

Polynomial Computer Algebra and implementation of Wilf-Zeilberger's method

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Overview

- 1 Introduction
- 2 Wilf-Zeilberger's method (Wilf, 1990)
- 3 Gosper's algorithm (Gosper, 1978)
- 4 Implementation
- 5 Results
- 6 Discussion and conclusions

What is the thesis about?

Polynomial Computer Algebra and implementation of
Wilf-Zeilberger's method

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What and Why?

WHAT

Show that

$$\sum_{k=n}^{\infty} \frac{1}{\binom{k}{n}} = \frac{n}{n-1}.$$

Characteristics

- Summation on one side.
- Show that...
- Often binomial coefficients.

WHY

- Wilf-Zeilberger's method → not so much
- Automized proof generation → a lot
- Computer Algebra → a lot

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Short version of the thesis

- Historical background
- Polynomials
- Wilf-Zeilberger's method
- Gosper's algorithm
- Results
- Conclusions

Important findings

- 1960s: Computer Algebra
- 1978: Gosper's Algorithm
- 1990: Wilf-Zeilberger's method
- 1994: WZ implemented in Mathematica

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- Used for implementation of WZ
- Polynomial

$$p(k) = a_0 + a_1k + \dots + a_mk^m$$

is stored as

$$[a_0, a_1, \dots, a_m]$$

- Coefficients a_i can be integers or polynomials

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- Used to prove identities on the form

$$\sum_k F(n, k) = 1$$

- Does this by proving

$$\sum_k F(n+1, k) = \sum_k F(n, k)$$

- Which is done by "changing variables"

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- An algorithm to find a function S such that

$$a_k = S_k - S_{k-1}$$

- Finds the change of variables needed in WZ

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- Historical background
 - Polynomials
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 - Results
 - Conclusions
- The program writes formal proofs
 - Proves 80% of the examples
 - The remaining seem impossible to prove by WZ method

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- Historical background
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 - Results
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- The program seems to work well, although cannot solve all examples
 - Computer Algebra quickly gets complicated

What problems can be solved?

Problems on the form

$$\sum_k F(n, k) = 1$$

can be solved. Problems on the form

$$\sum_k A(n, k) = B(n)$$

can get converted to the right form.

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Want to prove

$$\sum_k F(n, k) = 1$$

Find $G(n, k)$ such that $F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k)$
and $\lim_{k \rightarrow \pm\infty} G(n, k) = 0$. Now

$$\sum_k F(n+1, k) - F(n, k) = \sum_k G(n, k+1) - G(n, k) = 0.$$

Therefore

$$\sum_k F(n, k)$$

is constant for all n , therefore if we can evaluate for one n then we are done.

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Steps of the method

- ① Start with

$$\sum_k A(n, k) = B(n)$$

- ② Let $F(n, k) = \frac{A(n, k)}{B(n)}$
- ③ Find $G(n, k)$ such that the conditions are satisfied
- ④ Show that $\sum_k F(n', k) = 1$ for some n'

- ① We have

$$\sum_k \binom{n}{k} = 2^n$$

- ② $F(n, k) = \frac{\binom{n}{k}}{2^n}$
- ③ Let $G(n, k) = -\frac{\binom{n}{k-1}}{2^{n+1}}$. Now the conditions are satisfied.
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What problems can be solved?

Given a polynomial a_k , Gosper's algorithm finds a polynomial S_k such that

$$a_k = S_k - S_{k-1}.$$

With $a_k = F(n+1, k) - F(n, k)$ we get that $G(n, k) = S_{k-1}$ makes the first condition in Wilf-Zeilberger's method fulfilled.

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Steps of the algorithm

1 Find polynomials

p_k, q_k, r_k such that

$$\gcd(q_k, r_{k+j}) = 1$$

$\forall j \geq 0$ and

$$\frac{a_k}{a_{k-1}} = \frac{p_k}{p_{k-1}} \frac{q_k}{r_k}$$

2 Find polynomial f_k such that

$$p_k = q_{k+1} f_k - r_k f_{k-1}$$

3 Let $S_k = \frac{q_{k+1}}{p_k} f_k a_k$

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Now we see that

$$\begin{aligned} S_k - S_{k-1} &= \frac{q_{k+1}}{p_k} f_k a_k - \frac{q_k}{p_{k-1}} f_{k-1} a_{k-1} = \\ &= \frac{a_k}{p_k} \left(q_{k+1} f_k - \frac{q_k}{p_{k-1}} f_{k-1} p_k \frac{a_{k-1}}{a_k} \right) = \\ &= \frac{a_k}{p_k} \left(q_{k+1} f_k - \frac{q_k}{p_{k-1}} f_{k-1} p_k \frac{p_{k-1}}{p_k} \frac{r_k}{q_k} \right) = \\ &= \frac{a_k}{p_k} \left(q_{k+1} f_k - r_k f_{k-1} \right) = \frac{a_k}{p_k} p_k = a_k, \end{aligned}$$

which means that this S_k indeed is a solution.

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- ① For $\sum_k \binom{n}{k} = 2^n$ we get

$$\frac{a_k}{a_{k-1}} = \frac{(2k-n-1)(n+2-k)}{k(2k-n-3)}$$

which gives us $p_k = 2k - n - 1$,

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- ② In

$$2k - n - 1 = (n + 1 - k)f_k - kf_{k-1}$$

we see that $f_k = -1$ gives a solution.

- ③ Now we get $S_k = -\frac{n+1-k}{2k-n-1} a_k = -\frac{\binom{n}{k}}{2^{n+1}}$, which corresponds to the $G(n, k)$ we got in the previous example.

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Results as statistics

- 10 examples for training, 10 for validation
- The automatic solver manages to prove 8 of each
- The remaining examples seem to be unsolvable using Wilf-Zeilberger's method

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Results as an example

Proof

Automatic WZ-method prover

2019-11-25

$k < 1$ and
 $k > n + 1$
 gives that

$$\binom{n}{k-1} = 0,$$

 \Rightarrow

$$\lim_{k \rightarrow \pm \infty} G(n, k) = 0.$$

We want to prove that

$$\sum (-1)^k \cdot \binom{n}{k} \binom{2k}{k} 4^{n-k} = \binom{2n}{n} \quad (1)$$

holds. By dividing equation 1 by the right hand side we get

$$F(n, k) = \frac{(-1)^k \cdot \binom{n}{k} \binom{2k}{k} 4^{n-k}}{\binom{2n}{n}} \quad (2)$$

We use proof certificate

$$R(n, k) = \frac{2k-1}{2n+1}, \quad (3)$$

which is the same as using

$$G(n, k) = \frac{2k-1}{2n+1} \cdot \frac{(-1)^{k-1} \cdot \binom{n}{k-1} \binom{2(k-1)}{k-1} 4^{n-(k-1)}}{\binom{2n}{n}}, \quad (4)$$

the automatic solver has verified that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \quad (5)$$

Thereafter user now has to verify that

$$\lim_{k \rightarrow \pm \infty} G(n, k) = 0 \quad \forall n. \quad (6)$$

Then we get

$$\sum_k F(n+1, k) - F(n, k) = \sum_k G(n, k+1) - G(n, k) = 0 \quad (7)$$

Lastly equation 1 needs to be verified for some n , for instance $n = 0$. Thereafter the identity is shown.For $n = 0$ we get

$$\sum_k (-1)^k \binom{n}{k} \binom{2k}{k} 4^{n-k} =$$

$$(-1)^0 \binom{0}{0} \binom{0}{0} 4^0 = 1$$

Also $\binom{0}{0} = 1$.

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the automatic solver has verified that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k). \quad (5)$$

Thereafter user now has to verify that

$$\lim_{k \rightarrow \pm \infty} G(n, k) = 0 \quad \forall n. \quad (6)$$

Then we get

$$\sum_k F(n+1, k) - F(n, k) = \sum_k G(n, k+1) - G(n, k) = 0 \quad (7)$$

Lastly equation 1 needs to be verified for some n , for instance $n = 0$. Thereafter the identity is shown.

For $n = 0$ we get

$$\sum_k (-1)^k \binom{n}{k} \binom{2k}{k} 4^{n-k} =$$

$$(-1)^0 \binom{0}{0} \binom{0}{0} 4^0 = 1$$

Also $\binom{0}{0} = 1$.

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- Cannot come up with solution, only prove
- Some parts are left for the user
- Similar examples with different results

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- Solves most examples
- Gives a solution quickly (in seconds)

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Future work

- Can Wilf-Zeilberger's method be used in other fields? (not binomial coefficients)
- Combine the program with guessing solution to identity
- Computer algebra in general

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Thank you for listening!

Polynomials – Representation

Polynomial

$$p(k) = a_0 + a_1k + \dots + a_mk^m$$

is stored as

$$[a_0, a_1, \dots, a_m].$$

Polynomials 1 – Example

The polynomial

$$p(k, m) = 1 + k^2 + km - m^2 + km^2 + k^2m^2$$

is stored as

$$\left[[1, 0, 1], [0, 1, 0], [-1, 1, 1] \right].$$

Polynomials – Addition

Assume we want to add $f = [f_0, \dots, f_{m_f}]$ and $g = [g_0, \dots, g_{m_g}]$. Then we get $h = [h_0, \dots, h_m]$ where $m = \max(m_f, m_g)$. Then we have that

$$h_i = f_i + g_i,$$

if f_i and g_i are of one and the same variable. Otherwise we get

$$h_i = \text{ADD}(f_i, g_i).$$

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Polynomials – Division

Usually division (a divided by b) is done by finding polynomials q, r such that

$$a = q \cdot b + r,$$

and $\deg(r) < \deg(b)$. This is not possible in integer coefficients. Therefore we use q, r, f such that

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Polynomials – GCD

We get \gcd by Euclid's algorithm. With division as

$$a = q \cdot b + r$$

\gcd is usually done by

$$\gcd(a, b) = a \text{ if } b = 0 \text{ else } \gcd(b, r).$$

With division as

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we get \gcd by

$$\gcd(a, b) = a \text{ if } b = 0 \text{ else } \frac{g}{\bar{g}} \gcd(\bar{a}, \bar{b}),$$

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Proof generation

Steps of proof generation

- Get input and parser
- Parse input \rightarrow get $F(n, k)$ and $\frac{a_k}{a_{k-1}}$
- Get $G(n, k)$ from Gosper's algorithm
- Write proof in \LaTeX format
- Highlight parts that the user need to complete

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Dependencies of the code

