT and scale by M to yield the $(2NM-1)$-point cross-correlation sequence $c_{yx}[m]$ with temporal resolution of T/M seconds.

The Matlab code fragment at the bottom of the page plus the common fragment computes the interpolated discrete-time cross-correlation sequence. Note that the scaling by factor M compensates for the $1/M$ factor used in the computation of the inverse DTFT.

IV. Estimating Discrete-Time Signal Group and Phase Delays to Sample Interval Resolution

The following procedure will generate estimates of the group and phase delays from finite sampled data records based on an estimate of the discrete-time "analytic" cross-correlation sequence:

- Compute the $2N$-point DTFT's $\mathrm{X}[k]$ and $\mathrm{Y}[k]$ using the N-point data sequences, which have been zero padded with N zeros each.
- Form the $2N$-point discrete-time "analytic" cross-correlation transform sequence

$$\mathrm{C}^a_{yx}[k] = \begin{cases} \mathrm{Y}[0]\mathrm{X}^*[0], & k = 0 \\ 2\mathrm{Y}[k]\mathrm{X}^*[k], & 1 \le k \le N-1 \\ \mathrm{Y}[N]\mathrm{X}^*[N], & k = N \\ 0, & N+1 \le k \le 2N-1 \end{cases} \tag{17}$$

- Compute the $2N$-point inverse DTFT to yield the complex-valued $(2N-1)$-point "analytic" cross-correlation sequence $c^a_{yx}[m]$ with temporal resolution of T seconds.
- Equate the group delay estimate to $t_g = m'T$, where m' is the time index that maximizes $|c^a_{yx}[m]|$.
- Compute the phase delay estimate as

$$t_p = \arg\{\mathrm{Im}(c^a_{yx}[m'])/\mathrm{Re}(c^a_{yx}[m'])\}/2\pi f_c. \tag{18}$$

The first Matlab code fragment shown at the top of the next page plus the common fragment computes estimates of the group and phase delays to a temporal resolution of T seconds. The motivation in the above procedure for not using a factor of 2 with the terms that define $\mathrm{C}^a_{yx}[k]$ at $k=0$ and $k=N$ was covered in a companion correspondence [5] that discussed proper procedures for generating discrete-time "analytic" signals. The heuristic explanation is the observation that the transform values at DC and Nyquist (foldover) frequencies are shared at the boundary frequencies of the periodic spectra of sampled data signals, and therefore, half of the value of these terms go to each of the adjoining spectral replicants. These adjustments to the transform values also correct for bias effects observed in prior procedures that apply a factor of 2 to all $C^a_{yx}[k]$ terms, including those at $k=0$ and $k=N$. The procedure provided

```
C_yx_interp=[C_yx(1:N);C_yx(N+1)/2;zeros((M-1)*2*N-1,1),C_yx(N+1)/2;C_yx(N+2:2*N)];
c_yx_interp=fftshift(ifft(C_yx_interp))*M*T;
```

```
C_yx_anal=[C_yx(1);2*C_yx(2:N);C_yx(N+1);zeros(N-1,1)];
c_yx_anal=fftshift(ifft(C_yx_anal))*T;
[peak,m]=max(abs(c_yx_anal));
t_g=(m-N-1)*T;
t_p=arg(imag(c_yx_anal(m),real(c_yx_anal(m))/(2*pi*f_c);
```

```
C_yx_interp_anal=[C_yx(1);2*C_yx(2:N);C_yx(N+1);zeros(2*M*N-N-1,1)];
c_yx_interp_anal=fftshift(ifft(C_yx_interp_anal))*M*T;
[peak,m]=max(abs(c_yx_interp_anal));
t_g=(m-M*N-1)*(T/M);
t_p=arg(imag(c_yx_interp_anal(m),real(c_yx_interp_anal(m))/(2*pi*f_c);
```

here works particularly well when there is a DC component in the signal; there is no need to eliminate the DC component by subtracting the sample mean from the data values prior to transformation.

V. Estimating Discrete-Time Signal Group and Phase Delays to Sub-Sample Interval Resolution

In order to obtain subsample temporal resolution of T/M seconds, the discrete-time "analytic" cross-correlation sequence needs to be interpolated by a factor of M. The procedure of Section IV is modified by simply zero padding the the transform sequence $C_{yx}[k]$ with $2N(M-1)$ additional zeros. The second Matlab code fragment shown at the top of the page, together with the common fragment, will compute estimates of the group and phase delays to a temporal resolution of T/M seconds.

VI. Conclusion

This correspondence has shown that we cannot simply use the continuous-time infinite-duration analytic formulation when creating a discrete-time analytic-like cross correlation from a discrete-time finite-duration signal for use in group and phase delay estimation. When estimated using a frequency-domain approach, the DC and Nyquist frequency terms of the discrete-time Fourier transform of the crosscorrelation each require a scaling factor that is half that of the other transform terms, which is not required in or anticipated by the continuous-time case. This is a subtle difference that has eluded distinction in the DSP community. Two cases were elucidated here. One provides a resolution of the delay estimates to within the original sample interval, and the second provides an interpolated resolution to a subsample time interval.

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